SHELLS ON ELASTIC FOUNDATIONS

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SUMMARY

No realistic analytical work in the area of Shells on Elastic Foundations has been reported in the literature. Most of the work in this area has been done on two assumptions, namely, (a) shells behave as membranes, and (b) foundation reactions on shells have preassigned values.

Various foundation models have been proposed by several authors. These models involve one or more than one parameters to characterise the foundation medium. Some of these models cannot be used to derive the basic equations governing the behaviour of shells on elastic foundations. In the present work, starting from an elastic continuum hypothesis, a mathematical model for foundation has been derived in curvilinear orthogonal coordinates by the help of principle of virtual displacements, treating one of the virtual displacements as known to satisfy certain given conditions at its edge surfaces. In this model, several foundation parameters can be considered and it can also be used for layered medium of both finite and infinite thickness.

Classical equations of thin shells on elastic foundations are derived using the proposed foundation model. Considering shells of revolution with loads that are independent of polar angle as example, shell-foundation equations are solved numerically by multi-segment method. In this method, the boundary value problem of shells is converted into initial value problem with first order differential equations which are integrated numerically by Gaussian elimination technique.

Numerical values of displacements, stress resultants and stress couples of shells are presented in dimensionless form for various values of shell and foundation parameters for a shallow spherical shell with free edge conditions. In this example, the effect of flank (portion of foundation beyond shell) on shell-foundation system is also considered. An effective flank width is suggested for design purposes. Variation of shell-foundation interaction as function of above parameters is also presented.

Theoretical studies of this paper can be applied in designing 'Tube Shell' which can be used in pressure vessels.

1. Introduction : Shells on elastic foundation are of recent origin and no realistic analytical work has been done as yet although some experimental work has been reported in the literature. A comprehensive review of various foundation models has been done by Kerr [1]. After the simplest one parameter foundation model, given by Winkler [2], several authors have proposed foundation models which involve more than one parameters for the characterisation of foundation medium, such as those of Wieghardt [3], Filonenko-Borodich [4], Pasternak [5], Reissner [6], Hetenyi [7,8], Vlasov and Leontov [9] and Kameswara Rao et al [10].

An attempt has been made here to improve upon the existing foundation models by formulating the model in orthogonal curvilinear coordinates and considering displacements in all the three coordinates directions. The Mathematical model is derived using generalised variational method to elastic continuum. The proposed model is simple and useful for the analysis of any type of structure on elastic foundation and in any set of orthogonal coordinate system. The model is capable of analysing layerd foundations of both finite and infinite thicknesses. This model has been used in deriving the thin shallow spherical shells on elastic foundation. Vlasov's [11] shell equations have been used in the above formulation. These equations have been solved for simply supported boundary conditions. Foundation medium has been assumed to be of finite thickness and of single elastic material. Normal load transferred by the column on shell have been considered. Kalnins [12] multisegment technique has been used to solve the boundary value problem. Results have been presented for two parameters (h/R) and (R/a) where (h/R) accounts for the thinness of the shell and (R/a) represents the shallowness of the shell. Two ratios of E_s/E_o have been considered. Only axisymmetric case has been reported here.

2. Basic Equations of Elastic Foundation

The virtual work equations for a set of normal stresses σ_α, σ_β and σ_γ and shear stresses $\sigma_{\alpha\beta}$, $\sigma_{\beta\gamma}$ and $\sigma_{\gamma\alpha}$ which are in equilibrium at any point in an elastic body considered in curvilinear orthogonal coordinates α, β and γ, are [11],

$$\int_\Omega \Big[\frac{\partial}{\partial\alpha}(\sigma_\alpha H_2 H_3) + \frac{\partial}{\partial\beta}(\sigma_{\alpha\beta} H_1 H_3) + \frac{\partial}{\partial\gamma}(\sigma_{\gamma\alpha} H_1 H_2) - \sigma_\beta H_3 \frac{\partial H_2}{\partial\alpha} - \sigma_\gamma H_2 \frac{\partial H_3}{\partial\alpha} + \sigma_{\beta\gamma} H_3 \frac{\partial H_1}{\partial\beta} + \sigma_{\gamma\alpha} H_2 \frac{\partial H_1}{\partial\gamma} \Big] \, \delta u \, . \, d\Omega = 0 \tag{1a}$$

where $d\Omega = H_1 H_2 H_3 d\alpha \, d\beta \, d\gamma$ is the elemental volume and δu is the virtual displacement. The other two equations are obtained from equation (1a) by cyclic permutation of coordinates and indices. Here H_1, H_2 and H_3 are the coefficients of the first quadratic

$$ds^2 = H_1^2 \, d\alpha^2 + H_2^2 \, d\beta^2 + H_3^2 \, d\gamma^2$$

where ds is the elemental distance and

$$H_1^2 = \frac{d\bar{r}}{d\alpha} \cdot \frac{d\bar{r}}{d\alpha} \, ; \; H_2^2 = \frac{d\bar{r}}{d\beta} \cdot \frac{d\bar{r}}{d\beta} \text{ and } H_3^2 = \frac{d\bar{r}}{d\gamma} \cdot \frac{d\bar{r}}{d\gamma}$$

where $\bar{r}$ is the radius vector. Prime denotes the derivative with respect to γ.

The displacements can be presented in a finite series in the form,

$$u(\alpha,\beta,\gamma) = \sum_{i=1}^{\ell} u_i(\alpha,\beta) \, \phi_i(\gamma) \tag{2a}$$

$$v(\alpha, \beta, \gamma) = \sum_{j=1}^{m} v_i(\alpha,\beta) \cdot \chi_j(\gamma) \tag{2b}$$

$$w(\alpha,\beta, \gamma) = \sum_{k=1}^{n} w_k(\alpha,\beta)\, \psi_k(\gamma) \tag{2c}$$

where ϕ_i, χ_j and ψ_k are assumed dimensionless functions consistant with the constraints of the problem. The unknown functions u_i, v_j and w_k are the generalised displacements. On substitution of displacement functions (2) into virtual work equations, equations (1) can expressed in the form,

$$\int_\gamma \Big[\frac{\partial}{\partial\alpha} (\sigma_\alpha \cdot H_2H_3)\, \phi_f + \frac{\partial}{\partial\beta} (\sigma_{\alpha\beta}\, H_1\, H_3)\, \phi_f -$$

$$\sigma_{\alpha\beta}\, H_1H_2(H_3\, \phi_f)' - \sigma_\beta\, H_3 \frac{\partial H_2}{\partial\alpha} \phi_f - \sigma_\gamma\, H_2 \frac{\partial H_3}{\partial\alpha} \phi_f$$

$$- \sigma_{\alpha\beta} H_3 \frac{\partial H_1}{\partial\beta} \phi_f + \sigma_\gamma \cdot H_3 \frac{\partial H_1}{\partial\gamma} \phi_f \Big]\, H_3\, d\gamma$$

$$+ \big[\sigma_{\alpha\gamma} H_1\, H_2\, H_3\, \phi_f \big]_\gamma = 0 \tag{3a}$$

$(f = 1,2,\ldots,l)$.

The other two equations are obtained by cyclic permutation. With the help of stress-strain relations for an isotropic, elastic body [11],

$$\sigma_\alpha = \lambda\Delta + 2G\, e_\alpha \tag{4a}$$

$$\sigma_{\alpha\beta} = G\, e_{\alpha\beta} \tag{4d}$$

and the strain displacement relations [11],

$$\sigma_\alpha = \frac{1}{H_1} \frac{\partial u}{\partial\alpha} + \frac{1}{H_1H_2} \frac{\partial H_1}{\partial\beta} v + \frac{1}{H_1H_3} \frac{\partial H_2}{\partial\gamma} w \tag{5a}$$

$$\sigma_{\alpha\beta} = \frac{H_1}{H_2} \frac{\partial}{\partial\beta} \left(\frac{u}{H_1}\right) + \frac{H_2}{H_1} \frac{\partial}{\partial\alpha} \left(\frac{v}{H_2}\right) \tag{5d}$$

equations (3) can be expressed completely in terms of displacements. The other four stress-strain relations and the strain-displacement relations can be obtained by the cyclic permutations of equations (4) and (5) respectively. In the equations (4), λ and μ are Iame's constants which could be expressed in terms of Engineering constant. Using the displacement functions (2), equations (3) can be integrated with respect to γ and the resulting equations will be only functions of α and β. These will be the elastic foundation equations the accuracy of which depends on the number of terms and the choice of the functions ϕ_i, χ_j and ψ_k guided by the experimental results or theoretical considerations, in equations (2).

2.1 <u>Choice of the functions ϕ, χ and ψ</u> :

The dimensionless functions ϕ, χ and ψ have to be chosen from the physical constraints of the problem. For elastic foundations of finite thickness fixed on a rigid base, the distribution of the displacements can be assumed to be linearly decreasing through the thickness, especially for thin layers

$$\phi = \chi = \psi = \frac{H - \gamma}{H} \tag{6}$$

where H is the thickness of the foundation layer. If the foundation layer is relatively thick, the choice can be made as

$$\phi = \chi = \psi = \frac{\sinh c(H - \gamma)}{\sinh cH} \tag{7}$$

where c denotes the rate of decrease of displacement with depth. Another choice for a foundation extending to infinity can be

$$\phi = \chi = \psi = e^{-c\gamma} \tag{8}$$

Many such expressions can be selected depending on experimental and theoretical considerations.

2.2 Foundation equations for axially symmetric case

For axially symmetric case, the first quadratic equation is taken in the form

$$ds^2 = dr^2 + r^2\, d\beta^2 + d\gamma^2 \tag{9}$$

In this case, $\alpha = r$, $H_1 = H_3 = 1$, $H_2 = r$. Taking the displacement functions in the form

$$u(r,\gamma) = u_1(r)\ \phi_1(\gamma)$$

$$w(r,\gamma) = w_1(r)\ \psi_1(\gamma) \tag{10}$$

the foundation equations can be expressed as

$$a_{11}\frac{d^2u_1}{dr^2} + \frac{a_{11}}{r}\frac{du_1}{dr} - \frac{a_{11}}{r^2}u_1 - b_{11}\,u_1 + d_{11}\frac{dw_1}{dr} - c_{11}\frac{dw_1}{dr} + P_r = 0 \tag{11a}$$

$$r_{11}\frac{d^2w_1}{dr^2} + \frac{r_{11}}{r}\frac{dw_1}{dr} - s_{11}\,w_1 - d_{11}\frac{du_1}{dr} + c_{11}\frac{du_1}{dr} + c_{11}\frac{u_1}{r} - d_{11}\frac{u_1}{r} + P_\gamma = 0 \tag{11b}$$

where for $\phi_1(\gamma) = \psi_1(\gamma) = \frac{H - \gamma}{H}$,

$$a_{11} = \int_0^H \frac{E_o}{(1 - \nu_o^2)}\phi_1^2\, d\gamma = \frac{E_o}{(1 - \nu_o^2)}\frac{H}{3} \tag{12a}$$

$$b_{11} = \int_0^H \frac{E_o}{2(1 + \nu_o)}\ \phi_1'^2\, d\gamma = \frac{E_o}{2(1 + \nu_o)}\frac{1}{H} \tag{12b}$$

$$c_{11} = \int_0^H \frac{E_o}{2(1 + \nu_o)}\psi_1\,\phi_1'\, d\gamma = \frac{E_o}{2(1 + \nu_o)}\frac{1}{2} \tag{12c}$$

$$d_{11} = \int_0^H \frac{E_o\,\nu_o}{(1 - \nu_o^2)}\ \psi_1'\,\phi_1\, d\gamma = \frac{E_o\,\nu_o}{(1 - \nu_o^2)}\frac{1}{2} \tag{12d}$$

$$r_{11} = \int_0^H \frac{E_o}{2(1 + \nu_o)}\psi_1^2\, d\gamma = \frac{E_o}{2(1 + \nu_o)}\frac{H}{3} \tag{12e}$$

$$s_{11} = \int_0^H \frac{E_o}{(1 - \nu_o^2)}\ \psi_1'^2\, d\gamma = \frac{E_o}{(1 - \nu_o^2)}\ \frac{1}{H} \tag{12f}$$

$$P_r = [\,\sigma_{r\gamma}\,\phi_1(\gamma)\,]_0^H = 0 \tag{12g}$$

$$p_\gamma = [\sigma_\gamma \, \phi_1(\gamma)]_0^H = q_\gamma(r) \tag{12h}$$

Here, $q_\gamma(r)$ will be the interaction between the foundation and the shell resting on it and E_o and ν_o are Young's modulus and Poisson's ratio of the foundation material respectively.

3. Shallow Shell Equations on Elastic Foundations

Equations governing the deflections u_s and w_s of a shallow thin spherical shell on an elastic foundation with axially symmetric loading in the lateral directions are [11]

$$\frac{d^2u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2} + \frac{(1-\nu_s)}{R^2}u_s + \frac{(1+\nu_s)}{R}\frac{dw_s}{dr} + \frac{(1-\nu_s^2)}{E_s h}X = 0 \tag{13a}$$

$$\frac{h^2}{12}\left[\frac{d^4w_s}{dr^4} + \frac{2}{r}\frac{d^3w_s}{dr^3}\left(\frac{2}{R^2}-\frac{1}{r^2}\right)\frac{d^2w_s}{dr^2} + \frac{1}{r}\left(\frac{1}{r^2}+\frac{2}{R^2}\right)\frac{dw_s}{dr}\right] + \frac{2(1+\nu_s)}{R^2}w_s$$

$$+\frac{(1+\nu_s)}{R}\frac{du_s}{dr} + \frac{(1+\nu_s)}{Rr}u_s + \frac{(1-\nu_s^2)}{E_o h}(Z - q_\gamma) = 0 \tag{13b}$$

where u_s and w_s are the circumferential and lateral displacements of the shell respectively, E_s = Young's modulus of shell, ν_s = poisson's ratio and h = shell thickness, X and Z are the external loads on the shell in the circumferential and lateral directions respectively. q_γ is the foundation reaction on the shell.

The dimensionless forces and moments in the spherical shell are given by

$$\bar{N}_1 = \frac{1}{(1-\nu_s^2)}\left(\frac{h}{R}\right)\left[\frac{R}{a}\frac{d\bar{u}_s}{d\bar{r}} + (1+\nu_s)\bar{w}_1 + \nu_s\left(\frac{R}{a}\right)\frac{\bar{u}_s}{\bar{r}}\right] \tag{14a}$$

$$\bar{N}_2 = \frac{1}{(1-\nu_s^2)}\left(\frac{h}{R}\right)\left[\frac{R}{a}\frac{\bar{u}_s}{\bar{r}} + (1+\nu_s)\bar{w}_1 + \nu_s\frac{d\bar{u}_s}{d\bar{r}}\right] \tag{14b}$$

$$\bar{M}_1 = \left(\frac{h}{a}\right)\left[\frac{d^2\bar{w}_1}{d\bar{r}^2} + \frac{\nu_s}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}}\right] \tag{14c}$$

$$\bar{M}_2 = \left(\frac{h}{a}\right)\left[\frac{1}{\bar{r}}\frac{d\bar{w}_1}{d\bar{r}} + \nu_s\frac{d^2\bar{w}_1}{d\bar{r}^2}\right] \tag{14d}$$

where $\bar{N}_1 = \frac{N_1}{E_s h}$, $\bar{N}_2 = \frac{N_2}{E_s h}$, $\bar{M}_1 = \frac{M_1 a}{D}$

$\bar{M}_2 = \frac{M_2 a}{D}$, $\bar{u}_s = \frac{u_s}{h}$, $\bar{w}_s = \frac{w_s}{h}$ and $\bar{r} = \frac{r}{a}$

N_1 and N_2 are the stress-resultants in radial and circumferential directions, M_1 and M_2 are the stress - couples in radial and circumferential directions respectively. D is the flexural rigidity of the shell.

For a shell with simple supports along r = a the boundary conditions are,

$$u_1 = w_1 = M_1 = 0 \tag{15a}$$

The boundary conditions at r = 0 are written with the assumption that a column of radius b is transfering a vertical load P on the shell and that the column is much stiffer than the shell. Then the conditions at r = b (b << a) are

$$u_1 = \frac{dw_1}{dr} = 0 \tag{15b}$$

$$Q_1 + q_\gamma = \frac{P}{2\pi b} \tag{15c}$$

where Q_1 is the transverse shear force in the shell along $r = b$.

NUMERICAL CALCULATIONS AND CONCLUSIONS

In all the numerical examples that have been solved, following parameter values have been kept constant :

$H/a = 2.0$; $b/a = 0.2$; $\mu_s = 0.25$; $\mu_o = 0.17$

In some cases $(R/a) = 4.0$ has been kept constant while (h/R) and (E_s/E_o) are varied and in other cases $(h/R) = 0.0333$ has been kept constant while (R/a) and (E_s/E_o) are varied. Results have been presented in plots drawn between $\bar{r}$ and functional values per unit load parameter. Load parameter has been taken as

$$\tilde{P} = (\frac{a}{b}((\frac{P}{E_s h^2}) \tag{16}$$

Functions per unit load parameter have been redefined as follow :

$$\tilde{u}_1 = \bar{u}_1/\tilde{P} \tag{17a}$$

$$\tilde{u}_s = \bar{u}_s/\tilde{P} \tag{17b}$$

$$\tilde{w}_1 = \bar{w}_1/\tilde{P} \tag{17c}$$

$$\tilde{N}_1 = \bar{N}_1/\tilde{P} \tag{17d}$$

$$\tilde{N}_2 = \bar{N}_2/\tilde{P} \tag{17e}$$

$$\tilde{M}_1 = \bar{M}_1/\tilde{P} \tag{17f}$$

$$\tilde{M}_2 = \bar{M}_2/\tilde{P} \tag{17g}$$

$$\tilde{q} = \bar{q}/\tilde{P} \tag{17h}$$

Figs. (1) and (2) represent variation of $\tilde{u}_1$ and $\tilde{u}_s$ for different (h/R) and (E_s/E_o) values. It can be observed from these two figures that $\tilde{u}_s$ is positive whereas $\tilde{u}_1$ is negative all along $\bar{r}$. Also $\tilde{u}_s$ is much larger than $\tilde{u}_1$. This happens because the slip is permitted between shell and foundation. $\tilde{u}_s$, the response representing radial displacement of the shell, is positive due to stretching of the shell. Foundation being a semi-infinite continuum will have negative radial displacement at the surface.

Fig. (3) represents w_1 along $\bar{r}$ for different values of (h/R) and (E_s/E_o). It can be observed from Fig. (3) that the normal deflection $\tilde{w}_1$ is maximum at the centre of the shell and also it is greater when E_s/E_o ratio is more, that is, when shells Young's Modulus is relatively more than foundation's Young's modulus. It can also be observed that $\tilde{w}_1$ is greater for small (h/R) ratio which means $\tilde{w}_1$ is greater for thinner shells.

Fig. (4) shows the variation of $\tilde{N}_1$ for different (h/R) and (E_s/E_o) values whereas Fig. (5) shows the variation of $\tilde{N}_1$ for different (R/a) and (E_s/E_o) values. It can be observed that $\tilde{N}_1$ is negative all along $\bar{r}$ for all values of (h/R), (R/a) and $E_s/E_o)$. Magnitude of $\tilde{N}_1$ increases with increase in (h/R), (R/a) and (E_s/E_o) values.

Fig. (6) shows the variation of $\bar{M}_1$ for different values of (R/a) and (E_s/E_o) values along $\bar{r}$. Fig. (7) and (8) represent variation of $\bar{q}$, the foundation reaction, for different values of (h/R) and (R/a) respectively.

The proposed foundation model can be used as supporting medium for varities of structures. Various problems have been solved using this model [13].

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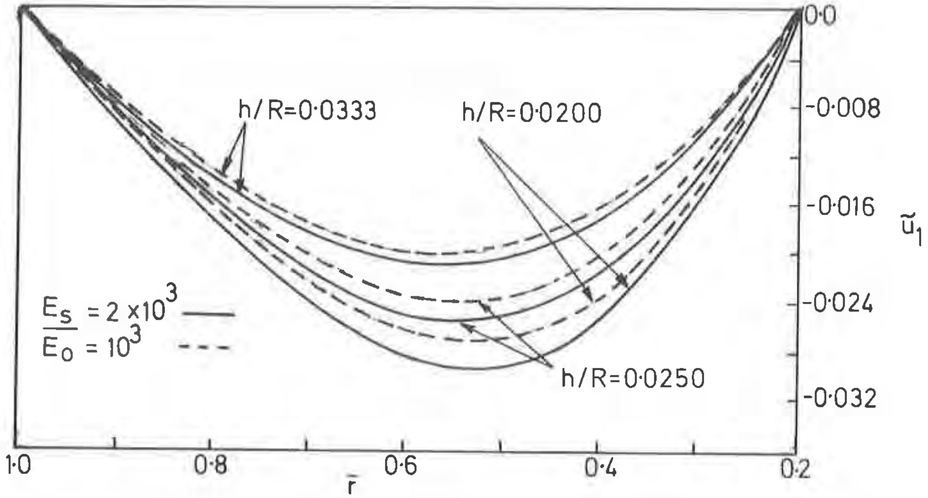


FIG.1 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_1$ FOR DIFFERENT (h/R) AND (E_s/ E_o) VALUES

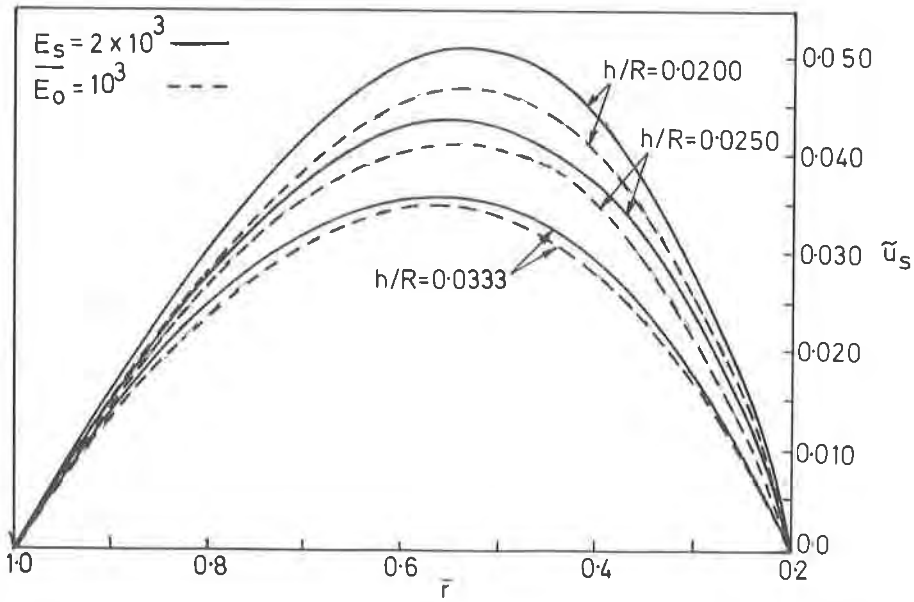


FIG.2 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{u}_s$ FOR DIFFERENT (h/R) AND (E_s/ E_o) VALUES

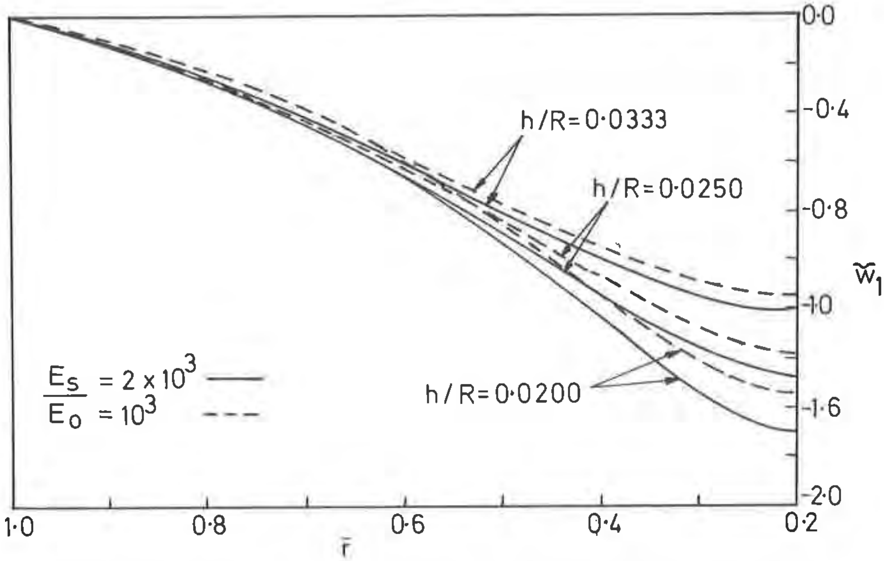


FIG.3 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{w}$ FOR DIFFERENCE (h/R) AND (E_s/E_o) VALUES

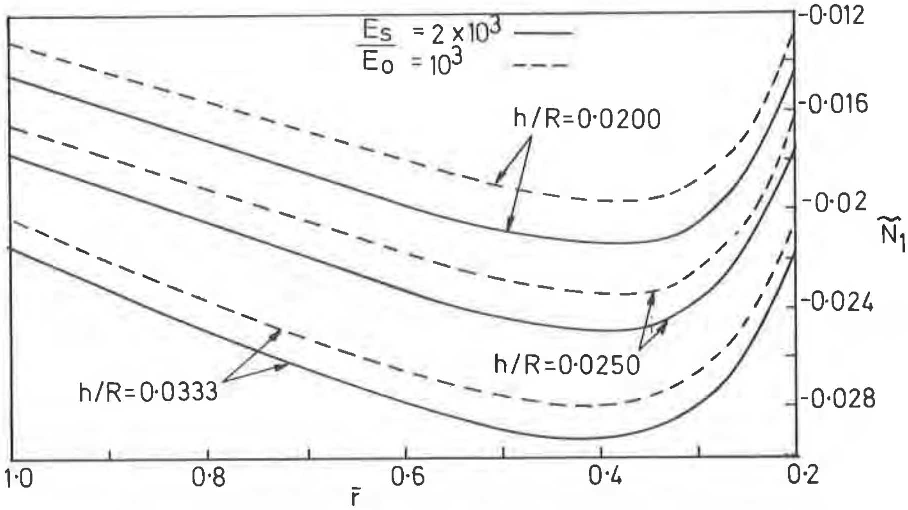
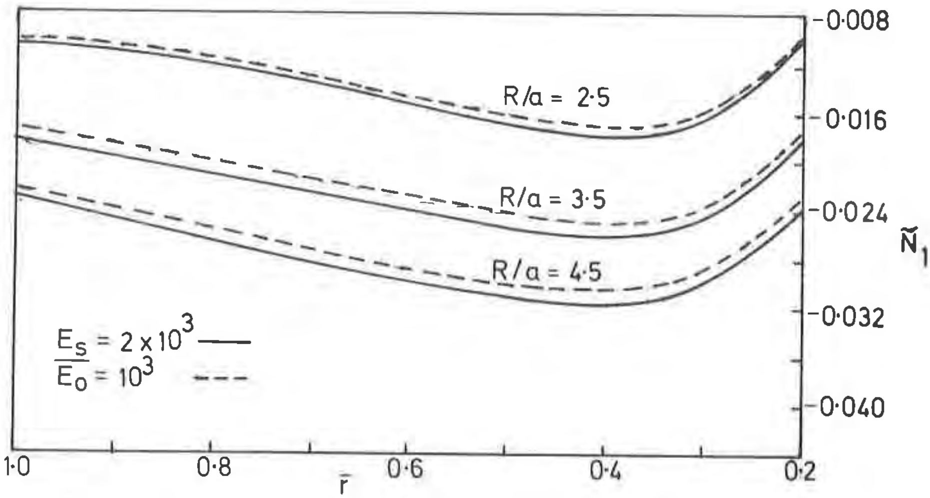
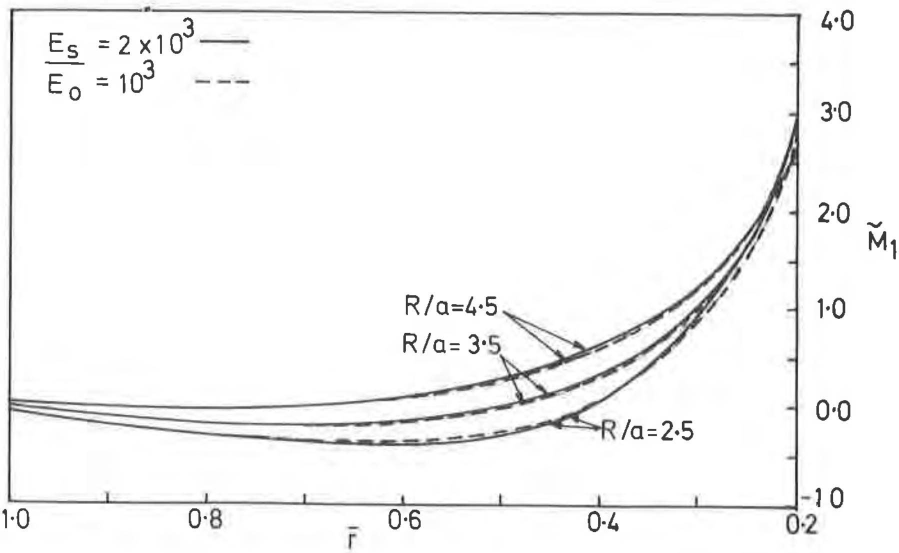


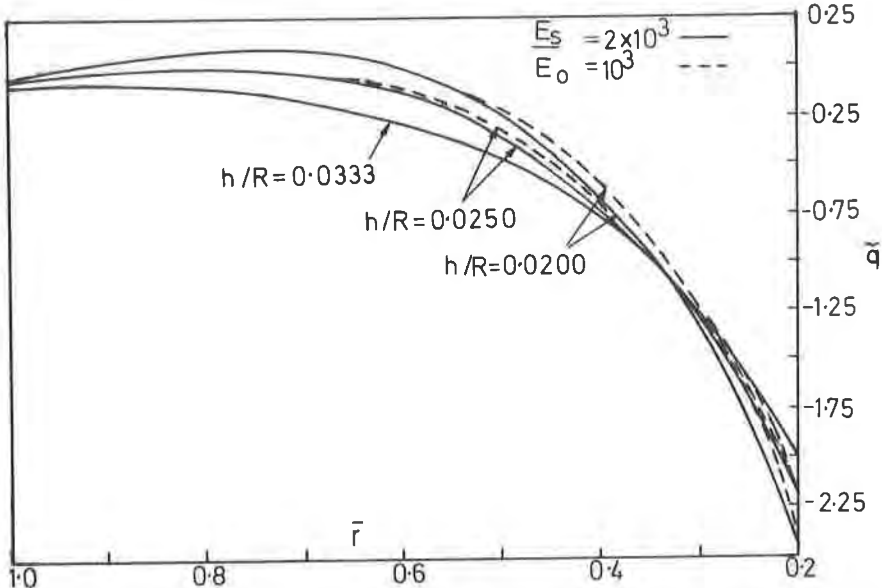
FIG. 4 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (h/R) AND (E_s/E_o) VALUES



FIG·5 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{N}_1$ FOR DIFFERENT (R/a) AND (E_s/E_o) VALUES



FIG·6 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{M}_1$ FOR DIFFERENT (R/a) AND (E_s/E_o) VALUES



FIG·7 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{q}$ FOR DIFFERENT (h/R) AND (E_S/E_O) VALUES

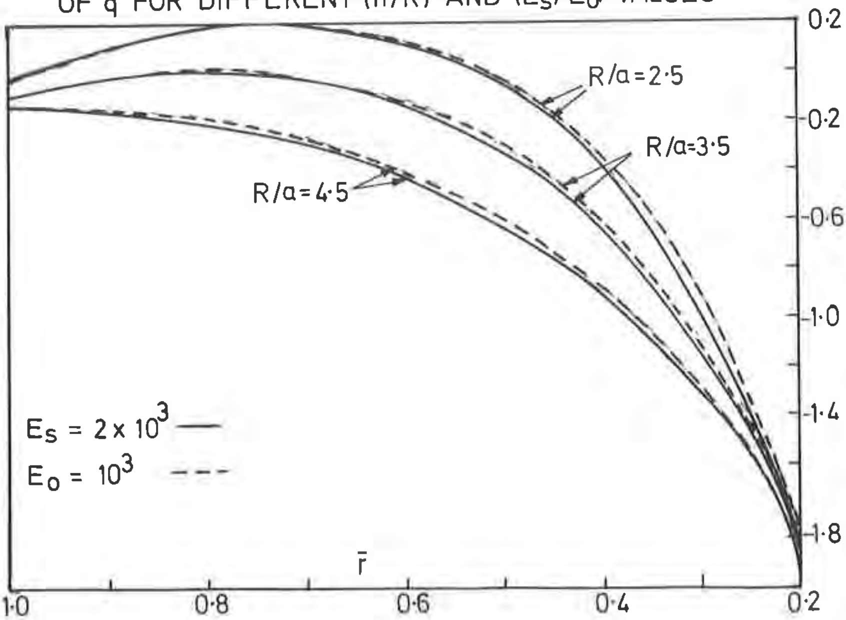


FIG.8 SIMPLY SUPPORTED SPHERICAL SHELL; NORMAL LOAD VARIATION OF $\tilde{q}$ FOR DIFFERENT (R/a) AND (E_S/E_O) VALUES