EVALUATION OF STATE DETERMINATION CALCULATIONS IN NONLINEAR ANALYSIS

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SUMMARY

The nonlinear stress analysis problem involves three major phases, namely (1) linearization, (2) equation solution, and (3) state determination. The evaluation of tangent stiffness relations for the structure constitutes the linearization phase; the solution of a set of linear equations to determine a displacement increment constitutes the equation solution phase; and computation of the corresponding increments of stress and strain to obtain a new state is the state determination phase. The linearization and equation solution phases have received wide coverage in the literature. The intention of this paper is to study the state determination computations.

Materials may be classified as elastic, hyperelastic or hypoelastic. Elastic and hyperelastic materials are relatively easy to consider, because their constitutive relationships can be derived from conditions of uniqueness and reversibility, and there is a one-to-one correspondence between stress and strain. Hypoelastic materials are more complex because the state of stress also depends on the history of deformation. For the often-occurring elasto-plastic materials, with either isotropic or kinematic hardening, the hardening rule can be specified by a loading function, such that any stress state enclosed by the current function is elastic, whereas any stress state on or beyond the function is plastic. Two important cases need to be considered in the state determination for these models, namely (a) the transitional case, in which the current state is elastic and the new state is plastic, and (b) the plastic loading case, in which both the states are plastic.

In the transitional case it is first necessary to compute that proportion of the strain increment which places the state of stress on the current loading function. The remaining strain increment is then covered by the plastic loading case. The calculations require an assumption regarding the strain path during the strain increment, usually that of portional straining. Determination of the strains required to reach the loading surface then involves the solution of an intersection problem involving the corresponding stress path and the loading function. State determination for the plastic loading case involves evaluation of an implicit nonlinear integral, which can be solved only by numerical integration. For analysis with small load increments, simple Euler integration is often adequate, although higher order schemes may be warranted. For larger load increments the authors have found that a variable increment Euler scheme is efficient, in which the strain increment is automatically divided into subincrements, with subincrement sizes selected to satisfy an error tolerance. A further important class of material models is that in which linear behavior is assumed between discrete "events", corresponding typically to different loading surfaces. The state determination problem in this case involves successive solutions of the intersection problem.

The computer implementation of these procedures in the general purpose nonlinear code ANSR is reviewed.

1. Introduction

Analysis of structures in the nonlinear regime has gained considerable importance, especially in the fields of nuclear reactor component design, seismic analysis, etc. The equations of motion describing the nonlinear behavior of a structure can be derived by examining the incremental deformation from the current deformed state of the structure (at time t) to the neighboring deformed state (at time $t + \Delta t$); see, for example, references [1,2,3]. The nonlinear stress analysis problem then involves three major computational phases, namely, (1) linearization, (2) equation solution, and (3) state determination. The evaluation of the tangent stiffness relations for the structure in any given state of deformation constitutes the linearization phase; the solution of the incremental equations of motion to compute the deformation increments (i.e. the displacement, velocity and acceleration increments) constitutes the equation solution phase; and the computation of the corresponding increments of stress and strain to obtain a new state constitutes the state determination phase.

Solution procedures for nonlinear structures generally involve applying loads in a step-by-step manner, and in many cases equilibrium iterations are necessary to obtain accurate results. The equilibrium iterations may be of Newton-Raphson type or of Constant Stiffness type. In both a simple step-by-step solution procedure (without iteration) and the Newton-Raphson iteration procedure, it is necessary to repeat all the three computational phases in each step or iteration. In the Constant Stiffness iteration procedure, the linearization phase and part of the equation solution phase (i.e. decomposition of the tangent stiffness matrix) are performed only once every few load steps or iterations; however, the state determination phase needs to be carried out repeatedly. The computational effort for the state determination phase is likely to be at least as important as for linearization and equation solution, but this phase has received considerably less attention in the literature. The intention of this paper is to study the state determination computations.

2. State Determination Concept

The state determination phase involves computation of the increments of stress and strain given a set of nodal displacement increments. The computation of the strain increments corresponding to the given displacement increments is purely a kinematic process, and has been adequately dealt with in the literature (see, for example, reference [3]). It is necessary, therefore, to consider only the computation of stress increments given the strain increments. This specifically involves the material characterization.

Materials may be classified as elastic, hyperelastic or hypoelastic. The state determination calculations for elastic and hyperelastic materials are relatively easy to consider, because their constitutive relationships can be derived from conditions of uniqueness and reversibility, and there is a one-to-one correspondence between stress and strain. For a comprehensive treatment of elastic and hyperelastic materials in finite elasticity, reference may be made to the text by Oden [3].

Hypoelastic materials, such as elasto-plastic materials, pose more complex state determination problems because the state of stress also depends on the history of deformation. For idealization of the elasto-plastic behavior of metals, plasticity models with either isotropic or kinematic hardening have been widely used (see Fig. 1). For these material models, the flow theory of plasticity involves specification of an initial yield condition, a flow rule and a hardening rule [4], which completely determine the material characterization. The hardening rule defines subsequent yield conditions (loading functions) after first yield, such that any stress state enclosed by the current function value is elastic, whereas any stress state on or outside the loading function is plastic. Generally, the loading function can be specified to be of the form

$$f(\sigma_{ij},\ \varepsilon^P_{ij},\ H) = 0 \tag{1}$$

in which σ_{ij} $(i,j = 1,2,3)$ are the current stresses; ε^P_{ij} are the plastic strains; and H is a work hardening parameter which depends on the history of deformation. In most cases, dependence on H is assumed to be implicit through the plastic strains. The form of the loading function corresponding to von Mises yield criterion has been most often used, and has been implemented in many inelastic analysis computer programs.

Two important cases need to be considered in the state determination for these material models, namely, (a) the transitional case, in which the current state is elastic and the new state is plastic, and (b) the plastic loading case, in which both states are plastic.

The transitional case is shown in Fig. 2, in which the current stress state ${}^1\underset{\sim}{\sigma}$ (point A) is within the loading surface, and therefore elastic, whereas the new stress state ${}^2\underset{\sim}{\sigma}$ (point C) is beyond the loading surface, and therefore plastic. Clearly the transitional case is obtained only if

$$f({}^1\sigma_{ij} + D_{ijkl}\ d\varepsilon_{kl},\ {}^1\varepsilon^P_{ij}) > 0 \tag{2}$$

where D_{ijkl} is the elasticity constitutive tensor. In Fig. 2, point D represents the stress state corresponding to the left hand side of equation (2). To find an approximation to the actual stress state, point C, it is required to make an assumption regarding the strain path during the strain increment. Usually an assumption of proportional straining will be reasonable. With such an assumption it is necessary first to compute the proportion, r, of the strain increment $d\underset{\sim}{\varepsilon}$ such that an intermediate state of stress on the current loading surface is reached (point B in Fig. 2). This involves the solution of a geometric intersection problem between the stress path (which is linear for proportional straining) and the loading surface (which is typically nonlinear). The proportion factor, r, is obtained by solving the equation

$$f({}^1\sigma_{ij} + r\ D_{ijkl}\ d\varepsilon_{kl},\ {}^1\varepsilon^P_{ij}) = 0, \quad 0 \le r \le 1 \tag{3}$$

For a loading function of von Mises type, r is simply the positive root of a quadratic equation.

3. Continuous Loading Functions

The stress increment for the remainder of the strain increment, $(1-r)d\underset{\sim}{\varepsilon}$, is now covered by the plastic loading case. This stress increment must be computed from the integral

$$d\sigma'_{ij} = \int_0^{\Delta\varepsilon'_{kl}} C_{ijkl}\ (\sigma_{ij},\ \varepsilon^P_{ij})\ d\varepsilon_{kl} \tag{4}$$

where C_{ijkl} is the tangent stress-strain relationship, which is a function of both stress and plastic strain; and $\Delta\varepsilon'_{kl} = (1-r)d\varepsilon_{kl}$ is the remaining strain increment. For materials with continuous loading functions, there is a continuous change in C_{ijkl} along the stress path (B-C in Fig. 2), and it is necessary to compute the integral of eq. (4) numerically. A problem in computational logic arises in performing this integration, because the change in C_{ijkl} along the integration path may either be small (for strain increments essentially normal to the loading function for typical hardening rules) or may be large (for large strain increments making a large angle with the normal to the loading function). Hence, for some strain increments a simple Euler integration may be sufficient, whereas for others it may be necessary to use a higher order integration scheme, or to subdivide the strain increment into several subincrements. An "adaptable" Euler integration procedure, as follows, has been found to be useful in this respect.

For a material with zero strain hardening, a part of the loading function is shown in Fig. 3 (the procedure for nonzero strain hardening is similar in principle). If the value of C_{ijkl} is established at B, then a single Euler integration step will predict stresses at point D_1. For this point, the error, measured by the distance of the point from the loading function, is as shown. To apply the adaptable Euler method, an error tolerance is specified by the computer program user, and a fraction of the elasto-plastic strain increment is determined such that the error equals (or, more strictly, approximates) this tolerance. This leads to point E_1 in Fig. 3. Note that the error does not vary in direct proportion to the fraction of the strain increment, so that the fraction to be taken is not simply the ratio of the tolerance to the error. Note also that an exact relationship between the strain fraction and the error may be difficult to establish, especially for a complicated loading function. However, since the user-specified tolerance is arbitrary, it is sufficient to establish an approximate relationship, and hence to obtain a point E_1 for which the error only approximates the tolerance. At point E_1, the stress values are adjusted to place the stress point exactly on the yield surface, at point B_1. A new value for C_{ijkl} is then established, and the process is repeated with the remainder of the strain increment, until the complete strain increment has been considered.

The advantages of this "adaptable" procedure are (1) that the number of strain subincrements is chosen automatically, and will be large only for large strain increments making large angles with the normal to the loading function; and (2) that the tolerance may be specified by the analyst at execution time, and may be chosen to provide any required degree of "accuracy". Note that a high degree of accuracy is likely to be fictitious, because the basic assumption of proportional straining is only approximate. The adaptable Euler procedure has been applied successfully for isotropically and kinematically hardening metals [5] and for soils [6].

4. Inconsistency Due to Error Correction

Regardless of how accurately the numerical integration is carried out, it will always be necessary to make adjustments of the type represented by the correction from E_1 to B_1 in Fig. 3. For strain hardening materials this introduces an interesting inconsistency in the results, as follows.

The new state will typically be defined by the new stresses, total strains and loading function. If the stresses are adjusted to move the stress point onto the loading function, then the elastic part of the total strain is adjusted, and hence also the plastic part of this strain. Because the new loading function for a strain hardening material will depend on the plastic strains, the loading function will no longer be consistent with the adjusted strains. That is, the strain hardening rule will not have been exactly followed. For most practical purposes, of course, such inconsistencies will be small and can be ignored. However, if large adjustments in the stresses and strains are made, the error could be substantial.

5. Discontinuous Loading Functions

In some cases it may be necessary or convenient to use loading functions which are not continuous. In such cases, it may be necessary to determine, for the plastic loading case, the proportion of the strain increment at which a discontinuity is reached. As with the transitional case this will involve the intersection of a stress path with a loading function, in this case the loading function which follows the discontinuity. Unlike the transitional case, however, the stress path up to the discontinuity will not generally be linear, and hence the solution of the intersection problem may be difficult. This is a reason against using a Tresca type of loading function, which has curved facets in general stress space.

The use of discontinuous loading functions can be convenient, however, if C_{ijkl} can be kept constant between discontinuities, so that the assumption of proportional strain leads to a linear stress path. The integration of eq. (4) reduces to the solution of intersection problems between a linear stress path and a nonlinear loading function. An idealization of this type is currently under investigation for soils, using a variable modulus procedure (as distinct from a flow rule procedure) to establish C_{ijkl} [7].

6. Concluding Remarks

It has been shown that the problem of state determination for inelastic materials reduces to the solution of (a) intersection problems between straight (or possibly curvilinear) stress paths and nonlinear loading functions in stress space, and (b) evaluation of an implicitly nonlinear integral. The possible intersection problems are of such wide variety that no general solution scheme can be proposed. For linear stress paths, however, the intersection problem is usually easy to solve. The numerical integration problem is complicated by the fact that the degree of nonlinearity may vary from problem to problem, and also from point to point or load step to load step within a single problem. Hence, it is desirable to use an integration scheme which achieves computational efficiency by adjusting automatically for the degree of nonlinearity. Such a scheme has been outlined in this paper.

The problem of state determination computation has received little attention in the literature. For the nonlinear integration, most workers appear to use a single-step Euler integration scheme. This paper suggests that a more sophisticated scheme may be desirable, and notes that for strain hardening materials the use of an overly simple scheme could lead to errors.

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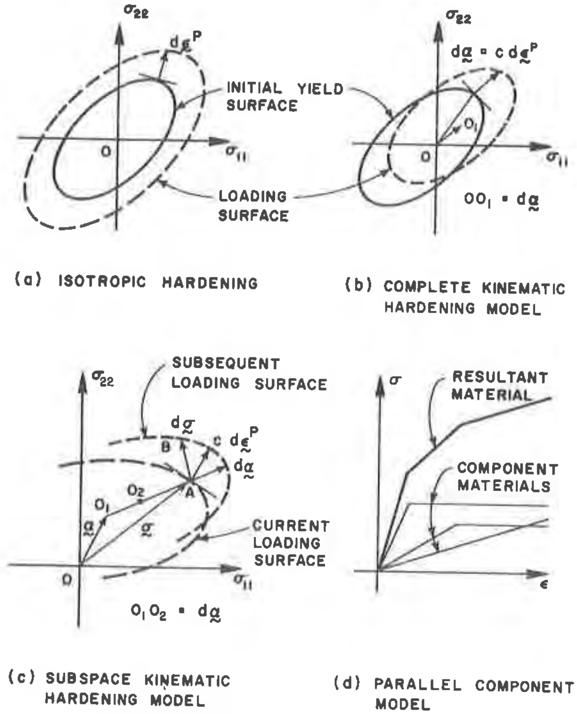

FIGURE 1 PLASTICITY MODELS WITH STRAIN HARDENING

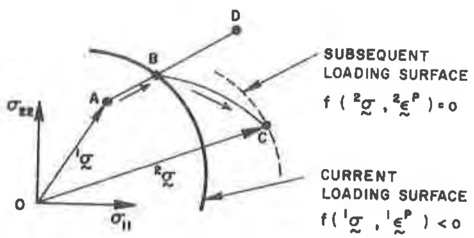

FIGURE 2 STATE DETERMINATION FOR PLASTICITY MODELS

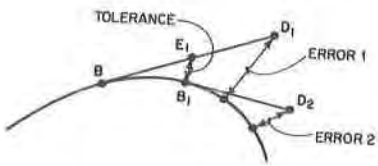

FIGURE 3 ADAPTABLE EULER INTEGRATION PROCEDURE