ABSTRACT

KESSLER, SARAH. Fraction Errors in a Digital Mathematics Environment: Latent Class and Transition Analysis. (Under the direction of Dr. Teomara A. Rutherford).

Student struggles with fractions are well documented, and due to fractions' importance to later mathematics achievement, identification of the errors students make when solving fraction problems is an area of interest for both researchers and teachers. Data were student errors on fraction problems in pre and post-quizzes in a digital mathematics environment. Students (N=1,431) were grouped by prevalence of error types using latent class analysis. Four different classes of error profiles were identified in the pre-quiz data. A latent transition analysis was then used to determine if class membership and class structure changed from pre- to post-quiz. Both the types of errors made and the frequency that they were made transformed from pre- to post-quiz. Identification of co-occurrence of and changes to fraction errors has implications for curricular design and pedagogical decisions.

Fraction Errors in a Digital Mathematics Environment: Latent Class and Transition Analysis.

by
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BIOGRAPHY

Sarah Kessler earned her B.S. in Psychology and B.S. in Mathematics from Longwood University in 2011. She is currently a Masters student in the department of Curriculum and Instruction and will continue her education as a doctoral student in the Teacher Education and Learning Sciences department.

During her undergraduate career, Sarah tutored middle and high schoolers in a variety of mathematics topics. Additionally, she tutored undergraduates in introductory and advanced statistics. It was her experience tutoring students with a range of mathematical knowledge that inspired her to pursue Educational Psychology and research mathematics education.

Since entering graduate school, Sarah has grown immensely as a researcher. She has worked on a multitude of research projects with topics on help seeking, spatial ability, problem-solving, and fraction performance. Her work as been presented at annual conferences for the Society for Research on Educational Effectiveness (2016), the American Educational Research Association (2016, 2017), the Association for Psychological Science (2016), and the American Psychological Association (2016, 2017). Also, she is a co-author on a paper currently under review. Sarah has worked as a graduate research assistant, math interventionist, and study skills instructor.

Sarah earned Honorable Mention on her application to the National Science Foundation's Graduate Research Fellowship Program in 2016 and in 2017, she was awarded the fellowship.

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CHAPTER 1

INTRODUCTION

For over twenty years, educators have been concerned with students' struggles with fractions (Brown & Quinn, 2012). An analysis of the 1990 National Assessment of Educational Progress revealed that less than half of 12th graders were successful with fractions, percentages, decimals, and simple algebra (Mullis, Dossey, Owen, & Phillips, 1991). These rates have not improved in more recent years across a number of U.S. states (e.g., California Department of Education, 2008; Higgins, 2008; Kim, Schneider, Engec, & Siskind, 2006; Pennsylvania Department of Education, 2011). Struggle with fractions is especially concerning because fraction knowledge is important for later success in mathematics (NMAP, 2008). In a study by Siegler and colleagues (2012), fraction and whole-number division knowledge were the strongest predictors of both algebra achievement and overall mathematics achievement five to six years later, even after controlling for a host of demographic and prior achievement factors. Similarly, Bailey, Hoard, Nugent, and Geary (2012) found that fraction achievement in sixth grade predicted general mathematics achievement in seventh grade, also after controlling for demographic and prior achievement. Additionally, when comparing associations between math achievement and fraction knowledge to associations between math achievement and whole number knowledge, fraction knowledge is a stronger predictor of math achievement than whole number knowledge (e.g., Bailey et al., 2012; Booth & Newton, 2012; Siegler et al, 2012). This supports the hypothesis that certain early math skills, especially fraction skills, may be more predictive of later mathematics achievement than are other early skills (e.g., Bailey et al., 2012).

With fractions critical to later mathematics achievement, it is important to understand some of the reasons students struggle with fractions. Additionally, there is a continued interest in understanding the common errors students make when solving mathematics problems, especially with fractions (Bottge, Ma, Gassaway, Butler, & Toland, 2014; Voza, 2011). These error analyses can help researchers and educators address fraction misunderstandings. This paper will contribute to our understanding of students' fraction errors by identifying which errors students in a new environment (ST Math) make as well as identifying whether students can be grouped according to these errors and their co-occurrence.

1.1. Why Do Students Struggle with Fractions?

Prior research has offered a number of reasons why students struggle with fractions: fractions are both one-dimensional and two-dimensional (e.g., Pantziara & Philippou, 2012), fraction algorithms are far more complex than natural numbers (e.g., Hiebert, 1992; Stafyliduo & Vosnidou, 2004), and students often experience a gap between conceptual and procedural knowledge (e.g., Bailey, Siegler, & Geary, 2014; Jordan, Hansen, Fuchs, Siegler, Gersten, & Micklos, 2013; Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2016). The following sections explain how these factors contribute to student difficulty with fraction understanding.

1.1.1 Properties of fractions. Fractions are thought to have five subconstructs: part-whole, ratio, quotient, operator, and measure (Pantziara & Philippou, 2012). Measure, which references where a fraction lies on the number line, is the only subconstruct that is one-dimensional (Pantziara & Philippou, 2012). The other four subconstructs are two-dimensional, because they consider the relationship between the numerator and denominator (Pantziara & Philippou, 2012). Lamon (2012) defined each of these constructs: part-whole is the "number of equal parts of a unit out of the total number of equal parts into which the unit is divided" (p. 145); ratio is the "comparison of any two quantities" (p. 225); quotient is the "result of a division" (p. 171); and operator is the "measure of some change in quantity from a prior state" (p. 47). For example, 2/3 could be considered two of three equal parts (part-whole), two parts to three parts (ratio), two divided by three (quotient), or two-thirds of something (operator).

Although many children may understand fractions as a part-whole, this concept alone is not enough for complete understanding of fractions (Lamon, 2012; Pantziara & Philippou, 2012). Charalambous and Pantazi (2007) included knowledge of a fraction's measure as necessary for developing proficiency in fractions and note that performance on fraction measure tasks is poor compared to tasks that represent the other constructs. One reason may be because having a two-dimensional number makes it harder for students to grasp the concept of measure. Determining the relationship between the numerator and denominator makes it harder for students to perform simple algorithms (like ordering or placing a fraction on the number line). Once again, consider 2/3 and compare it to 5/6. It difficult to immediately know which fraction is larger. Seeing that both five and six are larger than two and three, students may believe that 5/6 is larger. However, students are taught that the bigger the denominator, the smaller the part. This may lead to students saying 5/6 is the smaller number because it has the bigger denominator. It is not until we look at the relationship between two and three (for 2/3) and five and six (for 5/6), that we can be sure which is larger, 5/6.

1.1.2. Fractions algorithms are more complex than natural numbers. Fractions are far more complex than natural numbers both in their properties and in the algorithms associated with fractions. As described above, fractions can be both one- and two-dimensional, whereas natural numbers are solely one-dimensional. This dimensionality also places a critical role in the algorithms used for arithmetic (i.e., adding, subtraction, multiplication, and division). With a simple form, ordering natural numbers is easy—all one must do is compare the numbers in the respective place value; however, ordering fractions does not solely rely on either the numerator or the denominator. Instead, it is the relationship between the two numbers that determines where a fraction falls on the number line (Hiebert, 1992; Stafyliduo & Vosnidou, 2004).

The algorithms used for addition, subtraction, multiplication, and division are also more complex for fractions than for natural numbers. When adding or subtracting natural numbers, one can simply combine the digits of the same place value and regroup if necessary. However, when adding or subtracting fractions, one must first find the least common denominator, transform the fractions so they will have the same common denominator, then add/subtract the numerators. Although multiplication of fractions has the potential to be the simplest of the algorithms (multiply the numerators together and multiply the denominators together), the result of multiplying fractions is harder to understand than natural number multiplication. Fraction multiplication can make numbers either bigger or

Table 1. *Comparison of fractions and natural numbers*

	Natural Numbers	*Fractions*	*Arithmetic*
Form	Takes the form ab	Takes the form a/b	
Ordering	Ordering number depends on comparing similar place values	Ordering depends on the relationship between numerator and denominator	
Quantities	Discrete	Discrete and continuous	
Addition/ Subtraction	Combining digits of same place value	Use equivalent fractions with common denominator then combine numerators	a/b + c/d = da/db + bc/db = (da+bc)/db
Multiplication	Multiplication makes number bigger	Multiplication/division makes number bigger or smaller	a/b × c/d = ac/bd
Division	Division makes number smaller		a/b ÷ c/d = a/b × d/c = ad/bc

Note. Differences as stated in prior research

smaller, instead of just bigger, as when multiplying natural numbers other than one (Hiebert, 1992; Stafyliduo & Vosnidou, 2004). Similarly, division of fractions can either make the quotient bigger or smaller, instead of just smaller as with natural numbers (Stafyliduo & Vosnidou, 2004). Division of fractions adds further complexity in that division is a multiple-step process as typically carried out—requiring the inversion of the second fraction and then multiplication of the fractions together (Hiebert, 1992).

The typical order of mathematics instruction may compound student difficulties understanding fraction complexity. In most curricula, students learn natural numbers then fractions (e.g., Common Core State Standards Initiative, 2016). Although learning fractions after learning natural numbers is a logical sequence, students often have a hard time reconciling the schema of natural numbers with that of fractions (Hiebert, 1992; Mack, 1995; Stafyliduo & Vosnidou, 2004). Following from above, the algorithmic symbols for fractions do not have the same meaning as the algorithmic symbols for whole numbers. The change in the procedures associated with these symbols can be confusing for children who need to modify their schema for the symbols (e.g., the procedure for adding natural numbers compared to the procedure for adding fractions; Hiebert, 1992, p. 294). Thus, it is not just the concepts of fractions that are complex, their procedures are complex as well and often do not correspond to the procedures with whole numbers. Table 1 summarizes the differences between natural numbers and fractions.

1.1.3. Conceptual versus procedural knowledge. Each of the aforementioned challenges to understanding fractions highlight the difficulties students may have in gaining conceptual knowledge of fractions. Conceptual and procedural knowledge have a bidirectional relationship but are separate entities that contribute to fraction success (Baroody & Ginsburg, 1986; Rittle-Johnson & Alibali, 1999; Siegler & Stern, 1998). Thus, researchers sometimes split fraction knowledge into concepts and procedures (e.g., Bailey, Siegler, & Geary, 2014; Jordan et al., 2013; Jordan et al., 2016). Conceptual knowledge is "not any tidbit of knowledge in and of itself but rather the understanding of how these tidbits of knowledge are interrelated to each other" (Hallet, Nunes, Bryant, & Thorpe, 2012, p. 470). By comparison, procedural knowledge is "the knowledge of how to perform mathematical tasks…meant to generate the right answer to a given type of problem" (Hallet et al., 2012, p. 470). Although both conceptual and procedural knowledge are important to fraction achievement, reliance on procedural knowledge alone is unlikely to result in true understanding and advanced achievement—students who rely primarily on procedural knowledge may apply it incorrectly because they lack the conceptual knowledge that tells them when and why to use specific procedures (Hallet, 2008).

1.2. What Types of Errors Do Students Make?

It is important to understand both *why* students struggle with fractions and *how* they struggle with fractions. To understand how students struggle with fractions, researchers have looked at the errors students make when solving fraction problems (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016). To do this, researchers give participants a test/activity and observe errors made. Most tests given in this manner have been free response (e.g., 1/2 + 3/4 = ______), although some have included multiple choice (e.g., Brown & Quinn, 2006). Although this procedure is used among all fraction error researchers, there are differences in methods in terms of what skills are tested and how errors are addressed. For example, some researchers have focused on specific fraction knowledge, like ordering or adding/subtracting fractions (Bottge et al., 2014; Malone & Fuchs, 2016); whereas other focused on a broad range of fraction knowledge (Ashlock, 2001; Brown & Quinn, 2006). Additionally, some researches define categories of errors a priori and then examine tests to see how often students make these errors (Bottge et al., 2014; Malone & Fuchs, 2016); whereas other researchers first give tests and then code and define errors from student answers (Ashlock, 2001; Brown & Quinn, 2006). Below, I will describe the types of fraction errors identified by prior research.

1.2.1. Writing fractions. In one error analysis, Ashlock (2001) found students made two types of errors when writing a fraction to represent the shaded part of a figure. In one, the student would write the fraction as the shaded part of the figure over the unshaded part of the figure (ratio). For example, if a figure was split into four equal parts and one was shaded, the error would be to write 1/3 instead of 1/4. This is incorrect because the denominator represents the whole, not just another part. In another, the student would not realize that all parts needed to be of equal size. For example, even if the figure had four unequal parts, the student would still consider the denominator to be four.

1.2.2. Reducing fractions. Ashlock's (2001) test also had students reduce fractions. There were also two main errors students made with reducing fractions to their simplest form (Ashlock, 2001). Some students would try to simplify when impossible—they would try to simplify a fraction that was already in lowest terms (e.g., 3/8 to 1/4). Others would reduce the numerator but not the denominator and thus generate a non-equivalent fraction. For example, students may reduce 2/4 to 1/4 instead of 1/2, reducing only the numerator.

1.2.3. Ordering fractions. Malone and Fuchs (2016) gave students nine sets of three fractions to order, and identified two categories of errors students made when ordering fractions. Whole number ordering is when the student orders fractions based on the numerator or denominator instead of the relationship between the two. Within whole number ordering, the error could be denominator or numerator specific or a combination of the two. Thus, Malone and Fuchs (2016) considered three categories of whole number ordering—whole number ordering, whole number ordering-numerator specific, and whole number ordering-denominator specific. The other ordering error was called smallest denominator-biggest fraction. This is a more complex error that stems from the idea that the smaller denominators mean fewer but larger parts. Thus, if a student recognizes that the smaller denominator has the bigger parts, they may believe that the smaller denominator fraction is larger because it has bigger parts. However, this error happens when the students fail to recognize that fractions are the relationship between two numbers and that just because it has bigger parts, does not mean it is larger (e.g., $1/3 < 3/5$).

1.2.4. Fraction addition/subtraction. There were three common errors identified in the literature in problems involving adding and subtracting fractions. The most common error was when the students would add/subtract the two numerators and the two denominators (e.g., $1/2 + 3/4 = 4/6$ instead of $1/2 + 3/4 = 2/4 + 3/4 = 5/4$). This error was addressed by Ashlock (2001), Bottge et al. (2014), and Brown and Quinn (2006), and has been called the independent whole number strategy because it treats the numerator and denominator as independent whole numbers (Tian & Siegler, 2016). Another mistake was that the students would fail to find equivalent fractions to add/subtract that have the same denominator (e.g., $1/2 + 3/4 = 1/4 + 3/4 = 4/4$ instead of $1/2 + 3/4 = 2/4 + 3/4 = 5/4$; Ashlock, 2001; Bottge et al., 2014). Lastly, Bottge et al. (2014) also found that students would sometimes add all the numbers in both the numerators and denominators to create one total sum (e.g., $1/2 + 3/4 = 1 + 2 + 3 + 4 = 10$).

1.2.5. Fraction multiplication/division. When multiplying fractions, Ashlock (2001) noted two common errors. If multiplying two fractions, some students cross-multiplied wrong (e.g., $a/b \times c/d = (a \times d)/(b \times c)$ instead of $a/b \times c/d = (a \times c)/(b \times d)$). If multiplying a whole number by a fraction, the students multiplied both the numerator and denominator by the whole number, which would create an equivalent fraction (e.g., $3 \times 3/4 = (3 \times 3)/(3 \times 4) = 9/12 = 3/4$). Similarly, Ashlock (2001) found one error when students were dividing fractions. Other times, the students did not follow all of the steps of dividing fractions (keep the first fraction, change division to multiplication, and flip the second fraction). They might flip the first, instead of the second, fraction or they might incorrectly carry out the multiplication.

1.3. ST Math

ST Math, created by MIND Research Institute, is an interactive instructional software for computers and tablets that is based on visual instruction (MIND Research, 2016). ST Math is based on theory that suggests that the ability to visualize mathematics concepts leads to better conceptual knowledge and performance (see Geary, 1995; National Research Council, 2005; Shaw & Peterson, 2000) and has been previously shown to result in small improvements in math achievement, especially on topics involving number sense (Rutherford et al., 2014; Schenke, Rutherford, & Farkas, 2014). ST Math is currently used in 45 states and aligns with both Common Core and relevant state standards. It is intended to be supplemental to the schools' mathematics curriculum and is intended to be used twice a week for 45 minutes.

Within ST Math, students progress through a number of objectives focused on specific math concepts. Before students begin the objective, they must complete a five-question multiple choice pretest on that objective's content. The objective's content follows a hierarchical pattern: objective, sub-objective (optional), game, level, puzzle (see Figure 1). The objective may include sub-objectives that further denote specific math skills. Within the objective/sub-objective, there are a variety of games that use the same imagery and design throughout their levels. There are between one and 10 levels per game that increase in difficulty. Within each level, students complete interactive puzzles, which are the delivery method for the mathematics content. For each level, the student has

Figure 1. Structure of ST Math. The first figure is an objective. Within the objective (top-right figure), there are quizzes, represented by paper and pencil, and sub-objectives, the boxes. Within the sub-objectives (bottom-left), there are games (circles) and within games (bottom-right), there are levels (bars). Lastly, within the levels are puzzles.

between one and three lives—if they answer more puzzles incorrectly than they have lives, they are removed from the level and can chose to replay it or replay a previously-passed level. After demonstrating mastery of the content within an objective, the student then completes a five-question multiple choice posttest that mirrors the pretest, question-for-question in topic, but uses different specific examples and numbers. The pre- and post-quizzes have either three or four answer choices for each question.

1.4. Current Study

There is an extant body of working examining what errors students make (e.g., Ashlock, 2001; Brown & Quinn, 2006). However, there is little research about what errors tend to be made together. Understanding what errors students tend to make together can enhance how the errors are addressed. To this end, I have the following aims:

1. Determine errors that can be made in ST Math quizzes
2. Develop classes of student struggles based on what errors they make.
3. Determine if these classes change between pre-quiz and post-quiz.

CHAPTER 2

METHOD

2.1. Participants

Participants are third graders from a school district in central Florida participating in a larger NSF-funded project (Grant Number 1544273) relating gameplay within ST Math to student achievement and motivation. Students within the district played ST Math as part of their normal instruction during the 2015-2016 school year. Although there were data on 4,370 third grade students within the district that played ST Math, the current sample was limited to third (N = 1,431) graders with pre- and post-quiz data from the fraction objectives. The 1,431 students attended 75 different schools, 54% were male, 60% were white, and 46% qualified for free or reduced lunch. The sample was statistically different on most demographics from the larger sample of third graders (except for male, Hispanic, and Other

Table 2. *Demographics*

Variable	Percent of Sample (N=1,431)	Percent of Total (N=4,370)
Male	54%	52%*
Student with Disability	10%	17%
Free/Reduced Lunch	46%	58%
English Language Learner	11%	13%
Gifted	19%	13%
Race		
Asian	6%	4%
Black	13%	21%
Hispanic	16%	18%*
White	60%	53%
Other	5%	4%*

Note. Differences between total and analysis samples were determined using chi-squared tests. There were 75 schools in the sample and 95 schools in total.
*All significantly different at $p < 0.01$ unless marked (percent of male, Hispanic, and other).

races). This may be because the sample was limited to students playing on-grade level. Thus, the total sample may have a higher percentage of students using ST Math for remedial reasons. For more details on the demographics, see Table 2.

2.2. Measures of Fraction Errors

As implemented in Florida, the ST Math curriculum aligns with the Florida Standards, which introduce fractions in the third grade (Mathematics Florida Standards, 2014; MIND Research, 2016). The ST Math third grade curriculum has three fraction objectives—Fraction Concepts, Fractions on the Number Line, and Comparing Fractions. The pre-quiz and post-quiz from the same objective had parallel questions of the same format but with different numbers. Each quiz (pre- and post-quiz) had five questions with three or four multiple choice answers. Thus, there were 30 fraction questions in third grade.

2.2.1. Fraction Concepts. The Fraction Concepts objective quizzes had two types of questions. The first type of question showed a figure split into equal parts with some of the parts shaded. The student then matched the figure to a written fraction. The other type of question involves matching whole numbers to their equivalent fractions (e.g., 3 = 3/1). See Figure 2 for examples.

2.2.2. Fractions on the Number Line. Fractions on the Number Line had three different types of quiz questions. First, a number line was shown with a point marked. The student must match the point to a fraction. Second, a student was given three fractions (1/2, 1/4, and 1/3) and must identify the appropriate number line. Although the question asked specifically for 1/2, 1/4, and 1/3, there are other fractions on the number lines as well. Third,

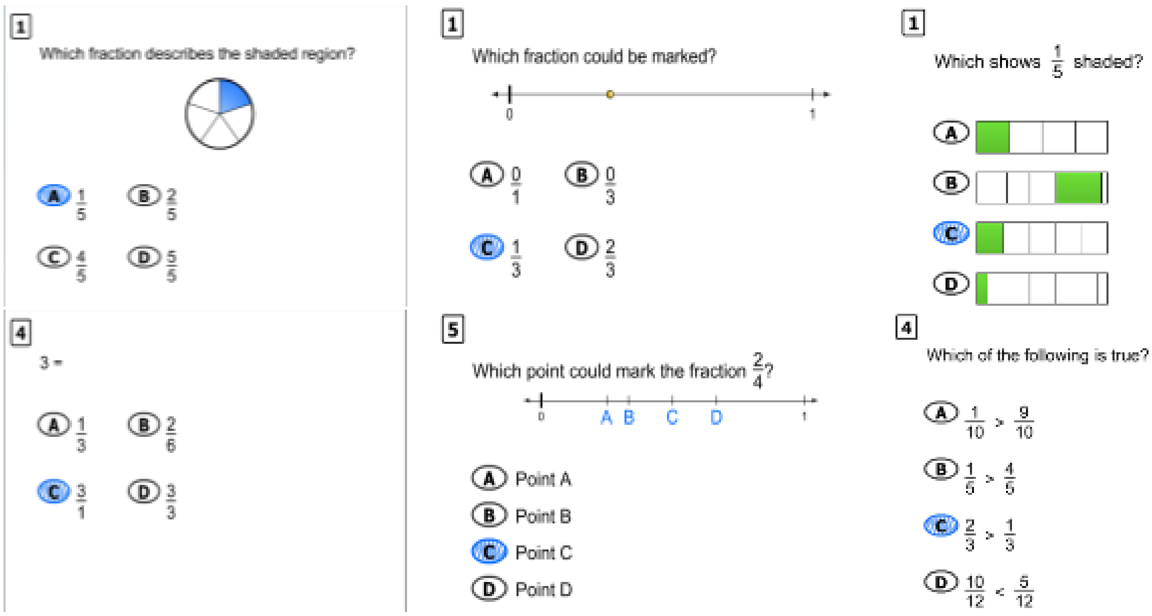

Figure 2. Example of quiz questions. The two questions on the left are from Fraction Concepts, the two in the middle are from Fractions on the Number Line, and the two on the right are from Comparing Fractions.

the student was given a fraction and must match it to a point on the number line. See Figure 2 for examples.

2.2.3. Comparing Fractions. The last third grade fraction objective, Comparing Fractions, also had three different types of quiz questions. The first was similar to the Fraction Concept questions where the student matches a figure to a written fraction. The second was the same question from Fractions on the Number Line where the student was given three fractions (1/2, 1/4, and 1/3) and must identify the appropriate number line. Additionally, there were questions where students must identify the correct statement of magnitude comparisons. This had two different forms. The first had the student identify the correct fraction sentence. For example, the student would have to pick which of the following statements were true: A: 1/10 > 9/10; B: 1/5 > 4/5; C: 2/3 > 1/3; D: 10/12 < 5/12 (pre-quiz, question four). The other form provided the student with a fraction and the answer choices were other fractions. For example, question three of the post-quiz asked the student to pick the fraction that completed the statement “1/7 < ___” with the options of 1/2, 1/8, 1/9, and 1/12. See Figure 2 for examples.

2.3. Method

2.3.1. Error coding. To code the errors students made, only incorrect answer choices were included in the analysis. Overall, there were 86 incorrect answer choices for third grade. To analyze the fraction errors, there were three key steps. First, answer choices were qualitatively coded using a priori codes and a posteriori codes. The a priori codes were taken from previous research on fraction errors (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016; see Appendix Table 1 for codes). After the first round of coding was finished, I collaborated with personnel at MIND Research to develop more accurate a posteriori codes for the incorrect answer choices. Incorrect answer choices were often created to represent common errors MIND had observed students make in prior gameplay; thus, error codes were adjusted and/created from these as well. Lastly, codes were refined through a final round of coding. This final round was to determine consistent names for codes and to find similar errors that could be categorized together (e.g., complement and reciprocal errors). Even after the three rounds of coding, there were still some answer choices that did not fit with a known or postulated error pattern. These answer choices were deemed filler (or random) choices. For example, one question asked what fraction represented a circle with five parts that has one part shaded (e.g., 1/5). One answer choice was 2/5, which does not follow any identifiable error logic. Thus, this answer choice was coded as “filler.”

From the 86 incorrect answer choices, I coded eight different error types and the filler choice, for a total of nine error categories. The type of error that could be made depended on the question being asked. From the three objectives used, there were four question types. The first is matching written fractions to a visual model. This includes questions such as "Which fraction describes the shaded region?" and "Which shows ___ shaded?" for both mixed numbers and proper fractions. There were 12 questions that have the student match written fractions to a visual model. Similarly, there was a question type of finding equivalent fractions. The difference between this question type and the previous question is that these only have written numbers and fractions, there are no visual models. There were two formats of questions for finding equivalent fractions. The first asked the student to match a whole number to a fraction. The other asked "Which number makes the number sentence true?" (e.g., 1/2 = ___/4). For this question type, there were eight questions. Details and examples of all error types are described in the results section.

2.3.2. Data cleaning. To make the error codes usable data, several steps were followed in the statistics analysis software *Stata* (version 14; StataCorp, 2015). First, each question was turned into its own variable. The values of the question variable were its answer choices (A, B, C, or D). Then, each answer choice was recoded as correct or as its attributed error category. For example, one question had the following answers—A: correct; B: filler; C: complement; D: inequivalent. At this point, the data were separated into two subsets—third grade pre-quiz and third grade post-quiz. New variables were created that represented the proportion of times the student made that specific error to the total errors the student made. In this manner, if a student made the complement/reciprocal error twice and the filler error three times, they would have a total of five errors. The proportion for the complement/reciprocal error would be 0.4 (2/5) and the proportion for the filler error would be 0.6 (3/5). All other errors would have proportions of zero because they did not make the errors. Errors are represented in this way to clearly depict how often the student made that particular error. The last step was creating an ordinal scale to be used in the latent class and transition analyses:

0. the student did not make the error;
1. the student made the error between 0% and 25% of the time ($0 < x \leq 0.25$);
2. the student made the error between 25% and 50% of the time ($0.25 < x \leq 0.50$);
3. the student made the error between 50% and 75% of the time ($0.50 < x \leq 0.75$);
4. the student made the error between 75% and 100% of the time ($0.75 < x \leq 1$).

2.4.3. Latent class and transition analysis. Latent class analysis (LCA) is a mixture modeling method used with categorical variables. It attributes the relationship between variables to an unobserved, latent variable (Collins & Lanza, 2010; Goodman, 2002; Nylund, Asparouhov, & Muthen, 2007). The goal of LCA is to group people into classes based on their observed variables. These classes represent the underlying categorical latent variable. This procedure is similar to factor analysis; however, it provides a person-centered approach instead of a variable-centered approach. Thus, the classes are created to group similar participants together, rather than similar variables (Collins & Lanza, 2010; Goodman, 2002). The person-centered approach allows for generalizations of the patterns of behavior. In this study, LCA was used to determine fraction error patterns. Models with k versus $k + 1$ classes were tested iteratively to determine best fit. To determine best fit, sample size adjusted

Bayesian Information Criterion (adjusted BIC), Bayes Factor, Entropy, and the bootstrap likelihood ratio test (BLRT) were used (Nylund et al., 2007). After conducting the LCA, class membership was regressed on demographics and game-play variables to further understand the composition of each class and external variables that predict class membership.

To determine if class membership changes between time points, such as between a pre and posttest, latent transition analysis (LTA) is used (Rindskopf, 2010). Rindskopf (2010) defines LTA

> as a statistical model in which (i) latent categorical constructs are defined at two or more time points, (ii) parameters are included that assess initial status and transition probabilities from time i to $i + 1$..., and (iii) observed variables are imperfect indicators of the hypothesized latent variables. (p. 199)

LTA regresses the class variable at the second time point onto the first time point to determine the likelihood of the classes remaining the same. LTA was used to determine if classes remained the same pre- and post-quiz. The *Mplus* software (version 7; Múthen & Múthen, 2016) was used to run both the LCA and LTA.

CHAPTER 3

RESULTS

3.1. Fraction Errors

Four of the eight error codes were a priori—inappropriate sizing/spacing (Ashlock, 2001), inequivalent fractions (Bottge et al., 2014), ratio (Ashlock, 2001), and whole number ordering (Malone & Fuchs, 2016)—and the other five were a posteriori—complement, reciprocal, ignoring information, midpoint, and same numerator/denominator ordering error. Table 3 summarizes the types of questions and the errors possible within each question type. Overall, students made the whole number ordering error the most in the pre-quiz and the same numerator/denominator ordering error most in the post-quiz. Table 4 details the proportions of times each error was made and how many students made it.

3.1.1 Complement. The first error was the complement of the correct fraction. The complement is when the sum of the correct fraction and incorrect fraction equals one. For example, if the question asked what fraction represents the shaded part of a shape and the answer was 2/3, the complement would be 1/3.

3.1.2. Reciprocal. The reciprocal of the correct fraction is when the student flips the numerator and denominator. Following the same example as above, the correct answer would be 2/3 and the reciprocal error would be 3/2.

3.1.3. Ignores information. This error had two different characterizations. The first was when the answer was partially correct but missing a critical part that would make it correct. The second was when the answer matched part of the question. For example, if the student was trying to identify the mixed number 1 3/4, they may only answer 3/4 or only 1,

Table 3. *Summary of question types and their errors*

Question Type	Number of Questions	Errors
Matching written fractions to visual models	8	complement reciprocal ignore information inappropriate sizing/spacing inequivalent numbers ratio
Making equivalent fractions	4	inequivalent numbers ignore information complement reciprocal
Placing fraction(s) on a number line	12	complement reciprocal ignore information inappropriate sizing/spacing midpoint reciprocal whole number ordering
Comparing fractions	6	same numerator/denominator ordering whole number ordering

Note. Question types are from the three fraction objectives.

ignoring the totality of the original prompt.

3.1.4. Midpoint of the number line. This error was found in questions where students had to match fractions to a place on the number line. Even when the fraction was not 1/2 or the midpoint, many students still picked the midpoint of the number line. This may be because 1/2 is the most common fraction and it seemed familiar.

3.1.5. Misconception of equivalent fractions. Unlike whole numbers, fractions can be equivalent to both other fractions and to whole numbers (Bottge et al., 2014). For example, any fraction with the numerator of zero is equivalent to zero (i.e., 0/a = 0) and any fraction with the denominator of one is equivalent to the numerator (i.e., a/1 = a). Students made errors in both of these forms of fraction/whole number equivalence within the ST Math quizzes.

3.1.6. Ratio. Some students selected fractions incorrectly as ratios. In this case, the student selected the fraction as the shaded parts over the unshaded parts instead of the shaded parts over all of the parts (Ashlock, 2001). Thus, if a circle is split into four equal parts with one of the parts shaded, a student would select 1/3 instead of 1/4.

3.1.7. Illogical sizing/spacing. This error represented the misunderstanding that fractions parts must have equal sizes and consistent spacing (Ashlock, 2001). There are two ways this error was reflected. The first was when students were presented with figures divided into unequal parts and were asked to match this figure to a fraction. Some students' answers included the number of parts in the denominator, indicating a failure to recognize that parts of a fraction must be the same size. The second was when students were presented with number lines upon which fractions were placed with incorrect space. For example, there would be an unequal distance between 0 and 1/4, 1/4 and 1/2, and 1/2 and 3/4.

3.1.8. Same numerator/denominator ordering error. This was an ordering error committed when the fractions had the same numerator or denominator. For these errors, the student would only have to compare the numerator or denominator but still made an error. For example, a student may say 1/3 > 2/3 (same denominator).

3.1.9. Whole number ordering. Whole number ordering was an error where the student ordered the fractions based on their numerator or the denominator, without looking at the relationship between the two of them (Malone & Fuchs, 2016). For example, a student might order 1/2, 1/3, and 1/4 in that order, because two is less than three and three is less than four. In this case, they are only ordering based on the denominators and not considering the

Table 4. *Summary of error types*

Description	Proportion of Times Error is Made			Max Times Error Was Made	Number of Students Who Made This Error
	Mean	Min	Max		
Complement of correct fraction (e.g., correct: 2/3, incorrect 1/3)	0.048	0	0.5	3	421
	0.012	0	0.2	1	134
Reciprocal of correct fraction (e.g., correct: 2/3, incorrect 3/2)	0.088	0	1	3	705
	0.109	0	0.5	3	761
Ignores part of the information (e.g., leaving out the whole number in a mixed fraction)	0.154	0	1	4	953
	0.061	0	0.5	3	494
The answer is the midpoint of the number line*	0	0	0	0	0
	0.080	0	0.5	3	614
Misconception of what fractions are equivalent †	0.132	0	1	4	938
	0.086	0	0.5	3	635
Ratio error—the fraction is the shaded part over the unshaded parts†*	0.008	0	0.333	1	99
	0.009	0	0.4	2	82
Unequal parts of a shape or illogical spacing on the number line†	0.056	0	1	3	422
	0.257	0	1	3	1,359
Same numerator/denominator ordering error	0.047	0	1	1	498
	0.202	0	0.5	1	1,399
Whole number ordering (ordering only the numerator or denominators) †	0.386	0	1	6	1,282
	0.123	0	1	4	789
Incorrect answer but no obvious error	0.081	0	1	5	622
	0.061	0	0.5	3	509

Note. N=1,431. Statistics for pre-quiz errors are on top and statistics for post-quiz errors are below
† a priori code * The midpoint error was dropped from analysis because it was not made in both the pre-quiz and post-quiz. The ratio error was dropped because very few students made it.

Table 5. *Total errors made in quizzes*

	Mean	Standard Deviation	Minimum	Maximum
Pre-Quiz	6.621	2.757	0	13
Post-Quiz	5.849	2.316	1	15

fraction as one number.

3.1.10. Filler. The filler error encompasses answer choices that did not appear to follow a known error logic. As an example, consider a question that had students identify a fraction equivalent to zero (Fraction Concepts, pre-quiz question 5 and post-quiz question 4). The answer choices were 1/1, 0/1, 1-0, and 0-1. The correct answer is 0/1 and 1/1 was coded as inequivalent. However, 1-0 and 0-1 were coded as filler. These answer choices were coded as filler because they did not fit with any other error and they address whole number subtraction, more so than fractions.

3.2. Pre-Quiz Latent Class Analysis

As noted above, 10 types of errors could be made in the third-grade quizzes—complement, reciprocal, ignore information, inequivalent, ratio, illogical sizing/spacing, same numerator/denominator ordering, whole number ordering, midpoint, and filler. However, only eight errors were used in the analysis. The midpoint error was dropped from analysis because it was not made by any students during the pre-quiz. The ratio error was dropped because the vast majority of students did not make this error. Analyses were run in *Mplus* using the proportion of times each of the eight errors were made to identify categorical latent variables (see Appendix Figure 1 for sample input). Each model (other than the one class model) used bootstrapping to test the difference between k and $k - 1$ class models. To do this, *Mplus* estimates the $k - 1$ model simultaneously with the k class model. *Mplus Version 7* generates data to fit both models and calculates the bootstrap likelihood ratio value (BLRT; two times the loglikelihood difference). This step is iterated multiple times to obtain the correct BLRT value (Asparouhov & Muthén, 2012).

Table 6. *Table of third grade pre-quiz LCA values*

Class	**Sample Size Adjusted BIC**	**BLRT *p*-value**	**Bayes Factor**
1	20,559		
2	19,476	<0.001	>10
3	19,245	<0.001	>10
4	19,194	<0.001	<1
5	19,148	<0.001	<1

Note. Entropy for three-class model is 0.968.

3.2.1. Fit criteria. The best model was determined using sample size adjusted Bayesian Information Criterion (adjusted BIC), Bayes Factor, and the bootstrap likelihood ratio test (BLRT) (Nylund et al., 2007). A comparison of models with one to five classes is found in Table 6. There was no model that displayed the best fit across every fit statistic. However, the three-class model had the lowest adjusted BIC value of the models that had strong Bayes Factors and was therefore determined to be the best-fitting (above 10; Kass & Raftery, 1995). Although the three-class model did not have the highest entropy, it was still above 0.90 (i.e., students were correctly placed into classes at least 90% of the time).

3.2.2. Pre-quiz error classes. Figure 3 shows the structures of fraction errors in each class. Classes are placed within a consistent position (left to right) within each error grouping. Colors indicate the percent of times, on average, students in that class made the particular error out of their total errors made, as defined in section 2.4.2. Green indicates that the students did not make that error. Blue indicates that, of the errors that student made, they made this error more than 0% and25% of the time. Yellow indicates that, of the errors that student made, they made this error 25%-50% of the time. This pattern follows for orange and red bars as well. The percentages on the y-axis represent how many students in that class had that proportion.

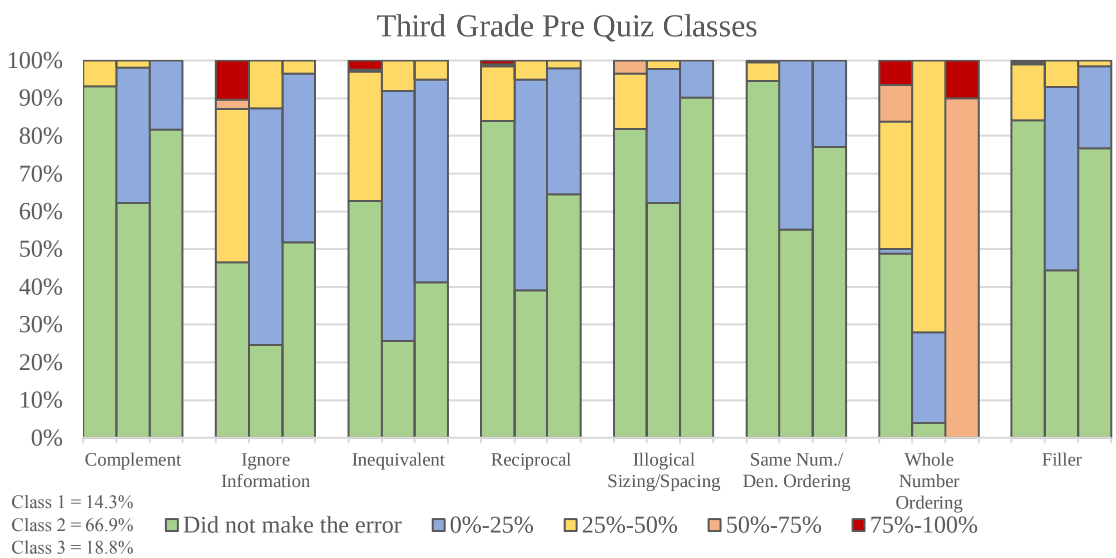

Figure 3. Class structures for pre-quiz errors

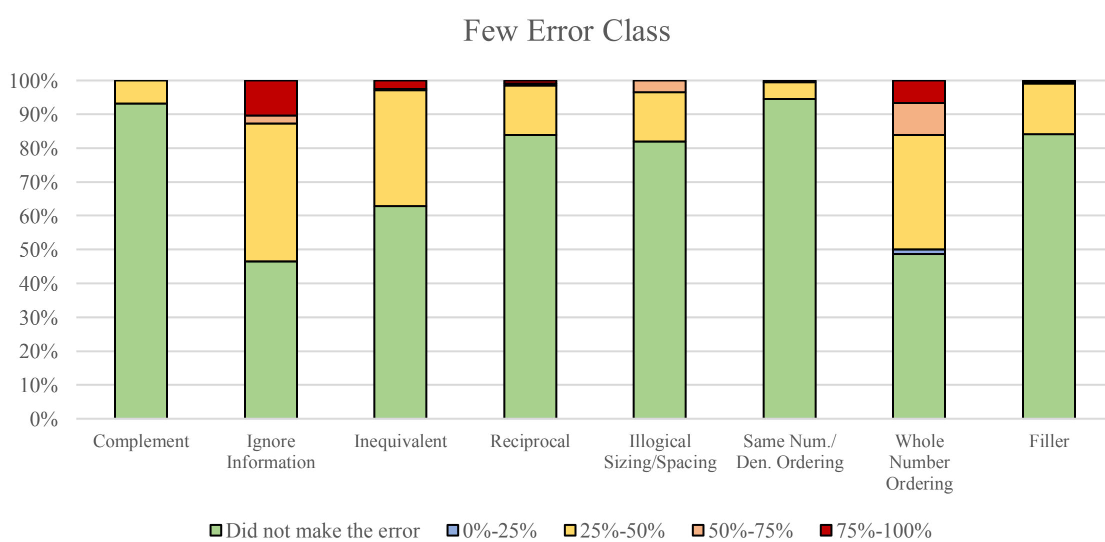

Figure 4. Few errors class

3.2.2.1. Few error class. At first, this pattern may appear puzzling—students in this class have both more green than the other classes and more yellow, orange, and red, indicating that students that share this class membership both made none of certain errors and many of those same errors. On closer observation, this is because the commonality between these students is that they made few errors overall, but when errors were made, they were constrained to one type or a few types of errors. This class made up 14.3% of students.

3.2.2.2. Distributed error class. Class two had the widest distribution of errors, meaning that they made few types of errors more than 25% of the time. The seemingly random nature of their errors may indicate that most of their errors are due to carelessness or general lack of fraction knowledge. This Distributed Error Class makes up the majority of the sample with 66.9%.

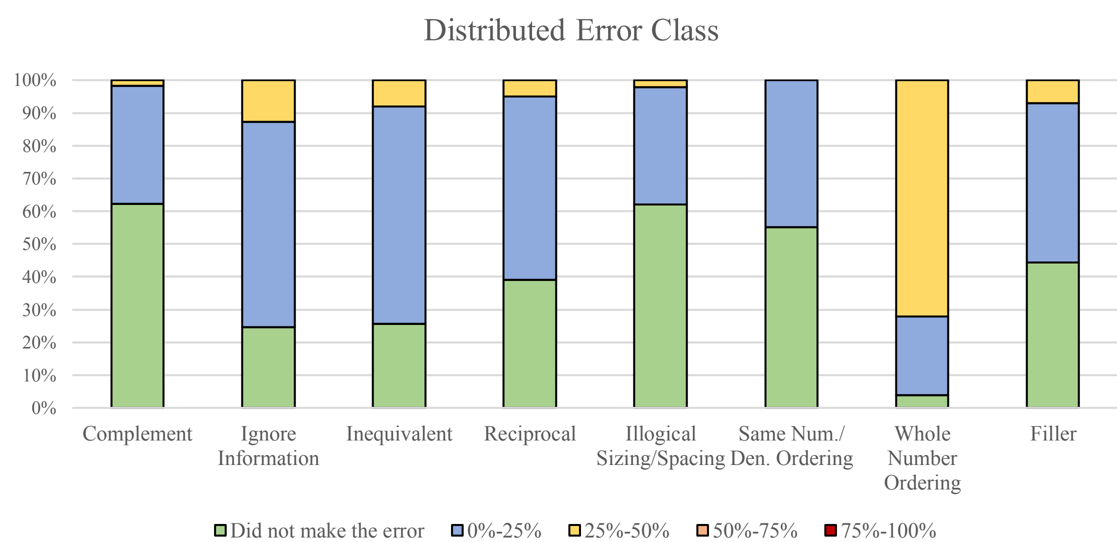

Figure 5. Distributed errors class

3.2.2.3. Whole number ordering error class. The last class made most errors less than 25% of the time. However, the students in this class made the whole number ordering error for at least 50% of their errors. This Whole Number Ordering Error Class makes up 18.8% of the sample.

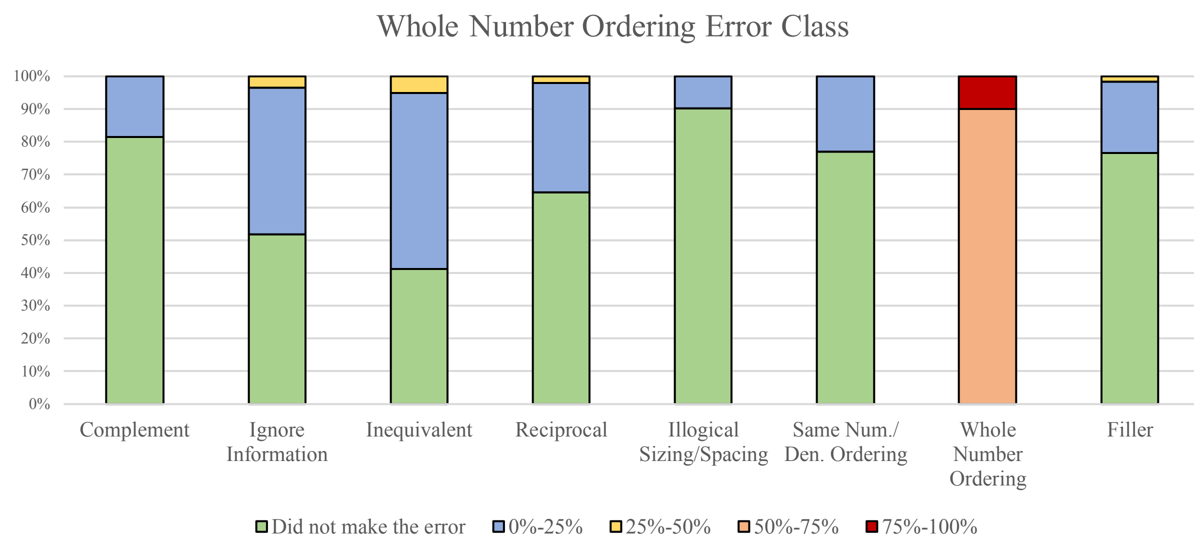

Figure 6. Whole number ordering error class

3.2.3. Differences in the classes. To determine who was in each class, logistic regressions were run on class membership. The odds ratios represent that an increase of one point of the predictor is associated with an increase in odds of class membership, independent of other predictors. For each point increase in average pre-quiz score, students had a 1.37-fold increase in the odds of being classified into the Few Error class. However, for each point increase in average pre-quiz score, students had a 1.08-fold increase of in the odds of not bring classified in the Distributed Error class. Therefore, students who had higher pre-quiz averages tend to be in the Few Errors class, whereas students who had lower pre-quiz averages tend to be in the Distributed Error class (see Table 7).

3.3. Latent Transition Analysis

Constraining the three-class pre-quiz to remain constant, a three-class post-quiz model was estimated by regressing the post-quiz model onto the pre-quiz model. By having the constrained pre-quiz model and unconstrained post-quiz model, the pre-quiz class

Table 7. *Logistic regressions for pre-quiz class membership*

	Few Error	Distributed Error	Whole Number
Male	0.93	1.11	0.76
	-0.29	*0.72*	*-1.78*
Content Progress	1.00	1.00	1.00
	0.56	*-1.31*	*1.46*
Average pre-quiz	1.37[c]	0.93[c]	1.00
	10.15	*-16.48*	*-1.22*
Disability	1.14	1.07	0.85
	0.26	*0.30*	*-0.62*
Free Lunch	0.92	0.16	0.83
	-0.30	*1.08*	*-1.17*
ELL	1.96	0.87	1.10
	-0.09	*-0.61*	*0.40*
Gifted	1.00	1.10	0.72
	-0.01	*0.56*	*0.53*
Race			
Asian	0.67	0.76	1.18
	-0.70	*-0.96*	*0.53*
Black	1.31	0.92	1.29
	0.51	*-0.40*	*1.14*
Hispanic	0.43	1.12	1.02
	-3.01	*0.60*	*0.08*
Other	2.23	1.38	0.61
	1.76	*1.01*	*-1.11*
Constant	<0.01[c]	134.04[c]	<0.15[b]

Note. [a] $p < 0.05$, [b] $p < 0.01$, [c] $p < 0.001$. Odds ratios and z-scores are provided (z-scores are italicized). Clustered by teacher

thresholds do not change because of the regression, whereas the regression determines the class thresholds for post-quiz errors. Additionally, the regression determines the transition probabilities based on the models, i.e. how likely a student is to remain in class one/two/three for pre-quiz and post-quiz.

3.3.1. Post-quiz model. See Figure 7 for structures of fraction errors in each class.

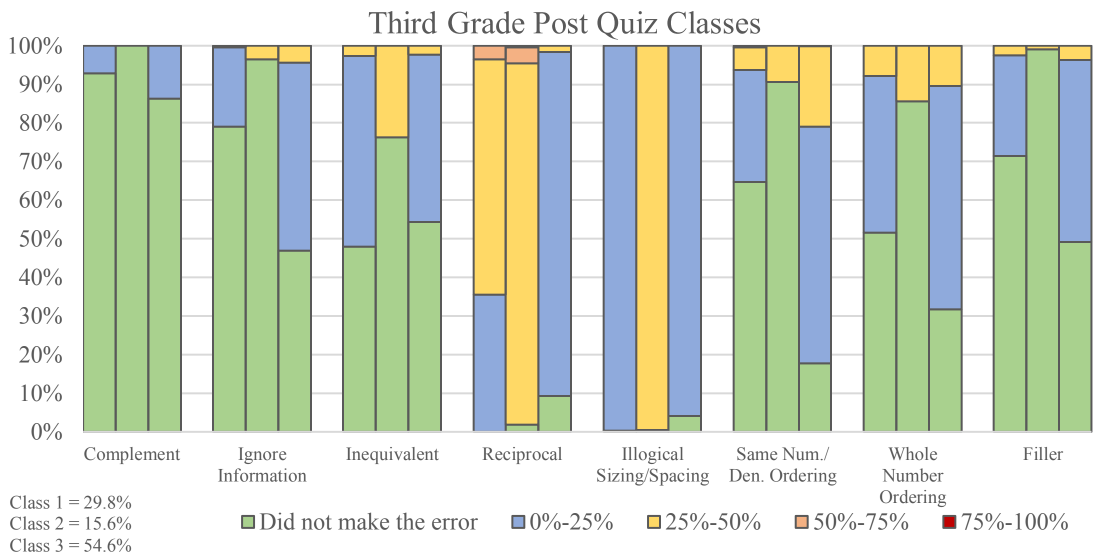

Figure 7. Class structures for post-quiz errors

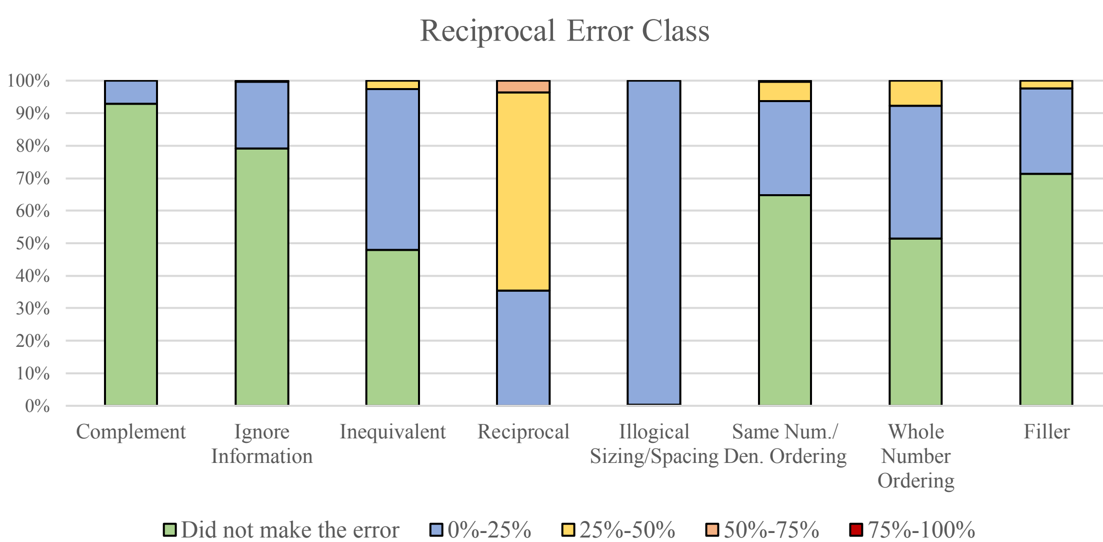

Figure 8. Reciprocal error class

3.3.1.1. Reciprocal error class. The first class, students primarily made errors less than 25% of the time, except for the reciprocal error, which most students made at least once and many made more than 25% of the time. In addition, students within this class made the

Illogical Sizing/Spacing error at least once, but not as much as those in the Class 2. Lastly, this class made more errors 25%-50% of the time than Class 2. This class, Reciprocal Error Class, was 29.8% of the sample (see Figure 8).

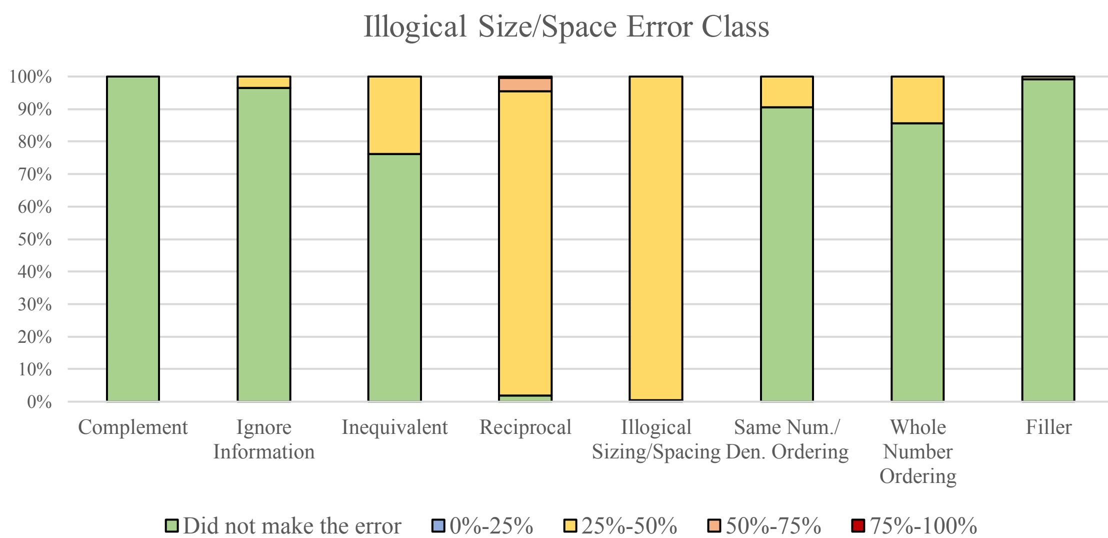

Figure 9. Illogical size/space error class

3.3.1.2. Illogical size/spacing error class. The students in the second class tend to make two errors more than others—illogical sizing/spacing and reciprocal error. Class two primarily made these errors 25%-50% of the time. This class, which was 15.6% of the sample, will be called the Illogical Size/Spacing Class (see Figure 9).

3.3.1.3. Distributed errors class. Similar to the Distributed Error Class in the pre-quiz, the Distributed Error Class in the other classes, students in class three rarely made an error more than 25% of the time. Similar to the previous classes, the reciprocal and illogical size/spacing errors occurred more often than other errors. However, in this class, they are primarily made between 0 and 25 percent of the time (instead of 25%-50% in the other classes). Of the sample, 54.6% were in the Ignoring Information Class (see Figure 10).

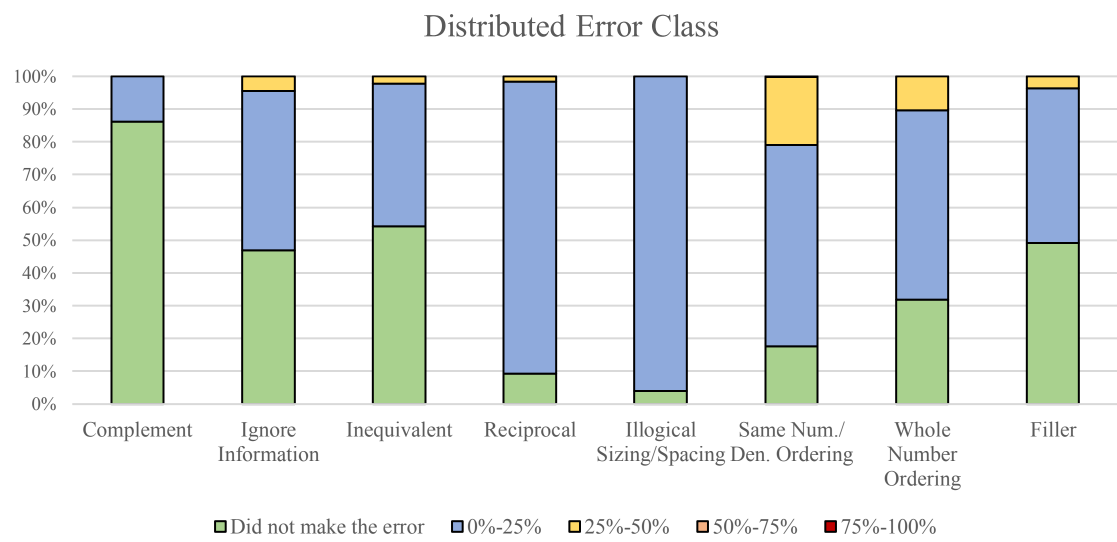

Figure 10. Distributed errors class

3.3.3. Differences in the classes. In the same fashion as the pre-quiz classes, logistic regressions were run to determine which demographic and game-play variables predicted class membership. A student had higher odds of being placed in the Illogical Size/Space Error Class if they had a higher pre-quiz average. Conversely, students were *less* likely to be placed in the Distributed Errors Class if they had a higher pre-quiz average. Unlike the pre-quiz classes, membership to the post-quiz classes was predicted by more than average pre-quiz score. Students who qualified for free or reduced lunch had 1.33-fold increase in odds of not being placed in the Reciprocal Error Class. In addition to higher pre-quiz averages, students were more likely to be placed in the Illogical Size/Spacing Error Class if they had a disability or were considered gifted. Lastly, students who were English Language Learners or were not gifted, were more likely to be in the Distributed Error Class. See Table 8 for logistic regression statistics.

Table 8. *Logistic regressions for post-quiz class membership*

	Reciprocal Error	Illogical Size/Spacing Error	Distributed Error
Male	0.95	0.94	1.02
	-0.38	*-0.40*	*0.17*
Content Progress	1.00	1.00	1.00
	0.04	*0.82*	*-0.58*
Average pre-quiz	1.01	1.08[c]	0.96[c]
	1.69	*10.16*	*-11.14*
Disability	0.93	2.11[b]	0.74
	-0.32	*2.82*	*-1.53*
Free Lunch	0.75[a]	1.18	1.18
	-2.19	*0.81*	*1.28*
ELL	0.80	0.50	1.57[a]
	-1.06	*-1.77*	*2.21*
Gifted	0.84	2.17[c]	0.69[a]
	-1.09	*3.68*	*-2.33*
Race			
Asian	1.17	0.75	0.96
	0.64	*-0.73*	*-0.17*
Black	1.07	0.56	1.22
	0.34	*-1.79*	*1.11*
Hispanic	1.04	1.17	0.89
	0.19	*0.63*	*-0.64*
Other	0.89	0.56	1.51
	-0.38	*-1.31*	*1.39*
Constant	0.89	<0.01[c]	<14.13[c]

Note. [a] $p < 0.05$, [b] $p < 0.01$, [c] $p < 0.001$. Odds ratios and z-scores are provided (z-scores are italicized). Clustered by teacher

3.3.4. Transition probabilities. Although there was a fair amount of movement between the classes, those who were in the Few Errors pre-test class moved only to the Illogical Size/Spacing and Distributed Error post-test classes, among which they were almost evenly split at 55% and 45% respectively.

Students who were in the pre-quiz Distributed Error Class moved to each of the three post-quiz classes. Primarily, these students were placed in the post-quiz Reciprocal Error Class (55%), followed by the post-quiz Distributed Error Class (37%), with only 8% moving to the post-quiz Illogical Size/Spacing Class.

Lastly, the students in the pre-quiz Whole Numbering Ordering Class (those who almost exclusively made the whole number ordering error), followed a similar transition pattern to those in the pre-quiz Distributed Error Class. Of the Whole Number Ordering Class, 47% moved to the post-quiz Reciprocal Error Class, 9% moved to the Illogical Size/Spacing Class, and 44% moved to the Distributed Error Class. See Table 9 for a summary of the latent transition probabilities.

Table 9. *Latent transition probabilities*

		Post-Quiz Classes		
		Reciprocal Error	Illogical Size/Spacing Error	Distributed Error
Pre-Quiz Classes	Few Errors	0.000	0.553	0.447
	Distributed Errors	0.551	0.081	0.369
	Whole Number	0.468	0.091	0.441

Note. Cells represent the probability that a student would be in the post-quiz class given that they were in the designated pre-quiz class.

CHAPTER 4

DISCUSSION

This study had three main aims—(1) determine errors that can be made in ST Math curriculum; (2) develop models of classes of student struggles based on what errors they make; (3) determine if these classes change between pre-quiz and post-quiz.

4.1. Research Aim 1: Fraction Errors in St Math

This study expanded on previous fraction error analyses in two ways. First, it used a different test of fraction knowledge than usual. To determine what errors students make when solving fraction problems, most researchers use a free response test (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016). However, the test used in this study was both multiple choice and on-line. This introduces a more efficient way to understand the errors students make—if researchers can make multiple choice answers with specific errors in mind, they can assume the error by the answer choice instead of coding each written answer. However, because students were required to pick an answer choice, selecting a random wrong (or right) answer was more likely to occur. It is possible that a student would pick an answer choice that matched an error code without actually following the logic to get that error. This may be especially true of students new to fraction problems, who, on open-ended questions, might not provide any answer. By forcing an answer choice, students who are new to fractions may make multiple errors (see Distributed Errors Classes)

In addition to a novel error coding method, this study also described new categories of errors. Of the eight errors coded, six of them were a posteriori as they were not previously described in research on fraction errors—complement, reciprocal, ignoring information, midpoint, and same numerator/denominator ordering error. Excluding the midpoint error, which was not made during the pre-quiz, these a posteriori errors were made just as often, if not more than those assigned a priori codes from prior research. It remains to be seen if these errors would be found outside of the context of ST Math, potentially generalizing outside of the multiple-choice digital mathematics environment.

As the multiple choice nature of the ST Math quiz questions may have influenced the errors that arose, so too may have the visual nature of the questions. The focus within ST Math on visual-spatial instruction is reflected in a number of quiz questions that use visual representations, such as shaded figures or number lines. These questions may elicit different errors than those from the prior studies, which relied primarily on questions that were symbolic (e.g., 1/2 + 3/4 = ____; Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016).

4.2. Research Aim 2: Models of Struggling Students

Three classes of errors were identified using LCA at pre-quiz—Few Errors, Distributed Errors, and Whole Number Ordering Error. Students in the Few Errors Class tended to have the highest pre-quiz averages. The students in this class made few errors, and when they did make errors, tended to make the same error the majority of the time. Although the LCA identified this group as a class, many of the students had little in common regarding the type of errors they made—instead, they were joined together merely by their propensity to make few errors and to make few different types of errors.

The Whole Number Ordering Class contained students who made the whole number ordering error more than 50% of the time. The whole number ordering error is when students did not consider the relationship between the numerator and denominator when ordering fractions, instead they ordered the fractions based on their numerator or the denominator (Malone & Fuchs, 2016). Although other errors were made by students in this class, these other errors were often made less than 25% of the time. None of the demographic or game-play variables measured predicted membership to the Whole Number Ordering Class. This may be because most students will have a similar lack of fraction knowledge when taking the pre-quiz and the whole number ordering error relies on the schema of ordering whole numbers.

Lastly, the largest group of students were placed in the pre-quiz Distributed Error Class. This class made a wide range of errors but did not make most errors more than 25% of the time. Students in this class had the lowest pre-quiz averages, indicating a lower level of fraction knowledge. This may indicate that students with little fraction knowledge do not make errors that indicate one type or a few types of conceptual or procedural misunderstandings, but instead, exhibit a pattern that may be more indicative of guessing. It may be that these students would be those that would leave open-ended questions blank, and therefore would not be attributed to a specific error or classified into an error profile on prior tests of fraction errors.

4.3. Research Aim 3: Class Membership Changes Pre to Post

Both class membership and class structure differed between pre- and post-quizzes. This was expected, because the number of total errors being made decreased as the students learned from ST Math. Only one of the three pre-quiz classes remained in the post-quiz—the

Distributed Error Class. Figure 11 depicts how the class changed from pre-quiz (left columns) to post-quiz (right columns). In general, the main differences between the pre- and

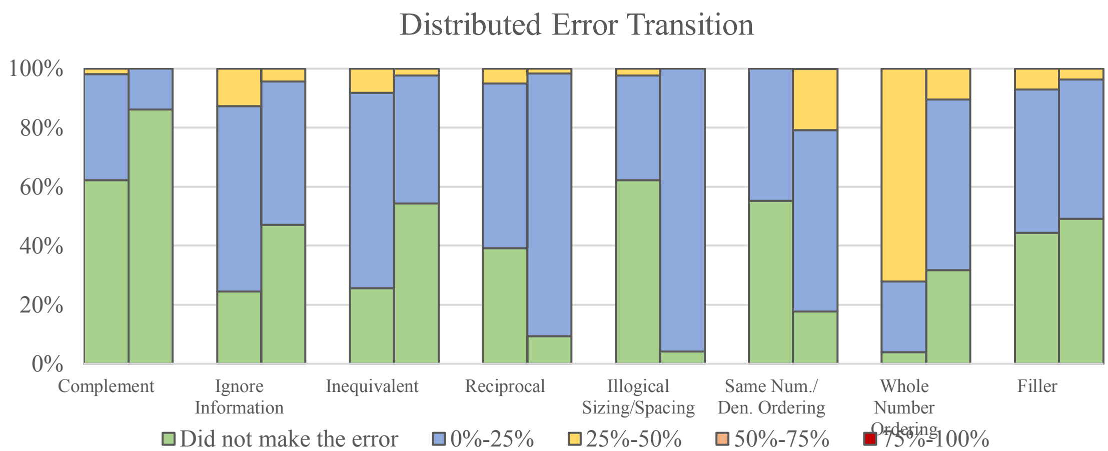

Figure 11. Distributed error class transition

post-quiz class is that errors were made less often in the post-quiz. Given that the students were new fraction learners, it is expected that errors would reduce from pre to post-quiz. Interestingly, for this class, the proportion of errors increased for the reciprocal, illogical size/spacing, and same numerator/denominator ordering categories, which is in line with the higher prevalence of those errors in the other two classes.

4.4. Limitations

A major limitation of this study is determining the accuracy of the errors that were coded. Although many followed obvious error logic, there is the possibility of errors being miscoded because we are not able to know what the student was thinking as they solved the problem. Multiple choice answers provided efficiency in terms of coding; however, without full knowledge of why the student chose the answer choice they did, it is uncertain if the error codes are accurate.

4.5. Implications

By identifying classes of students based on their common fraction errors and how these classes change before and after a learning event, researchers and teachers can better help students within each of the classes. The classes from the pre-quiz may represent a general misunderstanding of fraction concepts (e.g., the Few Errors and Whole Number Ordering classes). On the other hand, classes from the post-quiz may indicate some understanding of fractions but with a critical gap in knowledge (e.g., reciprocal error and illogical size/spacing). Additionally, the Distributed Error classes from both the pre- and post-quiz may show what the average students learns. This class contains the majority of students fall into this class in both the pre- and post-quiz and it is the only class that remains constant. The errors that were more common in the post-quiz than the pre-quiz suggest that more instruction is needed in regards to placing fractions on the number line (illogical size/spacing error) and comparing magnitudes of fractions (same numerator/denominator ordering error).

4.6. Conclusion

Overall, this research adds to the field in multiple ways. First, it has a novel approach to error analysis by examining errors through multiple choice questions. Second, it allows for a deeper understanding of the occurrence and co-occurrences of errors by using a person-center approach. Lastly, it enhances previous research by observing how these error classes change after a learning event. Although this research is novel in many respects, further research is necessary to delve deeper into fraction error profiles, especially with more complex fraction concepts (e.g., fraction multiplication and division).

Because fractions are crucial to later mathematics achievement (e.g., Bailey et al., 2012; Booth & Newton, 2012; Siegler et al, 2012), it is important to understand both how students struggle with fractions. Understanding the error patterns in fractions and how to predict them can identify and help struggling students. By providing this knowledge to teachers, they can help multiple students master fractions and subsequent math concepts, opening a path to all that math success provides.

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APPENDICES

Table 1.

Obj	Quiz	Number	A	B	C	D
SO7	Pre	1	correct	filler	complement/reciprocal	inequivalent
SO7	Pre	2	correct	ratio	filler	complement/reciprocal
SO7	Pre	3	correct	ignore information	filler	complement/reciprocal
SO7	Pre	4	complement/reciprocal	complement/reciprocal	correct	inequivalent
SO7	Pre	5	inequivalent	correct	filler	filler
SO7	Post	1	correct	ratio	complement/reciprocal	filler
SO7	Post	2	ratio	ratio	correct	complement/reciprocal
SO7	Post	3	ignore information	ignore information	complement/reciprocal	correct
SO7	Post	4	inequivalent	correct	filler	filler
SO7	Post	5	complement/reciprocal	complement/reciprocal	correct	inequivalent
SO8	Pre	1	inequivalent	inequivalent	correct	complement/reciprocal
SO8	Pre	2	ignore information	filler	filler	correct
SO8	Pre	3	correct	whole number ordering	inappropriate sizing/spacing	
SO8	Pre	4	whole number ordering	complement/reciprocal	ignore information	correct
SO8	Pre	5	filler	whole number ordering	correct	whole number ordering
SO8	Post	1	inequivalent	inequivalent	correct	complement/reciprocal
SO8	Post	2	correct	midpoint	inequivalent	filler
SO8	Post	3	correct	whole number ordering	inappropriate sizing/spacing	

SO8	Post	4	correct	midpoint	filler	filler
SO8	Post	5	midpoint	complement/reciprocal	complement/reciprocal	correct
SO9	Pre	1	ignore information	inappropriate sizing/spacing	correct	inappropriate sizing/spacing
SO9	Pre	2	correct	whole number ordering	inappropriate sizing/spacing	
SO9	Pre	3	correct	whole number ordering	whole number ordering	whole number ordering
SO9	Pre	4	same num/denom ordering	same num/denom ordering	correct	same num/denom ordering
SO9	Pre	5	ignore information	whole number ordering	whole number ordering	correct
SO9	Post	1	ignore information	correct	inappropriate sizing/spacing	inappropriate sizing/spacing
SO9	Post	2	correct	whole number ordering	inappropriate sizing/spacing	
SO9	Post	3	correct	whole number ordering	whole number ordering	whole number ordering
SO9	Post	4	correct	same num/denom ordering	same num/denom ordering	same num/denom ordering
SO9	Post	5	same num/denom ordering	whole number ordering	whole number ordering	correct

Figure 1.

```
TITLE:

Third Grade Pre-Quiz Fraction Errors

DATA:

File = g3 pre errors 4.3.txt;

VARIABLE:

Name = rsid school male grade disab nslp ell gifted math16 asian black hispan
   white other comp iginfo ineq size sndoe wno reci filler error;

Usevariables = comp iginfo ineq size sndoe wno reci filler;

Categorical = comp iginfo ineq size sndoe wno reci filler;

Idvariable = rsid;

Missing = all(999);

Classes = c(3); //three class model

ANALYSIS:

Estimator = MLR; //MLR for LCA

Type = mixture;

Starts = 100 50;

Processors = 4;

LRTstarts = 10 2 100 50; //runs bootstrap analysis

LRTbootstrap = 50;

MODEL: //if left blank, Mplus automatically uses the categorical variables

OUTPUT:
```

Tech1 Tech7 Tech10 Tech11 Tech14

residual patterns svalues;

PLOT:

Type = plot3;

Series = comp iginfo ineq size sndoe wno reci filler (*);

SAVEDATA:

File is step1save.dat;

SAVE=CPROB; //saves what classes students are in and the likelihood they are placed correctly

Figure 2.

```
TITLE:

Latent Transition Analysis Grade 3

DATA:

File = g3 errors 4.3.txt;

VARIABLE:

Name = rsid school male grade disab nslp ell gifted math16 asian black hispan
   white other comp1 info1 ineq1 reci1 size1 sndoe1 wno1 filler1 prepr1
   prepr2 prepr3 premod comp2 info2 ineq2 reci2 size2 sndoe2 wno2 filler2
   error2;

Usevariables = premod comp2 info2 ineq2 reci2 size2 sndoe2 wno2 filler2;

Categorical = comp2 info2 ineq2 reci2 size2 sndoe2 wno2 filler2; //post-quiz errors

Nominal = premod; //pre-quiz model

Idvariable = rsid;

Missing = all(999);

Classes = pre(3) post(3); //three classes for the pre- and post-quiz models

ANALYSIS:

Type = mixture; //mixture for LTA

Processors = 4;

MODEL:

%overall%
post on pre //regression post-quiz model on pre-quiz model to obtain LTA probabilities

Model pre: //constrain pre-quiz model to remain constant
%pre#1%
```

```
[premod#1@4.446];
[premod#2@-9.358];

%pre#2%
[premod#1@8.742];
[premod#2@13.809];

%pre#3%
[premod#1@-2.714];
[premod#2@-13.751];


Model post: //indicate variables and thresholds for post-quiz errors
%post#1%
  [comp2$1];
  [info2$1];
  [info2$2];
  [ineq2$1];
  [ineq2$2];
  [reci2$1];
  [reci2$2];
  [size2$1];
  [size2$2];
  [sndoe2$1];
  [sndoe2$2];
  [wno2$1];
  [wno2$2];
  [filler2$1];
  [filler2$2];

%post#2%
  [comp2$1];
  [info2$1];
  [info2$2];
  [ineq2$1];
  [ineq2$2];
  [reci2$1];
  [reci2$2];
  [size2$1];
  [size2$2];
  [sndoe2$1];
  [sndoe2$2];
  [wno2$1];
```

```
  [wno2$2];
  [filler2$1];
  [filler2$2];

%post#3%
  [comp2$1];
  [info2$1];
  [info2$2];
  [ineq2$1];
  [ineq2$2];
  [reci2$1];
  [reci2$2];
  [size2$1];
  [size2$2];
  [sndoe2$1];
  [sndoe2$2];
  [wno2$1];
  [wno2$2];
  [filler2$1];
  [filler2$2];

OUTPUT:

Tech1 Tech8 Tech15

residual patterns svalues;

PLOT:

Type = plot3;

Series = comp2 info2 ineq2 reci2 size2 sndoe2 wno2 filler2 (*);
```