Solution of Pressure-Displacement Systems for Fluid Structure Interaction Problems

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SUMMARY

The finite element formulation of fluid-structure interaction problems in terms of fluid pressure and structural displacement is considered. The resulting system of equations is the unsymmetric form presented by Zienkiewicz and Newton in 1969. Instead of using a scheme to transform the system equations into a symmetric form which is appropriate to any computer code, this paper chooses the implementation of unsymmetric equation and eigen-procedures to solve the problems directly.

The implementation of an equation solver for the unsymmetric pressure-displacement system equations is found to be straightforward, since there is no pivoting required during the Gaussian elimination. The eigen-problem for the fluid-structure system amounts to the solution of a generalized eigenvalue problem $Ax = \lambda Bx$, where A and B are unsymmetric real matrices. Because the eigen-pairs associated with the fluid-structure system are always real and the eigenvalues are positive semi-definite, a special procedure can be developed to extract the eigen-solution. The eigenvalue procedure introduced in this paper is the real form of the QZ-algorithm developed by Moler and Stewart. The basic approach is to transform the A and B matrices into upper Hessenberg and upper triangular forms respectively, before iterative transformations are used to make both A and B upper triangular. The application of the procedure is illustrated by a numerical example. The free vibration of a beam in a fluid chamber filled with compressible or incompressible fluid is analyzed. The natural frequencies determined from the finite element method are presented, along with the analytic solution.

1. Introduction

The finite element formulation of fluid-structure interaction problems where the fluid is represented by a linear acoustic medium was first presented by Zienkiewicz and Newton [1] in 1969. By taking the fluid pressure and the structural displacement as nodal variables, they arrived at an unsymmetric system of equations. Due to the lack of symmetry, this formulation does not receive wide application in general purpose finite element codes. In recent years, there has been a growing interest in solving fluid-structure interaction problems in industry. Research in finite element solution, using the pressure-displacement formulation, has also been oriented toward the recasting of the system equations in a symmetric form which is appropriate to any finite element code [2,3,4]. Many of the procedures suggested today involve complicated matrix manipulations, which are not easily adapted into a general purpose finite element code.

In this paper, the solution of the fluid-structure interaction problems is re-examined using the unsymmetric formulation originated by Zienkiewicz and Newton. A finite element code will then require an unsymmetric equation solver, as well as an unsymmetric eigen-routine suitable for the generalized eigenvalue problem $Ax = \lambda Bx$, where A and B are unsymmetric matrices. The implementation of an unsymmetric equation solver is straightforward for the fluid-structure system, since there is no pivoting required during elimination. A symmetric solver can easily be extended to an unsymmetric one with only slight modification in the elimination process. Unlike the equation solver, the eigen-routine for an unsymmetric system is much more complicated than that for a symmetric one, since the eigen-pairs are generally complex and many solution algorithms suffer from computational inefficiencies and/or numerical instabilities. Fortunately, since the eigen-pairs associated with the fluid-structure system are always real and the eigenvalues are always non-negative, a special, simpler procedure such as the real form of the QZ-algorithm [5] can be introduced to extract the eigen-solution.

The application of the above procedures is illustrated by a numerical example. The free vibration of a beam submerged in a fluid chamber is considered. The natural frequencies of the beam in both compressible and incompressible fluids are presented.

2. Governing Equations for Fluid-Structure Systems

Consider a fluid-structure system in which the fluid can be represented as a linear acoustic medium. In terms of the perturbation pressure p, the following wave equation is satisfied

$$\nabla^2 p = \frac{1}{c^2}\ddot{p} \tag{1}$$

The boundary conditions specified on the free surface and the fluid-structure interface are

$$p = 0 \tag{2}$$

$$\partial p/\partial n = -\rho\ddot{\delta}_n \tag{3}$$

where c is the sound speed, ρ is the density, n is the interface normal (outward from the fluid), and $\ddot{\delta}_n$ is the structural acceleration in the n-direction.

Following the finite element formulation [1] by choosing the fluid pressure and the structure displacement as nodal variables, the following eigen-problem describing the free vibration of an undamped fluid-structure system can be obtained.

$$\left(\left[\begin{array}{c|c} K & -L \\ \hline 0 & H \end{array} \right] - \omega^2 \left[\begin{array}{c|c} M & 0 \\ \hline \rho L^T & Q \end{array} \right] \right) \left\{ \begin{array}{c} \delta \\ \hline p \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \hline 0 \end{array} \right\} \tag{4}$$

where $\{\delta\}$ and $\{\ddot{\delta}\}$ are the structural displacement and acceleration vectors; [M] and [K] are the mass and stiffness matrices; {p} is the fluid pressure vector; [Q] and [H] are compliance and susceptance matrices; [L] is the interface area matrix; and ω is the circular frequency. The resulting system of equations (4) is not symmetric and a special eigenvalue subroutine is needed for the solution.

If the fluid is incompressible, i.e., c approaches infinity and [Q] = [0], eq. (4) reduces to the added mass fluid-structure system described by

$$[K]\,\{\delta\} = \omega^2\,([M] + [M^*])\,\{\delta\} \tag{5}$$

where [M*] is the symmetric hydrodynamic mass determined by

$$[M^*] = \rho[L][H]^{-1}[L]^T \tag{6}$$

provided that the matrix is invertible. In cases where the fluid is entirely surrounded by structures the matrix [H] is singular and the determination of the hydrodynamic mass can follow a modified procedure given by Yu [6].

3. QZ Algorithm for Eigen-Solution

From the theory of matrix algebra, the normal mode analysis of an acousto-elastic vibration (4) represents a generalized eigenvalue problem of the form

$$[A]\,\{x\} = \lambda[B]\{x\} \tag{7}$$

where [A] and [B] are real unsymmetric matrices. The eigenvalue λ and eigenvector {x} for such systems are generally complex. The solution algorithms for such problems are often inefficient and may also suffer from numerical instability. Fortunately, the eigen-problem (4) representing an acousto-elastic vibration can be transformed into a symmetric form by matrix manipulations that yield [1]

$$\left[\begin{array}{c|c} K & 0 \\ \hline 0 & Q/\rho \end{array} \right] \left\{ \begin{array}{c} \delta \\ \hline p \end{array} \right\} = \omega^2 \left[\begin{array}{c|c} M+M^* & LH^{-1}Q \\ \hline QH^{-1}L^T & QH^{-1}Q/\rho \end{array} \right] \left\{ \begin{array}{c} \delta \\ \hline p \end{array} \right\} \tag{8}$$

This will guarantee that the eigenvalues and eigenvectors are real. Moreover, since the two coefficient matrices are all positive semi-definite, the eigenvalues will be non-negative.

With these subsidiary conditions, a special eigenvalue procedure can be introduced to extract the solution of eq. (4). Although the symmetric form of eq. (8) appears to be a perfect expression for an eigenvalue problem, the computational scheme needed to arrive at such an expression is not easily implemented into a large scale computer program.

The special eigenvalue procedure chosen for the solution of acouto-elastic vibration problems as described by eq. (4) is the QZ algorithm developed by Moler and Stewart [5]. In reference to eq. (7), and with the restriction of the eigen-solution to a real space, the QZ algorithm can be described by a sequence of matrix transformations as follows: firstly, the matrices A and B are transformed into upper Hessenberg and upper triangular forms by QA and QB operations, where Q represents a sequence of Householder orthogonal matrices. Secondly, the Hessenberg-triangular forms of the two matrices are transformed iteratively into upper triangular matrices through QAZ and QBZ operations, where Q and Z represent other sequences of Householder matrices. Finally, the eigenvalues and eigenvectors are computed from the two triangular matrices.

4. Free Vibration of a Beam in a Fluid Chamber

Consider the free vibration of a beam submerged in a fluid chamber as shown in Figure 1. Let the beam be simply supported at the top and rotationally restrained at the bottom. For the problem data shown in Table 1, the natural frequencies of the beam in a vacuum and in both compressible and incompressible fluids will be determined.

4.1 Analytic Solution

Consider the normal mode vibration of a beam in a compressible fluid shown in Figure 1. Let the displacement of the beam and the pressure of the fluid be represented by $u(y)\cos\omega t$ and $p(x,y)\cos\omega t$. The conditions that u and p must satisfy are the following:

Differential equation of the beam:

$$EI(d^4u/dy^4) - \rho_s A\omega^2 u = p_{x=0^-} - p_{x=0^+} \tag{9}$$

Boundary conditions of the beam:

$$u(0) = (du/dy)_{y=h} = (d^3u/dy^3)_{y=h} = 0 \tag{10}$$

Differential equation of the fluid:

$$\partial^2 p/\partial x + \partial^2 p/\partial y^2 + (\omega/c)^2 p = 0 \tag{11}$$

Fluid boundary conditions:

$$p(x,0) = (\partial p/\partial x)_{x=\pm a} = (\partial p/\partial y)_{y=h} = 0 \tag{12}$$

Fluid-beam interface conditions:

$$(\partial p/\partial x)_{x=0^+} = (\partial p/\partial x)_{x=0^-} = \rho\omega^2 u \tag{13}$$

where $x=0^-$ and $x=0^+$ imply the left and right surfaces of the beam.

The eigen-solution of the problem can be obtained by assuming that

$$u = B\sin(\beta y) \tag{14}$$

$$p = R[\cosh(\alpha x) - \tanh(\alpha a)\sin(\alpha x)]\sin(\beta y) \tag{15}$$

for $x \geqslant 0$, and

$$p = L[\cosh(\alpha x) + \tanh(\alpha a)\sin(\alpha x)]\sin(\beta y) \tag{16}$$

for $x \leqslant 0$, and in which

$$\beta = (2n-1)\pi/2h, \quad n = 1,2,3,... \tag{17}$$

$$\alpha^2 = \beta^2 - (\omega/c)^2 \tag{18}$$

Upon the substitution of eqs. (14) - (16) into (9) through (13) the following frequency equation can be obtained:

$$\omega^2 = \beta^4 EI/(\rho_s A + \rho A^*) \tag{19}$$

The same frequency equation will also express the in-vacuo and the in-incompressible fluid vibrations if A* takes the following definitions.

$$A^* = \begin{cases} 2\coth(\alpha a) & \text{for compressible fluids} \\ 2\coth(\beta a) & \text{for incompressible fluids} \\ 0 & \text{in vacuum} \end{cases} \tag{20}$$

4.2 Finite Element Solution

Because of the symmetry of the two fluid compartments about the beam, only one side of the fluid is considered in the finite element model. The thickness of the fluid is doubled to account for the effect from the other side. The finite element model for the beam in a compressible fluid is shown in Figure 2. The fluid region is modelled by 32 6-node pressure elements, and the structure is modelled by 16 2-node beam elements. At the fluid-structure interface there are eight 3-node interface elements to couple the motion between the two. Using the WECAN [7] computer program to assemble the elements, an eigenvalue problem as shown in (4) can be obtained. By implementing the QZ algorithm in the program, both the frequencies and the mode shapes are determined. The results will be discussed later.

If the fluid is assumed to be incompressible, the normal modes of vibration can be determined using the WECAN computer code. First, the fluid and the fluid-structure interface are modelled by pressure and interface elements which form the following system of equations [6].

$$\left[\begin{array}{c|c} 0 & -L \\ \hline -L^T & -H/\rho \end{array}\right] \left\{\begin{array}{c} \ddot{\delta} \\ \hline p \end{array}\right\} = \left\{\begin{array}{c} f \\ \hline 0 \end{array}\right\} \tag{21}$$

Performing a substructural analysis by retaining only the accelerational degrees-of-freedom in (21), the reduced coefficient matrix is the hydrodynamic mass defined by eq. (6). Secondly, the substructural hydrodynamic mass is incorporated into the structural model for the normal analysis. The natural frequencies and mode shapes are determined using the Householder-QR procedure.

The finite element analysis is also performed for the beam vibration in a vacuum. Without any fluid-structure interaction, the acousto-fluid modes in a rigid compartment are also determined.

4.3 Analytical and Finite Element Results

The natural frequencies determined for the beam in a vacuum and for the fluid in a rigid compartment are shown in Table 2. The analytic result for the fluid is obtained by finding the non-trivial solution for eq. (15) when the pressure gradient normal to the beam is zero. For the first four frequencies, the finite element results are in excellent agreement with the analytic solution.

With consideration of fluid-structure coupling, the natural frequencies determined for the beam in the compressible and incompressible fluids are shown in Table 3. The frequencies determined for the compressible case are generally lower than those for the incompressible case, but for the problem considered the fluid compressibility doesn't show an appreciable effect for the first two modes. In comparing the finite element results with the analytic solution, the agreement between the two is remarkable for the first four frequencies.

By comparing the results in Tables 2 and 3, the vibration of the beam in a fluid shows a substantial drop in natural frequencies compared to the beam in-vacuo.

5. Conclusion

It was mentioned earlier that the finite element solution of a fluid-structure interaction problem using the unsymmetric pressure-displacement formulation may prove to be simple and efficient. By introducing the real form of the QZ-algorithm as an eigen-procedure a numerical example was presented. It should be noted that the QZ-procedure (in complex space) can also be used in solving a more general class of fluid-structure interaction problems describing flow-induced vibration.

The use of an unsymmetric equation solver is applicable to steady state and transient dynamic fluid-structure interaction problems. Due to the lack of space, no numerical examples are presented.

References

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TABLE 1 - PROBLEM DATA

GEOMETRY			
Chamber Depth (or beam span)		h	$=$ 120 in
Compartment width		a	$=$ 30 in
Beam cross-sectional area		A	$=$ 4 in^2
MECHANICAL PROPERTIES			
BEAM:	Young's modulus	E	$= 3 \times 10^7$ psi
	Density	ρ_s	$= 5 \times 10^{-4}$ $\text{lb-sec}^2/\text{in}^4$
	Moment of inertia	I	$=$ 16/3 in^4
FLUID:	Density	ρ	$= 8.22 \times 10^{-5}$ lb-sec /in^4
	Sound speed	c	$=$ 42,718 in/sec

TABLE 2 - NATURAL FREQUENCIES (Hz)

(Beam and Fluid are Uncoupled)

Mode	Beam		Fluid	
	Analytic	WECAN	Analytic	WECAN
1	7.7133	7.7133	88.996	88.996
2	69.420	69.420	266.99	267.01
3	192.83	192.84	444.98	445.21
4	377.95	378.01	622.97	624.15

TABLE 3 - NATURAL FREQUENCIES (Hz)

(Beam and Fluid are Coupled)

Mode	Beam in Incompressible Fluid		Beam in Compressible Fluid	
	Analytic	WECAN	Analytic	WECAN
1	2.0564	2.0564	2.0564	2.0564
2	38.317	38.339	38.142	38.164
3	128.01	128.41	126.40	126.78
4	274.32	276.79	267.52	269.72

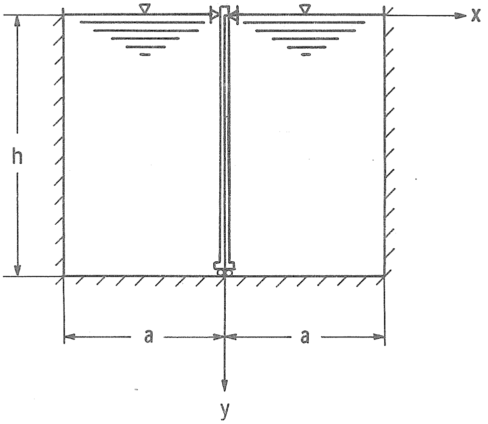

Fig. 1 — A beam in a fluid chamber

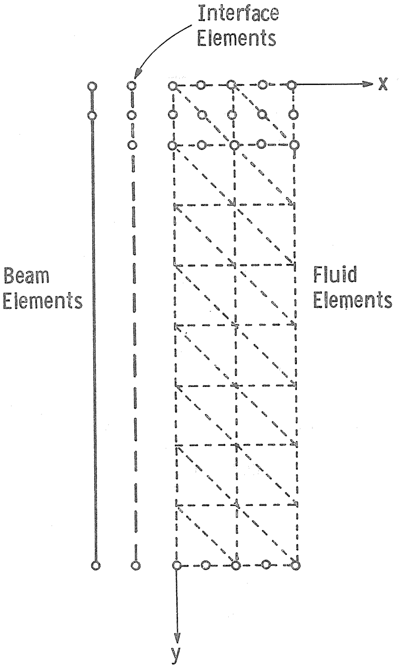

Fig. 2 — Finite element model