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Abstract

This paper analyzes a discrete-time, finite capacity, deterministic service time queueing system ($MMBP + MMBP^{[X]}/D(N)/1/K$) with superposing non-renewal arrival processes. This queueing system models a switch in high speed networks, where multiple applications which generate correlated and bursty traffic are supported simultaneously using fast packet switching. A designated arrival stream (VC-stream) which is modeled as a Markov Modulated Bernoulli Process (MMBP) is considered for analyzing the effect of the network traffic characteristics on its performance measures. The aggregate of the remaining arrival streams (EXT-stream) is modeled as an MMBP with batch arrivals. Using exact analysis we derive the queue length distribution at the VC-stream arrival instants, and also obtain the interdeparture time distribution of the VC-stream.

1 Introduction

Discrete-time queueing models have received considerable attention in communications modeling due to the increasing focus on issues related to high speed networks. Future Broadband Integrated Services Digital Networks (B-ISDNs) will support data, voice and video applications that generate traffic which is typically bursty and highly correlated ([12, 3, 11]). Complex, non-renewal processes are now needed to characterize these traffic streams, in contrast to the Poisson process which has traditionally been used to model teletraffic.

The Markov Modulated Bernoulli Process (MMBP), a non-renewal process which quantitatively models the time-varying behavior of arrival rates, and MMPP (Markov Modulated Poisson Process, the continuous time analogue of MMBP) have been used extensively to model traffic in communication networks. These processes capture the burstiness of a source and also some of the important correlations between the interarrival times, while still remaining analytically tractable. The Interrupted Poisson Process (IPP) a special case of MMPP, has long been used to model the overflow traffic in finite trunk systems ([5]). These processes can also be used to represent superposed traffic streams. For example, the superposition of packetized voice and related processes has been accurately characterized by an MMPP ([12, 3]). Fischer and Meier-Hellstern summarize a number of useful results on queueing models involving MMPP in [2]. A review of discrete-time models of superposing traffic streams can be found in [10] and [11].

MMPP is a special case of the Batch Markovian Arrival Process (BMAP), presented by Lucantoni in [6]. The BMAP in a simple generalized notation represents a rich class of point processes and is equivalent to Neut's versatile Markovian point process ([8]). Practical solutions of BMAP/G/1 queueing system are presented using matrix analytic approach in [6]. These results in algorithmic form can be applied to study many queueing systems of interest. A method is outlined in [2] to obtain per-stream performance measures in a superposition of many independent streams, each of which is represented by an MMPP. The superposition of these streams is modeled as a single aggregate MMPP which is obtained using Kronecker

sums. From the results of MMPP/G/1 (a special case of BMAP/G/1) system the per-stream quantities are then obtained using simple matrix manipulations. This method does not directly extend itself to discrete-time systems. If we represent the cumulative arrivals from all sources as a batch MMBP (denoted by $MMBP^{[X]}$) we can use the results from [6] to analyze the system since $MMBP^{[X]}$ is a special case of discrete-time BMAP. However, no method is known to obtain per stream quantities from this analysis.

An analysis of a discrete-time system with deterministic service time, where a GI-stream with general and independent arrivals, and a Bernoulli batch arrival process($M^{[X]}$) are superposed, is performed by Murata et al in [7]. They derive the queue length distribution for both finite and infinite buffer cases to study the effect of GI-stream arrival statistics on network performance. Ohba et al in [9] studied an extension of this system for an infinite buffer case, where they also consider a third arrival stream which represents the aggregate of many identical Interrupted Bernoulli Processes(IBPs). They obtain the interdeparture time distribution of the GI-stream and present approximate analysis of tandem queueing networks using decomposition. However, neither of these analyses account for correlation in the arrival streams. A recent paper by Herrmann [4] considers correlated arrival processes in a superposition. He considers superposition of two Discrete Markovian Arrival Processes (DMAPs) as a Semi-Markov Process ($SMP^{[2]}$) and provides exact analysis of the $DMAP + DMAP/D/1/K$ queue to obtain per-stream quality of service (QOS) parameters. He extends the solution to the $DMAP^{[X]} + DMAP/D/1/K$ queue where batch arrivals occur in one of the arrival streams. We consider a similar queueing system but use an alternate approach.

In this paper we present an exact analysis of a discrete-time finite capacity queueing system with superposing non-renewal arrival processes which finds applications in high speed network modeling. In order to study the effect of the arrival characteristics of network traffic on the performance of a single arrival stream, we model the queueing system as having two arrival processes. One of the arrival processes represents the designated traffic stream of interest, referred to as VC-stream (for Virtual Circuit stream), and the other process,

referred to as EXT-stream (for External traffic), represents the cumulative arrivals from the remaining sources. A Markov Modulated Bernoulli Process (MMBP) is used to model the VC-stream in order to capture the burstiness and correlation of the traffic. The EXT-stream inherits the burstiness and correlation between packet arrivals from its component streams ([3, 12]). Also, an arrival from the EXT-stream represents an arrival from one or more individual streams i.e., multiple arrivals can occur in the same time slot. To capture these properties of the aggregate process we model the EXT-stream as an $MMBP^{[z]}$. The queueing model analyzed can be denoted as $MMBP + MMBP^{[X]}/D(N)/1/K$ system. The analysis presented in this paper can be considered as an extension of those presented in [7] and [9]. We consider a finite buffer system where the arrival processes are bursty as well as correlated. We analyze the system assuming that a packet (which is an arriving unit) takes N units of server time for processing ($N \geq 1$). This queueing model simplifies to represent an ATM cell scale model as in [7] and [9] when $N = 1$.

This paper is organized as follows. In section 2, we describe the analytical model and the assumptions made. In section 3, by exploiting the Markov renewal property of an MMBP, we present an exact analysis to derive the steady state probability distribution of the number of packets in system (which we will refer to as NOPS in our discussion), as seen by the arrivals from the VC-stream. In section 4, we use the distribution of number of packets in system to obtain the interdeparture time distribution of the VC-stream packets. In section 5, we present some numerical results obtained from our analysis for different traffic parameters. We make our concluding remarks in section 6.

2 Analytical Model

In this section we present the defining parameters and the assumptions of the discrete-time, finite buffer capacity, single server queueing system. The queueing model is shown in Figure 1. The packets from the two streams arrive at a common buffer of size K (packets). A packet is discarded if it arrives when the buffer is full. If packets arrive from both the streams in

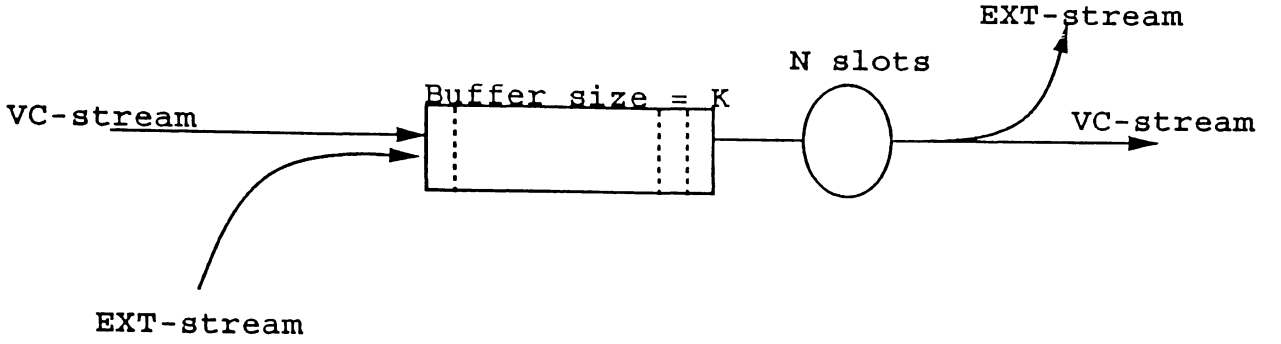


Figure 1: The queuing model

the same slot, then the VC-stream packet is given priority for buffer allocation. An arriving packet is not considered for service in the same slot i.e., a packet arriving in a slot which finds the server to be idle, will begin to receive service only in the next slot. Each packet takes N slots of service time and is processed in FIFO order of its placement in the buffer. A packet departs at the slot boundary.

The VC-stream traffic is modeled as a Markov Modulated Bernoulli Process (MMBP). An MMBP is a Bernoulli process where the arrival rate is varied according to an m -state Markov chain. This variation using state changes in an MMBP helps capture the notion of burstiness and correlation properties of the arrival stream. An MMBP is characterized by the state-transition probability matrix P_t of the Markov chain and the arrival rate descriptor Λ defined as follows:

$$P_t = \begin{bmatrix} p_{11} & & p_{1m} \\ & \ddots & \\ p_{m1} & & p_{mm} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix},$$

where p_{ij} , $1 \leq i, j \leq m$, is the transition probability that the state changes from i to j and $\sum_{j=1}^m p_{ij} = 1$ for $1 \leq i \leq m$. α_i is the probability of having an arrival in a slot when the MMBP is in state i , $1 \leq i \leq m$.

We characterize the burstiness of an arrival process by the squared coefficient of variation

of interarrival time, C^2 . It is defined as the ratio of variance over square of the mean interarrival time. By varying p_{ij} and α_i ($1 \leq i, j \leq m$), we can have different traffic loads and at the same time change the burstiness and correlation of the arrival process. Given certain values of load, burstiness and autocorrelation we can obtain the inter-arrival time distribution of the VC-stream arrival process conditioned on the MMBP state of the previous arrival (used in this form in our derivations) which is given as

$$a_v(v', j) = \Pr[\text{next VC-stream arrival occurs in MMBP state } v' \text{ and after } j \text{ slots} \mid \text{a VC-stream arrival occurred when MMBP state was } v].$$

The derivation of $a_v(v', j)$ for a 2-state MMBP and the definitions of parameters such as burstiness and autocorrelation are given in Appendix A.

The EXT-stream traffic is modeled as a Markov Modulated Bernoulli Process with batch arrivals ($MMBP^{[x]}$). An m state Markov chain is used for the MMBP as described above for VC-stream traffic. Each arrival brings into the system a batch of packets. The batch size (no. of packets in an arrival) follow a general distribution. Given that the MMBP is in a particular state x , the probability density of the number of packets arriving in that slot $b_x(j) = \Pr[EXT\text{-stream packets arriving in a slot} = j \mid MMBP \text{ state} = x]$, $j \geq 0$, is given as

$$b_x(j) = \begin{cases} 1 - \alpha_x & \text{if } j = 0 \\ \alpha_x s(j) & \text{otherwise} \end{cases}$$

where α_x is the probability of having an arrival when the EXT-stream MMBP is in state x and $s(j), j > 0$ is the probability density that the batch size is j . For a geometric distribution the density function is given as $s(j) = (1 - x)^{j-1}x$ where $j > 0$, $0 < x < 1$ and $x = \frac{1}{\text{Mean Batch Size}}$.

In the analysis we use the following constants: ST_V and ST_X represent the number of states in the VC-stream and EXT-stream MMBP respectively.

3 Distribution of number of packets in system seen by a VC-stream arrival

We derive the probability density function (p.d.f.) f_{NOPS} , of the *number of packets in system* (NOPS) seen by a VC-stream arrival. We use f_{NOPS} to obtain the delay distribution, loss probability and the inter-departure time distribution of VC-stream packets.

The NOPS in a given slot depend upon the NOPS in the preceding slot and also the arrival and departure events in the preceding slot. We present relations between the distribution of NOPS in a slot and that of the preceding slot. We use these relations to obtain the distribution of NOPS seen by a VC-stream arrival, from the distribution of NOPS seen by the previous VC-stream arrival. This is discussed in detail in the later part of this section. If the VC-stream arrival process is considered as a renewal process with the arrival epochs (T_n) as renewal points, then the interarrival times ($T \in \{I_n : I_n = T_n - T_{n-1}, n \in IN\}$) are identically and independently distributed (i.i.d). Hence, if steady state exists then the distribution of NOPS seen is the same for any VC-stream arrival and

$$f_{NOPS} = f_{NOPS} \mathbf{P}.$$

where array \mathbf{P} is stochastic with elements $p_{i,j}$ which represent the probability that at steady state the NOPS seen by a successive VC-stream arrival is j given that the *NOPS* seen by a VC-stream arrival is i . We solve for f_{NOPS} using power iterations (making use of relations between distribution of NOPS seen by two consecutive VC-stream arrivals) and without having to compute \mathbf{P} . This method is used in [7] and [9] where a GI process (a renewal process) is used for VC-stream arrivals.

In order to determine the probability distribution of the NOPS seen by a VC-stream arrival for the case of an MMBP arrival process we make use of the Markov renewal property of the MMBP ([1, 2]). We know that an MMBP is not a *renewal process* but a *Markov renewal process*: the distribution of the time between $(n-1)^{st}$ arrival and n^{th} arrival depends on the MMBP state at the times of the $(n-1)^{st}$ and n^{th} arrivals. The sequence $(V, T) =$

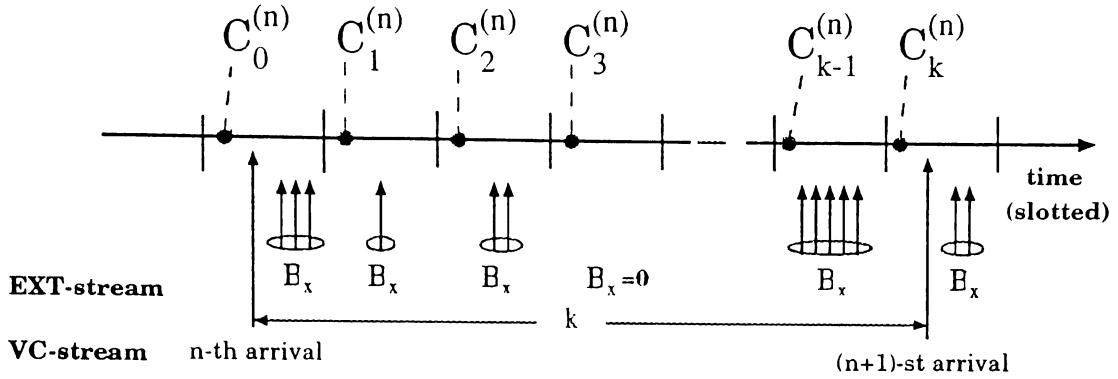


Figure 2: Relating system states observed by two consecutive VC-stream arrivals

$\{(V_n, T_n); n \in \mathbb{N}\}$, is a Markov renewal process, where V is the MMBP state and T is the arrival epoch. The interarrival time $I_{v,v'}$ between consecutive arrivals where v is the state of the MMBP at the first arrival and v' is the state at the second arrival, clearly represents a renewal process. To derive f_{NOPS} we need to consider the MMBP states for the VC-stream arrival process. Therefore, we consider the conditional interarrival time p.d.f. $a_v(v', k)$ (defined in section 2). In the remainder of this section we describe the method to obtain steady-state distribution of NOPS seen by a VC-stream arrival.

We are interested in relating the NOPS at the n^{th} arrival instant to the NOPS at the $(n+1)^{st}$ arrival instant of VC-stream packets. The random variable for NOPS alone does not completely represent the state of the system at an arrival instant. We define a random variable $C_k^{(n)}$ to represent the system state in the k^{th} slot following the n^{th} VC-stream arrival given that the $(n+1)^{st}$ VC-stream arrival does not occur in the preceding $k-1$ slots. $C_k^{(n)}$ is a 4-tuple, $C_k^{(n)} = (V_k^{(n)}, X_k^{(n)}, S_k^{(n)}, J_k^{(n)})$, $k \geq 0, n \geq 1$, where,

$V_k^{(n)}$ is the state of the VC-stream MMBP when the n^{th} VC-stream arrival occurred.

Note that this variable doesn't change with k .

$X_k^{(n)}$ is the state of the EXT-stream MMBP in the k^{th} slot following the n^{th} VC-stream arrival.

$S_k^{(n)}$ is the state of the server in the k^{th} slot following the n^{th} VC-stream arrival. The

server is either in an idle state ($S_k^{(n)} = 0$) or is in a busy state given by a positive value $S_k^{(n)} = i$, which indicates that the current slot is the i^{th} service slot for the packet in service. A departure occurs at the end of slot k if $S_k^{(n)}$ is equal to N .

$J_k^{(n)}$ is the number of packets in system (NOPS) in the beginning of the k^{th} slot following the n^{th} VC-stream arrival.

$C_0^{(n)}$ represents the system state as seen by n^{th} arrival of the VC-stream. We will determine the distribution of $C_0^{(n)}$. To do this, we need to relate the system states in two consecutive slots. The events of interest in representing the change in system state between two consecutive slots (slot $k-1$ and slot k) are (i) the VC-stream packet arrival in slot $k-1$ if $k=1$, (ii) a batch arrival of packets from the EXT-stream in slot $k-1$, (iii) the server state change at the end of slot $k-1$, (iv) a departure from the system at the end of the slot $k-1$ and (v) the EXT-stream MMBP state change at the beginning of slot k .

We consider the effect of these events on the system state in two phases:

phase (1): we consider the arrivals which occur in the slot $k-1$

phase (2): the state transitions at the slot boundary between slots $k-1$ and k and the departure (if $S_{k-1}^{(n)} = N$) at the end of slot $k-1$.

The order in which these events are considered is shown in figure 3. In the figure, $\bar{C}_{k-1}^{(n)}$ is the system state after phase (1). $\bar{C}_{k-1}^{(n)} = (\bar{V}_{k-1}^{(n)}, \bar{X}_{k-1}^{(n)}, \bar{S}_{k-1}^{(n)}, \bar{J}_{k-1}^{(n)})$ where,

$$\bar{V}_{k-1}^{(n)} = V_{k-1}^{(n)}, \quad \bar{X}_{k-1}^{(n)} = X_{k-1}^{(n)}, \quad \bar{S}_{k-1}^{(n)} = S_{k-1}^{(n)} \quad (1)$$

$$\bar{J}_{k-1(=0)}^{(n)} = \begin{cases} \min(B_x + 1, K) & \text{if } J_0^{(n)} = 0, \quad k = 1 \\ \min(J_0^{(n)} + B_x + 1, K + 1) & \text{if } J_0^{(n)} \neq 0, \quad k = 1 \end{cases} \quad (2)$$

$$\bar{J}_{k-1}^{(n)} = \begin{cases} \min(B_x, K) & \text{if } J_{k-1}^{(n)} = 0, \quad k > 1 \\ \min(J_{k-1}^{(n)} + B_x, K + 1) & \text{if } J_{k-1}^{(n)} \neq 0, \quad k > 1 \end{cases} \quad (3)$$

where $k \geq 1$, $n > 0$, K is the buffer size and B_x is the r.v. for the number of packets arriving in a slot from EXT-stream when the MMBP state is x . The p.d.f. of B_x is given by $b_x(j)$ (defined in the previous section). Equation (2) corresponds to a VC-stream arrival slot

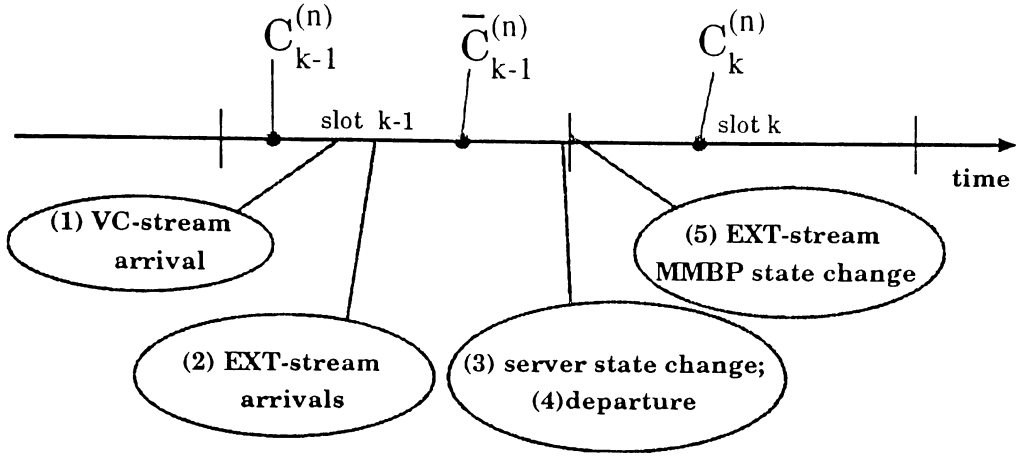


Figure 3: Relative order of events and observations in a slot

i.e., $k - 1 = 0$ where one VC-stream packet and B_x EXT-stream packets arrive. Equation (3) corresponds to slot $k - 1$ ($k > 1$) following a VC-stream arrival where B_x EXT-stream packets arrive. To determine NOPS, two cases are considered in equations (2) and (3):

case(i): If the system is not empty ($J_{k-1}^{(n)} \neq 0$), then the server must be busy servicing a packet. So, the buffer can accommodate $K + 1 - J_{k-1}^{(n)}$ new arrivals before packets are discarded.

case(ii): If the system is empty ($J_{k-1}^{(n)} = 0$ and the server is idle), then the buffer can accommodate up to K new arrivals before packets are discarded; i.e., an arriving packet cannot be placed in service in the same slot.

We derive expressions for $V_k^{(n)}$, $X_k^{(n)}$, $S_k^{(n)}$ and $J_k^{(n)}$ from $\bar{V}_{k-1}^{(n)}$, $\bar{X}_{k-1}^{(n)}$, $\bar{S}_{k-1}^{(n)}$, $\bar{J}_{k-1}^{(n)}$ and the relations defined by the events in phase (2). The EXT-stream MMBP state changes are described by its state transition probability matrix P_t . A departure is associated with the status of the server. A departure occurs if the previous slot ($k-1$) was the last slot of the packet in service i.e., $\bar{S}_{k-1}^{(n)} = N$. If the server was idle ($\bar{S}_{k-1}^{(n)} = 0$) or completed service of a packet ($\bar{S}_{k-1}^{(n)} = N$) in the preceding slot ($k - 1$) then it changes state according to the queue status in the current slot (k). It changes its state to idle ($S_k^{(n)} = 0$) if the buffer is empty or

else changes state to indicate first service slot ($S_k^{(n)} = 1$) of the next packet scheduled. If a packet is in service and a departure does not occur, then the server state changes to indicate the next service slot ($S_k^{(n)} = \bar{S}_{k-1}^{(n)} + 1$) of the packet in service. The relations are as follows:

$$\begin{aligned} V_k^{(n)} &= \bar{V}_{k-1}^{(n)}, \\ X_k^{(n)} &= x \quad \text{w.p. } p_{i,x} \quad \text{where } \bar{X}_{k-1}^{(n)} = i, \quad x \in \{1, \dots, ST_X\} \end{aligned} \quad (4)$$

$$S_k^{(n)} = \begin{cases} \bar{S}_{k-1}^{(n)} + 1 & \text{if } 0 < \bar{S}_{k-1}^{(n)} < N \\ 1 & \text{if } \bar{S}_{k-1}^{(n)} = 0, \bar{J}_{k-1}^{(n)} > 1 \quad \text{or} \quad \bar{S}_{k-1}^{(n)} = N, \bar{J}_{k-1}^{(n)} > 0 \\ 0 & \text{if } \bar{S}_{k-1}^{(n)} = 0, \bar{J}_{k-1}^{(n)} = 1 \quad \text{or} \quad \bar{S}_{k-1}^{(n)} = N, \bar{J}_{k-1}^{(n)} = 0 \end{cases} \quad (5)$$

$$J_k^{(n)} = \begin{cases} \bar{J}_{k-1}^{(n)} & \text{if } 0 \leq \bar{S}_{k-1}^{(n)} < N \\ \bar{J}_{k-1}^{(n)} - 1 & \text{if } \bar{S}_{k-1}^{(n)} = N \end{cases} \quad (6)$$

where $k \geq 1$ and $n \geq 1$.

Let $c_k^{(n)}(v, x, s, j)$ be the joint probability density function of the random variables which describe the system state in the k^{th} slot following the n^{th} VC-stream arrival, i.e., $c_k^{(n)}(v, x, s, j) = Pr[V_k^{(n)} = v, X_k^{(n)} = x, S_k^{(n)} = s, J_k^{(n)} = j]$. The corresponding density function of the system state in slot k after phase(1) (consideration of arrivals) is given as $\bar{c}_k^{(n)}(v, x, s, j) = Pr[\bar{V}_k^{(n)} = v, \bar{X}_k^{(n)} = x, \bar{S}_k^{(n)} = s, \bar{J}_k^{(n)} = j]$. To express the relations (1) to (6), in terms of the density functions, we define a function

$$CONV_{v,x,s,k}^{(n)}(j) = \sum_{i=0}^{\min\{j, K+1\}} c_k^{(n)}(v, x, s, i) b_x(j - i)$$

where $j \geq 0$. $CONV_{v,x,s,k}^{(n)}(j)$ is the convolution of $c_k^{(n)}(v, x, s, j)$ and $b_x(j)$ where $j \geq 0$.

The equations corresponding to phase(1) for the different cases are given below

case: $k = 0, J_k^{(n)} = 0$

$$\bar{c}_0^{(n)}(v, x, s, j) = \begin{cases} CONV_{v,x,s,0}^{(n)}(j - 1) & \text{if } 0 \leq j < K \\ \sum_{i=K-1}^{\infty} CONV_{v,x,s,0}^{(n)}(i) & \text{if } j = K \end{cases} \quad (7)$$

case: $k = 0, J_k^{(n)} > 0$

$$\bar{c}_0^{(n)}(v, x, s, j) = \begin{cases} CONV_{v,x,s,0}^{(n)}(j-1) & \text{if } 0 \leq j < K+1 \\ \sum_{i=K}^{\infty} CONV_{v,x,s,0}^{(n)}(i) & \text{if } j = K+1 \end{cases} \quad (8)$$

case: $k > 0, J_k^{(n)} = 0$

$$\bar{c}_k^{(n)}(v, x, s, j) = \begin{cases} CONV_{v,x,s,k}^{(n)}(j) & \text{if } 0 \leq j < K \\ \sum_{i=K}^{\infty} CONV_{v,x,s,k}^{(n)}(i) & \text{if } j = K \end{cases} \quad (9)$$

case: $k > 0, J_k^{(n)} > 0$

$$\bar{c}_k^{(n)}(v, x, s, j) = \begin{cases} CONV_{v,x,s,k}^{(n)}(j) & \text{if } 0 \leq j < K+1 \\ \sum_{i=K+1}^{\infty} CONV_{v,x,s,k}^{(n)}(i) & \text{if } j = K+1 \end{cases} \quad (10)$$

The equations corresponding to phase(2) where state transitions and a possible departure can occur are given below

$$c_k^{(n)}(v, x, s, j) = \begin{cases} \sum_{x'=1}^{ST-X} p_{x',x} (\bar{c}_{k-1}^{(n)}(v, x', 0, 0) + \bar{c}_{k-1}^{(n)}(v, x', N, 1)) & \text{if } s = 0, j = 0 \\ \sum_{x'=1}^{ST-X} p_{x',x} (\bar{c}_{k-1}^{(n)}(v, x', 0, j) + \bar{c}_{k-1}^{(n)}(v, x', N, j+1)) & \text{if } s = 1, 1 \leq j \leq K \\ \sum_{x'=1}^{ST-X} p_{x',x} \bar{c}_{k-1}^{(n)}(v, x', s-1, j+1) & \text{if } 2 \leq s \leq N, 1 \leq j \leq K+1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

where $k \geq 1, n \geq 1, 0 \leq x', x < ST-X$ and $p_{x',x}$ is the probability of transition from state x' to state x for the EXT-stream MMBP.

Starting with a distribution of system state seen by n^{th} VC-stream arrival ($c_0^{(n)}(v, x, s, j)$), we can apply equations (7) to (11) corresponding to phase(1) and phase(2) repeatedly to

obtain the system state distribution at subsequent slots ($c_k^{(n)}(v, x, s, j)$, for $k = 1, 2, \dots$). We relate this information to obtain the probability distribution of the system state seen by the next i.e., the $(n + 1)^{th}$ VC-stream arrival using the following equation

$$c_0^{(n+1)}(v', x, s, j) = \sum_k \sum_v a_v(v', k) c_k^{(n)}(v, x, s, j), \quad (12)$$

where $k \geq 1$, $1 \leq v, v' \leq ST_V$, $1 \leq x \leq ST_X$, $0 \leq s \leq N$ and $0 \leq j \leq K + 1$.

If steady state exists then the steady state p.d.f. of the system state seen by a VC-stream arrival is given as

$$c_0(v, x, s, j) = \lim_{n \rightarrow \infty} c_0^{(n)}(v, x, s, j).$$

Using the system state distribution we can now obtain the density functions for the queue length at VC-stream arrivals, the delay in system and the interdeparture time of VC-stream packets. The distribution of NOPS seen by any VC-stream arrival ($f_{NOPS}(j) = \Pr[\text{no. of packets in system} = j \text{ just before a VC-stream arrival}]$), is given as

$$f_{NOPS}(j) = \sum_{v=1}^{ST_V} \sum_{x=1}^{ST_X} \sum_{s=0}^N c_0(v, x, s, j) \quad \text{for } 0 \leq j \leq K + 1$$

Let $qld(j)$ represent the probability density function of the queue length seen by a VC-stream arrival. Using the above relations it is given as

$$qld(j) = \begin{cases} f_{NOPS}(0) + f_{NOPS}(1) & \text{if } j = 0 \\ f_{NOPS}(j + 1) & \text{if } 1 \leq j \leq K \end{cases}$$

The probability of loss for VC-stream packets ($vcloss = \Pr[\text{VC-stream packet is lost}]$) is given as

$$vcloss = \sum_{v=1}^{ST_V} \sum_{x=1}^{ST_X} \sum_{s=0}^N c_0(v, x, s, K + 1).$$

The density function of the delay in system for VC-stream packets $delay(j) = \Pr[VC\text{-stream packet delay} = j \text{ slots}]$, is given as

$$delay(j) = \begin{cases} \sum_{v=1}^{ST_V} \sum_{x=1}^{ST_X} c_0(v, x, 0, 0) & \text{if } j = N \\ \sum_{v=1}^{ST_V} \sum_{x=1}^{ST_X} c_0(v, x, s, l) & \text{if } N < j \leq N(K+1) \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

where $s = N - (j - 1) \bmod N$ and $l = \lfloor j/N \rfloor - 1$, i.e., s and l satisfy $j = N - s + 1 + lN$ where the term $N - s + 1$ represents delay caused by a packet in service at the VC-stream arrival instant and the part lN represents delay caused by buffered packets and the current VC-stream arrival.

4 Interdeparture Time Distribution of VC-stream Packets

We obtain the interdeparture time distribution of the VC-stream packets using the system state distribution derived in the previous section. To do this, we condition the r.v. for system state in the slots following a VC-stream arrival with the known system state in the arrival slot. Let the system state seen by a VC-stream arrival be $q = (q_v, q_x, q_s, q_j)$ where $V_0 = q_v$, $X_0 = q_x$, $S_0 = q_s$, and $J_0 = q_j$. The conditional random variable which represents the system state in the k^{th} slot following this arrival is represented as $C_{q,k} = (V_{q,k}, X_{q,k}, S_{q,k}, J_{q,k})$ and its probability density function is represented as $c_{q,k}(v, x, s, j)$. The definitions of the conditional random variables and the relations between the density functions of system states in consecutive slots are the same as in previous section except that here they are conditioned on q , see Appendix B for additional details. For each possible initial system state (q) we obtain the conditional system state p.d.f. $c_{q,k}(v, x, s, j)$ at subsequent slots ($k = 1, 2, \dots$) following an arrival.

Consider a system where $N = 1$ (server takes 1 slot to serve a packet). Whenever there

are packets in the system, a departure occurs at the end of the slot. It means that at least one arriving packet can be buffered in every slot. In a slot, a VC-stream arrival is assumed to have priority for buffering over EXT-stream arrivals. Hence, a VC-stream arrival will always be accepted (buffered) in an $N = 1$ system. In a queueing system where $N > 1$, it is possible that no packet arrival in a slot is buffered i.e., all arrivals in a slot are rejected. This occurs when the system is full and no departure occurred in the previous slot. Therefore, packets arriving from the VC-stream are subject to loss only when $N > 1$. The VC-stream departures consist of buffered (accepted) packets i.e., the ones which were not lost. The time interval between the arrivals of two consecutive accepted packets from the VC-stream, depends not only on the MMBP states at arrival times but also on the system state q , seen by the first accepted packet. The system state q influences the VC-stream packets lost between the two accepted arrivals. Hence, we consider the interarrival time distribution of the accepted VC-stream packets conditioned on the system state (q) seen by an accepted arrival which is given by

$$a'_q(v', k) = \Pr[\text{An accepted VC-stream arrival occurs in MMBP state } v' \text{ and after } k \text{ slots} \mid \text{previous accepted VC-stream arrival occurred when system state was } q].$$

In the derivation of this density function we make use of the following functions:

$$a''_q(v', k) = \Pr[\text{An accepted VC-stream arrival occurs in MMBP state } v' \text{ and after } k \text{ slots} \mid \text{a VC-stream arrival was rejected and the system state was } q],$$

$$\alpha_q(q', k) = \Pr[\text{system is full in } k^{\text{th}} \text{ slot following any VC-stream arrival and the system state is } q' = (q'_v, q'_x, q'_s, q'_j) \mid \text{system state was } q \text{ in the arrival slot and no further VC-stream arrivals have occurred}] \text{ and}$$

$$\beta_q(k) = \Pr[\text{system is full in } k^{\text{th}} \text{ slot following any VC-stream arrival} \mid \text{system state was } q \text{ in the arrival slot and no further VC-stream arrivals have occurred}].$$

Note that the functions $a'_q(v', k)$ and $a''_q(v', k)$ are p.d.f.s where as $\alpha_q(q', k)$ and $\beta_q(k)$ are not. The interval of k slots considered in $a'_q(v', k)$ and $a''_q(v', k)$ may include other VC-stream

arrivals which are lost. The functions $\alpha_q(q', k)$ and $\beta_q(k)$ are given as

$$\alpha_q(q' = (q_v, x, s, j), k) = \begin{cases} c_{q,k}(q_v, x, s, K + 1) & \text{if } j = K + 1 \\ 0 & \text{if } j < K + 1 \end{cases}$$

where $k \geq 1$, $1 \leq q_v \leq ST_V$, $1 \leq q_x, x \leq ST_X$, $0 \leq q_s, s \leq N$, $0 \leq j \leq K + 1$,

$$\beta_q(k) = \sum_{x=1}^{ST_X} \sum_{s=0}^N c_{q,k}(q_v, x, s, K + 1)$$

where $1 \leq k \leq IAS$, $1 \leq q_v \leq ST_V$, $1 \leq q_x \leq ST_X$ and $0 \leq q_s \leq N$.

Now consider two consecutive accepted VC-stream arrivals. Let the system state seen by the first arrival be $q = (q_v, q_x, q_s, q_j)$ and the time duration between these two arrivals be k slots. Let the MMBP state at the second arrival be v' . The interval (k slots) between these accepted arrivals can have the characteristics of either of the following two cases:

case(a): *no VC-stream packets are lost in this interval.* Here the interval is between two consecutive arrivals both of which are accepted. The probability of having the next arrival in slot k and in MMBP state v' given that an arrival occurs when the systems state is q , is $a_{q_v}(v', k)$, and the probability that the system is not full in slot k is $1 - \beta_q(k)$. The probability of having such accepted arrivals (case(a)) is represented by

$$\tilde{P}_k^{(a)} = a_{q_v}(v', k)(1 - \beta_q(k)) \quad (14)$$

case(b): *one or more VC-stream packets are lost in this interval.* In this case the interval of k slots is split into two intervals of length l and m slots (see figure 4), where l is the interval between the first arrival and the first VC-stream arrival which is lost and m is the interval between the first lost arrival and the next accepted arrival following it.

Let the system state when the first loss occurs (in the interval of length k) be $q' = (q'_v, q'_x, q'_s, q'_j)$, and the random variables which represent the length of the two intervals be L and M respectively. The probability of having the first lost arrival after l slots (i.e., $L=l$), is represented as

$$f_L(q', l) = a_{q_v}(q'_v, l)\alpha_q(q', l).$$

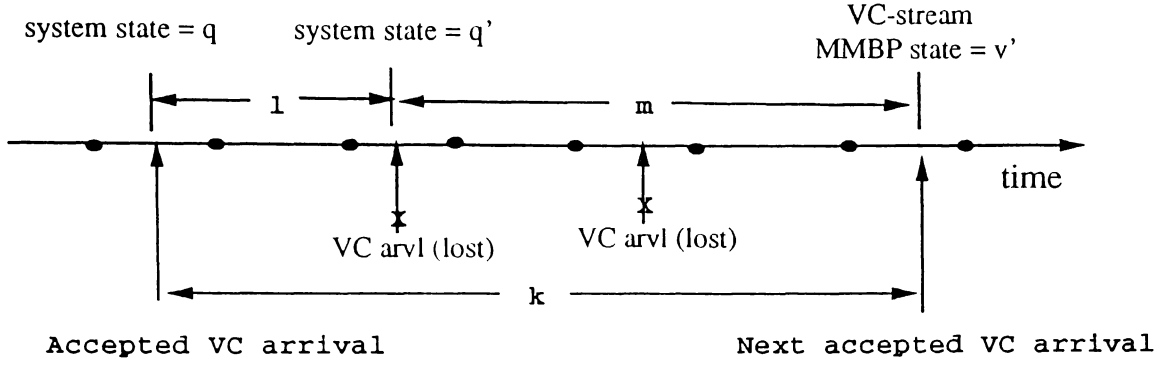


Figure 4: The interval between accepted arrivals from VC-stream with lost VC-stream arrivals

The density function of the r.v. M which is the interval between a lost arrival (when system state is q') and a next accepted arrival (in MMBP state v'), is represented by

$$f_{M_{q'}}(m) = a''_{q'}(v', m).$$

The probability of having two accepted arrivals k slots apart given that at least one arrival is lost in the interval k ($k = L + M$, $L \geq 1$, $M \geq 1$) and the system state seen by the first lost arrival is q' , is represented as

$$P_{q',k}^{(b)} = \sum_{l=1}^{k-1} a_{q'}(q'_v, l) \alpha_{q'}(q', l) a''_{q'}(v', k-l).$$

To find the probability of case(b), we consider all possible system states for q' (seen by the first lost arrival), which are all instances of system states where the buffer is full. The probability of case(b) is given by

$$P_k^{(b)} = \sum_{q'} P_{q',k}^{(b)} = \sum_{q'} \sum_{l=1}^{k-1} a_{q'}(q'_v, l) \alpha_{q'}(q', l) a''_{q'}(v', k-l) \quad (15)$$

where $q' = (q'_v, q'_x, q'_s, q'_j)$ such that $1 \leq q'_v \leq ST_V$, $1 \leq q'_x \leq ST_X$, $0 \leq q'_s \leq N$ and $q'_j = K + 1$.

The probability of having two accepted arrivals k slots apart and the MMBP state at the second arrival is v' given that the system state seen by the first arrival is q , is given as

$$a'_q(v', k) = P_k^{(a)} + P_k^{(b)}.$$

By substituting the derived expressions (equations (14) and (15)) for the r.h.s. terms in the above equation, the p.d.f. of interarrival time between accepted arrivals from VC-stream is given as

$$\begin{aligned} a'_q(v', k) &= a_{q_v}(v', k)(1 - \beta_q(k)) \\ &\quad + \sum_{q'} \sum_{l=1}^{k-1} a_{q_v}(q'_v, l) \alpha_q(q', l) a''_{q'}(v', k - l) \end{aligned} \quad (16)$$

where $k \geq 1$, $1 \leq q_v, q'_v, v' \leq ST-V$, $1 \leq q_x, q'_x \leq ST-X$, $0 \leq q_s, q'_s \leq N$, $0 \leq q_j \leq K$ and $q'_j = K + 1$.

To derive $a''_q(v', k)$, we consider the same scenario with the first VC-stream arrival as a lost arrival instead of an accepted arrival. The procedure to derive $a''_q(v', k)$ is exactly the same as the one used above to calculate $a'_q(v', k)$, the only difference being the initial system state, q . The resulting expressions for $a''_q(v', k)$ are

$$\begin{aligned} a''_q(v', k) &= a_{q_v}(v', k)(1 - \beta_q(k)) \\ &\quad + \sum_{q'} \sum_{l=1}^{k-1} a_{q_v}(q'_v, l) \alpha_q(q', l) a''_{q'}(v', k - l) \end{aligned} \quad (17)$$

where $k \geq 1$, $1 \leq q_v, q'_v, v' \leq ST-V$, $1 \leq q_x, q'_x \leq ST-X$, $0 \leq q_s, q'_s \leq N$, $q_j = K + 1$ and $q'_j = K + 1$.

We are interested in the distribution of system state seen by an accepted VC-stream arrival and the distribution of system state in k^{th} slot ($k > 0$) following an accepted VC-stream arrival. These distributions are later used in deriving the interdeparture time distribution of VC-stream packets.

Let $c'_o(q_v, x, s, j)$ represent the probability density function of the system state seen by an accepted arrival. In deriving $c'_o(q_v, x, s, j)$ we make note of the following: The fraction of the

total VC-stream arrivals which occur in a particular MMBP state is not necessarily the same as the fraction of the total accepted VC-stream arrivals which occur in that state, i.e., it is possible that arrivals in a particular MMBP state are more likely (or less likely) to be lost than arrivals in other states. Let $frac_0(v) = \Pr[\text{MMBP state is } v \text{ when VC-stream arrival occurs}]$ and $accfrac_0(v) = \Pr[\text{MMBP state is } v \text{ when a VC-stream arrival occurs which is accepted}]$. The two probabilities are given as

$$\begin{aligned}
frac_0(v) &= \sum_{\mathbf{x}, s, j} c_0(v, \mathbf{x}, s, j) \\
accfrac_0(v) &= \frac{\Pr[\text{MMBP arrival state is } v \text{ and arrival is accepted}]}{\Pr[\text{arrival is accepted}]} \\
&= \frac{\sum_{\mathbf{x}, s} \sum_{l=0}^K c_0(v, \mathbf{x}, s, l)}{\sum_{v', \mathbf{x}, s} \sum_{l=0}^K c_0(v', \mathbf{x}, s, l)} \tag{18}
\end{aligned}$$

The p.d.f. of the system state seen by an accepted VC-stream arrival is derived as follows:

$$\begin{aligned}
&\Pr[\text{system state seen by an arrival} = (v, \mathbf{x}, s, j) \mid \text{MMBP state on arrival} = v] \\
&= \frac{\Pr[\text{system state seen by an arrival} = (v, \mathbf{x}, s, j)]}{\Pr[\text{MMBP state on arrival} = v]} = \frac{c_0(v, \mathbf{x}, s, j)}{frac_0(v)}
\end{aligned}$$

$$\begin{aligned}
&\Pr[\text{arrival is lost} \mid \text{MMBP state on arrival} = v] \\
&= \sum_{\mathbf{x}, s} \frac{c_0(v, \mathbf{x}, s, K+1)}{frac_0(v)}
\end{aligned}$$

$$\begin{aligned}
&\Pr[\text{system state seen by accepted arrival} = (v, \mathbf{x}, s, j) \mid \text{MMBP state of accepted arrival} = v] \\
&= \frac{\Pr[\text{system state seen by an arrival} = (v, \mathbf{x}, s, j) \mid \text{MMBP state on arrival} = v]}{1 - \Pr[\text{arrival is lost} \mid \text{MMBP state on arrival} = v]} \\
&= \begin{cases} \frac{\frac{c_0(v, \mathbf{x}, s, j)}{frac_0(v)}}{1 - \sum_{\mathbf{x}, s} \frac{c_0(v, \mathbf{x}, s, K+1)}{frac_0(v)}} & \text{if } j \leq K \\ 0 & \text{otherwise} \end{cases} \tag{19}
\end{aligned}$$

$$\begin{aligned}
c'_0(v, x, s, j) &= Pr[\text{system state seen by accepted arrival} = (v, x, s, j)] \\
&= Pr[\text{system state seen by accepted arrival} = (v, x, s, j) \mid \text{MMBP state} \\
&\quad \text{on accepted arrival} = v] Pr[\text{MMBP state on accepted arrival} = v]
\end{aligned}$$

By substituting the probability values from equations (18) and (19) and simplifying we have

$$c'_0(v, x, s, j) = \begin{cases} \frac{\text{accfrac}_0(v) c_0(v, x, s, j)}{\text{frac}_0(v) - \sum_{x,s} c_0(v, x, s, K+1)} & \text{if } j < K+1 \\ 0 & \text{if } j = K+1 \end{cases} \quad (20)$$

where $1 \leq v \leq ST_V$, $1 \leq x \leq ST_X$, $0 \leq s \leq N$, $0 \leq j \leq K+1$.

Let $c'_{q,k}(q_v, x, s, j)$ represent the probability density function of the system state which a possible accepted VC-stream arrival would see if it occurs in the k^{th} slot following a previous accepted arrival, given that the previous accepted arrival saw the system state to be q and no further accepted arrivals occur in the interval of $k-1$ slots. In our previous definitions for p.d.f. of system state in the k^{th} slot following an arrival, i.e., $c_k^{(n)}(v, x, s, j)$ and $c_{q,k}(v, x, s, j)$, the interval k is assumed to have no VC-stream arrivals. These density functions remain the same if we have VC-stream arrivals in this interval but all of which are lost, since lost VC-stream arrivals do no influence the system state in slots subsequent to their arrival slot. Hence, to derive $c'_{q,k}(q_v, x, s, j)$ we make use of the density function $c_{q,k}(v, x, s, j)$. The derivation and the definitions of fractions are similar to that presented above (for the derivation of equation (20)). Let $\text{frac}_{q,k}(v)$ and $\text{accfrac}_{q,k}(v)$ be defined as

$$\begin{aligned}
\text{frac}_{q,k}(v) &= \sum_{x,s,j} c_{q,k}(v, x, s, j) \\
\text{accfrac}_{q,k}(v) &= \frac{\sum_{x,s} \sum_{l=0}^K c_{q,k}(v, x, s, l)}{\sum_{v',x,s} \sum_{l=0}^K c_{q,k}(v', x, s, l)}
\end{aligned}$$

The p.d.f. of system state seen by a possible accepted arrival in the k^{th} slot following an

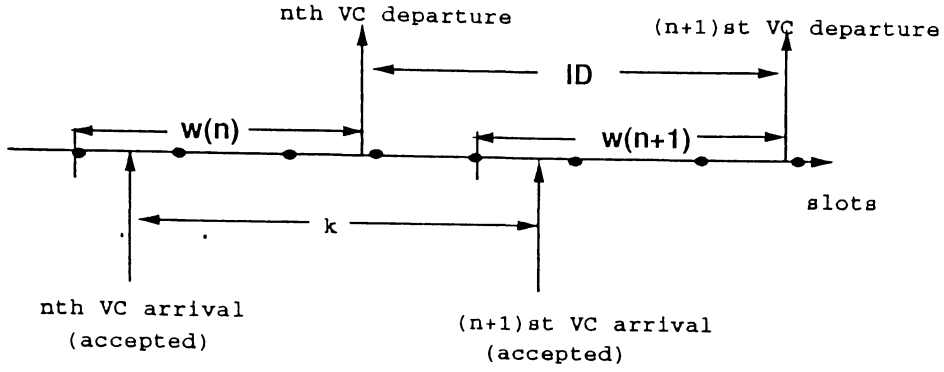


Figure 5: The interdeparture time interval

accepted arrival is given as

$$c'_{q,k}(v, x, s, j) = \begin{cases} \frac{\text{accfrac}_{q,k}(v) c_{q,k}(v, x, s, j)}{\text{frac}_{q,k}(v) - \sum_{x,s} c_{q,k}(v, x, s, K+1)} & \text{if } 0 \leq j, q_j \leq K \\ 0 & \text{if otherwise} \end{cases}$$

where $1 \leq v, q_v \leq ST_V$, $1 \leq x, q_x \leq ST_X$, $0 \leq s, q_s \leq N$, $0 \leq j, q_j \leq K + 1$.

Let $w(s, j)$ denote the delay of a packet in the system i.e., the time interval between its arrival and departure, when the server state (S_0) was s and the system length (J_0) was j upon arrival. If the system is not empty ($j \neq 0$), then the packet has to wait for $N - S + 1$ slots for the packet in server to finish service and $(j - 1)N$ slots for the service of the packets already in buffer and N slots for its own service, before it departs the system. If the system is empty ($j = 0$), it takes N slots of service time starting from the next slot (total of $N + 1$ slots) before it departs the system. $w(s, j)$ is given by

$$w(s, j) = \begin{cases} (j - 1)N + (N - s + 1) + N & \text{if } 1 \leq j \leq K \text{ and } 1 \leq s \leq N \\ N + 1 & \text{if } j = 0 \text{ and } s = 0 \end{cases}$$

Let $ID_{q,k}$ be the conditional random variable which represents the interdeparture time. It is the time interval between departures of two consecutive VC-stream packets, given that the system state was q on the arrival of the first packet, and the second packet arrived in the k^{th} following slot, as shown in figure 5. The delay in system for the first packet is $w(q_s, q_j)$

and for the next packet will be $w(S_{q,k}, J_{q,k})$. Then we have

$$ID_{q,k} = k + w(S_{q,k}, J_{q,k}) - w(q_s, q_j) \quad (21)$$

The random variables $S_{q,k}$ and $J_{q,k}$ represent the server state and the number of packets in system, seen by an accepted VC-stream arrival which occurs k slots after a previous accepted VC-stream arrival which saw the system state to be q . These random variables are defined by the joint p.d.f. $c'_{q,k}(q_v, x, s, j)$, where $S_{q,k} = s$, $J_{q,k} = j$, $v = q_v$ and $1 \leq x \leq ST_X$. Using the function $c'_{q,k}(q_v, x, s, j)$, the p.d.f. of the conditional random variable for interdeparture time ($ID_{q,k}$) is given by

$$id_{q,k}(l) = \sum_{x=1}^{ST_X} \sum_{s,j} c'_{q,k}(q_v, x, s, j) \quad \text{if } q_j > 0 \quad (22)$$

where $0 < l \leq N * (K + 1)$, and (s, j) satisfy $l = k + w(s, j) - w(q_s, q_j)$.

We use the interarrival time distribution of accepted VC-stream arrivals (equation (16)) and the p.d.f. of system state (q) seen by an accepted arrival (equation (20)) in order to uncondition equation (22). Finally, the interdeparture time distribution is given as

$$id(j) = \sum_k \sum_{v'} \sum_q a'_q(v', k) c'_0(q_v, q_x, q_s, q_j) id_{q,k}(j) \quad (23)$$

where $k \geq 1$, $1 \leq j \leq N(K + 1)$, $1 \leq q_v, v' \leq ST_V$, $1 \leq q_x \leq ST_X$, $0 \leq q_s \leq N$ and $0 \leq q_j \leq K + 1$.

5 Numerical Results

For using our analysis to obtain numerical solutions we need the following two constants: IAS represents the finite maximum for the interarrival time in slots for VC-stream arrivals and MAS is the finite maximum for number of EXT-stream packet arrivals in a slot. In theory, the values for IAS and MAS could be infinite but for practical solutions of exact analysis of the system we need finite values. By using a finite value for IAS we get exact results when the density function($a(j)$) of interarrival time for VC-stream arrivals is completely

specified with the chosen constant i.e., $\sum_{j=1}^{IAS} a(j) = 1$. Almost all distributions have infinite tail values, so we consider a tolerance δ for specifying the arrival p.d.f. with reasonable accuracy i.e., the value used for IAS is k which is the least value which satisfies the inequality $1 - \delta \leq \sum_{j=1}^k a(j) \leq 1$. The values of IAS for a few arrival processes corresponding to a tolerance of $\delta = 10^{-6}$ are given in figure 6. In the figure, we observe that on increasing the burstiness of a process while keeping the mean interarrival time as constant, the value of IAS becomes large. For specifying an arrival process we will need a very large value for the constant IAS (for a given δ) when the burstiness increases or when the mean interarrival time gets larger. This results in a larger state space and hence drastic increase in computational cost. To avoid infeasibility of a numerical solution due to excessive state space we limit the value of IAS to a fixed maximum value ($IAS = 750$ slots is used for our analysis) and trade with the accuracy in specifying the arrival process. We modify the arrival process by truncating the p.d.f. of interarrival times ($a(j)$) at $j = IAS$, and add all the truncated tail values ($a(j)$, $j > IAS$) to the truncated point, i.e.,

$$a(j) = \begin{cases} a(j) & \text{if } j < IAS \\ a(j) + \delta' & \text{if } j = IAS \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

where $\delta' = \sum_{j=IAS+1}^{\infty} a(j)$ (or $\delta' = 1 - \sum_{j=1}^{IAS} a(j)$). Since we use the conditional p.d.f. for the arrival process ($a_v(v', j)$), it is modified as $a_v(v', IAS) = a_v(v', IAS) + frac_0(v')\delta'$, where $\delta' = \sum_{v'=1}^{ST-X} \sum_{i=IAS+1}^{\infty} a_v(v', i)$. Similarly, $a'_v(v', j)$ and $a''_v(v', j)$ are modified using $accfrac_0(v')$ instead of $frac_0(v')$. Figure 7 shows the value of truncated tail (δ') of some VC-stream arrival processes for varying burstiness. For bursty arrival processes we notice that the values of δ' are much larger than a reasonable tolerance value of $\delta = 10^{-6}$. By modifying an arrival process as above (equation (24)) the characteristics of the arrival process change significantly (see figure 8). This results in error in the numerical solutions. Hence, to study the accuracy of the obtained numerical results from the analysis we compare them with simulation results.

By using a finite value for MAS we modify the EXT-stream arrival process $b_x(j)$ as

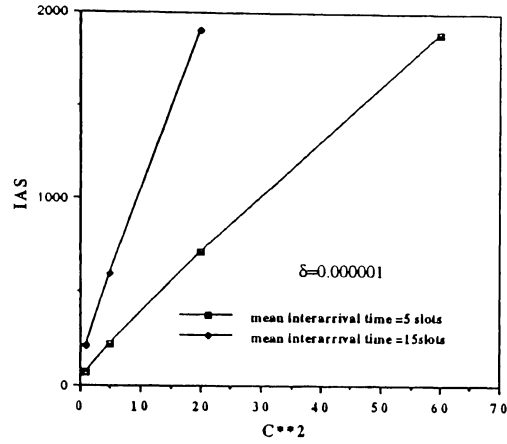


Figure 6: Value of IAS for the given tolerance $\delta = 1.0e - 6$ for VC-stream arrival process

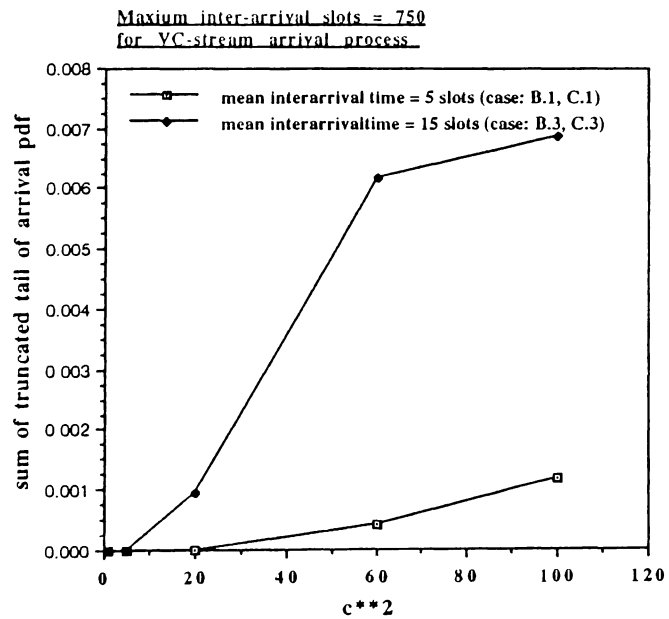


Figure 7: Truncated tail value (δ') of VC-stream arrival process

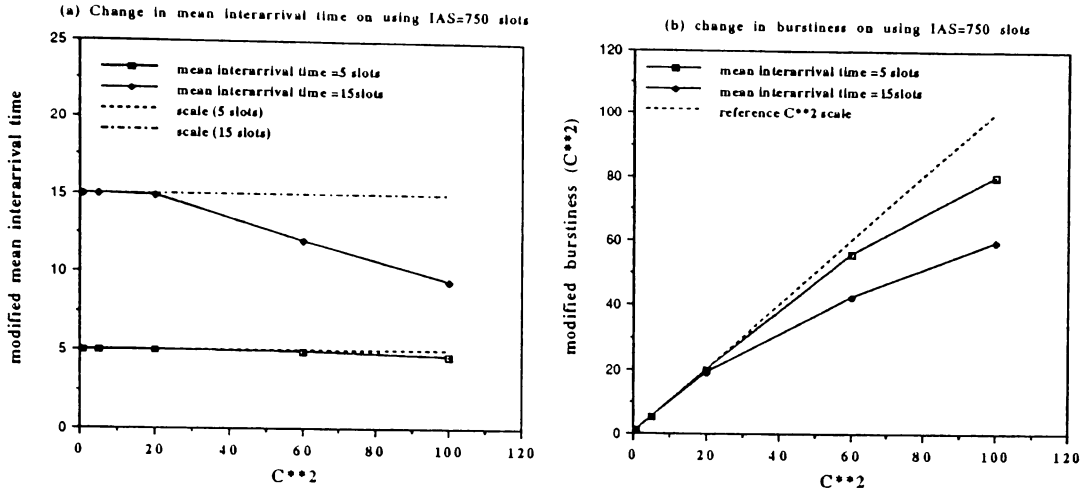


Figure 8: Change in arrival characteristics on using $IAS = 750$ slots

$b_x(MAS) = b_x(MAS) + \delta'$ (as in equation (24) where $\delta' = \sum_{i=MAS+1}^{\infty} b_x(i)$). Choosing a value for MAS such that $MAS \geq K + 1$ will not introduce error in analytical results.

We consider 2-state MMBPs to model the arrival process of both streams i.e., the values of ST_V and ST_X is 2. In both the MMBPs the probability of having an arrival when the MMBP state is 1 is considered to be 1, i.e., $\alpha_1 = 1.0$. This causes arrivals to occur in continuous slots as long as the MMBP remains in state 1. The batch size of an EXT-stream arrival is considered to be geometrically distributed with a mean of 3 packets per arrival. For both arrival streams given specific values of effective load on the server, the burstiness and autocorrelation between successive interarrival times (i.e. of lag 1, which is denoted as ψ_1) we compute P_t and Λ which define an MMBP (by making use of expressions (35) to (37)). The maximum interarrival time (IAS) used for the analysis is 750 slots. For the VC-stream arrival process we derive the p.d.f. $a_v(v', k)$ as described in Appendix A and modify it as described above.

Starting with initial values for the p.d.f. of system state seen by a VC-stream arrival i.e.,

$c_0^{(1)}(v, x, s, j)$, such that they satisfy

$$\sum_{v=1}^{ST_V} \sum_{x=1}^{ST_X} \sum_{s=0}^N \sum_{j=0}^{K+1} c_0^{(n+1)}(v, x, s, j) = 1,$$

we calculate the p.d.f. of the system state ($c_0^{(n+1)}(v, x, s, j)$) seen by the next arrival using equation (12). We repeatedly apply equation (12) until the p.d.f. $c_0^{(n+1)}(v, x, s, j)$ converges to give the steady state p.d.f. of system state seen by a VC-stream arrival, $c_0(v, x, s, j)$. Using the derived results in sections 3 and 4, we can obtain the performance measures for the VC-stream.

We study the performance measures of the system mainly for the following two types of traffic conditions.

1. Type B traffic: Varying burstiness of VC-stream traffic.

The offered load on the server due to the VC-stream and EXT-stream is held constant at 0.2 and 0.5 respectively. The EXT-stream is considered to be non-bursty ($C^2 = 1$) and both streams have arrivals where the interarrival times are slightly correlated. The parameters are given as

VC-stream: Load = 0.2, autocorrelation (ψ_1) = 0.01.

EXT-stream: Load = 0.5, burstiness (C^2)=1, autocorrelation (ψ_1) = 0.01 and mean batch size = 3.

2. Type C traffic: Varying autocorrelation of VC-stream traffic.

As in the above case, the offered load on the server due to the VC-stream and EXT-stream is 0.2 and 0.5 respectively. Here both streams are considered to be moderately bursty. Also, there is low correlation between interarrival times of EXT-stream arrivals. The parameters are

VC-stream: Load = 0.2, burstiness (C^2)=20.

EXT-stream: Load = 0.5, burstiness (C^2)=20, autocorrelation (ψ_1) = 0.01 and mean batch size = 3 packets.

We obtain results for the following two systems. In the first system we have the service

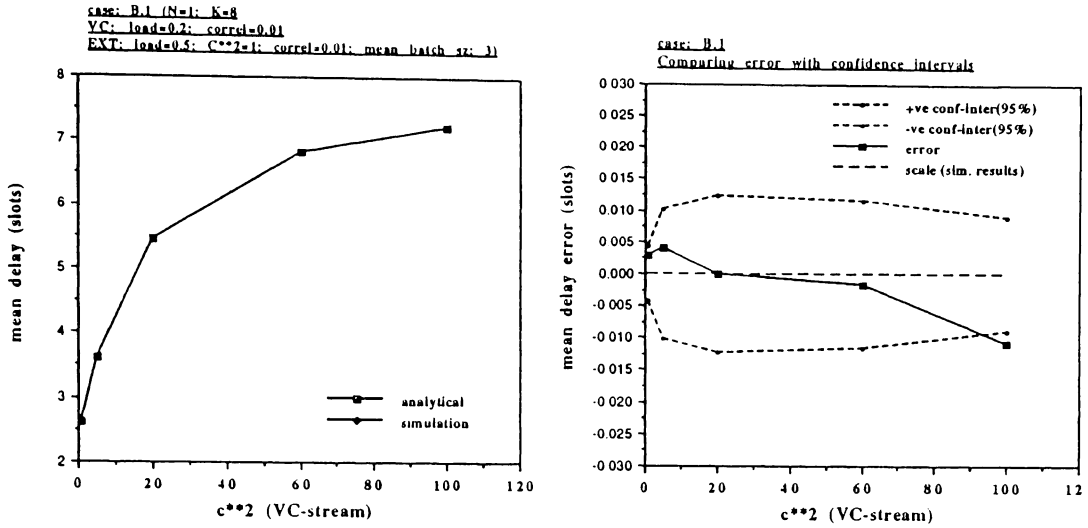


Figure 9: Mean delay at node for VC-stream packet in B.1 system

time per packet as one slot ($N=1$) and the buffer capacity to be 8 packets or equivalently slots. In the other system we consider the service time to be 3 slots and the buffer capacity as 16 packets (or equivalently 48 slots). We will refer to these systems in combination with the above two types of traffic as B.1, B.3, C.1 and C.3. In the above systems since we consider the offered load due VC-stream arrivals as 0.2, the corresponding mean interarrival time is 5 or 15 slots when N is 1 or 3 respectively. Figures 9,10, 11 and 12 show the mean delay for a VC-stream packet in systems B.1, B.3, C.1 and C.3 respectively. The probability of loss for a VC-stream packet is plotted in figures 13 and 14, for B.3 and C.3 systems respectively. In figures 15 and 16 we plot the density functions for buffer occupancy (queue length) at the VC-stream arrival instants, and the p.d.f. of interdeparture time for VC-stream packets are shown in figures 17 and 18. For a better perspective of the results, in all the above mentioned figures we also present the analytical results with respect to the 95th percentile confidence intervals of the simulation results.

In figure 9, the analytical results for mean delay for a VC-stream packet in a B.1 system are presented along with the simulation results. All analytical values obtained agree with the

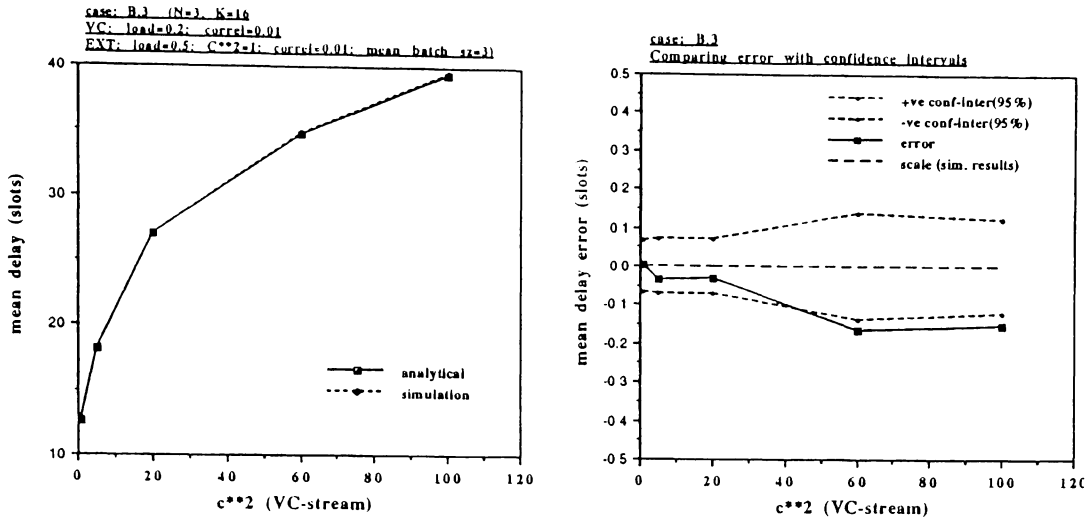


Figure 10: Mean delay at node for VC-stream packet in B.3 system

simulation results except for the case of a bursty VC-stream (corresponding to $C^2 = 100$ in the figure), where the analytical value of mean delay is outside the 95th percentile confidence interval indicating a small error. Also for B.3 system (see figure 10) analytical results for mean delay are outside the confidence intervals when $C^2 > 60$ i.e., when the VC-stream arrival process becomes bursty. Even though exact analysis is used for analytical results we observe that with increase in burstiness there is a small error when compared with simulation results. This is due to the approximation of modifying the VC-stream arrival process by using a finite maximum for interarrival times.

The following observations can be made about the influence of the approximation on the obtained results. Increasing burstiness causes the truncated tail of the density function to be significant (as seen in fig. 7). It results in lower values for mean interarrival time and the squared coefficient of variation (C^2) of the modified arrival process i.e., the offered load increases while the burstiness decreases. This influences the steady state results by giving higher estimates to initial part of the system state p.d.f. ($c_0(v, x, s, j)$) i.e., the probability density for values of J_0 closer to 0 (corresponding to empty system) have higher estimates

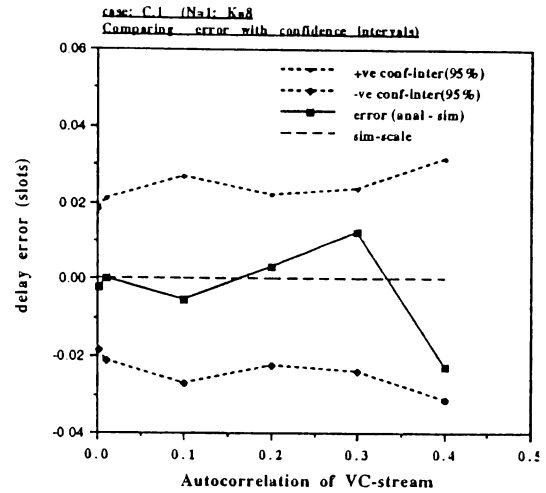
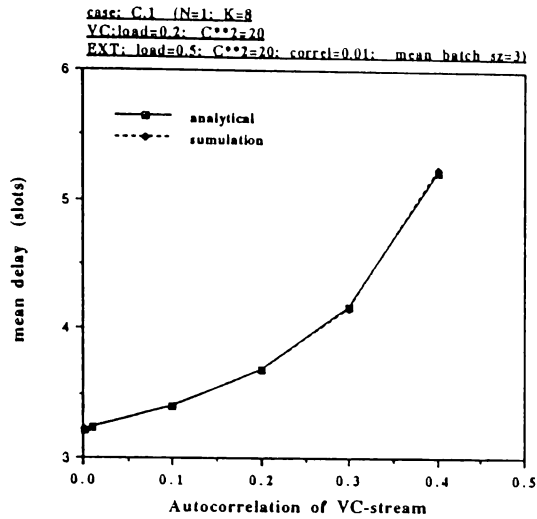


Figure 11: Mean delay at node for VC-stream packet in C.1 system

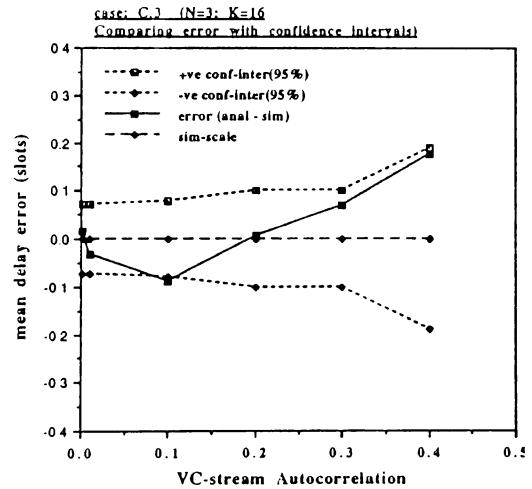
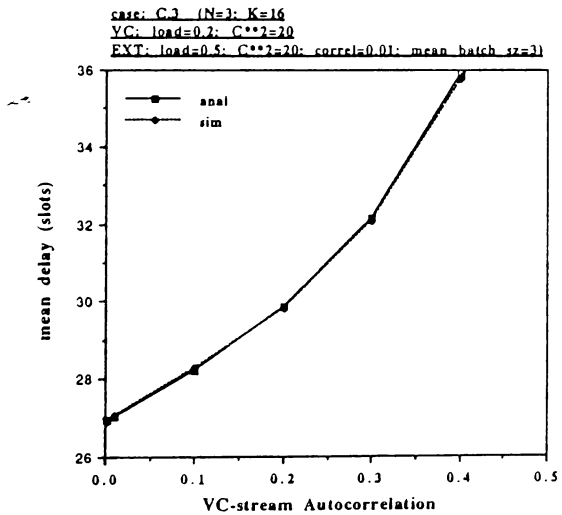


Figure 12: Mean delay at node for VC-stream packet in C.3 system

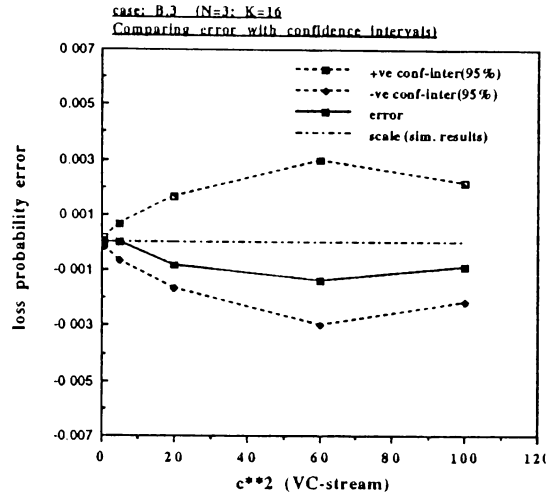
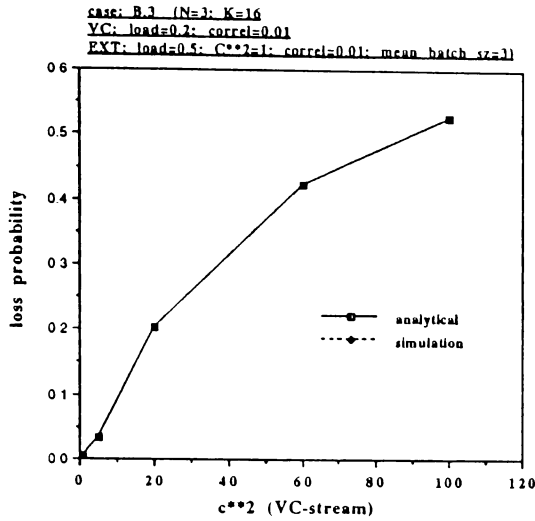


Figure 13: Loss probability of VC-stream packets in B.3 system

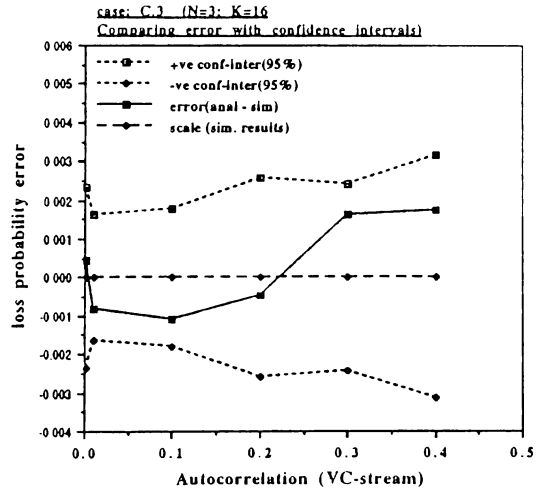
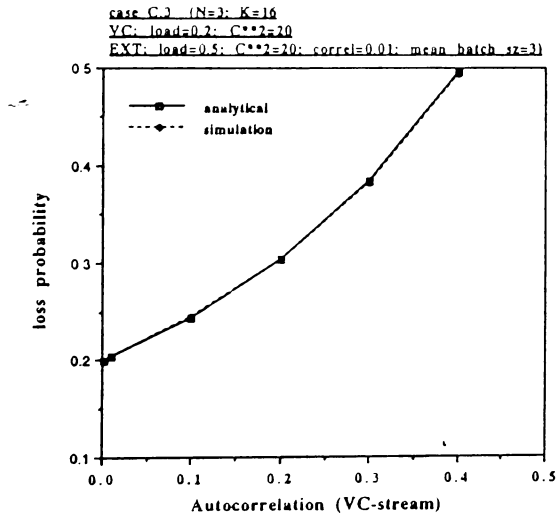


Figure 14: Loss probability of VC-stream packets in C.3 system

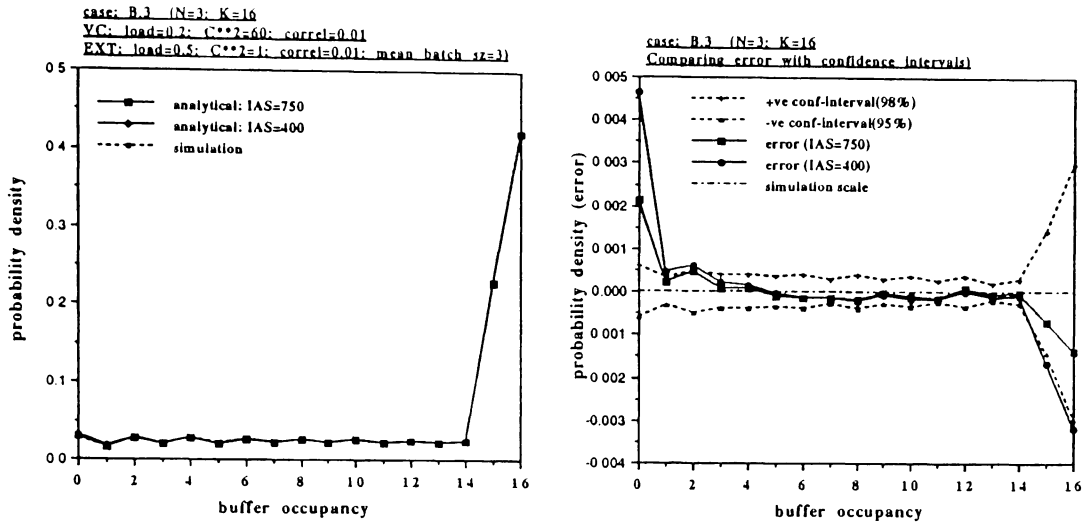


Figure 15: Comparison of p.d.f. of buffer occupancy (at VC-stream arrival instants) for different values of IAS

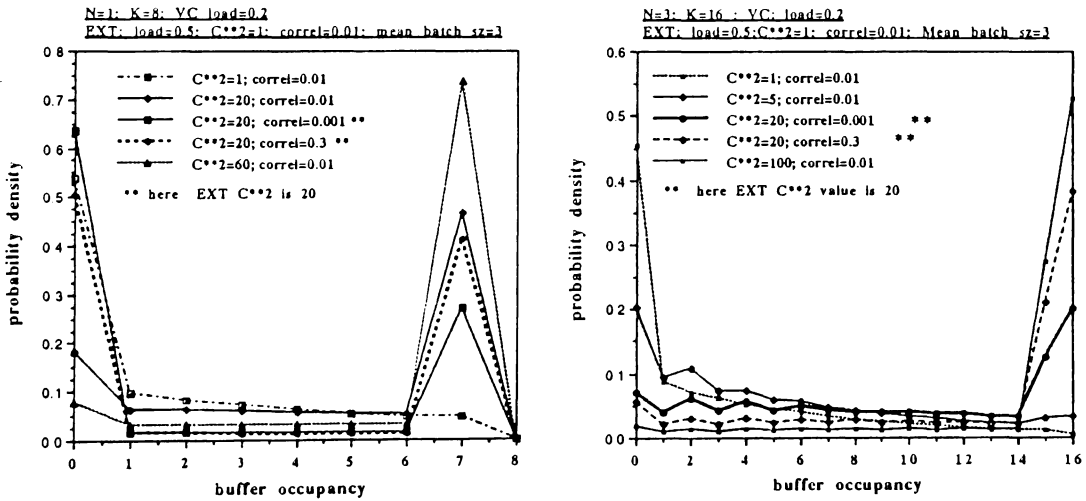


Figure 16: Buffer occupancy seen by VC-stream arrivals

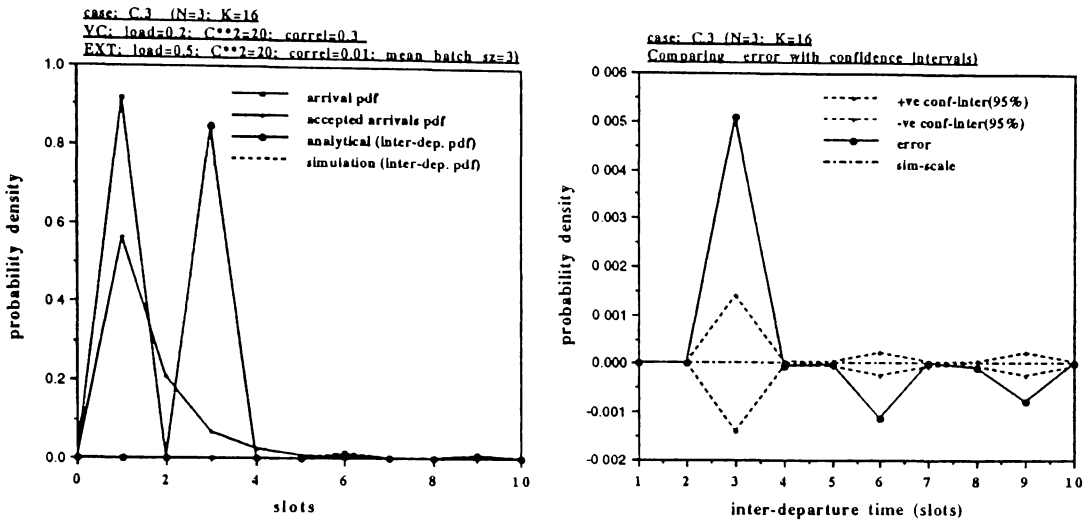


Figure 17: Arrival and departure process of VC-stream with correlated and bursty arrivals

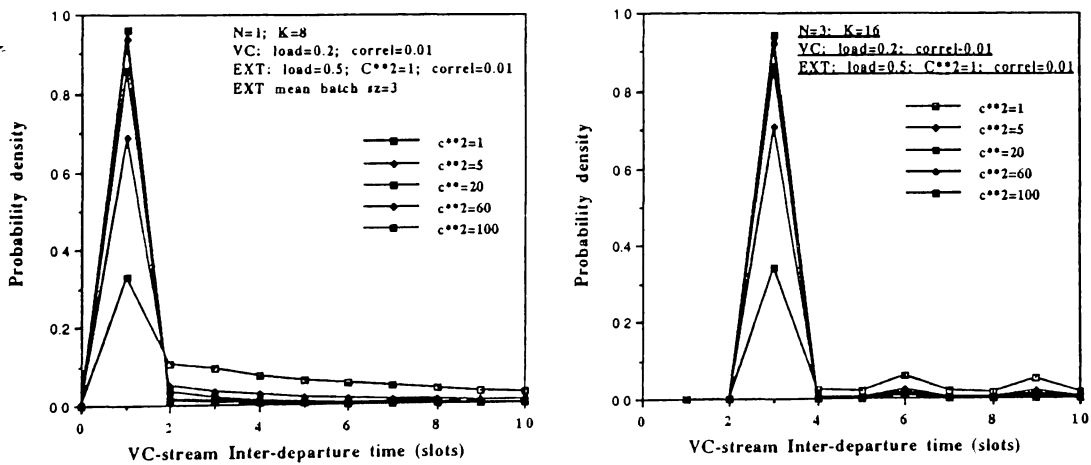


Figure 18: Departure process of VC-stream packets

while the others (corresponding to larger values of system length) have a lower estimate as observed in figure 15. This results in lower analytical values (compared to simulation) for mean delay and also for packet loss as seen in figures 9, 10 and 13.

To study the influence of autocorrelation we fix the values of C^2 and mean interarrival time, and vary ψ_1 , the parameters of an arrival process (as in C.1 and C.3 systems). The truncated tail value (δ') of the arrival process (for a fixed value of IAS) does not change with change in ψ_1 ; but the accuracy of the steady state results is altered. Increasing autocorrelation influences the steady state p.d.f. of system state by giving lower estimates for values when J_0 is closer to 0 and higher estimates for the other values. This counters the effect due to increase in burstiness. This phenomenon can be observed in figures 11, 12 and 14, where, even though the analytical values are within the confidence intervals the observed trend is that they get closer to the upper bounds of the confidence interval with increase in autocorrelation. The combined influence of burstiness and autocorrelation helps in obtaining very accurate results for moderately bursty and correlated traffic by using small values for IAS .

We observe in the numerical results that the mean delay of a packet and packet loss for VC-stream seem to increase logarithmically with increase in burstiness and exponentially with increase in autocorrelation. This shows that with a small increase in burstiness of a source which is initially non-bursty has a significant impact on the performance measures, whereas, the influence due to increase in autocorrelation is more significant when the traffic stream is initially correlated.

The buffer length distribution for different systems are given in figure 16. In a system where $N=1$ slot a VC-stream arrival never sees the buffer to be full (in figure 16 probability density corresponding to buffer size=8 is zero). When a buffer gets full in a slot due to arrivals, there is always one space emptied in the buffer at the end of the slot since a packet is taken for transmission in the next slot. Since we assumed that VC-stream arrival is given priority over EXT-stream arrivals in the same slot, we always have a buffer space available for a VC-stream arrival since a packet is taken into service at the end of every slot emptying

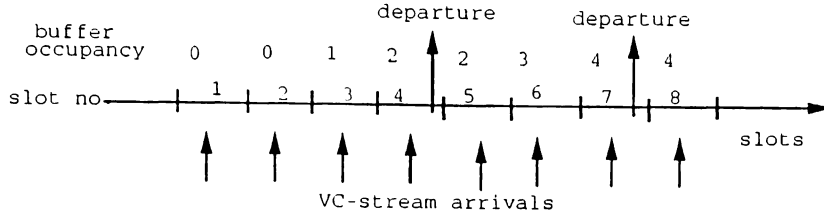


Figure 19: A sample busy period with only VC-stream arrivals

a buffer space for the next slot whenever the buffer is not empty. Due to this reason there is no packet loss in VC-stream when $N=1$. With increase in burstiness or correlation the probability density for a full or a near full buffer increases.

For the case when $N=3$ the buffer length probability density function (figure 16) shows an oscillatory behavior. Whenever the buffer length is a factor of $(N-1)$ it has a higher density and decreases for consecutive buffer lengths until the length is again a factor of $N-1$. This behavior can be explained using the following scenario: Let the first n ($n > N$) arrivals in a busy period be from the VC-stream and all of which occur in consecutive slots (see figure 19).

The first packet begins service in slot 2 and departs at the end of slot 4 ($N = 3$). The arrivals which occur in slot 4 and 5 respectively see the buffer length to be 2. Also for each departure i which occurs at the end of slot $iN + 1$ in this interval of n arrivals, the queue length seen by an arrival in slots $iN + 1$ and $iN + 2$ is $i(N - 1)$ packets. Hence, the p.d.f. of queue length has higher values for $i(N - 1)$. Such scenarios are very likely to occur since we assume $\alpha_1 = 1$ (probability of having an arrival in a slot when MMBP state is 1) for both streams.

In figure 18, the departure processes of B.1 and B.3 systems are plotted. For the B.1 system we notice a smooth slope pattern for the density functions. For the B.3 system we see that the function has a higher probability density instances which occur periodically every $N (=3)$ slots. Within a busy period (the interval of time the server status (S_k) is non-zero), departures can occur only at intervals of 3 slots. The minimum slots between two departures

is three slots. This corresponds to the cases where two departing VC-stream packets received service consecutively i.e., without any EXT-stream packets in between. Since both stream can be active at the same time, VC-stream packets can have EXT-stream packets interleaved between them when accepted into the system. Inside a busy period, inter-departure time of 6 slots corresponds to having one EXT-stream packet departure between two VC-stream departures and inter-departure time of 9 slots corresponds to 2 EXT-stream departures in between. Since under bursty conditions the busy periods are longer, two consecutive VC-stream departures are more likely to occur (in the same busy period). For this reason the interdeparture time of VC-stream packets is more likely to be multiples of 3 slots. Similar oscillatory behavior can also be seen for correlated traffic (see figure 18).

6 Conclusions

In this paper we presented exact analysis to obtain the quality of service (QoS) parameters of a single stream in a superposition of many arrival streams in a queueing system. For the analysis we considered a discrete time finite capacity queueing system with deterministic service time and two independent arrival processes. An arrival stream of interest (VC-stream) is modeled by an MMBP. The other arrival stream (EXT-stream) is modeled by a MMBP with batch arrivals denoted as $MMBP^{[X]}$. This queueing system models a switch in a high speed network where multiple applications which generate bursty and correlated traffic are supported simultaneously; the VC-stream corresponds to one of the input streams whose performance measures are of interest to us while the EXT-stream corresponds to the cumulative of remaining input streams. We derived the performance measures such as the queue length at VC-stream arrival instants, mean delay and packet loss probability of VC-stream packets using exact analysis. We also derived the departure process of VC-stream which is useful in end-to-end analysis of the VC-stream in a tandem queueing network. We presented numerical results by considering two-state MMBPs for the arrival streams. A constant *maximum interarrival slots (IAS)* was used for specifying the VC-stream arrival process. When

the VC-stream is very bursty and its offered load is very low a very large state space is required for the analysis to obtain exact results i.e., it becomes computationally expensive. We reduce the state space by choosing a lower value for IAS than the value required to specify the arrival process with the desired accuracy, for which we modify the arrival process. This changes the arrival characteristics (mean interarrival time and burstiness) and hence the numerical results. By comparing with simulation results we showed that the obtained analytical results are very accurate even when an arrival process was modified significantly.

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A Appendix: Interarrival time distribution of a 2-state MMBP

A 2-state MMBP is characterized by the state-transition probability matrix P_t of the Markov chain and the arrival rate descriptor Λ defined as follows:

$$P_t = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

The MMBP process stays in a particular state (state 1) for a period which is geometrically distributed during which arrivals occur in a Bernoulli fashion with a specific probability α . This period is followed by another geometrically distributed period (state 2) during

which arrivals occur in a Bernoulli fashion with a different probability β . These periods continuously alternate representing the state transition between the two states. Given that the MMBP is in state 1(or state 2) at slot i , it will remain in the same state in the next slot $i + 1$ with probability (w.p.) p (or q), or changes its state w.p. $1 - p$ (or $1 - q$).

Let t be the interarrival time between two successive arrivals, and t_{ij} be the time interval starting from a particular slot when the arrival process is in state i and ending at a slot when the next arrival occurs and the arrival process is in state j . The time interval t_{ij} , $1 \leq i, j \leq 2$, can be expressed as the following:

$$t_{11} = \begin{cases} 1 & w.p. \quad \alpha p \\ 1 + t_{11} & w.p. \quad (1 - \alpha)p \\ 1 + t_{21} & w.p. \quad (1 - \beta)(1 - p), \end{cases}$$

$$t_{21} = \begin{cases} 1 & w.p. \quad \alpha(1 - q) \\ 1 + t_{11} & w.p. \quad (1 - \alpha)(1 - q) \\ 1 + t_{21} & w.p. \quad (1 - \beta)q, \end{cases}$$

$$t_{12} = \begin{cases} 1 & w.p. \quad \beta(1 - p) \\ 1 + t_{12} & w.p. \quad (1 - \alpha)p \\ 1 + t_{22} & w.p. \quad (1 - \beta)(1 - p), \end{cases}$$

$$t_{22} = \begin{cases} 1 & w.p. \quad \beta q \\ 1 + t_{12} & w.p. \quad (1 - \alpha)(1 - q) \\ 1 + t_{22} & w.p. \quad (1 - \beta)(1 - q). \end{cases}$$

Let S_n be the state of the MMBP when the n^{th} arrival occurs, and let $T_{n,j}$ be the interarrival time between the $(n - 1)^{th}$ and n^{th} arrivals while the n^{th} arrival occurs in state j . If we define

$$A_{ij}(z) \equiv E[z^{T_{n,j}} | S_{n-1} = i],$$

then from the definition of t_{ij} and $T_{n,j}$ we have

$$A_{ij}(z) = E[z^{t_{ij}}], \quad \text{where } 1 \leq i, j \leq 2.$$

Therefore,

$$A_{11}(z) = \alpha pz + (1 - \alpha)pzA_{11}(z) + (1 - \beta)(1 - p)zA_{21}(z), \quad (25)$$

$$A_{21}(z) = \alpha(1 - q)z + (1 - \alpha)(1 - q)zA_{11}(z) + (1 - \beta)qzA_{21}(z), \quad (26)$$

$$A_{12}(z) = \beta(1 - p)z + (1 - \alpha)pzA_{12}(z) + (1 - \beta)(1 - p)zA_{22}(z), \quad (27)$$

$$A_{22}(z) = \beta qz + (1 - \alpha)(1 - q)zA_{12}(z) + (1 - \beta)qzA_{22}(z). \quad (28)$$

By sloving equations (25) (28), we have

$$A_{11}(z) = \frac{\alpha pz + \alpha(1 - \beta)(1 - p - q)z^2}{1 - ((1 - \alpha)p + (1 - \beta)q)z - (1 - \alpha)(1 - \beta)(1 - p - q)z^2}, \quad (29)$$

$$A_{12}(z) = \frac{\beta(1 - p)z}{1 - ((1 - \alpha)p + (1 - \beta)q)z - (1 - \alpha)(1 - \beta)(1 - p - q)z^2}, \quad (30)$$

$$A_{21}(z) = \frac{\alpha(1 - q)z}{1 - ((1 - \alpha)p + (1 - \beta)q)z - (1 - \alpha)(1 - \beta)(1 - p - q)z^2}, \quad (31)$$

$$A_{22}(z) = \frac{\beta qz + (1 - \alpha)\beta(1 - p - q)z^2}{1 - ((1 - \alpha)p + (1 - \beta)q)z - (1 - \alpha)(1 - \beta)(1 - p - q)z^2}. \quad (32)$$

Here $A_{ij}(z)$ represents the generating function of the conditional p.d.f. of interarrival time of VC-stream packets, $a_i(j, k)$. We obtain the relation between the generating functions of the p.d.f. of the interarrival time ($A(z)$) and the conditional interarrival times ($A_{ij}(z)$) of VC-stream packets as follows:

$$\begin{aligned} Pr[t = k] &= \sum_{i=1}^2 Pr[t = k | S_{n-1} = i]Pr[S_{n-1} = i] \\ &= \sum_{j=1}^2 \sum_{i=1}^2 Pr[t = k, S_n = j | S_{n-1} = i]Pr[S_{n-1} = i] \\ &= \sum_{j=1}^2 \sum_{i=1}^2 Pr[t_{ij} = k]\pi'_i \end{aligned} \quad (33)$$

where π'_i is the steady state probability of having an arrival in state i , and is given by $\pi' = [\frac{\alpha(1-q)}{\alpha(1-q)+\beta(1-p)}, \frac{\beta(1-p)}{\alpha(1-q)+\beta(1-p)}]$. By applying z-transform to the above equation, $A(z) \equiv$

$E\{z^t\}$ is given as

$$A(z) = \frac{\alpha(1-q)(A_{11}(z) + A_{12}(z)) + \beta(1-p)(A_{21}(z) + A_{22}(z))}{\alpha(1-q) + \beta(1-p)}. \quad (34)$$

We obtain the interarrival time distribution $a(i)$ ($=\Pr[\text{Interarrival time} = i \text{ slots}]$) by inverting the generating functions $A(z)$.

$$A(z) = \frac{\sum_{i=0}^2 b_i z^i}{\sum_{i=0}^2 c_i z^i} = \sum_{i=0}^{\infty} a(i) z^i.$$

Multiplying both sides by the denominator and then equating the coefficients of z^i on both sides we have the following set of linear difference equations:

$$\sum_{j=0}^{\min\{2,i\}} c_j a(i-j) = \begin{cases} b_i & i = 0, 1, 2 \\ 0 & i > 2 \end{cases}$$

We can now obtain the distribution $a(i), i = 0, 1, \dots$, using the recurrence relation:

$$a(i) = \frac{1}{c_0} \left[b_i - \sum_{j=1}^{\min\{2,i\}} c_j a(i-j) \right], \quad i = 0, 1, \dots,$$

where $b_i = 0$ for $i > 2$.

Similarly we obtain the conditional interarrival time p.d.f. $a_v(v', k)$ by inverting the generating functions $A_{vv'}(z)$, $1 \leq v, v' \leq 2$ individually.

For an MMBP, the mean interarrival time $E\{t\}$ and the squared coefficient of variation of the time between successive arrivals C^2 are as follows:

$$\begin{aligned} E\{t\} &= \frac{2-p-q}{\alpha(1-q) + \beta(1-p)}, \\ C^2 &= \frac{\text{Var}(t)}{E\{t\}^2} \\ &= \frac{2[\alpha(1-q) + \beta(1-p)]}{\alpha(1-q) + \beta(1-p) + \alpha\beta(p+q-1)} - \frac{\alpha(1-q) + \beta(1-p)}{2-p-q} \\ &\quad + \frac{2[\alpha(1-p) + \beta(1-q)][\alpha(1-q) + \beta(1-p)](p+q-1)}{(2-p-q)^2[\alpha(1-q) + \beta(1-p) + \alpha\beta(p+q-1)]} - 1. \end{aligned} \quad (35)$$

The average arrival rate, i.e. the probability that a slot contains a cell, λ is

$$\lambda = \frac{\alpha(1 - q) + \beta(1 - p)}{2 - p - q}. \quad (36)$$

The autocorrelation coefficient of the interarrival time of MMBP with lag 1 is given by

$$\begin{aligned} \psi_1 &= \frac{Cov(T_{n-1}, T_n)}{Var(T_n)} \\ &= \frac{E[T_{n-1} T_n] - E(T_{n-1})E(T_n)}{Var(T_n)} \\ &= \frac{\alpha\beta(\alpha - \beta)^2(1 - p)(1 - q)(p + q - 1)^2}{C^2(2 - p - q)^2[\alpha(1 - q) + \beta(1 - p) + \alpha\beta(p + q - 1)]^2}. \end{aligned} \quad (37)$$

B Appendix: Conditional distribution of system state seen by VC-stream arrival

Let the system state seen by an arrival be $q = (q_v, q_x, q_s, q_j)$ where $V_0 = q_v$, $X_0 = q_x$, $S_0 = q_s$ and $J_0 = q_j$. The random variable corresponding to C_k which represents the system state in the k th slot following a VC-stream arrival which saw the system state as q is represented as $C_{q,k} = (V_{q,k}, X_{q,k}, S_{q,k}, J_{q,k})$ where these random variables are defined as follows

$\tilde{V}_{q,k}$: r.v. which represents the state of the VC-stream MMBP when the VC-stream arrival which saw the system state as q occurred. The value of $V_{q,k}$ is always q_v .

$X_{q,k}$: r.v. which represents the state of the EXT-stream MMBP in the k th slot following a VC-stream arrival which occurred with system state as q .

$S_{q,k}$: r.v. which represents the state of the server in the k th slot following a VC-stream arrival which occurred with system state as q .

$J_{q,k}$: r.v. which represents the NOPS in the beginning of k th slot following a VC-stream arrival which occurred with system state as q .

The r.v. corresponding to $C_{q,k}$ at the second observation point i.e., after phase(1) is represented as $\bar{C}_{q,k} = (\bar{V}_{q,k}, \bar{X}_{q,k}, \bar{S}_{q,k}, \bar{J}_{q,k})$

The equations corresponding to transition due to events of phase(1) (described in section 3) are

$$\bar{V}_{q,k-1} = V_{q,k-1}, \quad \bar{X}_{q,k-1} = X_{q,k-1}, \quad \bar{S}_{q,k-1} = S_{q,k-1} \quad (38)$$

where $k \geq 1$, and

$$\bar{J}_{q,0} = \begin{cases} \min(B_x + 1, K) & \text{if } q_j = 0 \\ \min(q_j + B_x + 1, K + 1) & \text{if } q_j \neq 0 \end{cases} \quad (39)$$

$$\bar{J}_{q,k-1} = \begin{cases} \min(B_x, K) & \text{if } J_{q,k-1} = 0 \\ \min(J_{q,k-1} + B_x, K + 1) & \text{if } J_{q,k-1} \neq 0 \end{cases} \quad (40)$$

where $k > 1$. The relationship between the random variable corresponding to events of phase(2) can be given as

$$V_{q,k} = \bar{V}_{q,k-1}$$

$$X_{q,k} = x \quad \text{w.p. } p_{i,x} \quad \text{where } \bar{X}_{q,k-1} = i$$

$$S_{q,k} = \begin{cases} \bar{S}_{q,k-1} + 1 & \text{if } 0 < \bar{S}_{q,k-1} < N \\ 1 & \text{if } \bar{S}_{q,k-1} = 0, \bar{J}_{q,k-1} > 0 \quad \text{or} \quad \bar{S}_{q,k-1} = N, \bar{J}_{q,k-1} > 0 \\ 0 & \text{if } \bar{S}_{q,k-1} = 0, \bar{J}_{q,k-1} = 0 \quad \text{or} \quad \bar{S}_{q,k-1} = N, \bar{J}_{q,k-1} = 0 \end{cases}$$

$$J_{q,k} = \begin{cases} \bar{J}_{q,k-1} & \text{if } 0 \leq \bar{S}_{q,k-1} < N \\ \bar{J}_{q,k-1} - 1 & \text{if } \bar{S}_{q,k-1} = N \end{cases}$$

where $1 \leq k \leq IAS$.

Let $c_{q,k}(v, x, s, j)$ be the joint probability density function of the random variables which describe the system state at the first observation point in the k th slot following a VC-stream arrival which saw the system state to be q , i.e. $c_{q,k}(v, x, s, j) = Pr[V_{q,k} = v, X_{q,k} = x, S_{q,k} = s, J_{q,k} = j]$. Similarly, the density function of the system state at the second observation point is $\bar{c}_{q,k}(v, x, s, j) = Pr[\bar{V}_{q,k} = v, \bar{X}_{q,k} = x, \bar{S}_{q,k} = s, \bar{J}_{q,k} = j]$.

The equations which relate the probability density functions at two consecutive observation points are the same as the ones derived in the previous section and are given below.

The equations corresponding to (7) to (10) are

case: $k = 0, J_{q,k} = 0$

$$\bar{c}_{q,0}(q_v, x, s, j) = \begin{cases} CONV_{q,x,s,0}(j-1) & \text{if } 0 \leq j < K \\ \sum_{i=K-1}^{\infty} CONV_{q,x,s,0}(i) & \text{if } j = K \end{cases} \quad (41)$$

case: $k = 0, J_{q,k} > 0$

$$\bar{c}_{q,0}(q_v, x, s, j) = \begin{cases} CONV_{q,x,s,0}(j-1) & \text{if } 0 \leq j < K+1 \\ \sum_{i=K}^{\infty} CONV_{q,x,s,0}(i) & \text{if } j = K+1 \end{cases} \quad (42)$$

case: $k > 0, J_{q,k} = 0$

$$\bar{c}_{q,k}(q_v, x, s, j) = \begin{cases} CONV_{q,x,s,k}(j) & \text{if } 0 \leq j < K \\ \sum_{i=K}^{\infty} CONV_{q,x,s,k}(i) & \text{if } j = K \end{cases} \quad (43)$$

case: $k > 0, J_{q,k} > 0$

$$\bar{c}_{q,k}(q_v, x, s, j) = \begin{cases} CONV_{q,x,s,k}(j) & \text{if } 0 \leq j < K+1 \\ \sum_{i=K+1}^{\infty} CONV_{q,x,s,k}(i) & \text{if } j = K+1 \end{cases} \quad (44)$$

here $CONV_{q,x,s,k}(i)$ is the modified conditional convolution function which is defined as

$$CONV_{q,x,s,k}^{(n)}(j) = \sum_{i=0}^{\min\{j,K+1\}} c_{q,k}^{(n)}(q_v, x, s, i) b_x(j-i).$$

The equation corresponding to (11) which relates the density function at the second observation point of slot $k-1$ with the density function at the first observation point of slot k is

$$c_{q,k}(v, x, s, j) = \begin{cases} \sum_{x'=1}^{ST_X} p_{x',x}(\bar{c}_{q,k-1}(v, x', 0, 0) + \bar{c}_{q,k-1}(v, x', N, 1)) & \text{if } s = 0, j = 0 \\ \sum_{x'=1}^{ST_X} p_{x',x}(\bar{c}_{q,k-1}(v, x', 0, j) + \bar{c}_{q,k-1}(v, x', N, j + 1)) & \text{if } s = 1, 1 \leq j \leq K \\ \sum_{x'=1}^{ST_X} p_{x',x} \bar{c}_{q,k-1}(v, x', s - 1, j + 1) & \text{if } 2 \leq s \leq N, 1 \leq j \leq K + 1 \\ 0 & \text{otherwise} \end{cases} \quad (45)$$

where $0 < k \leq IAS$ and $0 \leq x', x < ST_X$.

For each possible initial system state (q) we obtain the system state distribution ($c_{q,k}(v, x, s, j)$) at subsequent slots following an arrival (for $k = 1, 2, \dots$), by applying equations (41) to (45) repeatedly. The initial condition being $Pr[V_{q,0} = q_v, X_{q,0} = q_x, S_{q,0} = q_s, J_{q,0} = q_j] = 1$ i.e.,

$$c_{q,0}(v, x, s, j) = \begin{cases} 1 & \text{if } v = q_v, x = q_x, s = q_s, j = q_j \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq v, q_v \leq ST_V, 1 \leq x, q_x \leq ST_X, 0 \leq s, q_s \leq N, 0 \leq j, q_j \leq K + 1$.