Transient Thermal Stress Analysis of Multi-Layered Composite Laminate Cylinder and its Analytical Extension to Non-Homogeneous Materials

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1 INTRODUCTION

Many kinds of composite materials, which consists of composite structure of matrix material and mixture for reinforcement, have been developed recently. And evaluation or research of their mechanical behaviors have been carried out widely (Vinson et al, 1986). As one of composite materials, Metal Matrix Composites have been developed as new materials which may be adaptable for the super high-temperature environment, and it would be predicted that they show the complicated and characteristic mechanical behaviors, these phenomena could not be shown in isothermal problems due to the external loading.

In the present paper, we have analyzed the transient plane thermal stress problem of a multi-layered composite laminated cylinder due to symmetrical and asymmetrical heating. So far as analytical developments for the multi-layered composite cylinder, we have introduced the method of Laplace transform to the temperature field and Airy's stress function method to the thermoelastic field, and then evaluated the temperature and thermal stress distributions in a transient state. Moreover, we have applied the theoretical developments proposed in the present paper into the analysis of a cylinder with nonhomogeneous material properties such as a functionally gradient material.

2 ANALYSIS

2.1 Heat Conduction Problem

We consider here a multi-layered composite laminate cylinder made of numerous layers of different materials, as shown in Fig.1, having its inner and outer radii a and b. Throughout the paper, the indices $i(1,2,\cdots,n)$ are associated with layers of composite cylinder from inner side, respectively. Let t_i be the thickness of the i-th layer. We assume that the multi-layered composite laminate cylinder is suddenly heated from the inner and outer surfaces by surrounding media with relative heat transfer coefficients h_a and h_b. Here we consider the analytical developments for the case of asymmetrical heating. Then, we denote the temperature of the surrounding media by the symmetrical functions $F_a(\theta)$ and $F_b(\theta)$ with respect to $\theta=0$, and assume that the composite cylinder is initially at zero temperature. The transient heat conduction equation for the i-th layer in dimensionless form can be shown as

$$\overline{T}_{i},_{\tau} = \overline{\kappa}_i \overline{\Delta}\, \overline{T}_i \qquad ; \; i=1\sim n \qquad (1)$$

and the initial and thermal boundary conditions in dimensionless form are taken in the following forms:

$$\tau=0 \;\; : \; \overline{T}_i=0 \qquad ; \; i=1\sim n \qquad (2)$$

$$\rho=\bar{a} \quad : \bar{T}_{1,\rho}-H_a\bar{T}_1=-H_a f_a(\theta) \tag{3}$$

$$\rho=R_i \quad : \bar{T}_i=\bar{T}_{i+1} \quad , \quad \bar{\lambda}_i\bar{T}_{i,\rho}=\bar{\lambda}_{i+1}\bar{T}_{i+1,\rho} \qquad ; \; i=1\sim(n-1) \tag{4}$$

$$\rho=1 \quad : \bar{T}_{n,\rho}+H_b\bar{T}_n=H_b f_b(\theta) \tag{5}$$

where

$$R_i=\bar{a}+\sum_{m=1}^{i}\bar{t}_m \quad , \quad \bar{\Delta}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \tag{6}$$

In expressions (1)-(6) we introduced the following dimensionless values:

$$\begin{aligned}&\bar{T}_i=T_i/T_0,\ \rho=r/b,\ \bar{t}_i=t_i/b,\ \tau=\kappa_0 t/b^2,\ \bar{\kappa}_i=\kappa_i/\kappa_0,\ H_a=bh_a,\ H_b=bh_b\\&\bar{\lambda}_i=\lambda_i/\lambda_0,\ \bar{a}=a/b,\ f_a(\theta)=F_a(\theta)/T_0,\ f_b(\theta)=F_b(\theta)/T_0\end{aligned} \tag{7}$$

where T_i is temperature change, κ_i is thermal diffusivity, λ_i is thermal conductivity, t is time, and T_0 , κ_0 and λ_0 are typical values of temperature, thermal diffusivity, and thermal conductivity, respectively. Throughout the paper, a comma denotes a partial differentiation with respect to a variable it follows. Now, we expand the temperature functions $f_a(\theta)$ and $f_b(\theta)$ into the following series forms

$$f_a(\theta)=\sum_{m=0}^{\infty}a_m cosm\theta, \quad f_b(\theta)=\sum_{m=0}^{\infty}b_m cosm\theta \tag{8}$$

$$\left\{\begin{matrix}a_m\\b_m\end{matrix}\right\}=\frac{\varepsilon_m}{\pi}\int_0^{\pi}\left\{\begin{matrix}f_a(\theta)\\f_b(\theta)\end{matrix}\right\} cosm\theta d\theta, \quad \varepsilon_m=\left\{\begin{matrix}1 \; ; \; m=0\\2 \; ; \; m\geqq 1\end{matrix}\right. \tag{9}$$

Introducing a method of Laplace transform, the solution of equation (1) can be obtained with aid of residue theorem so as to satisfy the conditions (2)-(5). Then, the transient part of temperature solution is given as follows:

$$\bar{T}_{iu}=\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\frac{2exp(-\omega_{mj}^2\tau)}{\omega_{mj}\Delta'_m(\omega_{mj})}[\bar{A}_{im}(\omega_{mj})J_m(\beta_i\omega_{mj}\rho)+\bar{B}_{im}(\omega_{mj})Y_m(\beta_i\omega_{mj}\rho)]\; cosm\theta \tag{10}$$

where

$$\begin{aligned}&\{\Delta'_m(\omega_{mj}),\bar{A}_{im}(\omega_{mj}),\bar{B}_{im}(\omega_{mj})\}=\{d\Delta_m/d\omega,\bar{A}_{im},\bar{B}_{im}\}\big|_{\omega\to\omega_{mj}}\\&\beta_i^2=1/\bar{\kappa}_i\end{aligned} \tag{11}$$

where Δ_m is a determinant of the $2n$X$2n$ matrix $[a^m{}_{kl}]$, and the coefficients $\bar{A}_{im}$ and $\bar{B}_{im}$ are defined as the determinant of the matrix similar to the coefficient matrix $[a^m{}_{kl}]$, in which the $(2i-1)$-th column and $2i$-th column are each exchanged by the constant vector $\{c^m{}_k\}$. The nonzero elements $a^m{}_{kl}$ and $c^m{}_k$ among the coefficient matrix $[a^m{}_{kl}]$ and the constant vector $\{c^m{}_k\}$, are given as follows:

$$\begin{aligned}
&a^m{}_{11}=\beta_1\omega\, J_{m+1}(\beta_1\omega\bar{a})+(H_a-m/\bar{a})J_m(\beta_1\omega\bar{a})\\
&a^m{}_{12}=\beta_1\omega\, Y_{m+1}(\beta_1\omega\bar{a})+(H_a-m/\bar{a})Y_m(\beta_1\omega\bar{a})\\
&a^m{}_{2n,2n-1}=(H_b+m)J_m(\beta_n\omega)-\beta_n\omega\, J_{m+1}(\beta_n\omega)\\
&a^m{}_{2n,2n}=(H_b+m)Y_m(\beta_n\omega)-\beta_n\omega\, Y_{m+1}(\beta_n\omega)\\
&a^m{}_{2i,2i-1}=J_m(\beta_i\omega\, R_i), \quad a^m{}_{2i,2i}=Y_m(\beta_i\omega\, R_i)\\
&a^m{}_{2i,2i+1}=-J_m(\beta_{i+1}\omega\, R_i), \quad a^m{}_{2i,2i+2}=-Y_m(\beta_{i+1}\omega\, R_i)\\
&a^m{}_{2i+1,2i-1}=\bar{\lambda}_i[mR_i^{-1}J_m(\beta_i\omega\, R_i)-\beta_i\omega\, J_{m+1}(\beta_i\omega\, R_i)]\\
&a^m{}_{2i+1,2i}=\bar{\lambda}_i[mR_i^{-1}Y_m(\beta_i\omega\, R_i)-\beta_i\omega\, Y_{m+1}(\beta_i\omega\, R_i)]
\end{aligned}$$

$$a^m_{2i+1,2i+1}=-\bar{\lambda}_{i+1}[mR_i^{-1}J_m(\beta_{i+1}\omega R_i)-\beta_{i+1}\omega J_{m+1}(\beta_{i+1}\omega R_i)]$$
$$a^m_{2i+1,2i+2}=-\bar{\lambda}_{i+1}[mR_i^{-1}Y_m(\beta_{i+1}\omega R_i)-\beta_{i+1}\omega Y_{m+1}(\beta_{i+1}\omega R_i)] \ ; \ i=1\sim(n-1)$$
$$c^m_1=H_a a_m, \quad c^m_{2n}=H_b b_m \tag{12}$$

in which $J_m(x)$ and $Y_m(x)$ are the m-th order Bessel function of the first and second kind of argument x. And ω_{mj} represent the j-th positive roots of the following transcendental equation:

$$\Delta_m(\omega)=0 \tag{13}$$

On the other hand, for the sake of brevity, the steady part of temperature solution is omitted here.

2.2 Thermal Stress Problem

In order to find the thermal stress distributions for the foregoing asymmetric temperature distributions, we shall introduce the Airy's thermal stress function χ_i. The fundamental equation governing the plane-strain problem may be expressed in the known form.

$$\bar{\Delta}\bar{\Delta}\bar{\chi}_i=-\bar{k}_i\bar{\Delta}\bar{T}_i, \quad \bar{k}_i=\bar{\alpha}_i\bar{E}_i/(1-\nu_i) \quad ; \ i=1\sim n \tag{14}$$

The stress components in dimensionless form can be given by the stress function as follows:

$$\bar{\sigma}_{rri}=\rho^{-1}\bar{\chi}_{i,\rho}+\rho^{-2}\bar{\chi}_{i,\theta\theta}, \quad \bar{\sigma}_{\theta\theta i}=\bar{\chi}_{i,\theta\theta}, \quad \bar{\sigma}_{r\theta i}=-(\rho^{-1}\bar{\chi}_{i,\theta})_{,\rho} \tag{15}$$

The solution $\bar{\chi}_i$ of Eq.(14) can be obtained as to satisfy the conditions of single valuedness for displacement components and rotation, called Michell's conditions. From Eq.(15), the thermal stress components can be evaluated with aid of the stress function χ_i. Assuming that the composite cylinder is in the plane strain state that axial strain is not bounded, the axial strain component is given in dimensionless form as

$$\bar{\varepsilon}_{zzi}=\bar{E}_i^{-1}\{\bar{\sigma}_{zzi}-\nu_i(\bar{\sigma}_{rri}+\bar{\sigma}_{\theta\theta i})\}+\bar{\alpha}_i\bar{T}_i\equiv C_z \tag{16}$$

Using the stress-strain and strain-displacement relations, the displacement components are obtained. For example, the expression for the radial displacement component $\bar{u}_{ri}$ is

$$\begin{aligned}
\bar{E}_i(1+\nu_i)^{-1}\bar{u}_{ri}=&2(1-2\nu_i)D_{i0}\rho-\bar{E}_i(1+\nu_i)^{-1}\nu_iC_z\rho-E_{i0}\rho^{-1}\\
&+\{D_{i1}\rho^{-2}+(1-4\nu_i)E_{i1}\rho^2+g_i\}\cos\theta-\sum_{m=2}^{\infty}\{mC_{im}\rho^{m-1}\\
&-mD_{im}\rho^{-m-1}+(m-2+4\nu_i)E_{im}\rho^{m+1}-(m+2-4\nu_i)F_{im}\rho^{-m+1}\}\cos m\theta\\
&+\frac{\bar{k}_i}{4D_0}\{2\bar{A}'_{i0}\rho+\bar{B}'_{i0}\rho(2\ln\rho-1)\}+\frac{\bar{k}_i}{8D_1}(3\bar{A}'_{i1}\rho^2+4\bar{B}_{i1}\ln\rho)\cos\theta\\
&+\frac{\bar{k}_i}{4}\sum_{m=2}^{\infty}D_m^{-1}\{\frac{m+2}{m+1}\bar{A}'_{im}\rho^{m+1}+\frac{m-2}{m-1}\bar{B}'_{im}\rho^{-m+1}\}\cos m\theta\\
&-\bar{k}_i\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\frac{2\exp(-\omega_{mj}^2\tau)}{\beta_i^2\omega_{mj}^3\Delta'_m(\omega_{mj})\rho}[\bar{A}_{im}(\omega_{mj})\{mJ_m(\beta_i\omega_{mj}\rho)\\
&-\beta_i\omega_{mj}\rho J_{m+1}(\beta_i\omega_{mj}\rho)\}+\bar{B}_{im}(\omega_{mj})\{mY_m(\beta_i\omega_{mj}\rho)\\
&-\beta_i\omega_{mj}\rho Y_{m+1}(\beta_i\omega_{mj}\rho)\}]\cos m\theta
\end{aligned} \tag{17}$$

In Eq.(17), D_{i0}, E_{i0}, D_{i1}, E_{i1}, C_{im}, D_{im}, E_{im}, F_{im}, g_i and C_z are unknown constants, which should be determined from mechanical conditions at the boundaries and the interfaces. If both the cylindrical surfaces are traction free, then the boundary conditions and the conditions of continuity at the interfaces can be represented as follows:

$$
\left.
\begin{array}{ll}
\rho=\bar{a} & : \bar{\sigma}_{rr1}=0,\ \bar{\sigma}_{r\theta 1}=0 \\
\rho=R_i & : \bar{\sigma}_{rri}=\bar{\sigma}_{rri+1},\ \bar{\sigma}_{r\theta i}=\bar{\sigma}_{r\theta i+1},\ \bar{u}_{ri}=\bar{u}_{ri+1},\ \bar{u}_{\theta i}=\bar{u}_{\theta i+1} \quad ;\ i=1\sim(n-1) \\
\rho=1 & : \bar{\sigma}_{rrn}=0,\ \ \bar{\sigma}_{r\theta n}=0 \\
\multicolumn{2}{l}{\sum_{i=1}^{n} \int_0^{2\pi} \int_{R_{i-1}}^{R_i} \bar{\sigma}_{zzi}\rho \, d\rho \, d\theta, \quad R_0=\bar{a}}
\end{array}
\right\} \quad (18)
$$

In expressions (14)-(18) the following dimensionless quantities are introduced:

$$
\left.
\begin{array}{l}
\bar{\chi}_i=\chi_i/(\alpha_0 E_0 T_0 b^2),\ \bar{\alpha}_i=\alpha_i/\alpha_0,\ \bar{E}_i=E_i/E_0 \\
(\bar{\sigma}_{rri},\bar{\sigma}_{\theta\theta i},\bar{\sigma}_{r\theta i},\bar{\sigma}_{zzi})=(\sigma_{rri},\sigma_{\theta\theta i},\sigma_{r\theta i},\sigma_{zzi})/(\alpha_0 E_0 T_0) \\
\bar{\varepsilon}_{zzi}=\varepsilon_{zzi}/(\alpha_0 T_0),\ (\bar{u}_{ri},\bar{u}_{\theta i})=(u_{ri},u_{\theta i})/(\alpha_0 T_0 b)
\end{array}
\right\} \quad (19)
$$

where α_i is the coefficient of linear thermal expansion, E_i is Young's modulus of elasticity, ν_i is Poisson's ratio and α_0 and E_0 are typical values of the coefficient of linear thermal expansion and Young's modulus of elasticity, respectively.

In the above mentioned formulations, we have discussed the problem for asymmetrical heating. For the sake of brevity, the analysis for the case of symmetrical heating is omitted here. Because the temperature solution for such case corresponds to the one for the case of $m=0$ in Eq.(10). Then, the thermal displacement and thermal stress components, u_{ri}, σ_{rri}, $\sigma_{\theta\theta i}$ and σ_{zzi} can be obtained by using Airy's thermal stress function method as well as the case of asymmetrical heating.

3 NUMERICAL RESULTS AND DISCUSSION

For illustrative purpose to the foregoing analysis, we have performed the numerical calculation for asymmetrical heating. The numerical results are presented for the following values

$$
\begin{array}{l}
H_a=H_b=10.0,\ \bar{a}=0.5,\ \bar{t}_i=(1-\bar{a})/n,\ \theta_0=30° \\
f_a(\theta)=H(\theta-|\theta_0|),\ f_b(\theta)=0 \ \text{(heat supply from inner surface)}
\end{array} \quad (20)
$$

where $H(x)$ is Heaviside's function. From a viewpoint of heat-resistant composite material, we assumed that metal matrix composite materials are made of alumina and aluminum alloy. These material properties are shown in Table 1. And, the typical values of material properties, used to normalize the numerical data, are based on those of aluminum alloy.

Figures.2-4 show the results of the 5-layered composite hollow cylinder laminated alternately by alumina and aluminum alloy in which the inner material is alumina in order to enhance an adiabatic effect. The variations of temperature are shown in Figs.2 and 3, and the variations of thermal stress are shown in Fig.4. From Fig.2, the adiabatic effect of alumina can be seen clearly by the alumina's low thermal diffusivity. From Fig.3, on account of the different of material properties such as the coefficient of linear thermal expansion and Young's modulus, the thermal stress vary characteristically in each layer, and especially large discontinuities occur at the interface of each layer.

Next, we applied the theoretical developments proposed in our present paper into the analysis for nonhomogeneous material of the thermal stress relaxation type (functionally gradient material). We considered the transient thermal stress problem of nonhomogeneous cylinder, and assumed that the cylinder has the material properties for alumina at heated cylindrical surface and the one for aluminum-alloy at cooled surface, and then their material properties vary continuously in radial direction of the cylinder. We can analyze the thermoelastic behavior for such nonhomogeneous material under the assumption that the nonhomogeneous cylinder is composed of numerous hypothetical layers with homogeneous and different material properties. We carried out numerical calculations assuming that material properties vary linearly in radial direction of the cylinder. In the following numerical results, we assigned n=20 for the number of hypothetical layer. Figures.5 and 6 show the variations of temperature and thermal stress distributions along the thickness direction, and these numerical results show the smooth and continuously distributed data.

On the other side, in order to compare the results of nonhomogeneous cylinder shown in Fig.6, the numerical results of the 2-layered composite hollow cylinder, made of alumina and aluminum alloy in which the inner material is alumina, are shown in Fig.7. The effect of relaxation of stress distributions for nonhomogeneous cylinder can be clearly seen compared with the 2-layered composite laminate cylinder.

We can conclude that the analytical procedure proposed in this paper can be extended to the transient thermal stress problems of multi-layered composite laminate cylinders under the perfectly three-dimensional temperature conditions or multi-layered composite laminate spheres.

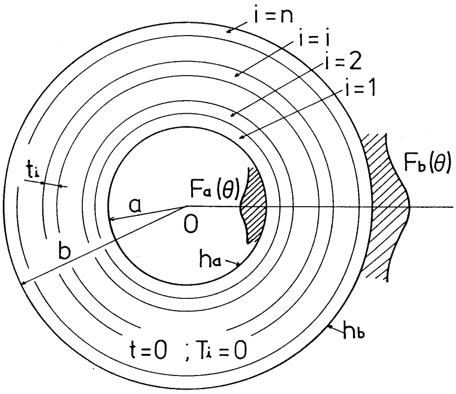

Fig.1. Coordinate system and thermal boundary conditions

Table 1. Material properties

Property	Al.Alloy	Al_2O_3
κ_i [m^2/s]	90.6×10^{-6}	11.9×10^{-6}
λ_i [W/mK]	22.2×10	3.6×10
α_i [1/K]	23.6×10^{-6}	8.0×10^{-6}
E_i [GPa]	7.0×10	34.3×10
ν_i	0.33	0.22

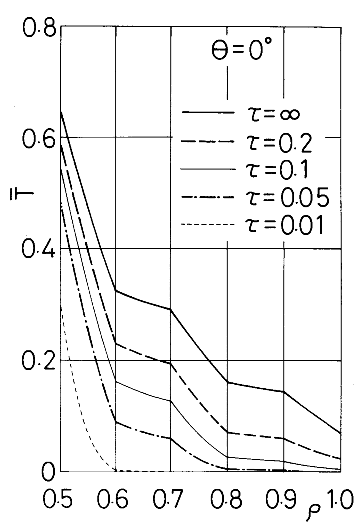

Fig.2. Temperature variation in the radial direction (θ=0°)

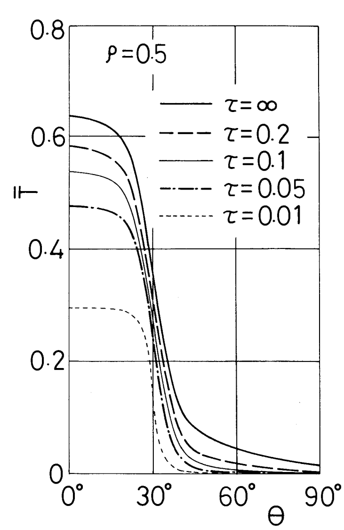

Fig.3. Temperature variation in the circumferential direction (ρ=0.5)

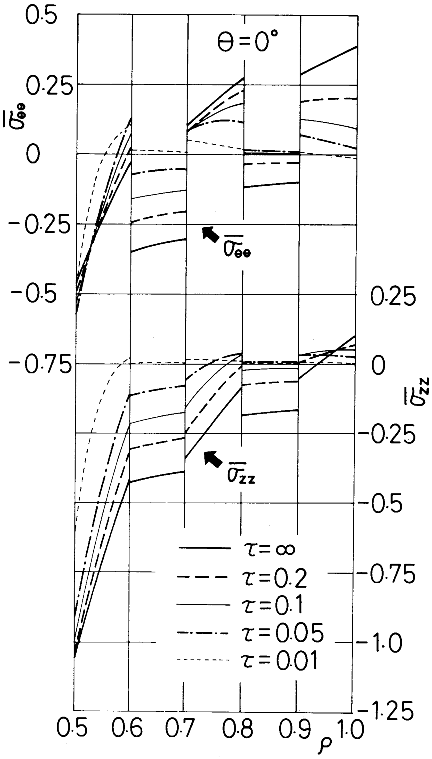

Fig.4. Thermal stress variations in the radial direction (θ=0°)

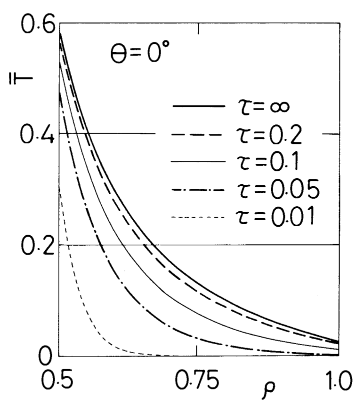

Fig.5. Temperature variation in the radial direction (θ=0°)

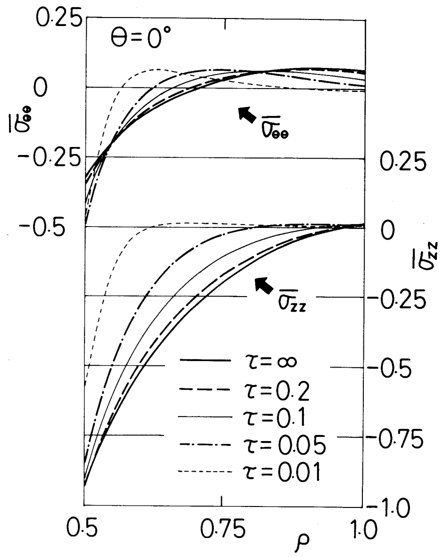

Fig.6. Thermal stress variations in the radial direction (θ=0°)

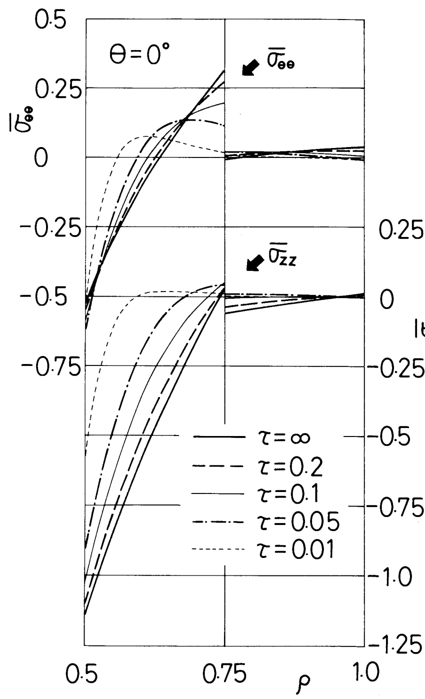

Fig.7. Thermal stress variations in the radial direction (θ=0°)

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