

ABSTRACT

POPLIN, PHILLIP L.. The Semiring of Multisets. (Under the direction of Professor R. Hartwig.)

The aim of this paper is to develop the basic theory of the Multiset Algebra. Multisets provide a connection between the eigenvalue/eigenvector equations for the Max-Plus and nonnegative real number systems. Multisets provide a more general setting than either of these two algebras. Results about these two algebras will be developed from the theory of Multisets.

THE SEMIRING OF MULTISSETS

by

Phillip L. Poplin

A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh

2000

APPROVED BY:

Chair of Advisory Committee

BIOGRAPHY

Phillip Poplin received his A.S. degree from Gaston College, B.S. degree from the University of North Carolina at Charlotte, and M.A. degree from Appalachian State University. Phillip began pursuing his Ph.D. degree in Mathematics at North Carolina State University in January 1992. While studying at NCSU Phillip worked as an instructor at Wake Technical Community College and also as a Teaching Assistant at NCSU. Phillip has spent his summers working at outdoor backpacking camps for youth. His other interests include chess, working with computers, and, of course, teaching and studying mathematics.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Robert Hartwig, for his encouragement in the preparation of this thesis. I would like to thank him for his support, guidance, and friendship throughout my studies at North Carolina State University.

Contents

1	Introduction and Motivation	1
2	Semirings	3
2.1	Axioms	3
2.2	Matrices	4
2.3	Graphs	6
2.4	Polynomial Forms	9
3	Zerosumfree Semirings	11
4	Idempotent Semirings (Path Algebras)	15
5	Max-Plus Algebra	16
5.1	Example	16
5.2	Definitions	18
	Elementary Properties	19
	Matrices	20
	Graphs	20
5.3	Eigenvalue / Eigenvector Analysis in Max-Plus algebra	22
	Definite Matrices	23
	General Matrices	26
5.4	Symmetrization of the Max-Plus Algebra	35
	The new system: \mathbb{S}_{\max}	35
	Equations to Balances	39
	The scalar linear balance	40
	Determinants in Symmetrized Algebra	41
6	Max-Times Algebra	44
6.1	Basics	44
6.2	Spectral Radius of A and the Maximum Geometric Mean of A	48
7	Cancellative Semirings	54
7.1	Embeddings and Morphisms of Semirings	54
8	Multiset Algebra	58
8.1	Notation and Definitions	58
8.2	Properties	59
8.3	Mappings	62

8.4	Matrices over $3^{\mathbb{S}}$	62
8.5	Partial Ordering and Lattice Structure	64
8.6	Equations in $3^{\mathbb{S}}$	64
8.7	Matrix Equations over $3^{\mathbb{S}}$	75
	Homogeneous Systems	75
	Nonhomogeneous Systems	79
	Another Mapping	86
8.8	Polynomials in $3^{\mathbb{S}}$	87
8.9	The Characteristic Polynomial in the Mset Algebra	89
	Notation	90
	Partial Permutations	90
	Applications	97
	Minimal Equations in the Mset Algebra	98
8.10	Linear Independence	101
8.11	Embeddings and Morphisms of Semirings	102
8.12	Ring of differences for the mset algebra over \mathbb{R}^+	104
8.13	Application.	112
8.14	Cramer's Rule	113
8.15	A Connection between \mathfrak{M}^* and \mathbb{R}^+	120
8.16	End Notes	127
9	Characteristic Polynomial in Max-Times Algebra	130
9.1	Algorithm	130
9.2	Examples	131
9.3	Eigenvalues and Roots of the Characteristic Equation	136
9.4	Algorithm for $\nu(A)$	137
9.5	Companion Matrix	143
9.6	Characteristic Equation of Block Matrices in \mathfrak{M}^*	146
	Partitioned Matrices (Reducible Matrix)	146
	Partitioned Matrices (Block Diagonal Matrix)	149
	Bordered Matrices	152
	Partitioned Matrices and Recursion	155
10	Algebra of Polynomial Pairs in the Max-Times Algebra	156
10.1	Evaluation Mapping	156
	(Max, \times)-Algebra	156
10.2	Equations in $(\mathbb{R}[x], \boxplus, \boxtimes)$	158
	Equivalence of equations	159

10.3	$(\mathbb{P}^2, \oplus, \otimes)$	159
	Equivalence of polynomial pairs	162
	Division	162
10.4	Ideal	165
11	Parallels	167
12	Conclusions and Open Questions	169
12.1	Open Questions	169
12.2	Questions to be answered about Polynomial Pairs	170
A	Appendix	172
A.1	Basic Properties	172
A.2	Matlab programs for the Max-Plus Algebra System	174
A.3	Matlab programs for the Max-Times Algebra System	179
A.4	Matlab programs for the mset Algebra System	182
A.5	UBASIC programs for Characteristic Polynomial	183

List of Figures

1	Semirings Interconnectedness	2
2	Graph for Transportation Network	17
3	Relationship between semiring embeddings.	57
4	Graph with first property	95
5	Graph with second property	96
6	Relationship between semirings and eigenvalue equations	169

List of Tables

1	Examples of Path Algebras	15
2	The family of (max,+) and tropical semirings	16
3	Examples in (max,+) algebra	19
4	Partial Permutations in 3×3 characteristic polynomial	139
5	Partial Permutations in 4×4 characteristic polynomial	141
6	Parallels between Nonnegative and \mathfrak{M}^+ matrices	168
7	Basic facts about some specific path algebras	173

1 Introduction and Motivation

The Max-Plus algebra is one example of the class of Path Algebras. Path algebras have evolved over the past few decades to help answer questions that arise in weighted graph theory. At first several different Path Algebras were investigated separately but as research evolved it was found that the same theory held for all Path Algebras. This was the beginning of the field of idempotent semirings, which is a subclass of semirings. This theory covers areas of path algebras as well as lattices and weighted graph theory.

The use of the Max-Plus (max,+) algebra has allowed many problems which are non-linear in the usual algebra system to be linearized. This has allowed difficult problems to be solved by elementary methods. The (max,+)-algebra system has evolved to solve a multitude of problems in seemingly different fields of study. As such the (max,+) algebra has been used in Discrete Event Systems, transportation networks, parallel computations, project management, machine scheduling, and many other fields. For a detailed introduction we refer to [13].

Semirings provide the global setting of the present thesis. Within this context we study some matrix identities which can be developed analogously from the theory of rings. In particular we show that some identities in a ring can be generalized to an analogous identity in a semiring. We begin with an introduction to semirings. The semirings we study can be separated into three special types; idempotent semirings (which are Path Algebras), zerosumfree semirings, and cancellative semirings. The Max-Plus and Max-Times Algebras are subclasses of the class of Path Algebras.

As we will see in Example 5.1 the Max-Plus Algebra system can be used to solve problems of discrete event systems using linear algebra. One aim of this thesis is to obtain a better understanding of the relative relationship between the Max-Plus, Max-Times, Multiset, and Nonnegative number systems. Indeed we show that \mathfrak{M}^+ , \mathfrak{M}^* , and the nonnegative real numbers can all be viewed as a special case of multisets. We give an overview of the Max-Plus Algebra, including many of the theorems used in connection with these problems. The Max-Times Algebra is isomorphic to the Max-Plus Algebra system, but since it is closer to the usual algebra, it is somewhat easier to develop. The \mathfrak{M}^+ algebra is more natural for applications but \mathfrak{M}^* is easier to use. We survey the material known about these systems and add several new results.

A search for a common enveloping theory between the Max-Times algebra and the non-negative real number system leads to the semiring of multisets, and, subsequently, to its ring of differences. The Multiset Algebra will allow us to recover all the cases above by performing suitable elementwise computations.

The multiset algebra is developed. We give a detailed introduction to multisets, then proceed to equations with multisets and systems of equations involving multisets. We study the multiset algebra system and its relationship to the other algebras. The multiset algebra allows us to keep an optimal amount of information throughout the calculations. Keeping more information, however,

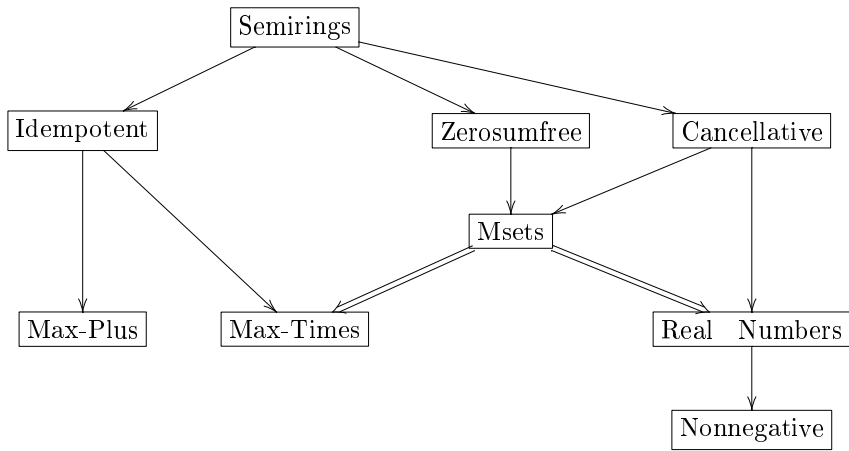


Figure 1: Semirings Interconnectedness

also has disadvantages in that some problems become much more complicated. We show how multi-sets allow us to connect the eigenvalue/eigenvector equations for the Max-Times algebra with that of our conventional algebra system. Figure 1 shows the interconnectedness of these structures.

2 Semirings

A semiring is one of the fundamental structures in mathematics. The set most people first encounter is the natural numbers, which is a semiring, as is the nonnegative real numbers. Since a semiring is a more general structure than the familiar ring structure, the theory is less developed and more fractured in its development.

2.1 Axioms

We begin with a definition of the semiring structure

Definition 2.1. A **semiring** is a set R , equipped with two binary operations, $+$ and \times , which have the following properties.

1. The $+$ operation is commutative and associative.
 - (a) $a + b = b + a$ for all $a, b \in R$
 - (b) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$
2. The \times operation is associative and distributive over $+$.
 - (a) $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$
 - (b) $a \times (b + c) = (a \times b) + (a \times c)$ for all $a, b, c \in R$
 - (c) $(b + c) \times a = (b \times a) + (c \times a)$ for all $a, b, c \in R$
3. The set R contains a zero element θ such that:
 - (a) $\theta + a = a$ for all $a \in R$
 - (b) $\theta \times a = a \times \theta = \theta$ for all $a \in R$ (absorption)
4. The set R contains a unit element e such that
$$e \times a = a \times e = a \text{ for all } a \in R$$

The zero element is absorbing, which has now become an axiom instead of a theorem. The above axioms define a semiring. If $e = \theta$ then $r = re = r\theta = 0$ for all r in R , so $R = \{\theta\}$. We add the following axiom to avoid this trivial case.

5. $e \neq \theta$

We add further axioms to define specific types of semirings.

Definition 2.2. A **commutative semiring** is a semiring, R , with the additional axiom:

6. $a \times b = b \times a$ for all $a, b \in R$

Definition 2.3. An **additively idempotent semiring** has the additional axiom:

7. $a + a = a$ for all $a \in R$

Although one could also define a semiring which is multiplicatively idempotent, we will have no need for this definition in the present setting. As such we will refer to an idempotent semiring as meaning additively idempotent.

Definition 2.4. A **cancellative semiring**, R , satisfies the additional axiom:

8. If $a + b = a + c$ then $b = c$ for all $a, b, c \in R$

Definition 2.5. A **zerosumfree semiring**, R , satisfies the additional axiom:

9. If $a + b = \theta$ then $a = b = \theta$.

Definition 2.6. A **semifield** is a semiring in which the nonzero elements of R have a multiplicative inverse. It is noted that this definition uses the existence of the multiplicative identity as an axiom of semirings.

If $a \in R$, and $a \neq \theta$, then a is a **zero divisor** in R if there exists a $b \neq \theta$ in R such that $a \times b = \theta$. Although one could define a left and right zero divisor separately, the semirings in this thesis will be commutative, so the definitions are the same.

We see from the above axioms that the important difference between these semiring structures and that of a ring is the lack of an additive inverse. It is this fact that separates rings and semirings. Any theory that requires the existence of an additive inverse can not be handled by a semiring structure and must be embedded in a ring structure.

If S is a subset of R which is also a semiring, then S is called a subsemiring of R .

Example 2.7. $(\mathbb{N}, +, \times)$, the natural numbers, with the usual addition and multiplication is a familiar example of a semiring. We also note that \mathbb{N} is a subsemiring of \mathbb{Q}^+ , the semiring of nonnegative rational numbers, which is a subsemiring of \mathbb{R}^+ , the semiring of nonnegative real numbers. ■

2.2 Matrices

We let R_n equal the set of all $n \times 1$ vectors with elements from R . We also let $R_{n \times n}$ equal the set of all $n \times n$ matrices with elements from R .

Definition 2.8. Suppose $A, B \in R_{n \times n}$, $\alpha \in R$. The following are the definitions of the basic operations of matrices in the semiring.

$$(1) \ I_n \equiv I = \begin{cases} I_{ii} = e & \forall i = 1 \dots n \\ I_{ij} = \theta & \forall i \neq j \end{cases} = \text{Identity matrix in } R_{n \times n}.$$

- (2) $(\alpha \times A)_{ij} \equiv \alpha \times a_{ij}$
- (3) $[A + B]_{ij} = a_{ij} + b_{ij}$
- (4) $[A \times B]_{ij} = [\sum_{k=1}^n (a_{ik} \cdot b_{kj})]$
- (5) The zero matrix, $\theta_{n \times n}$, is defined as the matrix with all elements equal to the θ -element.
- (6) The ones-matrix, $J_{n \times n}$, is defined as the matrix with all elements equal to the e -element.

We may omit the “ \times ” when it is understood, or we may replace it by the “ \cdot ”, as the need arises.

Definition 2.9 (Eigenvalue-Eigenvector Equations).

Given a matrix A, if $\lambda \in \mathbb{R}$ and $\underline{x} \in R_n$ exists such that Equation (1) holds, then λ is called an eigenvalue of A and \underline{x} is called a right eigenvector of A.

$$A\underline{x} = \lambda\underline{x} \implies \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (1)$$

$$\implies \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n \end{cases} \quad (2)$$

Definition 2.10. The matrix A is said to be reducible if there exists a permutation matrix P such that $PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$.

If the matrix A is not reducible then we say that A is irreducible.

Theorem 2.11. If $A \neq \theta$ and A is reducible, then

- (a) A^k is reducible for all k
- (b) $\sum_k A^k$ is reducible.

Proof. Suppose $A \approx \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$.

$$(a) \quad A^k \approx \begin{bmatrix} B^k & ? \\ 0 & D^k \end{bmatrix},$$

and

$$(b) \quad \sum_k A^k \approx \begin{bmatrix} \sum_k B^k & ? \\ 0 & \sum_k D^k \end{bmatrix}. \quad \square$$

The following Corollary follows immediately from the above theorem.

Corollary 2.12. If $A \geq 0$, and there exists k such that A^k is irreducible, then A is irreducible.

2.3 Graphs

A **weighted directed graph** is a triplet (V, E, W) . V is a set of vertices, called **nodes**. The nodes will be labeled as S_i or simply as i , where it is unambiguous. E is the set of edges, or **arcs**, given as a directed line segment from node S_i to S_j . W is the set of weights associated with each arc. Since all graphs in this thesis are weighted directed graphs, we simply refer to these as **graphs**.

A path from node S_{i_0} to S_{i_t} is a sequence: $(S_{i_0}, S_{i_1}, \dots, S_{i_t})$, or referred to simply as the sequence: (i_0, i_1, \dots, i_t) . The nodes i_k are said to be on the path. The path is said to pass through the nodes. The **length** of the path is t . The path is said to be an **elementary path** if the nodes (i_1, i_2, \dots, i_t) are pairwise distinct. (No repeated nodes.) The length of an elementary path is at most $t - 1$. A path is called a **circuit** if $i_0 = i_t$, and a path is called a **cycle** if it is an **elementary circuit**. A **loop** is a cycle which consists of a single arc. The length of a circuit is the length of its path. Elementary circuits have length of at most n , in a graph with n nodes. When listing the nodes of a circuit, the first node can be any node on the graph; the other nodes are given in the order around the circuit.

The **adjacency matrix** of a graph is the matrix A with elements made of the weights of the arcs of the graph.

$$a_{ij} = \begin{cases} w_{ij} & \text{if an arc exists from node } i \text{ to node } j \\ \theta & \text{otherwise.} \end{cases} \quad (3)$$

Definition 2.13. $G(B)$ is the graph which has B as its adjacency matrix. A matrix B , has exactly one graph associated with it, and a graph has a unique adjacency matrix.

We can associate a **path product** to each path (i_0, i_1, \dots, i_t) as: $w_{i_0 i_1} \times w_{i_1 i_2} \cdots \times w_{i_{t-1} i_t}$

If τ is a path, say i_1, i_2, \dots, i_t , then we will write

$$\omega(\tau) = w_{i_0 i_1} \times w_{i_1 i_2} \times \cdots \times w_{i_{t-1} i_t}, \text{ or using the elements of the matrix } A \text{ as}$$

$$\omega(\tau) = a_{i_0 i_1} \times a_{i_1 i_2} \times \cdots \times a_{i_{t-1} i_t}. \text{ We write } \ell(\tau) \text{ for the length of the path } \tau.$$

The following theorems are from linear algebra and carry over analogously to \mathfrak{M}^+ .

Theorem 2.14 (Frobenius Normal Form).

Suppose M contains k strongly connected components. Then there exists a permutation matrix P such that M takes the following block triangular form:

$$\overline{M} = P M P^{-1} = \begin{bmatrix} \overline{M}_1 & \overline{M}_{12} & \cdots & \overline{M}_{1k} \\ & \overline{M}_2 & \cdots & \overline{M}_{2k} \\ & & \ddots & \vdots \\ 0 & & & \overline{M}_k \end{bmatrix} \quad (4)$$

where $\overline{M}_1, \overline{M}_2, \dots, \overline{M}_k$ are all irreducible submatrices which correspond to k strongly connected components of $G(\overline{M})$.

Theorem 2.15 (Refined Frobenius Normal Form).

Suppose M contains k strongly connected components. Then there exists a permutation matrix P such that M takes the following block triangular form:

$$\overline{M} = P M P^{-1} = \left[\begin{array}{cccc|ccc} \overline{M}_1 & \overline{M}_{12} & \cdots & \overline{M}_{1k} & & & \\ & \overline{M}_2 & \cdots & \overline{M}_{2k} & & & \\ & & \ddots & \vdots & & & ? \\ 0 & & & \overline{M}_k & & & \\ \hline & & & & \overline{M}_{k+1} & 0 & 0 \\ & & 0 & & & \ddots & 0 \\ & & & & 0 & & \overline{M}_s \end{array} \right] \quad (5)$$

where $\overline{M}_1, \overline{M}_2, \dots, \overline{M}_k$ are all irreducible submatrices which correspond to k strongly connected components of $G(\overline{M})$.

The following fact shows how elements in the power of a matrix, A , are related to the graph associated with A .

Fact 2.16. The sum of all of all t -step path-products from node S_i to node S_j can be represented by:

$$(B^t)_{ij} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{t-1}} b_{i,k_1} b_{k_1,k_2} \cdots b_{k_{t-1},j} \quad (6)$$

The proof of the following theorem is graph-theoretically based, and therefore the proof is valid in any semiring.

Theorem 2.17. The following are equivalent:

- (1) A is reducible.
- (2) $G(A)$ has an absorbing class.
- (3) $G(A)$ is not strongly connected.

Proof.

$$(1 \implies 2) \quad \text{Suppose } A \underset{\text{per}}{\approx} B = \begin{array}{c} S_1 \\ \vdots \\ S_r \\ S_{r+1} \\ \vdots \\ S_n \end{array} \left[\begin{array}{ccc|ccc} S_1 & \cdots & S_r & S_{r+1} & \cdots & S_n \\ \hline & & ? & & & ? \\ \hline & & \theta & & & ? \\ \hline & & & & & \end{array} \right]$$

Let $G=G(B)$.

Let $W=\{S_{r+1}, \dots, S_n\}$, and let $W^c=\{S_1, \dots, S_r\}$. Since $(B)_{ij}=\theta$ for $r+1 \leq i \leq n$ and $1 \leq j \leq r$, then no arc exists between S_i and S_j . Therefore W is an absorbing class.

(2 \implies 3) Suppose that B has an absorbing class, say $W=\{S_{r+1}, \dots, S_n\}$. Then G cannot be strongly connected since there is no path from S_{r+1} to S_1 .

(3 \implies 2) Suppose that G is not strongly connected. Then there exists nodes S_i and S_j such no path exists from S_i to S_j . Let $W=\{S_k \mid S_i \rightarrow S_k\}$, and let $W^c = \{S_k \mid S_k \notin W\}$. We claim that there is no path from W to W^c . If such a path exists, then there is a $u \in W$, and $v \in W^c$, with $u \rightarrow v$. Then $S_i \rightarrow u \rightarrow v$, and $S_i \rightarrow v$ and $v \in W$, which is a contradiction. Therefore no path exists from W to W^c .

(2 \implies 1) Let W be the set of nodes in the absorbing class. Since there is no path from a node in W to a node in W^c then $B_{ij}=\theta$ if $S_i \in W$ and $S_j \in W^c$. So $B = \left[\begin{array}{c|c} B_1 & B_3 \\ \hline 0 & B_4 \end{array} \right]$. □

Lemma 2.18 is from [9, page 50].

Lemma 2.18. If G is a graph with no cycles then the vertices of G can be relabeled so that there is no arc from S_i to S_j if $i < j$.

Corollary 2.19. If A is the adjacency matrix associated with the graph G and G has no cycles

$$\text{then } A \underset{\text{per}}{\approx} B = \left[\begin{array}{cccccc} \theta & * & \cdots & \cdots & * \\ \theta & \theta & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \theta & \cdots & \cdots & \ddots & * \\ \theta & \cdots & \cdots & \cdots & \theta \end{array} \right]$$

The following theorem is also graph-theoretically based and holds for any weighted graph over a semiring.

Theorem 2.20. If G be a graph in which every vertex has out degree at least one, then G has a circuit.

Proof. If G has no circuit, then as noted in 2.18, we can relabel the vertices of G so that there is no edge from S_i to S_j if $i < j$. But then vertex S_n must have out-degree zero, which is a contradiction. Therefore G must have a circuit. \square

2.4 Polynomial Forms

The definitions in this section are the same as for rings. Let $(\mathbf{R}, +, \times)$ be a commutative semiring with identity. Let $\mathbf{R}[x]$ be the collection of all finite ordered subsets (a_0, a_1, \dots, a_n) of \mathbf{R} where

- (1) n is an arbitrary nonnegative integer.
- (2) Either $a_n \neq 0$, or $n=0$ and $a_0=0$.

To make $\mathbf{R}[x]$ into a semiring we define addition and multiplication.

Definition 2.21 (Polynomial forms over $(\mathbf{R}, +, \times)$).

1. $p = (p_0, p_1, p_2, \dots, p_n)$, with $p_i \in \mathbf{R}$.
2. $\partial p = n$
3. The zero polynomial has no degree.

Let p and q be polynomial forms, then we define the operation of addition of these two polynomials.

Definition 2.22 (Addition). If $p=(p_0, p_1, \dots, p_n)$ and $q=(q_1, q_2, \dots, q_m)$ with $n > m$, then $p \boxplus q = (p_0 + q_0, p_1 + q_1, \dots, p_m + q_m, p_{m+1}, \dots, p_n)$.

Proposition 2.23 (Properties of \boxplus).

1. $\mathbf{R}[x]$ is closed under \boxplus since \mathbf{R} is closed.
2. \boxplus is commutative since \mathbf{R} is commutative.
3. \boxplus is associative since \mathbf{R} is associative.
4. $\theta = (\theta, \theta, \theta, \dots, \theta)$ is the additive identity.

Definition 2.24 (Scalar Multiplication). If $\alpha \in R$ and $p=(p_0, p_1, \dots, p_n)$, then $(\alpha \boxtimes p) = (\alpha \times p_0, \alpha \times p_1, \dots, \alpha \times p_n)$. We use, here, that $\alpha \boxtimes \theta = \theta$.

Definition 2.25 (Multiplication). If $p=(p_0, p_1, \dots, p_n)$, and $q=(q_0, q_1, \dots, q_m)$, then $p \boxtimes q = (c_0, c_1, c_2, \dots, c_r)$, where $c_i = \sum_{j+k=i} a_j b_k$.

This multiplication is just the normal Cauchy product of two polynomials.

Proposition 2.26 (Properties of Multiplication).

Let $\alpha, \beta \in R$, and p, q, r are polynomial forms.

1. $\alpha \boxtimes (p \boxplus q) = \alpha \boxtimes p \boxplus \alpha \boxtimes q$
2. $(\alpha + \beta) \boxtimes p = \alpha \boxtimes p \boxplus \beta \boxtimes p$
3. $e \boxtimes p = p \boxtimes e = p$
4. $R[x]$ is closed under \boxtimes since R is closed.
5. \boxtimes is commutative since R is commutative and associative.
6. \boxtimes is distributive over \boxplus .

With the properties above, we have

Theorem 2.27. The system $(R[x], \boxplus, \boxtimes)$ is a semiring.

The notation which is sometimes used for polynomial forms is

$p=(p_0, p_1, \dots, p_n) = p_0 + p_1x + p_2x^2 + \dots + p_nx^n$, where 'x' is a place holder.

There is no division algorithm for a general semiring.

3 Zerosumfree Semirings

A zerosumfree semiring is a semiring that obeys the additional axiom that if $a+b = \theta$ then $a = b = \theta$. This condition says that no element of the semiring R has an additive inverse. The Path Algebras and multiset algebra are examples of zerosumfree semirings.

Definition 3.1. If G is a graph, then E is called the **connectivity matrix** associated with G .

$$E \text{ is defined as } (E)_{ij} = \begin{cases} e & \text{if an arc exists from } S_i \text{ to } S_j \\ \theta & \text{otherwise} \end{cases}$$

Let R be a zerosumfree semiring. We note that since $e \neq \theta$ in R , then $e + e \neq \theta$ and $e \times e = e$. We will use the notation $B > \theta$ to mean that every element of $B \neq \theta$.

Note 3.2. We will assume that R contains no zero divisors. This is certainly true for the semirings presented in this thesis. In applications this is required for otherwise a path could exist in a graph with a path product equal to zero. This would defeat the use of path products as we require that if a path exists then its path product is not zero.

Theorem 3.3. G is strongly connected if and only if $I + E + \dots + E^{n-1} > \theta$.

Proof.

(\implies) Suppose G is strongly connected and let E be its connectivity matrix. Let $B = I + E + \dots + E^{n-1}$. For all i, j , there is a path from S_i to S_j . Suppose that $i \neq j$ and $(B)_{ij} = \theta$. Then $((E)_{ij} + (E^2)_{ij} + \dots + (E^{n-1})_{ij}) = \theta$. Since R is zerosumfree we have $(E^k)_{ij} = \theta$ for all k . This implies that the sum of all path products must be zero. Again, zerosumfree implies that all path products must be zero. This is a contradiction since G is strongly connected and a t -step path exists from S_i to S_j and $(E^t)_{ij} = e \neq \theta$. Therefore $B > \theta$.

(\impliedby) Suppose that $B > \theta$. Fix i and j . Then $B_{ij} \neq \theta$ shows that there exists a k such that $(E^k)_{ij} \neq \theta$. So at least one path product is not zero. Therefore there is a path from S_i to S_j . Since this is true for every pair (i, j) , the result follows. □

The theorem above, as well as the theorem below, can be generalized to the adjacency matrix of G provided that the semiring R has no zero divisors (which we will assume).

Theorem 3.4. G is strongly connected if and only if $(I + E)^{n-1} > \theta$.

Proof. $(I + E)^{n-1} = I + \binom{n-1}{1}E + \binom{n-1}{2}E^2 + \dots + E^{n-1}$. Let $B = I + \binom{n-1}{1}E + \binom{n-1}{2}E^2 + \dots + E^{n-1}$. Here, we use multiplication as a shortened form of repeated addition, in other words, $\binom{n-1}{1}E = E + E + \dots + E$.

(\implies) Suppose that G is strongly connected and E is the connectivity matrix of G . Fix i, j , with $i \neq j$. Since G is strongly connected then there is a k -cycle from S_i to S_j with $1 \leq k \leq n-1$. $(E^k)_{ij} = e$, and $\binom{n-1}{k}E_j^k > \theta$. Therefore $(B)_{ij} \neq \theta$.

(\Leftarrow) Suppose $(I + E)^{n-1} > \theta$. So $(B)_{ij} \neq \theta$ for all i, j . Since R is zerosumfree then $(B)_{ij} \neq \theta$ if and only if $(I + E + E^2 + \cdots + E^{n-1})_{ij} \neq \theta$. And by Theorem 3.3 we have that G is strongly connected. □

The following theorem is a generalization of the scaling that is done in \mathfrak{M}^+ , or in \mathbb{R}^+ .

Lemma 3.5. If $A^n = \theta$, then

$$A(\lambda^{n-1}I + \lambda^{n-2}A + \cdots + A^{n-1}) + \lambda^n I = \lambda^n I + \lambda^{n-1}A + \lambda^{n-2}A^2 + \cdots + \lambda A^{n-1} \quad (7)$$

We make the following definition to aid the writing of the above lemma.

Definition 3.6. Let $A^\dagger = \lambda^{n-1}I + \lambda^{n-2}A + \cdots + A^{n-1}$

With this definition Lemma 3.5 becomes

Lemma 3.7. $A A^\dagger + \lambda^n I = \lambda A^\dagger$

Theorem 3.8 illustrates how a nilpotent matrix and its associated graph relate to the eigenvalues of the matrix. This theorem is a nice generalization of the theorems in \mathfrak{M}^+ . We use that R is zerosumfree and has no zero divisors.

Theorem 3.8. Suppose $A \neq \theta$, then the following are equivalent:

(a) $G(A)$ has no cycles.

(b) $A \underset{\text{per}}{\approx} B = \begin{bmatrix} \theta & * & \cdots & \cdots & * \\ \theta & \theta & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \theta & \cdots & \cdots & \ddots & * \\ \theta & \cdots & \cdots & \cdots & \theta \end{bmatrix}$

(c) A is nilpotent. If so then θ is the only eigenvalue.

(d) If, in addition, any irreducible matrix of size at least 2×2 has a nonzero eigenvalue, then θ is the only eigenvalue of A

Item (d) above is not true for matrices over a general semiring. It is true for \mathbb{R}^+ , and Max-Times, but not for msets.

Proof. We will prove (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftarrow (d), and (b) \Rightarrow (d)

((a) \Rightarrow (b)) This follows from Lemma 2.18.

((b) \Rightarrow (a)) This is clear, as there is no path from S_i to S_j if $i \leq j$.

((b) \Rightarrow (c)) This, too, is seen from the fact that $B^n = \theta$.

((c) \implies (b)) Suppose A is nilpotent. Without loss of generality, we may assume that

$$A \underset{\text{per}}{\approx} B = \begin{bmatrix} A_1 & ? & ? & ? \\ 0 & A_2 & ? & ? \\ 0 & 0 & \ddots & ? \\ 0 & 0 & \cdots & A_q \end{bmatrix}$$

which is the Frobenius Normal Form (FNF) of A, where $A_i = [\theta]_{1 \times 1}$, or A_i is irreducible.

$$\text{So, } A = PBP^{-1}. \text{ So there exist } k \text{ such that } A^k = \theta = PB^kP^{-1} \approx \begin{bmatrix} A_1^k & ? & ? & ? \\ 0 & A_2^k & ? & ? \\ 0 & 0 & \ddots & ? \\ 0 & 0 & \cdots & A_q^k \end{bmatrix}$$

Since $A_i^k = \theta$ and A_i is irreducible. Consider the first block A_1 . Suppose that A_1 contains a t-cycle through the first t nodes. Then $(A_1^t)_{11} \neq \theta$. Therefore the index of nilpotency must be greater than t. But $(A_1^t)^\ell \neq \theta$ for all ℓ . Therefore there is no number such that $A_1^m = \theta$.

Since $A_i^k = \theta$ and A_i is irreducible, then A_i is $\theta_{1 \times 1}$.

Therefore the FNF is strictly upper triangular.

((b) \implies (c))

$$A \underset{\text{per}}{\approx} \begin{bmatrix} \theta & a_{12} & \cdots & \cdots & a_{1n} \\ \theta & \theta & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta & \theta & \cdots & \theta & a_{n-1,n} \\ \theta & \theta & \cdots & \cdots & \theta \end{bmatrix}. \quad (8)$$

$\underline{x} = [e, \theta, \theta, \dots, \theta]^T$ is an eigenvector for the eigenvalue θ .

Therefore θ is an eigenvalue.

Now, suppose the $\lambda \neq \theta$ is an eigenvalue of A. So $Ax = \lambda x$ with $x \neq \theta$.

$$\begin{bmatrix} \theta & a_{12} & \cdots & \cdots & a_{1n} \\ \theta & \theta & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta & \theta & \cdots & \theta & a_{n-1,n} \\ \theta & \theta & \cdots & \cdots & \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \quad (9)$$

The last equation of (9) yields $\theta = \lambda x_n \implies x_n = \theta$, since $\lambda \neq \theta$.

The next to last equation of (9) yields: $a_{n-1,n}x_n = \theta = \lambda x_{n-1} \implies x_{n-1} = \theta$.

Continuing backwards to the first equation, we get:

$$a_{12}x_2 = a_{12}(\theta) = \theta = \lambda x_1 \implies x_1 = \theta.$$

Therefore $x = [\theta, \theta, \dots, \theta]^T$, which is a contradiction to the fact that $\underline{x} \neq \theta$.

Therefore θ is the only eigenvalue of A.

((d) \implies (c)) We need to show that if θ is the only eigenvalue of A then A is nilpotent.

Suppose that A is not nilpotent. Then A can be permuted to the form

$$A \underset{\text{per}}{\approx} M = \left[\begin{array}{c|c} \theta & ? \\ \hline \ddots & D \\ \hline 0 & ? \end{array} \right] = \left[\begin{array}{c|c} M_1 & M_3 \\ \hline 0 & M_4 \end{array} \right], \text{ where } D \text{ is not } \theta_{1 \times 1}. \text{ } M_1 \text{ can be written as}$$

$$M_1 = \left[\begin{array}{c|c} \theta & ? \\ \hline \ddots & ? \\ \hline 0 & D \end{array} \right] = \left[\begin{array}{c|c} A & C \\ \hline 0 & D \end{array} \right].$$

Since D is irreducible then, by assumption, D has a nonzero eigenvalue, say λ . Let $\underline{u} \neq \theta$ be an associated eigenvector for D .

Define $\underline{w} = \begin{bmatrix} A^\dagger C \underline{u} \\ \lambda^n \underline{u} \end{bmatrix}$. We know that $\underline{w} \neq \theta$ since $\lambda \neq \theta$, and $\underline{u} \neq \theta$.

Consider $M_1 \underline{w}$.

$$\begin{aligned} M_1 \underline{w} &= \begin{bmatrix} A & C \\ 0 & D \end{bmatrix} \begin{bmatrix} A^\dagger C \underline{u} \\ \lambda^n \underline{u} \end{bmatrix} = \begin{bmatrix} AA^\dagger C \underline{u} + \lambda^n C \underline{u} \\ \lambda^n D \underline{u} \end{bmatrix} = \begin{bmatrix} (AA^\dagger + \lambda^n I) C \underline{u} \\ \lambda^n D \underline{u} \end{bmatrix} \\ &= \begin{bmatrix} \lambda A^\dagger C \underline{u} \\ \lambda^{n+1} \underline{u} \end{bmatrix} = \lambda \cdot \begin{bmatrix} A^\dagger C \underline{u} \\ \lambda^n \underline{u} \end{bmatrix} = \lambda \underline{w} \end{aligned} \tag{10}$$

Therefore, \underline{w} is an eigenvector for M_1 and $\lambda \neq \theta$ is an eigenvalue for M_1 . So

$$M \begin{bmatrix} \underline{u} \\ \underline{\theta} \end{bmatrix} = \lambda \begin{bmatrix} \underline{u} \\ \underline{\theta} \end{bmatrix}$$

and $\lambda \neq \theta$ is an eigenvalue of M , which contradicts the fact that M has only θ as an eigenvalue.

Therefore A must be nilpotent. \square

We note that not all matrices in a particular semiring possess an eigenvalue.

4 Idempotent Semirings (Path Algebras)

Path Algebras have been developed to answer questions pertaining to weighted graphs. Examples include finding the shortest path or longest path in a graph. A Path Algebra is a specific subclass of idempotent semirings. Recall that an idempotent semiring satisfies axioms (1)-(4) along with axiom (6) on page 3. Some authors also refer to these structures as idempotent dioids. Included in this section are some examples of the different types of path algebras that are used to work with weighted graphs. Table 1 lists some of these different algebra systems. Two particular path algebras are presented in this thesis. We investigate the Max-Plus algebra and the Max-Times algebra. These systems are used to determine the longest path of a weighted graph.

Set	$x \oplus y$	$x \otimes y$	θ	e	Applications
$\{0,1\}$	max	min	0	1	Existence of paths
$\mathbb{R} \cup \{\infty\}$	min	+	∞	0	Shortest path
$\mathbb{R} \cup \{-\infty\}$	max	+	$-\infty$	0	Longest path (critical path)
$\{x \in \mathbb{R} \mid x \geq 0\}$	max	x	0	1	critical path
$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$	max	x	0	1	Most reliable path network transmissions
$\mathbb{R}^+ \cup \{\infty\}$	max	min	0	∞	Path of greatest capacity
$\mathfrak{P}(\Sigma^*)$	$x \cup y$	concat	Φ	Λ	Listing all paths Σ =alphabet, Λ =empty word
$\mathfrak{P}(S)$	$x \cup y$	concat	Φ	Λ	Listing simple paths
B	$b(x \cup y)$	concat	Φ	Λ	Listing elementary paths computer logic circuits $b(x \cup y)$ = basic (no abbrev)

Table 1: Examples of Path Algebras

Table 2, on page 16 lists several useful variants of the $(\max,+)$ semiring. For details of these semiring see [19].

We shall investigate two particular path algebras in this thesis: the Max-Plus algebra and the Max-Times algebra. These two algebras are quite similar. As mentioned before they are isomorphic.

Notation	Semiring	Name	Description
\mathbb{R}_{\max}	$(\mathbb{R} \cup \{-\infty\}, \max, +)$	$(\max, +)$ - semiring max algebra	idempotent semifield
$\overline{\mathbb{R}}_{\max}$	$(\mathbb{R} \cup \{\pm\infty\}, \max, +)$	completed $(\max, +)$ - semiring	$-\infty + (+\infty) = -\infty$ for $\theta \otimes a = \theta$
$\mathbb{R}_{\max, \times}$	$(\mathbb{R}^+, \max, \times)$	(\max, \times) semiring	isomorphic to \mathbb{R}_{\max} ($x \rightarrow \log x$)
\mathbb{R}_{\min}	$(\mathbb{R} \cup \{+\infty\}, \min, +)$	$(\min, +)$ semiring	isomorphic to \mathbb{R}_{\max} ($x \rightarrow -x$)
\mathbb{N}_{\min}	$(\mathbb{N} \cup \{+\infty\}, \min, +)$	tropical semiring	(famous in Language Theory)
$\mathbb{R}_{\max, \min}$	$(\mathbb{R} \cup \{\pm\infty\}, \min, \min)$	Bottleneck algebra	
\mathbb{B}	$(\{\text{false}, \text{true}\}, \text{or}, \text{and})$	Boolean semiring	isomorphic to $(\{\theta, e\}, +, \times)$, for any of the above semirings
\mathbb{R}_h	$(\mathbb{R} \cup \{-\infty\}, \oplus_h, +)$ $a \oplus_h b = h \log(e^{a/h} + e^{b/h})$	Maslov semirings	isomorphic to $(\mathbb{R}^+, +, \times)$ $\lim_{h \rightarrow 0^+} \mathbb{R}_h = \mathbb{R}_0 = \mathbb{R}_{\max}$

Table 2: The family of $(\max, +)$ and tropical semirings

5 Max-Plus Algebra

The Max-Plus Algebra has been developed to answer questions pertaining to the longest path in a graph. The following example shows one application of this field.

5.1 Example

We begin with an example to introduce the subject. This will allow us to illustrate many of the different topics that will be covered in this thesis. This is an example of using the Max-Plus algebra system to linearize a problem which is nonlinear in the conventional algebra.

Example 5.1. Suppose we look at a simple train network. The train network is between three cities with intercity routes also. Suppose that there are two trains at each station; one for the intracity track and one for the intercity track. Figure 2 shows a diagram which represents the train network. To simplify the problem we will assume that the times given include the time needed to board the train, the time required for exiting, and the time required for change-overs. ■

Suppose we let x_i represent the time that a train leaves station i , for $i = 1, 2, 3$. We let $x_i(k)$ represent the k^{th} time a train leaves station i . Let us suppose, first of all, that each of the trains leave at time=0. Therefore we have $x_1(1)=0$, $x_2(1)=0$, and $x_3(1)=0$. Then we can see that the time

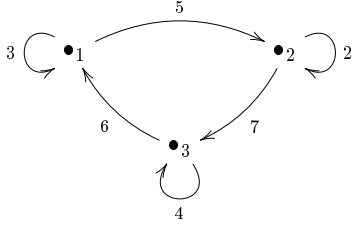


Figure 2: Graph for Transportation Network

at which the trains leave the station the second time is:

$$\begin{cases} x_1(2) = \max\{x_1(1) + 3, x_3(1) + 6\} \\ x_2(2) = \max\{x_2(1) + 2, x_1(1) + 5\} \\ x_3(2) = \max\{x_3(1) + 4, x_2(1) + 7\} \end{cases}$$

Note the long delay due to waiting for change-overs. This set of equations is certainly nonlinear. The question, now, is: Can this set of equations be made into a linear system? The answer is “yes” if we change to a different algebra system. Define $a \oplus b$ to be the $\max(a,b)$ and $a \otimes b$ to be $a + b$ in the conventional algebra. Then the set of equations above can be written as:

$$\begin{cases} x_1(2) = 3 \otimes x_1(1) \oplus 6 \otimes x_3(1) \\ x_2(2) = 2 \otimes x_2(1) \oplus 5 \otimes x_1(1) \\ x_3(2) = 4 \otimes x_3(1) \oplus 7 \otimes x_2(1) \end{cases}$$

This looks very similar to a linear algebra problem. We will assume the usual laws of arithmetic (and matrix arithmetic) apply as usual in this example. If we let $\underline{x}(k) = [x_1(k), x_2(k), x_3(k)]^T$ then $\underline{x}(k)$ represents the k^{th} time that each train leaves the station. The above equations

can be written in matrix form as $\underline{x}(2)^T = \underline{x}(1)^T \otimes A$, where $A = \begin{bmatrix} 3 & 5 & -\infty \\ -\infty & 2 & 7 \\ 6 & -\infty & 4 \end{bmatrix}$. Here, $a_{i,j}$ is

the travel time from station i to station j . Note that $-\infty$ is included in the matrix where no path (route) exists, since $\theta \oplus a = a$, and $\theta \otimes a = -\infty$ for all $a \in \mathfrak{M}^+$.

Note 5.2. We may use multiplication by \underline{x} on the right if we are willing to use A^T instead of A .

This yields: $\underline{x}(2)^T = \underline{x}(1)^T \otimes A$, or $\underline{x}(2) = A^T \otimes \underline{x}(1)$.

From above we have $\underline{x}(2)^T = [6, 5, 7]$. For $\underline{x}(3)$, we have

$\underline{x}(3) = A^T \otimes \underline{x}(2) = A^T \otimes (A^T \otimes \underline{x}(1)) = (A^T)^2 \otimes \underline{x}(1)$., and in general:

$\underline{x}(k) = A^T \otimes \underline{x}(k-1) = (A^T)^{k-1} \otimes \underline{x}(1)$.

Starting with the initial value $\underline{x}(1)=[0,0,0]^T$, we iterate to get:

$$\begin{bmatrix} \underline{x}(1) \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(2) \\ 6 \\ 5 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(3) \\ 13 \\ 11 \\ 12 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(4) \\ 18 \\ 18 \\ 18 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(5) \\ 24 \\ 23 \\ 25 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(6) \\ 31 \\ 29 \\ 30 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(7) \\ 36 \\ 36 \\ 36 \end{bmatrix} \quad (11)$$

This sequence of iterates has a period of 3.

If we start with $\underline{x}(1)=[1,0,1]^T$, we get:

$$\begin{bmatrix} \underline{x}(1) \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(2) \\ 7 \\ 6 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(3) \\ 13 \\ 12 \\ 13 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(4) \\ 19 \\ 18 \\ 19 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(5) \\ 25 \\ 24 \\ 25 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(6) \\ 31 \\ 30 \\ 31 \end{bmatrix} \rightarrow \begin{bmatrix} \underline{x}(7) \\ 37 \\ 36 \\ 37 \end{bmatrix} \quad (12)$$

Notice from this sequence that the iterates have a period=1.

This leads to the question: For what starting values does the sequence of iterates lead to a period of 1?

As we look at this example in more depth we notice that there are four cycles in the graph. $1 \rightarrow 1$, $2 \rightarrow 2$, $3 \rightarrow 3$, and $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. The average length of these cycles are, respectively, $\frac{3}{1} = 3$, $\frac{2}{1} = 2$, $\frac{4}{1} = 4$, $\frac{5+7+6}{3} = 6$. The largest average cycle mean is 6. We also note from the iterates above that $[1 \ 0 \ 1] \otimes A = 6 \otimes [1 \ 0 \ 1]$. Therefore, in the terms of linear algebra, $[1 \ 0 \ 1]^T$ is a left eigenvector of A corresponding to the eigenvalue $\lambda = 6$.

In the conventional linear algebra, this would lead to the eigenvalue/eigenvector equation $\underline{x} A = \lambda \underline{x}$, or $A^T \underline{x} = \lambda \underline{x}$. We see from the above problem that $\underline{x} = [1 \ 0 \ 1]^T$ and $\lambda = 6$ satisfies this equation.

5.2 Definitions

We begin with definitions and basic properties of this system. We then present some of the basic theorems of this algebra system. Many good references exist, including [13], [2], and [4].

Definition 5.3. The Max-Plus algebra is defined as:

- (i) $\mathfrak{M} = \mathbb{R} \cup \{-\infty\}$, where \mathbb{R} represents the Real Numbers.
- (ii) $a \oplus b \equiv \max(a,b)$
- (iii) $a \otimes b \equiv a + b$

Note 5.4. We will use the notation \mathfrak{M}^+ to represent the algebraic system $(\mathfrak{M}, \oplus, \otimes)$. Many authors refer to this as the (max,+)-algebra.

In defining the (max,+) algebra we have the following:

- (a) $\theta = -\infty$ is the additive identity of \mathfrak{M}^+ , i.e. $-\infty \oplus a = a$ for all $a \in \mathfrak{M}^+$.
- (b) $e = 0$ is the multiplicative identity of \mathfrak{M}^+ , i.e. $-\infty \otimes a = a$ for all $a \in \mathfrak{M}^+$.

Table 3 contains some examples of the basic mathematical operations in \mathfrak{M}^+ .

Problem in \mathfrak{M}^+	Problem in usual algebra	Answer
$2 \oplus 3$	$\max(2,3)$	3
$3 \otimes 5$	$3 + 5$	8
3^2	$2 \cdot 3$	6
$10^{1/2}$	$\frac{1}{2} \cdot 10$	5
$\sqrt[5]{15}$	$\frac{15}{5}$	3
$3^{-1} \otimes 6$	$\frac{6}{3}$	2

Table 3: Examples in $(\max,+)$ algebra

In what follows we will use \oplus and \otimes for the Max-Plus Algebra, and, $+$ and \cdot for the conventional algebra.

Proposition 5.5. \mathfrak{M}^+ is a zerosumfree idempotent semiring.

In particular, we add that \mathfrak{M}^+ is a semifield since all nonzero elements have a multiplicative inverse.

Elementary Properties

We begin by listing some of the elementary properties of the Max-Plus Algebra system.

Proposition 5.6.

- (a) \mathfrak{M}^+ is idempotent, i.e. $a \oplus a = a$ for all $a \in \mathfrak{M}^+$
- (b) $a^k \equiv k \cdot a \quad \forall a \in \mathfrak{M}^+, \text{ and } \forall k \in \mathbb{N}$.
- (c) $k > 0 \implies \theta^k = \theta$.
- (d) θ^θ is undefined.
- (e) $\lambda \otimes \theta = \theta$, for all λ
- (f) $\theta \oplus \theta = \theta$
- (g) $\theta \otimes \theta = \theta$

- (h) $\frac{\theta}{\theta} = \text{undefined}$
- (i) If $\lambda > \theta$, then $\lambda^{-1}\theta = \frac{\theta}{\lambda} = \theta$
- (j) $\alpha^{(1/\lambda)} \equiv \frac{1}{\lambda} \cdot \alpha = \frac{\alpha}{\lambda}$ if $\lambda \neq \theta$.
- (k) $a \leq b \iff a \oplus b = b$
- (l) $\theta \leq a$ for all $a \in \mathfrak{M}^+$
- (m) The normal hierarchy from the conventional algebra is used, namely:
 - (i) Exponents
 - (ii) Multiplication
 - (iii) Addition

Matrices

The usual properties of matrices from the previous sections on semirings and zerosumfree semirings also hold for matrices in \mathfrak{M}^+ . Definition 5.7 allows for a translation of the matrix A.

Definition 5.7. If $A \in \mathfrak{M}_{n \times n}^+$, $\lambda \in \mathfrak{M}^+$, then $\lambda^{-1}A \equiv A - \lambda J$, where J is the $n \times n$ matrix of all ones.

We include an example to show the matrix operations in \mathfrak{M}^+ .

Example 5.8.
$$\begin{bmatrix} 2 & 3 \\ -\infty & 4 \end{bmatrix} \otimes \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The work of this example in the conventional algebra is shown:

$$\begin{bmatrix} 2 & 3 \\ -\infty & 4 \end{bmatrix} \otimes \begin{bmatrix} -3 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \max(2 + (-3), 3 + 0) \\ \max(-\infty + (-3), 4 + 0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \blacksquare$$

Graphs

From Section 2 we have our basic definitions of graphs. We continue with our investigation of graphs in the \mathfrak{M}^+ algebra.

Definition 5.9. A **circuit mean** is the **average weight of a circuit** which is expressed as $w(\tau)^{\frac{1}{\ell(\tau)}}$ in the $(\max, +)$ algebra. This is equivalent to calculating the arithmetic mean of the circuit; $\frac{w(\tau)}{\ell(\tau)}$ in the conventional algebra.

Given a circuit τ , we will denote the average weight of a circuit τ by $\overline{\omega(\tau)}$. We will denote the set of circuits of $G(A)$ by $C(A)$.

Note 5.10. If $G(B)$ contains at least one circuit then $C(B) \neq \emptyset$.

Definition 5.11. We define the **maximal circuit mean of A**, denoted by $\mu(A)$ as:

$$\mu(A) = \begin{cases} \max_{\tau \in C(A)} \overline{\omega(\tau)} & \text{if } C(A) \neq \emptyset \\ -\infty & \text{if } C(A) = \emptyset \end{cases} \quad (13)$$

A circuit is called a **critical circuit** if $\overline{\omega(\tau)} = \mu(A)$. We will use $CN(A)$ to denote the set of all nodes of all of the critical circuits of A.

A directed graph is **strongly connected** if there exists a path from any node in the graph to any other node in the graph. We say that A is **irreducible** if and only if $G(A)$ is strongly connected. If A is not irreducible then we say that it is reducible. Therefore, A is reducible if and only if $G(A)$ is not strongly connected.

We include the following lemma which will be useful in our development of cycles.

Lemma 5.12. Suppose that a,b,c,d are real numbers, with b,d>0, then

$$\max \left\{ \frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d} \right\} = \max \left\{ \frac{a}{b}, \frac{c}{d} \right\} \quad (14)$$

Proof. Suppose, without loss of generality, $\frac{a}{b} \geq \frac{c}{d}$. Then

$$\begin{aligned} \frac{a}{b} &\geq \frac{c}{d} \\ ad &\geq bc \\ ab + ad &\geq ab + bc \\ a(b+d) &\geq b(a+c) \\ \frac{a}{b} &\geq \frac{a+c}{b+d} \end{aligned}$$

The result follows. □

With the use of the lemma above, we have

Lemma 5.13. We only need to consider the cycles of the graph in the calculation of the maximum circuit mean.

Proof. Suppose we have a circuit τ which is made up of two separate circuits, say γ and σ . We write $\tau = \gamma + \sigma$.

$$\begin{aligned} \overline{\omega(\tau)} &= \frac{\sum_{i,j \in \gamma} a_{i,j} + \sum_{p,q \in \sigma} a_{p,q}}{\ell(\gamma) + \ell(\sigma)} \\ \overline{\omega(\gamma)} &= \frac{\sum_{i,j \in \gamma} a_{i,j}}{\ell(\gamma)} \end{aligned}$$

$$\overline{\omega(\sigma)} = \frac{\sum_{p,q \in \sigma} a_{p,q}}{\ell(\sigma)}$$

By Lemma 5.12, above, we have that

$$\max\{\overline{\omega(\gamma)}, \overline{\omega(\sigma)}, \overline{\omega(\tau)}\} = \max\{\overline{\omega(\gamma)}, \overline{\omega(\sigma)}\}$$

□

5.3 Eigenvalue / Eigenvector Analysis in Max-Plus algebra

We say that $\lambda \in \mathfrak{M}^+$ is an eigenvalue of $A \in \mathfrak{M}_{n \times n}^+$ if there exists a vector $\underline{x} \neq \theta$ such that $A \otimes \underline{x} = \lambda \otimes \underline{x}$. In this case we say that \underline{x} is an eigenvector of A corresponding to the eigenvalue λ .

Definition 5.14 (Eigenvalue-Eigenvector equations in the Max-Plus Algebra system).

$$A \otimes \underline{x} = \lambda \otimes \underline{x} \implies \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \otimes \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (15)$$

$$\implies \begin{cases} \max\{a_{11} + x_1, a_{12} + x_2, \dots, a_{1n} + x_n\} = \lambda + x_1 \\ \max\{a_{21} + x_1, a_{22} + x_2, \dots, a_{2n} + x_n\} = \lambda + x_2 \\ \vdots \\ \max\{a_{n1} + x_1, a_{n2} + x_2, \dots, a_{nn} + x_n\} = \lambda + x_n \end{cases} \quad (16)$$

We begin with a few definitions and some properties of matrices in the Max-Plus algebra system. We use J for the matrix of all ones, i.e. $J = \underline{e}\underline{e}^T$, where \underline{e} is the vector of all ones.

Definition 5.15. Suppose $A \in \mathfrak{M}_{n \times n}^+$ and $\lambda \in \mathfrak{M}^+$, then

- (a) $(\lambda \otimes A)_{ij} \equiv \lambda + a_{ij}$
- (b) $\lambda^{-1} \otimes A = \frac{A}{\lambda} \equiv A - \lambda J$, if $\lambda \neq \theta$.

Lemma 5.16. Suppose $A, B \in \mathfrak{M}_{n \times n}^+$, $\lambda \in \mathfrak{M}^+$, then

- (a) $\lambda \otimes \left(\frac{A}{\lambda}\right) = \lambda \otimes (\lambda^{-1} \otimes A) = A$
- (b) If $\lambda > \theta$ and A is nilpotent then $\left(\frac{A}{\lambda}\right)$ is nilpotent.
- (c) $\frac{A}{\lambda} \frac{B}{\lambda} = \frac{AB}{\lambda^2}$

(d) If $\lambda > \theta$ and $A^n = \theta$, then $\left(\frac{A}{\lambda}\right)^n = \theta$

Proof.

(a) $[\lambda + (a_{i,j} - \lambda)] = [a_{ij}]$.

(b)

$$A = \begin{bmatrix} \theta & & ? \\ & \ddots & \\ \theta & & \theta \end{bmatrix} \Rightarrow \begin{bmatrix} \theta/\lambda & & ? \\ & \ddots & \\ \theta/\lambda & & \theta/\lambda \end{bmatrix} = \begin{bmatrix} \theta & & ? \\ & \ddots & \\ \theta & & \theta \end{bmatrix} \quad (17)$$

(c)

$$\bigoplus_k \left(\frac{A}{\lambda}\right)_{ik} \otimes \left(\frac{B}{\lambda}\right)_{kj} \equiv \max_k \{(a_{ik} - \lambda) + (b_{kj} - \lambda)\} = \max_k \{(a_{ik} + b_{kj})\} - 2\lambda \equiv \frac{AB}{\lambda^2} \quad (18)$$

(d) If $\lambda \neq \theta$, then

$$\left(\frac{A}{\lambda}\right)^n = \frac{A^n}{\lambda^n} = \frac{\theta}{\lambda^n} \equiv (-\infty - n \cdot \lambda) = -\infty = \theta \quad (19)$$

□

Definite Matrices

In this section we give a brief overview of the subject of definite matrices. See [13] for a thorough introduction to the theory of definite matrices in the Max-Plus algebra. The theory of definite matrices is important as a first step in the development of eigenvalue-eigenvector equations. These matrices are used to simplify the equations, for, as we will see, the eigenvectors are easily found for a matrix in the Max-Plus algebra.

Definition 5.17. A matrix, B, is said to be **definite** if $C(B) \neq \emptyset$, $w(\gamma) \leq e$, for all circuits $\gamma \in G(B)$, and suppose that there exists at least one circuit, τ , with $w(\tau) = e$.

We include some of the basic theorems on definite matrices from [13]. We present no proofs for these theorems but refer to [13] for details. These theorems are included for completeness.

Definition 5.18. Suppose B is definite. Let

1. $B^+ = B \oplus B^2 \oplus \dots \oplus B^n$
2. $B^\ddagger = I \oplus B \oplus \dots \oplus B^{n-1}$

Theorem 5.19. Suppose B is definite. Then:

- (1) $BB^\dagger = B^+$
- (2) $B^r \leq B^+ \quad \forall r \geq 1$
- (3) $I \oplus B \oplus B^2 \oplus \cdots \oplus B^{n-1} = I \oplus B \oplus B^2 \oplus \cdots \oplus B^n$
- (4) $I \oplus B^+ = B^\dagger$
- (5) $I \oplus B^+ = I \oplus B^+ \oplus B^k, \quad \forall k > n$
- (6) $BB^\dagger \oplus I = B^\dagger$
- (7) $(I \oplus B)^{n-1} = B^\dagger$
- (8) $B^+ = B(I \oplus B)^{n-1}$
- (9) $B(I \oplus B^+) \oplus I = I \oplus B^+$
- (10) $B^\dagger = \bigoplus_{k=0}^r B^k, \quad \text{for all } r \geq n-1$
- (11) B^r is definite $\forall r \geq 0$
- (12) B^+ is definite
- (13) B^\dagger is definite
- (14) There exist i such that $1 \leq i \leq n$ and $B_{ii}^\dagger = e$
- (15) $(B^+)^r \leq B^+, \quad \text{for all } r \geq 1$
- (16) $B^+ = B(I \oplus B)^r \quad \text{for all } r \geq (n-1)$
- (17) $BB^\dagger \underline{e}_i = B^+ \underline{e}_i \quad \text{for all } i \in CN(B)$
- (18) $B(B^+ \underline{e}_i) = e \otimes B^+ \underline{e}_i$
- (19) e is an eigenvalue of B
- (20) $B^+ \underline{e}_i$ is an eigenvector of B , for all $i \in CN(B)$.

Theorem 5.20. Suppose B is definite and suppose that node j is an eigennode, a node on a critical circuit. Then

- (1) B^+ is definite.
- (2) $(B^+)_{jj} = e$

Theorem 5.21. Suppose B is definite and $(B^+)_{jj} = e$ for some j . Then node j is an eigennode of $G(B)$.

Theorem 5.22. Suppose B is definite, and j is an eigennode of G(B). Then $B \otimes \xi(j) = \xi(j)$, where $\xi(j)$ is the j^{th} column of B^+ .

Proof. We need to show that $B \otimes B^+ \underline{e}_i = e \otimes B^+ \underline{e}_i$. This is the result of Theorem 5.19, equation (18). \square

Theorem 5.23. Suppose B is definite. If $\xi(h), \xi(j)$ are fundamental eigenvectors of B corresponding to equivalent eigen-nodes h and k, respectively, then $\xi(h)$, and $\xi(j)$ are finite multiples of one another.

Theorem 5.24. Let $M_\omega = \omega^{-1} \otimes M \equiv M - \omega J$, where $\omega = \mu(M)$. Then $\mu(M_\omega) = 0$.

Proof. $\omega = \mu(M)$

$$\omega = \frac{\sum_\gamma M_{ij}}{\ell(\gamma)}, \text{ and } \sum_\gamma M_{ij} = \omega \ell(\gamma) \quad (20)$$

$$\mu(M_\omega) = \frac{\sum_\gamma (M_{ij} - \omega)}{\ell(\gamma)} = \frac{\sum_\gamma M_{ij}}{\ell(\gamma)} - \frac{\sum_\gamma \omega}{\ell(\gamma)} \quad (21)$$

$$= \frac{\ell(\gamma) \cdot \omega}{\ell(\gamma)} - \frac{\ell(\gamma) \cdot \omega}{\ell(\gamma)} = \omega - \omega = 0. \quad (22)$$

\square

Theorem 5.25. Given A. If $w(\gamma) < e$ for all $\gamma \in G(A)$, then A^\ddagger exists, and the equation $AX+I=X$ has A^\ddagger as a solution.

Theorem 5.25 shows that A^\ddagger plays the role of $(I - A)^{-1}$ in this context.

Definition 5.26. We say that B is e-astic definite if B is definite, and $(B)_{ij} \leq e$, for $(i,j=1,2, \dots, n)$.

Theorem 5.27. B is e-astic definite if and only if B is definite and the elements of B form a e-astic set. In particular, if B is row- e-astic or column-e-astic, then B is e-astic definite.

Theorem 5.28. Suppose B is e-astic definite. Then:

- (i) B^+ is e-astic definite, and B^r is e-astic definite for each $r \geq 1$.
- (ii) If $\xi(h)$, and $\xi(k)$ are equivalent fundamental eigenvectors of B, corresponding to equivalent eigen-nodes h, k respectively, then $\xi(h) = \xi(k)$.
- (iii) Each fundamental eigenvector of B is column-e-astic.
- (iv) The fundamental eigenvector $\xi(j)$ corresponding to a given eigen-node j of G(B) contains components equal to e in all positions corresponding to eigen-nodes equivalent to j.

Theorem 5.29. Suppose B is e-astic definite. If $\xi(j), \xi(k)$ are fundamental eigenvectors corresponding to eigen-nodes j, and k of G(B), and j, k are not equivalent, then: either $(\xi(j))_k < e$ or $(\xi(k))_j < e$.

Theorem 5.30. Suppose B is e -astic definite. If $\xi(j_1), \xi(j_2), \dots, \xi(j_s)$ are fundamental eigenvectors corresponding to pairwise non-equivalent eigen-nodes j_1, j_2, \dots, j_s respectively, then no one of $\xi(j_1), \xi(j_2), \dots, \xi(j_s)$ is linearly dependent on the others.

Theorem 5.31. Suppose B is row- e -astic. Let $\xi(j_1), \xi(j_2), \dots, \xi(j_s)$ be a maximal set of non-equivalent fundamental eigenvectors for B and let $\Phi(B)$ be the matrix whose columns are $\xi(j_1), \xi(j_2), \dots, \xi(j_s)$, in that order. Then $\Phi(B)$ is doubly e -astic.

Theorem 5.32. Suppose B is definite. Then each element of the eigenspace of B is an eigenvector of B with corresponding eigenvalue equal to e .

Theorem 5.33. Suppose B is e -astic definite. Let $\underline{\eta}$ be an eigenvector of B having corresponding eigenvalue e . If h and k are (distinct) equivalent eigen-nodes in $G(B)$, then $(\underline{\eta})_h = (\underline{\eta})_k$.

Theorem 5.34. Suppose B is e -astic definite. Let $\underline{\eta}$ be a finite eigenvector of B having corresponding eigenvalue e . If j is a non-eigen-node in $G(B)$ then there is an eigen-node d of $G(B)$ such that $(\underline{\eta})_j = (B^+)_{jd} \otimes (\underline{\eta})_d$.

Theorem 5.35. Suppose B is e -astic definite. If $\underline{\eta}$ is a finite eigenvector having corresponding eigenvalue e , then $\underline{\eta}$ lies in the eigenspace of B .

General Matrices

In this section we investigate the theory for general matrices in \mathfrak{M}^+ .

The following theorem is analogous to the theorem from Nonnegative matrices.

Theorem 5.36. $\theta = -\infty$ is an eigenvalue of A if and only if A has an infinite column.

Proof. From [4].

If $(-\infty, x)$ is an eigenpair of A and if x_j is finite, then it follows from the eigenvalue-eigenvector equation that the j -th column of A is $-\infty$. Conversely, if the j -th column of A is $-\infty$, then a vector with all components $-\infty$ except the j -th component serves as an eigenvector of A with $-\infty$ as the eigenvalue. \square

Corollary 5.37. If A is irreducible, then any eigenvalue is finite, i.e. if $\lambda \in \sigma(A)$, then $\lambda > -\infty$.

Proof. This is a corollary of Theorem 3.8. If A is irreducible, then A cannot have an infinite column. Therefore $-\infty$ is not an eigenvalue of A . So any eigenvalue must be finite. \square

Theorem 5.38 (\mathfrak{M}^+). If A has a partly infinite eigenvector, then A is reducible.

Proof. The proof given here is essentially the proof given in [4]. Let (λ, x) be an eigenpair of A with x partly infinite. If $\lambda = -\infty$, then by Theorem 5.36, we conclude that A has an infinite column and hence A is reducible.

So suppose λ is finite. We assume, without loss of generality, that $x=[y, -\infty]^T$, where y is finite. Partition A conformally, so that we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \otimes \begin{bmatrix} y \\ -\infty \end{bmatrix} = \lambda \otimes \begin{bmatrix} y \\ -\infty \end{bmatrix}$$

Then $A_{21} \otimes y = \lambda \otimes -\infty$. Since y is finite, then $A_{21} = -\infty$, and hence A is reducible. \square

The following is a corollary of Theorem 5.38.

Corollary 5.39. If A is irreducible, then any eigenvector of A is finite.

The next theorem has a analogous theorem from the nonnegative matrices.

Theorem 5.40. For any matrix A , if $A \otimes x = \lambda \otimes x$, where $x > -\infty$ (x is finite), then $\lambda = \mu(A)$.

Proof. From [4].

If $\lambda = -\infty$, then as noted in Theorem 5.36, every column of A must be $-\infty$ and hence $\lambda = \mu(A)$. So assume that λ is finite. We have

$$\max_j (a_{ij} + x_j) = \lambda + x_i, \quad \text{for } i=1,2,\dots,n \quad (23)$$

and hence, in particular,

$$a_{ij} + x_j - x_i \leq \lambda, \quad \text{for } i,j=1,2,\dots,n. \quad (24)$$

Since λ is finite, we must have $C(A) \neq \emptyset$ by Theorem 3.8. Let $\tau \in C(A)$ be the circuit formed by the edges $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$ and add the resulting inequalities to get $M_A(\tau) \leq \lambda$.

Now construct a directed graph K with vertices $1, 2, \dots, n$ and an edge (i, j) if and only if there is equality in (24), i.e. $a_{ij} + x_j - x_i = \lambda$. By (23), each vertex in K has out-degree at least one and hence, by Theorem 2.20, K has a circuit, say, τ . Since λ is finite, K is clearly a subgraph of $G(A)$. Thus $\tau \in C(A)$, and furthermore, $M_A(\tau) = \lambda$. Therefore, $\lambda = \mu(A)$. \square

Work for Theorem 5.40 from (24)

$$a_{12} + x_2 \leq \lambda + x_1$$

$$a_{23} + x_3 \leq \lambda + x_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{k1} + x_1 \leq \lambda + x_k$$

$$(a_{12} + \cdots + a_{k1}) + (x_1 + \cdots + x_n) \leq \lambda^k + (x_1 + \cdots + x_n)$$

$$(a_{12} + \cdots + a_{k1}) \leq \lambda^k$$

$$M_A(\tau) \leq \lambda$$

or as

$$a_{12} \otimes x_2 \leq \lambda \otimes x_1$$

$$a_{23} \otimes x_3 \leq \lambda \otimes x_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{k1} \otimes x_1 \leq \lambda \otimes x_k$$

$$\bigotimes_{ij} a_{ij} \otimes \bigotimes_i x_i \leq \lambda^{\otimes k} \otimes \bigotimes_i x_i$$

$$\bigotimes_{ij} a_{ij} \leq \lambda^{\otimes k}$$

$$M_A(\tau) \leq \lambda$$

The following theorem is one of the main theorems in the theory of matrices over \mathfrak{M}^+ .

Theorem 5.41. If $A \in \mathfrak{M}_{n \times n}^+$, then $\mu(A)$ is an eigenvalue of A .

Proof. The proof given here is essentially that given in [3]. If $C(A) = \emptyset$, then $\mu(A) = -\infty$ and it is an eigenvalue of A by Theorem 3.8. So assume that $C(A) \neq \emptyset$, in which case $\mu(A)$ must be finite. Let B be the matrix with $b_{ij} = a_{ij} - \mu(A)$. (i.e. $B = (\mu(A))^{-1} \otimes A$.) for all i, j . and let $B^+ = B \oplus B^2 \oplus \cdots \oplus B^n$. Let $j \in \{1, 2, \dots, n\}$ be a vertex which belongs to a critical circuit in $C(B)$. Any path from j to itself in $G(B)$ must have weight at most zero and since j belongs to a critical circuit, there is at least one path from j to itself of weight zero. Thus $B_{jj}^+ = 0$ and, in particular, B_j^+ , the j -th column of B^+ , is not $-\infty$.

Now observe that for any i , the maximal weight of a path, of any length, from i to j may also be expressed as

$$\bigoplus_{k=1}^n b_{ik} \otimes B_{kj}^+ \quad (25)$$

since there exists such a path (in which a vertex may occur more than once) of length at least two, in view of the fact that j is on a critical circuit. However, (25) is precisely the i^{th} entry of $B \otimes B_j^+$. Therefore, $B \otimes B_j^+ = B_j^+ = 0 \otimes B_j^+$, and hence $(0, B_j^+)$ is an eigen-pair of B . Now

$$A \otimes B_j^+ = \mu(A) \otimes B \otimes B_j^+ = \mu(A) \otimes B_j^+ \quad (26)$$

and therefore $\mu(A)$ is an eigenvalue of A . \square

Theorem 5.42. If A is irreducible, then $\mu(A)$ is only eigenvalue of A and every eigenvector corresponding to $\mu(A)$ is finite.

Proof. The proof given here is from [4] and is included for completeness. Since A is irreducible, using an argument as in Theorem 3.8, we see that $C(A) \neq \emptyset$ and hence $\mu(A)$ is finite. Let B be the $n \times n$ matrix with $b_{ij} = a_{ij} - \mu(A)$ for all i, j .

$$\text{i.e. } A = \mu(A) \otimes B, \text{ or } B = (\mu(A))^{-1} \otimes A.$$

Define B^+ by

$$B^+ = B \oplus B^2 \oplus \cdots \oplus B^n \quad (27)$$

Then:

$$B^+ \geq B^k, \forall k \geq n + 1 \quad (28)$$

Let $\tau \in C(A)$ be such that $M_A(\tau) = \mu(A)$ and let j be any vertex that belongs to τ . Any path from j to itself in $G(B)$ must have weight at most zero, and because j is in τ , there is at least one path from j to itself in $G(B)$ with weight zero. Thus $B_{jj}^+ = 0$ and, in particular, B_j^+ , the j -th column of B^+ , is not $-\infty$.

For any i , the maximal weight of a path in $G(B)$ (of any length) from i to j may also be expressed as

$$\bigoplus_k b_{ik} \otimes B_{kj}^+ \quad (29)$$

since there exists such a path (in which a vertex may occur more than once) of length at least two, in view of the fact that j is in τ . However (29) is precisely the i^{th} entry of $B \otimes B_j^+$. Therefore $B \otimes B_j^+ = B_j^+ = 0 \otimes B_j^+$, and hence $(0, B_j^+)$ is an eigenpair of B . Now

$$A \otimes B_j^+ = \mu(A) \otimes B \otimes B_j^+ = \mu(A) \otimes B_j^+ \quad (30)$$

and hence $\mu(A)$ is an eigenvalue of A . \square

The following is a well-known theorem from linear algebra.

Theorem 5.43. If A is a nonnegative matrix then A and A^T have the same eigenvalues.

In \mathfrak{M}^+ things differ from their nonnegative counterparts.

Note 5.44. If $A \in \mathfrak{M}_{n \times n}^+$, then A and A^T may have different eigenvalues.

The following example illustrates this fact for a small matrix over \mathfrak{M}^+ .

Example 5.45. If $A = \begin{bmatrix} 5 & -\infty \\ 2 & 6 \end{bmatrix}$, then

$$A \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix} = 6 \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix} \quad (31)$$

and it can be shown that 6 is the only eigenvalue for A .

For A^T , we see that

$$\begin{bmatrix} 5 & 2 \\ -\infty & 6 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -\infty \end{bmatrix} = 5 \otimes \begin{bmatrix} 0 \\ -\infty \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 2 \\ -\infty & 6 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 6 \otimes \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

and, thus, 5 and 6 are eigenvalues. ■

Note 5.46. Although a matrix, A , may have different eigenvalues than its transpose, A^T , we can say that $\mu(A) = \mu(A^T)$. This may be seen graphically since A and A^T are related by changing the direction of all paths in the graph. Therefore the same cycles are still critical cycles, since only the direction is changed. The maximum cycle mean of both graphs is the same.

The following three theorems come from Chen, et. al., [10]. These theorems provide a nice classification for the eigenvalues and eigenvectors of a matrix in the Max-Plus Algebra system.

Theorem 5.47 (Canonical Form). For the matrix \overline{M} in Theorem 2.14, we can recombine its blocks by a permutation matrix P_1 to make it in the following form:

$$\widehat{M} = P_1 \overline{M} P_1^{-1} = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1k_0} & A_{10} \\ & A_2 & \cdots & A_{2k_0} & A_{20} \\ & & \ddots & \vdots & \vdots \\ & 0 & & A_{k_0} & A_{k_0 0} \\ & & & & A_0 \end{bmatrix} \quad (32)$$

where A_j ($j=1,2,\dots,k_0, 0$; $k_0 \leq k$) is also a block triangular submatrix; $G(A_j)$ ($j=1,2,\dots,k_0,0$) consists of several strongly connected components; and \widehat{M} satisfies:

ordered as before. We can then create the matrix:

$$\begin{array}{l}
\widehat{V}_k \\
V_k \\
\overline{V}_k
\end{array}
\left[
\begin{array}{c|c|c}
\widehat{V}_k & V_k & \overline{V}_k \\
\hline
B_1 & & \\
\ddots & & ? \\
D_1 & B_p & \\
\hline
D_2 & M_{k_1} & \\
\hline
D_3 & D_4 & M_{k_1+1} \\
& & \ddots \\
& & D_5 & M_q
\end{array}
\right]. \quad (36)$$

To show that $D_i=0$, for $i=1,2,\dots,5$, we proceed as follows:

Claim: $D_1 = 0$. **Proof:** The vertices are written in the same numerical order as in the matrix \overline{M} , so there is no path from one node to a node that precedes it in the matrix \overline{M} . \blacktriangledown

Claim: $D_2 = 0$. **Proof:** If $S_i \in V_k$, and $S_j \in \widehat{V}_k$, then a path from S_i to S_j would imply that S_j is in V_k , which is not the case. \blacktriangledown

Claim: $D_3 = 0$. **Proof:** There is no path from a node in \overline{V}_k to any node in V_k . If there was a one-step path from a node in \overline{V}_k to a node in \widehat{V}_k , then there would be a path that goes from that node to a node in V_k , which can not happen, by definition of a node being in \overline{V}_k . \blacktriangledown

Claim: $D_4 = 0$. **Proof:** A node in \overline{V}_k cannot have a path to a node in V_k , so it certainly cannot have a one-step path to a node in V_k . \blacktriangledown

Claim: $D_5 = 0$. **Proof:** Reasoning similar to the proof that $D_1=0$. \blacktriangledown

So we end with a matrix of the form:

$$\begin{array}{l}
\widehat{V}_k \\
V_k \\
\overline{V}_k
\end{array}
\left[
\begin{array}{c|c|c}
\widehat{V}_k & V_k & \overline{V}_k \\
\hline
B_1 & & \\
\ddots & & \\
0 & B_p & \\
\hline
0 & M_{k_1} & \\
\hline
0 & 0 & M_{k_1+1} \\
& & \ddots \\
& & 0 & M_q
\end{array}
\right], \quad (37)$$

which has the form: $\begin{bmatrix} A_1 & B \\ 0 & D \end{bmatrix}$, where $\lambda_1 = \mu(\overline{M}) = \mu(\overline{M}_{k_1})$ which is in the lower right corner of A_1 .

We then continue the same process with the matrix D to arrive at the required form. \square

Theorem 5.48. Suppose λ is an eigenvalue of matrix M . There is at least one circuit in $G(M)$ whose average weight is λ .

Theorem 5.49.

1. For the matrix \widehat{M} in Theorem 5.47, $\lambda_1, \lambda_2, \dots, \lambda_{k_0}$ are its eigenvalues. If i_0 is a node belonging to a circuit of average weight λ_j then the i_0^{th} -row of $\widehat{M}_{\lambda_j}^+$ is an eigenvector of \widehat{M} that corresponds to λ_j .
2. M has no other eigenvalue than $\lambda_1, \lambda_2, \dots, \lambda_{k_0}$.

If V_i and V_j are classes then we say that V_j has access to V_i if either $i=j$ or there exists $u \in V_j$ and $v \in V_i$ such that there exists a path from u to v in $G(A)$. If $\mu(A_j) > \mu(A_i)$, then we say that class V_j **dominates** V_i . The following theorem from [3] gives the condition for a (reducible) matrix to have an eigenvalue.

Theorem 5.50. Let $A \in \mathfrak{M}_{n \times n}^+$, and suppose that A is in Frobenius Normal Form. Then λ is an eigenvalue of A if and only if there exists an i such that $\mu(A_i) = \lambda$ and no class that dominates A_i has access to A_i .

Example 5.51. For a simple example, consider the matrices $A = \begin{bmatrix} 5 & 2 \\ \theta & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 2 \\ \theta & 5 \end{bmatrix}$.

A has one eigenvalue, $\lambda = 5$, with

$$\begin{bmatrix} 5 & 2 \\ \theta & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \theta \end{bmatrix} = 5 \otimes \begin{bmatrix} 1 \\ \theta \end{bmatrix}$$

while B has two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 5$, with

$$\begin{bmatrix} 1 & 2 \\ \theta & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \theta \end{bmatrix} = 1 \otimes \begin{bmatrix} 1 \\ \theta \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ \theta & 5 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 5 \otimes \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

This is the case since in matrix A , the class with 5 has access to the class with 1, whereas in matrix B , the maximum class with 5 does not have access to the class with 1. ■

Theorem 5.52. If A is irreducible, then there exists $k \in \mathbb{N}$, and $c \in \mathbb{N} \setminus \{0\}$, such that $A^{k+c} = \lambda^c \otimes A^k$.

Example 5.53. $A = \begin{bmatrix} \theta & 2 & \theta \\ \theta & \theta & 3 \\ 4 & \theta & \theta \end{bmatrix}$
 $\mu(A) = 3$

With the help of Theorem 5.52, since A is irreducible, we note that $A^4 = 3^3 \otimes A^1$. ■

Example 5.54.

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 2 & 1 & 9 & 1 \end{bmatrix} \quad \begin{array}{l} \mu(A) = 5 \\ A^5 = A^{3+2} = 5^2 \otimes A^3 \end{array}$$

$$A^3 = \begin{bmatrix} 10 & 10 & 13 & 11 \\ 10 & 10 & 13 & 11 \\ 10 & 10 & 13 & 11 \\ 12 & 12 & 19 & 13 \end{bmatrix} \quad A^5 = \begin{bmatrix} 20 & 20 & 23 & 21 \\ 20 & 20 & 23 & 21 \\ 20 & 20 & 23 & 21 \\ 22 & 22 & 29 & 23 \end{bmatrix}$$

The following theorem illustrates a useful method to be used to calculate $\mu(A)$. ■

Theorem 5.55. A formula for the maximum cycle mean μ in the max-plus algebra notation is:

$$\mu = \bigoplus_{j=1}^n (\text{Trace}(A^j))^{(1/j)} \quad (38)$$

Justification. See [2] for more detail. Consider an $n \times n$ matrix A with corresponding precedence graph $G=(V,E)$. The maximum weight of all Circuits of length j which pass through node i of G can be written as $(A^j)_{ii}$. The maximum of these maximum weights over all nodes is $\bigoplus_{i=1}^n (A^j)_{ii}$ which can be written $\text{Trace}(A^j)$. The average weight is obtained by dividing this number by j in the usual algebra sense, but this can be written $(\text{Trace}(A^j))^{(1/j)}$ in the max-plus algebra notation. Finally, we have to take the maximum over all circuit lengths j. It is not necessary to consider lengths larger than the number n of nodes since it is enough to limit ourselves to elementary circuits. Therefore we get the above formula. □

The following theorem is due to Karp, and is included as an example of another method which can be used to calculate $\mu(A)$ for a matrix in \mathfrak{M}^+ .

Theorem 5.56. Given an $n \times n$ matrix A , the maximum cycle mean is given by:

$$\lambda = \max_{1 \leq i \leq n} \min_{0 \leq k \leq n-1} \frac{(A^n)_{ij} - (A^k)_{ij}}{n - k}, \forall j \quad (39)$$

5.4 Symmetrization of the Max-Plus Algebra

The Max-Plus algebra system lacks an additive inverse. Therefore some equations do not have a solution. For example, the equation $x \oplus 3 = 2$ has no solution since there is no number x such that $\max(x, 3) = 2$. One way of trying to solve this problem is to embed the Max-Plus algebra into a larger system which will have an additive inverse. We would like to be able to solve the equation: $a \oplus x = \theta$, where $a \in \mathfrak{M}^+$. This is known as *symmetrizing* a . The following proposition shows that this is impossible.

Proposition 5.57. Every idempotent group is reduced to the null element.

Proof. Assume that the group (G, \oplus) is idempotent with additive identity θ . Suppose that b is the symmetric element of $a \in G$, i.e., $a \oplus b = \theta$. Then we have:

$$a = a \oplus \theta = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b = \theta. \quad \square$$

Gaubert, [18], creates the S_{\max} system to extend \mathfrak{M}^+ to a system which include additive inverses. We give a brief overview of his technique. The technique presented there is a first attempt to symmetrize the Max-Plus Algebra system. This system is not as transparent as one would like. The basic operation used in this symmetrization is not transitive. One is able to get around this initial problem, but the result, although usable, is somewhat complicated. We present this material as a comparison to the method developed later in Section 8, where we are able to compute the solutions to equations in a manner which is considerably easier. The embedding here is a more difficult embedding.

The new system: S_{\max}

We attempt to extend the Max-Plus algebra to a larger system which will include an additive inverse in the same way that the natural numbers were extended to the larger system of integers. A thorough introduction and treatment of this symmetrization algebra is given in [18].

Note 5.58. We use the same symbols, \oplus and \otimes , that were used in \mathfrak{M}^+ because there is a natural correspondence between these “new” operations (on ordered pairs) and the original operations in \mathfrak{M}^+ . The context of the symbols should tell which system is being used in an unambiguous way.

Definition 5.59. We let $\mathfrak{P} \stackrel{def}{=} \mathfrak{M}^+ \times \mathfrak{M}^+$.

Let $x = (x_1, x_2)$, and $y = (y_1, y_2)$, then we define the new operations on \mathfrak{P} :

Definition 5.60. Let $x, y \in \mathfrak{P}$, then

$$x \oplus y = (x_1, x_2) \oplus (y_1, y_2) \equiv (x_1 \oplus x_2, y_1 \oplus y_2) \quad (40)$$

$$x \otimes y = (x_1, x_2) \otimes (y_1, y_2) \equiv (x_1 \otimes y_1 \oplus x_2 \otimes y_2, x_1 \otimes y_2 \oplus x_2 \otimes y_1) \quad (41)$$

Here we have (θ, θ) is the additive identity, and (e, θ) is the unit element (multiplicative identity) of the set \mathfrak{P} .

Definition 5.61. Let $x=(x_1, x_2)$, and define:

1. $\ominus x \equiv (x_2, x_1)$.
2. The absolute value of x : $|x| = x_1 \oplus x_2 \equiv \max\{x_1, x_2\}$.
3. The balance operator: $x^\bullet = x \ominus x = (|x|, |x|)$.

It can be verified that (\mathfrak{P}, \oplus) is associative, commutative, and idempotent, with additive identity (θ, θ) ; and (\mathfrak{P}, \otimes) is associative, commutative and distributive over \oplus , and the multiplicative identity is (e, θ) . The additive identity is absorbing under \otimes . Therefore the algebraic structure $(\mathfrak{P}, \oplus, \otimes)$ is a commutative dioid. We will call this the *algebra of pairs*.

We can also show that the following properties hold:

1. $u^\bullet = (\ominus u)^\bullet = (u^\bullet)^\bullet$
2. $u \otimes v^\bullet = (u \otimes v)^\bullet$
3. $\ominus(\ominus u) = u$
4. $\ominus(u \oplus v) = (\ominus u) \oplus (\ominus v)$
5. $\ominus(u \otimes v) = (\ominus u) \otimes v = u \otimes (\ominus v)$

In the conventional algebra, we have $a - a = 0$, but in the algebra of pairs, we have $a \ominus a = a^\bullet \neq (\theta, \theta)$ for all $a \in \mathfrak{P}$, unless $a=(\theta, \theta)$. So we introduce a new relation on \mathfrak{P} :

Definition 5.62 (balance relation). Let $x=(x_1, x_2)$ and $y=(y_1, y_2)$. We say that x balances y , written $x \Delta y$, if and only if $x_1 \oplus y_2 = x_2 \oplus y_1$.

With this new relation, we have $a \ominus a = a^\bullet = (|a|, |a|) \Delta (\theta, \theta)$ for all $a \in \mathfrak{P}$. We would like to form the quotient set using this new relation, but the big caveat is that the balance relation is not transitive! For example: $(3, 2) \Delta (3, 3)$, and $(3, 3) \Delta (2, 3)$, but $(3, 2) \not\Delta (2, 3)$. Therefore, the balance relation is not an equivalence relation, and we can not use it to form a quotient set. Therefore we define a new relation which is closely related to the balance operator, but that is transitive:

$$(x_1, x_2) \mathcal{R}(y_1, y_2) \quad \text{if} \quad \begin{cases} (x_1, x_2) \Delta (y_1, y_2) & \text{if } x_1 \neq x_2 \text{ and } y_1 \neq y_2, \\ (x_1, x_2) = (y_1, y_2) & \text{otherwise.} \end{cases} \quad (42)$$

This new relation is an equivalence relation on \mathfrak{P} . So we define the quotient set $\mathbb{S} = \mathfrak{P} / \mathcal{R}$. The algebraic structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the *symmetrized max-plus algebra*. The quotient set has three different types of equivalence classes:

- $\overline{(t, -\infty)} = \{(t, x) \in \mathfrak{P} \mid x < t\}$, called max - positive;
- $\overline{(-\infty, t)} = \{(x, t) \in \mathfrak{P} \mid x < t\}$, called max - negative;
- $\overline{(t, t)} = \{(t, t) \in \mathfrak{P}\}$, called balanced.

We see that $\overline{(t, -\infty)}$ are associated with the elements of \mathbb{R} . Let $\mathbb{S}^{\oplus} =$ the max-positive elements, and $\mathbb{S}^{\ominus} =$ the max-negative elements, and $\mathbb{S}^{\bullet} =$ the balanced elements. Also let $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$. We call \mathbb{S}^{\vee} the set of signed elements in \mathbb{S} . So we have $\mathbb{S} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet}$. Note that $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \overline{(\theta, \theta)}$, and $\theta = \ominus \theta = \theta^{\bullet}$.

Example 5.63. $2 \oplus (\ominus 3) = \overline{(2, -\infty)} \oplus \overline{(-\infty, 3)} = \overline{(2, 3)} = \overline{(-\infty, 3)} = \ominus 3$ ■

In general, if $a, b \in \mathbb{R}$, then we have:

- $a \oplus (\ominus b) = \begin{cases} a & \text{if } a > b \\ \ominus b & \text{if } b > a \end{cases}$
- $a \ominus a = a \oplus (\ominus a) = a^{\bullet}$

The set $\mathbb{S}^{\vee} \setminus \{\theta\}$ is the set of invertible elements in \mathbb{S} .

The main algebraic features of balances are:

Proposition 5.64.

- (i) $a \Delta a$ (reflexivity)
- (ii) $a \Delta b \iff b \Delta a$ (symmetry)
- (iii) $a \Delta b \iff a \ominus b \Delta \theta$
- (iv) $a \Delta b, c \Delta d \implies a \oplus c \Delta b \oplus d$
- (v) $a \Delta b \implies ac \Delta bc$

Although Δ is not transitive, we can manipulate balances in the cases where some variables are signed.

Proposition 5.65 (Weak substitution).

$$\begin{cases} x \Delta a \\ c x \Delta b \end{cases} \text{ and } x \in \mathbb{S}^{\vee} \implies c a \Delta b \quad (43)$$

Letting $c=e$ yields:

Proposition 5.66 (Weak transitivity).

$$a\Delta x, x\Delta b, \text{ and } x \in \mathbb{S}^\vee \implies a\Delta b \quad (44)$$

There is one case in which we can translate balances into equations:

Proposition 5.67 (Reduction of balances).

$$x \Delta y, \text{ and both } x, y \in \mathbb{S}^\vee \implies x = y$$

We can give some extra properties of balances.

Proposition 5.68.

$$\forall a, b, c \in \mathbb{S}, \quad a \ominus c \Delta b \quad \text{if and only if} \quad a \Delta b \oplus c.$$

Proposition 5.69. $\forall a, b \in \mathbb{S}^\vee, \quad a \Delta b \implies a = b.$

Proposition 5.70. If $x \in \mathbb{S}$, then

$$(a) \quad x \otimes b^\bullet = (x \otimes b)^\bullet$$

$$(b) \quad x \oplus b^\bullet = \begin{cases} x & \text{if } |x| > |b| \\ b^\bullet & \text{if } |x| < |b| \end{cases}$$

Example 5.71.

$$\begin{aligned} 3 \ominus 2^\bullet \oplus 4 \ominus 5 &= 3 \ominus (2 \ominus 2) \oplus 4 \ominus 5 \\ &= 3 \ominus 2 \oplus 2 \oplus 4 \ominus 5 \\ &= 3 \oplus 2 \oplus 4 \ominus 2 \ominus 5 \\ &= (3 \oplus 2) \oplus 4 \ominus (2 \oplus 5) \\ &= 3 \oplus 4 \ominus (2 \oplus 5) \\ &= 4 \ominus 5 \\ &= \ominus 5 \end{aligned}$$

■

The balance operation can be extended to vectors and matrices by applying the balance operator elementwise.

Proposition 5.72. For all $A, B, C \in \mathbb{S}^{m \times n}$, $A \ominus C \Delta B$ if and on if $A \Delta B \oplus C$.

Proposition 5.73. For all $A, B \in (\mathbb{S}^\vee)^{m \times n}$, $A \Delta B \implies A=B$.

Equations to Balances

Consider the equation in \mathbb{R}_{\max} :

$$Ax \oplus b = Cx \oplus d \quad (45)$$

in \mathbb{R}_{\max} . We can rewrite equation (45) in \mathbb{S}^{\max} as:

$$Ax \oplus b \Delta Cx \oplus d \quad (46)$$

by reflexivity, and by the previous properties as:

$$(A \ominus C)x \oplus (b \ominus d) \Delta \theta \quad (47)$$

We would like to solve some equations or balances when one of the elements is an element of \mathbb{S}^{\vee} . To this end we have one lemma.

Lemma 5.74. Suppose $d \in \mathbb{S}^{\bullet}$. $p = b \oplus d \implies p \Delta b$.

Proof. Let $p=(p_1,p_2)$, $b=(b_1,b_2)$, $d=(g,g)$, where $g=|d|$. If $(p_1,p_2)=(b_1,b_2) \oplus (g,g)$, then $p_1 = b_1 \oplus g$, and $p_2 = b_2 \oplus g$. To show that $p \Delta b$ we must show that $p_1 \oplus b_2 = p_2 \oplus b_1$. So suppose that $p_1 \oplus b_2 \neq p_2 \oplus b_1$, but by substitution, we get $b_1 \oplus g \oplus b_2 \neq b_2 \oplus g \oplus b_1$, which is a contradiction. Therefore, $p \Delta b$. \square

Lemma 5.74 allows us to remove a balanced element from an equation if we allow the equation to become a balance.

The following are some of the basic properties of \mathbb{S} .

Proposition 5.75. If $a,b \in \mathbb{S}$ and $d^{\bullet},g^{\bullet} \in \mathbb{S}^{\bullet}$, then

- (a) If $a \Delta \theta$, and $a \in \mathbb{S}^{\vee}$, then $a=\theta$.
- (b) If $a \Delta \theta$, then $a \Delta d^{\bullet}$.
- (c) $b \Delta d^{\bullet} \not\Rightarrow b \Delta \theta$.
- (d) $a \Delta d^{\bullet}$, and $|a| > |d^{\bullet}|$, then $a \in \mathbb{S}^{\bullet}$.
- (e) $d^{\bullet} = (\ominus d)^{\bullet} = (d^{\bullet})^{\bullet}$
- (f) $(\ominus a) \otimes (\ominus b) = a \otimes b$.
- (g) $a \otimes d^{\bullet} = (a \otimes d)^{\bullet}$.
- (h) $\ominus (\ominus a) = a$.

- (i) $\ominus (a \oplus b) = (\ominus a) \oplus (\ominus b)$.
- (j) $\ominus (a \otimes b) = (\ominus a) \otimes b = a \otimes (\ominus b)$.
- (k) $d^\bullet \oplus g^\bullet = (d \oplus g)^\bullet$.
- (l) $a \ominus d^\bullet = a \oplus d^\bullet$
- (m) $d^\bullet \oplus g^\bullet = (d \oplus g)^\bullet$
- (n) $d^\bullet \ominus g^\bullet = d^\bullet \oplus g^\bullet = (d \oplus g)^\bullet$
- (o) $a \oplus d^\bullet = \begin{cases} a & \text{if } |a| > |d^\bullet| \\ d^\bullet & \text{if } |a| \leq |d^\bullet| \end{cases}$.
- (p) $d^\bullet \ominus a = \begin{cases} \ominus a & \text{if } |a| > |d^\bullet| \\ d^\bullet & \text{if } |a| \leq |d^\bullet| \end{cases}$.

Proposition 5.76. The set of solutions of the general linear system (45) in \mathbb{R}_{\max} and the set of positive solutions of the associated linear balance (47) in \mathbb{S} coincide.

Proof. See [24]. □

The scalar linear balance

Theorem 5.77. Let $a \in \mathbb{S}^\vee \setminus \{\theta\}$ and $b \in \mathbb{S}^\vee$, then the balance

$$ax \oplus b\Delta\theta \tag{48}$$

has the unique signed solution: $x^b = \ominus a^{-1}b$.

Proof. See [24]. □

Note 5.78. Non-trivial linear balances always have solutions in \mathbb{S} , that is why \mathbb{S} may be considered as a *linear closure* of \mathbb{R}_{\max} .

Note 5.79. We can describe all the solutions of (48). For all $t \in \mathbb{R}_{\max}$, we have obviously $at^\bullet \Delta\theta$. Adding this balance to $ax^b \oplus b\Delta\theta$, we get $a(x^b \oplus t^\bullet) \oplus b\Delta\theta$. Thus,

$$x_t = x^b \oplus t^\bullet \tag{49}$$

is a solution to (48). If $t \geq |x^b|$, then $x_t = b^\bullet$ is balanced. Conversely, it can be checked that *every* solution of (48) may be written as in (49). The *unique* signed solution x^b is also the least solution.

Note 5.80. If $b \notin \mathbb{S}^\vee$, we lose uniqueness of signed solutions. Every x such that $|ax| \leq |b|$ (i.e. $|x| \leq |a^{-1}b|$) is a solution of balance (48).

Note 5.81. If $a \notin \mathbb{S}^\vee$, we again lose uniqueness. Assume $b \in \mathbb{S}^\vee$ (otherwise, the balance holds for all value of x), then every x such that $|ax| \geq |b|$ is a solution.

Determinants in Symmetrized Algebra

Definition 5.82 (Max-algebraic signature). If σ is a permutation, then the max-algebraic signature of σ is defined as:

$$\text{sgn}(\sigma) = \begin{cases} e & \text{if } \sigma \text{ is even} \\ \ominus e & \text{if } \sigma \text{ is odd} \end{cases} \quad (50)$$

Definition 5.83 (Max-algebraic determinant). Consider a matrix $A \in \mathbb{S}^{n \times n}$.

The max-algebraic determinant of A is defined as:

$$\det(A) = \bigoplus_{\sigma \in \mathfrak{P}} \text{sgn}(\sigma) \otimes \bigotimes_{i=1}^n a_{i\sigma(i)} \quad (51)$$

Note 5.84. As in the conventional algebra system, if $\alpha \in \mathbb{S}$ and $A \in \mathbb{S}^{n \times n}$ then $\det(A^T) = \det(A)$, and $\det(\alpha \otimes A) = \alpha^n \otimes \det(A)$.

To illustrate the use of definition 5.83, we include the following example.

Example 5.85. Find the determinant of $A = \begin{bmatrix} 2 & \ominus 1 & 3^\bullet \\ 1 & \theta & 2 \\ 3^\bullet & 2 & \ominus 5 \end{bmatrix}$

$$\begin{aligned}
\begin{vmatrix} 2 & \ominus 1 & 3^\bullet \\ 1 & \theta & 2 \\ 3^\bullet & 2 & \ominus 5 \end{vmatrix} &= 2 \otimes \begin{vmatrix} \theta & 2 \\ 2 & \ominus 5 \end{vmatrix} \ominus 1 \otimes \begin{vmatrix} \ominus 1 & 3^\bullet \\ 2 & \ominus 5 \end{vmatrix} \oplus 3^\bullet \otimes \begin{vmatrix} \ominus 1 & 3^\bullet \\ \theta & 2 \end{vmatrix} \\
&= 2 \otimes (\theta \otimes (\ominus 5) \ominus 2 \otimes 2) \\
&\quad \oplus (\ominus 1) \otimes (\ominus 1 \otimes \ominus 5 \ominus 2 \otimes 3^\bullet) \\
&\quad \oplus 3^\bullet \otimes (\ominus 1 \otimes 2 \ominus 3^\bullet \otimes \theta) \\
&= 2 \otimes (\theta \ominus 4) \oplus (\ominus 1) \otimes (6 \ominus 5^\bullet) \oplus 3^\bullet \otimes (\ominus 3 \ominus \theta) \\
&= 2 \otimes (\ominus 4) \oplus (\ominus 1) \otimes (6) \oplus 3^\bullet \otimes (\ominus 3) \\
&= (\ominus 6) \oplus (\ominus 7) \oplus (6^\bullet) \\
&= (\ominus 6) \oplus (\ominus 7) \oplus (6 \ominus 6) \\
&= 6 \ominus (6 \oplus 7 \oplus 6) \\
&= 6 \ominus 7 \\
&= \ominus 7
\end{aligned}$$

■

Theorem 5.86 (Cramer system). Let A be an $n \times n$ matrix with entries in \mathbb{S}_{\max} , and $b \in (\mathbb{S}_{\max}^n)$. Then every signed solution of

$$Ax \Delta b \tag{52}$$

satisfies

$$\det A \cdot x \Delta A^{\text{adj}} b. \tag{53}$$

Conversely, assume that $A^{\text{adj}} b$ is signed and $\det(A)$ is invertible, then the ‘‘Cramer solution’’ $x^b = \det(A)^{-1} A^{\text{adj}} b$ is the unique signed solution of (52).

The following example is included from [24, page 155]. We will work this example again after studying msets as a comparisons of the two methods, see Example 8.95.

Example 5.87. Find the solutions of:

$$\begin{cases} \max(x, y - 4, 1) = \max(x - 1, y + 1, 2) \\ \max(x + 3, y + 2, -5) = \max(y + 2, 7) \end{cases} \quad (54)$$

This can be written in matrix form as:

$$\begin{bmatrix} 0 & -4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\infty & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad (55)$$

The balance corresponding the equation (55) is

$$\begin{bmatrix} 0 & \ominus 1 \\ 3 & 2 \bullet \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \Delta \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad (56)$$

The $\det(A)=4$, which is invertible.

$$D_x = \begin{vmatrix} 2 & \ominus 1 \\ 7 & 2 \bullet \end{vmatrix} = 8, \quad D_y = \begin{vmatrix} 0 & 3 \\ 2 & 7 \end{vmatrix} = 7$$

$$A^{\text{adj}b} = \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix} \in (\mathbb{S}^\vee)^2$$

So, $x = \frac{D_x}{D} = 8 - 4 = 4$, and $y = \frac{D_y}{D} = 7 - 4 = 3$ gives the unique positive solution in \mathbb{S}_{\max} of Equation (56). Thus it is the unique solution of Equation (54) in \mathbb{S}_{\max} .

■

6 Max-Times Algebra

6.1 Basics

In the previous sections, we have been working within the Max-Plus Algebra system. All of the results in the Max-Plus Algebra system have analogs to corresponding theorems in the Max-Times Algebra. Many new results, and easier proofs, can be developed using this particular Path Algebra. The Max-Times algebra is closer to the conventional system of non-negative matrices, and thus easier to manipulate. The notation \mathbb{R}^+ will be used for the nonnegative real numbers.

We begin with a definition of the Max-Times algebra.

Definition 6.1. Let \mathbb{R}^+ represent the nonnegative number and define

- (i) $a \boxplus b = \max(a, b)$, where $a, b \in \mathbb{R}^+$.
- (ii) $a \boxtimes b = a \cdot b$ (The usual multiplication).

Note 6.2. We use the notation \mathfrak{M}^* for the algebraic system $(\mathbb{R}^+, \boxplus, \boxtimes)$.

Definition 6.3. If $A, B \in \mathfrak{M}_{n \times n}^*$, then \boxplus , and \boxtimes are extended to matrices as

- (i) $[A \boxplus B]_{ij} = \max(a_{ij}, b_{ij})$.
- (ii) $[A \boxtimes B]_{ij} = \max_{1 \leq k \leq n} \{a_{ik} \cdot b_{kj}\}$

There is an isomorphism between the Max-Plus Algebra and the Max-Times Algebra: $\varphi : \mathfrak{M}^+ \rightarrow \mathfrak{M}^*$, which is defined as

- (i) $\varphi(x) = e^x$, for all $x \in \mathbb{R}$, with $\varphi(-\infty) = 0$,
- (ii) $\varphi^{-1}(y) = \ln(y)$ for all $y \in \mathbb{R}_{\geq 0}$, with $\varphi^{-1}(0) = -\infty$.

For matrices, the isomorphism is defined elementwise as $[\varphi(A)]_{ij} = [e^{a_{ij}}]$

To verify that this is an isomorphism, we note that:

- (i) $\varphi(a \oplus b) = e^{a \oplus b} = e^{\max(a, b)} = \max(e^a, e^b) = e^a \boxplus e^b = \varphi(a) \boxplus \varphi(b)$.
 $\therefore \varphi(a \oplus b) = \varphi(a) \boxplus \varphi(b)$.
- (ii) $\varphi(a \otimes b) = e^{a \otimes b} = e^{a+b} = e^a \cdot e^b = \varphi(a) \boxtimes \varphi(b)$
 $\therefore \varphi(a \otimes b) = \varphi(a) \boxtimes \varphi(b)$.

With this isomorphism it follows that $A \otimes B = \varphi^{-1}(\varphi(A) \boxtimes \varphi(B))$.

Definition 6.4. Given a cycle in $G(A)$, we define, analogous to the Max-Plus definition, the maximum geometric circuit mean as:

$$\nu(A) = \max_{\gamma \in G(A)} \left(\prod_{(i,j) \in \gamma} a_{i,j} \right)^{\frac{1}{l(\gamma)}} \quad (57)$$

With this definition of $\nu(A)$, and the previous definition of $\mu(A)$, it can be seen that $\mu(A) = \varphi^{-1}(\nu(\varphi(A)))$, which we may write as $\nu(\varphi(A)) = e^{\mu(A)}$.

As in the case of the Max-Plus Algebra we only need to consider cycles in our calculations of the maximum cycle means. This is the analog of Lemma 5.12 in the Max-Plus Algebra. We present this theorem for later use.

Lemma 6.5. If $a, b \geq 0$, and $m, n > 0$, then $\max\{a^{1/m}, b^{1/n}, (ab)^{1/(m+n)}\} = \max\{a^{1/m}, b^{1/n}\}$

Proof. Suppose, without loss of generality, that $a^{\frac{1}{m}} \geq b^{\frac{1}{n}}$, then

$$\begin{aligned} a^{\frac{1}{m}} \geq b^{\frac{1}{n}} &\iff a^n \geq b^m &\iff a^m a^n \geq a^m b^m \\ &\iff a^{m+n} \geq (ab)^m &\iff a^{\frac{1}{m}} \geq (ab)^{\frac{1}{m+n}} \end{aligned}$$

□

We now give some of the basic results in the Max-Times algebra, and add proofs for completeness.

Proposition 6.6. Let $A, B \in \mathfrak{M}_{n \times n}^*$, and $\alpha \in \mathfrak{M}^*$ then:

- (i) If α is a scalar, then $\alpha \boxtimes A = \alpha \cdot A$.
where $[\alpha \cdot A]_{ij} \equiv \alpha \cdot A_{ij}$
Therefore, we will only write αA instead of $\alpha \boxtimes A$.
- (ii) If A has only one positive element in each row, then $A \boxtimes B = A \cdot B$.
- (iii) If B has only one positive element in each column, then $A \boxtimes B = A \cdot B$.
- (iv) If P is a permutation matrix then $P \boxtimes A \boxtimes P^T = P \cdot A \cdot P^T$.
- (v) If D is a diagonal matrix, $D = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$, where each element a_{ii} is nonzero, then D^{-1} exists in the Max-Times Algebra,
and $D^{-1} = \text{diag}(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{nn}})$.
- (vi) If D is a diagonal matrix in $\mathfrak{M}_{n \times n}^*$ and $d_{ii} > 0$,
then $D \boxtimes A \boxtimes D^{-1} = D \cdot A \cdot D^{-1}$.

- (vii) If $A=[a_{ij}]$ is a matrix with one nonzero element in each row and one nonzero element in each column, then A^{-1} exists in \mathfrak{M}^* , and $(A^{-1})_{ij} = \begin{cases} \frac{1}{a_{ji}} & \text{if } a_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$
- (viii) If P is a permutation matrix then P^{-1} exists in the Max-Times Algebra, and $P^{-1} = P^T$
- (ix) $A \boxtimes B \leq A \cdot B \leq n \cdot (A \boxtimes B)$.
- (x) $A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_k \leq A_1 \cdot A_2 \cdots \cdots A_k \leq n^{k-1} (A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_k)$.
- (xi) $A^{\boxtimes k} \leq A^k \leq n^{k-1} A^{\boxtimes k}$.
- (xii) $\frac{A^k}{n^{k-1}} \leq A^{\boxtimes k} \leq A^k$.
- (xiii) There exists k such that $A^k > 0 \iff$ there exists m such that $A^{\boxtimes m} > 0$.

Proof.

- (i) This result is obvious since calculations are element-wise.
- (ii) Consider $[A \boxtimes B]_{ij} = \boxplus_k (A_{ik} B_{kj})$. Since all elements of row i in A are zero except for one, then the sum (max) has only one nonzero product.
- (iii) Same as (ii) except that B_{jk} has only one nonzero element in each column.
- (iv) Corollary of (ii) and (iii).
- (v) This is obvious as it can be seen that $DD^{-1}=I$.
- (vi) Corollary of (ii) and (iii).
- (vii) It can be checked that $A A^{-1}=I$.
- (viii) Corollary of (vii).
- (ix)

$$\begin{aligned}
[A \boxtimes B]_{ij} &= \boxplus_k (A_{ik} \boxtimes B_{kj}) \\
&\equiv \max_k (A_{ik} \cdot B_{kj}) \\
&\leq \sum_k (A_{ik} B_{kj}) \\
&\leq n \cdot (\max_j (A_{ik} \cdot B_{kj})) \\
&= n \cdot (A \boxtimes B)
\end{aligned} \tag{58}$$

- (x) Use induction with (ix).
- (xi) Corollary of (x).
- (xii) Corollary of (xi).
- (xiii) Follows from (xii).

$$0 < A^{\boxtimes m} \leq A^m \tag{59}$$

$$\text{and } 0 < \frac{A^k}{n^{n-1}} \leq \frac{A^{\boxtimes k}}{n^{k-1}} \tag{60}$$

□

The results from the Max-Plus Algebra carry over to the Max-Times Algebra through the isomorphism. In particular we mention the following theorem.

Theorem 6.7. If A is irreducible, then $\nu(A)$ is only eigenvalue of A and every eigenvector corresponding to $\nu(A)$ is positive.

6.2 Spectral Radius of A and the Maximum Geometric Mean of A

Suppose $M \in \mathbb{R}_{n \times n}^+$. We use the following two definitions in this section.

$\rho(M)$ = the spectral radius of M (in the conventional algebra sense).

$\nu(M)$ = the max geometric mean of all circuits in $G(M)$.

$$\nu(M) = \max_{\gamma \in G(A)} \left(\prod_{(i,j) \in \gamma} m_{ij} \right)^{\frac{1}{l(\gamma)}} \quad (61)$$

This $\nu(M)$ is the definition of the maximum circuit mean used in the Max-Plus algebra translated to the Max-Times algebra. We have found that $\mu(M)$ is an eigenvalue of M in the max-plus algebra. This also carries over to the max-times algebra system in an analogous manner. The question that arises is when, in the conventional algebra, are the two definitions equivalent; i.e. when is the maximum geometric mean of the circuits of a graph equal to the maximum eigenvalue of the adjacency matrix of the graph. The aim of this section is to give a necessary and sufficient condition for the maximum geometric mean of a nonnegative matrix to equal the maximum eigenvalue of the matrix.

We begin with the following theorem.

Theorem 6.8. Let $A_{n \times n} \in \mathbb{R}_{n \times n}^+$. Then $\rho(A) \geq \nu(A)$.

Proof. Suppose that $A \geq \mathbb{R}_{n \times n}^+$. If A has no cycles then A is nilpotent and 0 is only eigenvalue. So $\rho=0=\nu$.

So assume that A has cycles. There exists a cycle, say $S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_t \rightarrow S_1$. $\rho > 0$, and $\nu(A) > 0$.

$$A\underline{x} = \rho\underline{x} \quad \implies \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ a_{21} & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tt} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} = \rho \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} \quad (62)$$

$$\rho x_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1t} x_t \geq a_{12} x_2$$

$$\rho x_2 = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2t} x_t \geq a_{23} x_3$$

$$\vdots$$

$$\rho x_t = a_{t1} x_1 + a_{t2} x_2 + \cdots + a_{tt} x_t \geq a_{t1} x_1$$

$$\rho^t (x_1 x_2 \cdots x_t) \geq (a_{12} a_{23} \cdots a_{t1}) (x_1 x_2 \cdots x_t)$$

$$\rho \geq (w(\gamma))^{(1/t)} = \overline{w(\gamma)} = \nu(A)$$

Therefore $\rho(A) \geq \nu(A)$. □

We see from this theorem that it must be a special matrix which satisfies the conditions. With this theorem complete, we may ask the following question: When is $\rho(A) = \nu(A)$? We will answer this question by finding the necessary and sufficient conditions for $\rho(A)=\nu(A)$. We begin with a sufficient condition.

Theorem 6.9 (Sufficient condition).

$$\text{If } A \underset{\text{per}}{\approx} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & + \\ + & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & + & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & + & 0 & 0 \\ 0 & \cdots & \cdots & 0 & + & 0 \end{bmatrix}, \text{ then } \rho(A) = \nu(A).$$

$$\text{Proof. } \Delta_A(\lambda) = \begin{vmatrix} \lambda & 0 & 0 & 0 & -a_{1n} \\ -a_{21} & \lambda & 0 & 0 & \cdots \\ 0 & -a_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -a_{n,n-1} & \lambda \end{vmatrix} = \lambda^n - \omega(\gamma)$$

and $\lambda^n - \omega(\gamma) = 0 \implies \lambda = \omega(\gamma)^{(1/n)} = \overline{\omega(\gamma)}$. □

So we see from Theorem 6.9 that if A is a matrix whose graph contains exactly one cycle, then $\rho(A)=\nu(A)$. In order to investigate the necessary condition for $\rho(A) = \nu(A)$, we need a few facts and lemmas from linear algebra.

Fact 6.10. Suppose that $A \geq 0$, and irreducible. Suppose that the cycle index of A is h, which is the greatest common divisor of the lengths of all cycles of A. Then we can permute the rows and columns of A to get a matrix in the form:

$$M = \begin{bmatrix} 0_{c_1 \times c_1} & \cdots & \cdots & \cdots & M_1 \\ M_h & 0_{c_2 \times c_2} & & \vdots & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & M_3 & 0_{c_{h-1} \times c_{h-1}} & \\ \cdots & \cdots & & M_2 & 0_{c_h \times c_h} \end{bmatrix},$$

which we usually write as $M = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & M_1 \\ M_h & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & M_3 & 0 & \\ 0 & \cdots & & M_2 & 0 \end{bmatrix},$

or just as $M = \begin{bmatrix} & & & M_1 \\ M_h & & & \\ & \ddots & & \\ & & M_3 & \\ & & & M_2 \end{bmatrix}$.

Fact 6.11. Suppose $M = \begin{bmatrix} & & & M_1 \\ M_h & & & \\ & \ddots & & \\ & & M_3 & \\ & & & M_2 \end{bmatrix}$, then $M^h = \begin{bmatrix} B_1 & & & \\ & \ddots & & \\ & & & \\ & & & B_h \end{bmatrix}$, where

$$B_1 = A_1 A_2 \cdots A_h \quad (63)$$

$$\text{and } B_j = A_j A_{j+1} \cdots A_h A_1 \cdots A_{j-1} \quad (64)$$

Fact 6.12. $\rho (M^h) = \rho (A_1 A_2 \cdots A_h)$

Fact 6.13. $\rho (M^h) = [\rho (M)]^h$.

Proof. $M x = \lambda x \implies M^k x = \lambda M^{k-1} x = \cdots = \lambda^k x. \implies \lambda^k$ is an eigenvalue of M^k . \square

Fact 6.14.

$$(A_1 A_2 \cdots A_h)_{ij} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{h-1}} (a_{ik_1} a_{k_1 k_2} \cdots a_{k_{h-1} j}) \quad (65)$$

Each product in the sum represents the product of the weights of the arcs of a path of length= h from S_i to S_j . So, this represents the sum of all the path-products of length= h from S_i to S_j .

The following facts from linear algebra will help with our investigation.

Fact 6.15. $\rho(A) \geq a_{ii}$, for all $1 \leq i \leq n$.

Proof. See [17].

$$\rho = \max_{\substack{x \geq 0 \\ x \neq 0}} \min_{1 \leq i \leq n} \frac{(Ax)_i}{x_i},$$

Letting $x_i = e_i$, we get $\rho \geq e_i^T A e_i = A_{ii}$ \square

Suppose that the matrix M is a matrix which contains at least one cycle. Let τ be a critical cycle in $G(M)$, and suppose, without loss of generality, that τ contains the node S_1 .

Theorem 6.16. $(\rho(M))^k \geq$ any one circuit-product through S_i of length=h, say τ

$$\begin{aligned}
(\rho(M))^k &= \rho(M^k) = \rho(A_1 A_2 \cdots A_h) \\
&\geq (A_1 A_2 \cdots A_h)_{ii} \\
&= \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{h-1}} (A_{i k_1} A_{k_1 k_2} \cdots A_{k_{h-1} i}) \\
&= \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_h} M_{i \ell_1} M_{\ell_1 \ell_2} \cdots M_{\ell_{h-1} i} \\
&\quad \sum_{1 \leq i \leq h-1} c_i \leq \ell_1 \leq \sum_{1 \leq i \leq h} c_i \\
&\quad \sum_{1 \leq i \leq h-2} c_i \leq \ell_2 \leq \sum_{1 \leq i \leq h-1} c_i \\
&\quad \vdots \\
&\geq \text{any one circuit-product through } S_i \text{ of length=h, say } \tau
\end{aligned}$$

Theorem 6.17 (Theorem 2). The length of τ must be h.

Proof. Since h=greatest common divisor of the circuit length of all circuits in $G(M)$, then all circuits lengths must be a multiple of h. Suppose that the length of τ , (one of the) maximal circuits, is d·h.

Then $M^d = \underbrace{(A_1 \cdots A_h A_1 \cdots A_h)}_{d \text{ times}} = (A_1 \cdots A_h)^d$, and

$(A_1 \cdots A_h)_{11}^d = \sum_{j_1, \dots, j_{d-1}} (a_{1j_1} \cdots a_{j_{d-1} 1} a_{1k_1} \cdots) = \sum(\text{all d·h circuit products from } S_i \text{ to } S_i) = \text{maximal circuit product.}$

Therefore, all d·h circuit products must be equal to zero, except for the circuit product corresponding to τ .

Claim. The maximal circuit cannot be an h-circuit repeated d times.

Proof of Claim. If τ were an h-circuit from S_i to S_i and γ was the circuit τ circuit repeated d times, then $w(\tau) = w(\gamma)$, because $\left(\prod_{\tau} (A_1 \cdots A_h) \right)^{(1/h)} = \left(\prod_{\gamma} (A_1 \cdots A_h)^d \right)^{(1/(dh))}$. So, if γ is a dh-circuit which leads to the maximum product, then τ is an h-circuit with the same maximum geometric mean. Therefore in the $\sum(\cdots)$, there would be the dh-circuit product and the h-circuit product repeated d times.

Hence, $\sum(\cdots) >$ maximal circuit product, which can not occur.

And we see that the maximal circuit must be of length=h. □

The same reasoning as above gives that the maximal circuit also can not be a k·h-circuit for any $k < d$.

Therefore, the length of τ must be h. □

Lemma 6.18. If $h \mid k$, then $\nu(M^k) \geq (\nu(M))^k$

$$A \underset{\text{per}}{\approx} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & + \\ + & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & + & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & + & 0 & 0 \\ 0 & \cdots & \cdots & 0 & + & 0 \end{bmatrix}$$

□

We have now proven the following Theorem.

Theorem 6.21. $\rho(A) = \mu(A)$ if and only if $A \underset{\text{per}}{\approx} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & + \\ + & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & + & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & + & 0 & 0 \\ 0 & \cdots & \cdots & 0 & + & 0 \end{bmatrix}$

Theorem 6.21, above, states that the Necessary and Sufficient condition for $\rho(A)=\nu(A)$ is that $G(A)$ consists of exactly one cycle. We include the following example to illustrate this equality.

Example 6.22.

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \in \mathbb{R}_{4 \times 4}^+$$

$$\begin{aligned} |\lambda I - A| &= \lambda^4 - 16 \\ \rho(A) &= \sqrt[4]{16} = 4 \\ \mu(A) &= \left(\prod_{(i,j) \in \gamma_0} a_{ij} \right)^{(1/\ell(\gamma_0))} \\ \mu(A) &= (1 \cdot 2 \cdot 2 \cdot 4)^{\frac{1}{4}} = \sqrt[4]{16} = 4 \end{aligned}$$

Here, according to Theorem 6.21, $\mu(A) = \rho(A)$ ■

7 Cancellative Semirings

As we recall a cancellative semiring is one which obeys the additional axiom that if $a+b=a+c$ then $a=c$. Our aim in studying Cancellative semirings is to generalize the theory of rings. We investigate the embedding theorems in this section. This will allow us to prove identities in Semirings which are analogous to theorems in rings. In particular we will look at the identities involving the Adjoint and the Determinant, which will allow us to solve problems in linear algebra using an analogy of Cramer's Rule.

We begin by discussing the Embeddings of Semirings into a "ring of Differences."

7.1 Embeddings and Morphisms of Semirings

What we would like to do is to embed the semiring, S , into a ring, R , such that R has a subset, S' , which is a semiring isomorphic to S .

This is not possible in general, but if S is a cancellative semiring with no zero divisors, then it is possible.

Golan, [20], contains the following two propositions.

Proposition 7.1. If S_1 is a semiring, then there exists a cancellative semiring, S , and a surjective morphism from S to S_1 .

Proposition 7.2. If S is a commutative, cancellative semiring, then S is isomorphic to a subsemiring, S_2 , of a ring, R^D ="ring of differences," such that every element of R^D is the difference between two elements in the image of S .

Proof. We include the construction of the Ring of Differences. Suppose that $(S, +, \cdot)$ is a commutative, cancellative semiring. Form $S^2 = S \times S$, and consider (S^2, \oplus, \odot) defined by:

Definition 7.3. If $(r,s), (r_1,s_1) \in S^2$, then

$$(a) (r,s) \oplus (r_1,s_1) \equiv (r+r_1, s+s_1)$$

$$(b) (r,s) \odot (r_1,s_1) \equiv (rr_1+ss_1, rs_1+sr_1)$$

It can be checked that S^2 is indeed a commutative semiring with $(0,0)$ as the additive identity and $(1,0)$ as the multiplicative identity.

To create "negative" elements, we look at Quotient spaces. Define the equivalence relation " \sim " on S^2 by

$$\text{Definition 7.4.} \quad (r,s) \sim (r_1,s_1) \iff r+s_1=s+r_1$$

Lemma 7.5. " \sim " is an equivalence relation.

Proof. (a) (Reflexive) $(r,s) \sim (r,s)$ is clear as $+$ is commutative.

$$(b) \text{ (Symmetric) } (r,s) \sim (r_1,s_1) \implies r+s_1=s+r_1 \implies r_1+s=s_1+r \implies (r_1,s_1) \sim (r,s)$$

(c) (Transitive) Suppose $(r,s) \sim (r_1,s_1)$, and $(r_1,s_1) \sim (r_2,r_2)$. So $r+s_1=s+r_1$, and $r_1+r_2=s_1+r_2$. Add these two equations together to get $r+s_1+r_1+r_2=s+r_1+s_1+r_2$. Since S is cancellative, we get $r+r_2=s+r_2$. Therefore, $(r,s) \sim (r_2,r_2)$. \square

Consider the equivalence classes: $\overline{(r,s)} = \{(r_1,s_1) \mid (r_1,s_1) \sim (r,s)\}$ and let $\frac{S^2}{\sim} \equiv \{\overline{(r,s)}\}$.

Define \boxplus and \boxminus on $\frac{S^2}{\sim}$ by :

Definition 7.6. Let $\overline{(r,s)}, \overline{(r_1,s_1)} \in \frac{S^2}{\sim}$, then

$$(a) \overline{(r,s)} \boxplus \overline{(r_1,s_1)} \equiv \overline{(r+r_1,s+s_1)}$$

$$(b) \overline{(r,s)} \boxminus \overline{(r_1,s_1)} \equiv \overline{(rr_1+ss_1,rs_1+sr_1)}$$

We need to show that these operations are well-defined.

(a) (\boxplus) Let $(r,s) \sim (r_1,s_1)$ and $(p,q) \sim (p_1,q_1)$. We need to show that

$$\overline{(r,s)} \boxplus \overline{(p,q)} = \overline{(r_1,s_1)} \boxplus \overline{(p_1,q_1)} \quad (66)$$

$$\overline{(r+p,s+q)} = \overline{(r_1+p_1,s_1+q_1)} \quad (67)$$

$$(r+p,s+q) \sim (r_1+p_1,s_1+q_1) \quad (68)$$

To this end, we have:

$$r+s_1 = s+r_1 \quad (69)$$

$$p+q_1 = q+p_1 \quad (70)$$

Add equations (69) and (70) together. Since S is associative, and commutative, we get:

$$(r+p) + (s_1+q_1) = (s+q) + (r_1+p_1) \quad (71)$$

Therefore $(r+p,s+q) \sim (r_1+p_1,s_1+q_1)$, and Equation (68) is verified, and \boxplus is well-defined.

(b) (\boxminus) Let $(r,s) \sim (r_1,s_1)$ and $(p,q) \sim (p_1,q_1)$. We need to show that

$$\overline{(r,s)} \boxminus \overline{(p,q)} = \overline{(r_1,s_1)} \boxminus \overline{(p_1,q_1)} \quad (72)$$

$$\overline{(rp+sq,rq+sp)} = \overline{(r_1p_1+s_1q_1,r_1q_1+s_1p_1)} \quad (73)$$

$$(rp+sq,rq+sp) \sim (r_1p_1+s_1q_1,r_1q_1+s_1p_1) \quad (74)$$

To this end, we have:

$$r + s_1 = s + r_1 \quad (75)$$

$$p + q_1 = q + p_1 \quad (76)$$

We multiply:

(a) Equation (75) by “p”

(b) Equation (76) by “q”

(c) Equation (75) by “ r_1 ”

(d) Equation (76) by “ s_1 ”

To get, respectively:

$$pr + ps_1 = ps + pr_1 \quad (77)$$

$$qr + qs_1 = qs + qr_1 \quad (78)$$

$$pr_1 + q_1 r_1 = qr_1 + p_1 r_1 \quad (79)$$

$$ps_1 + q_1 s_1 = qs_1 + p_1 s_1 \quad (80)$$

Add the equations by setting:

$$\text{LHS}_{(77)} + \text{RHS}_{(78)} + \text{LHS}_{(79)} + \text{RHS}_{(80)} = \text{RHS}_{(77)} + \text{LHS}_{(78)} + \text{RHS}_{(79)} + \text{LHS}_{(80)}$$

Using the commutative and associative properties of \mathbb{S} , we get:

$$\begin{aligned} & (rp + sq) + (r_1 q_1 + s_1 p_1) + (ps_1 + qr_1 + pr_1 + qs_1) \\ &= (ps + qr) + (r_1 p_1 + s_1 q_1) + (pr_1 + qs_1 + qr_1 + ps_1) \end{aligned} \quad (81)$$

and using the cancellative property of \mathbb{S} , we get:

$$(rp + sq) + (r_1 q_1 + s_1 p_1) = (ps + qr) + (r_1 p_1 + s_1 q_1) \quad (82)$$

Therefore, $(rp+sq, r_1 q_1 + s_1 p_1) \sim (r_1 p_1 + s_1 q_1, r_1 q_1 + s_1 p_1)$, and Equation (74) is verified. We have shown that \boxtimes is well-defined.

Let $R^D \equiv (\frac{S^2}{\sim}, \boxtimes, \boxdot)$

In R^D $\overline{(a, a)} = \overline{(0, 0)}$ is the additive identity, as:

$$(a, a) \oplus (r, s) = (a + r, a + s) \sim (r, s) \quad (83)$$

$$\implies \overline{(a, a)} \boxplus \overline{(r, s)} = \overline{(a + r, a + s)} = \overline{(r, s)} \quad (84)$$

And $\overline{(a, a)}$ is absorbing under \boxtimes , as:

$$(a, a) \otimes (r, s) = (ar + as, as + ar) \sim (0, 0) \quad (85)$$

$$\implies \overline{(a, a)} \boxtimes \overline{(r, s)} = \overline{(0, 0)} \quad (86)$$

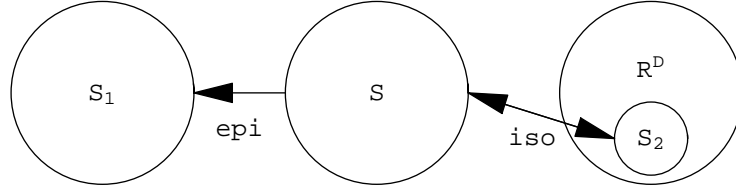


Figure 3: Relationship between semiring embeddings.

Let $S_2 = \{\overline{(a, 0)} \in \frac{S^2}{\sim} \mid a \in S\}$.

We now consider the mapping: $\varphi: S \rightarrow R^D$, defined by $\varphi(a) = \overline{(a, 0)}$.

$$\varphi(a + b) = \overline{(a + b, 0)} = \overline{(a, 0)} \boxplus \overline{(b, 0)} = \varphi(a) \boxplus \varphi(b) \quad (87)$$

$$\varphi(ab) = \overline{(ab, 0)} = \overline{(a, 0)} \boxtimes \overline{(b, 0)} = \varphi(a) \boxtimes \varphi(b) \quad (88)$$

Therefore φ is a homomorphism from S to R^D .

$$\begin{aligned} \varphi(a) = \varphi(b) &\implies \overline{(a, 0)} = \overline{(b, 0)} \\ \implies (a, 0) \sim (b, 0) &\implies a + 0 = b + 0 \implies a = b \end{aligned}$$

Hence, φ is one-to-one, and we have an isomorphism between S and S_2 . □

With this embedding we can identify a cancellative semiring with its isomorphic image in its “ring of differences.” We can consider the semiring as a subsemiring of R^D .

Figure 3 is a representation of these relationships. We use φ_1 as the isomorphism between S and S_2 , and φ_2 as the epimorphism from S to S_2 .

With the use of Propositions 7.1 and 7.2, we are able to conclude things about the semiring S_1 , by considering the ring R^D . In particular any identity in R^D which is wholly in S_2 can be carried over to S_1 by going, first, through the semiring S . We will consider identities in rings which have an analogous identity in semirings. We refer to section 8.11 for these identities. First, we study the new algebra of multisets.

8 Multiset Algebra

In this section we consider the algebraic structure of the Multiset algebra, abbreviated as the Mset Algebra. A multiset, or mset, is a collection of elements in which repetition is allowed. An mset can be thought of a “set” in which multiplicities are allowed. For example, $\{1,2,2\} = \{2,1,2\} \neq \{1,2\}$. The operations of intersection and union work similarly with msets as they do with sets. If α and β are two msets then $\alpha \cap \beta = \delta$, where δ is the smallest mset which is contained in α and β .

The motivation that inspires us to consider msets over \mathbb{S} , or $3^{\mathbb{S}}$, is that this allows much more information to be kept throughout the problem. As we see throughout the following sections there is much more available information accessible for our use. However, keeping these extra pieces of information results in some equations which are not solvable in the sense of being able to write down an explicit solution to the equation. All that is possible is a “consistency condition” that allows us to characterize the solution to the equation. In some cases we are able to give an algorithmic method to solve the equation. If \mathbb{S} is a semiring, then if the problem is translated back to the system \mathbb{S} , then in many cases, a solution is available. Throughout this section, it may be realized that the field \mathbb{R} is an underlying structure which motivates the development of the subject of the mset algebra.

8.1 Notation and Definitions

We begin with a set \mathbb{S} , along with two binary operations:

- (a) \oplus , which is associative and commutative, with additive identity= θ .
- (b) \odot , which is associative and commutative, with multiplicative identity= e .
- (c) \odot is distributive over \oplus .

Consider the set $3^{\mathbb{S}}$ derived from the basic set \mathbb{S} above. $3^{\mathbb{S}}$ is the “power set” of all finite msets with elements from \mathbb{S} . We use $3^{\mathbb{S}}$ for the power set instead of the more familiar $2^{\mathbb{S}}$ to highlight the difference between sets and multisets.

Definition 8.1. With the new system $3^{\mathbb{S}}$, we have the operations of:

1. \uplus , called ‘fusion’, an ‘addition’
2. \boxtimes , a ‘multiplication’

We use the name ‘fusion’ to signify that this operation is different from ‘union’ which is used with sets, and ‘concatenation’ which implies that order matters.

The symbol $4^{\mathbb{S}}$ can be used as the power set of \mathbb{S} which allows infinite sets. The multisets in this thesis will be assumed finite, so there is no need for this power set.

Definition 8.2. We make the following definitions in the mset algebra:

Let $\alpha = \{a_1, a_2, \dots, a_k\}$, and $\beta = \{b_1, b_2, \dots, b_m\}$, with $a_i, b_j \in \mathbb{S}$. Then

1. $\{a\} \uplus \{b\} \equiv \{a, b\}$, with $a, b \in \mathbb{S}$.
2. $\alpha \uplus \beta = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m\}$
3. $\{a\} \boxtimes \{b\} \equiv \{a \odot b\}$ (product with \odot from the set \mathbb{S}) usually written ab .
4. If $\{a\} \in 3^{\mathbb{S}}$, then $\{a\} \boxtimes \{b_1, b_2, \dots, b_n\} = \{a \odot b_1, a \odot b_2, \dots, a \odot b_n\}$.
5. $\alpha \boxtimes \beta = \{a_i \odot b_j \mid a_i \in \alpha, b_j \in \beta\}$.
6. $\alpha \boxtimes \beta = \uplus_i \uplus_j \alpha_i \odot \beta_j$
7. The elements of \mathbb{S} are the scalars of $3^{\mathbb{S}}$.
8. $|\alpha| \equiv$ the number of elements in α .
9. $|\alpha \boxtimes \beta| = |\alpha| \cdot |\beta|$
10. If $\alpha \in 3^{\mathbb{S}}$, then $\alpha^n = \underbrace{\alpha \boxtimes \alpha \boxtimes \dots \boxtimes \alpha}_{n \text{ times}}$.
11. If $|\alpha|=k$, then $|\alpha^n| = k^n$

As a matter of illustration, a multiset can be thought of as a linear combination of the distinct elements of the multiset. For example, the multiset $\{a, a, a, b, b, c\}$ can be written as $\{3a, 2b, c\}$. This allows a nice formulation of our 'fusion' operation. As an example, we could define $\{2a, 3b, c\} \uplus \{4a, 2b, c\} = \{6a, 5b, 2c\}$. The elements of \mathbb{Z} are used to 'keep track' of the number of times an element appears in the multiset. The "mset intersection" can be formulated as "take each element appearing in both msets and keep the smaller coefficient." The "mset union" can be formulated as "take each element appearing in both msets and keep the largest coefficient." In this thesis, we will not use this method of 'keeping track' of the elements, but will only list all the elements in the multiset explicitly; although the mset union and mset intersection can be calculated with these definitions in mind.

8.2 Properties

In this section we list the basic properties of the operations of the mset algebra.

Proposition 8.3 (Properties of \uplus under $3^{\mathbb{S}}$). If $\alpha, \beta, \gamma \in 3^{\mathbb{S}}$, then

- (1) $3^{\mathbb{S}}$ is closed under \uplus

- (2) \uplus is associative. $\alpha \uplus (\beta \uplus \gamma) = (\alpha \uplus \beta) \uplus \gamma$
String the sets together, and rearrange in the appropriate order. Order does not matter in the multisets.
- (3) \uplus is commutative. $\alpha \uplus \beta = \beta \uplus \alpha$
There is no order in the sets, just rearrange the sets.
- (4) \emptyset , the empty set, is the additive identity.
 $\emptyset \uplus \alpha = \alpha$.
- (5) $|\alpha \uplus \beta| = |\alpha| + |\beta|$.
- (6) $3^{\mathbb{S}}$ is cancellative.
If α, β , and γ are msets, and $\alpha \uplus \beta = \alpha \uplus \gamma$, then $\beta = \gamma$.
- (7) \uplus is NOT idempotent.
 $\alpha \uplus \alpha \neq \alpha$
 $\{a\} \uplus \{a\} = \{a, a\}$.
- (8) There are NO additive inverses in $3^{\mathbb{S}}$.
If $\alpha \in 3^{\mathbb{S}}$ and $\alpha \neq \emptyset$, then there is no element, β of $3^{\mathbb{S}}$ such that $\alpha \uplus \beta = \emptyset$.
- (9) $3^{\mathbb{S}}$ is zerosumfree.
If $\alpha \uplus \beta = \emptyset$ then $\alpha = \beta = \emptyset$.

Example 8.4. $\{1,2,2\} \uplus \{1,2,3\} = \{1,1,2,2,2,3\}$ ■

Proposition 8.5 (Properties of \boxtimes under $3^{\mathbb{S}}$). If $\alpha, \beta, \gamma \in 3^{\mathbb{S}}$, then

- (1) $3^{\mathbb{S}}$ is closed under \boxtimes
- (2) \boxtimes is associative.
 $\alpha \boxtimes (\beta \boxtimes \gamma) = (\alpha \boxtimes \beta) \boxtimes \gamma = \uplus_{i,j,k} \alpha_i \odot \beta_j \odot \gamma_k$
- (3) \boxtimes is commutative.
 $\alpha \boxtimes \beta = \beta \boxtimes \alpha$
- (4) \boxtimes is distributive over \uplus .
- (5) $|\alpha \boxtimes \beta| = |\alpha| \cdot |\beta|$.

- (6) $\{1\}$, the multiplicative identity of \mathbb{S} , is the multiplicative identity.
 $\{1\} \boxtimes \{a\} = \{a\}$.
- (7) \emptyset is absorbing under \boxtimes .
 $\emptyset \boxtimes \{a\} = \emptyset$.
- (8) If $\alpha = \beta$, then $\alpha \boxtimes \gamma = \beta \boxtimes \gamma$.
- (9) If $\alpha \boxtimes \gamma = \beta \boxtimes \gamma$, then it does not follow that $\alpha = \beta$. Example: $\{1\} \boxtimes \{2, -2\} = \{-1\} \boxtimes \{2, -2\}$.
- (10) $\{0\} \boxtimes \alpha = \{0, \dots, 0\}_{1 \times k}$, where $k = |\alpha|$.
- (11) $\mathbb{S} \subset 3^{\mathbb{S}}$. We can associate the elements of \mathbb{S} with the singletons, or scalars, of $3^{\mathbb{S}}$.
- (12) Multiplicative inverses in $3^{\mathbb{S}}$:
If $a \in \mathbb{S}$, and a is invertible in \mathbb{S} , then $\{a\} \boxtimes \{a^{-1}\} = \{1\}$.

Example 8.6. $\{1, 2, 3\} \boxtimes \{4, 5\} = \{4, 5, 8, 10, 12, 15\}$. ■

Proposition 8.7. If $\alpha = (\alpha^+, \theta_\alpha, \alpha^-)$, and $\beta = (\beta^+, \theta_\beta, \beta^-)$, then we can more rigorously define the size of the product of these two msets. Suppose $\alpha \boxtimes \beta = \gamma$, and let

$$\begin{array}{lll}
|\alpha| = k & |\beta| = \ell & |\gamma| = u \\
|\alpha^+| = m & |\beta^+| = q & |\gamma^+| = v \\
|\alpha^-| = n & |\beta^-| = r & |\gamma^-| = w \\
|\theta_\alpha| = p & |\theta_\beta| = t & |\theta_\gamma| = y
\end{array} \tag{89}$$

$$\text{Then } |\alpha \boxtimes \beta| = k \cdot \ell = (m + n + p) \cdot (q + r + t)$$

In what follows we will consider a multiset as consisting of two parts: a multiset with all positive elements, and a multiset with all negative elements. We may write this as $\alpha = \{\alpha^+, \alpha^-\}$. The element $\theta \in \mathbb{S}$ warrants special attention. At the beginning of this development, a multiset was split into 3 parts: $\alpha = \{\alpha^+, \theta_\alpha, \alpha^-\}$, where θ_α is a multiset consisting of all the zeros of α (see Proposition 8.7). As the work progressed it was seen that the θ_α -multiset does not affect the outcome. It only adds a consistency condition which must be satisfied. As we see from Equation (91), in using $\varphi_s^{-1}(\alpha)$, the zeros of \mathbb{S} do not affect the sum, since θ is the additive identity of \mathbb{S} .

With the properties illustrated above, we have the following lemma.

Lemma 8.8. If \mathbb{S} is a semiring then $3^{\mathbb{S}}$ is a cancellative zerosumfree semiring with no zero divisors.

It should be noticed, here, that $3^{\mathbb{S}}$ is a semiring with a cancellation property (where cancellation is defined as dropping elements of a multiset from each side of an equation), whereas the set $\mathbb{S} = \mathfrak{M}^*$, as a path algebra, is a semiring with idempotency. In this respect we have traded the idempotency property for a cancellation property. As we compare these two semirings, we also note that the addition of \mathbb{S} (\oplus) does not play a role in $3^{\mathbb{S}}$; only the multiplication of \mathbb{S} is used in the development of the mset algebra.

8.3 Mappings

In this section we define mappings from \mathbb{S} to $3^{\mathbb{S}}$, and from $3^{\mathbb{S}}$ to \mathbb{S} .

Definition 8.9. Define $\varphi_{\mathbb{S}}: \mathbb{S} \rightarrow 3^{\mathbb{S}}$ by:

$$\varphi_{\mathbb{S}}(a) \begin{cases} \{a\} & \text{if } a \neq \theta \\ \emptyset & \text{if } a = \theta \end{cases} \quad (90)$$

Definition 8.10. Define $\varphi_{\mathbb{S}}^{-1}: 3^{\mathbb{S}} \rightarrow \mathbb{S}$ by:

$$\varphi_{\mathbb{S}}^{-1}(\alpha) \begin{cases} \bigoplus_i \alpha_i & \text{if } \alpha \neq \emptyset \\ \theta & \text{if } \alpha = \emptyset \end{cases} \quad (91)$$

Proposition 8.11. If $a \in \mathbb{S}$, then $\varphi_{\mathbb{S}}^{-1}(\varphi_{\mathbb{S}}(a)) = a$

Note 8.12. If $\alpha \in 3^{\mathbb{S}}$, then $\varphi_{\mathbb{S}}(\varphi_{\mathbb{S}}^{-1}(\alpha)) \neq \alpha$, unless α consists of only one element.

In our development of the mset algebra, the addition operation of \mathbb{S} is not used. Definition 8.10 shows how the addition operation of \mathbb{S} is connected to the mset algebra. See Section 8.16 for more the mapping of Definition 8.10.

Definition 8.13. If $A \in \mathbb{S}_{n \times n}$, then

$$[\varphi_{\mathbb{S}}(A)]_{ij} \begin{cases} \{a_{ij}\} & \text{if } a_{ij} \neq \theta \\ \emptyset & \text{if } a_{ij} = \theta \end{cases} \quad (92)$$

8.4 Matrices over $3^{\mathbb{S}}$

Suppose $A, B \in 3^{\mathbb{S}}_{n \times n}$, and $\alpha \in 3^{\mathbb{S}}$, then we define:

1. $(A \uplus B)_{ij} \equiv a_{ij} \uplus b_{ij}$.

2. $(A \boxtimes B)_{ij} \equiv \uplus_k \{a_{ik} \boxtimes b_{kj}\} = \{a_{i1} \odot b_{1j}, a_{i2} \odot b_{2j}, \dots, a_{in} \odot b_{nj}\}.$
3. $(\alpha \boxtimes A)_{ij} \equiv \alpha \boxtimes a_{ij}$

Example 8.14. If $A \in 3_{n \times n}^{\mathbb{S}}$ then

1. $a_{ij} = \text{mset}$
2. $A_{ij}^2 = \{a_{i1} \boxtimes a_{1j}, a_{i2} \boxtimes a_{2j}, \dots, a_{in} \boxtimes a_{nj}\}.$
3. $A_{ij}^3 = \uplus_{k_1 k_2} \{a_{ik_1} \boxtimes a_{k_1 k_2} \boxtimes a_{k_2 j}\}.$
4. $A_{ij}^m = \uplus_{k_1 k_2} \dots \uplus_{k_{m-1}} \{a_{ik_1} \boxtimes a_{k_1 k_2} \boxtimes \dots \boxtimes a_{k_{m-1} j}\}.$

■

Proposition 8.15. $3_{n \times n}^{\mathbb{S}}$ is a cancellative zerosumfree semiring with multiplicative identity.

Proposition 8.16.

$$(1) \varnothing_n = \begin{bmatrix} \varnothing & \cdots & \varnothing \\ \vdots & & \vdots \\ \varnothing & \cdots & \varnothing \end{bmatrix} \text{ is the additive identity.}$$

$$(2) I_n = \begin{bmatrix} \{1\} & \cdots & \varnothing \\ \vdots & \ddots & \vdots \\ \varnothing & \cdots & \{1\} \end{bmatrix} \text{ is the multiplicative identity.}$$

Proposition 8.17. If $A \in 3_{n \times n}^{\mathbb{S}}$, then:

- (1) $\varnothing_n \uplus A = A$
- (2) $\varnothing_n \boxtimes A = A \boxtimes \varnothing_n = \varnothing_n$
- (3) $I_n \boxtimes A = A \boxtimes I_n = A$

8.5 Partial Ordering and Lattice Structure

Here, we give a brief result for msets.

Definition 8.18. We use the notation $\alpha \subseteq \beta$ to mean that each element (including multiplicities) of α is also in β .

Proposition 8.19. Mset-inclusion induces a partial order on 3^R .

Proof. Suppose $\alpha, \beta, \gamma \in 3^R$.

- (1) $\alpha \subseteq \alpha$.
- (2) If $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$, then $\alpha = \beta$.
- (3) If $\alpha \subseteq \beta$ and $\beta \subseteq \gamma$, then $\alpha \subseteq \gamma$.

□

Definition 8.20. We say that $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

We use the usual meaning for vectors, i.e. $\underline{x} \leq \underline{y}$ means elementwise. Likewise $\underline{x} \geq \varnothing$ means $x_i \geq \varnothing$, and at least one element of \underline{x} is greater than \varnothing .

The partial ordering on 3^R creates a lattice structure on 3^R , where

1. $\alpha \wedge \beta = \alpha \cap \beta$, where ‘ \cap ’ represents mset intersection.
2. $\alpha \vee \beta = \alpha \cup \beta$, where ‘ \cup ’ represents the mset union.

8.6 Equations in $3^{\mathbb{S}}$

In this section we investigate equations with coefficients in $3^{\mathbb{S}}$. After developing the algebra of $3^{\mathbb{S}}$, we are interested in the solvability of certain types of equations. We assume throughout this section that \mathbb{S} has a total order and that the nonzero elements of \mathbb{S} have multiplicative inverses. It is noted that this is true for \mathbb{R} which is a field, and for the algebras \mathfrak{M}^+ and \mathfrak{M}^* which have been used earlier in this thesis.

Lemma 8.21. Suppose \mathbb{S} is a semiring in which the nonzero elements have multiplicative inverses, and suppose that \mathbb{S} is totally ordered. Let \mathbb{P} be the set of “positive” elements of \mathbb{S} . $\mathbb{P} = \{a \mid a > \theta\}$. If $\alpha, \beta, \gamma \in \mathbb{P}$ and $\alpha \boxtimes \beta = \alpha \boxtimes \gamma$, then $\beta = \gamma$

Proof. Write the elements of α, β, γ in increasing order. Then $a_1 b_1 = a_1 c_1$, since each product is the minimal element of the multiset. Since $a_1 > \theta$, and a_1^{-1} exists, then multiply both sides by a_1^{-1} , and conclude that $b_1 = c_1$. Form $\beta' = \beta \setminus \{b_1 \boxtimes \alpha\}$, and $\gamma' = \gamma \setminus \{c_1 \boxtimes \alpha\}$, and repeat. □

Example 8.22. Lemma 8.21 is not valid if “negative” numbers are allowed in α . Consider $\mathbb{S} = \mathbb{R}$, with the usual multiplication. $\{1\} \{1, -1\} = \{-1\} \{-1, 1\}$, but $\{1\} \neq \{-1\}$. ■

Lemma 8.23. If $\alpha, \beta, \gamma \in 3^{\mathbb{S}}$, and $\alpha < 0, \beta > 0, \gamma > 0$, and $\alpha \boxtimes \beta = \alpha \boxtimes \gamma$, then $\beta = \gamma$

Proof. The proof is similar to the proof of Lemma 8.21 □

Note 8.24. We use $3^{\mathbb{R}}$, msets over the real numbers in the rest of this section.

For msets, $\gamma > 0$ means that every element of the mset α is positive.

Proposition 8.25. Solve the equation: $\alpha \boxtimes x = \emptyset$.

If $\alpha \neq \emptyset$, then $x = \emptyset$ is the unique solution.

The above proposition may be restated as:

$$\alpha \boxtimes x = \emptyset \implies \alpha = \emptyset, \text{ or } x = \emptyset$$

Lemma 8.26. If $\alpha > 0, \beta > 0$, and $\alpha^2 = \beta^2$, then $\alpha = \beta$.

Proof. Suppose $\alpha > 0$ and $\beta > 0$. Assume α , and β are in ascending order. $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$. Then $a_1^2 = b_1^2$. So $a_1 = b_1$. Form $\alpha' = \alpha \setminus (a_1 \boxtimes \alpha)$, and $\beta' = \beta \setminus (a_1 \boxtimes \alpha)$. Continue to get $\alpha = \beta$. □

Lemma 8.27. If $\alpha u \uplus \beta v = \beta u \uplus \alpha v$, and $\alpha, \beta, u, v > 0$, then $\alpha = \beta$, or $u = v$.

Proof. Write $u = u_1 \uplus u_2$, and $v = v_1 \uplus v_2$ with $u_1 = u \cap v = v_1$.

$$\alpha u_1 \uplus \alpha u_2 \uplus \beta v_1 \uplus \beta v_2 = \beta u_1 \uplus \beta u_2 \uplus \alpha v_1 \uplus \alpha v_2 \tag{93}$$

$$\implies \alpha u_2 \uplus \beta v_2 = \beta u_2 \uplus \alpha v_2 \tag{94}$$

In Equation (94), u_2 , and v_2 , have no elements in common. So we will assume that in Lemma 8.27, that this is also the case. So suppose we have

$$\alpha u \uplus \beta v = \beta u \uplus \alpha v \tag{95}$$

with u , and v having no common elements. Assume α, β, u , and v are written with their elements in ascending order. Consider the minimal element on each side of Equation (95):

$$\min\{\alpha_1 u_1, \beta_1 v_1\} = \min\{\beta_1 u_1, \alpha_1 v_1\} \tag{96}$$

There are four cases to be investigated:

$$(1) \alpha_1 u_1 = \beta_1 u_1 \implies \alpha_1 = \beta_1$$

$$(2) \alpha_1 u_1 = \alpha_1 v_1 \implies u_1 = v_1$$

$$(3) \beta_1 v_1 = \beta_1 u_1 \implies u_1 = v_1$$

$$(4) \beta_1 v_1 = \alpha_1 v_1 \implies \alpha_1 = \beta_1$$

We can now reduce the size of the appropriate mset, and continue. \square

The constraints of Lemma 8.27 can be relaxed to the following lemma, whose proof is similar to the proof of Lemma 8.27

Lemma 8.28. If $\alpha u \uplus \beta v = \beta u \uplus \alpha v$, and $\alpha, \beta > 0$, then $\alpha = \beta$, or $u = v$.

The proof of the following is also similar to the proof of Lemma 8.27.

Lemma 8.29. If $\alpha \boxtimes x = \beta \boxtimes x$, and $x > 0$, then $\alpha = \beta$

Example 8.30. Solve the equation:

$$\alpha \boxtimes x = \beta \tag{97}$$

The consistency condition for this equation is: $|\alpha| \cdot |x| = |\beta|$.

This equation is separated into three cases

1. $\alpha > 0$ and $\beta > 0$.

Then $x > 0$, and this is just Lemma 8.21.

2. $\beta > 0$. Then there are two cases, as α and x cannot contain both positive and negative elements.

- (a) $\alpha > 0$, and $x > 0$.

This is just the previous case.

- (b) $\alpha < 0$, and $x < 0$.

The proof of this is similar to the previous case, or to that of Lemma 8.21.

3. (General Case) $(\alpha^+, \alpha^-)(x^+, x^-) = (\beta^+, \beta^-)$

This leads to the two equations:

$$\alpha^+ x^+ \uplus \alpha^- x^- = \beta^+ \tag{98}$$

$$\alpha^- x^+ \uplus \alpha^+ x^- = \beta^- \tag{99}$$

Here, we let $a = \alpha^+$, $b = -\alpha^-$, $x = x^+$, $y = -x^-$, $c = \beta^+$, and $d = -\beta^-$ to yield:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \quad (100)$$

where all entries are positive.

This can be written as:

$$ax \uplus by = c \quad (101)$$

$$bx \uplus ay = d \quad (102)$$

If we multiply equation (101) by 'a' and equation (102) by 'b', combine the two equations and cancel the common term of 'aby', then we get:

$$a^2x \uplus bd = b^2x \uplus ac \quad (103)$$

Likewise, we multiply equation (101) by 'b', and equation (102) by 'a', combine the two equations and cancel the common term of 'aby', and get:

$$a^2y \uplus bc = b^2y \uplus ad \quad (104)$$

Equations (103), and (104) are equivalent to the solutions associated with Cramer's Rule. Each of these equations is an equation with only one variable. We are not able to solve these systems for 'x' or 'y'.

An "algebraic" solution for $x=\{x^+, x^-\}$ is not possible in this case as the problem depends greatly on β .

■

It is possible to use an algorithm to solve for $x=\{x^+, x^-\}$, but the available algorithm grows exponentially.

Algorithm 8.31. To solve $\alpha x = \beta$

(1) Set $n = \frac{|\beta|}{|\alpha|}$ If $|\alpha| \nmid |\beta|$ then no solution exists.

(2) Calculate:

$$(2a) \hat{x} = x_{\max}^+ = \frac{\beta_{\min}^-}{\alpha_{\min}^-}$$

$$(2b) x' = x_{\min}^- = \frac{\beta_{\max}^-}{\alpha_{\max}^+}$$

(3) Calculate:

$$(3a) \alpha \hat{x}$$

(3b) $\alpha x'$

(4) For each of parts (3a) and (3b) above, if all elements of the set are elements of β , then cancel these elements from β . Replace β by $\beta \setminus (\alpha \hat{x})$ or $\beta \setminus (\alpha x')$

(5) Repeat at step (2) until $\beta = \emptyset$.

If, at any step, a contradiction occurs, then back out to the next branch. If all branches occur with no solution, then no solution exists.

The following is another description of the algorithm. Consider $(\alpha^-, \alpha^+)(x^-, x^+) = (\beta^-, \beta^+)$. Then β_1^- is obtained from either (1) $\alpha_1^- x_n^+$ or (2) $\alpha_n^+ x_1^-$. From these two equations, we find either (1) $x_n^+ = \frac{\beta_1^-}{\alpha_1^-}$, or (2) $x_1^- = \frac{\beta_1^-}{\alpha_n^+}$. We pick one of these two choices, multiply by α , and cancel the appropriate terms from β , and continue to find the next element of “x”. If all the terms of $\alpha * x_n^+$ (or the other) are not present, then we try the other choice. If it does not work then there is not solution. This algorithm will continue until we have solved for “x”, or until we have tried all 2^n choices for the elements of “x”.

Example 8.32. Use Algorithm 8.31 to solve: $\{2,3\}_x = \{-9, -6, 4, 6, 6, 9\}$.

$$|x| = 3, \implies x = \{x_1, x_2, x_3\}$$

(1) $\{2,3\}_{\{x_1, x_2, x_3\}} = \{-9, -6, 4, 6, 6, 9\}$.

(a) $\hat{x} = \frac{-9}{2} = -4.5$

(i) $\alpha \hat{x} \notin \beta$

(ii) no good.

(b) $x' = \frac{-9}{3} = -3$

(i) $\alpha x' = \{-9, -6\}$

(ii) β replaced by $\beta \setminus (\alpha x')$

(2) $\{2,3\}_{\{x_2, x_3\}} = \{4, 6, 6, 9\}$.

(a) $\hat{x} = \frac{4}{2} = 2$

(i) $\alpha \hat{x} = \{4, 6\}$

(ii) β replaced by $\beta \setminus (\alpha \hat{x})$

(b) $x' = \frac{4}{3}$

(i) $\alpha x' \notin \beta$

(ii) no good.

(3) $\{2,3\}_{\{x_3\}} = \{6, 9\}$.

- (a) $\hat{x} = \frac{6}{2} = 3$
- (i) $\alpha \hat{x} = \{6,9\}$
 - (ii) β replaced by $\beta \setminus (\alpha \hat{x})$
 - (iii) $\beta = \emptyset$.
 - (iv) Complete.
- (b) $x' = \frac{6}{3} = 2$
- (i) $\alpha x' \not\subseteq \beta$
 - (ii) no good.

For this example, we see that $x = \{-3, 2, 3\}$. ■

We can also use Algorithm 8.39 to solve Equations (103) and (104) to arrive at a solution to Equation (97).

Example 8.33. Solve the equation: $\alpha \boxtimes x = \beta \boxtimes x$.

The consistency condition for this equation is: $|\alpha| \cdot |x| = |\beta| \cdot |x|$, or $|\alpha| = |\beta|$.

We consider three different cases to analyze this equation.

- (1) $\alpha > 0, \beta > 0, x > 0$.

Assume α, β , and x are written in increasing order. Then $\alpha_1 \boxtimes x_1 = \beta_1 x_1$. Multiply by x_1^{-1} to get $\alpha_1 = \beta$. Form $\beta' = \beta \setminus \beta_1, \alpha' = \alpha \setminus \alpha_1$. The new equation is $\alpha' \boxtimes x = \beta'$. Continue to find $\alpha_2 = \beta_2, \alpha_3 = \beta_3, \dots$. Therefore, we get that $\alpha = \beta$.

- (2) $\beta > 0$.

Write $\{\alpha^+, \alpha^-\} \{x^+, x^-\} = \beta \{x^+, x^-\}$.

$$\alpha^+ x^+ \uplus \alpha^- x^- = \beta x^+ \tag{105}$$

$$\alpha^- x^+ \uplus \alpha^+ x^- = \beta x^- \tag{106}$$

Multiply Equation (105) by x^- , and Equation (106) by x^+ . Combine the equations and cancel $\alpha^+ x^+ x^-$ to get:

$$\alpha^- (x^-)^2 = \alpha^- (x^+)^2 \tag{107}$$

If we let $a = x^+, b = -x^-$, and $d = -\alpha^-$, then the equation becomes:

$$d(b)^2 = d(a)^2 \tag{108}$$

By Lemma 8.23,

$$(b)^2 = (a)^2 \quad (109)$$

So by Lemma 8.26, $b=a$.

So, $x^+ = -x^-$. This is the first condition to solve the equation.

Equation (105) becomes

$$\begin{aligned} \alpha^+ x^+ \uplus (-\alpha^-)(-x^-) &= \beta x^+ \\ \implies \alpha^+ x^+ \uplus (-\alpha^-)x^+ &= \beta x^+ \\ \implies (\alpha^+ \uplus (-\alpha^-))x^+ &= \beta x^+ \\ \implies \alpha^+ \uplus (-\alpha^-) &= \beta \end{aligned} \quad (110)$$

So, we have that $x^+ = -x^-$, and $\beta = \alpha^+ \uplus \alpha^-$.

(3) (General case).

$$\{\alpha^+, \alpha^-\}\{x^+, x^-\} = \{\beta^+, \beta^-\}\{x^+, x^-\} \quad (111)$$

The two equations are:

$$\alpha^+ x^+ \uplus \alpha^- x^- = \beta^+ x^+ \uplus \beta^- x^- \quad (112)$$

$$\alpha^+ x^- \uplus \alpha^- x^+ = \beta^+ x^- \uplus \beta^- x^+ \quad (113)$$

If we let $a=\alpha^+$, $b=-\alpha^-$, $c=\beta^+$, $d=-\beta^-$, $x=x^+$, $y=-x^-$, then Equation (112) can be written as:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (114)$$

This system can be written as:

$$ax \uplus by = cx \uplus dy \quad (115)$$

$$bx \uplus ay = dx \uplus cy \quad (116)$$

Multiply Equation (115) by 'a', and Equation (116) by 'b' to get

$$a^2 x \uplus aby = acx \uplus ady \quad (117)$$

$$b^2 x \uplus aby = bdx \uplus bcy \quad (118)$$

We add these equations together and cancel the term of 'aby' to get

$$a^2x \uplus bdx \uplus bcy = b^2x \uplus acx \uplus ady \quad (119)$$

And, we can also multiply Equation (115) by 'c', and Equation (116) by 'd', add together and cancel a common term of 'cdy' to get:

$$acx \uplus d^2x \uplus bcy = c^2x \uplus bdx \uplus ady \quad (120)$$

We then add Equations (119) and (120) and cancel the common terms of 'ady' and 'bcy' to get the equation:

$$(a^2 \uplus bd \uplus bd \uplus c^2)x = (b^2 \uplus ac \uplus ac \uplus d^2)x \quad (121)$$

since $x > 0$. Then by Lemma 8.29,

$$a^2 \uplus bd \uplus bd \uplus c^2 = b^2 \uplus ac \uplus ac \uplus d^2 \quad (122)$$

This is the same equation one would get if the variable 'x' was eliminated from Equations (115) and (116), using the fact the $y > 0$.

Replacing the variables by the respective elements, we get:

$$\begin{aligned} (\alpha^+)^2 \uplus (-\alpha^-)(-\beta^-) \uplus (-\alpha^-)(-\beta^-) \uplus (\beta^+)^2 \\ = (-\alpha^-)^2 \uplus \alpha^+ \beta^+ \uplus \alpha^+ \beta^+ \uplus (-\beta^-)^2 \end{aligned} \quad (123)$$

Equation (123) is a consistency condition; but it is not the consistency condition that we seek, since it does not allow us to proceed further.

So, we return to Equations (115) and (116): Multiply Equation (115) by 'y', multiply Equation (116) by 'x', combine the two equations, and cancel the common terms 'axy' and 'cxy' to get the equation:

$$dx^2 \uplus by^2 = bx^2 \uplus dy^2 \quad (124)$$

We can also multiply Equation (115) by 'x', multiply Equation (116) by 'y', combine the two equations, and cancel the common terms 'bxy' and 'dxy' to get the equation:

$$ax^2 \uplus cy^2 = cx^2 \uplus ay^2 \quad (125)$$

By applying Lemma 8.27 to Equations (124) and (125) we get conditions for the solution to the Equations (115) and (116) : either $x^+ = -x^-$, or $\alpha^+ = \beta^+$ and $\alpha^- = \beta^-$.

So we have either:

(a) $x^+ = -x^-$. In this case we go back to to equation (112) and get:

$$\begin{aligned}
& \alpha^+ x^+ \uplus (-\alpha^-)(-x^-) = \beta^+ x^+ \uplus (-\beta^-)(-x^-) \\
\implies & \alpha^+ x^+ \uplus (-\alpha^-)x^+ = \beta^+ x^+ \uplus (-\beta^-)x^+ \\
\implies & (\alpha^+ \uplus (-\alpha^-))x^+ = (\beta^+ \uplus (-\beta^-))x^+ \\
\implies & \alpha^+ \uplus (-\alpha^-) = \beta^+ \uplus (-\beta^-)
\end{aligned} \tag{126}$$

This can be stated as: $|\alpha| = |\beta|$, and the msets composed of the absolute values of the elements of α and β are equal msets.

(b) $\alpha = \beta$. Then x can be any mset. ■

Example 8.34. Suppose that $\alpha x \subseteq \beta x$. Suppose, further, that $\alpha, \beta, x > 0$.

The following example shows that there does not need to be any relation between α and β . In particular $\alpha \not\subseteq \beta$! This is a catastrophic example for it spoils many possible results involving eigenvalue equation proofs in the mset algebra.

$$\{2, 5\} \boxtimes \{3, 4, 7\} \subseteq \left\{ \frac{8}{7}, \frac{3}{2}, \frac{15}{7}, \frac{14}{3}, \frac{20}{3}, \frac{35}{4} \right\} \boxtimes \{3, 4, 7\} \tag{127}$$

■

The previous example proves the following important lemma.

Lemma 8.35.

$$\alpha x \subseteq \beta x \quad \not\Rightarrow \quad \alpha \subseteq \beta \tag{128}$$

Although the following constraints are more strict than necessary, we have a more general case of Example 8.34.

Example 8.36. If $a, b, c, p, q > 0$ and a, b, c, p, q are all pairwise relatively prime then

$$\{p, q\} \boxtimes \{a, b, c\} \subseteq \left\{ \frac{pb}{a}, \frac{qb}{a}, \frac{pc}{b}, \frac{qc}{b}, \frac{pa}{c}, \frac{qa}{c} \right\} \boxtimes \{a, b, c\} \tag{129}$$

■

Example 8.37. The general linear equation over msets is

$$\alpha \boxtimes x \uplus \beta = \gamma \boxtimes x \uplus \delta. \quad (130)$$

This equation is too general for a full investigation of its general solutions. The consistency condition for this equation is $|\alpha| \cdot |x| + |\beta| = |\gamma| \cdot |x| + |\delta|$. This is equivalent to the condition that $|x| = \frac{|\delta| - |\beta|}{|\alpha| - |\gamma|}$. This condition is not sufficient to determine whether a solution exists to the equation. ■

Example 8.38. Let us consider the equation, with $\alpha, \beta, \gamma, \delta > 0$.

$$\alpha \boxtimes x \uplus \beta = \gamma \boxtimes x \uplus \delta. \quad (131)$$

with $\alpha, \beta, \gamma, \delta > 0$.

We assume that all elements are positive, and that they are in increasing order. The assumption that $\alpha, \beta, \gamma, \delta > 0$ is not too restrictive in the sense that Equation (131) is the equation which results from using Cramer's Rule to solve a system of equations. The following algorithm can be used to find x . If the msets, $\alpha, \beta, \gamma, \delta$, are of the appropriate sizes to determine the exact size of x , then we know the exact number of steps required to solve for x . If $|\alpha| = |\beta|$, then there is no definite number of steps known, a priori. We know that a finite number of steps will be required. This derives from the fact that at each stage of the algorithm, at least one element from each side is removed. So after a finite number of steps, $x^{(N)} = \emptyset$. As stated we do not know how many steps will be required.

Algorithm 8.39.

Suppose we want to solve $\alpha \boxtimes x \uplus \beta = \gamma \boxtimes x \uplus \delta$. Assume that $\alpha \cap \gamma = \emptyset$, and $\beta \cap \delta = \emptyset$. Otherwise, these terms can be canceled from each side of the equation.

- (1) $x = x^{(0)}$, $\beta^{(0)} = \beta$, $\delta^{(0)} = \delta$
- (2) $k=1$
- (3) Let $\hat{x} = \max\left(\frac{\delta_{\max}}{\alpha_{\max}}, \frac{\beta_{\max}}{\gamma_{\max}}\right)$
- (4) We rewrite $x = (x^{(1)}, \hat{x})$
- (5) Substitute into the equation $\alpha \boxtimes x \uplus \beta = \gamma \boxtimes x \uplus \delta$ to get
- (6) $\alpha \boxtimes (x^{(k)}, \hat{x}) \uplus \beta^{(k-1)} = \gamma \boxtimes (x^{(k)}, \hat{x}) \uplus \delta^{(k-1)}$
- (7) $\alpha x^{(k)} \uplus (\alpha \hat{x} \uplus \beta^{(k-1)}) = \gamma x^{(k)} \uplus (\gamma \hat{x}) \uplus \delta^{(k-1)}$
Cancel common elements.
- (8) Rewrite as $\alpha \boxtimes x^{(k)} \uplus \beta^{(k)} = \gamma \boxtimes x^{(k)} \uplus \delta^{(k)}$

(9) If $\beta^{(k)} = \delta^{(k)} = \emptyset$ then stop.

(10) If, at any time, the consistency condition fails then there is no solution. Stop.

(11) $k=k+1$

(12) Goto Step (3).

■

This algorithm may be improved by also considering the minimum element of each side. This could improve the time of computation and also aid in approximating the number of steps necessary to complete the algorithm.

Example 8.40. Use Algorithm 8.39 to solve:

$$\{7, 9\}x \uplus \{15, \frac{63}{4}, 20\} = \{5, \frac{21}{4}\}x \uplus \{27, 28, 36\} \quad (132)$$

We go through the steps of the algorithm to get the solution.

$$\hat{x} = \max\left(\frac{36}{9}, \frac{20}{\frac{21}{4}}\right) = \max\left(4, \frac{80}{21}\right) = 4 \quad (133)$$

$$\{7, 9\}\{x^{(1)}, \hat{x}\} \uplus \{15, 20, \frac{63}{4}\} = \{5, \frac{21}{4}\}\{x^{(1)}, \hat{x}\} \uplus \{27, 28, 36\} \quad (134)$$

$$\{7, 9\}x^{(1)} \uplus \{7, 9\}\{4\} \uplus \{15, \frac{63}{4}, 20\} = \{5, \frac{21}{4}\}x^{(1)} \uplus \{5, \frac{21}{4}\}\{4\} \uplus \{27, 28, 36\} \quad (135)$$

$$\{7, 9\}x^{(1)} \uplus \{15, \frac{63}{4}, 20, 28, 36\} = \{5, \frac{21}{4}\}x^{(1)} \uplus \{5, \frac{21}{4}\}\{20, 21, 27, 28, 36\} \quad (136)$$

$$\{7, 9\}x^{(1)} \uplus \{15, \frac{63}{4}\} = \{5, \frac{21}{4}\}x^{(1)} \uplus \{21, 27\} \quad (137)$$

$$\hat{x} = \max\left(\frac{27}{9}, \frac{\frac{63}{4}}{\frac{21}{4}}\right) = \max(3, 3) = 3 \quad (138)$$

$$\{7, 9\}\{x^{(2)}, \hat{x}\} \uplus \{15, \frac{63}{4}\} = \{5, \frac{21}{4}\}\{x^{(2)}, \hat{x}\} \uplus \{21, 27\} \quad (139)$$

$$\{7, 9\}x^{(2)} \uplus \{7, 9\}\{3\} \uplus \{15, \frac{63}{4}\} = \{5, \frac{21}{4}\}x^{(2)} \uplus \{5, \frac{21}{4}\}\{3\} \uplus \{21, 27\} \quad (140)$$

$$\{7, 9\}x^{(2)} \uplus \{15, \frac{63}{4}, 21, 27\} = \{5, \frac{21}{4}\}x^{(2)} \uplus \{15, \frac{63}{4}, 21, 27\} \quad (141)$$

$$\{7, 9\}x^{(2)} = \{5, \frac{21}{4}\}x^{(2)} \quad (142)$$

$$x^{(2)} = \emptyset \quad (143)$$

Therefore, we have $x = \{3, 4\}$ as the solution to equation (132) ■

8.7 Matrix Equations over $3^{\mathbb{S}}$

In this section we investigate systems of equations over $3^{\mathbb{S}}$. The natural question may arise as to the amount of information which can be developed from the known results of Linear systems in the conventional algebra. In this section we investigate some of the elementary properties of Linear systems in $3^{\mathbb{S}}$. We start with a short overview of homogeneous systems and continue with an investigation of Cramer's Rule and its relation to the system of nonhomogeneous equations.

Homogeneous Systems

Proposition 8.41. If $a_1x_1 \uplus a_2x_2 \uplus \dots \uplus a_nx_n = \emptyset$, and $x_i \neq \emptyset$ for all i , then $a_i = \emptyset$ for all i .

There is a very close analogy between solving systems of equations over $3^{\mathbb{S}}$ and solving systems of equations over the nonnegative real numbers. Many of these results will involve graph theory as the matrix A is viewed as the adjacency matrix of a graph. The mset can be considered as the "weight" of the path in which the time for a "process" is broken into its constituent parts. For example, $\{2, 1\}$ is viewed as a weight of "3" which is separated into two sub-processes, one of weight=2 and one of weight=1.

The first system to consider is the homogeneous system of equations:

$$Ax = \emptyset \quad (144)$$

Theorem 8.42. If A is an irreducible $n \times n$ mset matrix and $A\underline{x} = \emptyset$, then $\underline{x} = \emptyset$.

Proof. Suppose $x_k \neq \emptyset$. Then, for each i ,

$$\emptyset = (A\underline{x})_i = \uplus_{j=1}^n a_{ij}x_j \geq a_{ik}x_k$$

Therefore, $a_{ik}=0$, for all i . So A has a column equal to \emptyset , and, hence, it is reducible. Therefore $\underline{x} = \emptyset$. \square

We include a few results which are quite useful in the theory of mset matrices. As we see the proofs of these results are almost identical with the proofs given in the theory of nonnegative matrices.

Theorem 8.43. If $A \geq \emptyset$ and A is reducible, then

- (a) A^k is reducible for all k
- (b) $\uplus_k A^k$ is reducible.

Proof. Suppose $A \approx \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$.

$$(a) A^k \approx \begin{bmatrix} B^k & ? \\ 0 & D^k \end{bmatrix}$$

and

$$(b) \uplus_k A^k \approx \begin{bmatrix} \uplus_k B^k & ? \\ 0 & \uplus_k D^k \end{bmatrix} \quad \square$$

Theorem 8.44. If A is an irreducible mset $n \times n$ matrix, $n \geq 2$, and $\underline{x} \geq \emptyset$ with exactly k elements with are not \emptyset , $1 \leq k \leq n-1$, then $(I \uplus A)\underline{x}$ has more than k nonempty elements.

Proof. We suppose that A and \underline{x} have been permuted so that \underline{x} has the form $\begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where $x_i \neq \emptyset$,

for $1 \leq i \leq k$.

So \underline{x} has $n-k$ "zero" elements. $(I \uplus A)\underline{x} = \uplus A\underline{x}$ cannot have more than $n-k$ zero elements. Suppose that $(I \uplus A)$ has exactly $n-k$ elements equal to \emptyset . Therefore $(A\underline{x})_i = \emptyset$ whenever $x_i = \emptyset$. Therefore $(I \uplus A)\underline{x}$ has as many zero elements as \underline{x} . So $(A\underline{x})_i = \emptyset$, for $i=k+1, k+2, \dots, n$.

$$(A\underline{x})_i = \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^k a_{ij}x_j = \emptyset \quad (145)$$

Since $x_j = \emptyset$, for $1 \leq j \leq k$ then $a_{ij} = \emptyset$ for $j=k+1, k+2, \dots, n$. Therefore A is reducible, which is a contradiction. So $(I \uplus A)\underline{x}$ has more than k nonzero elements. \square

$$A \underset{\text{per}}{\approx} \begin{bmatrix}
M_{11} & & & & & & & & ? \\
& \ddots & & & & & & & \\
& & M_{1k_1} & & & & & & \\
0 & & & \alpha_{12} & & \alpha_{13} & & \alpha_{14} & \\
M_{21} & & & & & & & & \\
& \ddots & & & & & & & \\
& & M_{2k_2} & & & & & & \\
& & & 0 & & \alpha_{23} & & \alpha_{24} & \\
& & & & \ddots & & & & \\
& & & & & 0 & & \alpha_{34} & \\
& & & & & & \ddots & & \\
& & & & & & & 0 & \\
0 & & & & & & & & M_{\ell 1} \quad \ddots \\
& & & & & & & & & M_{\ell k_\ell}
\end{bmatrix} \tag{150}$$

The vector \underline{x} is partitioned conformally with A: $\underline{x} = \left[\begin{array}{c|c|c|c|c} x_1 & y_1 & x_2 & y_2 & x_\ell \end{array} \right]^T$
Consider the last equation in (146)

$$\begin{bmatrix} M_{\ell 1} & & \\ & \ddots & \\ & & M_{\ell k_\ell} \end{bmatrix} \begin{bmatrix} x_\ell \\ \\ \end{bmatrix} = 0 \tag{151}$$

Since M_ℓ is irreducible then $x_\ell=0$.
Continuing backwards, we get that all x_i 's must equal 0.
Hence, the matrix A must be nilpotent and can be replaced by the matrix

$$\begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1\ell} \\ & 0 & \alpha_{23} & \cdots & \alpha_{2\ell} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \alpha_{\ell-1,\ell} \\ & & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad (152)$$

So if there is a path from \underline{Q}_i to \underline{Q}_j then the associated y_i 's are equal to zero. For example if there is a path from \underline{Q}_3 to \underline{Q}_4 then $\alpha_{34} \neq 0$ which implies that $y_4=0$. So if there is a path from \underline{Q}_i to \underline{Q}_j then $y_j=0$.

Therefore a column of zeros imply that a nonzero solution exists.

Nonhomogeneous Systems

In this section we consider matrices over $3^{\mathbb{R}^+}$. The aim is to simplify the work in the \mathfrak{M}^* algebra. Many results can be shown in a general semiring. In particular, see Theorem 8.101.

Suppose that we are given a mset equation

$$A\underline{x} = \underline{b}. \quad (153)$$

One of our main methods of solution for the nonhomogeneous equation, (153), is through Cramer's Rule.

Example 8.47 (Cramer's Rule). We look at the 2×2 system of msets over $3^{\mathbb{R}^+}$ first. All elements of each mset is positive in this first example.

Solve the following system of equations.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (154)$$

This leads to the system:

$$a_{11}x \uplus a_{12}y = b_1 \quad (155)$$

$$a_{21}x \uplus a_{22}y = b_2 \quad (156)$$

Multiply Equation 155 by a_{22} , and multiply Equation 156 by a_{12} .

$$a_{22}a_{11}x \uplus a_{22}a_{12}y = a_{22}b_1 \quad (157)$$

$$a_{12}a_{21}x \uplus a_{12}a_{22}y = a_{12}b_2 \quad (158)$$

Combining these equations yields:

$$a_{22}a_{11}x \uplus a_{22}a_{12}y \uplus a_{12}b_2 = a_{12}a_{21}x \uplus a_{12}a_{22}y \uplus a_{22}b_1 \quad (159)$$

We can cancel $a_{12}a_{22}y$ and get the equation:

$$a_{22}a_{11}x \uplus a_{12}b_2 = a_{12}a_{21}x \uplus a_{22}b_1 \quad (160)$$

Likewise, we can multiply Equation 155 by a_{21} , and multiply Equation 156 by a_{11} . Combine the two equations, and cancel the term $a_{11}a_{21}x$ from each side. This leads to the equation:

$$a_{11}a_{22}y \uplus a_{21}b_1 = a_{12}a_{21}y \uplus a_{11}b_2 \quad (161)$$

So, we get the system of equations:

$$a_{11}a_{22}x \uplus a_{12}b_2 = a_{12}a_{21}x \uplus a_{22}b_1 \quad (162)$$

$$a_{11}a_{22}y \uplus a_{21}b_1 = a_{12}a_{21}y \uplus a_{11}b_2 \quad (163)$$

Equation (162) is an equation in “x” only , and Equation (163) is an equation in “y” only.

This can be viewed, at least for now, as Consistency Conditions for the system. If we use the sizes of the msets, we get four consistency conditions from Equations (155) and (156), and Equations (162) and (163).

$$|a_{11}| \cdot |x| + |a_{12}| \cdot |y| = |b_1| \quad (164)$$

$$|a_{21}| \cdot |x| + |a_{22}| \cdot |y| = |b_2| \quad (165)$$

$$|a_{11}a_{22}| \cdot |x| + |a_{12}b_2| = |a_{12}a_{21}| \cdot |x| + |a_{22}b_1| \quad (166)$$

$$|a_{11}a_{22}| \cdot |y| + |a_{21}b_1| = |a_{12}a_{21}| \cdot |y| + |a_{11}b_2| \quad (167)$$

So, we have taken the system of equations and separated the equations so that each equation has only one unknown. The equations seem beyond an algebraic solution although Algorithm (8.39) can be used to solve for the unknown.

The next part of this problem is to show that if we are given the system of equations, Equations (162) and (163), then we can go back to the original system of equations, namely Equation (154).

Given the equations in (162) and (163), we multiply equation (162) by a_{11} , and multiply equation (163) by a_{12} to get:

$$a_{11}a_{11}a_{22}x \uplus a_{11}a_{12}b_2 = a_{11}a_{12}a_{21}x \uplus a_{11}a_{22}b_1 \quad (168)$$

$$a_{11}a_{22}a_{12}y \uplus a_{12}a_{21}b_1 = a_{12}a_{12}a_{21}y \uplus a_{11}a_{12}b_2 \quad (169)$$

Add the left sides of the equations and add the right sides of the equations, cancel the like term of $a_{11}a_{12}b_2$, and factor the remaining terms to get:

$$a_{11}a_{22}(a_{11}x \uplus a_{12}y) \uplus a_{12}a_{21}b_1 = a_{12}a_{21}(a_{11}x \uplus a_{12}y) \uplus a_{11}a_{22}b_1 \quad (170)$$

Likewise, we could also multiply equation (162) by a_{21} , multiply equation (163) by a_{22} , add together as above, cancel the common term of $a_{21}a_{22}b_1$, and factor the remaining terms to get:

$$a_{11}a_{22}(a_{21}x \uplus a_{22}y) \uplus a_{12}a_{21}b_2 = a_{12}a_{21}(a_{21}x \uplus a_{22}y) \uplus a_{11}a_{22}b_2 \quad (171)$$

Consider Equation (170). Using Lemma 8.27, with $\alpha = a_{11}a_{22}$, $\beta = a_{12}a_{21}$, $u = a_{11}x \uplus a_{12}y$, and $v = b_1$, we get that either $a_{11}a_{22} = a_{12}a_{21}$, or $a_{11}x \uplus a_{12}y = b_1$.

Likewise for Equation (171), we get that either $a_{11}a_{22} = a_{12}a_{21}$, or $a_{21}x \uplus a_{22}y = b_2$.

So, we see that starting with equations (162) and (163), we have either that $a_{11}a_{22} = a_{12}a_{21}$ or $Ax=b$, the system in Equation (154). ■

The 3×3 case of Cramer's Rule is quite illustrative. We now consider this case.

Example 8.48. Solve the following system of equations of msets over $3^{\mathbb{R}^+}$. Again, we note that all elements are msets of positive elements.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \boxtimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (172)$$

The three equations are:

$$a_{11}x_1 \uplus a_{12}x_2 \uplus a_{13}x_3 = b_1 \quad (173)$$

$$a_{21}x_1 \uplus a_{22}x_2 \uplus a_{23}x_3 = b_2 \quad (174)$$

$$a_{31}x_1 \uplus a_{32}x_2 \uplus a_{33}x_3 = b_3 \quad (175)$$

If we multiply Equation (173) by a_{23} , and multiply Equation (174) by a_{13} , combine the two equations, and cancel the common multisets of $a_{13}a_{23}x_3$ to get:

$$a_{11}a_{23}x_1 \uplus a_{12}a_{23}x_2 \uplus a_{13}b_2 = a_{13}a_{21}x_1 \uplus a_{13}a_{22}x_2 \uplus a_{23}b_1 \quad (176)$$

Likewise, using Equations (174) and (175), we get:

$$a_{21}a_{33}x_1 \uplus a_{22}a_{33}x_2 \uplus a_{23}b_3 = a_{23}a_{31}x_1 \uplus a_{23}a_{32}x_2 \uplus a_{33}b_2 \quad (177)$$

Likewise, using Equations (173) and (175), we get:

$$a_{13}a_{31}x_1 \uplus a_{13}a_{32}x_2 \uplus a_{33}b_1 = a_{11}a_{33}x_1 \uplus a_{12}a_{33}x_2 \uplus a_{13}b_3 \quad (178)$$

Multiplying Equation (176) by a_{32} , Equation (177) by a_{12} , and Equation (178) by a_{22} yields:

$$a_{11}a_{23}a_{32}x_1 \uplus a_{12}a_{23}a_{32}x_2 \uplus a_{13}a_{32}b_2 = a_{13}a_{21}a_{32}x_1 \uplus a_{13}a_{22}a_{32}x_2 \uplus a_{23}a_{32}b_1 \quad (179)$$

$$a_{21}a_{33}a_{12}x_1 \uplus a_{22}a_{33}a_{12}x_2 \uplus a_{23}a_{12}b_3 = a_{23}a_{31}a_{12}x_1 \uplus a_{23}a_{32}a_{12}x_2 \uplus a_{33}a_{12}b_2 \quad (180)$$

$$a_{13}a_{31}a_{22}x_1 \uplus a_{13}a_{32}a_{22}x_2 \uplus a_{33}a_{22}b_1 = a_{11}a_{33}a_{22}x_1 \uplus a_{12}a_{33}a_{22}x_2 \uplus a_{13}a_{22}b_3 \quad (181)$$

If we add the LHS of Equations (179), (180), and (181), and set this equal to the sum of the RHS of the same equations and cancel equal multisets on each side then we get the equation:

$$\begin{aligned} & (a_{11}a_{23}a_{32} \uplus a_{22}a_{13}a_{31} \uplus a_{33}a_{12}a_{21})x_1 \uplus a_{22}a_{33}b_1 \uplus a_{13}a_{32}b_2 \uplus a_{12}a_{23}b_3 \\ & = (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21})x_1 \uplus a_{23}a_{32}b_1 \uplus a_{12}a_{33}b_2 \uplus a_{13}a_{22}b_3 \end{aligned} \quad (182)$$

We note that this is an equation with only one unknown, namely x_1 .

We can continue to do the same calculations with the original equations ((173), (174), (175)) to eliminate x_2 , and again to eliminate x_3 , and we get the following three equations:

$$\begin{aligned} & (a_{11}a_{23}a_{32} \uplus a_{22}a_{13}a_{31} \uplus a_{33}a_{12}a_{21})x_1 \uplus a_{22}a_{33}b_1 \uplus a_{13}a_{32}b_2 \uplus a_{12}a_{23}b_3 \\ & = (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21})x_1 \uplus a_{23}a_{32}b_1 \uplus a_{12}a_{33}b_2 \uplus a_{13}a_{22}b_3 \end{aligned} \quad (182)$$

$$\begin{aligned} & (a_{11}a_{23}a_{32} \uplus a_{22}a_{13}a_{31} \uplus a_{33}a_{12}a_{21})x_2 \uplus a_{23}a_{31}b_1 \uplus a_{11}a_{33}b_2 \uplus a_{13}a_{21}b_3 \\ & = (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21})x_2 \uplus a_{21}a_{33}b_1 \uplus a_{13}a_{31}b_2 \uplus a_{11}a_{23}b_3 \end{aligned} \quad (183)$$

$$\begin{aligned} & (a_{11}a_{23}a_{32} \uplus a_{22}a_{13}a_{31} \uplus a_{33}a_{12}a_{21})x_3 \uplus a_{21}a_{32}b_1 \uplus a_{31}a_{12}b_2 \uplus a_{11}a_{22}b_3 \\ & = (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21})x_3 \uplus a_{22}a_{31}b_1 \uplus a_{11}a_{32}b_2 \uplus a_{12}a_{21}b_3 \end{aligned} \quad (184)$$

Note 8.49. The three equations (182), (183), (184), derived from Cramer's Rule can not be simplified further. Each equation is an equation in only one variable, and once the equation is translated back into the semiring \mathbb{R}^+ , it can be solved for x_i .

Although \mathbb{R}^+ and $3^{\mathbb{R}^+}$ do not have a "minus", we can manipulate the expressions identically with expressions in $(\mathbb{R}, +, \times)$ in calculating the determinants. When complete the terms containing a minus-sign are moved to the opposite side of the equation. This is equivalent to the ordinary

Cramer's Rule if we write:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad (185)$$

where, in calculating the “determinants”, we keep the “positive” and “negative” terms separate. In this manner we get:

$$x_i = \frac{N_i^+ - N_i^-}{M^+ - M^-} \iff M^+ x_i + N_i^- = M^- x_i + N_i^+ \quad (186)$$

Here, we have $M^+ = |A|^+$, $M^- = |A|^-$, $N_i^+ = |A_i(b)|^+$, $N_i^- = |A_i(b)|^-$. We use the notation $A_i(b)$ to represent the matrix A where the i^{th} column is replaced by b.

So, we can write Equation (186) as:

$$|A|^+ x_i + |A_i(b)|^- = |A|^- x_i + |A_i(b)|^+ \quad (187)$$

We briefly describe the system of algebraic manipulations required to show that Equations (182), (183), (184) are equivalent to the system of equations in Equation (172).

Multiply:

- (1) Equation (182) by a_{11} ,
- (2) Equation (183) by a_{12} , and
- (3) Equation (184) by a_{13}

Set the sum of the left sides of the resulting equations equal to the sum of the right sides of the equations. Cancel the terms which are the same on each side, and factor the result. This leads to the following equation:

$$\begin{aligned} & a_{11} a_{11} a_{23} a_{32} x_1 \uplus a_{11} a_{22} a_{13} a_{31} x_1 \uplus a_{11} a_{33} a_{12} a_{21} x_1 \uplus a_{11} a_{22} a_{33} b_1 \uplus a_{11} a_{13} a_{32} b_2 \uplus a_{11} a_{12} a_{23} b_3 \\ & \uplus a_{12} a_{11} a_{23} a_{32} x_2 \uplus a_{12} a_{22} a_{13} a_{31} x_2 \uplus a_{12} a_{33} a_{12} a_{21} x_2 \uplus a_{12} a_{23} a_{31} b_1 \uplus a_{12} a_{11} a_{33} b_2 \uplus a_{12} a_{13} a_{21} b_3 \\ & \uplus a_{13} a_{11} a_{23} a_{32} x_3 \uplus a_{13} a_{22} a_{13} a_{31} x_3 \uplus a_{13} a_{33} a_{12} a_{21} x_3 \uplus a_{13} a_{21} a_{32} b_1 \uplus a_{13} a_{31} a_{12} b_2 \uplus a_{13} a_{11} a_{22} b_3 \\ & = a_{11} a_{11} a_{22} a_{33} x_1 \uplus a_{11} a_{12} a_{23} a_{31} x_1 \uplus a_{11} a_{13} a_{32} a_{21} x_1 \uplus a_{11} a_{23} a_{32} b_1 \uplus a_{11} a_{12} a_{33} b_2 \uplus a_{11} a_{13} a_{22} b_3 \\ & \uplus a_{12} a_{11} a_{22} a_{33} x_2 \uplus a_{12} a_{12} a_{23} a_{31} x_2 \uplus a_{12} a_{13} a_{32} a_{21} x_2 \uplus a_{12} a_{21} a_{33} b_1 \uplus a_{12} a_{13} a_{31} b_2 \uplus a_{12} a_{11} a_{23} b_3 \\ & \uplus a_{13} a_{11} a_{22} a_{33} x_3 \uplus a_{13} a_{12} a_{23} a_{31} x_3 \uplus a_{13} a_{13} a_{32} a_{21} x_3 \uplus a_{13} a_{22} a_{31} b_1 \uplus a_{13} a_{11} a_{32} b_2 \uplus a_{13} a_{12} a_{21} b_3 \end{aligned} \quad (188)$$

which can be factored as:

$$\begin{aligned}
& (a_{11}a_{23}a_{32} \uplus a_{13}a_{22}a_{31} \uplus a_{12}a_{21}a_{33}) (a_{11}x_1 \uplus a_{12}x_2 \uplus a_{13}x_3) \\
& \quad \uplus (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21}) b_1 \\
& = (a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21}) (a_{11}x_1 \uplus a_{12}x_2 \uplus a_{13}x_3) \\
& \quad \uplus (a_{11}a_{23}a_{32} \uplus a_{13}a_{22}a_{31} \uplus a_{12}a_{21}a_{33}) b_1 \quad (189)
\end{aligned}$$

Again, using Lemma 8.27, we see that either

$$a_{11}a_{23}a_{32} \uplus a_{13}a_{22}a_{31} \uplus a_{12}a_{21}a_{33} = a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21} \quad (190)$$

or

$$a_{11}x_1 \uplus a_{12}x_2 \uplus a_{13}x_3 = b_1 \quad (191)$$

Equation (190) corresponds to having a “determinant” equal to zero, and Equation (191) is equivalent to the first equation of the system of equations in Equation (172).

The other two equations can be found using manipulations similar to those above.

This example has shown that the 3×3 system of positive-multiset elements is equivalent to the equations defined by Cramer’s Rule or

$$a_{11}a_{23}a_{32} \uplus a_{13}a_{22}a_{31} \uplus a_{12}a_{21}a_{33} = a_{11}a_{22}a_{33} \uplus a_{12}a_{23}a_{31} \uplus a_{13}a_{32}a_{21}.$$

■

Definition 8.50. Here, we define:

$$\text{par}(n) = \begin{cases} “+” & \text{if } n \text{ is even} \\ “-” & \text{if } n \text{ is odd} \end{cases}$$

Example 8.51. Let us consider Example 8.48 over \mathfrak{R}^+ once again. We consider the system of manipulations required in the second part of that example in a much more systematic way that can easily be extended to larger systems of equations. You may see Theorem 8.101 for the general case over semirings.

$$\text{Expand by first column: } \begin{cases} M^+ = a_{11}M_{11}^+ \uplus a_{21}M_{21}^- \uplus a_{31}M_{31}^+ = \biguplus_{i=1}^3 a_{i1}M_{i1}^{\text{par}(i+1)} \\ M^- = a_{11}M_{11}^- \uplus a_{21}M_{21}^+ \uplus a_{31}M_{31}^- = \biguplus_{i=1}^3 a_{i1}M_{i1}^{\text{par}(i)} \\ N_1^+ = b_1M_{11}^+ \uplus b_2M_{21}^- \uplus b_3M_{31}^+ = \biguplus_{i=1}^3 b_iM_{i1}^{\text{par}(i+1)} \\ N_1^- = b_1M_{11}^- \uplus b_2M_{21}^+ \uplus b_3M_{31}^- = \biguplus_{i=1}^3 b_iM_{i1}^{\text{par}(i)} \end{cases} \quad (192)$$

$$\text{Expand by second column: } \begin{cases} M^+ = a_{12}M_{12}^- \uplus a_{22}M_{22}^+ \uplus a_{32}M_{32}^- = \biguplus_{i=1}^3 a_{i2}M_{i2}^{par(i)} \\ M^- = a_{12}M_{12}^+ \uplus a_{22}M_{22}^- \uplus a_{32}M_{32}^+ = \biguplus_{i=1}^3 a_{i2}M_{i2}^{par(i+1)} \\ N_2^+ = b_1M_{12}^- \uplus b_2M_{22}^+ \uplus b_3M_{32}^- = \biguplus_{i=1}^3 b_iM_{i2}^{par(i)} \\ N_2^- = b_1M_{12}^+ \uplus b_2M_{22}^- \uplus b_3M_{32}^+ = \biguplus_{i=1}^3 b_iM_{i2}^{par(i+1)} \end{cases} \quad (193)$$

$$\text{Expand by third column: } \begin{cases} M^+ = a_{13}M_{13}^+ \uplus a_{23}M_{23}^- \uplus a_{33}M_{33}^+ = \biguplus_{i=1}^3 a_{i3}M_{i3}^{par(i+1)} \\ M^- = a_{13}M_{13}^- \uplus a_{23}M_{23}^+ \uplus a_{33}M_{33}^- = \biguplus_{i=1}^3 a_{i3}M_{i3}^{par(i)} \\ N_3^+ = b_1M_{13}^+ \uplus b_2M_{23}^- \uplus b_3M_{33}^+ = \biguplus_{i=1}^3 b_iM_{i3}^{par(i+1)} \\ N_3^- = b_1M_{13}^- \uplus b_2M_{23}^+ \uplus b_3M_{33}^- = \biguplus_{i=1}^3 b_iM_{i3}^{par(i)} \end{cases} \quad (194)$$

Rewriting Equation (186) as three equations, we get:

$$M^+x_1 \uplus N_1^- = M^-x_1 \uplus N_1^+ \quad (195a)$$

$$M^+x_2 \uplus N_2^- = M^-x_2 \uplus N_2^+ \quad (195b)$$

$$M^+x_3 \uplus N_3^- = M^-x_3 \uplus N_3^+ \quad (195c)$$

If we want to use these equations ((195a),(195b), and (195c)) to get the first equation of the system (172), then we consider:

$$a_{11} \boxtimes (195a) \uplus a_{12} \boxtimes (195b) \uplus a_{13} \boxtimes (195c) \quad (196)$$

where the left sides of the three products are added together and the right sides of the equations are added together.

This leads to the following very long equation:

$$\begin{aligned} & a_{11}a_{11}M_{11}^+ \uplus a_{11}a_{21}M_{21}^- \uplus a_{11}a_{31}M_{31}^+ \uplus a_{12}a_{12}M_{12}^- \uplus a_{12}a_{22}M_{22}^+ \uplus a_{12}a_{32}M_{32}^- \\ & \uplus a_{13}a_{13}M_{13}^+ \uplus a_{13}a_{23}M_{23}^- \uplus a_{13}a_{33}M_{33}^+ \uplus a_{11}b_1M_{11}^- \uplus a_{11}b_2M_{21}^+ \uplus a_{11}b_3M_{31}^- \\ & \uplus a_{12}b_1M_{12}^+ \uplus a_{12}b_2M_{22}^- \uplus a_{12}b_3M_{32}^+ \uplus a_{13}b_1M_{13}^- \uplus a_{13}b_2M_{23}^+ \uplus a_{13}b_3M_{33}^- \\ & = a_{11}a_{11}M_{11}^- \uplus a_{11}a_{21}M_{21}^+ \uplus a_{11}a_{31}M_{31}^- \uplus a_{12}a_{12}M_{12}^+ \uplus a_{12}a_{22}M_{22}^- \uplus a_{12}a_{32}M_{32}^+ \\ & \uplus a_{13}a_{13}M_{13}^- \uplus a_{13}a_{23}M_{23}^+ \uplus a_{13}a_{33}M_{33}^- \uplus a_{11}b_1M_{11}^+ \uplus a_{11}b_2M_{21}^- \uplus a_{11}b_3M_{31}^+ \\ & \uplus a_{12}b_1M_{12}^- \uplus a_{12}b_2M_{22}^+ \uplus a_{12}b_3M_{32}^- \uplus a_{13}b_1M_{13}^+ \uplus a_{13}b_2M_{23}^- \uplus a_{13}b_3M_{33}^+ \end{aligned} \quad (197)$$

In this case the terms with b_2 and b_3 must be the same. If we consider the part of Equation (197) that have the terms containing b_2 , we get the following subequation:

$$(a_{11}M_{21}^+ \uplus a_{12}M_{22}^- \uplus a_{13}M_{23}^+)b_2 = (a_{11}M_{21}^- \uplus a_{12}M_{22}^+ \uplus a_{13}M_{23}^-)b_2 \quad (198)$$

These two sides are equal from the method of False Cofactor Expansion, and cancel from each side of Equation (197). Likewise, we can show, similarly, that the terms involving b_3 will also cancel from each side of the equation. This same method works for the other two equations in the system. Therefore, we have seen that Equation (172) is equivalent to the set of solutions to Equations ((182), (183), (184)). ■

Another Mapping

As we look at the progression from Equation (111) to Equation (114), we notice a method that can be used to convert an equation with one variable (an mset) into a 2×2 system which has all mset elements positive. This same method can be used to change a $n \times n$ -system into a $(2n) \times (2n)$ -system which has all elements positive. With this in mind, we may further our investigation by considering the following function $\Psi(\alpha)$.

Definition 8.52. Suppose $\alpha = (\alpha^+, \alpha^-) \in \mathbb{R}$. Define:

$$\Psi(\alpha) = \begin{bmatrix} \alpha^+ & -\alpha^- \\ -\alpha^- & \alpha^+ \end{bmatrix} \quad (199)$$

Proposition 8.53. We list some properties of this mapping. Let $\alpha, \beta \in \mathbb{3}^{\mathbb{R}}$, and $c \in \mathbb{R}$.

(1) $\Psi(\alpha \uplus \beta) = \Psi(\alpha) \uplus \Psi(\beta)$

(2) $\Psi(\alpha \boxtimes \beta) = \Psi(\alpha) \boxtimes \Psi(\beta)$

(3) $\Psi(c \boxtimes \alpha) = \begin{cases} \Psi(\{c\alpha^+, c\alpha^-\}) = c\Psi(\alpha) & \text{if } c > 0 \\ \Psi(\{c\alpha^-, c\alpha^+\}) = c\Psi(-\alpha) & \text{if } c < 0 \end{cases}$

(4) $\Psi(\alpha) = \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \iff \alpha = \emptyset$

(5)

$$\begin{aligned} \Psi(\alpha) = \Psi(\beta) &\iff \begin{bmatrix} \alpha^+ & -\alpha^- \\ -\alpha^- & \alpha^+ \end{bmatrix} = \begin{bmatrix} \beta^+ & -\beta^- \\ -\beta^- & \beta^+ \end{bmatrix} \\ &\iff \alpha = \beta \end{aligned}$$

With the results of this section at our disposal we can show that Cramer's Rule works in the case of general systems of equations over $\mathbb{3}^{\mathbb{R}}$ with the aid of the following theorem.

Theorem 8.54. If $Ax=b$ has a solution, then the system is equivalent to the system of Cramer's Rule equations, and, conversely, a system of Cramer's Rule equations is equivalent to the system $Ax=b$. Here $A \in \mathbb{3}_{n \times n}^{\mathbb{S}}$, $x \in \mathbb{3}_{n \times 1}^{\mathbb{S}}$, and $b \in \mathbb{3}_{n \times 1}^{\mathbb{S}}$

Proof. Suppose

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \boxtimes \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad (200)$$

We apply Ψ to the system to get the $(2n) \times (2n)$ system:

$$\begin{bmatrix} a_{11}^+ & -a_{11}^- & \cdots & a_{1n}^+ & -a_{1n}^- \\ -a_{11}^- & a_{11}^+ & & -a_{1n}^- & a_{1n}^+ \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1}^+ & -a_{n1}^- & \cdots & a_{nn}^+ & -a_{nn}^- \\ -a_{n1}^- & a_{n1}^+ & & -a_{nn}^- & a_{nn}^+ \end{bmatrix}_{2n \times 2n} \boxtimes \begin{bmatrix} x_1^+ & -x_1^- \\ -x_1^- & x_1^+ \\ \vdots & \vdots \\ x_n^+ & -x_n^- \\ -x_n^- & x_n^+ \end{bmatrix}_{2n \times 2n} = \begin{bmatrix} b_1^+ & -b_1^- \\ -b_1^- & b_1^+ \\ \vdots & \vdots \\ b_n^+ & -b_n^- \\ -b_n^- & b_n^+ \end{bmatrix}_{2n \times 2n} \quad (201)$$

Although Equation (201) represents two separate systems of equations, it can be shown that the two systems are equivalent by considering the block-diagonal permutation matrix $P=$

$$\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & 1 & 0 \end{bmatrix}_{2n \times 2n}$$

From this we get a system of $2n$ equations associated with Cramer's Rule, in which each equation contains only 1 variable. So we see that:

Equation (200) \iff Equation (201) \iff Cramer's Rule system of equations – Equation (187). \square

8.8 Polynomials in $3^{\mathbb{S}}$

In this section we investigate some of the properties of polynomial forms over $3^{\mathbb{S}}$, or $3^{\mathbb{S}}[x]$.

Definition 8.55. A polynomial $p \in 3^{\mathbb{S}}[x]$ is an ordered list, with each element from $3^{\mathbb{S}}$.

$p = (p_0, p_1, \dots, p_n, \emptyset, \dots, \emptyset)$, where $p_i \in 3^{\mathbb{S}}$.

We may write the polynomial as $p_0 \uplus p_1 \lambda \uplus p_2 \lambda^2 \uplus \cdots \uplus p_n \lambda^n$, where λ is a “place holder”.

The degree of p , written ∂p , is n .

Definition 8.56 (Addition). If $p, q \in 3^{\mathbb{S}}[x]$, then:

$p \oplus q \equiv (p_0 \uplus q_0, p_1 \uplus q_1, \dots, p_n \uplus q_n, \dots)$.

Proposition 8.57. The following are properties of addition in $3^{\mathbb{S}}[x]$.

- (1) Addition is closed in $3^{\mathbb{S}}[x]$.
- (2) Addition is commutative, since \uplus is commutative.
- (3) The empty polynomial $\emptyset(x) = (\emptyset, \emptyset, \dots, \emptyset) = \emptyset \uplus \emptyset \lambda \uplus \emptyset \lambda^2, \dots$ is the additive identity.
- (4) Addition is associative, since \uplus is associative.

Definition 8.58 (Scalar Multiplication). If $\gamma \in 3^{\mathbb{S}}$, and $p \in 3^{\mathbb{S}}[x]$, then

$$\gamma \boxtimes p = (\gamma \boxtimes p_0, \gamma \boxtimes p_1, \dots, \gamma \boxtimes p_n, \emptyset, \dots)$$

Proposition 8.59. If $\gamma, \delta \in 3^{\mathbb{S}}$, and $p, q \in 3^{\mathbb{S}}[x]$, then

- (1) $\gamma \boxtimes (p \oplus q) = \gamma \boxtimes p \oplus \gamma \boxtimes q$
- (2) $(\gamma \uplus \delta) \boxtimes p = \gamma \boxtimes p \oplus \delta \boxtimes p$
- (3) $\{1\} \boxtimes p = p$

Definition 8.60 (Polynomial Multiplication). If $p, q \in 3^{\mathbb{S}}[x]$, then $p \star q = s = (s_1, s_2, \dots, s_n, \dots)$, where $s_k = p_0 \boxtimes q_k \uplus p_1 \boxtimes q_{k-1} \uplus \dots \uplus p_k \boxtimes q_0$

This is just the usual definition of polynomial multiplication.

Example 8.61.

$$(p_0, p_1, p_2) \star (q_0, q_1, q_2) = (p_0 \boxtimes q_0, p_0 \boxtimes q_1 \uplus p_1 \boxtimes q_0, p_0 \boxtimes q_2 \uplus p_1 \boxtimes q_1 \uplus p_2 \boxtimes q_0, p_1 \boxtimes q_2 \uplus p_2 \boxtimes q_1, p_2 \boxtimes q_2) \quad \blacksquare$$

Proposition 8.62. The following are some properties of multiplication of polynomials in $3^{\mathbb{S}}[x]$.

- (1) \star is commutative since \boxtimes is commutative.

$$p \star q = q \star p.$$
- (2) \star is associative.

$$p \star (q \star r) = (p \star q) \star r$$

This follows from the associativity of matrix multiplication.
- (3) \star is distributive over \oplus .

$$p \star (q \oplus r) = p \star q \oplus p \star r$$
- (4) If $\gamma \in \mathbb{S}$, then $\gamma \star (p \star q) = (\gamma \star p) \star q$

We can define the following Evaluation Map.

Definition 8.63. Given $A \in \mathbb{S}_{n \times n}$, define

$$\varphi_A(p) = p_0 \boxtimes I_N \uplus p_1 \boxtimes A \uplus p_2 \boxtimes A^2 \cdots p_n \boxtimes A^n \quad (202)$$

Proposition 8.64. The following are some properties of the evaluation map.

(1)

$$\begin{aligned} \varphi_A(p \oplus q) &= \varphi_A(p_0 \uplus q_0, p_1 \uplus q_1, \dots) \\ &= (p_0 \uplus q_0)I \uplus (p_1 \uplus q_1)A \uplus \dots \\ &= (p_0I \uplus p_1A \uplus p_2A^2 \uplus \dots) \uplus (q_0I \uplus q_1A \uplus q_2A^2 \uplus \dots) \\ &= \varphi_A(p) \uplus \varphi_A(q) \end{aligned} \quad (203)$$

(2)

$$\begin{aligned} \varphi_A(\gamma \star p) &= \varphi_A(\gamma \boxtimes p_0, \gamma \boxtimes p_1, \dots) \\ &= (\gamma \boxtimes p_0)I \uplus (\gamma \boxtimes p_1)A \uplus \dots \\ &= \gamma \boxtimes (p_0I \uplus p_1A \uplus p_2A^2 \uplus \dots) \\ &= \gamma \boxtimes \varphi_A(p) \end{aligned} \quad (204)$$

(3)

$$\varphi_A(p \star q) = \varphi_A(p) \boxtimes \varphi_A(q) \quad (205)$$

8.9 The Characteristic Polynomial in the Mset Algebra

We can convert the problem of equating the sums on each side of an equation into a problem of set inclusion, namely, showing that the msets on each side of the equation sign equal (contain exactly the same elements). We will use the mset Algebra from the previous section to prove the main result of this section.

Suppose that we are given a set \mathbb{S} , along with two binary operations:

- (a) \oplus , which is associative and commutative, with additive identity.
- (b) \odot , which is associative and commutative, with multiplicative identity.
- (c) \odot is distributive over \oplus .

A matrix, $A \in \mathbb{S}_{n \times n}$, can be viewed as the adjacency matrix of a weighted directed graph. Given two matrices $A, B \in \mathbb{S}_{n \times n}$, then define:

1. $[A \oplus B]_{ij} = A_{ij} \oplus B_{ij}$
2. $[A \odot B]_{ij} = \bigoplus_{k=1}^n (A_{ik} \odot A_{kj})$
3. $[A^m]_{ij} = \bigoplus_{k_1=1}^n \bigoplus_{k_2=1}^n \cdots \bigoplus_{k_{m-1}=1}^n (A_{ik_1} \odot A_{k_1 k_2} \odot \cdots \odot A_{k_{m-1} j})$

Notation

Let $A \in \mathbb{S}_{n \times n}$. Let $S_n = \{1, 2, \dots, n\}$. Let C_k^n be the set of all permutations of k elements from S_n . Let A_ϕ^φ be the submatrix of A obtained by taking the rows of A indexed by φ and the columns of A indexed by ϕ , $\varphi \subseteq S_n$, and $\phi \subseteq S_n$. The principal submatrices of A are denoted by A_ϕ^ϕ , where $\phi \subseteq S_n$.

Partial Permutations

A *partial permutation* of S_n is a bijection, σ , of a subset of S_n onto itself. The domain of σ is denoted by $\text{dom}(\sigma)$, and the cardinality of σ is denoted by $|\sigma|$. A complete permutation is defined as a partial permutation σ with $\text{dom}(\sigma) = S_n$. The completion of a partial permutation, denoted as $\hat{\sigma}$, is defined by:

$$\hat{\sigma} = \begin{cases} \sigma(i) & \text{if } i \in \text{dom}(\sigma) \\ i & \text{if } i \in S_n \setminus \text{dom}(\sigma) \end{cases} \quad (206)$$

We use the conventional definition of the signature of a permutation:

$$\text{sgn}(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is an odd permutation} \\ 1 & \text{if } \sigma \text{ is an even permutation} \end{cases} \quad (207)$$

The signature of a partial permutation, denoted as $\text{sig}(\sigma)$, is defined by:

$$\text{sig}(\sigma) = (-1)^k \text{sgn}(\sigma) \quad (208)$$

where $\text{sgn}(\sigma)$ is the conventional signature of $\hat{\sigma}$.

Every partial permutation has a unique representation as a set of disjoint cycles. For example:

$\sigma = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$, we have cycle representation $\{(2,4)\}$.

$\hat{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 4 & 3 & 2 & 5 & \cdots & n \end{pmatrix}$, which has cycle representation $\{(1)(2,4)(3)(5) \dots (n)\} = \{(2,4)\}$.

For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, the cycle representation is $\{(1)(2,3)\} = \{(2,3)\}$.

This shows that the parity of σ is the same as the parity of $\hat{\sigma}$, i.e. $\text{sgn}(\hat{\sigma}) = \text{sgn}(\sigma)$.

The unique partial permutation of degree zero has the empty set as its cycle representation. Here we define $\text{sig}(\sigma)=1$.

If σ is a *cyclic* partial permutation of degree k , then $\text{sig}(\sigma)=(-1)^k(-1)^{k-1} = -1$. It follows that for any partial permutation, σ , $\text{sig}(\sigma)=(-1)^r$, where r is the number of cycles appearing in the cycle representation of σ .

If σ is a partial permutation, then the weight of σ is defined as: $\omega(\sigma) = \bigodot_{i \in \text{dom}(\sigma)} a_{i, \sigma(i)}$.

Definition 8.65. Define the following msets:

$$(a) \quad \sigma_0^e = \{1\} \quad \sigma_0^o = \emptyset \quad \sigma_1^o = \emptyset$$

$$(b) \quad \sigma_k^e = \{\omega(\sigma) \mid \sigma \in C_k^n \text{ and } \text{sgn}(\sigma) = \text{even}\} \quad \forall k=1, \dots, n$$

$$(c) \quad \sigma_k^o = \{\omega(\sigma) \mid \sigma \in C_k^n \text{ and } \text{sgn}(\sigma) = \text{odd}\} \quad \forall k=2, \dots, n$$

Define the following two formal polynomials, where each element of the sequence is viewed as a multiset.

$$p_A^+(\lambda) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q)} \boxtimes \lambda^{n-q} \right) = \left(\sigma_n^{\text{par}(n)}, \sigma_{n-1}^{\text{par}(n-1)}, \dots, \sigma_1^o, \sigma_0^e \right) \quad (209)$$

$$p_A^-(\lambda) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q+1)} \boxtimes \lambda^{n-q} \right) = \left(\sigma_n^{\text{par}(n+1)}, \sigma_{n-1}^{\text{par}(n)}, \dots, \sigma_1^e, \sigma_0^o \right) \quad (210)$$

By definition, we have $\sigma_1^o = \emptyset$, $\sigma_0^e = \{1\}$, and $\sigma_0^o = \emptyset$.

Since “ \boxtimes ” is replaced by “ \odot ” from the set \mathbb{S} , then we will drop “ \boxtimes ”, and simply write “ \odot ” instead. Or we may simply drop the symbol altogether and assume the proper multiplication symbol.

Example 8.66. Consider the matrix $A = \begin{bmatrix} \{1\} & \{2\} & \{3\} \\ \{4\} & \{7\} & \{9\} \\ \{5\} & \{1\} & \{3\} \end{bmatrix}$, with $\oplus \equiv \text{“}\uplus\text{”}$, and $\odot \equiv \text{“}\cdot\text{”}$ ■

We have $\sigma_0^e = \{1\}$, $\sigma_0^o = \emptyset$,
 $\sigma_1^e = \{1, 3, 7\}$, $\sigma_1^o = \emptyset$,
 $\sigma_2^e = \{3, 7, 21\}$, $\sigma_2^o = \{8, 9, 15\}$,
 $\sigma_3^e = \{12, 21, 90\}$, $\sigma_3^o = \{9, 24, 105\}$

$p_A^+(\lambda) = \{1\} \boxtimes \lambda^3 \uplus \emptyset \boxtimes \lambda^2 \uplus \{3, 7, 21\} \boxtimes \lambda \uplus \{9, 24, 105\} = \{1\} \lambda^3 \uplus \{3, 7, 21\} \lambda \uplus \{9, 24, 105\}$ and
 $p_A^-(\lambda) = \emptyset \boxtimes \lambda^3 \uplus \{1, 3, 7\} \boxtimes \lambda^2 \uplus \{8, 9, 15\} \boxtimes \lambda \uplus \{12, 21, 90\} = \{1, 3, 7\} \lambda^2 \uplus \{8, 9, 15\} \lambda \uplus \{12, 21, 90\}$

To define an evaluation map, we make the following definitions:

$$p_A^+(\{a\}) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q)} \boxtimes \{a\}^{n-q} \right) \quad \text{and} \quad p_A^-(a) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q+1)} \boxtimes \{a\}^{n-q} \right) \quad (211)$$

If $A_{n \times n} \in \mathbb{S}_{n \times n}$ is a matrix, then

$$p_A^+(A) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q)} \boxtimes A^{n-q} \right) \quad \text{and} \quad p_A^-(A) = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q+1)} \boxtimes A^{n-q} \right) \quad (212)$$

We note here that each evaluation in Equation (212) is a matrix a mset in each entry.

$$[p_A^+(A)]_{ij} = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q)} \boxtimes [A^{n-q}]_{ij} \right) \quad \text{and} \quad [p_A^-(A)]_{ij} = \bigoplus_{q=0}^n \left(\sigma_q^{\text{par}(q+1)} \boxtimes [A^{n-q}]_{ij} \right) \quad (213)$$

Example 8.67. Consider the matrix in Example (8.66) along with $p_A^+(\lambda)$ and $p_A^-(\lambda)$. ■

1. Calculate $p_A^+(\{7\})$, and $p_A^-(\{7\})$.

$$\begin{aligned} p_A^+(\{7\}) &= \{1\} \boxtimes \{7\}^3 \uplus \{3, 7, 21\} \boxtimes \{7\} \uplus \{9, 24, 105\} \\ &= \{343\} \uplus \{21, 49, 147\} \uplus \{9, 24, 105\} = \{9, 21, 24, 49, 105, 147, 343\} \end{aligned}$$

$$\begin{aligned} p_A^-(\{7\}) &= \{1, 3, 7\} \boxtimes \{7\}^2 \uplus \{8, 9, 15\} \boxtimes \{7\} \uplus \{12, 21, 90\} \\ &= \{49, 147, 343\} \uplus \{56, 63, 105\} \uplus \{12, 21, 90\} = \{12, 21, 49, 56, 63, 90, 105, 147, 343\} \end{aligned}$$

2. Calculate $p_A^+(A)$, and $p_A^-(A)$.

$$A^2 = \begin{bmatrix} \{1, 8, 15\} & \{2, 3, 14\} & \{3, 9, 18\} \\ \{4, 28, 45\} & \{8, 9, 49\} & \{12, 27, 63\} \\ \{4, 5, 15\} & \{3, 7, 10\} & \{9, 9, 15\} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \begin{Bmatrix} 1 & 8 & 15 \\ 8 & 56 & 90 \\ 12 & 15 & 45 \end{Bmatrix} & \begin{Bmatrix} 2 & 3 & 14 \\ 16 & 18 & 98 \\ 9 & 21 & 30 \end{Bmatrix} & \begin{Bmatrix} 3 & 9 & 18 \\ 24 & 54 & 126 \\ 27 & 27 & 45 \end{Bmatrix} \\ \begin{Bmatrix} 4 & 32 & 60 \\ 28 & 196 & 315 \\ 36 & 45 & 135 \end{Bmatrix} & \begin{Bmatrix} 8 & 12 & 56 \\ 56 & 63 & 343 \\ 27 & 63 & 90 \end{Bmatrix} & \begin{Bmatrix} 12 & 36 & 72 \\ 84 & 189 & 441 \\ 81 & 81 & 135 \end{Bmatrix} \\ \begin{Bmatrix} 5 & 40 & 75 \\ 4 & 28 & 45 \\ 12 & 15 & 45 \end{Bmatrix} & \begin{Bmatrix} 10 & 15 & 70 \\ 8 & 9 & 49 \\ 9 & 21 & 30 \end{Bmatrix} & \begin{Bmatrix} 15 & 45 & 90 \\ 12 & 27 & 63 \\ 27 & 27 & 45 \end{Bmatrix} \end{bmatrix}$$

$$p_A^+(A) = \{1\} \boxtimes A^3 \uplus \{3, 7, 21\} \boxtimes A \uplus \{9, 24, 105\} \boxtimes I$$

$$= \left[\begin{array}{ccc} \begin{pmatrix} 1 & 8 & 15 \\ 8 & 56 & 90 \\ 12 & 15 & 45 \\ 3 & 7 & 21 \\ 9 & 24 & 105 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 14 \\ 16 & 18 & 98 \\ 9 & 21 & 30 \\ 6 & 14 & 42 \end{pmatrix} & \begin{pmatrix} 3 & 9 & 18 \\ 24 & 54 & 126 \\ 27 & 27 & 45 \\ 9 & 21 & 63 \end{pmatrix} \\ \begin{pmatrix} 4 & 32 & 60 \\ 28 & 196 & 315 \\ 36 & 45 & 135 \\ 12 & 28 & 84 \end{pmatrix} & \begin{pmatrix} 8 & 12 & 56 \\ 56 & 63 & 343 \\ 27 & 63 & 90 \\ 21 & 49 & 147 \\ 9 & 24 & 105 \end{pmatrix} & \begin{pmatrix} 12 & 36 & 72 \\ 84 & 189 & 441 \\ 81 & 81 & 135 \\ 27 & 63 & 189 \end{pmatrix} \\ \begin{pmatrix} 5 & 40 & 75 \\ 4 & 28 & 45 \\ 12 & 15 & 45 \\ 15 & 35 & 105 \end{pmatrix} & \begin{pmatrix} 10 & 15 & 70 \\ 8 & 9 & 49 \\ 9 & 21 & 30 \\ 3 & 7 & 21 \end{pmatrix} & \begin{pmatrix} 15 & 45 & 90 \\ 12 & 27 & 63 \\ 27 & 27 & 45 \\ 9 & 21 & 63 \\ 9 & 24 & 105 \end{pmatrix} \end{array} \right]$$

$$p_A^-(A) = \{1, 3, 7\} \boxtimes A^2 \uplus \{8, 9, 15\} \boxtimes A \uplus \{12, 21, 90\} \boxtimes I$$

$$= \left[\begin{array}{ccc} \begin{pmatrix} 1 & 8 & 15 \\ 3 & 24 & 45 \\ 7 & 56 & 105 \\ 8 & 9 & 15 \\ 12 & 21 & 90 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 14 \\ 6 & 9 & 42 \\ 14 & 21 & 98 \\ 16 & 18 & 30 \end{pmatrix} & \begin{pmatrix} 3 & 9 & 18 \\ 9 & 27 & 54 \\ 21 & 63 & 126 \\ 24 & 27 & 45 \end{pmatrix} \\ \begin{pmatrix} 4 & 28 & 45 \\ 12 & 84 & 135 \\ 28 & 196 & 315 \\ 32 & 36 & 60 \end{pmatrix} & \begin{pmatrix} 8 & 9 & 49 \\ 24 & 27 & 147 \\ 56 & 63 & 343 \\ 56 & 63 & 105 \\ 12 & 21 & 90 \end{pmatrix} & \begin{pmatrix} 12 & 27 & 63 \\ 36 & 81 & 189 \\ 84 & 189 & 441 \\ 72 & 81 & 135 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 15 \\ 12 & 15 & 45 \\ 28 & 35 & 105 \\ 40 & 45 & 75 \end{pmatrix} & \begin{pmatrix} 3 & 7 & 10 \\ 9 & 21 & 30 \\ 21 & 49 & 70 \\ 8 & 9 & 15 \end{pmatrix} & \begin{pmatrix} 9 & 9 & 15 \\ 27 & 27 & 45 \\ 63 & 63 & 105 \\ 24 & 27 & 45 \\ 12 & 21 & 90 \end{pmatrix} \end{array} \right]$$

Looking closely at the two matrices above, it may be noted that the two matrices have the same

elements in each mset in each element of the corresponding matrices. The following theorem shows that this is not a coincidence.

Theorem 8.68. $p_A^+(A) = p_A^-(A)$, with the definitions of Equations (212).

Proof. The following proof is adapted from [30].

A path $\pi \in \{1, 2, \dots, n\}$ is a sequence of pairs $\{(i_0, i_1), (i_1, i_2), \dots, (i_{t-1}, i_t)\}$ where each i_k belongs to S_n . The number t is called the length of π and is denoted by $|\pi|$. The i_j s are called the vertices of π and the pairs (i_{k-1}, i_k) are the edges of π . The vertex i_0 , denoted $\alpha(\pi)$, is the beginning of the path, and the vertex i_t , denoted $\beta(\pi)$, is the end of the path π . The weight of the path π , denoted $\omega(\pi)$, is $\omega(\pi) = \prod_{k=0}^{t-1} a_{i_k i_{k+1}}$. We make the convention that for each $i \in S_n$, there is a unique (empty) path π_i such that $|\pi| = 0, \alpha(\pi_i) = \beta(\pi_i) = i$, and $\omega(\pi) = 1$.

Let $1 \leq i, j \leq n$ and let T_{ij}^+ be the set of pairs (σ, π) , where σ is a partial permutation and π is a path, which satisfy

$|\sigma| + |\pi| = n, \alpha(\pi) = i, \omega(\pi) = j, \text{sig}(\sigma) = 1$. The set T_{ij}^- is defined analogously as, except the conditions satisfied are

$|\sigma| + |\pi| = n, \alpha(\pi) = i, \omega(\pi) = j, \text{sig}(\sigma) = -1$. Denote $T_{ij}^+ \cup T_{ij}^-$ by T_{ij} .

For $k \geq 0$,

$$(A^k)_{ij} = \bigsqcup_{\left\{ \begin{array}{l} |\pi| = k \\ \alpha(\pi) = i \\ \beta(\pi) = j \end{array} \right\}} \omega(\pi)$$

(The convention regarding paths of length zero makes this formula valid for $k=0$ as well.) It follows that

$$p^+(A)_{ij} = \bigsqcup_{(\sigma, \pi) \in T_{ij}^+} \omega(\sigma) \boxtimes \omega(\pi) \quad p^-(A)_{ij} = \bigsqcup_{(\sigma, \pi) \in T_{ij}^-} \omega(\sigma) \boxtimes \omega(\pi)$$

The theorem asserts that these two sums are equal for all $\{i, j\} \in S_n$. This follows from the following lemma. □

Lemma 8.69. For each pair (i, j) , with $1 \leq i, j \leq n$, there is a bijection $\eta_{ij}: T_{ij}^+ \rightarrow T_{ij}^-$ in such a way that $\eta_{ij}(\sigma, \pi) = (\sigma', \pi')$ implies $|\sigma| \odot |\pi| = |\sigma'| \odot |\pi'|$.

Proof. Represent each pair $(\sigma, \pi) \in T_{ij}$ by a directed graph with vertices $\{1, 2, \dots, n\}$, and two classes of edges:

$U = \{(i, \sigma(i)) \mid i \in \text{dom}(\sigma)\}$, and $V = \{(k, \ell) \mid (k, \ell) \text{ is an edge of } \pi\}$

This graph will, in general, contain multiple edges, since π may traverse the same edges more than once, or the same edge may appear in both U and V . $\omega(\sigma)\omega(\pi)$ is the product, with multiplicities taken into account, of all the $a_{k\ell}$, where (k, ℓ) is an edge of the graph associated to (σ, π) .

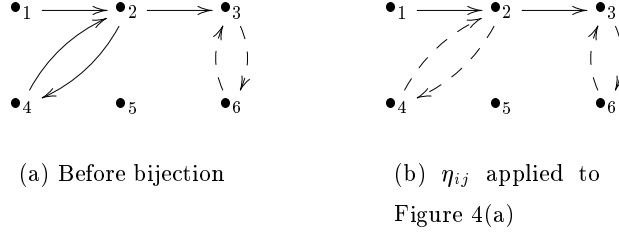


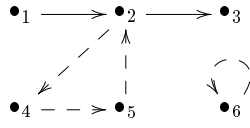
Figure 4: Graph with first property

Two examples are illustrated below (see Figure 4). Edges of U are denoted by dashed lines, and the edges of V are denoted by solid lines.

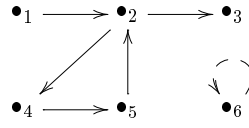
Consider the nodes $\{i_0, i_1, \dots, i_q\}$. Then π is a path (set of edges) connecting the nodes in that order — i.e. let $\pi = \{(i_0, i_1), \dots, (i_{q-1}, i_q)\}$. There is a smallest integer $v \geq 0$ such that either $i_u = i_v$ for some u less than v or $i_v \in \text{dom}(\sigma)$. So there exists a smallest integer v such that either $i_u = i_v$ for some u less than v or $i_v \in \text{dom}(\sigma)$. We let $v =$ smallest of the two cases if both exist. Indeed, if there is no pair of distinct indices u and v such that $i_u = i_v$, then π has $|\pi| + 1$ distinct vertices; the edges $(i, \sigma(i))$ of U account for $|\sigma|$ different vertices, and $|\sigma| + |\pi| + 1 = n + 1$, thus there must be a vertex of π in $\text{dom}(\sigma)$. Furthermore, this smallest v cannot have both properties, for if $i_v \in \text{dom}(\sigma)$ and $i_u = i_v$ for some u with $u < v$, then we have an index u smaller than v with $i_u \in \text{dom}(\sigma)$. In Figure 4(a), $v = 3$, since $i_0 = 1, i_1 = 2, i_2 = 4, i_3 = 2 = i_1$. In Figure 5(a) $v = 1$, since $i_0 = 1, i_1 = 2 \in \text{dom}(\sigma)$. *Case 1:* If $i_u = i_v$ for $u < v$, then the loop $\{(i_u, i_{u+1}), \dots, (i_{v-1}, i_v)\}$ is removed from π and adjoined as a new cycle to σ . (Observe that none of the vertices in this loop already belongs to $\text{dom}(\sigma)$.) In this instance we define $\eta_{ij}(\sigma, \pi) = (\sigma', \pi')$, where σ' is the resulting longer partial permutation and π' is the resulting shorter path. An example of this is given in Figure 4(a), with the result of applying η_{ij} given in Figure 4(b). Here, $\eta_{ij}(\sigma, \pi) = (\sigma', \pi')$. In Figure 4(a), $\sigma = \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}$, and $\pi = \{(1,2), (2,4), (4,2), (2,3)\}$, and in Figure 4(b), $\sigma' = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 4 & 6 & 2 & 3 \end{pmatrix}$, and $\pi' = \{(1,2), (2,3)\}$.

Case 2: If $i_v \in \text{dom}(\sigma)$, then we define $\eta_{ij}(\sigma, \pi) = (\sigma', \pi')$ where σ' is formed by removing the cycle containing i_v from σ , and where π' is formed by inserting this cycle into π . The results of applying η_{ij} to the graph in Figure 5(a) is illustrated in Figure 5(b). In this example $\eta_{ij}(\sigma, \pi) = (\sigma', \pi')$, where: $\sigma = \begin{pmatrix} 2 & 4 & 5 & 6 \\ 4 & 5 & 2 & 6 \end{pmatrix}$, and $\pi = \{(1,2), (2,3)\}$, and in Figure 4(b), $\sigma' = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, and $\pi' = \{(1,2), (2,3), (2,4), (4,5), (5,2)\}$.

Since the number of cycles of σ' differs by 1 from the number of cycles in σ , $\text{sig}(\sigma) = (-1) \cdot \text{sig}(\sigma')$, and thus η_{ij} maps T_{ij}^+ into T_{ij}^- and vice versa. It is easy to see that the composition $\eta_{ij} \circ \eta_{ij}$ is the identity map on T_{ij} , consequently the restriction of η_{ij} to T_{ij}^+ is a bijection of T_{ij}^+ onto T_{ij}^- .



(a) Before bijection



(b) η_{ij} applied to Figure 5(a)

Figure 5: Graph with second property

Finally, $\omega(\sigma)\omega(\pi) = \omega(\sigma')\omega(\sigma')$ — indeed, nothing has changed in the graph of (σ, π) except the “colors” of the edges. The proof is complete. \square

Applications

From the previous section we have the fact that, for a given matrix $A \in \mathbb{S}_{n \times n}$, $p_A^+(A) = p_A^-(A)$ using the definitions of Equations 212. This is a very strong result in that it is true for a general semiring. The list of applications can be seen for any set \mathbb{S} , with two appropriate operations, which is a semiring, ring, or even a field.

Theorem 8.68 states that the two sides of the equation are matrices of msets, and the two matrices are identical. We can apply this theorem to the different algebra systems in which we have been working. Theorem 8.68 is a generalized version of the Cayley-Hamilton Theorem.

Real Numbers System In the Real number system we use $\uplus \equiv "+"$, and $\odot \equiv "-"$. Here, Theorem 8.68 becomes an alternative form of the Cayley-Hamilton Theorem.

Example 8.70. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 5 & 1 & 3 \end{bmatrix} \in \mathbb{R}_{n \times n}$. ■

In this example:

$$\begin{aligned} p_A^+(A) &= \{1\} \boxtimes A^3 \uplus \{3, 7, 21\} \boxtimes A \uplus \{9, 24, 105\} \boxtimes I \\ p_A^-(A) &= \{1, 3, 7\} \boxtimes A^2 \uplus \{8, 9, 15\} \boxtimes A \uplus \{12, 21, 90\} \boxtimes I \\ A^3 + (31)A + (138)I &= (11)A^2 + (32)A + (123)I \end{aligned}$$

and it can be checked that each side of the equation is: $\begin{bmatrix} 419 & 273 & 426 \\ 975 & 1073 & 1410 \\ 424 & 252 & 582 \end{bmatrix}$

Since this example is in the conventional algebra, with subtraction, the above equation becomes $A^3 - 11A^2 - 1A + 15I = 0$, which is the equation given by the ordinary Cayley-Hamilton Theorem.

Max-Times Algebra When Theorem 8.68 is applied to the Max-Times Algebra, we get an equation which is satisfied by the matrix. In \mathfrak{M}^* ,

Theorem 8.71 (Cayley Hamilton Theorem in Max-Times).

In Max-Times, $p^+(A) = p^-(A)$, where the equation $p_A^+(x) = p_A^-(x)$ is given by definitions of Equations 212.

Proof. Use of Theorem 8.68 with $\uplus \equiv "max"$, and $\odot \equiv "x"$ □

Example 8.72. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 5 & 1 & 3 \end{bmatrix} \in \mathfrak{M}_{n \times n}^*$. ■

In this example:

$$\begin{aligned} p_A^+(A) &= \{1\} \boxtimes A^3 \uplus \{3, 7, 21\} \boxtimes A \uplus \{9, 24, 105\} \boxtimes I \\ p_A^-(A) &= \{1, 3, 7\} \boxtimes A^2 \uplus \{8, 9, 15\} \boxtimes A \uplus \{12, 21, 90\} \boxtimes I \end{aligned}$$

$$A^3 \oplus (21)A \oplus (105)I = (7)A^2 \oplus (15)A \oplus (90)I$$

For this example, it can be checked that each side of the above equation is:

$$\begin{bmatrix} 105 & 98 & 126 \\ 315 & 343 & 441 \\ 105 & 70 & 105 \end{bmatrix}$$

This can be simplified to: $A^3 \oplus 21A \oplus 105I = 7A^2$.

Max-Plus Algebra The Max-Plus Algebra was investigated in the first part of this thesis. We did not give a Characteristic Polynomial Equation for the Max-Plus Algebra system at that time, but now we have the tools to do so. Here we use $\uplus \equiv \text{“max”}$, and $\odot \equiv \text{“+”}$

Example 8.73. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 5 & 1 & 3 \end{bmatrix} \in \mathfrak{M}_{n \times n}^+$. ■

In this example:

$$p_A^+(A) = \{1\} \boxtimes A^3 \uplus \{4, 8, 10\} \boxtimes A \uplus \{9, 11, 15\} \boxtimes I$$

$$p_A^-(A) = \{1, 3, 7\} \boxtimes A^2 \uplus \{6, 8, 10\} \boxtimes A \uplus \{8, 11, 16\} \boxtimes I$$

$$A^3 \oplus (10)A \oplus (15)I = (7)A^2 \oplus (10)A \oplus (16)I$$

For this example, it can be checked that each side of the above equation is:

$$\begin{bmatrix} 16 & 16 & 18 \\ 21 & 21 & 23 \\ 15 & 15 & 17 \end{bmatrix}$$

Min-Times Algebra The Min-Times Algebra is another example of a Path Algebra. Here, we are able to give a Characteristic Polynomial Equation for this path algebra. Here, we use $\uplus \equiv \text{“min”}$, and $\odot \equiv \text{“.”}$, and the zero element is ∞ , and the multiplicative identity is 1.

Example 8.74. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 5 & 1 & 3 \end{bmatrix}$. ■

In this example:

$$p_A^+(A) = \{1\} \boxtimes A^3 \uplus \{4, 8, 10\} \boxtimes A \uplus \{9, 11, 15\} \boxtimes I$$

$$p_A^-(A) = \{1, 3, 7\} \boxtimes A^2 \uplus \{6, 8, 10\} \boxtimes A \uplus \{8, 11, 16\} \boxtimes I$$

$$A^3 \oplus (4)A \oplus (9)I = (1)A^2 \oplus (6)A \oplus (8)I$$

For this example, it can be checked that each side of the above equation is:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Minimal Equations in the Mset Algebra

The natural question that arises is whether a minimal equation exists for matrices in the Mset algebra as they do for the conventional algebra.

Example 8.75. Consider the matrix: $A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}$. ■

$$\text{Let } B = \varphi_{\mathbb{S}}(A) = \begin{bmatrix} \{1\} & \emptyset & \{b\} \\ \emptyset & \{1\} & \{c\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix}$$

$$\text{Then } B^2 = \begin{bmatrix} \{1\} & \emptyset & \{b, b\} \\ \emptyset & \{1\} & \{c, c\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix} \text{ and}$$

$$B^3 = \begin{bmatrix} \{1\} & \emptyset & \{b, b, b\} \\ \emptyset & \{1\} & \{c, c, c\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix}$$

The Characteristic Equation for B is:

$$B^3 \uplus \{1, 1, 1\}B \uplus \emptyset I = \{1, 1, 1\}B^2 \uplus \emptyset B \uplus \{1\}I \quad (214)$$

which can be simplified to

$$B^3 \uplus \{1, 1, 1\}B = \{1, 1, 1\}B^2 \uplus \{1\}I \quad (215)$$

It can be seen that an equation of smaller degree exists, namely,

$$B^2 \uplus \{1\}I = \{1, 1\}B \quad (216)$$

Is it possible to say that the minimal equation is a divisor of the characteristic equation?

Definition 8.76. We say that equation $p^+ = p^-$ divides equation $h^+ = h^-$ if there exists a polynomial $q^+ = q^-$ such that

$$h^+ = p^+ q^+ \uplus p^- q^- \quad (217)$$

$$h^- = p^+ q^- \uplus p^- q^+ \quad (218)$$

Here, $p^+ = \{1\} B^2 \uplus \{1\} I$, and $p^- = \{1, 1\} B$, and $h^+ = B^3 \uplus \{1, 1, 1\} B$, and $h^- = \{1, 1, 1\} B^2 \uplus \{1\} I$. For this example we note that if $q^+ = B$, and $q^- = I$, then

$$h^+ = B^3 \uplus \{1, 1, 1\}B = (\{1\}B^2 \uplus \{1\}I) \boxtimes (\{1\}B) \uplus (\{1, 1\}B) \boxtimes (\{1\}I) = p^+ q^+ \uplus p^- q^- \quad (219)$$

$$h^- = \{1, 1, 1\}B^2 \uplus \{1\}I = (\{1\}B^2 \uplus \{1\}I) \boxtimes (\{1\}I) \uplus (\{1, 1\}B) \boxtimes (\{1\}B) = p^+ q^- \uplus p^- q^+ \quad (220)$$

If we apply φ_s to Equation (214), we get

$$A^3 \oplus 3A = 3A^2 \oplus I \quad (221)$$

Example 8.77. Consider the matrix: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{R}$. ■

$$\text{Let } E = \varphi_s(A) = \begin{bmatrix} \emptyset & \{1\} & \{1\} \\ \{1\} & \emptyset & \{1\} \\ \{1\} & \{1\} & \emptyset \end{bmatrix}$$

$$\text{Then } E^2 = \begin{bmatrix} \{1, 1\} & \{1\} & \{1\} \\ \{1\} & \{1, 1\} & \{1\} \\ \{1\} & \{1\} & \{1, 1\} \end{bmatrix} \text{ and}$$

$$E^3 = \begin{bmatrix} \{1, 1\} & \{1, 1, 1\} & \{1, 1, 1\} \\ \{1, 1, 1\} & \{1, 1\} & \{1, 1, 1\} \\ \{1, 1, 1\} & \{1, 1, 1\} & \{1, 1\} \end{bmatrix}$$

The Characteristic Equation for E is:

$$E^3 \uplus \emptyset E \uplus \emptyset I = \emptyset E^2 \uplus \{1, 1, 1\} E \uplus \{1, 1\} I \quad (222)$$

which can be simplified to

$$E^3 = \{1, 1, 1\} E \uplus \{1, 1\} I \quad (223)$$

It can be seen that an equation of smaller degree exists, namely,

$$E^2 = \{1\} E \uplus \{1, 1\} I \quad (224)$$

It is noted that the minimal equation divides the characteristic polynomial is we let $q^+ = E \uplus I$, and $q^- = \emptyset$.

In this case we get:

$$h^+ = E^3 \uplus E^2 \quad (225)$$

$$h^- = E^2 \uplus \{1, 1, 1\} E \uplus \{1\} E \uplus \{1, 1\} I \quad (226)$$

We see that h^+ , and h^- have a common term of E^2 , which can be removed to get the characteristic equation in Equation (223).

Example 8.78. Consider the matrix: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}$.

In the system \mathbb{R} , we have that the characteristic equation is $\Delta_A = \lambda^3 - 5\lambda^2 + 7\lambda - 3$ And the minimal polynomial is $\psi_A = \lambda^2 - 4\lambda + 3$.

Now consider the same matrix $B = \begin{bmatrix} \{2\} & \{1\} & \{1\} \\ \{1\} & \{2\} & \{1\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix} \in 3^{\mathbb{R}^+}$.

The characteristic polynomial for B is $\lambda^3 \uplus \{4, 2, 2\} \lambda \uplus \{1\} = \{2, 2, 1\} \lambda^2 \uplus \{1\} \lambda \uplus \{4\}$

Looking for an annihilating polynomial of degree two, we consider the polynomial

$\alpha \lambda^2 \uplus \beta \lambda \uplus \gamma = \delta \lambda \uplus \varepsilon$. If we substitute B into the equation, we look for the msets that satisfy:

$$\alpha B^2 \uplus \beta B \uplus \gamma = \delta B \uplus \varepsilon \quad (227)$$

$$B^2 = \begin{bmatrix} \{4, 1\} & \{2, 2\} & \{1, 1, 2\} \\ \{2, 2\} & \{1, 4\} & \{1, 1, 2\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix}.$$

$$(1) \text{ (1,1)-element of (227)} \implies \{\alpha, 4\alpha, 2\beta, \gamma\} = \{2\delta, \varepsilon\}$$

$$(2) \text{ (1,2)-element of (227)} \implies \{2\alpha, 2\alpha, \beta\} = \{\delta\}$$

$$(3) \text{ (1,3)-element of (227)} \implies \{\alpha, \alpha, 2\alpha, \beta\} = \{\delta\}$$

Examining the above statements, Equations (2) and (3), imply that $\alpha = \emptyset$. So the only msets for the above polynomial are $\alpha = \emptyset$, $\beta = \delta$, and $\gamma = \varepsilon$

This example illustrates that a polynomial of degree smaller (than the characteristic polynomial) exists for the matrix A over \mathbb{R} , but no polynomial of smaller degree exists for matrix B over $3^{\mathbb{R}}$.

We will return to this example again in Example 8.91 to see why this occurs. ■

8.10 Linear Independence

Suppose we are given a set of vectors, $\underline{x}_1, \dots, \underline{x}_n$, with elements in 3^S . Under what conditions is this set of vectors linearly independent? Adapting the definition from linear algebra we would have the following definition.

Definition 8.79. The set of vectors $\underline{x}_1, \dots, \underline{x}_n$ is **linearly independent** if and only if

$$c_1 \underline{x}_1 \uplus c_2 \underline{x}_2 \uplus \dots \uplus c_n \underline{x}_n = d_1 \underline{x}_1 \uplus d_2 \underline{x}_2 \uplus \dots \uplus d_n \underline{x}_n \implies c_i = d_i \quad \forall i \quad (228)$$

A set of vectors is said to be **linearly dependent** if it is not linearly independent.

This definition is easily stated, but it is not easy, in practice, to show that a set is linearly independent.

8.11 Embeddings and Morphisms of Semirings

With the use of Propositions 7.1 and 7.2, we are able to conclude things about the semiring S_1 , by considering the ring R^D . In particular any identity in R^D which is wholly in S_2 can be carried over to S_1 by going, first, through the semiring S .

We will use the notation for a “positive” and “negative” determinant of a matrix.

Definition 8.80. If A is a matrix then we define:

1. $|A|^+ = \sigma_n^e$
2. $|A|^- = \sigma_n^o$

Proposition 8.81. For the mapping φ_2 from S to S_1 , if $A \in S$, then

- (a) $\varphi_2(A^{\text{adj}+}) = (\varphi_2(A))^{\text{adj}+}$
- (b) $\varphi_2(|A|^+) = |\varphi_2(A)|^+$
- (c) $\varphi_2(I) = I$

Proof. (a) and (b) follow from the fact that the epimorphism, φ_2 , from S to S_2 is applied to a matrix elementwise. So if $A = [a_{ij}]$, then $[\varphi_2(A)]_{ij} = [\varphi_2(a_{ij})]$.

(c) is obvious since the epimorphism is applied elementwise. □

Example 8.82. As an example of this relationship we will consider the identity in a ring:

$$|AB| = |A||B|.$$

Since R^D is such a ring, we write the identity in R^D as:

$$|AB|^+ - |AB|^- = (|A|^+ - |A|^-)(|B|^+ - |B|^-) \quad (229)$$

This becomes:

$$|AB|^+ + |A|^+|B|^- + |A|^-|B|^+ = |AB|^- + |A|^+|B|^+ + |A|^-|B|^- \quad (230)$$

which is an identity wholly in S_2 , and therefore is true for all matrices over any semiring. ■

Example 8.83. As another example we consider the ring identity:

$$(\lambda I - A)\text{adj}(\lambda I - A) = \Delta_A I \quad (231)$$

In a semiring, this becomes:

$$\begin{aligned} \lambda I \text{adj}(\lambda I - A)^+ + A \text{adj}(\lambda I - A)^- + \Delta^-(\lambda) I \\ = \lambda I \text{adj}(\lambda I - A)^- + A \text{adj}(\lambda I - A)^+ + \Delta^+(\lambda) I \end{aligned} \quad (232)$$

■

Recall that:

Fact 8.84.

$$\text{adj}(\lambda I - A) = \sum_{k=0}^{n-1} A_k \lambda^k \quad (233)$$

where $A_k = a_{k+1}I + a_{k+2}A + \dots + a_n A^{n-(k+1)}$. The a_k 's are defined by:

$a_{k+1} = (-1)^{n-(k+1)} \sigma_{n-(k+1)}$, where the σ_k 's are the coefficients of the characteristic polynomial.

Example 8.85. We verify the identity of Example 8.83, in particular Equation (232).

Consider the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 3 & 10 & 16 \end{bmatrix} \in \mathbb{R}^+$.

For this matrix, we have $\Delta_A^+(\lambda) = \lambda^3 + 101\lambda + 129$, and $\Delta_A^-(\lambda) = 22\lambda^2 + 57\lambda + 130$.

We have

$$\begin{array}{cccc} a_0^+ = 129 & a_1^+ = 101 & a_2^+ = 0 & a_3^+ = 1 \\ a_0^- = 130 & a_1^- = 57 & a_2^- = 22 & a_3^- = 0 \end{array}$$

$$A_0^+ = a_1^+ I + a_2^+ A + a_3^+ A^2 = \begin{bmatrix} 109 & 22 & 27 \\ 27 & 180 & 107 \\ 71 & 216 & 410 \end{bmatrix} \quad A_0^- = a_1^- I + a_2^- A + a_3^- A^2 = \begin{bmatrix} 79 & 44 & 22 \\ 44 & 167 & 110 \\ 66 & 220 & 409 \end{bmatrix}$$

$$A_1^+ = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 3 & 10 & 16 \end{bmatrix} \quad A_1^- = \begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}$$

$$A_2^+ = I_{3 \times 3} \quad A_2^- = 0_{3 \times 3}$$

$$\begin{aligned} \text{adj}(\lambda I - A)^+ &= A_0^+ + A_1^+ \lambda + A_2^+ \lambda^2 = \begin{bmatrix} \lambda^2 + \lambda + 109 & 2\lambda + 22 & \lambda + 27 \\ 2\lambda + 27 & \lambda^2 + 5\lambda + 180 & 5\lambda + 107 \\ 3\lambda + 71 & 10\lambda + 216 & \lambda^2 + 16\lambda + 410 \end{bmatrix} \\ \text{adj}(\lambda I - A)^- &= A_0^- + A_1^- \lambda + A_2^- \lambda^2 = \begin{bmatrix} 22\lambda + 79 & 44 & 22 \\ 44 & 22\lambda + 167 & 110 \\ 66 & 220 & 22\lambda + 409 \end{bmatrix} \end{aligned}$$

And it can be verified that Equation (232) holds, as each side of the equation is:

$$\begin{bmatrix} \lambda^3 + 23\lambda^2 + 188\lambda + 363 & 2\lambda^2 + 66\lambda + 598 & \lambda^2 + 49\lambda + 651 \\ 2\lambda^2 + 71\lambda + 708 & \lambda^3 + 27\lambda^2 + 347\lambda + 2153 & 5\lambda^2 + 217\lambda + 2639 \\ 3\lambda^2 + 137\lambda + 1733 & 10\lambda^2 + 436\lambda + 5322 & \lambda^3 + 38\lambda^2 + 819\lambda + 7840 \end{bmatrix}$$

■

Example 8.86. As another example we consider the ring identity:

$$A_{k-1} - AA_k = a_k I \quad (234)$$

This can be written in a semiring as

$$A_{k-1}^+ + AA_k^- + a_k^- I = A_{k-1}^- + AA_k^+ + a_k^+ I \quad (235)$$

■

8.12 Ring of differences for the mset algebra over \mathbb{R}^+

We may recall that the ring of differences is used to create the integers from the nonnegative integers. In this abstract algebra (a,b) is the equivalence class associated with $a - b$.

Let us consider the ring of differences created from $3^{\mathbb{R}^+}$. The equivalence relation for msets is $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$ if and only if $(\alpha_1 \uplus \beta_2) = (\alpha_2 \uplus \beta_1)$. As an example we may notice that $(\{2,3\}, \{3,4\}) \sim (\{1,2\}, \{1,4\})$. This has a very nice relation to the ring of differences created from the nonnegative integers. For the example above, if we apply the sum to each mset in the pairs, as in Definition (8.10), we get that $(5,7) \sim (3,5)$, which is the equivalence class for -2 in the integers.

We may also notice that the equivalence class denoted by $(\{2,3\}, \{3,4\})$ can be written in an "easier" mset as $\{2,3,-3,-4\}$. In this manner we see that using the set \mathbb{R}^+ as our underlying semiring, we can embed this semiring in the ring \mathbb{R} .

With this method, we have:

Note 8.87. If $\alpha, \beta \in 3^{\mathbb{R}^+}$, then the element $(\alpha, \beta) \in R^D$ over $3^{\mathbb{R}^+}$ can be written as $(\alpha \uplus (-\beta)) \in 3^R$ as a convenience.

Lemma 8.88. The ring R^D , for msets over \mathbb{R}^+ , is an integral domain.

Proof. Suppose that $\overline{(r, s)} \boxtimes \overline{(r_1, s_1)} = \overline{(0, 0)}$. Then $(rr_1 + ss_1, rs_1 + sr_1) \sim (0, 0)$. Therefore $rr_1 + ss_1 = rs_1 + sr_1$. From Lemma 8.27, we have that either (a) $r=s$, or (b) $r_1=s_1$. In either case, we have that one of these two elements is equivalent to $\overline{(0, 0)}$. □

The following lemma gives the invertible elements in $(\alpha, \beta) \in R^D$ over $3^{\mathbb{R}^+}$.

Lemma 8.89. If $(\alpha, \beta) (\gamma, \delta) \sim (\{1\}, \emptyset)$, with $\alpha \cap \beta = \emptyset$, and $\gamma \cap \delta = \emptyset$, then either:

(1) $\beta = \delta = \emptyset$, and $|\alpha| = |\gamma| = 1$, with $\gamma_1 = \frac{1}{\alpha_1}$,

(2) $\alpha = \gamma = \emptyset$, and $|\beta| = |\delta| = 1$, with $\delta_1 = \frac{1}{\beta_1}$.

Proof. Suppose $(\alpha, \beta) (\gamma, \delta) \sim (\{1\}, \emptyset)$, with $\alpha \cap \beta = \emptyset$, and $\gamma \cap \delta = \emptyset$.

Then

$$\alpha\gamma \uplus \beta\delta = \alpha\delta \uplus \beta\gamma \uplus \{1\}. \quad (236)$$

We consider two cases for the proof.

(I.) One mset is not empty.

(a) $\beta \neq \emptyset, \delta = \emptyset$

(b) $\beta = \emptyset, \delta \neq \emptyset$

(c) $\alpha \neq \emptyset, \gamma = \emptyset$

(d) $\alpha = \emptyset, \gamma \neq \emptyset$

(II.) None of the msets are empty.

Let $|\alpha| = a, |\beta| = b, |\gamma| = c,$ and $|\delta| = d$.

(I.) The four subcases are all similar, so we will prove (I.a). Suppose that $\beta \neq \emptyset$, and $\delta = \emptyset$. Then Equation (236) becomes $\alpha\gamma = \beta\gamma \uplus \{1\}$. The consistency condition implies that $ac=bc+1$, or $c(a-b)=1$, since this condition is in the nonnegative integers then $c=1$ and $a=b+1$. The equation becomes

$$\{a_1, a_2, \dots, a_{k+1}\} \{c\} = \{b_1, b_2, \dots, b_k\} \{c\} \uplus \{1\}$$

which can be expanded as

$$\{a_1c, a_2c, \dots, a_{k+1}c\} = \{b_1c, b_2c, \dots, b_kc, 1\}$$

Since these two msets are equal then either: (a) $a_1c = 1$, or (b) $a_1c = b_m c$ for some m . In the latter case we have that $a_1 = b_m$ which is a contradiction. In the former case, we use the same reasoning for a_2c to get a contradiction. Therefore case (I) cannot occur.

(II) Suppose that $\alpha \neq \emptyset$, $\beta \neq \emptyset$, $\gamma \neq \emptyset$, and $\delta \neq \emptyset$.

The consistency condition implies that

$$\begin{aligned} ac + bd &= ad + bc + 1 \\ (a - b)(c - d) &= 1 \end{aligned}$$

Again, since we are working over the nonnegative integers, we have that $a-b=1$ and $c-d=1$, which implies that $a=b+1$, and $c=d+1$. We may also have that $b=a+1$, and $d=c+1$, but the proof of this case is similar. We suppose that all the elements are positive, and that they are in increasing (nondecreasing) order. So Equation (236) becomes:

$$\begin{aligned} \{a_1, a_2, \dots, a_{k+1}\}\{c_1, c_2, \dots, c_{m+1}\} \uplus \{b_1, b_2, \dots, b_k\}\{d_1, d_2, \dots, d_m\} \\ = \{a_1, a_2, \dots, a_{k+1}\}\{d_1, d_2, \dots, d_m\} \uplus \{b_1, b_2, \dots, b_k\}\{c_1, c_2, \dots, c_{m+1}\} \uplus \{1\} \end{aligned}$$

which is expanded to:

$$\begin{aligned} \{a_1c_1, \dots, a_1c_{m+1}, \dots, a_{k+1}c_1, \dots, a_{k+1}c_{m+1}, b_1d_1, \dots, b_1d_m, \dots, b_kd_1, \dots, b_kd_m\} \\ = \{a_1d_1, \dots, a_1d_m, \dots, a_{k+1}d_1, \dots, a_{k+1}d_m, b_1c_1, \dots, b_1c_{m+1}, \dots, b_kc_1, \dots, b_kc_{m+1}, 1\} \end{aligned} \quad (237)$$

Taking the maximum element of each side, we get that

$$\max(a_{k+1}c_{m+1}, b_kd_m) = \max(a_{k+1}d_m, b_kc_{m+1}, 1) \quad (238)$$

and comparing the minimum elements of each side gives:

$$\min(a_1c_1, b_1d_1) = \min(a_1d_1, b_1c_1, 1) \quad (239)$$

Equation (238) leads to either (a) $a_{k+1}c_{m+1} = 1$, or (b) $b_kd_m = 1$, for otherwise we get a contradiction of the empty “intersection” of pairwise msets. Similarly, if we compare the minimum elements of each mset, we get either (1) $a_1c_1 = 1$, or (2) $b_1d_1 = 1$.

This leads to a combination of 4 separate cases.

1. $a_{k+1}c_{m+1} = 1$, and $a_1c_1 = 1$.
2. $b_kd_m = 1$, and $b_1d_1 = 1$.

3. $a_{k+1}c_{m+1} = 1$, and $b_1d_1 = 1$.

4. $b_kd_m = 1$, and $a_1c_1 = 1$.

Cases (1) and (2) are similar as are cases (3) and (4). Each case leads to the same conclusion, so we will illustrate the procedure for cases (2) and (4). Consider case (2). Suppose $b_kd_m = 1$, and $b_1d_1 = 1$. We also know that $b_kd_m \geq a_{k+1}c_{m+1}$, and $a_1c_1 \geq b_1d_1$. These statements lead to:

$$1 = b_kd_m \geq a_{k+1}c_{m+1} \geq a_1c_1 \geq b_1d_1 \geq 1 \quad (240)$$

Therefore all products in the above statement must equal 1. Therefore we have that $a_ic_j = 1$ for all i and j , and $b_id_j = 1$ for all i and j . Since we know that $b_1 \leq b_k \leq$, then $\frac{1}{b_1} \geq \frac{1}{b_k} \geq 1$, which implies that $d_1 \geq d_m \geq 1$. Since $d_m \geq d_1$ then $d_1 = d_m$. Therefore $d_1 = d_2 = \dots = d_m$, and $b_1 = b_2 = \dots = b_k$.

For case (4), suppose $b_kd_m = 1$, and $a_1c_1 = 1$. We also know that $b_kd_m \geq a_{k+1}d_{m+1}$, and $b_1d_1 \geq a_1c_1$. From these conditions, we get the following set of inequalities:

$$1 = b_kd_m \geq a_{k+1}c_{m+1} \geq a_1c_1 = 1 \quad (241)$$

$$1 = b_kd_m \geq b_1d_1 \geq a_1c_1 = 1 \quad (242)$$

Equations (241) and (242) imply that $a_ic_j = 1$ for all i and j , and $b_id_j = 1$ for all i and j . Equation (242) implies that $b_1d_1 = 1$, and this case becomes part of the previous case.

These cases show that every element on the LHS of Equation (237) must be “1”. Therefore every element on the RHS of Equation (237) must also be “1”. We also have that $b_1 = b_2 = \dots = b_k$, and $d_1 = d_2 = \dots = d_m$. Therefore we also have that $a_1 = a_2 = \dots = a_{k+1}$, and $c_1 = c_2 = \dots = c_{m+1}$. The LHS of Equation (237) has two separate elements, say “ac” which occurs $(k+1)(m+1)$ times, and “bd” which occurs $(k)(d)$ times. Likewise the RHS of Equation (237) has “ad” which occurs $(k+1)(m)$ times, and “bc” which occurs $(k)(m+1)$ times, plus a separate “1”. Therefore either $ac=ad$ which leads to a contradiction, or $ac=bc$ which also leads to a contradiction.

Therefore case (II) is also impossible. The consistency condition implies the sizes of the msets for the result. \square

We revisit a past example, once again, to see how it can be worked with the ring of difference.

Example 8.90. Consider the matrix of Example (8.78): $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^+$.

We translate this matrix into the Multiset Algebra: $B = \begin{bmatrix} \{2\} & \{1\} & \{1\} \\ \{1\} & \{2\} & \{1\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix} \in 3^{\mathbb{R}^+}$.

We then translate this matrix into the Ring of Differences:

$$E = \begin{bmatrix} (\{2\}, \emptyset) & (\{1\}, \emptyset) & (\{1\}, \emptyset) \\ (\{1\}, \emptyset) & (\{2\}, \emptyset) & (\{1\}, \emptyset) \\ (\emptyset, \emptyset) & (\emptyset, \emptyset) & (\{1\}, \emptyset) \end{bmatrix} \in R^D.$$

We note that the Characteristic Polynomial for the matrix E is:

$$(\{1\}, \emptyset)\lambda^3 \uplus (\emptyset, \{1, 2, 2\})\lambda^2 \uplus (\{2, 2, 4\}, \{1\})\lambda \uplus (\{1\}, \{4\}) \quad (243)$$

It can be checked that

$$(\{1\}, \emptyset)E^3 \uplus (\emptyset, \{1, 2, 2\})E^2 \uplus (\{2, 2, 4\}, \{1\})E \uplus (\{1\}, \{4\})I \equiv \overline{(\emptyset, \emptyset)} \quad (244)$$

as the left side of Equation (244) is

$$\left[\begin{array}{ccc} \left(\left(\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 8 & 8 & \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 8 & 8 & \end{array} \right) \right) & \left(\left(\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & & \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & & \end{array} \right) \right) & \left(\left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 4 \\ 4 & & \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 4 \\ 4 & & \end{array} \right) \right) \\ \left(\left(\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & & \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 8 & 8 & \end{array} \right) \right) & \left(\left(\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 8 & 8 & \end{array} \right), \left(\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 8 & 8 & \end{array} \right) \right) & \left(\left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 4 \\ 4 & & \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 4 \\ 4 & & \end{array} \right) \right) \\ (\emptyset, \emptyset) & (\emptyset, \emptyset) & \left(\left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 4 \\ \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 4 \\ \end{array} \right) \right) \end{array} \right] \quad (245)$$

■

Example 8.91. Once again we consider Example 8.78 and the matrix $A \in \mathbb{R}^+$: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

We convert this matrix to $3^{\mathbb{R}^+}$ as:

$$\widehat{A} = \begin{bmatrix} \{2\} & \{1\} & \{1\} \\ \{1\} & \{2\} & \{1\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix}$$

Then we convert this matrix to the ring of differences, R^D :

$$\overline{A} = \begin{bmatrix} (\{2\}, \emptyset) & (\{1\}, \emptyset) & (\{1\}, \emptyset) \\ (\{1\}, \emptyset) & (\{2\}, \emptyset) & (\{1\}, \emptyset) \\ (\{0\}, \emptyset) & (\{0\}, \emptyset) & (\{1\}, \emptyset) \end{bmatrix}$$

The Characteristic Polynomial for A is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = (\lambda - 1)^2(\lambda - 3)$.

The Minimal Polynomial for A is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$.

In $3^{\mathbb{R}^+}$, the Characteristic Equation is $\lambda^3 \uplus \{2, 2, 4\}\lambda \uplus \{1\} = \{1, 2, 2\}\lambda^2 \uplus \{1\}\lambda \uplus \{4\}$.

In R^D the characteristic equation is: $(\{1\}, \emptyset)\lambda^3 + (\emptyset, \{1, 2, 2\})\lambda^2 + (\{2, 2, 4\}, \{1\})\lambda + (\{1\}, \{4\})$.

Is there a division algorithm for the polynomials over R^D ? Given $p(\lambda)$, and $h(\lambda)$, can $q(\lambda)$ and $r(\lambda)$ be found such that $h(\lambda) = q(\lambda)p(\lambda) + r(\lambda)$, with $\partial(r) < \partial(p)$? In order to have a division algorithm we must have the leading coefficient of $p(\lambda)$ to be an invertible element. As we have seen

this is the case if the polynomial is monic, where we define a monic polynomial in R^D as a polynomial with leading coefficient equal to $(\{1\}, \emptyset)$.

We note that $(\{1\}, \emptyset)$ is a “root” of the Characteristic Polynomial. We divide by

$$\lambda - (\{1\}, \emptyset) = \lambda + (\emptyset, \{1\})$$

$$\begin{array}{r} \lambda^2 + (\emptyset, \{2, 2\})\lambda + (\{4\}, \{1\}) \\ \lambda + (\emptyset, \{1\}) \overline{\lambda^3 + (\emptyset, \{1, 2, 2\})\lambda^2 + (\{2, 2, 4\}, \{1\})\lambda + (\{1\}, \{4\})} \\ \lambda^3 + (\emptyset, \{1\})\lambda^2 \\ \hline (\emptyset, \{2, 2\})\lambda^2 + (\{2, 2, 4\}, \{1\})\lambda \\ (\emptyset, \{2, 2\})\lambda^2 + (\{2, 2\}, \emptyset)\lambda \\ \hline (\{4\}, \{1\})\lambda + (\{1\}, \{4\}) \\ (\{4\}, \{1\})\lambda + (\{1\}, \{4\}) \\ \hline (\emptyset, \emptyset) \end{array} \quad (246)$$

So, we have:

$$\begin{aligned} & (\{1\}, \emptyset)\lambda^3 + (\emptyset, \{1, 2, 2\})\lambda^2 + (\{2, 2, 4\}, \{1\})\lambda + (\{1\}, \{4\}) \\ &= \left(\lambda - (\{1\}, \emptyset) \right) \left(\lambda^2 + (\emptyset, \{2, 2\})\lambda + (\{4\}, \{1\}) \right) \end{aligned} \quad (247)$$

We also note that $(\{2\}, \{1\})$ is a “root”.

$$\begin{array}{r} \lambda + (\emptyset, \{1, 2\}) \\ \lambda + (\{1\}, \{2\}) \overline{\lambda^2 + (\emptyset, \{2, 2\})\lambda + (\{4\}, \{1\})} \\ \lambda^2 + (\{1\}, \{2\})\lambda \\ \hline (\emptyset, \{1, 2\})\lambda + (\{4\}, \{1\}) \\ (\emptyset, \{1, 2\})\lambda + (\{2, 4\}, \{1, 2\}) \\ \hline (\emptyset, \emptyset) \end{array} \quad (248)$$

The Characteristic Polynomial becomes:

$$\begin{aligned} & (\{1\}, \emptyset)\lambda^3 + (\emptyset, \{1, 2, 2\})\lambda^2 + (\{2, 2, 4\}, \{1\})\lambda + (\{1\}, \{4\}) \\ &= \left(\lambda - (\{1\}, \emptyset) \right) \left(\lambda - (\{2\}, \{1\}) \right) \left(\lambda - (\{1, 2\}, \emptyset) \right) \end{aligned} \quad (249)$$

We notice that the Characteristic Polynomial factors into linear factors. Since the three factors are distinct the minimal polynomial is the same as the characteristic polynomial.

The roots of the Characteristic Polynomial are eigenvalues of the given matrix. Below we list the eigenvectors for the matrix B above.

(A) $(\{1,2\},\emptyset)$ is an eigenvalue.

$$\begin{aligned}\bar{A} - (\{1,2\},\emptyset)I &= \bar{A} + (\emptyset,\{1,2\})I \\ &= \begin{bmatrix} (\{2\},\{1,2\}) & (\{1\},\emptyset) & (\{1\},\emptyset) \\ (\{1\},\emptyset) & (\{2\},\{1,2\}) & (\{1\},\emptyset) \\ (\emptyset,\emptyset) & (\emptyset,\emptyset) & (\{1\},\{1,2\}) \end{bmatrix} \\ &= \begin{bmatrix} (\emptyset,\{1\}) & (\{1\},\emptyset) & (\{1\},\emptyset) \\ (\{1\},\emptyset) & (\emptyset,\{1\}) & (\{1\},\emptyset) \\ (\emptyset,\emptyset) & (\emptyset,\emptyset) & (\emptyset,\{2\}) \end{bmatrix}\end{aligned}$$

Letting $\underline{x} = \begin{bmatrix} x_1, x_2 \\ y_1, y_2 \\ z_1, z_2 \end{bmatrix}$, and setting $(\bar{A} - \lambda I) \underline{x} = \bar{0}$ yields the three equations:

- (1) $(x_2, x_1) + (y_1, y_2) + (z_1, z_2) \sim (\emptyset, \emptyset) \iff (x_2 + y_1 + z_1, x_1 + y_2 + z_2) \sim (\emptyset, \emptyset)$
- (2) $(x_1, x_2) + (y_2, y_1) + (z_1, z_2) \sim (\emptyset, \emptyset) \iff (x_1 + y_2 + z_1, x_2 + y_1 + z_2) \sim (\emptyset, \emptyset)$
- (3) $(2z_2, 2z_1) \sim (\emptyset, \emptyset)$

These equations lead to the eigenvector $\begin{bmatrix} (\{1\},\emptyset) \\ (\{1\},\emptyset) \\ (\emptyset,\emptyset) \end{bmatrix}$.

It can be checked that $\bar{A} \begin{bmatrix} (\{1\},\emptyset) \\ (\{1\},\emptyset) \\ (\emptyset,\emptyset) \end{bmatrix} = (\{1,2\},\emptyset) \begin{bmatrix} (\{1\},\emptyset) \\ (\{1\},\emptyset) \\ (\emptyset,\emptyset) \end{bmatrix} = \begin{bmatrix} (\{1,2\},\emptyset) \\ (\{1,2\},\emptyset) \\ (\emptyset,\emptyset) \end{bmatrix}$.

(B) $(\{1\},\emptyset)$ is an eigenvalue.

The corresponding eigenvector is: $\begin{bmatrix} (\{1\},\emptyset) \\ (\{1\},\emptyset) \\ (\emptyset,\{2\}) \end{bmatrix}$

(C) $(\{2\},\{1\})$ is an eigenvalue.

The corresponding eigenvector is: $\begin{bmatrix} (\{1\},\emptyset) \\ (\emptyset,\{1\}) \\ (\emptyset,\emptyset) \end{bmatrix}$

So, although in \mathbb{R} the characteristic equation for A is $\Delta(\lambda) = (\lambda - 1)^2(\lambda - 3)$, and the minimal polynomial for A is $\psi(\lambda) = (\lambda - 1)(\lambda - 3)$, we see that the “1” which is a repeated root in \mathbb{R} has become two different elements in R^D , name $(\{1\},\emptyset)$, and $(\{2\},\{1\})$. ■

It may be noticed that

$$\sum \lambda_i = (\{1\}, \emptyset) + (\{2\}, \{1\}) + (\{1, 2\}, \emptyset) = (\{1, 2, 2\}, \emptyset) = \text{Trace}(\overline{A}) \quad (250)$$

and

$$\prod \lambda_i = (\{1\}, \emptyset) (\{2\}, \{1\}) (\{1, 2\}, \emptyset) = (\{4\}, \{1\}) = (-1)^3 |\overline{A}| \quad (251)$$

As we see from the above example, the roots of the characteristic equation are eigenvalues of the given matrix. The following theorem shows that this is true for the case in which the ring of differences is an integral domain, as is the case for the ring of differences, R^D , for \mathbb{R}^+ .

The following lemma is from [25, page 161].

Lemma 8.92. A matrix, A , over a ring $R_{n \times n}$ is a divisor of zero in $R_{n \times n}$ if and only if $|A|$ is a divisor of zero in R .

With this lemma, we may now prove

Lemma 8.93. If R^D is an integral domain, then $Ax=0$ has nonzero solutions if and only if $|A| \neq 0$.

Proof. Suppose that $A\underline{x} = 0$ has a nonzero solution. Let $X = [\underline{x} \ * \ \cdots \ *] \in R_{n \times n}$. Then A is a divisor of zero. Therefore by Lemma 8.92, $|A|$ is a divisor of zero. If $|A|$ is a divisor of zero then there exists an element, d , such that $|A| \cdot d = 0$. Since R^D is an integral domain then $|A| = 0$. Conversely, suppose that $|A| = 0$. Then there exists a nonzero matrix X such that $AX=0$. Let \underline{x} equal one of the nonzero columns of X . Then $A\underline{x} = 0$, with $\underline{x} \neq 0$. □

Now we can prove

Theorem 8.94. An eigenvalue of A is a root of the Characteristic Polynomial of A , and every root of the Characteristic Polynomial is an eigenvalue of A .

Proof. Suppose that $Ax = \lambda x$ has a nonzero solution \underline{x} (an eigenvector).

$$\begin{aligned} A\underline{x} &= \lambda \underline{x} \\ \iff (\lambda I - A)\underline{x} &= 0 \\ \iff |\lambda I - A| &= 0 \\ \iff \Delta_A(\lambda) &= 0 \end{aligned} \quad (252)$$

□

We note that $(\{1,2\}, \emptyset)$ is the “largest” “positive” eigenvalue, and the eigenvector corresponding to this eigenvalue is strictly “nonnegative.” Can we say that the maximum “nonnegative” eigenvalue is the only eigenvalue with a strictly nonnegative eigenvector?

8.13 Application.

We include the following example to show how the mset algebra may be used to solve a problem from the Max-Plus Algebra system. The following problem is from [24], where the system of equations is solved using the symmetrized algebra. The mset algebra offers an alternative method of solution.

Example 8.95. Find the solutions of:

$$\begin{cases} \max(x, y - 4, 1) = \max(x - 1, y + 1, 2) \\ \max(x + 3, y + 2, -5) = \max(y + 2, 7) \end{cases} \quad \text{This can be written in matrix form as:}$$

$$\begin{bmatrix} 0 & -4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\infty & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad (253)$$

To solve this system of equations, we first translate the problem to the Max-Times algebra to get:

$$\begin{bmatrix} e^0 & e^{-4} \\ e^3 & e^2 \end{bmatrix} \boxtimes \begin{bmatrix} e^x \\ e^y \end{bmatrix} \boxplus \begin{bmatrix} e^1 \\ e^{-5} \end{bmatrix} = \begin{bmatrix} e^{-1} & e^1 \\ 0 & e^2 \end{bmatrix} \boxtimes \begin{bmatrix} e^x \\ e^y \end{bmatrix} \boxplus \begin{bmatrix} e^2 \\ e^7 \end{bmatrix} \quad (254)$$

This equation is translated into the mset algebra using \mathbb{R} as the set \mathcal{S} . We then translate the system into an equation of the form $Ax=b$:

$$\begin{bmatrix} \{1, -e^{-1}\} & \{e^{-4}, -e^1\} \\ \{e^3\} & \{e^2, -e^2\} \end{bmatrix} \boxtimes \begin{bmatrix} e^x \\ e^y \end{bmatrix} = \begin{bmatrix} \{e^2, -e^1\} \\ \{e^7, -e^{-5}\} \end{bmatrix} \quad (255)$$

We now apply Cramer's Rule to this matrix equation. We use the equation 186 from Note 8.49.

$$\{e^2, -e^2, -e^1, e^1\}e^x \boxplus \{e^3, -e^{-9}, -e^8, e^{-4}\} = \{e^{-1}, -e^4\}e^x \boxplus \{e^4, -e^4, -e^3, e^3\} \quad (256)$$

$$\{e^2, -e^2, -e^1, e^1\}e^y \boxplus \{e^5, -e^4\} = \{e^{-1}, -e^4\}e^y \boxplus \{e^7, -e^{-5}, -e^6, e^{-6}\} \quad (257)$$

or

$$\{e^2e^x, -e^2e^x, -e^1e^x, e^1e^x, e^3, -e^{-9}, -e^8, e^{-4}\} = \{e^{-1}e^x, -e^4e^x, e^4, -e^4, -e^3, e^3\} \quad (258)$$

$$\{e^2e^y, -e^2e^y, -e^1e^y, e^1e^y, e^5, -e^4\} = \{e^{-1}e^y, -e^4e^y, e^7, -e^{-5}, -e^6, e^{-6}\} \quad (259)$$

We translate back to \mathbb{R}^+ , by moving all negative terms to the other side of the equation:

$$\{e^2e^x, e^1e^x, e^4e^x, e^3, e^4, e^3, e^{-4}\} = \{e^{-1}e^x, e^2e^x, e^1e^x, e^4, e^3, e^8, e^{-9}\} \quad (260)$$

$$\{e^2e^y, e^1e^y, e^4e^y, e^5, e^{-5}, e^6\} = \{e^{-1}e^y, e^2e^y, e^1e^y, e^7, e^{-6}, e^4\} \quad (261)$$

Translating back to Max-Times algebra by applying "max" to each mset, we get:

$$e^4e^x \boxplus e^4 = e^2e^2 \boxplus e^8 \implies e^4e^x = e^8 \implies e^x = e^4 \quad (262)$$

$$e^4e^y \boxplus e^6 = e^2e^y \boxplus e^7 \implies e^4e^y = e^7 \implies e^y = e^3 \quad (263)$$

And, finally translating back into the Max-Plus Algebra, we get:

$$\begin{array}{|c|} \hline x = 4 \\ \hline y = 3 \\ \hline \end{array}$$

■

8.14 Cramer's Rule

We have seen, in the previous sections, that Cramer's Rule is a viable method for working with systems of equations over msets. In this section we will further refine the methods used previously. As we examine Example (8.95) closely, we see that we have taken the algebra system $\mathbb{S}_{n \times n}$ and imbedded it into a system $\mathbb{S}_{n \times n}^*$, which has negatives. In this example, where we use \mathbb{R}^+ and \mathbb{R} , it seems natural to use negative signs, but in a general semiring, we may use an “ \ominus ” instead. By imbedding the semiring $\mathbb{S}_{n \times n}$ in $\mathbb{S}_{n \times n}^*$, the notation becomes somewhat easier to use. Most of the results of this section could be modified to include only $\mathbb{S}_{n \times n}$.

We begin with some of the results of [27], after some the definitions of the notation used in this section.

Definition 8.96 (Determinants). For a matrix $A \in R^{n \times n}$, we define

$$(1) |A|^+ = \bigoplus_{\sigma=\text{even}} \left(\bigotimes_{1 \leq i \leq n} A_{ij} \right)$$

$$(2) |A|^- = \bigoplus_{\sigma=\text{odd}} \left(\bigotimes_{1 \leq i \leq n} A_{ij} \right)$$

Definition 8.97. If A is a square matrix, then the *minor*, M_{ij} , of the element a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A .

Recall that, as in \mathbb{R} , the sign pattern for cofactors is:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{264}$$

We now define the “positive adjoint matrix” and “negative adjoint matrix” by:

Definition 8.98 (Adjoint Matrix). Let $A \in \mathbb{S}_{n \times n}$, then

$$(1) A^{\text{adj}+} = \begin{bmatrix} M_{11}^+ & M_{21}^- & M_{31}^+ & \cdots \\ M_{12}^- & M_{22}^+ & M_{32}^- & \cdots \\ M_{13}^+ & M_{23}^- & M_{33}^+ & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$(2) A^{\text{adj}-} = \begin{bmatrix} M_{11}^- & M_{21}^+ & M_{31}^- & \cdots \\ M_{12}^+ & M_{22}^- & M_{32}^+ & \cdots \\ M_{13}^- & M_{23}^+ & M_{33}^- & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Fact 8.99. If $A, B \in 3^{\mathbb{S}}$, and $A=B$, then $|A|^+ = |B|^+$, and $|A|^- = |B|^-$.

The following condition holds for some semirings. We have seen in Lemma 8.27 that this condition holds for multisets over the nonnegative numbers.

Definition 8.100 (Switching Condition).

In semiring \mathbb{S} , if $\alpha u \oplus \beta v = \beta u \oplus \alpha v$, then $\alpha = \beta$, or $u = v$.

We can now consider Cramer's Rule in its entirety.

Theorem 8.101 (Cramer's Rule). Let \mathbb{S} be a semiring and suppose $A \in \mathbb{S}_{n \times n}$. If \underline{x} satisfies the system of equations

$$A\underline{x} = \underline{b} \tag{265}$$

then \underline{x} satisfies the Cramer's Rule equations stated as:

$$|A|^+ \underline{x} + A^{\text{adj}-} \underline{b} = |A|^- \underline{x} + A^{\text{adj}+} \underline{b} \tag{266}$$

If, in addition, \mathbb{S} is a cancellative semiring in which the switching condition holds then if \underline{x} satisfies Equation (266) then either:

1. $|A|^+ = |A|^-$, or
2. $A\underline{x} = \underline{b}$.

Proof. We prove this for the first element, x_1 . The proofs for the other variables are similar.

Suppose \underline{x} is a solution to $A\underline{x} = \underline{b}$ and consider the matrix equation:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} b & a_2 & \cdots & a_n \end{bmatrix} \tag{267}$$

We write this equation as:

$$AX_1 = B_1 \tag{268}$$

Take the determinant of each side of Equation (268). Use Equation (230), and note that $|X_1|^+ = x_1$, and $|X_1|^- = 0$, to get

$$|AX_1|^+ + |A|^- x_1 = |AX_1|^- + |A|^+ x_1 \tag{269}$$

Using Fact 8.99, we see that $|AX_1|^+ = |B_1|^+$, and $|AX_1|^- = |B_1|^-$.

So Equation (269) becomes

$$|A|^+x_1 + |B_1|^- = |A|^-x_1 + |B_1|^+ \quad (270)$$

Finally we claim that $|B_1|^+ = (A^{\text{adj}+\underline{b}})_1$, and $|B_1|^- = (A^{\text{adj}-\underline{b}})_1$. To show that this is true, we have:

$$|B_1| = \begin{vmatrix} b & a_2 & \cdots & a_n \end{vmatrix} = \sum_{i=1}^n b_i A_{i1} = \sum_{i=1}^n b_i (-1)^{i+1} M_{i1} \quad (271)$$

So,

$$\begin{aligned} |B_1|^+ - |B_1|^- &= \sum_{i=1}^n (-1)^{i+1} b_i M_{i1} \\ &= b_1 M_{11} - b_2 M_{21} + b_3 M_{31} - \cdots \\ &= b_1 (M_{11}^+ - M_{11}^-) - b_2 (M_{21}^+ - M_{21}^-) + b_3 (M_{31}^+ - M_{31}^-) - \cdots \\ &= (b_1 M_{11}^+ + b_2 M_{21}^- + b_3 M_{31}^+ + \cdots) - (b_1 M_{11}^- + b_2 M_{21}^+ + b_3 M_{31}^- + \cdots) \\ &= (A^{\text{adj}+\underline{b}})_1 - (A^{\text{adj}-\underline{b}})_1 \end{aligned} \quad (272)$$

So Equation (270) can be written as Equation (266) for the variable x_1 . The other variables are proved analogously. We have shown that if \underline{x} is a solution to Equation (265) then \underline{x} is also a solution to Equation (266).

We would like to be able to start with a solution to Equation (266) and show that this is a solution to the system given by Equation (265). In order to obtain this result, we must assume that the semiring \mathbb{S} is cancellative and also that the Switching condition holds in \mathbb{S} .

Equation (266) is equivalent to

$$\sum_j a_{ij} (|A|^+ \underline{x} + A^{\text{adj}-\underline{b}}) = \sum_j a_{ij} (|A|^- \underline{x} + A^{\text{adj}+\underline{b}}) \quad (273)$$

as we simply multiply by a_{ij} and sum.

We will need the ring identity

$$\sum_j a_{ij} |B_j| = |A| b_i \quad (274)$$

which can be rewritten in the semiring as

$$\sum_j a_{ij} |B_j|^+ + b_i |A|^- = \sum_j a_{ij} |B_j|^- + b_i |A|^+ \quad (275)$$

We notice that the expansion of Equation (273) has terms similar to Equation (275), except it is missing a term on each side; so we add the corresponding sides of the equation (identity)

$b_i (|A|^+ + |A|^-) = b_i (|A|^+ + |A|^-)$ to Equation (273) to get

$$\begin{aligned} \sum_j a_{ij}|A|^+x_j + \sum_j a_{ij}|B_j|^- + b_i|A|^+ + b_i|A|^- \\ = \sum_j a_{ij}|A|^-x_j + \sum_j a_{ij}|B_j|^+ + b_i|A|^+ + b_i|A|^- . \end{aligned} \quad (276)$$

Using the identity from Equation (275) yields the equation

$$\sum_j a_{ij}|A|^+x_j + b_i|A|^- = \sum_j a_{ij}|A|^-x_j + b_i|A|^+ . \quad (277)$$

which can be written as

$$\sum_j a_{ij}|A|^+x_j + b_i|A|^- = \sum_j a_{ij}|A|^-x_j + b_i|A|^+ . \quad (278)$$

and Equation (278) is equivalent to

$$|A|^+(Ax) + |A|^-b = |A|^- (Ax) + |A|^+b \quad (279)$$

The switching condition implies that either

- (1) $|A|^+ = |A|^-$, or
- (2) $A\underline{x} = \underline{b}$

□

Note 8.102. In a general semiring, it is possible to have a solution to the Cramer's Rule equations in Equation (266) that is not satisfied by the corresponding system since ψ_2 is an epimorphism from S to S_1 .

The following is an example of the above note.

Example 8.103. Let $S = (\mathbb{R}, \cup, \cap)$ be the semiring of sets using "union" as the sum, and "intersection" as the product. Consider the system:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad (280)$$

This equation has the corresponding Cramer's Rule equations:

$$(a \cap d \cap x) \cup (f \cap c) = (b \cap c \cap x) \cup (e \cap d) \quad (281)$$

$$(a \cap d \cap y) \cup (b \cap e) = (b \cap c \cap y) \cup (a \cap f) \quad (282)$$

Consider the sets:

$$\begin{aligned}
a &= \{1, 2\} & b &= \{0, 1\} \\
c &= \{3, 5\} & d &= \{2, 3\} \\
e &= \{3, 5\} & f &= \{3, 4\} \\
x &= \{2, 4\} & y &= \{3, 4\}
\end{aligned} \tag{283}$$

It can be seen that the sets given in Equation (283) make Equations (281) and (282) true, yet they do not satisfy Equation (280). ■

We include, here, an example which illustrates how we can use the adjoint matrix to solve a system of linear equations in the (Max, \times)-algebra system.

Example 8.104. In the Max-Times algebra system, solve the system:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 16 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 2 \\ 16 \end{bmatrix} \tag{284}$$

We translate this problem into the mset algebra, and imbedding the system \mathbb{R}^+ into \mathbb{R} .

$$\begin{bmatrix} \{1, -5\} & \{2, -4\} \\ \{3, -3\} & \{4, -1\} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \{2, -16\} \\ \{16, -3\} \end{bmatrix} \tag{285}$$

We then form the following four quantities:

$$(1) |A|^+ = \{1, -5\}\{4, -1\} = \{-20, -1, 4, 5\}$$

$$(2) |A|^- = \{3, -3\}\{2, -4\} = \{-12, -6, 6, 12\}$$

$$(3) A^{\text{adj}+\underline{b}} = \begin{bmatrix} \{4, -1\} & \emptyset \\ \emptyset & \{1, -5\} \end{bmatrix} \begin{bmatrix} \{2, -16\} \\ \{16, -3\} \end{bmatrix} = \begin{bmatrix} \{-64, -2, 8, 16\} \\ \{-80, -3, 15, 16\} \end{bmatrix}$$

$$(4) A^{\text{adj}-\underline{b}} = \begin{bmatrix} \emptyset & \{2, -4\} \\ \{3, -3\} & \emptyset \end{bmatrix} \begin{bmatrix} \{2, -16\} \\ \{16, -3\} \end{bmatrix} = \begin{bmatrix} \{-64, -6, 12, 32\} \\ \{-48, -6, 6, 48\} \end{bmatrix}$$

Using Equation (266) we get

$$\{-20, -2, 4, 5\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{-64, -6, 12, 32\} \\ \{-48, -6, 6, 48\} \end{bmatrix} = \{-12, -6, 6, 12\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{-64, -2, 8, 16\} \\ \{-80, -3, 15, 16\} \end{bmatrix} \tag{286}$$

After moving “negative” terms to the opposite side of the equation, we get the following equation with all terms positive:

$$\{4, 5, 6, 12\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{2, 12, 32, 64\} \\ \{3, 6, 48, 80\} \end{bmatrix} = \{1, 6, 12, 20\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{6, 8, 16, 64\} \\ \{6, 15, 16, 48\} \end{bmatrix} \quad (287)$$

If we now translate back into the \mathfrak{M}^* -system, we get:

$$12 \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 64 \\ 80 \end{bmatrix} = 20 \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 64 \\ 48 \end{bmatrix} \quad (288)$$

And each of these equations contain one variable, for which we can solve.

$$\begin{array}{ll} 12x \boxplus 64 = 20x \boxplus 64 & 12y \boxplus 80 = 20y \boxplus 48 \\ 64 = 20x \boxplus 64 & 80 = 20y \\ x \leq \frac{16}{5} & y = 4 \end{array}$$

So, the solutions to the original system of equations is:

$$\boxed{\begin{array}{l} x \leq 3.2 \\ y = 4 \end{array}} \quad (289)$$

■

We include, here, a list of the cases that can occur in the \mathfrak{M}^* algebra.

Proposition 8.105 (\mathfrak{M}^*). If $Ax=b$ is a system of equations over the Max-Times algebra then the Cramer's Rule equations are

$$|A|^+ \underline{x} \boxtimes A^{\text{adj}-} \underline{b} = |A|^- \underline{x} \boxtimes A^{\text{adj}+} \underline{b}. \quad (266)$$

For each variable, we arrive at an equation of the form

$$ax_i \boxtimes b_i = cx_i \boxtimes d_i. \quad (290)$$

The solution to this equation leads to nine cases, which can be considered independently:

- | | | |
|------------------------|------------------------|------------------------|
| (1) $a > c, b_i > d_i$ | (4) $a < c, b_i > d_i$ | (7) $a = c, b_i > d_i$ |
| (2) $a > c, b_i < d_i$ | (5) $a < c, b_i < d_i$ | (8) $a = c, b_i < d_i$ |
| (3) $a > c, b_i = d_i$ | (6) $a < c, b_i = d_i$ | (9) $a = c, b_i = d_i$ |

Cases (4)-(6) need not be considered as they are symmetric to cases (1)-(3).

For each of these cases, we analyze the solution to Equation (290).

- (1) Equation (290) has no solution for x_i .
- (2) $x_i = \frac{d_i}{a}$
- (3) $x_i \leq \frac{d_i}{a}$
- (7) $x_i \geq \frac{b_i}{c}$
- (8) $x_i \geq \frac{d_i}{a}$
- (9) In this case there are infinitely many solutions. x_i becomes an arbitrary variable. Use ratios to find solutions.

We include, here, an example for which case (9), above, occurs and in which case we get a parameter of solutions.

Example 8.106. In the Max-Times algebra system, solve the system:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad (291)$$

We translate this problem into the mset algebra, and imbedding the system \mathbb{R}^+ into \mathbb{R} .

$$\begin{bmatrix} \{2, -4\} & \{3, -3\} \\ \{4, -1\} & \{5, -5\} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \{2, -3\} \\ \{7, -2\} \end{bmatrix} \quad (292)$$

We then form the following four quantities:

- (1) $|A|^+ = \{-20, -10, 10, 20\}$
- (2) $|A|^- = \{-12, -3, 3, 12\}$
- (3) $A^{\text{adj}+\underline{b}} = \begin{bmatrix} \{5, -5\} & \emptyset \\ \emptyset & \{2, -4\} \end{bmatrix} \begin{bmatrix} \{2, -3\} \\ \{7, -2\} \end{bmatrix} = \begin{bmatrix} \{-15, -10, 10, 15\} \\ \{-28, -4, 8, 14\} \end{bmatrix}$
- (4) $A^{\text{adj}-\underline{b}} = \begin{bmatrix} \emptyset & \{3, -3\} \\ \{4, -1\} & \emptyset \end{bmatrix} \begin{bmatrix} \{2, -3\} \\ \{7, -2\} \end{bmatrix} = \begin{bmatrix} \{-21, -6, 6, 21\} \\ \{-12, -2, 3, 8\} \end{bmatrix}$

Using Equation (266) we get

$$\{-20, -10, 10, 20\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{-21, -6, 6, 21\} \\ \{-12, -2, 3, 8\} \end{bmatrix} = \{-12, -3, 3, 12\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{-15, -10, 10, 15\} \\ \{-28, -4, 8, 14\} \end{bmatrix} \quad (293)$$

After moving “negative” terms to the opposite side of the equation, we get the following equation with all terms positive:

$$\{3, 10, 12, 20\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{6, 10, 15, 21\} \\ \{3, 4, 8, 28\} \end{bmatrix} = \{3, 10, 12, 20\} \begin{bmatrix} x \\ y \end{bmatrix} \uplus \begin{bmatrix} \{6, 10, 15, 21\} \\ \{2, 8, 12, 14\} \end{bmatrix} \quad (294)$$

If we now translate back into the \mathfrak{M}^* -system, we get:

$$20 \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 21 \\ 28 \end{bmatrix} = 20 \begin{bmatrix} x \\ y \end{bmatrix} \boxplus \begin{bmatrix} 21 \\ 14 \end{bmatrix} \quad (295)$$

And each of these equations contain one variable, for which we can solve.

$$\begin{array}{ll} 20y \boxplus 28 = 20y \boxplus 14 & 20x \boxplus 21 = 20x \boxplus 21 \\ 20y \boxplus 28 = 20y & x \geq \frac{21}{20}, \text{ and use ratios} \\ y \geq \frac{7}{5} = 1.4 & \end{array}$$

The equation for “x” implies there are infinitely many solutions, so we use ratios to determine the relationship between the variables.

$$\left(x = \frac{21}{20} \text{ and } y = \frac{7}{5} \right) \implies \left(x = \frac{3}{4}y \right)$$

So, the solutions to the original system of equations is:

$$\boxed{\begin{array}{l} y \geq 1.4 \text{ and} \\ x = \frac{3}{4}y \end{array}} \quad (296)$$

■

8.15 A Connection between \mathfrak{M}^* and \mathbb{R}^+ .

In this section we attempt to show a connection between the two different algebras: \mathfrak{M}^* and \mathbb{R}^+ . This connection can be used to find the eigenvalue/eigenvectors of a matrix when considered as a matrix over either \mathbb{R}^+ or \mathfrak{M}^* . Msets plays the crucial role of connecting these two different algebras.

The goal is to find an eigenvalue/eigenvector equation over msets which contains the information for both the \mathfrak{M}^* -algebra, and \mathbb{R} -algebra.

Preliminaries We begin our discussion with a few definitions. First is the concept of a mset norm. I will use the notation “ $\|\cdot\|$ ” for the norm. For the present discussion this norm will represent either the usual max-norm, $\|\cdot\|_\infty$, sometimes called the infinity norm, and the “1-norm”, $\|\cdot\|_1$.

If α is an mset then we define:

$$\|\alpha\| = \begin{cases} \|\alpha\|_\infty \equiv \max_i(\alpha_i) & \text{if } \alpha \in \mathfrak{M}^* \\ \|\alpha\|_1 \equiv \sum_i(\alpha_i) & \text{if } \alpha \in \mathfrak{R}^+ \end{cases} \quad (297)$$

Let us verify that both norms are multiplicative and additive.

Lemma 8.107. $\|\cdot\|$ is multiplicative and additive for the two norms of this section. In particular,

- (a) $\|\cdot\|_\infty$ is multiplicative over \mathfrak{M}^* .
- (b) $\|\cdot\|_\infty$ is additive over \mathfrak{M}^* .
- (c) $\|\cdot\|_1$ is multiplicative over \mathbb{R} .
- (d) $\|\cdot\|_1$ is additive over \mathbb{R} .

Proof.

(a)

$$\begin{aligned} \|\alpha \uplus \beta\| &= \|\alpha \uplus \beta\|_\infty \\ &= \max(\alpha \uplus \beta) \\ &= \max(\max_i(\alpha_i), \max_j(\beta_j)) \\ &= \max(\|\alpha\|_\infty, \|\beta\|_\infty) \\ &= \|\alpha\| \oplus \|\beta\| \end{aligned} \quad (298)$$

(b)

$$\begin{aligned} \|\alpha \boxtimes \beta\| &= \|\alpha \boxtimes \beta\|_\infty \\ &= \max_{i,j}(\alpha_i \beta_j) \\ &= \max(\alpha_i) \cdot \max(\beta_j) \\ &= \|\alpha\|_\infty \cdot \|\beta\|_\infty \\ &= \|\alpha\| \otimes \|\beta\| \end{aligned} \quad (299)$$

(c)

$$\begin{aligned}
\|\alpha \uplus \beta\| &= \|\alpha \uplus \beta\|_1 \\
&= \sum_k (\alpha \uplus \beta) \\
&= \sum_i (\alpha_i) + \sum_j (\beta_j) \\
&= \|\alpha\|_1 + \|\beta\|_1 = \|\alpha\| + \|\beta\|
\end{aligned} \tag{300}$$

(d)

$$\begin{aligned}
\|\alpha \boxtimes \beta\| &= \|\alpha \boxtimes \beta\|_1 \\
&= \sum_{i,j} (\alpha_i \beta_j) \\
&= \sum_i (\alpha_i) \cdot \sum_j (\beta_j) \\
&= \|\alpha\|_1 \cdot \|\beta\|_1 \\
&= \|\alpha\| \otimes \|\beta\|
\end{aligned} \tag{301}$$

□

Definition 8.108. If \underline{x} is a vector with msets as elements then $n(\underline{x}) = \begin{bmatrix} \|x_1\| \\ \|x_2\| \\ \vdots \\ \|x_n\| \end{bmatrix}$.

The Connection The following theorem is important in that it allows us to find some of the necessary requirements for an irreducible mset matrix to possess an eigenvalue/eigenvector equation.

Theorem 8.109. If A is an irreducible mset matrix and $A\underline{x} = \lambda \underline{x}$, then $\|\lambda\|n(\underline{x}) = n(A)n(\underline{x})$.

Proof. Suppose A is an irreducible mset matrix, and $A\underline{x} = \lambda \underline{x}$ is an mset eigenvalue equation. Then, for each i ,

$$\lambda x_i = \biguplus_{k=1}^n a_{ik} \boxtimes x_k \tag{302}$$

We consider two cases:

(Case I) For \mathbb{R} , we apply the “1-norm” to each side of Equation (302).

$$\begin{aligned}
\|\lambda \underline{x}_i\| &= \|\lambda\| \|x_i\| \\
&= \left\| \biguplus_{k=1}^n a_{ik} x_k \right\| \\
&= \sum_{k=1}^n \|a_{ik} x_k\| \\
&= \sum \|a_{ik}\| \cdot \|x_k\| \\
&= \sum (n(A))_{ik} (n(\underline{x}))_k
\end{aligned} \tag{303}$$

Therefore, Equation (302) becomes

$$\|\lambda\|_1 n(\underline{x}) = n(A) n(\underline{x}) \tag{304}$$

(Case II) For \mathfrak{M}^* , we apply the “ ∞ -norm” to each side of Equation (302):

$$\begin{aligned}
\|\lambda \underline{x}_i\|_\infty &= \max\{\lambda \boxtimes x_i\} \\
&= \max\{\lambda\} \cdot \max\{x_i\} \\
&= \max_k \{a_{ik} \cdot x_k\} \\
&= \max_k \{\max\{a_{ik}\} \cdot \max\{x_k\}\} \\
&= \max \|a_{ik}\| \cdot \|x_k\|
\end{aligned} \tag{305}$$

Therefore, Equation (302) becomes:

$$\|\lambda\|_\infty n(\underline{x}) = n(A) n(\underline{x}) \tag{306}$$

So Equation (302) is true for both cases. □

Corollary 8.110. If A is an irreducible mset matrix and $A\underline{x} = \lambda \underline{x}$, with $\underline{x} \geq \underline{\emptyset}$, then

(a) $\|\lambda\|_\infty = \nu(n(A))$

(b) $\|\lambda\|_1 = \rho(n(A))$

Proof. We shall assume that $n(\underline{x}) > 0$, since otherwise we could multiply by $(I \uplus A)^{n-1} > 0$.

Equation (304) is a matrix equation in \mathbb{R} , while Equation (306) is a matrix equation in \mathfrak{M}^* . In each case $n(A) \geq 0$ and irreducible, and $n(\underline{x}) > 0$. In \mathbb{R} , we know that ρ is eigenvalue of $n(A)$. For \mathfrak{M}^* , we know that $\nu(A)$ is the unique eigenvalue of $n(A)$. In either case the result follows. □

Corollary 8.111. If A is an irreducible matrix over \mathbb{R}^+ , or \mathfrak{M}^* , and $A\underline{x} = \lambda \underline{x}$, with $\underline{x} \geq \underline{\emptyset}$, then

(a) $\|\lambda\|_\infty = \nu(A)$

(b) $\|\lambda\|_1 = \rho(A)$

Proof. If A is a matrix in \mathfrak{M}^* or \mathbb{R} , then the mset matrix associated with A has a one-element mset in each entry. So $n(A)=A$, and the result follows. □

Example 8.112. Consider the matrix $\begin{bmatrix} 6 & 1 \\ 4 & 6 \end{bmatrix}$ as a matrix in either \mathfrak{M}^* or \mathbb{R} . The mset eigenvalue equation is

$$\begin{bmatrix} \{6\} & \{1\} \\ \{4\} & \{6\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \{2\} \end{bmatrix} = \{6, 2\} \boxtimes \begin{bmatrix} \{1\} \\ \{2\} \end{bmatrix} \tag{307}$$

In \mathbb{R} , we get the conventional eigenvalue equation

$$\begin{bmatrix} 6 & 1 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 8 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{308}$$

And, in \mathfrak{M}^* , we get the eigenvalue equation

$$\begin{bmatrix} 6 & 1 \\ 4 & 6 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 6 \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{309}$$

■

Corollary 8.111 gives the fact that if a matrix has an eigenvalue equation over the mset algebra, then we can apply $\|\cdot\|_1$ to get the Real eigenvalue equation, and we can apply $\|\cdot\|_\infty$ to get the Max-Times equation. Although the theory is nice, the caveat is that a general mset matrix – even one with only one element in each mset – may not have an mset eigenvalue equation. The following example illustrates this fact.

Example 8.113. Consider the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ as a matrix in either \mathfrak{M}^* or \mathbb{R} .

Then $\rho(A)=9$, and $\nu(A)=7$.

Therefore the mset eigenvalue equation for A must be:

$$\begin{bmatrix} \{4\} & \{2\} & \{1\} \\ \{2\} & \{7\} & \{2\} \\ \{1\} & \{2\} & \{4\} \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \{7, \lambda_2, \lambda_3\} \boxtimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{310}$$

where x,y,z are msets, and $\lambda_2 + \lambda_3 = 2$ and $7 \geq \lambda_2 \geq \lambda_3$.

If one of x, y , or z equals \emptyset , then all msets must equal \emptyset . Hence all three elements must be nonempty. So $|\lambda| = 3$.

This leads to the following equations:

$$4x \uplus 2y \uplus z = \{7, \lambda_2, \lambda_3\}x \quad (311)$$

$$2x \uplus 7y \uplus 2z = \{7, \lambda_2, \lambda_3\}y \quad (312)$$

$$x \uplus 2y \uplus 4z = \{7, \lambda_2, \lambda_3\}z \quad (313)$$

Equation (312) is simplified as

$$2x \uplus 2z = \{\lambda_2, \lambda_3\}y \quad (314)$$

$$4x \uplus 4z = 2\{\lambda_2, \lambda_3\}y \quad (315)$$

$$x \uplus z = \frac{1}{2}\{\lambda_2, \lambda_3\}y \quad (316)$$

If we add the LHS and RHS of Equations (311) and (313), and replace $(x \uplus z)$ by Equation (316), and replace $(4x \uplus 4y)$ by Equation (315), then we get:

$$\left\{ \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, 2, 2, 2\lambda_2, 2\lambda_3 \right\} y = \frac{1}{2} \{7\lambda_2, 7\lambda_3, \lambda_2^2, \lambda_3^2, \lambda_2\lambda_3, \lambda_2\lambda_3\} y \quad (317)$$

If $y = \emptyset$ then $x = z = \emptyset$, which can not occur in an eigenvalue equation. Therefore $y \neq \emptyset$.

Equation (317) becomes:

$$\{\lambda_2, \lambda_3, 4, 4, 4\lambda_2, 4\lambda_3\} = \{7\lambda_2, 7\lambda_3, \lambda_2^2, \lambda_3^2, \lambda_2\lambda_3, \lambda_2\lambda_3\} \quad (318)$$

It can be shown that there is no λ_2 and λ_3 that will satisfy Equation (318).

Therefore the matrix A has no mset eigenvalue equation. ■

Example 8.114. We return once more to Example 8.78. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. For this example,

$\rho(A) = 3$ and $\nu(A) = 2$. The eigenvalue equation for matrix A is:

$$\begin{bmatrix} \{2\} & \{1\} & \{1\} \\ \{1\} & \{2\} & \{1\} \\ \emptyset & \emptyset & \{1\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \{1\} \\ \emptyset \end{bmatrix} = \{2, 1\} \boxtimes \begin{bmatrix} \{1\} \\ \{1\} \\ \emptyset \end{bmatrix} \quad (319)$$

This mset eigenvalue equation will collapse to either the Real eigenvalue equation or to the Max-eigenvalue equation, depending upon which norm is used. ■

The question of which matrices possess an mset eigenvalue equation remains open. It will most likely involve higher combinatorics.

We will classify the simple 2×2 matrices which possess an eigenvalue equation.

Theorem 8.115. If A is a 2×2 matrix with one element in each mset, then A has an mset eigenvalue equation if and only if

(1) A is a diagonal matrix.

(2) A is reducible.

(3) A is singular.

(4) A has the form. $\begin{bmatrix} \{a\} & \{c\} \\ \{b\} & \{a\} \end{bmatrix}$

Proof. Assume that $a \geq d$ in the proofs.

$$(1) \begin{bmatrix} \{a\} & \emptyset \\ \emptyset & \{d\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \emptyset \end{bmatrix} = \{a\} \boxtimes \begin{bmatrix} \{1\} \\ \emptyset \end{bmatrix}$$

$$(2) \begin{bmatrix} \{a\} & \{c\} \\ \emptyset & \{d\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \emptyset \end{bmatrix} = \{a\} \boxtimes \begin{bmatrix} \{1\} \\ \emptyset \end{bmatrix}$$

$$(3) \begin{bmatrix} \{a\} & \{c\} \\ \{ka\} & \{kc\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \{k\} \end{bmatrix} = \{a, ck\} \boxtimes \begin{bmatrix} \{1\} \\ \{k\} \end{bmatrix}$$

The proof of part (3) is sufficient for any rank one matrix as

$$\begin{bmatrix} \{1\} \\ \{k\} \end{bmatrix} \boxtimes \begin{bmatrix} \{a\} & \{c\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \{k\} \end{bmatrix} = \{a, ck\} \boxtimes \begin{bmatrix} \{1\} \\ \{k\} \end{bmatrix}$$

$$(4) \begin{bmatrix} \{a\} & \{c\} \\ \{b\} & \{a\} \end{bmatrix} \boxtimes \begin{bmatrix} \{1\} \\ \{\frac{\sqrt{bc}}{c}\} \end{bmatrix} = \{a, \sqrt{bc}\} \boxtimes \begin{bmatrix} \{1\} \\ \{\frac{\sqrt{bc}}{c}\} \end{bmatrix}$$

For the necessity part in (4), consider the matrix $A = \begin{bmatrix} \{a\} & \{c\} \\ \{b\} & \{d\} \end{bmatrix}$. Assume that $a > d$, and

$a^2 > bc$. Then $\nu(A) = a$, and $\rho(A) = \frac{(a+d) + \sqrt{(a-d)^2 + 4bc}}{2}$. $\mu = \rho - \nu = \rho - a = \frac{(a-d) + \sqrt{(a-d)^2 + 4bc}}{2}$.

So we require

$$\begin{bmatrix} \{a\} & \{c\} \\ \{b\} & \{d\} \end{bmatrix} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} = \{a, \mu\} \boxtimes \begin{bmatrix} x \\ y \end{bmatrix} \quad (320)$$

So we have

$$ax \uplus cy = ax \uplus \mu x \implies cy = \mu x \quad (321)$$

$$bx \uplus dy = ay \uplus \mu y \quad (322)$$

If $c = \emptyset$, then the matrix is reducible, otherwise we have $y = \frac{\mu}{c}x$.

Therefore Equation (322) becomes

$$x \left\{ b, \frac{d\mu}{c} \right\} = x \left\{ \frac{a\mu}{c}, \frac{\mu^2}{c} \right\} \quad (323)$$

If $x=\emptyset$, then from Equation (321) $y=\emptyset$, which can not occur. Therefore Equation (323) implies that

$$\left\{ b, \frac{d\mu}{c} \right\} = \left\{ \frac{a\mu}{c}, \frac{\mu^2}{c} \right\} \quad (324)$$

So there are two cases to consider:

$$(I) \begin{cases} bc = a\mu \\ d\mu = \mu^2 \end{cases}$$

Therefore, $\mu=d$ or $\mu=\emptyset$.

$d=\mu \implies ad=bc$. In this case the matrix A is singular. If $\mu=\emptyset$ then $d=\emptyset$ and $bc=\emptyset$.

So either $b=\emptyset$ or $c=\emptyset$, and the matrix A is reducible.

$$(II) \begin{cases} bc = \mu^2 \\ d\mu = a\mu \end{cases}$$

Either $a=d$ or $\mu=\emptyset$. If $\mu=\emptyset$ then $bc=\emptyset$ and the matrix A is is reducible. Suppose $a=d$.

So $\mu = \sqrt{bc}$ and $y = \frac{\sqrt{bc}}{c}x$. The eigenvalue equation of part (4) follows.

One last form to consider is $\begin{bmatrix} a & c \\ b & \emptyset \end{bmatrix}$. It can be shown that, in this case, the matrix must either

be reducible or it must be of the form $\begin{bmatrix} \emptyset & c \\ b & \emptyset \end{bmatrix}$.

□

For matrices with more than one element in each mset or larger than 2×2 , the theory becomes more difficult.

8.16 End Notes

The study of msets is a method to incorporate the theory of convex cones into linear algebra. The mset algebra can be used in much the same way that the ordinary algebra is used. The mset algebra is an attempt to keep as much information as possible about a particular problem available throughout the analysis. This certainly makes the investigation into these topics more complicated than the usual (conventional) algebra system. As may have been noticed in some of the examples earlier in this section, and particularly in Section 8.3, one may wish to discuss a homomorphism from $3^{\mathbb{S}}$ to \mathbb{S} . This is the same definition given in Definition 8.10.

Definition 8.116. Define the function $\psi : \mathfrak{3}^{\mathbb{S}} \rightarrow \mathbb{S}$

$$\psi(\alpha) = \bigoplus_k \alpha_k \quad (325)$$

Proposition 8.117. ψ is a homomorphism from $\mathfrak{3}^{\mathbb{S}}$ to \mathbb{S} .

(1) $\psi(\alpha \uplus \beta) = \psi(\alpha) \oplus \psi(\beta)$

(2) $\psi(\alpha \boxtimes \beta) = \psi(\alpha) \otimes \psi(\beta)$

(3) $\psi(\emptyset) = 0$

Proof.

(8.117.1)

$$\psi(\alpha \uplus \beta) = \bigoplus_k (\alpha \uplus \beta) \quad (326)$$

$$= \left(\bigoplus_i \alpha_i \right) \oplus \left(\bigoplus_j \beta_j \right) \quad (327)$$

$$= \psi(\alpha) \oplus \psi(\beta) \quad (328)$$

(8.117.2)

$$\psi(\alpha \boxtimes \beta) = \bigoplus_k (\alpha \boxtimes \beta) \quad (329)$$

$$= \bigoplus_k (\uplus_i \uplus_j \alpha_i \beta_j) \quad (330)$$

$$= \left(\bigoplus_k (\uplus_i \alpha_i) \right) \otimes \left(\bigoplus_k (\uplus_j \beta_j) \right) \quad (331)$$

$$= \left(\bigoplus_i \alpha_i \right) \otimes \left(\bigoplus_j \beta_j \right) \quad (332)$$

$$= \psi(\alpha) \otimes \psi(\beta) \quad (333)$$

(8.117.3) is follows from the definition. □

By using the homomorphism, some information is lost in going from $\mathfrak{3}^{\mathbb{S}}$ to \mathbb{S} , but the algebra may become much easier. As an example we consider the set $\mathbb{S} = \mathbb{R}$ with the usual operations.

Example 8.118. Consider the equation: $\alpha x = \beta$.

$$\alpha x = \beta \tag{334}$$

$$\psi(\alpha x) = \psi(\beta) \tag{335}$$

$$\left(\sum_i \alpha_i \right) \cdot \left(\sum_j x_j \right) = \left(\sum_k \beta_k \right) \tag{336}$$

$$\left(\sum_j x_j \right) = \frac{\left(\sum_k \beta_k \right)}{\left(\sum_i \alpha_i \right)} \tag{337}$$

In the conventional algebra this would be written as $x = \frac{\beta}{\alpha}$. ■

Note 8.119. We cannot take a matrix equation in \mathbb{R} , translate it to the Mset algebra, and get a true mset equation.

Example 8.120. Consider the matrix equation: $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \end{bmatrix}$

This equation is true in \mathbb{R} , but the corresponding equation is not true in the Mset algebra.

$$\begin{bmatrix} \{2\} & \{3\} \\ \{5\} & \{4\} \end{bmatrix} \begin{bmatrix} \{2\} \\ \{4\} \end{bmatrix} \neq \begin{bmatrix} \{16\} \\ \{26\} \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} \{4, 12\} \\ \{10, 16\} \end{bmatrix} \neq \begin{bmatrix} \{16\} \\ \{26\} \end{bmatrix}$$

■

It should be pointed out though that, if we apply the map ψ to the last equation above, we get a true statement.

9 Characteristic Polynomial in Max-Times Algebra

9.1 Algorithm

In the past sections, we have developed an equation which is satisfied by the given equation. Although this equation can be applied to many different algebra systems, we concentrate on the Max-Times Algebra system in this section.

We would like to find an algorithm which can be used to find the characteristic equation of a matrix in the Max-Times Algebra system without using all of the knowledge of the mset theory. We begin with the algorithm to find the Characteristic Equation for a Matrix in \mathfrak{M}^* . The proof of the algorithm comes later in the section.

Algorithm 9.1. Given matrix $A_{n \times n} \in \mathfrak{M}_{n \times n}^*$.

1. For $k = 1$ to n
2. Let $\gamma_k^e =$ set of all terms in $|A_\phi^e|$ which are obtained from even permutations, where $\phi \in C_k^n$.
3. Let $\gamma_k^o =$ set of all terms in $|A_\phi^o|$ which are obtained from odd permutations, where $\phi \in C_k^n$.
4. If $\gamma_k^e = \gamma_k^o = \{\}$ then $c_k = 0$, and goto Step (13).
5. Let $\Gamma_k^e = \max\{\gamma_k^e\}$. If $\gamma_k^e = \emptyset$ then $\Gamma_k^e = 0$.
6. Let $\Gamma_k^o = \max\{\gamma_k^o\}$. If $\gamma_k^o = \emptyset$ then $\Gamma_k^o = 0$.
7. If $\Gamma_k^e \neq \Gamma_k^o$ then goto Step (9)
8. If $\Gamma_k^e = \Gamma_k^o$ then $\gamma_k^e := \gamma_k^e \setminus \Gamma_k^e$ and $\gamma_k^o := \gamma_k^o \setminus \Gamma_k^o$. Goto (4).
9. Let $\Gamma_k^* = \max_{\zeta \in \Gamma_k^e \cup \Gamma_k^o} \{\zeta \mid \zeta \notin \Gamma_k^e \cap \Gamma_k^o\}$.
10. Let $\zeta_k =$ the permutation in $\Gamma_k^e \cup \Gamma_k^o$ which yields the maximum.
11. Let $\text{sgn}(\zeta) = \begin{cases} 1 & \text{if } \zeta \text{ is an even permutation} \\ -1 & \text{if } \zeta \text{ is an odd permutation} \end{cases}$
12. Let $c_k = (-1)^k \cdot \text{sgn}(\zeta_k) \cdot \Gamma_k^*$.
13. Next k
14. The terms with $c_k > 0$ are on the LHS of the equation, and terms with $c_k < 0$ are on the RHS of the equation. This yields an equation of the form:

$$\lambda^k \boxplus \left(\boxplus_{c_k > 0} c_k \lambda^{n-k} \right) = \boxplus_{c_k < 0} (-1 \cdot c_k) \lambda^{n-k} \quad (338)$$

Justification. The justification for the above algorithm will be developed shortly, in particular, see Theorem 9.11.

First, let us look at a few examples.

9.2 Examples

A few examples will be quite illustrative.

Example 9.2. Find the characteristic equation for: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 5 & 1 & 3 \end{bmatrix} \in \mathfrak{M}_{n \times n}^*$ ■

$$\Gamma_1^e = \{1, 7, 3\}, \quad \Gamma_2^e = \{3, 7, 21\}, \quad \Gamma_2^o = \{8, 9, 15\}$$

$$\Gamma_3^e = \{12, 21, 90\} \quad \Gamma_3^o = \{9, 24, 105\}$$

$$\Gamma_1^* = 7, \quad \Gamma_2^* = 21, \quad \Gamma_3^* = 105$$

$$c_1 = (-1)^1 * (1) * 7 = -7 \quad \text{Note: } \zeta_7 \in \Gamma_1^e$$

$$c_2 = (-1)^2 * (1) * 21 = 21 \quad \text{Note: } \zeta_{21} \in \Gamma_2^e$$

$$c_3 = (-1)^3 * (-1) * 105 = 105 \quad \text{Note: } \zeta_{105} \in \Gamma_3^o$$

Therefore we have $P(\lambda) = \lambda^3 \boxplus 21 \boxtimes \lambda \boxplus 105$, and

$$Q(\lambda) = 7 \boxtimes \lambda^2.$$

And the characteristic equation is:

$$\lambda^3 \boxplus 21 \boxtimes \lambda \boxplus 105 = 7 \boxtimes \lambda^2.$$

We can see that the matrix A satisfies:

$$A^3 \boxplus 21 \boxtimes A \boxplus 105 \boxtimes I = 7 \boxtimes A^2, \text{ as each side of the equation is: } \begin{bmatrix} 105 & 98 & 126 \\ 315 & 343 & 441 \\ 105 & 70 & 105 \end{bmatrix}$$

Example 9.3. Consider the matrix: $A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 3 & 2 \\ 3 & 0 & 3 \end{bmatrix} \in \mathfrak{M}_{n \times n}^*$ ■

$$\Gamma_1^e = \{1, 3, 3\}, \quad \Gamma_2^e = \{3, 3, 9\}, \quad \Gamma_2^o = \{0, 6, 10\}$$

$$\Gamma_3^e = \{0, 9, 30\}, \quad \Gamma_3^o = \{0, 18, 30\}$$

$$\Gamma_1^* = 3, \quad \Gamma_2^* = 10, \quad \Gamma_3^* = 18$$

$$c_1 = (-1)^1 * (1) * 3 = -3 \quad \text{Note: } \zeta_3 \in \Gamma_1^e$$

$$c_2 = (-1)^2 * (-1) * 10 = -10 \quad \text{Note: } \zeta_{10} \in \Gamma_2^o$$

$$c_3 = (-1)^3 * (-1) * 18 = 18 \quad \text{Note: } \zeta_{18} \in \Gamma_3^o$$

There we see that the characteristic equation is:

$$\lambda^3 \boxplus 18 = 3 \boxtimes \lambda^2 \boxplus 10 \boxtimes \lambda,$$

and we note that the matrix A satisfies: $A^3 \boxplus 18I = 3A^2 \boxplus 10A$. In this case we have 30 as the maximum from both an even permutation and an odd permutation of length 3, so we use 18 which is the next largest permutation product.

Define the degree of an equation in the most natural way: the degree of an equation is the largest degree of the polynomial on either side of the equal sign.

As it is with the conventional linear algebra, we ask the natural question of whether we can find a polynomial equation of smaller degree which is satisfied by the matrix.

Example 9.4. For the matrix $A=J_{3 \times 3}$, the characteristic equation using the above algorithm is $x^3 = x^2$, but it can be checked that $x^2 = x$ is also an equation of smaller degree. ■

Example 9.5. For the matrix $A=J_{n \times n}$, the characteristic equation using the above algorithm is $x^n = x^{n-1}$, but it can be checked that $x^2 = x$ is also an equation of smaller degree. ■

Example 9.6. For the matrix $A=J_{3 \times 3}-I_{3 \times 3}$, the characteristic equation using the above algorithm is $x^3 = x \boxplus 1$, but it can be checked that $x^2 = x \boxplus 1$ is also an equation of smaller degree. ■

Example 9.7. For the matrix $A=J_{n \times n}-I_{n \times n}$, the characteristic equation using the above algorithm is $x^n = \boxplus_{k=2}^n x^{n-k}$, but it can be checked that $x^2 = x \boxplus 1$ is also an equation of smaller degree. ■

The above four examples show that there may very well exist a polynomial equation of smaller degree which is satisfied by the given matrix. This, of course, leads to the questions:

1. How can we determine whether an equation of lower degree exists?
2. If another equation exists, how can we find it?

The following will help clarify some of the information about σ .

Proposition 9.8. (1) If k is even, and σ is an even permutation,
then $\text{sig}(\sigma)=(-1)^k \text{sgn}(\sigma) = 1$

(2) If k is even, and σ is an odd permutation,
then $\text{sig}(\sigma)=(-1)^k \text{sgn}(\sigma) = -1$

(3) If k is odd, and σ is an even permutation,
then $\text{sig}(\sigma)=(-1)^k \text{sgn}(\sigma) = -1$

(4) If k is odd, and σ is an odd permutation,
then $\text{sig}(\sigma)=(-1)^k \text{sgn}(\sigma) = 1$

Note 9.9. If $a, b \in \mathbb{R}^+$, and $a \leq b$, then the expression $ax^\ell \boxplus bx^\ell$ can be reduced to the expression bx^ℓ , i.e. $ax^\ell \boxplus bx^\ell = bx^\ell$.

To see how Proposition 9.8 relates to the proof of Algorithm 9.1 it can be seen that the parity of k and the parity of σ determine which side of the equation the term will occur. This means that $p_A^+(x)$ has the terms of x^{n-q} which have a coefficient in which σ is an permutation of $\text{par}(q)$, and $p_A^-(x)$ has the terms of x^{n-q} which have a coefficient in which σ is an permutation of $\text{par}(q+1)$. This is yet another, possibly easier, way of classifying the terms of the equation. $p_A^+(x) = p_A^-(x)$.

Example 9.10. Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{aligned} \det(xI - A) &= 0 \\ x^3 - (a_{11} + a_{22} + a_{33})x^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})x \\ &\quad + (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) = 0 \\ x^3 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})x + (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}) \\ &= (a_{11} + a_{22} + a_{33})x^2 + (a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32})x \\ &\quad + (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \end{aligned}$$

■

Theorem 9.11 (Algorithm). Algorithm 9.1 is correct.

Proof. It was shown in Theorem 8.71, that a polynomial exists. We need to show that the part of the algorithm which requires dropping terms with equal coefficients from both sides, and take the largest coefficient for that term which is not shared by both sides.

Consider the set equation in Theorem 8.68.

Suppose that there exists σ_1 as an “even” permutation, and σ_2 as an “odd” permutation, both of length k , with $\omega(\sigma_1) = \omega(\sigma_2)$, and $|\sigma_1| = |\sigma_2|$.

If $|\sigma_1| = k$, then there is a permutation π_1 associated with σ_1 with $|\pi_1| = n - k$. We can assume, without loss of generality, that $(\sigma_1, \pi_1) \in T_{ij}^+$, and $(\sigma_2, \pi_2) \in T_{ij}^-$. There exists $i_1, j_1 \in S_n$ such that $\eta_{i_1, j_1}(\sigma_1, \pi_1) = (\sigma'_1, \pi'_1)$, with $\omega(\sigma_1)\omega(\pi_1) = \omega(\sigma'_1)\omega(\pi'_1)$, and there exists $i_2, j_2 \in S_n$ such that $\eta_{i_2, j_2}(\sigma_2, \pi_2) = (\sigma'_2, \pi'_2)$, with $\omega(\sigma_2)\omega(\pi_2) = \omega(\sigma'_2)\omega(\pi'_2)$.

This means that the element of the set corresponding to (σ_1, π_1) on one side and (σ'_1, π'_1) on the other side occur is the same element.

Therefore we can delete these two elements from both sides from the equation (Equation (8.68)).

Likewise, we can drop the element on each side corresponding to (σ_2, π_2) on one side and (σ'_2, π'_2) on the other side. After dropping the identical elements from both sides of the equation, we apply the maximum-function to both sides to get another equation which must hold. □

Note 9.12. Although the equation given by Theorem 8.71 is a polynomial equation which is satisfied by the matrix A, it may have terms of the same degree on each side of the equation. In this thesis we refer to the polynomial equation derived from Algorithm 9.1 as THE CHARACTERISTIC POLYNOMIAL of the matrix A.

As an example we can show equality of the sets in the 2x2 case.

Example 9.13. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. ■

$$A^2 = \begin{bmatrix} \{a_{11}a_{11}, a_{12}a_{21}\} & \{a_{11}a_{12}, a_{12}a_{22}\} \\ \{a_{11}a_{21}, a_{22}a_{21}\} & \{a_{21}a_{12}, a_{22}a_{22}\} \end{bmatrix}.$$

We show equality of the sets in the identity (equation):

$$\begin{array}{cc} \biguplus_{|\sigma|=0} \omega(\sigma) \odot A^2 & \biguplus_{|\sigma|=2} \omega(\sigma) \odot I = & \biguplus_{|\sigma|=1} \omega(\sigma) \odot A & \biguplus_{|\sigma|=2} \omega(\sigma) \odot I \\ |\pi|=2 & |\pi|=0 & |\pi|=1 & |\pi|=0 \\ \text{sig}(\sigma) = 1 & \text{sig}(\sigma) = 1 & \text{sig}(\sigma) = -1 & \text{sig}(\sigma) = -1 \end{array}$$

where

$$\begin{array}{cc} \biguplus_{|\sigma|=0} \omega(\sigma) \odot A^2 = \{1\} \odot A^2 & , & \biguplus_{|\sigma|=2} \omega(\sigma) \odot I = \{a_{11}a_{22}\} \odot I \\ |\pi|=2 & & |\pi|=0 \\ \text{sig}(\sigma) = 1 & & \text{sig}(\sigma) = 1 \end{array}$$

$$\begin{array}{cc} \biguplus_{|\sigma|=1} \omega(\sigma) \odot A = \{a_{11}, a_{22}\} \odot A & , & \biguplus_{|\sigma|=2} \omega(\sigma) \odot I = \{a_{12}a_{21}\} \odot I \\ |\pi|=1 & & |\pi|=0 \\ \text{sig}(\sigma) = -1 & & \text{sig}(\sigma) = -1 \end{array}$$

So,

$$\begin{aligned}
& \begin{array}{cc} \bigoplus & \omega(\sigma) \odot A^2 \\ |\sigma| = 0 & \\ |\pi| = 2 & \\ \text{sig}(\sigma) = 1 & \end{array} & \begin{array}{cc} \bigoplus & \omega(\sigma) \odot I \\ |\sigma| = 2 & \\ |\pi| = 0 & \\ \text{sig}(\sigma) = 1 & \end{array} \\
& = \{1\} \odot \begin{bmatrix} \{a_{11}a_{11}, a_{12}a_{21}\} & \{a_{11}a_{12}, a_{12}a_{22}\} \\ \{a_{11}a_{21}, a_{22}a_{21}\} & \{a_{21}a_{12}, a_{22}a_{22}\} \end{bmatrix} \bigoplus \{a_{11}a_{22}\} \odot \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix} \\
& = \begin{bmatrix} \{a_{11}a_{11}, a_{12}a_{21}\} & \{a_{11}a_{12}, a_{12}a_{22}\} \\ \{a_{11}a_{21}, a_{22}a_{21}\} & \{a_{21}a_{12}, a_{22}a_{22}\} \end{bmatrix} \bigoplus \begin{bmatrix} \{a_{11}a_{22}\} & \{0\} \\ \{0\} & \{a_{11}a_{22}\} \end{bmatrix} \\
& = \begin{bmatrix} \{a_{11}a_{11}, a_{12}a_{21}, a_{11}a_{22}\} & \{a_{11}a_{12}, a_{12}a_{22}\} \\ \{a_{11}a_{21}, a_{22}a_{21}\} & \{a_{21}a_{12}, a_{22}a_{22}, a_{11}a_{22}\} \end{bmatrix} \\
& = \begin{bmatrix} \{a_{11}, a_{22}\} \odot \{a_{11}\} \uplus \{a_{12}a_{21}\} & \{a_{11}, a_{22}\} \uplus \{a_{12}\} \\ \{a_{11}, a_{22}\} \odot \{a_{21}\} & \{a_{22}, a_{11}\} \odot \{a_{22}\} \uplus \{a_{21}a_{12}\} \end{bmatrix} \\
& = \{a_{11}, a_{22}\} \odot A \uplus \{a_{12}a_{21}\} \odot I \\
& = \begin{array}{cc} \bigoplus & \omega(\sigma) \odot A \\ |\sigma| = 1 & \\ |\pi| = 1 & \\ \text{sig}(\sigma) = -1 & \end{array} & \begin{array}{cc} \bigoplus & \omega(\sigma) \odot I \\ |\sigma| = 2 & \\ |\pi| = 0 & \\ \text{sig}(\sigma) = -1 & \end{array}
\end{aligned}$$

Using the notation of σ_k^e , and σ_k^o , the above equation can be written briefly as:

$$\sigma_0^e \odot A^2 \uplus \sigma_1^o \odot A \uplus \sigma_2^e \odot I = \sigma_0^o \odot A^2 \uplus \sigma_1^e \odot A \uplus \sigma_2^o \odot I$$

where

$$\begin{aligned}
\sigma_0^e &= \{1\} & \sigma_0^o &= \{0\} \\
\sigma_1^e &= \{a_{11}, a_{22}\} & \sigma_1^o &= \{0,0\} \\
\sigma_2^e &= \{a_{11}a_{22}\} & \sigma_2^o &= \{a_{12}a_{21}\}
\end{aligned}$$

Example 9.14. For the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. ■

The characteristic equation can be written as:

$$\begin{array}{ccc}
\begin{array}{l} \uplus \\ |\sigma| = 0 \\ |\pi| = 3 \\ \text{sig}(\sigma) = 1 \end{array} & \omega(\sigma) \odot A^3 & \begin{array}{l} \uplus \\ |\sigma| = 2 \\ |\pi| = 1 \\ \text{sig}(\sigma) = 1 \end{array} & \omega(\sigma) \odot A & \begin{array}{l} \uplus \\ |\sigma| = 3 \\ |\pi| = 0 \\ \text{sig}(\sigma) = 1 \end{array} & \omega(\sigma) \odot I \\
= & \begin{array}{l} \uplus \\ |\sigma| = 1 \\ |\pi| = 2 \\ \text{sig}(\sigma) = -1 \end{array} & \omega(\sigma) \odot A^2 & \begin{array}{l} \uplus \\ |\sigma| = 2 \\ |\pi| = 1 \\ \text{sig}(\sigma) = -1 \end{array} & \omega(\sigma) \odot A & \begin{array}{l} \uplus \\ |\sigma| = 3 \\ |\pi| = 0 \\ \text{sig}(\sigma) = -1 \end{array} & \omega(\sigma) \odot I
\end{array}$$

which simplifies to:

$$\begin{aligned}
& \{1\} \odot A^3 \uplus \{a_{11}a_{22}, a_{11}a_{33}, a_{22}a_{33}\} \odot A \uplus \{a_{13}a_{22}a_{31}, a_{11}a_{23}a_{32}, a_{12}a_{21}a_{33} \odot I \\
& = \{a_{11}, a_{22}, a_{33}\} \odot A^2 \uplus \{a_{12}a_{21}, a_{13}a_{31}, a_{23}a_{32}\} \odot A \uplus \{a_{11}a_{22}a_{33}, a_{12}a_{23}a_{31}, a_{13}a_{21}a_{32} \odot I
\end{aligned}$$

This can be written more compact using the previous definitions as:

$$\sigma_0^e \odot A^3 \uplus \sigma_2^e \odot A \uplus \sigma_3^e \odot I = \sigma_1^e \odot A^2 \uplus \sigma_2^e \odot A \uplus \sigma_3^e \odot I$$

With the theorems above we can add the following theorem.

Theorem 9.15. Given a matrix A, and a permutation matrix P, the characteristic polynomials of A and M=P⁻¹⊠ A ⊠ P are the same.

Proof. The matrix M has the same graph as A, except that the nodes have been relabeled. The set of “partial determinants” of the principal minors of M and the set of “partial determinants” of the principal minors of A are the same set. □

9.3 Eigenvalues and Roots of the Characteristic Equation

Example 9.16. Consider the matrix: $A = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$. ■

Matrix A has the characteristic equation: $\lambda^3 \boxplus 6\lambda = 3\lambda^2 \boxplus 6$. This equation has $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$ has roots. We see that for this matrix 1,2, and 3 are all eigenvalues for the matrix.

$$\begin{bmatrix} 1 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Example 9.17. Consider the matrix: $B = \begin{bmatrix} 3 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$. ■

This matrix has the same characteristic equation as the matrix in Example 9.16, namely $\lambda^3 \boxplus 6\lambda = 3\lambda^2 \boxplus 6$. Also the equation has the same roots, namely, $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. But this matrix has $\lambda = 3$ as the only eigenvalue. Here it can be checked that

$$\begin{bmatrix} 3 & 4 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \lambda = 3 \text{ is the only eigenvalue.}$$

Notice that both matrices above, Examples 9.16 and 9.17, have $\lambda = 3$ as an eigenvalue, and $\lambda = 3$ is a root of the characteristic equation of both matrices. Is it true that the maximum eigenvalue of a matrix, $\nu(A)$, is always a solution to the characteristic equation? The answer is certainly yes, just as it is in the conventional algebra. The proof is given in Theorem 9.27.

9.4 Algorithm for $\nu(A)$

In this section, we propose an algorithm which can be used to calculate the maximum eigenvalue of a matrix, denoted as $\nu(A)$, from the characteristic equation in the Max-Times Algebra.

Algorithm 9.18.

Suppose that $p_A^+(x) = p_A^-(x)$ is the characteristic equation of A .

(a) $p_A^+(x) = x^n \boxplus a_{n_1} x^{n_1} \boxplus a_{n_2} x^{n_2} \boxplus \dots \boxplus a_{n_k} x^{n_k}$

(b) $p_A^-(x) = a_{m_1} x^{m_1} \boxplus a_{m_2} x^{m_2} \boxplus \dots \boxplus a_{m_\ell} x^{m_\ell}$

(c) n_i s and m_p s are all distinct, and $m_1 = n - 1$

(2) For $i = 1 \dots \ell$

set $x^n = a_{m_i} x^{m_i}$

divide both sides by x^{m_i} , yielding $x^{n-m_i} = a_{m_i}$

set $b_i = \sqrt[n-m_i]{a_{m_i}} = (a_{m_i})^{\frac{1}{n-m_i}}$

end

(3) $\nu(A) = \boxplus_{i=1}^{\ell} b_i = b_1 \boxplus b_2 \boxplus \dots \boxplus b_\ell = \max_i(b_i)$.

(4) End.

Theorem 9.26 provides the proof of the above algorithm. Included here is the proof of the 2×2 , and 3×3 cases. Also included is the 4×4 case to show more detail of the cycle structure of the coefficients of the terms of polynomial equation. This will help lead to the proof of the general case.

Example 9.19. $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ ■

The characteristic equation for the matrix is:

$$x^2 \boxplus ad = (a \boxplus d)x \boxplus bc$$

This leads to the two equations:

$$(1) \quad x^2 = (a \boxplus d)x \quad \implies \quad b_1 = (a \boxplus d)$$

$$(2) \quad x^2 = bc \quad \implies \quad b_2 = \sqrt{bc}.$$

$$\text{Therefore, } \nu(A) = b_1 \boxplus b_2 = a \boxplus d \boxplus \sqrt{bc}.$$

Before showing the 3×3 case, we need a lemma, whose proof is quite simple.

Lemma 9.20. If $a, b, c \geq 0$, and $\max(a, b, c) = a$, then $a \geq (abc)^{\frac{1}{3}}$.

Proof. The proof follows from the fact that $a^3 \geq abc$. □

Example 9.21. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ■

The characteristic equation for the matrix is:

$$\begin{aligned} x^3 \boxplus (a_{11}a_{22} \boxplus a_{11}a_{33} \boxplus a_{22}a_{33})x \boxplus (a_{13}a_{22}a_{31} \boxplus a_{11}a_{23}a_{32} \boxplus a_{12}a_{21}a_{33}) \\ = (a_{11} \boxplus a_{22} \boxplus a_{33})x^2 \boxplus (a_{12}a_{21} \boxplus a_{13}a_{31} \boxplus a_{23}a_{32})x \\ \boxplus (a_{11}a_{22}a_{33} \boxplus a_{12}a_{23}a_{31} \boxplus a_{13}a_{21}a_{32}) \end{aligned}$$

This leads to the three equations:

$$(1) \quad x^3 = (a_{11} \boxplus a_{22} \boxplus a_{33})x^2,$$

$$\text{so } b_1 = (a_{11} \boxplus a_{22} \boxplus a_{33})$$

$$(2) \quad x^3 = (a_{12}a_{21} \boxplus a_{13}a_{31} \boxplus a_{23}a_{32})x,$$

$$\text{so } b_2 = \sqrt{(a_{12}a_{21} \boxplus a_{13}a_{31} \boxplus a_{23}a_{32})} = \sqrt{a_{12}a_{21}} \boxplus \sqrt{a_{13}a_{31}} \boxplus \sqrt{a_{23}a_{32}}$$

$$(3) \ x^3 = (a_{11}a_{22}a_{33} \boxplus a_{12}a_{23}a_{31} \boxplus a_{13}a_{21}a_{32}),$$

so

$$b_3 = \sqrt[3]{(a_{11}a_{22}a_{33} \boxplus a_{12}a_{23}a_{31} \boxplus a_{13}a_{21}a_{32})}$$

$$= (\sqrt[3]{a_{11}a_{22}a_{33}} \boxplus \sqrt[3]{a_{12}a_{23}a_{31}} \boxplus \sqrt[3]{a_{13}a_{21}a_{32}})$$

Therefore,

$$\begin{aligned} \nu(A) &= b_1 \boxplus b_2 \boxplus b_3 \\ &= a_{11} \boxplus a_{22} \boxplus a_{33} \boxplus \sqrt{a_{12}a_{21}} \boxplus \sqrt{a_{13}a_{31}} \boxplus \sqrt{a_{23}a_{32}} \\ &\quad \boxplus \sqrt[3]{a_{11}a_{22}a_{33}} \boxplus \sqrt[3]{a_{12}a_{23}a_{31}} \boxplus \sqrt[3]{a_{13}a_{21}a_{32}} \\ &= a_{11} \boxplus a_{22} \boxplus a_{33} \boxplus \sqrt{a_{12}a_{21}} \boxplus \sqrt{a_{13}a_{31}} \boxplus \sqrt{a_{23}a_{32}} \\ &\quad \boxplus \sqrt[3]{a_{12}a_{23}a_{31}} \boxplus \sqrt[3]{a_{13}a_{21}a_{32}} \end{aligned}$$

The last step in the above equation results from removing $\sqrt[3]{a_{11}a_{22}a_{33}}$ by use of Lemma 9.20, where $a=a_{11}$, $b=a_{22}$, and $c=a_{33}$.

The partial permutations can be split into even or odd permutations by looking at the number of transpositions needed to achieve the permutation. Here we are looking for permutations on 3 elements, S_3 . Table 4 on page 139 shows the “cycles” for the 3×3 matrix.

Side of equation	σ	trans-positions	Cycles
RHS	σ_1^e	0	(1), (2), (3)
RHS	σ_2^o	1	(1,2) , (1,3) , (2,3)
LHS	σ_2^e	0	(1)(2), (1)(3), (2)(3)
LHS	σ_3^o	1	(1)(2,3), (2)(1,3), (3)(1,2)
RHS	σ_3^e	0	(1)(2)(3)
	σ_3^e	2	(1,2,3), (1,3,2)

Table 4: Partial Permutations in 3×3 characteristic polynomial

To gain more insight into the problem the 4×4 case is included. It is quite illustrative.

The following is the (long) characteristic equation for the 4×4 matrix.

$$\begin{aligned}
& x^4 + (a_{11}a_{22} + a_{11}a_{33} + a_{11}a_{44} + a_{22}a_{33} + a_{22}a_{44} + a_{33}a_{44})x^2 \\
& \quad + (a_{11}a_{23}a_{32} + a_{11}a_{24}a_{42} + a_{11}a_{34}a_{43} + a_{13}a_{22}a_{31} + a_{14}a_{22}a_{41} \\
& \quad + a_{22}a_{34}a_{43} + a_{12}a_{21}a_{33} + a_{14}a_{33}a_{41} + a_{24}a_{33}a_{42} + a_{12}a_{21}a_{44} \\
& \quad + a_{13}a_{31}a_{44} + a_{23}a_{32}a_{44})x + (a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} \\
& \quad + a_{11}a_{24}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\
& \quad + a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\
& \quad + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{23}a_{32}a_{41}) \\
= & (a_{11} + a_{22} + a_{33} + a_{44})x^3 + (a_{12}a_{21} + a_{13}a_{31} + a_{14}a_{41} + a_{23}a_{32} + a_{24}a_{42} + a_{34}a_{43})x^2 \\
& \quad + (a_{11}a_{22}a_{33} + a_{11}a_{22}a_{44} + a_{11}a_{33}a_{44} + a_{12}a_{23}a_{31} + a_{12}a_{24}a_{41} + a_{13}a_{21}a_{32} \\
& \quad + a_{13}a_{34}a_{41} + a_{14}a_{21}a_{42} + a_{14}a_{31}a_{43} + a_{22}a_{33}a_{44} + a_{23}a_{34}a_{42} + a_{24}a_{32}a_{43})x \\
& \quad + (a_{11}a_{22}a_{34}a_{43} + a_{11}a_{23}a_{32}a_{44} + a_{11}a_{24}a_{33}a_{42} + a_{12}a_{21}a_{33}a_{44} + a_{12}a_{23}a_{34}a_{41} \\
& \quad + a_{12}a_{24}a_{31}a_{43} + a_{13}a_{21}a_{34}a_{42} + a_{13}a_{22}a_{31}a_{44} + a_{13}a_{24}a_{32}a_{41} + a_{14}a_{21}a_{32}a_{43} \\
& \quad + a_{14}a_{22}a_{33}a_{41} + a_{14}a_{23}a_{31}a_{42})
\end{aligned}$$

The partial permutations can be split up into even or odd permutations by looking at the number of transpositions needed to achieve the permutation. Table 5 on page 141 summarizes the results.

Side of equation	σ	trans-positions	Cycles
RHS	σ_1^e	0	(1), (2), (3), (4)
RHS	σ_2^o	1	(1,2) , (1,3) , (1,4) , (2,3) , (2,4) , (3,4)
LHS	σ_2^e	0	(1)(2), (1)(3), (1)(4), (2)(3), (2)(4), (3)(4)
LHS	σ_3^o	1	(1)(2,3), (1)(2,4), (1)(3,4), (2)(1,3), (2)(1,4), (2)(3,4), (3)(1,2), (3)(1,4), (3)(2,4), (4)(1,2), (4)(1,3), (4)(2,3)
RHS	σ_3^e	0	(1)(2)(3), (1)(2)(4), (1)(3)(4), (2)(3)(4)
	σ_3^e	2	(1,2,3), (1,2,4), (1,3,2), (1,3,4), (1,4,2), (1,4,3), (2,3,4), (2,4,3)
RHS	σ_4^o	1	(1)(2)(3,4), (1)(3)(2,4), (1)(4)(2,3), (1,2)(3)(4), (1,3)(2)(4), (1,4)(2)(3)
	σ_4^o	3	(1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (1,4,3,2)
LHS	σ_4^e	0	(1)(2)(3)(4)
	σ_4^e	2	(1)(2,3,4), (1)(2,4,3), (1,2)(3,4), (2)(1,3,4), (2)(1,4,3), (1,3)(2,4), (3)(1,2,4), (3)(1,4,2), (1,4)(2,3), (4)(1,2,3), (4)(1,3,2)

Table 5: Partial Permutations in 4×4 characteristic polynomial

We need the following lemmas to prove the general case. Lemma 9.22 is a general lemma which covers the case for all diagonal elements.

Lemma 9.22. If $\max_{1 \leq i \leq k} (a_i) = a_*$, then $a_* \geq \left(\prod_{i=1}^k a_i \right)^{\frac{1}{k}}$

Proof.

$$(a_*)^k \geq \prod_{i=1}^k a_i$$

□

Lemma 9.23 is more general than Lemma 9.22. It states that the maximal element of a set is at least as large as the q^{th} root of any product of q elements of the set, i.e. the largest element is larger than the geometric mean of any q elements.

Lemma 9.23. If $\max_{i \in \Omega} (a_i) = a_*$, then $a_* \geq \left(\prod_{\substack{j=1 \\ i_j \in \Omega}}^q a_{i_j} \right)^{\frac{1}{q}}$, $\forall q \leq k, \forall i_j \in \Omega$

Proof. $a_*^q \geq \prod_{\substack{j=1 \\ i_j \in \Omega}}^q a_{i_j}$

□

Lemma 9.24 states that the maximum geometric mean of any cycle of $G(A)$ is greater than the geometric mean of two separate cycles concatenated together.

Lemma 9.24. If τ is the k -cycle of maximum geometric mean of the graph $G(A)$, i.e. $\nu(A) = (\omega(\tau))^{\frac{1}{k}}$, where $k = \ell(\tau)$, then for any m_1 -cycle ξ_1 , and m_2 -cycle ξ_2 , with $k = m_1 + m_2$,

$$(\omega(\tau))^{\frac{1}{k}} \geq (\omega(\xi_1)\omega(\xi_2))^{\frac{1}{m_1+m_2}}$$

Proof. We assume that $\omega(\tau) \geq 0, \omega(\xi_1) \geq 0, \omega(\xi_2) \geq 0$.

$$\begin{aligned} (\omega(\tau))^{\frac{1}{k}} &\geq (\omega(\xi_1))^{\frac{1}{m_1}} \quad \text{and} \quad (\omega(\tau))^{\frac{1}{k}} \geq (\omega(\xi_2))^{\frac{1}{m_2}} \\ (\omega(\tau))^{m_1} &\geq (\omega(\xi_1))^k \quad \text{and} \quad (\omega(\tau))^{m_2} \geq (\omega(\xi_2))^k \end{aligned}$$

$$\begin{aligned} (\omega(\tau))^{m_1+m_2} &\geq (\omega(\xi_1)\omega(\xi_2))^k \\ (\omega(\tau))^{\frac{1}{k}} &\geq (\omega(\xi_1)\omega(\xi_2))^{\frac{1}{m_1+m_2}} \end{aligned}$$

□

Note 9.25. The above lemma, Lemma 9.24, can be extended at once to more than just the concatenation of two cycles.

We are now ready to prove that the algorithm given in Section 9.4 is correct as a method to calculate $\nu(A)$.

Theorem 9.26. Algorithm 9.18 is correct.

Proof.

(1) First we must show that the coefficient of x^{n-q} on the RHS of the equation is the sum of all real “q-cycles” plus possibly some other dominated terms, and therefore all coefficients on the LHS are “pseudo q-cycles” which are “products” of cycles of length less than q whose sum is equal to q. Fix q, and suppose that σ is a cycle of length q, then $\text{sig}(\sigma) = (-1)^q \text{sig}(\sigma) = (-1)^q (-1)^{q-1} = -1$. Therefore $\omega(\sigma)$ appears on the RHS of the characteristic polynomial — i.e., $\omega(\sigma)x^{n-q}$ is a term of $p_A^-(x)$.

(2) Next, we must show that $\nu(A)$ is at least as large as any other of the b_i s. Fix q. Consider the coefficient, α_q , of the term $\alpha_q x^{n-q}$ of $p_A^-(x)$. We know from part (1) that all q-cycles are elements (summands) of α_q . What are the other elements of the α_q ? Since all elements of α_q are q-permutations of S_n , then the elements are either q-cycles or they are products of cycles of lengths less than q, but whose length sum q. For example, see Table 4, or Table 5. By Lemma 9.24, we know that $\nu(A)$ is greater than or equal to any other element of α_q . □

Theorem 9.27. $\nu(A)$ is a root of the characteristic equation.

Proof. Using the same reasoning of part (2) of the proof of Theorem 9.26, and using Lemma 9.24, $\nu(A)$ is also greater than any element of the coefficient of $\alpha_q x^k$ on the LHS of the characteristic equation. Therefore, we see, by Lemma 9.24, that $p_A^+(\nu(A)) = (\nu(A))^n$ because all other terms of the polynomial are dominated by $\nu(A)$.

Therefore $p_A^+(\nu(A)) = p_A^-(\nu(A))$ □

9.5 Companion Matrix

In the previous sections we have discussed the characteristic equation for matrices in the Max-Times Algebra system.

An obvious question to consider is whether for any given equation in the Max-Times Algebra system, does a matrix exist which has this given equation as its characteristic equation?

We answer this question in this section. As we see some equations do have an associated matrix, and some equations do not. We begin with the simple case of a class of equations for which an associated matrix exists.

If we are given an equation of the form

$$\lambda^n = \boxplus_{i=1}^n a_i \lambda^{n-i} \quad (339)$$

then the companion matrix for this equation is

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & & \cdots & a_2 & a_1 \end{bmatrix} \quad (340)$$

Example 9.28.

For the equation $\lambda^3 = 5\lambda^2 \boxplus 4\lambda \boxplus 3$, the companion matrix is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 5 \end{bmatrix}$. ■

Theorem 9.29. For the Companion matrix of a polynomial equation, there is no polynomial equation of smaller degree which is satisfied by the matrix.

Proof.

Consider the matrix $L_{n \times n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & & \cdots & a_2 & a_1 \end{bmatrix}$

Then we have the following:

$$e_1^T \boxtimes L = e_2^T \quad (341)$$

$$e_1^T \boxtimes L^2 = e_3^T \quad (342)$$

⋮

$$e_1^T \boxtimes L^{n-1} = e_n^T \quad (343)$$

Suppose that there exists a polynomial equation $p_L^+(x) = p_L^-(x)$ of smaller degree which is satisfied by the matrix. Therefore there exists a $k \in \mathbb{N}$, and $r \in \mathbb{N}$, such that $\text{degree}(p_L^+(x)) = k$, and $\text{degree}(p_L^-(x)) = r$, with $r < k < n$. Then from the Equations (341) above, $L_{1,k+1}^k = 1$, while $L_{1,k+1}^j = 0$, for all $j < k$, in particular, for all $j \leq r$. Therefore $[p_L^+(L)]_{1,k+1} = 1$, while $[p_L^-(L)]_{1,k+1} = 0$. So, no such equation can exist. □

The theory becomes more difficult for polynomial equations with more than one term on the LHS. The following examples illustrate some of the difficulty.

Example 9.30. Are there polynomial equations which are not the characteristic equation for any matrix?

This can certainly be answered in the affirmative, as the polynomial equation $x^2 + 4 = x$ does not have any solutions. Therefore it is not the characteristic equation for any matrix. Since $\nu(A)$ must be a root of the characteristic polynomial, then an equation with no solutions can not be the characteristic equation for a matrix. ■

Example 9.31. If a polynomial equation has a solution then is it the characteristic equation for some matrix?

A polynomial equation with a solution is not sufficient to have a companion equation. The polynomial equation $x^3 + 16x = 2x^2 + 1$ has $x = \frac{1}{16}$ as a solution. It can be shown that this equation can not be the characteristic equation for any 3×3 matrix. To see this, note that $\max(a_{11}, a_{22}, a_{33}) = 2$, and $\max(a_{11}a_{22}, a_{11}a_{33}, a_{22}a_{33}) = 16$, which is impossible in the set \mathbb{R}^+ . ■

9.6 Characteristic Equation of Block Matrices in \mathfrak{M}^* .

As we have seen in the previous sections, given a matrix A in the Max-Times Algebra, we can find an equation which is satisfied by the matrix.

Namely, for a matrix A, we have an equation $p_A^+(x) = p_A^-(x)$ such that $p_A^+(A) = p_A^-(A)$.

Partitioned Matrices (Reducible Matrix)

Let $M = \begin{bmatrix} A & 0 \\ B & D \end{bmatrix}$. In this section we develop a characteristic equation associated with the matrix M.

The following Theorem satisfies this problem.

Theorem 9.32. Let $M = \begin{bmatrix} A & 0 \\ B & D \end{bmatrix}$. Let $p_A^+(x) = p_A^-(x)$ be the characteristic equation for the submatrix A, and let $p_D^+(x) = p_D^-(x)$ be the characteristic equation for the submatrix D. Then the characteristic equation for M is: $p_M^+(x) = p_M^-(x)$, where $p_M^+(x) = p_A^+(x)p_D^+(x) \boxplus p_A^-(x)p_D^-(x)$, and $p_M^-(x) = p_A^-(x)p_D^-(x) \boxplus p_A^+(x)p_D^+(x)$.

Proof. For the proof of this theorem, we show that the msets associated with the matrix M, is connected in the proper way to the matrices A, and D. It can be seen in the following proof, that the coefficients of the separate polynomials interlace in just the right way.

Let $M = \begin{bmatrix} A_{m \times m} & 0 \\ B & D_{n \times n} \end{bmatrix}$. Let $N = m + n$. For the matrix M, we know that

$$p_M^+(x) = \biguplus_{k=0}^N \sigma_k^{\text{par}(k)} x^{N-k}, \text{ and } p_M^-(x) = \biguplus_{k=0}^N \sigma_k^{\text{par}(k+1)} x^{N-k}.$$

Recall that $\sigma_k(M) =$ "sum of all $k \times k$ principal minors of M". The only non-zero entries are those which have part of the principal minor in A and the rest of the principal minor is in D. Letting $\sigma_\ell = \sigma_\ell(A)$ and $\tau_{k-\ell} = \tau_{k-\ell}(D)$, we can write:

$$\sigma_k(M) = \biguplus_{\ell=0}^k (\sigma_\ell \boxtimes \tau_{k-\ell})$$

Each of the $\sigma_k(M)$ can be split into the even permutations and the odd permutations.

$$\sigma_k^e(M) = \biguplus_{\ell=0}^k (\sigma_\ell^e \tau_{k-\ell}^e) \oplus \biguplus_{\ell=0}^k (\sigma_\ell^o \tau_{k-\ell}^o) \quad \text{and} \quad \sigma_k^o(M) = \biguplus_{\ell=0}^k (\sigma_\ell^e \tau_{k-\ell}^o) \oplus \biguplus_{\ell=0}^k (\sigma_\ell^o \tau_{k-\ell}^e)$$

Now consider the parts of the characteristic equations of the submatrices, A and D.

$$\begin{aligned}
p_A^+(x) &= \biguplus_{k=0}^m \sigma_k^{\text{par}(k)} x^{m-k}, & \text{and} & & p_A^-(x) &= \biguplus_{k=0}^m \sigma_k^{\text{par}(k+1)} x^{m-k}, \\
p_D^+(x) &= \biguplus_{k=0}^n \sigma_k^{\text{par}(k)} x^{n-k}, & \text{and} & & p_D^-(x) &= \biguplus_{k=0}^n \sigma_k^{\text{par}(k+1)} x^{n-k}.
\end{aligned}$$

Consider $p_A^+(x)p_D^+(x) \oplus p_A^-(x)p_D^-(x)$.

$$\begin{aligned}
p_A^+(x)p_D^+(x) &= \left(\sigma_0^e x^m \oplus \sigma_1^o x^{m-1} \oplus \sigma_2^e x^{m-2} \oplus \sigma_3^o x^{m-3} \oplus \dots \oplus \sigma_m^{\text{par}(m)} \right) \\
&\quad \boxtimes \left(\tau_0^e x^n \oplus \tau_1^o x^{n-1} \oplus \tau_2^e x^{n-2} \oplus \tau_3^o x^{n-3} \oplus \dots \oplus \tau_n^{\text{par}(n)} \right) \\
&= x^{m+n} (\sigma_0^e \tau_0^e) \\
&\quad \oplus x^{m+n-1} (\sigma_0^e \tau_1^o \oplus \sigma_1^o \tau_0^e) \\
&\quad \oplus x^{m+n-2} (\sigma_0^e \tau_2^e \oplus \sigma_1^o \tau_1^o \oplus \sigma_2^e \tau_0^e) \\
&\quad \oplus x^{m+n-3} (\sigma_0^e \tau_3^o \oplus \sigma_1^o \tau_2^e \oplus \sigma_2^e \tau_1^o \oplus \sigma_3^o \tau_0^e) \\
&\quad \oplus x^{m+n-4} (\sigma_0^e \tau_4^e \oplus \sigma_1^o \tau_3^o \oplus \sigma_2^e \tau_2^e \oplus \sigma_3^o \tau_1^o \oplus \sigma_4^e \tau_0^e) \\
&\quad \oplus \dots
\end{aligned}$$

and

$$\begin{aligned}
p_A^-(x)p_D^-(x) &= \left(\sigma_0^o x^m \oplus \sigma_1^e x^{m-1} \oplus \sigma_2^o x^{m-2} \oplus \sigma_3^e x^{m-3} \oplus \dots \oplus \sigma_m^{\text{par}(m+1)} \right) \\
&\quad \boxtimes \left(\tau_0^o x^n \oplus \tau_1^e x^{n-1} \oplus \tau_2^o x^{n-2} \oplus \tau_3^e x^{n-3} \oplus \dots \oplus \tau_n^{\text{par}(n+1)} \right) \\
&= x^{m+n} (\sigma_0^o \tau_0^o) \\
&\quad \oplus x^{m+n-1} (\sigma_0^o \tau_1^e \oplus \sigma_1^e \tau_0^o) \\
&\quad \oplus x^{m+n-2} (\sigma_0^o \tau_2^o \oplus \sigma_1^e \tau_1^e \oplus \sigma_2^o \tau_0^o) \\
&\quad \oplus x^{m+n-3} (\sigma_0^o \tau_3^e \oplus \sigma_1^e \tau_2^o \oplus \sigma_2^o \tau_1^e \oplus \sigma_3^e \tau_0^o) \\
&\quad \oplus x^{m+n-4} (\sigma_0^o \tau_4^o \oplus \sigma_1^e \tau_3^e \oplus \sigma_2^o \tau_2^o \oplus \sigma_3^e \tau_1^e \oplus \sigma_4^o \tau_0^o) \\
&\quad \oplus \dots
\end{aligned}$$

So, adding the two previous results yields:

$$\begin{aligned}
& p_A^+(x)p_D^+(x) \oplus p_A^-(x)p_D^-(x) \\
&= x^{m+n} (\sigma_0^e \tau_0^e \oplus \sigma_0^o \tau_0^o) \\
&\quad \oplus x^{m+n-1} (\sigma_0^e \tau_1^o \oplus \sigma_1^o \tau_0^e \oplus \sigma_0^o \tau_1^e \oplus \sigma_1^e \tau_0^o) \\
&\quad \oplus x^{m+n-2} (\sigma_0^e \tau_2^e \oplus \sigma_1^o \tau_1^o \oplus \sigma_2^e \tau_0^e \oplus \sigma_0^o \tau_2^o \oplus \sigma_1^e \tau_1^e \oplus \sigma_2^o \tau_0^o) \\
&\quad \oplus x^{m+n-3} (\sigma_0^e \tau_3^o \oplus \sigma_1^o \tau_2^e \oplus \sigma_2^e \tau_1^o \oplus \sigma_3^o \tau_0^e \oplus \sigma_0^o \tau_3^e \oplus \sigma_1^e \tau_2^o \oplus \sigma_2^o \tau_1^e \oplus \sigma_3^e \tau_0^o) \\
&\quad \oplus x^{m+n-4} (\sigma_0^e \tau_4^e \oplus \sigma_1^o \tau_3^o \oplus \sigma_2^e \tau_2^e \oplus \sigma_3^o \tau_1^o \oplus \sigma_4^e \tau_0^e \oplus \sigma_0^o \tau_4^o \oplus \sigma_1^e \tau_3^e \oplus \sigma_2^o \tau_2^o \oplus \sigma_3^e \tau_1^e \oplus \sigma_4^o \tau_0^o) \\
&\quad \oplus \dots \\
&= x^N (\sigma_0^e \tau_0^e) \oplus x^N (\sigma_0^o \tau_0^o) \\
&\quad \oplus x^{N-1} (\sigma_0^e \tau_1^o \oplus \sigma_1^e \tau_0^o) \oplus x^{N-1} (\sigma_0^o \tau_1^e \oplus \sigma_1^o \tau_0^e) \\
&\quad \oplus x^{N-2} (\sigma_0^e \tau_2^e \oplus \sigma_1^e \tau_1^e \oplus \sigma_2^e \tau_0^e) \oplus x^{N-2} (\sigma_0^o \tau_2^o \oplus \sigma_1^o \tau_1^o \oplus \sigma_2^o \tau_0^o) \\
&\quad \oplus x^{N-3} (\sigma_0^e \tau_3^o \oplus \sigma_1^e \tau_2^o \oplus \sigma_2^e \tau_1^o \oplus \sigma_3^e \tau_0^e) \oplus x^{N-3} (\sigma_0^o \tau_3^e \oplus \sigma_1^o \tau_2^e \oplus \sigma_2^o \tau_1^e \oplus \sigma_3^o \tau_0^e) \\
&\quad \oplus x^{N-4} (\sigma_0^e \tau_4^e \oplus \sigma_1^e \tau_3^e \oplus \sigma_2^e \tau_2^e \oplus \sigma_3^e \tau_1^e \oplus \sigma_4^e \tau_0^e) \\
&\quad \quad \oplus x^{N-4} (\sigma_0^o \tau_4^o \oplus \sigma_1^o \tau_3^o \oplus \sigma_2^o \tau_2^o \oplus \sigma_3^o \tau_1^o \oplus \sigma_4^o \tau_0^o) \\
&\quad \oplus \dots \\
&= x^N (\sigma_0^e(M)) \oplus x^{N-1} (\sigma_1^o(M)) \\
&\quad \oplus x^{N-2} (\sigma_2^e(M)) \oplus x^{N-3} (\sigma_3^o(M)) \oplus x^{N-4} (\sigma_4^e(M)) \oplus \dots
\end{aligned}$$

Likewise, we can follow the same method to calculate $p_M^-(x)$.

Consider $p_A^+(x)p_D^-(x) \oplus p_A^-(x)p_D^+(x)$.

$$\begin{aligned}
p_A^+(x)p_D^-(x) &= \left(\sigma_0^e x^m \oplus \sigma_1^o x^{m-1} \oplus \sigma_2^e x^{m-2} \oplus \sigma_3^o x^{m-3} \oplus \dots \oplus \sigma_m^{\text{par}(m)} \right) \\
&\quad \boxtimes \left(\tau_0^o x^n \oplus \tau_1^e x^{n-1} \oplus \tau_2^o x^{n-2} \oplus \tau_3^e x^{n-3} \oplus \dots \oplus \tau_n^{\text{par}(n+1)} \right) \\
&= x^{m+n} (\sigma_0^e \tau_0^o) \\
&\quad \oplus x^{m+n-1} (\sigma_0^e \tau_1^e \oplus \sigma_1^o \tau_0^o) \\
&\quad \oplus x^{m+n-2} (\sigma_0^e \tau_2^o \oplus \sigma_1^o \tau_1^e \oplus \sigma_2^e \tau_0^o) \\
&\quad \oplus x^{m+n-3} (\sigma_0^e \tau_3^e \oplus \sigma_1^o \tau_2^o \oplus \sigma_2^e \tau_1^e \oplus \sigma_3^o \tau_0^o) \\
&\quad \oplus x^{m+n-4} (\sigma_0^e \tau_4^o \oplus \sigma_1^o \tau_3^e \oplus \sigma_2^e \tau_2^o \oplus \sigma_3^o \tau_1^e \oplus \sigma_4^e \tau_0^o) \\
&\quad \oplus \dots
\end{aligned}$$

and

$$\begin{aligned}
p_A^-(x)p_D^+(x) &= \left(\sigma_0^o x^m \oplus \sigma_1^e x^{m-1} \oplus \sigma_2^o x^{m-2} \oplus \sigma_3^e x^{m-3} \oplus \dots \oplus \sigma_m^{\text{par}(m+1)} \right) \\
&\quad \boxtimes \left(\tau_0^e x^n \oplus \tau_1^o x^{n-1} \oplus \tau_2^e x^{n-2} \oplus \tau_3^o x^{n-3} \oplus \dots \oplus \tau_n^{\text{par}(n)} \right) \\
&= x^{m+n} (\sigma_0^o \tau_0^e) \\
&\quad \oplus x^{m+n-1} (\sigma_0^o \tau_1^o \oplus \sigma_1^e \tau_0^e) \\
&\quad \oplus x^{m+n-2} (\sigma_0^o \tau_2^e \oplus \sigma_1^e \tau_1^o \oplus \sigma_2^o \tau_0^e) \\
&\quad \oplus x^{m+n-3} (\sigma_0^o \tau_3^o \oplus \sigma_1^e \tau_2^e \oplus \sigma_2^o \tau_1^o \oplus \sigma_3^e \tau_0^e) \\
&\quad \oplus x^{m+n-4} (\sigma_0^o \tau_4^e \oplus \sigma_1^e \tau_3^o \oplus \sigma_2^o \tau_2^e \oplus \sigma_3^e \tau_1^o \oplus \sigma_4^o \tau_0^e) \\
&\quad \oplus \dots
\end{aligned}$$

So, adding the two previous results yields:

$$\begin{aligned}
p_A^+(x)p_D^-(x) \oplus p_A^-(x)p_D^+(x) &= x^{m+n} (\sigma_0^e \tau_0^o \oplus \sigma_0^o \tau_0^e) \\
&\quad \oplus x^{m+n-1} (\sigma_0^e \tau_1^e \oplus \sigma_1^o \tau_0^o \oplus \sigma_0^o \tau_1^o \oplus \sigma_1^e \tau_0^e) \\
&\quad \oplus x^{m+n-2} (\sigma_0^e \tau_2^o \oplus \sigma_1^o \tau_1^e \oplus \sigma_2^o \tau_0^o \oplus \sigma_0^o \tau_2^e \oplus \sigma_1^e \tau_1^o \oplus \sigma_2^e \tau_0^e) \\
&\quad \oplus x^{m+n-3} (\sigma_0^e \tau_3^e \oplus \sigma_1^o \tau_2^o \oplus \sigma_2^e \tau_1^e \oplus \sigma_3^o \tau_0^o \oplus \sigma_0^o \tau_3^e \oplus \sigma_1^e \tau_2^e \oplus \sigma_2^o \tau_1^o \oplus \sigma_3^e \tau_0^e) \\
&\quad \oplus x^{m+n-4} (\sigma_0^e \tau_4^o \oplus \sigma_1^o \tau_3^e \oplus \sigma_2^e \tau_2^o \oplus \sigma_3^o \tau_1^e \oplus \sigma_4^o \tau_0^o \oplus \sigma_0^o \tau_4^e \oplus \sigma_1^e \tau_3^o \oplus \sigma_2^o \tau_2^e \oplus \sigma_3^e \tau_1^o \oplus \sigma_4^e \tau_0^e) \\
&\quad \oplus \dots \\
&= x^N (\sigma_0^e \tau_0^o) \oplus x^N (\sigma_0^o \tau_0^e) \\
&\quad \oplus x^{N-1} (\sigma_0^e \tau_1^e \oplus \sigma_1^e \tau_0^e) \oplus x^{N-1} (\sigma_0^o \tau_1^o \oplus \sigma_1^o \tau_0^o) \\
&\quad \oplus x^{N-2} (\sigma_0^e \tau_2^o \oplus \sigma_1^e \tau_1^e \oplus \sigma_2^e \tau_0^o) \oplus x^{N-2} (\sigma_0^o \tau_2^e \oplus \sigma_1^o \tau_1^e \oplus \sigma_2^o \tau_0^e) \\
&\quad \oplus x^{N-3} (\sigma_0^e \tau_3^e \oplus \sigma_1^e \tau_2^e \oplus \sigma_2^e \tau_1^e \oplus \sigma_3^e \tau_0^e) \oplus x^{N-3} (\sigma_0^o \tau_3^o \oplus \sigma_1^o \tau_2^o \oplus \sigma_2^o \tau_1^o \oplus \sigma_3^o \tau_0^o) \\
&\quad \oplus x^{N-4} (\sigma_0^e \tau_4^o \oplus \sigma_1^e \tau_3^o \oplus \sigma_2^e \tau_2^o \oplus \sigma_3^e \tau_1^e \oplus \sigma_4^e \tau_0^o) \\
&\quad \quad \oplus x^{N-4} (\sigma_0^o \tau_4^e \oplus \sigma_1^o \tau_3^e \oplus \sigma_2^o \tau_2^e \oplus \sigma_3^o \tau_1^e \oplus \sigma_4^o \tau_0^e) \\
&\quad \oplus \dots \\
&= x^N (\sigma_0^o(M)) \oplus x^{N-1} (\sigma_1^e(M)) \oplus x^{N-2} (\sigma_2^o(M)) \oplus x^{N-3} (\sigma_3^e(M)) \oplus x^{N-4} (\sigma_4^o(M)) \oplus \dots
\end{aligned}$$

The proof is complete. \square

Partitioned Matrices (Block Diagonal Matrix)

Theorem 9.33. Let $M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Let $p_A^+(x) = p_A^-(x)$ be the characteristic equation for the submatrix A, and let $p_D^+(x) = p_D^-(x)$ be the characteristic equation for the submatrix D. Then the

characteristic equation for M is: $p_M^+(x) = p_M^-(x)$, where $p_M^+(x) = p_A^+(x)p_D^+(x) \boxplus p_A^-(x)p_D^-(x)$, and $p_M^-(x) = p_A^+(x)p_D^-(x) \boxplus p_A^-(x)p_D^+(x)$.

Proof. This is simply a corollary of Theorem 9.32 □

Note 9.34. Recall that if M were a matrix in \mathbb{R} , then the characteristic polynomial for M is $\Delta_M = \Delta_A \Delta_D$. In the max-times algebra, we split each characteristic polynomial into the “positive part”, and the “negative part”. Write $\Delta_A = \Delta_A^+ - \Delta_A^-$, and $\Delta_D = \Delta_D^+ - \Delta_D^-$. Multiplying these together, and separating the parts again yields:

$$\Delta_M = \Delta_A \Delta_D = (\Delta_A^+ - \Delta_A^-)(\Delta_D^+ - \Delta_D^-) \quad (344)$$

$$= (\Delta_A^+ \Delta_D^+ + \Delta_A^- + \Delta_D^-) - (\Delta_A^+ \Delta_D^- + \Delta_A^+ + \Delta_D^-) \quad (345)$$

$$= \Delta_M^+ - \Delta_M^- \quad (346)$$

So $p_M^+(x) = p_A^+(x)p_D^+(x) \boxplus p_A^-(x)p_D^-(x)$, and $p_M^-(x) = p_A^+(x)p_D^-(x) \boxplus p_A^-(x)p_D^+(x)$.

A few examples are included to illustrate the method used above.

Example 9.35. Consider $M = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 7 & 0 \\ \hline 5 & 9 & 3 \end{array} \right]$ ■

For this example we have $A = \left[\begin{array}{cc} 2 & 1 \\ 4 & 7 \end{array} \right]$ and $D = [3]$. The characteristic equation for A is $x^2 \boxplus 14 = 7x$, and for D, $x = 3$. Therefore, $p_A^+(x) = x^2 \boxplus 14$, $p_A^- = 7x$, $p_D^+ = x$, and $p_D^- = 3$.

We now have:

$$p_M^+(x) = p_A^+ p_D^+ \boxplus p_A^- p_D^- = x^3 \boxplus 14x \boxplus 21x = x^3 \boxplus 21x$$

$$p_M^-(x) = p_A^+ p_D^- \boxplus p_A^- p_D^+ = 7x^2 \boxplus 3x^2 \boxplus 42 = 7x^2 \boxplus 42$$

This leads to the characteristic equation for M: $x^3 \boxplus 21x = 7x^2 \boxplus 42$, and it can be checked that:

$$M^3 \boxplus 21M = 7M^2 \boxplus 42I$$

We continue to the next example.

Example 9.36. Consider $M = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 4 & 7 & 0 & 0 \\ \hline 5 & 9 & 1 & 2 \\ 3 & 7 & 5 & 6 \end{array} \right]$ ■

For this example we have $A = \left[\begin{array}{cc} 2 & 1 \\ 4 & 7 \end{array} \right]$ and $D = \left[\begin{array}{cc} 1 & 2 \\ 5 & 6 \end{array} \right]$. The characteristic equation for A is $x^2 \boxplus 14 = 7x$, and for D, $x^2 = 6x \boxplus 10$. Therefore, $p_A^+(x) = x^2 \boxplus 14$, $p_A^- = 7x$, $p_D^+ = x^2$, and $p_D^- = 6x \boxplus 10$.

We now have:

$$p_M^+(x) = p_A^+ p_D^+ \boxplus p_A^- p_D^- = x^4 \boxplus 14x^2 \boxplus 42x^2 \boxplus 70x = x^4 \boxplus 42x^2 \boxplus 70x$$

$$p_M^-(x) = p_A^+ p_D^- \boxplus p_A^- p_D^+ = 7x^3 \boxplus 6x^3 \boxplus 10x^2 \boxplus 84x \boxplus 140 = 7x^3 \boxplus 10x^2 \boxplus 84x \boxplus 140$$

This leads to the characteristic equation:

$$x^4 \boxplus 42x^2 \boxplus 70x = 7x^3 \boxplus 10x^2 \boxplus 84x \boxplus 140$$

which can be simplified to:

$$x^4 \boxplus 42x^2 = 7x^3 \boxplus 84x \boxplus 140.$$

And it can be checked that:

$$M^4 \boxplus 42M^2 = 7M^3 \boxplus 84M \boxplus 140I.$$

And, finally, one more example:

Example 9.37. Consider $M =$

$$\left[\begin{array}{cccc|cc} 9 & 3 & 4 & 1 & 0 & 0 \\ 7 & 2 & 9 & 2 & 0 & 0 \\ 4 & 7 & 6 & 6 & 0 & 0 \\ 4 & 4 & 2 & 6 & 0 & 0 \\ \hline 2 & 7 & 8 & 3 & 2 & 6 \\ 6 & 2 & 6 & 5 & 3 & 0 \end{array} \right]$$

For this example we have $A =$

$$\left[\begin{array}{cccc} 9 & 3 & 4 & 1 \\ 7 & 2 & 9 & 2 \\ 4 & 7 & 6 & 6 \\ 4 & 4 & 2 & 6 \end{array} \right] \text{ and } D = \left[\begin{array}{cc} 2 & 6 \\ 3 & 0 \end{array} \right]. \text{ The characteristic equation for}$$

A is $x^4 \boxplus 567x = 9x^3 \boxplus 63x^2 \boxplus 3402$, and for D , $x^2 = 2x \boxplus 18$. Therefore, $p_A^+(x) = x^4 \boxplus 567x$, $p_A^- = 9x^3 \boxplus 63x^2 \boxplus 3402$, $p_D^+ = x^2$, and $p_D^- = 2x \boxplus 18$.

We now have:

$$\begin{aligned} p_M^+(x) &= p_A^+ p_D^+ \boxplus p_A^- p_D^- \\ &= x^6 \boxplus 567x^3 \boxplus 18x^4 \boxplus 126x^3 \boxplus 6804x \boxplus 162x^3 \boxplus 1134x^2 \boxplus 61236 \\ &= x^6 \boxplus 18x^4 \boxplus 567x^3 \boxplus 1134x^2 \boxplus 6804x \boxplus 61236 \\ p_M^-(x) &= p_A^+ p_D^- \boxplus p_A^- p_D^+ \\ &= 2x^5 \boxplus 18x^4 \boxplus 1134x^2 \boxplus 10206x \boxplus 9x^5 \boxplus 63x^4 \boxplus 3402x^2 \\ &= 9x^5 \boxplus 63x^4 \boxplus 3402x^2 \boxplus 10206x \end{aligned}$$

This leads to the characteristic equation:

$$x^6 \boxplus 18x^4 \boxplus 567x^3 \boxplus 1134x^2 \boxplus 6804x \boxplus 61236 = 9x^5 \boxplus 63x^4 \boxplus 3402x^2 \boxplus 10206x$$

which can be simplified to:

$$x^6 \boxplus 567x^3 \boxplus 61236 = 9x^5 \boxplus 63x^4 \boxplus 3402x^2 \boxplus 10206x.$$

And it can be checked that:

$$M^6 \boxplus 567M^3 \boxplus 61236I = 9M^5 \boxplus 63M^4 \boxplus 3402M^2 \boxplus 10206M.$$

Theorem 9.38. Let $M = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ C & & & A_k \end{bmatrix}$

The characteristic equation for M can be written as $p_M^+ = p_M^-$, where:

$p_M^+ = \boxplus(A_1^{(\pm)} A_2^{(\pm)} \cdots A_k^{(\pm)})$, where the sum is taken over all parts such that the total number of negative parts is even (zero is an even number here), and

$p_M^- = \boxplus(A_1^{(\pm)} A_2^{(\pm)} \cdots A_k^{(\pm)})$, where the sum is taken over all parts such that the total number of negative parts is odd.

Proof. This can be proved easily by induction, using Theorem 9.32. □

Example:

If $M = \begin{bmatrix} A & 0 \\ B & C \\ D & C \end{bmatrix}$, then the characteristic equation can be written as $p_M^+ = p_M^-$, where

$$p_M^+ = p_A^+ p_B^+ p_C^+ \boxplus p_A^+ p_B^- p_C^- \boxplus p_A^- p_B^+ p_C^- \boxplus p_A^- p_B^- p_C^+$$

and

$$p_M^- = p_A^+ p_B^+ p_C^- \boxplus p_A^+ p_B^- p_C^+ \boxplus p_A^- p_B^+ p_C^+ \boxplus p_A^- p_B^- p_C^-$$

Bordered Matrices

Theorem 9.39. If $M_{2 \times 2} = \left[\begin{array}{c|c} a & c \\ \hline b^T & d \end{array} \right]$, where a is a 1×1 matrix and d is 1×1 , then the

characteristic equation for M can be written as $p_M^+ = p_M^-$, where

$$p_M^+ = p_a^+ p_d^+ \boxplus p_a^- p_d^-,$$

and

$$p_M^- = p_a^+ p_d^- \boxplus p_a^- p_d^+ \boxplus (b^T c)$$

Proof. $p_a^+ = x$, $p_a^- = a$, $p_d^+ = x$, $p_d^- = d$.

Then $p_M^+ = x^2 \boxplus ad$, and $p_M^- = (a \boxplus b)x \boxplus bc$ □

The following facts are well-known from linear algebra.

Fact 9.40. $\left| \begin{array}{cc} A & y \\ x^T & z \end{array} \right| = \det(A) d - x^T \text{adj}(A) y.$

Fact 9.41.

$$\text{adj}(\lambda I - A) = \sum_{k=0}^{n-1} A_k \lambda^k \quad (347)$$

where $A_k = a_{k+1}I + a_{k+2}A + \dots + a_n A^{n-(k+1)}$. The a_k 's are defined by:

$a_{k+1} = (-1)^{n-(k+1)} \sigma_{n-(k+1)}$, where the σ_k 's are the coefficients of the characteristic polynomial.

Example 9.42. Consider the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}_{2 \times 2}$. ■

The characteristic polynomial of A is $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$

Here, $\sigma_0 = 1$, $\sigma_1 = (a_{11} + a_{22})$, $\sigma_2 = (a_{11}a_{22} - a_{12}a_{21})$

So

$$a_0 = (-1)^2 \sigma_2 = (a_{11}a_{22} - a_{12}a_{21}),$$

$$a_1 = (-1)^1 \sigma_1 = -(a_{11} + a_{22}), \quad \text{and}$$

$$a_2 = (-1)^0 \sigma_0 = 1.$$

$$A_0 = a_1 I + a_2 A = \begin{bmatrix} -a_{11} - a_{22} & 0 \\ 0 & -a_{11} - a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}$$

$$A_1 = a_2 I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{adj}(\lambda I - A) = A_0 + A_1 \lambda = \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} + \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda - a_{22} & a_{12} \\ a_{21} & \lambda - a_{11} \end{bmatrix}$$

Theorem 9.43. If $M_{3 \times 3} = \left[\begin{array}{c|c} A & c \\ \hline b^T & d \end{array} \right]$, where A is a 2×2 matrix and d is a scalar, then the

characteristic equation for M can be written as $p_M^+ = p_M^-$, where

$$p_M^+ = p_A^+ p_d^+ \boxplus p_A^- p_d^- \boxplus (a_{11} \boxplus a_{22})(b^T c),$$

and

$$p_M^- = p_A^+ p_d^- \boxplus p_A^- p_d^+ \boxplus (b^T c)x \boxplus (b^T A c)$$

Proof. The proof of this theorem is based on the determinant of a bordered matrix as given in

Lemma 9.40. Consider the matrix $M = \left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ \hline b_1 & b_2 & d \end{array} \right]$

We calculate the $\det(\lambda I - M)$, just as in the conventional algebra.

$$\det(\lambda I - M) = \left| \begin{array}{c|c} \lambda I - A & -c \\ \hline -b^T & \lambda - d \end{array} \right| = \det(\lambda I - A) * (\lambda - d) - b^T \text{adj}(\lambda I - A)c$$

From Example 9.42, we have that $\text{adj}(\lambda I - A) = a_1 I + a_2 A + a_2 I \lambda$

This leads to the equation

$$\begin{aligned} \text{adj}(\lambda I - M) &= \det(\lambda I - A) * (\lambda - d) - b^T \text{adj}(\lambda I - A)c \\ &= \det(\lambda I - A) * (\lambda - d) - b^T (a_1 I + a_2 A + a_2 I \lambda)c \\ &= \det(\lambda I - A) * (\lambda - d) - a_1 b^T c - a_2 b^T A c - a_2 b^T c \lambda \\ &= \det(\lambda I - A) * (\lambda - d) + (a_{11} + a_{22}) b^T c - b^T A c - b^T c \lambda \end{aligned}$$

Setting the above equation equal to zero and moving all negative terms to the other side of the equation leads to the equation given in the theorem.

An alternative method to calculate the equation using the methods of section 10 is:

$$(p^+, p^-) \otimes (\lambda, d) \oplus ((a_{11} \boxplus a_{22}) b^T c, 0) \oplus (0, b^T A c) \oplus (0, b^T c \lambda).$$

□

Note 9.44. Although this algorithm could be generalized to larger $n \times n$ bordered matrices with $n > 3$, the results get increasingly more complicated because the a_i s have to be split into a_i^+ and a_i^- , and no savings are realized over the algorithm given in section 9.1.

Example 9.45. Consider $M = \left[\begin{array}{c|c} 1 & 2 \\ \hline 3 & 4 \end{array} \right]$

■

If this case we have:

$$\begin{array}{rcl} x & = & 1 \\ x & = & 4 \\ \hline x^2 \boxplus 4 & = & 4x \boxplus x \end{array}$$

$$b^T c = 6.$$

This yields $x^2 \boxplus 4 = 4x \boxplus x \boxplus 6$. This can be simplified to $x^2 = 4x \boxplus 6$, and it can be checked that M satisfies the equation $M^2 = 4M \boxplus 6I$.

Partitioned Matrices and Recursion

The above sections lead to a recursive method to find the polynomial equation associated with some particular matrices.

Example 9.46. Consider $M = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 1 \end{bmatrix}$ ■

To calculate the characteristic equation for this polynomial, we start from the 1×1 principal submatrix, and continue with the next larger principal submatrix. We can use recursion (updates) on this 4×4 matrix because it is reducible and thus no 4×4 updates are necessary as noted in Note 9.44.

(a) $[0] \quad x = 0$

(b) $\left[\begin{array}{c|c} 0 & 0 \\ \hline 2 & 2 \end{array} \right] \quad \begin{array}{l} x = 0 \\ x = 2 \\ \hline x^2 \boxplus 0 = 2x \boxplus 0 \\ x^2 = 2x \end{array}$

(c) $\left[\begin{array}{cc|c} 0 & 0 & 2 \\ 2 & 2 & 0 \\ \hline 3 & 3 & 3 \end{array} \right] \quad \begin{array}{l} x^2 = 2x \\ x = 3 \\ \hline x^3 \boxplus 6x = 3x^2 \boxplus 2x^2 \\ x^3 \boxplus 6x = 3x^2 \end{array}$

$$b^T c = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 6$$

$$(a_{11} \boxplus a_{22})b^T c = (2)(6) = 12$$

$$b^T A c = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 12$$

$$\begin{array}{l} x^3 \boxplus 6x \boxplus 12 = 3x^2 \boxplus 6x \boxplus 12 \\ x^3 = 3x^2 \end{array}$$

(d) $\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ \hline 3 & 3 & 3 & 1 \end{array} \right] \quad \begin{array}{l} x^3 = 3x^2 \\ x = 1 \\ \hline x^4 \boxplus 3x^2 = x^3 \boxplus 3x^3 \\ x^4 \boxplus 3x^2 = 3x^3 \end{array}$

So in this example, we find the characteristic polynomial for the matrix A to be $x^4 \boxplus 3x^2 = 3x^3$, which agrees with the Algorithm 9.1.

10 Algebra of Polynomial Pairs in the Max-Times Algebra

10.1 Evaluation Mapping

Definition 10.1 (Evaluation Mapping for elements of \mathbf{R}).

If $\alpha \in \mathbf{R}$, and $p \in \mathbf{R}[x]$, then $F_\alpha(p) = "p(\alpha)" = p_0 + p_1\alpha + \cdots + p_n\alpha^n$

For the two specific algebras, we have:

$$(i) \ [(\text{Max}, \times)], p(\alpha) = \max_{0 \leq k \leq n} \{p_k \alpha^k\}$$

$$(ii) \ [3^{\mathbf{R}}], p(\alpha) = \bigoplus_{k=0}^n p_k \alpha^k$$

Proposition 10.2.

1. F_α is linear: $F_\alpha(p \boxplus q) = F_\alpha(p) + F_\alpha(q)$
2. $F_\alpha(\beta \boxtimes p \boxplus q) = \beta \times F_\alpha(p) + F_\alpha(q)$

Definition 10.3 (Evaluation Mapping for Matrices over $\mathbf{R}^{n \times n}$).

Let $A \in \mathbf{R}_{n \times n}$, $p \in \mathbf{R}[x]$, then

$$\phi_A(p) = "p(A)" = p_0 \times I + p_1 \times A + \cdots + p_n \times A^n$$

Proposition 10.4. ϕ_A is linear: $\phi_A(\beta p \boxplus q) = \beta \times \phi_A(p) + \phi_A(q)$

Proposition 10.5. ϕ_A is multiplicative. I.e. $\phi_A(p \boxtimes q) = \phi_A(p) \times \phi_A(q)$.

The proof of the two propositions above are identical to the proof over rings.

Definition 10.6. In $\mathbf{R}[x]$, we say that $f=g$ if and only if $F_x(f)=F_x(g)$ for all $x \in \mathbb{R}^+$

Note 10.7. In this respect, we say that two polynomials are equal if their valuations are equal.

We are considering, here, the polynomials as *polynomial functions* and not as *polynomial forms* which are ordered sequences of coefficients.

(Max, \times)-Algebra

Theorem 10.8. $(x \boxplus a)^k = x^k \boxplus a^k$, for all $a \in \mathbb{R}^+$, and for all $k \in \mathbb{N}$.

Proof. First, we note that if $x=a$ then the theorem is obvious since \boxplus is an idempotent operation. We prove this lemma by induction on k . For $k=1$, the result is obvious. Suppose that the lemma is true for k , we now show that it is true for $n=k+1$.

$$\begin{aligned} (x \boxplus a)^{k+1} &= (x \boxplus a) \boxtimes (x \boxplus a)^k \\ &= (x \boxplus a) \boxtimes (x^k \boxplus a^k) \\ &= x^{k+1} \boxplus ax^k \boxplus a^k x \boxplus a^{k+1} \end{aligned}$$

If $x < a$, then $ax^k < a^{k+1}$, and $a^k x < a^{k+1}$,
 and if $x > a$ then $ax^k < x^{k+1}$, and $a^k x < x^{k+1}$.
 Therefore, $(x \boxplus a)^{k+1} = x^{k+1} \boxplus a^{k+1}$ □

In \mathbb{R} , we know that if two polynomials are equal as evaluations, then they are also equal as formal polynomials. This may be stated as saying there is a bijection between polynomials as valuations and formal polynomials. In the max-times algebra this is not the case, although (Max, \times) is an infinite semiring.

Example 10.9. $(x \boxplus 2)^2 = x^2 \boxplus 2x \boxplus 2x \boxplus 4 = x^2 \boxplus 2x \boxplus 4 = x^2 \boxplus 4$ ■

In this example the term “ $2x$ ” is dominated by the other two terms for any nonnegative real number.

What can be said about two polynomials which are equal, as evaluations?

Lemma 10.10. If $f(x), g(x) \in \mathbb{R}[x]$, and $f(x) = g(x)$ (as evaluations), then

- (1) $\partial f(x) = \partial g(x)$ (degree($f(x)$) = degree($g(x)$)).
- (2) the leading coefficients of the two polynomials are equal.
- (3) the constant terms of the two polynomials are equal.

i.e. if $f(x) = \max_k \{a_k x^k \mid k = 0, 1, \dots, n\}$, and $g(x) = \max_\ell \{b_\ell x^\ell \mid \ell = 0, 1, \dots, m\}$, then

- (1) $m=n$
- (2) $a_n=b_m$
- (3) $a_0=b_0$

Proof. Suppose $f(x) = \max_k \{a_k x^k \mid k = 0, 1, \dots, n\}$, and $g(x) = \max_\ell \{b_\ell x^\ell \mid \ell = 0, 1, \dots, m\}$, with $a_n \neq 0$, and $b_m \neq 0$.

Suppose that $n > m$, then we have

$$a_n x^n \boxplus (\text{other stuff}) = b_m x^m \boxplus (\text{lower powers}).$$

If x is sufficiently large then $a_n x^n > b_m x^m \boxplus (\text{lower orders})$.

If $a_n \neq 1$, then divide both sides by a_n .

$$\begin{aligned} x^n &> b_i x^i \quad \text{for all } 0 \leq i \leq m \\ x^{n-i} &> b_i \\ x &> \sqrt[n-i]{b_i} \\ X_0 &= \max_{i=0,1,\dots,m} \{ \sqrt[n-i]{b_i} \} \end{aligned}$$

For $x > X_0$, $a_n x^n > b_m x^m \boxplus b_{m-1} x^{m-1} \boxplus \dots \boxplus b_0$, which can't happen. Therefore $\partial p(x) = \partial q(x)$. Likewise, since we have shown that the degree of the two polynomials are equal, we can use the same reasoning to show that the leading coefficients of the two polynomials must also be equal. Since the two polynomials evaluate identically for all $x \in \mathbb{R}^+$, then, in particular, $a_0 = f(0) = g(0) = b_0$. □

We would like to develop a method similar to the Euclidean Algorithm for the polynomials in $(\text{Max-}\times)$. The following example shows that this is futile.

Example 10.11. Consider the two polynomials, $(x^2 \boxplus 1)$, and $(x \boxplus 1)$. Can we write this as $(x^2 \boxplus 1) = q \boxtimes (x \boxplus 1) \boxplus r$, with $\partial r < \partial(x \boxplus 1)$? ■

Here we must have $\partial q = 1$, and $\partial r = 0$. Therefore we may write this equation as

$$\begin{aligned} x^2 \boxplus 1 &= (q_1 x \boxplus q_0) \boxtimes (x \boxplus 1) \boxplus r_0. \\ x^2 \boxplus 1 &= q_1 x^2 \boxplus q_1 x \boxplus q_0 x \boxplus q_0 \boxplus r_0 \end{aligned}$$

Comparing the left and right sides of the above equation, we see that q_1 must equal 1, and q_0 must be less than 1. If this is the case, then the right side of the equation above becomes $x^2 \boxplus x \boxplus q_0 \boxplus r$. This can not equal the left side of the equation. Therefore, we can not write $(x^2 \boxplus 1) = q \boxtimes (x \boxplus 1) \boxplus r$, with $\partial r < \partial(x \boxplus 1)$

Although Example 10.11 shows that we can not, in general, develop a division algorithm in $(\text{Max-}\times)$ which is similar to the Euclidean Algorithm, the following example shows that, for certain polynomials we can write an equation which is similar to the result of the Euclidean Algorithm.

Example 10.12. Consider the two polynomials, $(x^2 \boxplus 2x \boxplus 3)$, and $(x \boxplus 1)$. Can we write this as $(x^2 \boxplus 2x \boxplus 1) = q \boxtimes (x \boxplus 1) \boxplus r$, with $\partial r < \partial(x \boxplus 1)$? ■

Here, we can write the appropriate equation

$$(x^2 \boxplus 2x \boxplus 3) = (x \boxplus 2) \boxtimes (x \boxplus 1) \boxplus 3$$

10.2 Equations in $(\mathbf{R}[x], \boxplus, \boxtimes)$

In the previous sections we have considered polynomials in $\mathbf{R}[x]$. In this section we investigate equations in $\mathbf{R}[x]$. An equation is a relationship between two polynomials in $\mathbf{R}[x]$, i.e. $p(x) = q(x)$, with $p(x), q(x) \in \mathbf{R}[x]$. A “solution”, or the “solution set”, to an equation is the set of all elements, $a \in \mathbf{R}$, such that $p(a) = q(a)$.

Consider the equation, in the unknown variable x , in $\mathbf{R}[x]$, $p(x) = q(x)$. We are interested in the solution(s) to such equations. We note that not all equations have solutions. For example, $x \boxplus 4 = 2$ has no solutions in the (Max, \times) -algebra.

Example 10.13 ((Max, ×)). $x \boxplus 4 = 2$ has no solutions. ■

Example 10.14 ((Max, ×)). $x \boxplus 2 = 4$ has $x=4$ as the only solution. ■

Example 10.15 ((Max, ×)). $x \boxplus 2 = x \boxplus 1$ has the solution set $\{x \mid x \geq 2\}$ ■

Equivalence of equations

Definition 10.16. Given two polynomial equations, $p_1(x)=p_2(x)$, and $q_1(x)=q_2(x)$, we say that the two polynomials are equivalent, denoted $p_1=p_2 \sim q_1=q_2$, if the two equations have the same solution set.

Lemma 10.17. Definition 10.16 is an equivalence relation.

Proof. It is easy to see that the relation is Reflexive, Symmetric, and Transitive, since the sets associated with the equations are equal in the set-theoretical sense, i.e. the sets associated with each equation are equal. □

It is interesting to note that if $p(x)$ has terms dominated by all other terms in the polynomial, then the terms can be dropped. Example: $x^2 \boxplus 2x \boxplus 4$ can be replaced by $x^2 \boxplus 4$, since the term “ $2x$ ” is dominated by the other two terms for any number in (Max-×). This is not true, however,

for matrices. Consider the two equations from above, and the matrix $A = \begin{bmatrix} 1 & 7 \\ 4 & 1 \end{bmatrix}$.

For this matrix, we have :

$$A^2 \boxplus 2A \boxplus 4I = \begin{bmatrix} 28 & 7 \\ 4 & 28 \end{bmatrix} \boxplus \begin{bmatrix} 2 & 14 \\ 8 & 2 \end{bmatrix} \boxplus \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 28 & 14 \\ 8 & 28 \end{bmatrix}$$

while

$$A^2 \boxplus 4I = \begin{bmatrix} 28 & 7 \\ 4 & 28 \end{bmatrix} \boxplus \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 28 & 7 \\ 4 & 28 \end{bmatrix}$$

Therefore, for this matrix, $A^2 \boxplus 2A \boxplus 4I \neq A^2 \boxplus 4I$.

10.3 $(\mathbb{P}^2, \oplus, \otimes)$

Let $\mathbb{P}^2 = \mathbb{R}[x] \times \mathbb{R}[x]$.

In the previous section we considered solution sets for equations in the $(\mathbb{R}[x], \boxplus, \boxtimes)$ semiring. Let us look closer at these equations. To facilitate our investigation, we will consider polynomial equations as a pair of polynomials.

An equation $p_1(x)=p_2(x)$ can be associated with an ordered pair (p_1, p_2) . For reasons which will be apparent later, we will write the equations as $p^+(x)=p^-(x)$ and the ordered pairs as (p^+, p^-) . Let us define an algebra on these ordered pairs.

Definition 10.18.

- (1) For $\alpha \in \mathbf{R}$, $\alpha \otimes (p^+, p^-) = (\alpha p^+, \alpha p^-)$.
- (2) For $f(x) \in \mathbf{R}[x]$, $f(x) \otimes (p^+, p^-) = (f(x) \boxtimes p^+, f(x) \boxtimes p^-)$.
- (3) Let $p=(p^+, p^-)$, $q=(q^+, q^-) \in \mathbb{P}^2$. Then
- (a) $p \oplus q = (p^+, p^-) \oplus (q^+, q^-) \equiv (p^+ \boxplus q^+, p^- \boxplus q^-)$
 - (b) $p \otimes q = (p^+, p^-) \otimes (q^+, q^-) \equiv (p^+ q^+ \boxplus p^- q^-, p^+ q^- \boxplus p^- q^+)$

Theorem 10.19. The system $(\mathbb{P}^2, \oplus, \otimes)$ is an idempotent semiring.

Proof. We go through the proof in detail. We write $p, q \in \mathbf{R}[x]$ instead of $p(x)$, $q(x)$.

(A) \oplus

(1) Closure:

$$p \oplus q = (p^+, p^-) \oplus (q^+, q^-) = (p^+ \boxplus q^+, p^- \boxplus q^-)$$

$$p^+ \boxplus q^+ \in \mathbb{P}, \text{ and}$$

$$p^- \boxplus q^- \in \mathbb{P}. \text{ Therefore, } p \oplus q \in \mathbb{P}^2.$$

(2) Associativity:

$$\begin{aligned} [p \oplus q] \oplus r &= [(p^+, p^-) \oplus (q^+, q^-)] \oplus (r^+, r^-) \\ &= (p^+ \boxplus q^+, p^- \boxplus q^-) \oplus (r^+, r^-) \\ &= ((p^+ \boxplus q^+) \boxplus r^+, (p^- \boxplus q^-) \boxplus r^-) \\ &= (p^+ \boxplus (q^+ \boxplus r^+), p^- \boxplus (q^- \boxplus r^-)) \\ &= (p^+, p^-) \oplus ((q^+ \boxplus r^+, q^- \boxplus r^-)) \\ &= (p^+, p^-) \oplus [(q^+, q^-) \oplus (r^+, r^-)] \\ &= p \oplus [q \oplus r] \end{aligned}$$

(3) Commutativity: (\oplus is commutative because \boxplus is commutative.)

$$\begin{aligned} p \oplus q &= (p^+, p^-) \oplus (q^+, q^-) \\ &= (p^+ \boxplus q^+, p^- \boxplus q^-) \\ &= (q^+ \boxplus p^+, q^- \boxplus p^-) \\ &= (q^+, q^-) \oplus (p^+, p^-) \\ &= q \oplus p \end{aligned}$$

(4) Additive identity:

$$(0,0) \text{ is the additive identity of } \mathbb{P}^2.$$

$$(0,0) \oplus p = (0,0) \oplus (p^+, p^-) = (p^+, p^-) \oplus (0,0) = (p^+, p^-) = p$$

(B) \otimes

(a) Closure:

$$p \otimes q = (p^+, p^-) \otimes (q^+, q^-) = (p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+).$$

$$p^+ \boxtimes q^+ \boxplus p^- \boxtimes q^- \in R[x], \text{ and}$$

$$p^+ \boxtimes q^-, q^- \boxplus p^- \boxtimes q^+ \in R[x]$$

Therefore $p \otimes q \in \mathbb{P}^2$.

(b) Associativity:

$$\begin{aligned} [p \otimes q] \otimes r &= [(p^+, p^-) \otimes (q^+, q^-)] \otimes (r^+, r^-) \\ &= [(p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+)] \otimes (r^+, r^-) \\ &= ((p^+q^+ \boxplus p^-q^-)r^+ \boxplus (p^+q^- \boxplus p^-q^+)r^-, \\ &\quad (p^+q^+ \boxplus p^-q^-)r^- \boxplus (p^+q^- \boxplus p^-q^+)r^+) \\ &= (p^+q^+r^+ \boxplus p^-q^-r^+ \boxplus p^+q^-r^- \boxplus p^-q^+r^-, \\ &\quad p^+q^+r^- \boxplus p^-q^-r^- \boxplus p^+q^-r^+ \boxplus p^-q^+r^+) \\ &= (p^+q^+r^+ \boxplus p^+q^-r^- \boxplus p^-q^+r^- \boxplus p^-q^-r^+, \\ &\quad p^+q^+r^- \boxplus p^+q^-r^+ \boxplus p^-q^+r^+ \boxplus p^-q^-r^-) \\ &= (p^+(q^+r^+ \boxplus q^-r^-) \boxplus p^-(q^+r^- \boxplus q^-r^+), \\ &\quad p^+(q^+r^- \boxplus q^-r^+) \boxplus p^-(q^+r^+ \boxplus q^-r^-)) \\ &= (p^+, p^-) \otimes [(q^+r^+ \boxplus q^-r^-, q^+r^- \boxplus q^-r^+)] \\ &= (p^+, p^-) \otimes [(q^+, q^-) \otimes (q^+, q^-)] \\ &= p \otimes [q \otimes r] \end{aligned}$$

(c) Commutativity:

$$\begin{aligned} p \otimes q &= (p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+) \\ &= (q^+p^+ \boxplus q^-p^-, q^+p^- \boxplus q^-p^+) \\ &= q \otimes p \end{aligned}$$

(d) Multiplicative Identity:

(1,0) is the multiplicative identity.

$$(1,0) \otimes p = (1,0) \otimes (p^+, p^-) = (1 p^+ \boxplus 0 p^-, 0 p^- \boxplus 1 p^-) = (p^+, p^-) = p.$$

(e) Absorbing Property of Additive Identity:

$$(0,0) \otimes p = (0,0) \otimes (p^+, p^-) = (0,0).$$

$$p \otimes (0,0) = (p^+, p^-) \otimes (0,0) = (0,0).$$

(C) Distributivity of \otimes over \oplus .

We show left distributivity, and leave right distributivity for the reader.

$$\begin{aligned}
p \otimes [q \oplus r] &= (p^+, p^-) \otimes [(q^+, q^-) \oplus (r^+, r^-)] \\
&= (p^+, p^-) \otimes [(q^+ \boxplus r^+, q^- \boxplus r^-)] \\
&= (p^+(q^+ \boxplus r^+) \boxplus p^-(q^- \boxplus r^-), p^+(q^- \boxplus r^-) \boxplus p^-(q^+ \boxplus r^+)) \\
&= (p^+q^+ \boxplus p^+r^+ \boxplus p^-q^- \boxplus p^-r^-, p^+q^- \boxplus p^+r^- \boxplus p^-q^+ \boxplus p^-r^+) \\
&= (p^+q^+ \boxplus p^+q^- \boxplus p^-r^+ \boxplus p^-r^-, p^+q^- \boxplus p^+q^+ \boxplus p^-r^- \boxplus p^-r^+) \\
&= [(p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+)] \oplus [(p^+r^+ \boxplus p^-r^-, p^+r^- \boxplus p^-r^+)] \\
&= [(p^+, p^-) \otimes (q^+, q^-)] \oplus [(p^+, p^-) \otimes (r^+, r^-)] \\
&= (p \otimes q) \oplus (p \otimes r)
\end{aligned}$$

And the proof is complete. □

We can add the following lemma.

Lemma 10.20. If $R[x]$ is idempotent, then \mathbb{P}^2 is idempotent.

Proof. $p \oplus p = (p^+, p^-) \oplus (p^+, p^-) = (p^+ \boxplus p^+, p^- \boxplus p^-) = (p^+, p^-) = p$ □

Equivalence of polynomial pairs

Definition 10.21. (p^+, p^-) is said to be equivalent to (q^+, q^-) if and only if $p^+(x)=p^-(x)$ is equivalent to $q^+(x)=q^-(x)$ as equations from section 10.2.

We note here that if $p_1(x)=p_2(x) \sim q_1(x)=q_2(x)$ and $a \in R$ with $p_1(a)=p_2(a)$, then $q_1(a)=q_2(a)$. However, this is not true for matrices. As an example, consider the equivalence of the equations: $x^2 = 2x \boxplus 3 \sim x = 2$. They are equivalent since both equations have $x=2$ as the only solution. If

we consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, we note that $p_1(A)=p_2(A)$ but $q_1(A) \neq q_2(A)$. So,

$$A^2 = 2A \boxplus 3I, \text{ but } A \neq 2I$$

Another counterexample: $x^2 = 2x \boxplus 3 \sim x^2 = 2x \boxplus 1$. The matrix A above satisfies the equation $x^2 = 2x \boxplus 3$, but matrix A above does not satisfy the equation $x^2 = 2x \boxplus 1$.

Division

Our main motivation for considering division of polynomial pairs is to work with the Characteristic Polynomials associated with matrices. As such, we will assume, in the polynomial equation $p^+(x)=p^-(x)$, that $\partial p^+ > \partial p^-$

We define a division for these polynomial pairs. Suppose that $\partial p^+ > \partial p^-$, and $\partial h^+ > \partial h^-$.

Definition 10.22.

We say that $p \mid h$ (as polynomial pairs) if there exists a polynomial pair, $q, \in \mathbb{P}^2$, such that $p \otimes q = h$,

or

$(p^+, p^-) \mid (h^+, h^-)$ if there exists $(q^+, q^-) \in \mathbb{P}^2$, such that $(p^+, p^-) \otimes (q^+, q^-) = (h^+, h^-)$.

The definition above can be written as:

$$h^+ = \gamma^+ h^+ \boxplus \gamma^- h^- \quad (348)$$

$$h^- = \gamma^- h^+ \boxplus \gamma^+ h^- \quad (349)$$

Let us show that this (division) is a Reflexive, Transitive, Anti-symmetric relation on \mathbb{P}^2 . We assume that all equations in this section represent “identities”, equations which are true for all $x \in R$.

(A) Reflexive:

$$p \mid p, \text{ since } (1,0) \otimes p = p.$$

(B) Transitive:

Suppose $p \mid q$, and $q \mid h$.

Then there exists an α such that $\alpha \otimes p = q$, and there exists a β such that $\beta \otimes q = h$.

Then $h = \beta \otimes q = \beta \otimes [\alpha \otimes p] = [\beta \otimes \alpha] \otimes p$. Therefore $p \mid h$.

(C) Anti-Symmetric:

Suppose $p \mid h$ and $h \mid p$. Then $h = \alpha \otimes p$, and $p = \beta \otimes h$. Then

$h = \alpha \otimes [\beta \otimes h] = [\alpha \otimes \beta] \otimes h = \gamma \otimes h$, where $\gamma = [\alpha \otimes \beta]$. This leads to:

$$h^+ = \gamma^+ h^+ \boxplus \gamma^- h^- \quad (350)$$

$$h^- = \gamma^- h^+ \boxplus \gamma^+ h^- \quad (351)$$

By using Lemma 10.10, we get that $\gamma^+ = 1$ from Equation (350), and $\gamma^- = 0$ from Equation (351). Therefore $\gamma = (1,0)$, which implies that $\alpha \otimes \beta = (1,0)$.

$$\gamma = (1, 0)$$

$$\alpha \otimes \beta = (1, 0)$$

$$(\alpha^+, \alpha^-) \otimes (\beta^+, \beta^-) = (1, 0)$$

$$(\alpha^+ \beta^+ \boxplus \alpha^- \beta^-, \alpha^+ \beta^- \boxplus \alpha^- \beta^+) = (1, 0)$$

$$\begin{cases} \alpha^+ \beta^+ \boxplus \alpha^- \beta^- = 1 \\ \alpha^+ \beta^- \boxplus \alpha^- \beta^+ = 0 \end{cases} \quad (352)$$

Equation (352) leads to the following two cases:

(I) $\alpha^+ \beta^+ \neq 0$

$$\begin{aligned}
\alpha^+ \beta^+ \neq 0 &\implies \alpha^+ \neq 0 \text{ and } \beta^+ \neq 0 \\
&\implies \beta^- = 0 \text{ and } \alpha^- = 0 \text{ since } \alpha^+ \beta^- = 0 \text{ and } \alpha^- \beta^+ = 0 \\
&\implies \alpha^+ \beta^+ = 1 \\
&\implies \alpha^+, \text{ and } \beta^+ \text{ are constants by definition (of degree)} \\
&\implies (h^+, h^-) = (\alpha^+, 0) \otimes (p^+, p^-) = \alpha^+ \otimes (p^+, p^-) \\
&\implies h = \alpha \otimes p, \text{ where } \alpha \text{ is a constant.}
\end{aligned}$$

(II) $\alpha^- \beta^- \neq 0$

$$\begin{aligned}
\alpha^- \beta^- \neq 0 &\implies \alpha^- \neq 0 \text{ and } \beta^- \neq 0 \\
&\implies \beta^+ = 0 \text{ and } \alpha^+ = 0 \\
&\implies \alpha^- \beta^- = 1 \\
&\implies \alpha^-, \text{ and } \beta^- \text{ are constants by definition (of degree)} \\
&\implies (p^+, p^-) = (0, \beta^-) \otimes (h^+, h^-) = \beta^- \otimes (h^+, h^-) \\
&\implies p = \beta \otimes h, \text{ where } \beta \text{ is a constant.}
\end{aligned}$$

$$\begin{cases} h^+ = \alpha p^+ \\ h^- = \alpha p^- \end{cases} \quad \text{or} \quad \begin{cases} p^+ = \beta h^- \\ p^- = \beta h^+ \end{cases} \quad (353)$$

Since this division relation has been shown to be Reflexive, Transitive, and Anti-Symmetric, it induces a partial order on the system $(\mathbb{P}^2, \oplus, \otimes)$: $p \leq q \iff p|q$

Note 10.23.

(1) $(1,0) \otimes (p^+, p^-) = (p^+, p^-)$

(2) $(0,1) \otimes (p^+, p^-) = (p^-, p^+)$

(3) We can define: $e=(1,0)$, and $i=(0,1)$, then

(a) $i^2=e$

(b) $(p^+, p^-) = [p^+ \otimes e] \oplus [p^- \otimes i]$

(c) $(p^+, p^-) \otimes (q^+, q^-) = [p^+ \otimes (q^+, q^-)] \oplus [p^- \otimes (0, 1) \otimes (q^+, q^-)]$

10.4 Ideal

Definition 10.24. Given a matrix A , define

$$\eta_A \equiv \{ p=(p^+,p^-) \in \mathbb{P}^2 \mid p^+(A) = p^-(A), \text{ and } \partial p^+ > \partial p^- \}$$

From Section 8.9 we know that $\eta_A \neq \emptyset$ since the Characteristic Polynomial Equation Pair associated with the matrix A exists and is an element of η_A .

Lemma 10.25. If $p \in \eta_A$, and $\alpha \in \mathbb{R}$, then $\alpha p \in \eta_A$.

Proof. The proof of this lemma follows from the fact that multiplication by a constant is calculated element-wise. □

Lemma 10.26. If $p \in \eta_A$, and $f \in \mathbb{R}[x]$, then $(f \otimes p) \in \eta_A$.

Proof. Suppose $p^+(A)=p^-(A)$. Consider $f(A)p^+(A)$, and $f(A)p^-(A)$.

$$\begin{bmatrix} \dots & \dots & \dots \end{bmatrix} \otimes \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \end{bmatrix} \otimes \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \quad (354)$$

Look at the ij^{th} element of the product. Since $p^+(A)=p^-(A)$ then $[p^+(A)]_{ij}=[p^-(A)]_{ij}$ $\forall 1 \leq ij \leq n$, then $[f(A) p^+(A)]_{ij}=[f(A) p^-(A)]_{ij}$ □

Lemma 10.27. If $p \in \eta_A$, and $q \in \eta_A$, then $(p \oplus q) \in \eta_A$.

Proof. $(p^+ \boxplus q^+)(A) = p^+(A) \boxplus q^+(A)$, and

$(p^- \boxplus q^-)(A) = p^-(A) \boxplus q^-(A)$ from Lemma 10.2.

Consider the ij^{th} element of $p^+(A) \boxplus q^+(A)$. The ij^{th} element of $p^+(A) \boxplus q^+(A)$ is one of the elements $[p^+(A)]_{ij}$, or $[q^+(A)]_{ij}$. Suppose, without loss of generality, that it is $[q^+(A)]_{ij}$. Then

$[q^+(A)]_{ij} > [p^+(A)]_{ij}$. Therefore, $[q^-(A)]_{ij} > [p^-(A)]_{ij}$, since $[p^+(A)]_{ij} = [p^-(A)]_{ij}$, and

$[q^+(A)]_{ij} = [q^-(A)]_{ij}$. Therefore, $[p^-(A) \boxplus q^-(A)]_{ij} = [q^-(A)]_{ij}$, and

$[p^+(A) \boxplus q^+(A)]_{ij} = [p^-(A) \boxplus q^-(A)]_{ij}$. □

Lemma 10.28. If $p \in \eta_A$, and $q \in \eta_A$, then $(p^+ \boxtimes q^-)(A) = (p^- \boxtimes q^+)(A)$.

Proof. Proof is similar to Lemma 10.27.

Consider the equation $(p^+ \boxplus q^-)(A) = (p^- \boxplus q^+)(A)$. Consider the ij^{th} element of each side.

Without loss of generality, if $[p^+(A)]_{ij} > [q^-(A)]_{ij}$ then $[p^+(A)]_{ij} = [p^-(A)]_{ij} > [q^+(A)]_{ij}$, and the result follows. □

Lemma 10.29. If $p \in \eta_A$, and $q \in \eta_A$, then $(p \otimes q) \in \eta_A$.

Proof. Let $p=(p^+,p^-) \in \eta$, and $q=(q^+,q^-) \in \eta$. So $p^+(A) = p^-(A)$, and $q^+(A) = q^-(A)$.

$$(p \otimes q)(A) = (p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+)(A).$$

We need to show that: $p^+(A) q^+(A) \boxplus p^-(A) q^-(A) = p^+(A) q^-(A) \boxplus p^-(A) q^+(A)$ This follows from the identity, and use of Lemma 10.26 and Lemma 10.27: $(p^+q^+ \boxplus p^-q^-, p^+q^- \boxplus p^-q^+) = p^+ \otimes (q^+,q^-) \oplus p^- \otimes (q^-,q^+)$. □

Lemma 10.30. If $p \in \eta_A$, and $f \in \mathbb{P}^2$, then $(f \otimes p) \in \eta_A$.

Proof. Suppose $p^+(A)=p^-(A)$, and $f=(f^+, f^-) \in \mathbb{P}^2$. Then $f^+ \otimes p \in \eta_A$ and $f^- \otimes (p^+, p^-) \in \eta_A$, by Lemma 10.26. Therefore just as in 10.29, we have $(f^+p^+ \boxtimes f^-p^-)(A) = (f^+p^- \boxtimes f^-p^+)(A)$.

Therefore $(f \otimes p) \in \eta_A$. □

Theorem 10.31. η is an ideal of \mathbb{P}^2 .

Proof.

1. Lemma 10.27 provides the proof that η_A is closed under \oplus .
2. \otimes : Lemma 10.29 provides the proof that η_A is closed under \otimes .
3. Lemma 10.30 provides the proof that \otimes “captures” elements on \mathbb{P}^2 .

□

11 Parallels

Many parallels exist between the Max-Times / Max-Plus algebras and the algebra of nonnegative matrices. Table 6 lists a few of the many parallels between these algebras.

Nonnegative	\mathfrak{M}^+
$\rho(A)$ is eigenvalue of A .	$\mu(A)$ is an eigenvalue of A .
If A is irreducible, then $\rho(A)$ is only positive eigenvalue of A , and associated eigenvector is $v=\alpha u$, $u>0$.	If A is irreducible, then $\mu(A)$ is only eigenvalue of A and every eigenvector corresponding to $\mu(A)$ is finite.
If A has a partly non-positive eigenvector, then A is reducible.	If A has a partly infinite eigenvector, then A is reducible.
If $B \leq A \leq C$, then $\rho(B) \leq \rho(A) \leq \rho(C)$.	If $B \leq A \leq C$, then $\mu(B) \leq \mu(A) \leq \mu(C)$.
If $A \not\geq \underline{0}$, then $\rho(A)$ is eigenvalue of A and $A \underline{x} = \rho \underline{x}$ for some $\underline{x} \geq \underline{0}$	If $A \not\geq \underline{\theta}$, then $\mu(A)$ is eigenvalue of A and $A \underline{x} = \mu \underline{x}$ for some $\underline{x} \geq \underline{\theta}$
If A is nonnegative and irreducible, $n \geq 2$, and $y \geq 0$ with exactly k positive coordinates, $1 \leq k \leq n-1$, then $(I_n + A)y$ has more than k positive coordinates.	If A is nonnegative and irreducible, $n \geq 2$, and $y \geq 0$ with exactly k positive coordinates, $1 \leq k \leq n-1$, then $(I_n + A)y$ has more than k positive coordinates.
If A is irreducible, $n \geq 2$, and $y \geq -\infty$ with exactly k finite coordinates, $1 \leq k \leq n-1$, then $(I_n + A) \times y$ has more than k finite coordinates.	If A is irreducible, $n \geq 2$, and $y \geq -\infty$ with exactly k finite coordinates, $1 \leq k \leq n-1$, then $(I_n \oplus A) \otimes y$ has more than k finite coordinates.
$A \geq 0$, irreducible \Rightarrow there exists m such that $(I+A)^m > 0$.	$A \geq 0$, irreducible \Rightarrow there exists m such that $(I \oplus A)^m > 0$.
If A is non-negative and irreducible, and $\underline{y} \geq \underline{0}$, then $(I_n + A)^{n-1} \underline{y} > 0$.	If A is irreducible, and $\underline{y} \geq \underline{\theta}$, then $(I_n \oplus A)^{n-1} \underline{y} > \underline{\theta}$.
$A_{n \times n}$ is non-negative and irreducible \Rightarrow $A + A^2 + \dots + A^n > 0$	$A_{n \times n}$ irreducible \Rightarrow $A \oplus A^2 \oplus \dots \oplus A^n > -\infty$
$A_{n \times n}$ is non-negative and irreducible \Rightarrow $I + A + A^2 + \dots + A^{n-1} > 0$	$A_{n \times n}$ irreducible \Rightarrow $I \oplus A \oplus A^2 \oplus \dots \oplus A^{n-1} > -\infty$
If A has no cycles then 0 is the unique eigenvalue of A .	If A has no cycles then $-\infty$ is the unique eigenvalue of A .
0 is an eigenvalue of A if and only if A has an zero column.	$-\infty$ is an eigenvalue of A if and only if A has an infinite column.

Table 6: Parallels between Nonnegative and \mathfrak{M}^+ matrices

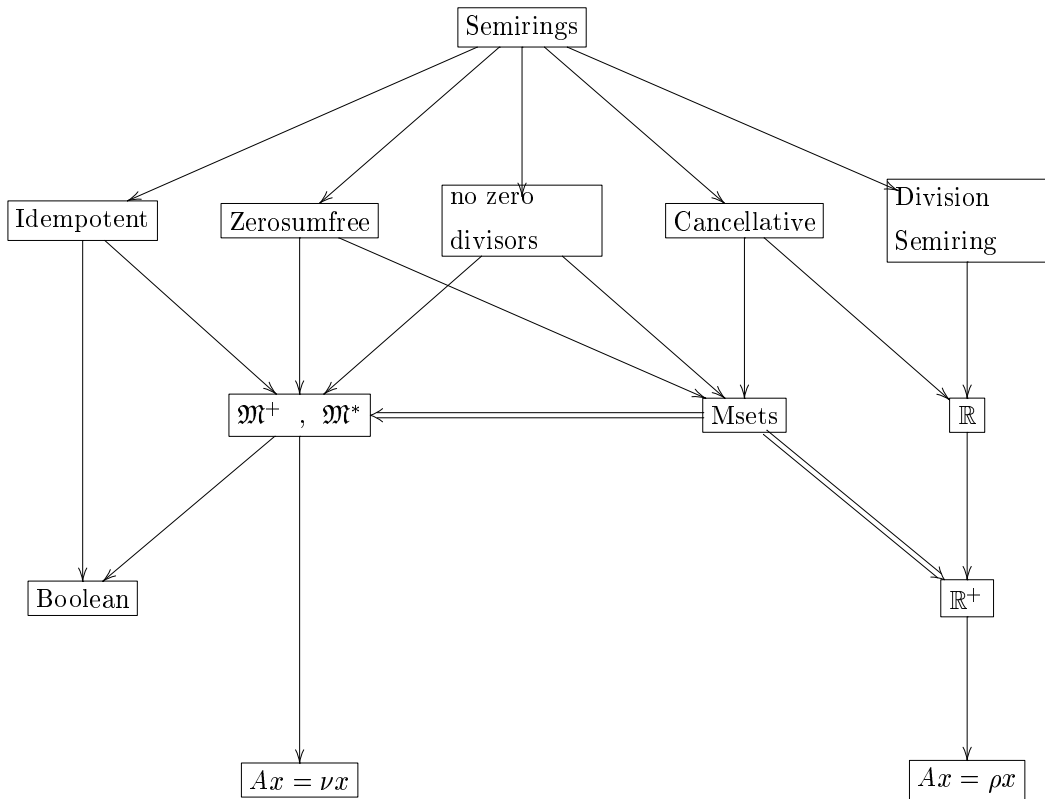


Figure 6: Relationship between semirings and eigenvalue equations

12 Conclusions and Open Questions

As we have seen in Section 8 there is a theory which can be considered as a connection between the nonnegative numbers and the Max-Times algebra. The mset algebra has allowed us to catch a glimpse at one theory which connects these algebras. We have seen that a connection exists between the eigenvalue/eigenvector equations for the two algebras. The multisets are a tool which can be used for calculations in any semiring (and ring).

12.1 Open Questions

There are still many open questions related to this connection. In this section some of the open questions that have appeared are presented as a step toward continuing this research.

Example 12.1. An interesting equation developed, which was investigated. Although no full investigation was completed an elegant conjecture evolved. The proof will require more time to develop. The problem is to find a relationship between the msets for the following equation to hold: $\alpha \boxtimes \beta = \gamma \boxtimes \delta$. ■

The consistency condition requires that $|\alpha| \cdot |\beta| = |\gamma| \cdot |\delta|$.

The following conjecture has been developed.

Conjecture 12.2. If $\alpha \boxtimes \beta = \gamma \boxtimes \delta$, then

$$\begin{aligned}\alpha &= \alpha_1 \otimes \alpha_2 \cdots \otimes \alpha_n, \\ \beta &= \beta_1 \otimes \beta_2 \cdots \otimes \beta_\ell, \\ \gamma &= \gamma_1 \otimes \gamma_2 \cdots \otimes \gamma_s, \\ \delta &= \delta_1 \otimes \delta_2 \cdots \otimes \delta_t.\end{aligned}\tag{355}$$

where the direct products of γ and δ are composed of combinations of the parts of α and β .

Although some easier examples are certainly possible the following example is a fairly general example.

Example 12.3. Consider the true equation: $\{1,2,4,8\} \boxtimes \{1,3,5,15\} = \{1,3,4,12\} \boxtimes \{1,2,5,10\}$
In this example we see that each of the multisets can be decomposed into a direct product of multisets:

$$\begin{aligned}\alpha &= \{1,2\} \boxtimes \{1,4\} \\ \beta &= \{1,3\} \boxtimes \{1,5\} \\ \gamma &= \{1,3\} \boxtimes \{1,4\} \\ \delta &= \{1,2\} \boxtimes \{1,5\}\end{aligned}$$

■

A full investigation of this relation involves a deep understanding of permutations of the elements of rank one matrices.

Conjecture 12.4. If $\alpha x \subseteq \beta x$, and $|\beta| < |\alpha| \cdot |x|$, then $\alpha \subseteq \beta$.

A full investigation of the analog to the matrix B^\ddagger over msets is not complete. The question still remains as to how this matrix should be defined. One answer would be to define this as the infinite sum $B^\ddagger = I \uplus B \uplus B^2 \uplus \cdots$ is okay, but this matrix is now over the power set 4^R of infinite msets, which is no longer cancellative. We may need to look at the conditions for which the sum of the elements exists, such as the elements should be in ℓ^1 .

12.2 Questions to be answered about Polynomial Pairs

1. If $f = (f^+, f^-) \in \mathbb{P}^2$, then $\nu(A)$ is a root of $(f \otimes p)$, where p is characteristic equation of A .
2. Is every root of a characteristic equation also a root of every equation in η_A ?

3. Would it be possible to define $(p^+, p^-) \sim (q^+, q^-)$ if and only if $p^+(x) \boxplus q^-(x) = p^-(x) \boxplus q^+(x) \quad \forall x \in \mathbb{R}^+$?
4. Does the ideal η_A have a generator? If so, is the generator unique? If so, can we say that every element of η_A is divisible by the generator? This would lead to an association much like that of a PID.
5. Can we find a matrix with a non-monic minimal poly? I don't think so, unless we go to a system which is not an integral domain, such as $Z/(12)$. For example see [25, p.164ff].

Conjecture 12.5. If (p^+, p^-) is the characteristic equation of A , and b is a root of (p^+, p^-) , then b is a root of $(f \otimes p)$ for any $f \in \mathbb{P}$

Conjecture 12.6. $\nu(A)$ is a root of every element of η_A .

Conjecture 12.7. If (p^+, p^-) is the characteristic equation of A , and b is a root of (p^+, p^-) , then b is a root of every element of η_A .

The next two examples concern the minimal polynomial of a matrix.

Example 12.8. Consider Example 9.4. If $A=J_{3 \times 3}$, the characteristic equation is $x^3 = x^2$, but it can be checked that $x^2 = x$ is also an equation of smaller degree. Can we say that the polynomial equation of smaller degree divides the polynomial equation of larger degree? ■

In this example we first go from equations to polynomial pairs, and rephrase the question above as: Does $(x^2, x) \mid (x^3, x^2)$?

This question can be answered in the affirmative, as it can be checked that $(x, 0) \otimes (x^2, x) = (x^3, x^2)$.

Example 12.9. If $A=J_{3 \times 3}-I_{3 \times 3}$, then the characteristic equation is $x^3 = x \boxplus 1$, but it can be checked that $x^2 = x \boxplus 1$ is also an equation of smaller degree which is satisfied by the matrix. ■

In this example we first go from equations to polynomial pairs, and rephrase the question above as: Does $(x^2, x \boxplus 1) \mid (x^3, x \boxplus 1)$?

This question can be answered in the affirmative, as it can be checked that $(x \boxplus 1, 0) \otimes (x^2, x \boxplus 1) = (x^3, x \boxplus 1)$.

A Appendix

A.1 Basic Properties

Table 7 lists some basic facts about the following four algebra systems.

(a) Non-negative Real numbers

(i) $a \oplus b = a + b$

(ii) $a \otimes b = a \times b$

(iii) $\theta = 0$ and $e = 1$

(iv) $\{0, +\} \iff \{0, (>0)\}$

(b) Boolean Algebra

(i) $a \oplus b = \max(a,b)$

(ii) $a \otimes b = \min(a,b)$

(iii) $\theta = 0$ and $e = 1$

(iv) $\{0, 1\} \iff \{0, (>0)\}$

(c) Max-Plus Algebra

(i) $a \oplus b = \max(a,b)$

(ii) $a \otimes b = a + b$

(iii) $\theta = -\infty$ and $e = 0$

(iv) $\{-\infty, (\neq -\infty)\} \leftrightarrow \{\theta, (>\theta)\}$

(d) Max-Times Algebra

(i) $a \oplus b = \max(a,b)$

(ii) $a \otimes b = a \times b$

(iii) $\theta = 0$ and $e = 1$

(iv) $\{0, +\} \iff \{0, (>0)\}$

Number	Statement	True/False
1	If $x_i = \theta, \forall i$, then $\oplus x_i = \theta$	True
2	If $x_i = \theta, \forall i$, then $\otimes x_i = \theta$	True
3	If $\oplus x_i \neq \theta$, then $\exists i \ni x_i \neq \theta$ (CP-1)	True
4	If $\otimes x_i \neq \theta$, then $\exists i \ni x_i \neq \theta$ (CP-2)	True
4.5	If $\otimes x_i \neq \theta$, then $x_i \neq \theta, \forall i$ (stronger than 4.)	True
5	If $\oplus x_i = \theta$, then $x_i = \theta, \forall i$ (Conv-1)	True
6	If $\otimes x_i = \theta$, then $x_i = \theta, \forall i$ (Conv-2) (only one $x_i = \theta$ needed)	False
6.5	If $\otimes x_i = \theta$, then $\exists i \ni x_i = \theta$ (Weaker than 6.) (No Zero-Divisors)	True
7	If $\exists i \ni x_i \neq \theta$, then $\oplus x_i \neq \theta$ (Inv-1)	True
8	If $\exists i \ni x_i \neq \theta$, then $\otimes x_i \neq \theta$ (Inv-2)	False
9	If $x_i \neq \theta, \forall i$, then $\oplus x_i \neq \theta$	True
10	If $x_i \neq \theta, \forall i$, then $\otimes x_i \neq \theta$	True

Table 7: Basic facts about some specific path algebras

A.2 Matlab programs for the Max-Plus Algebra System

This section contains some programs in Matlab to perform some of the elementary operations in the $(\max,+)$ - algebra.

```
=====
function cout = oeeye(n)
% Returns the nxn identity matrix in the max-plus system.

cout=zeros(n,n); for i=1:n
  for j=1:n
    if i==j
      cout(i,j)=0;
    else
      cout(i,j)=-inf;
    end
  end
end
=====
=====
function cout = otrace(a)
% This function returns the trace of the matrix a in the
% max-plus algebra system.
% Requires: none.

if size(a,1)~=size(a,2)
  disp('ERROR -- matrix must be square')
  return
end
  cout=-inf;
  for i=1:size(a,1)
    cout=max(cout,a(i,i));
  end
=====
=====
function cout = oplus(a,b)
% calculates a + b in the max-plus algebra system
% oplus(a,b)=max(a,b)
% Requires: none.

cout=max(a,b);
=====
=====
function cout = otimes(a,b);
% calculates the product of a and b in the max-times algebra.
% Requires: none.

if max(size(a))==1
  cout=a+b;
elseif size(a,2) ~= size(b,1)
  disp('??? Error using ==> otimes')
  disp('Inner matrix dimensions must agree')
```

```

else
    kc=size(a,2); ar=size(a,1); bc=size(b,2);
    cout=zeros(ar,bc);
    for r=1:ar
        for c=1:bc
            t=a(r,1) + b(1,c);
            if a(r,1)==-inf | b(1,c)==-inf
                t=-inf;
            end
            for k=2:kc
                s=a(r,k)+b(k,c);
                if a(r,k)==-inf | b(k,c)==-inf
                    s=-inf;
                end
                t=max(t,s);
            end
            cout(r,c)=t;
        end
    end
end
=====
function cout = opower(a,n)
% calculates the nth power of the input matrix
% in the max-plus algebra
% calculates the power using binary representation
% of the exponent -- very fast!
% Requires: oeye.m , otimes.m , oplus.m

if n==0
    cout=oeye(size(a));
elseif n==1
    cout=a;
elseif n==2
    cout=otimes(a,a);
else
    x=[];
    xw=n;
    while xw~=0
        x=[x rem(xw,2)];
        xw=fix(xw/2);
    end
    cout=a;
    lx=length(x)-1;
    for z=lx:-1:1
        if x(z)==0
            cout=otimes(cout,cout);
        else
            cout=otimes((otimes(cout,cout)),a);
        end
    end
end

```

```

end
=====
=====
function cout = omu(a);
% calculates the maximum cycle mean of a matrix
% in the max-plus algebra
% Requires: oeye.m , otimes.m , otrace.m

if size(a,1)~=size(a,2)
    disp('ERROR -- matrix must be square')
    return
end

cout = -inf; d=oeye(size(a,1));

for j=1:size(a,1)
    d=otimes(d,a);
    x=otrace(d);
    if x~-inf
        xx=(1/j)*x;
    else
        xx=x;
    end
    cout = max(cout,xx);
end
=====
=====
function cout = oddag(b)
% calculates b double dagger if it exists.
% oddag(B) = I + B + B^2 + ... + B^{n-1}
% max circuit mean must be <= 0 for convergence
% Requires: oeye.m , omu.m , opower.m , otimes.m , oplus.m

if size(b,1)~=size(b,2)
    disp('Error -- matrix must be square')
    return
end
z=omu(b); if z>0
    disp('ERROR -- max eigenvalue > 0 --- too large')
    disp('No convergence')
return
end

sz=size(b,1); cout=oeye(sz); cout=oplus(cout,b);
for x=2:sz-1
    cout=oplus(cout,opower(b,x));
end
=====
=====
function cout = odplus(b)
% calculates "b^plus" if it exists.

```

```

% odplus(B) = B + B^2 + ... + B^n
% max circuit mean must be <= 0 for convergence
% Requires: omu.m , opower.m , otimes.m , oplus.m

if size(b,1)~=size(b,2)
    disp('Error -- matrix must be square')
    return
end
z=omu(b);
if z>0
    disp('ERROR -- max eigenvalue > 0 --- too large')
    disp('No convergence')
    return
end

sz=size(b,1); cout=b;
for x=2:sz
    cout=oplus(cout,opower(b,x));
end
=====
function cout = omukarp(a)
% calculates the maximum circuit mean for the adjacency
% matrix A, using Karp's Algorithm.
% Requires: opower.m , otimes.m , oplus.m , oeye.m

N=size(a,1);
an=opower(a,N);
mnk=inf;
mxj=-inf;
for i=1:N
    for j=1:N
        for k=0:N-1
            ak=opower(a,k);
            z=(an(i,j)-ak(i,j))/(N-k);
            if an(i,j)==-inf & ak(i,j)==-inf
                z=-inf;
            end
            if z==NaN,z=-inf,end
            mnk=min(mnk,z);
        end
        mxj=max(mxj,mnk);
    end
end
cout=mxj;
=====
function cout = oplusm(a,b,c,d,e,f,g,h)
% This function calculates the sum of up to 8 matrices in the
% max-plus algebra system.
% oplus(a,b)=max(a,b)

```

```

% Requires: none.

if nargin==1
    cout=a;
elseif nargin==2
    cout=max(a,b);
elseif nargin==3
    cout=max(max(a,b),c);
elseif nargin==4
    cout=max(max(a,b),max(c,d));
elseif nargin==5
    cout=max(max(a,b),max(max(c,d),e));
elseif nargin == 6
    cout=max(max(a,b),max(c,d));
    cout=max(cout,max(e,f));
elseif nargin == 7
    cout=max(max(a,b),max(c,d));
    cout=max(cout,max(e,max(f,g)));
elseif nargin == 8
    cout=max(max(a,b),max(c,d));
    cout=max(cout,max(e,f));
    cout=max(cout,max(g,h));
else
    disp('ERROR using oplusm.m')
end
=====

```

A.3 Matlab programs for the Max-Times Algebra System

This section contains some programs in Matlab to perform some of the elementary operations in the (\max, \times) - algebra.

```
=====
function cout = btrace(a)
% This function returns the trace of the matrix a in the
% max-times algebra system.
% Requires: none.

if size(a,1)~=size(a,2)
    disp('ERROR -- matrix must be square')
    return
end cout=0; for i=1:size(a,1)
    cout=max(cout,a(i,i));
end
=====
function cout = bplus(a,b)
% returns the sum of two numbers (matrices) in the
% max-times algebra system.
% bplus(a,b)=max(a,b)
% Requires: none.

cout=max(a,b);
=====
function cout = btimes(a,b);
% calculates the product of two numbers (matrices) in the
% max-times algebra system.
% Requires: none.

if max(size(a))==1
    cout=a*b;
elseif size(a,2) ~= size(b,1)
    disp('??? Error using ==> btimes.m')
    disp('Inner matrix dimensions must agree')
else
    kc=size(a,2);
    ar=size(a,1);
    bc=size(b,2);
    cout=zeros(ar,bc);
    for r=1:ar
        for c=1:bc
            t=a(r,1)*b(1,c);
            for k=2:kc
                s=a(r,k)*b(k,c);
                t=max(t,s);
            end
            cout(r,c)=t;
        end
    end
end
=====
```

```

        end
    end
end
=====
=====
function cout = bpower(a,n)
% calculates the nth power of the input matrix
% in the max-times algebra
% calculates the power using binary representation
% of the exponent -- very fast!
% Requires: btimes.m , bplus.m

if n==0
    cout=eye(size(a));
elseif n==1
    cout=a;
elseif n==2
    cout=btimes(a,a);
else
    x=[];
    xw=n;
    while xw~=0
        x=[x rem(xw,2)];
        xw=fix(xw/2);
    end
    cout=a;
    lx=length(x)-1;
    for z=lx:-1:1
        if x(z)==0
            cout=btimes(cout,cout);
        else
            cout=gtimes((btimes(cout,cout)),a);
        end
    end
end
end
=====
=====
function cout = bmu(a);
% calculates the maximum cycle mean of a matrix
% in the max-times algebra
% Note: omu(a) is the same as: log(bmu(exp(a)))
% Requires: btimes.m , btrace.m , bplus.m

if size(a,1)~=size(a,2)
    disp('ERROR -- matrix must be square')
    return
end
cout =0;
d=eye(size(a,1));
for j=1:size(a,1)
    d=btimes(d,a);

```

```

    x=btrace(d);
    xx=x^(1/j);
    cout = max(cout,xx);
end
=====
=====
function cout = bddag(b)
% calculates b double dagger if it exists.
% bddag(B) = I + B + B^2 + ... + B^{n-1}
% max circuit mean must be <= 1 for convergence
% Requires bmu.m , bpower.m , btimes.m , bplus.m

z=gmu(b);
if z>1
    disp('ERROR')
    disp('max eigenvalue > 1 --- too large')
    disp('No convergence')
    return
end

sz=size(b,1);
cout=eye(sz);
cout=bplus(cout,b);
for x=2:sz-1
    cout=bplus(cout,bpower(b,x));
end
=====
=====
function cout = bdplus(b)
% calculates "b^plus" if it exists.
% bdplus(B) = B + B^2 + ... + B^n
% max circuit mean must be <= 1 for convergence
% Requires: bmu.m , bpower.m , btimes.m , bplus.m

z=gmu(b);
if z>1
    disp('ERROR')
    disp('max eigenvalue > 1 --- too large')
    disp('No convergence')
    return
end

sz=size(b,1);
cout=b;
for x=2:sz cout=bplus(cout,bpower(b,x)); end
=====

```

A.4 Matlab programs for the mset Algebra System

This section contains some programs in Matlab to perform some of the elementary operations in the mset - algebra.

```
=====
function cout=am(a,b)
% Multiply two msets (as row(a) and row(b)) and return
% the product as a sorted list
% a is '1xn' vector and b is '1xn' vector.
% Requires: none.

if ((size(a,1)==0)&(size(a,2)==0)|((size(b,1)==0)&(size(b,2)==0))
    cout=[];
    return
end

if (size(a,1)~=1)|(size(b,1)~=1)
    disp('Error -- both matrices must be 1xn')
    return
end

as=size(a,2); bs=size(b,2); cs=as*bs; cout=zeros(1,cs);

for i=1:as
    for j=1:bs
        cout(bs*(i-1)+j)=a(i)*b(j);
    end
end

cout=sort(cout);
=====
function [e,f]=amrd(a,b,c,d)
% Multiply (a,b)*(c,d) as msets over  $R^D$ . (Ring of Differences)
% Requires: am.m

e1=am(a,c); e2=am(b,d); f1=am(a,d); f2=am(b,c);

e=sort([e1,e2]); f=sort([f1,f2]);
=====
```

A.5 UBASIC programs for Characteristic Polynomial

This appendix contains a program written in UBASIC to calculate the characteristic polynomial in the Max-Times Algebra system. UBASIC is a fast, and free, BASIC-interpreter capable of large number arithmetic.

```

=====
10  ' This UBASIC program can be used to find the characteristic
20  ' equation for a matrix. It returns the list of
30  ' coefficients associated with the equation.
40  ' The program also returns the maximum eigenvalue for
42  ' the given matrix.
45  ' The program could be reworded to use
47  ' extended memory and increase the size of the matrix.
50  word 100 'Uses small numbers to get 5x5 matrices stored.
60  cls
70  Mx=5:Mxf=(Mx):Mxf2=Mxf//2
80  dim M(Mx,Mx),Sbm(Mx,Mx),Rwa(Mx),Cla(Mx),Charpoly(Mx+2),BNu(Mx+1)
90  dim I(Mx),Mat(Mx,Mx),Detlist(Mxf),Arr(Mx),Detodd(Mxf2),Deteven(Mxf2)
120  '
130  print "Enter number of Rows (and columns), Max size is ";
140  print Mx;"x";Mx;" : ":input Rws
150  Cols=Rws
160  ' === input matrix
170  for I=1 to Rws:for J=1 to Cols
180      print "element M(";I;",";J;") ":input M(I,J)
190  next J:next I
200  cls
210  '
220  print
230  gosub *Prtmtrx(M(),Rws,Cols)
240  print
250  for Cb=1 to Rws
260      Tcnt=0
270      gosub *Combinat(Rws,Cb) 'Calculates the rows/cols for princ det
280      gosub *Calccoef:'Calculates the coefficients
290  next Cb
300  print:print "charpoly coefficients"
310  for I=1 to Rws:print Charpoly(I),:next I
320  print:print:print:print "Characteristic polynomial"
330  print "x^";Rws;
340  for I=1 to Rws 'Print LHS of equation.
350      if Charpoly(I)>0 then print " + ";Charpoly(I);"* x^";Rws-I;
360  next I
370  print " = ";
380  for I=1 to Rws 'Prints RHS of equation.
390      if Charpoly(I)<0 then print " + ";(-1)*Charpoly(I);"* x^";Rws-I;
400  next I
410  gosub *Nu(&Charpoly(),Rws) ' Calculates max. eigenvalue.
420  print:print
430  print "Maximum eigenvalue of the matrix is $\nu$(A) = ";Mxnu

```

```

440 print "$\nu$(A) = (" ; (-1)*Charpoly(Svplc); ")^(1/" ; Svplc; ")"
450 print=print
460 end
470 '=====  
LIST OF SUBROUTINES.  
480 '=====  
print maxtrix subroutine  
490 *Prtmtrx(Xarr()),R,C)  
500 for I=1 to R:for J=1 to C:print Xarr(I,J);:next J:print:next I  
510 return  
520 '=====  
submatrix  
530 *Submatrix(Arr(),Rwa(),Cla(),Sbm(),Rn,Cn)  
540 local I,J,A,B  
550 for I=1 to Rn:for J=1 to Cn  
560     A=Rwa(I):B=Cla(J)  
570     Sbm(I,J)=Arr(A,B)  
580 next J:next I  
590 gosub *Det(Sbm(),Rn)  
600 return  
610 '=====  
combination routine  
620 *Combinat(N,M)  
630 local J  
640 for J=1 to M:I(J)=J:next J 'initialization  
650 '=====  
process selection == takes principal submatrix  
660 gosub *Submatrix(M(),I(),I(),Sbm(),M,M)  
670 '=====  
end process  
680 P=M  
690 if I(P)<N-(M-P) then  
700     :inc I(P):for K=P+1 to M:I(K)=I(P)+K-P:next K:goto 650  
710 :else  
720     :if I(1)<(N-M+1) then  
730         :dec P:goto 690  
740     :else  
750         :goto 780  
760     :endif  
770 :endif  
780 return  
790 '=====  
determinant  
800 *Det(Mat(),R)  
810 local I  
820 dim Arr(20)  
830 for I=0 to R-1:Arr(I)=I+1:next I 'initialize  
840 gosub *Workingpart  
850 gosub *Permute(&Arr(),R,0)  
860 return  
870 '=====  
permutation program  
880 *Permute(&Xrr(),PN,PS) 'this subroutine does permutations  
890 local Tmp,P,Q,T  
900 if PS>=PN-1 then return  
910 for P=N-2 to PS step -1  
920     for Q=P+1 to PN-1 step 1  
930         Tmp=Xrr(P):Xrr(P)=Xrr(Q):Xrr(Q)=Tmp  
940         gosub *Permute(&Xrr(),PN,P+1)

```

```

950         gosub *Workingpart
960     next Q
970     dec Q
980     Tmp=Xrr(P)
990     for Tiq=P to Q-1:Xrr(Tiq)=Xrr(Tiq+1):next Tiq
1000    Xrr(Q)=Tmp
1010    next P
1020    return
1030    *Workingpart
1040    inc Tcnt
1050    gosub *Evenodd(Arr(),R,&Eo)
1060    Prd=1
1070    for Zz=1 to R:Prd=Prd*Mat(Zz,Arr(Zz-1)):next Zz
1080    Detlist(Tcnt)=(-1)^Eo*Prd
1090    return
1100    '===== Even or odd permutation
1110    *Evenodd(A(),Ln,&Eo)
1120    local I,J
1130    Eo=0
1140    for I=0 to Ln-2:for J=I+1 to Ln-1
1150    if A(J)<A(I) then inc Eo:endif
1160    next J:next I
1170    return
1180    ' ===== calculate coefficient of polynomial
1190    *Calccoef
1200    Odcnt=1:Evcnt=1
1210    for I=1 to Tcnt
1220    if Detlist(I)>0 then
1230        :Deteven(Evcnt)=Detlist(I):inc Evcnt
1240    :elseif Detlist(I)<0 then
1250        :Detodd(Odcnt)=abs(Detlist(I)):inc Odcnt
1260    :endif
1270    next I
1280    dec Odcnt:dec Evcnt
1290    gosub *Sort(&Deteven(),Evcnt)
1300    print "E";Cb,Evcnt,
1305    for Zzz=1 to Evcnt:print Deteven(Zzz);:next Zzz:print
1310    gosub *Sort(&Detodd(),Odcnt)
1320    print "O";Cb,Odcnt,
1325    for Zzz=1 to Odcnt:print Detodd(Zzz);:next Zzz:print
1330    Si=0
1340    if Si>Odcnt or Si>Evcnt then Charpoly(Cb)=0:goto 1410
1350    if Deteven(Evcnt-Si)=Detodd(Odcnt-Si) then inc Si:goto 1340
1360    A=Deteven(Evcnt-Si):B=Detodd(Odcnt-Si)
1370    Mx=max(A,B)
1380    if Mx=A then Mxeo=1 else Mxeo=-1:endif
1390    Mxprd=(-1)^Cb*Mxeo*Mx
1400    Charpoly(Cb)=Mxprd
1410    return
1420    ' ===== sort algorithm    ( old slow sort )
1430    *Sort(&A(),N)

```

```

1440 local I,J
1450 for I=1 to N-1:for J=I+1 to N
1460   if A(I)>A(J) then swap A(I),A(J):endif
1470 next J:next I
1480 return
1490 ' ===== maximum eigenvalue of A,  $\nu$(A)
1500 *Nu(&C(),N)
1510 Plc=1:Svplc=0:Mxnu=0
1520 for I=1 to N
1530   if C(I)<=0 then
1540     :BNu(Plc)=((-1)*C(I))^(1/(I))
1550     :if BNu(Plc)>Mxnu then Svplc=I:Mxnu=BNu(Plc):endif:inc Plc
1560   endif
1570 next I
1580 return
=====

```

References

- [1] M. Akian, R. Bapat, S. Gaubert, *Asymptotics of the Perron eigenvalue and eigenvector using Max-algebra*, INRIA Report No. 3450, 1998.
- [2] F.L. Baccelli, G. Cohen, G.J. Olsder, J. - P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, John Wiley & Sons, 1992.
- [3] R.B. Bapat, D. P. Stanford, P. van den Driessche, *The Eigenproblem in Max Algebra*, Report DMS-631-IR, June 1993.
- [4] R.B. Bapat, T.E.S. Raghavan, *Nonnegative Matrices and Applications*, Cambridge University Press, 1997.
- [5] R.B. Bapat, *A max version of the Perron-Frobenius theorem*, Linear Algebra and Its Applications, 275:3 - 18, 1998.
- [6] A. Berman, M. Neumann, R.J. Stern, *Nonnegative Matrices in Dynamic Systems*, John Wiley and Sons, 1989.
- [7] A. Berman, R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
- [8] R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, 1991.
- [9] B. Carré, *Graphs and networks*, Clarendon Press – Oxford, 1979.
- [10] W. Chen, X. Qi, S. Deng, *The eigen-problem and period analysis of the discrete-event system*, Systems Science and Mathematical Sciences, 3:243-260, 1990.
- [11] G. Cohen, P. Moller, J. - P. Quadrat, M. Voit, *A linear-system-theoretic view of discrete event processes and its use for performance evaluation in manufacturing*, IEEE Trans. Automat. Control, 30:210-220, 1985.
- [12] G. Cohen, P. Moller, J. - P. Quadrat, M. Voit, *Algebraic tools for the performance evaluation of discrete event systems*, Proceedings of IEEE, 77:39-58, 1989.
- [13] R. Cuninghame - Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, no. 166, Springer Verlag, 1979.
- [14] W. E. Deskins, *Abstract Algebra*, Dover Publishing, Inc., 1992.
- [15] A.S. Dief, *Advanced Matrix Theory for Scientists and Engineers*, Abacus Press, 1982.
- [16] L. Elsner and P. van den Driessche, *On the Power Method in Max Algebra*, Linear Algebra and Its Applications, 302/303:17-32, 1999.

- [17] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Co., 1977.
- [18] S. Gaubert, *Théorie des Systèmes Linéaire dans les Dioides*, PhD Thèse, École Nationale Supérieure des Mine de Paris, Paris, 1992.
- [19] S. Gaubert, Max Plus, *Methods and Applications of $(\max, +)$ Linear Algebra*, Tech. Report no. 3088, INRIA, 1997.
- [20] J. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Longman Scientific and Technical, 1992.
- [21] J. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, 1999.
- [22] M. Gondran, M. Minoux, *Graphs and Algorithms*, John Wiley & Sons Ltd., 1984.
- [23] J. Gunawardena, *An Introduction to Idempotency*, in Idempotency, Publications of the Newton Institute, 1998.
- [24] Max Plus, (Akiam, Cohen, Gaubert, Nicoukhah, and Quadrat), “Linear systems in the $(\max, +)$ algebra”, in *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu, Hawaii, pp.151-156, December 1990.
- [25] N. McCoy, *Rings and Ideals*, Mathematical Association of America, 1948.
- [26] G.J. Olsder, C. Ross, *Cramer and Cayley-Hamilton in the max-algebra*, Linear Algebra and its Applications, 101:87 - 108, 1988.
- [27] C. Reutenauer, H. Straubing, *Inversion of Matrices over a Commutative Semiring* Journal of Algebra, 88:350-360, 1984.
- [28] B. Schneider, *The Influence of the Marked Reduced Graph of a Nonnegative Matrix on the Jordan Form and on Related Properties: A Survey*, Linear Algebra and Its Applications, 84:161 - 189, 1986.
- [29] B. De Schutter and B. De Moor, *A note on the characteristic equation in the max - plus algebra*, Linear Algebra and Its Applications, 261:237 - 250, 1997.
- [30] H. Straubing, *A Combinatorial Proof of the Cayley-Hamilton Theorem*, Discrete Mathematics, 43:273 - 279, 1983.