

A STUDY ON
THE LAW OF THE ITERATED LOGARITHM FOR ARRAYS OF RANDOM VARIABLES

By

Tien-chung Hu^{*}

Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan 30043
R.O.C.

Abstract: For sequences of independent, identically distributed random variables, it is well known that the existence of second moment implies the law of iterated logarithm. Can it be extended from sequences to arrays even under the stronger assumption of the existence of higher moments? In this note, we provide a counter example which tell us that the law of iterated logarithm for arrays is false even for bounded random variables.

^{*}The author now is visiting at the Department of Statistics, the University of North Carolina at Chapel Hill.

1. INTRODUCTION

In traditional probability theory, the strong limit laws refer to those theorems which deal with almost sure convergence of a sequence of random variables: Let $\{X_n\}$ be a sequence of independent, identically distributed random variables, let $S_n = \sum_{k=1}^n X_k$, $n = 1, 2, \dots$. Kolmogorov (1930) proved the following result, known as the strong law of large numbers,

$$(1) \quad \frac{1}{n} S_n \rightarrow EX_1 \text{ a.s. as } n \rightarrow \infty \text{ if and only if } E|X_1| < \infty.$$

If the $\{X_n\}$ are assumed, in addition, to satisfy the conditions $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geq 1$, Hartman and Wintner (1941) proved another fundamental result, known as the law of iterated logarithm,

$$(2) \quad P \left\{ \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2 \varphi_n^2 \log \log \varphi_n^2}} = 1 \right\} = 1,$$

where $\varphi_n^2 = \text{Var}(S_n)$.

However, many important statistical problems are concerned with arrays of random variables, (for example, kernel density estimation and bootstrap estimation) thus it is also natural to consider the strong limit laws for arrays of random variables. So it is important to extend the law of large number and the law of iterated logarithm from sequences of random variables to arrays of random variables. First, we have to ask can these be extended and what are the minimal moment conditions? Let $\{X_{nk}\}$ be an array of independent, identically distributed random variables with $EX_{nk} = 0$ for all n and k . Let $S_n = \sum_{k=1}^n X_{nk}$ for $n = 1, 2, \dots$. Hu, Moricz and Taylor (1989) obtained the strong law of large numbers for arrays $\{X_{nk}\}$. That is,

$$(3) \quad \frac{1}{n} \sum_{k=1}^n X_{nk} \rightarrow EX_{11} \text{ a.s. as } n \rightarrow \infty \text{ if and only if } EX_{11}^2 < \infty.$$

If we compare (1) and (3), it seems that if the finiteness condition for the first moment of a sequence of independent, identically distributed random variables is changed to the finiteness condition for the second moment of an array of independent, identically distributed random variables, then the law of large numbers can be extended from sequences to arrays. It is then natural to ask if the law of iterated logarithm holds for arrays of random variables if the r th moment is finite for some $r \geq 2$. That is, can $E|X_{11}|^r < \infty$ imply

$$(4) \quad P \left\{ \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2 \varphi_n^2 \log \log \varphi_n^2}} = 1 \right\} = 1,$$

where $\varphi_n^2 = \text{Var}(S_n)$? The answer is no. In fact, in this note, we claim that even if $\{X_{nk}\}$ are bounded random variables, (4) still does not hold. A theorem which can explain why the law of iterated logarithm is false for arrays of random variables is given in the section 2. The proof of the theorem is given in the section 3.

2. MAIN RESULTS (counter example)

In this section, we investigate the law of iterated logarithm for arrays of random variables. A result is found, which is different from traditional results of sequences. Also, it can easily be seen that the law of iterated logarithm will not hold even for arrays of bounded random variables. Now the result is stated in the following theorem.

Theorem Let $\{X_{nk}; k = 1, 2, \dots, n = 1, 2, \dots\}$ be an array of independent, identically distributed random variables with $P\{X_{nk} = \pm 1\} = \frac{1}{2}$ for all n and k .

Then

$$P \left[\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log n}} = 1 \right] = 1.$$

where $S_n = \sum_{k=1}^n X_{nk}$.

From the theorem we can easily see that

$$\limsup \frac{S_n}{\sqrt{2n \log \log n}} = \infty \text{ with probability 1.}$$

Therefore, the law of iterated logarithm cannot hold in general for the setup of arrays of random variables.

3. PROOF

The following lemmas are needed in the proof of our theorem and can be found in the literature (see Chow and Teicher, 1978). For the sake of brevity we state them without proof.

Lemma 1. For all $x > 0$.

$$\frac{x}{1+x^2} e^{-x^2/2} \leq \int_x^\infty e^{-u^2/2} du \leq \frac{1}{x} e^{-x^2/2}.$$

Lemma 2. If $\{B_n\}$ is a sequence of random variables having binomial distributions with parameter n and p , $0 < p < 1$, and $\{a_n\}$ is a sequence of real numbers with $a_n \rightarrow \infty$, $a_n = o(n^{1/6})$ as $n \rightarrow \infty$, then for all x ,

$$P\left[\frac{B_n - np}{\sqrt{npq}} \leq a_n\right] \sim \Phi(a_n)$$

where $q = 1-p$ and $\Phi(x)$ is the distribution function of standard normal random variable.

Proof of Theorem.

It suffices to show that for every $\epsilon > 0$

$$(5) \quad P\left[\frac{S_n}{\sqrt{2n \log n}} > 1+\epsilon \text{ i.o.}\right] = 0$$

and

$$(6) \quad P\left[\frac{S_n}{\sqrt{2n \log n}} > 1-\epsilon \text{ i.o.}\right] = 1$$

which are respectively equivalent to, by virtue of Borel-Cantelli lemma,

$$(7) \quad \sum_{n=1}^{\infty} P\left[\frac{S_n}{\sqrt{2n \log n}} > 1+\epsilon\right] < \infty$$

and

$$(8) \quad \sum_{n=1}^{\infty} P\left[\frac{S_n}{\sqrt{2n \log n}} > 1-\epsilon\right] = \infty.$$

Let $Y_{nk} = \frac{X_{nk}+1}{2}$ and $S_n^* = \sum_{k=1}^n Y_{nk}$. Then $\{S_n^*\}$ is a sequence of independent random variables having binomial distributions with parameter n and $\frac{1}{2}$. From Lemma 1 and Lemma 2, we have

$$\begin{aligned} P\{S_n > (1+\epsilon) \sqrt{2n \log n}\} &= P\{S_n^* > \frac{1}{2} (1 + \epsilon) \sqrt{2n \log n} + \frac{n}{2}\} \\ &= P\left[\frac{S_n^* - \frac{n}{2}}{\sqrt{\frac{n}{4}}} > (1 + \epsilon) \sqrt{2 \log n}\right] \\ &\sim \frac{1}{(1 + \epsilon) \sqrt{2 \log n}} e^{-(1+\epsilon)^2 \log n} \\ &= \frac{1}{(1+\epsilon) \sqrt{2 \log n} n^{(1+\epsilon)^2}}. \end{aligned}$$

Similarly, we have

$$P(S_n > (1 - \epsilon) \sqrt{2 n \log n}) \sim \frac{1}{(1+\epsilon) \sqrt{2 \log n} n^{(1+\epsilon)^2}}.$$

Thus, (7) and (8) hold. The proof is completed.

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