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Predictor-corrector methods in shakedown analysis of hardened structures

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ABSTRACT-A numerical approach is proposed to approximate the shakedown load multiplier for structures with nonlinear kinematic hardening material. The method starts from the upper bound statical shakedown theorem for nonassociated hardening materials. The initial optimization problem is suitably transformed to a set of nonlinear algebraic equations allowing for effective treatment by standard predictor-corrector iterative methods.

1. INTRODUCTION

Recent papers [1] and [2] clearly proves that the statical shakedown criterion which in associative plasticity is both sufficient and necessary has to be splitted into two separate criteria for complex material behaviour exhibiting nonlinear hardening or softening. The sufficient criterion leads to the lower bound on shakedown multiplier whereas the necessary criterion leads to the upper bound. It was shown in [2] that in the case of nonlinear hardening material of Chaboche type the gap between bounds is cased by a nonassociativity in the space of static and kinematic hardening variables. The gap can be quite loose in some circumstances where plastic potential and yield function differ significantly for static hardening variable and completely vanishes in the case of associativity (overlay model of material [3]). It seems that in the case of nonlinear nonassociative hardening the shakedown multiplier may depend, to some extend, on the initial conditions introduced to the structure in the form of initial residual stresses. The above phenomenon is not observed in the associated plasticity for elastic-perfectly plastic, linear hardening and overlay material models.

In the present paper a numerical procedure is proposed for effective treatment of structures with materials exhibiting nonlinear kinematic hardening as presented in [4]. The shakedown problem is described by a nonlinear set of relations between the loading multiplier and the plastic multipliers (appearing in the associated plastic flow rule). The size of the problem is varying since the incremental set of equations is formulated only within zones where plastic flow may occur on a given incremental step. The solution of incremental problem is repeated until the final shakedown load multiplier is reached. The linearized set of equations is characterized by the tangent matrix which becomes ill-conditioned at the end of the process. In order to avoid problems with singularity different predictor-corrector iterative schemes can be adopted. The most preferable are plastic strain control and the orthogonal plane scheme.

2. STRUCTURAL DISCRETIZATION

Consider a discrete elastic-plastic structure. Denote by $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ global vectors assembling all components of local stresses $\boldsymbol{\sigma}_G$, and local strains, $\boldsymbol{\epsilon}_G$, respectively, defined for Gauss points $\mathbf{x}_G$ of the finite element. Describe by $\mathbf{P}$ and $\mathbf{u}$ the global vectors of external loads and generalized displacements, respectively, defined in all nodal points of the structure. These quantities have to be chosen in such a way that the internal and external work increments are equal to each other

$$\boldsymbol{\sigma}\delta\boldsymbol{\epsilon} = \mathbf{P}\delta\mathbf{u} \tag{1}$$

Within the small displacement theory, the global vector of generalized elastic stresses $\boldsymbol{\sigma}^E$ is a linear function of external loads $\mathbf{P}$ applied to the nodal points of the discretized structure. The vectors of stresses, $\boldsymbol{\sigma}$, and elastic stresses, $\boldsymbol{\sigma}^E$, are assumed to be in equilibrium with the same external loads $\mathbf{P}$. Therefore, the residual stress vector being in the equilibrium with vanishing loading is defined as

$$\boldsymbol{\sigma}^{res} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^E \tag{2}$$

The vector of total strains is assumed to be decomposed into elastic and plastic parts $\boldsymbol{\epsilon}^e$ and $\boldsymbol{\epsilon}^p$, so that

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p \tag{3}$$

The stress-elastic strain relation is governed by the linear Hooke's law with $\mathbf{E}$ being the symmetric elasticity matrix, whereas the residual stresses are linearly related to the vector of plastic strains

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}^e, \qquad \boldsymbol{\sigma}^{res} = \mathbf{Z}\boldsymbol{\epsilon}^p \tag{4}$$

Here, $\mathbf{Z}$ is called the influence matrix (see, e.g. [6], [9]) and is assumed to be known.

3. NONLINEAR KINEMATIC HARDENING MODEL OF MATERIAL

Let us consider the material of the structure to be elastic-plastic with nonlinear hardening. The Frederick-Armstrong model [5] next extended and modified by Chaboche et. al. [4] is adopted in the paper. The material curve for this model, i.e. the relation between stresses and plastic strains, is exponentially approximated. Moreover, material ratcheting is manifested by model for closed stress paths.

For a specified Gauss point of the structure the material behaviour is described by the following equations:

1. yield surface: $\boldsymbol{\mathcal{F}}_G(\boldsymbol{\sigma}_G - \boldsymbol{\chi}_G) = 0$
2. associated flow rule: $\dot{\boldsymbol{\epsilon}}_G^p = \dot{\lambda}_G \dfrac{\partial \mathcal{F}_G^T}{\partial \boldsymbol{\sigma}_G} = \dfrac{1}{H_G}\left(\dfrac{\partial \mathcal{F}_G^T}{\partial \boldsymbol{\sigma}_G}\dot{\boldsymbol{\sigma}}_G\right)\dfrac{\partial \mathcal{F}_G^T}{\partial \boldsymbol{\sigma}_G}$
3. backstress evolution rule: $\dot{\boldsymbol{\chi}}_G = \frac{2}{3}C_G\dot{\boldsymbol{\epsilon}}_G^p - \dot{\lambda}_G\gamma_G\boldsymbol{\chi}_G$

4. plastic hardening modulus resulting from the consistency condition
$H_G = \frac{2}{3}C_G\left(\frac{\partial\mathcal{F}_G^T}{\partial\boldsymbol{\sigma}_G}\frac{\partial\mathcal{F}_G^T}{\partial\boldsymbol{\sigma}_G}\right) - \gamma_G\left(\frac{\partial\mathcal{F}_G^T}{\partial\boldsymbol{\sigma}_G}\boldsymbol{\chi}_G\right)$

In the above equations $\boldsymbol{\chi}_G$ is a local back stress vector, $\boldsymbol{\epsilon}_G^p$ is a local plastic strain deviator, $\dot{\gamma}_G$ and H_G are plastic multiplier and plastic hardening modulus, whereas C_G and γ_G describe material constants. All mentioned quantities are assigned to the specified Gauss point denoted by the subscript $_G$.

For clarity of theoretical divagations presented in the next section all equations listed above will be formulated on the basis of the thermodynamic framework ([7]) for the whole structure.

Assuming vector of free energy potentials in the form

$$\Psi(\boldsymbol{\epsilon}^e, \boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\epsilon}^{e^T}\mathbf{E}\boldsymbol{\epsilon}^e + \frac{1}{3}\boldsymbol{\eta}^T\mathbf{C}\boldsymbol{\eta} \tag{5}$$

the global stress and backstress vectors can be presented as

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\epsilon}^e} = \mathbf{E}\boldsymbol{\epsilon}^e \quad \boldsymbol{\chi} = \frac{\partial\Psi}{\partial\boldsymbol{\eta}} = \frac{2}{3}\mathbf{C}\boldsymbol{\eta} \tag{6}$$

The yield condition is defined as the vector assembling yield functions $\mathcal{F}$ for all Gauss points

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\sigma} - \boldsymbol{\chi}) \leq \mathbf{0} \tag{7}$$

The plastic potential is adopted in the following form

$$\boldsymbol{\phi}(\sigma, \boldsymbol{\chi}) = \boldsymbol{\mathcal{F}}(\boldsymbol{\sigma} - \boldsymbol{\chi}) + \boldsymbol{\phi}^D(\boldsymbol{\chi}) \tag{8}$$

with $\phi_G^D = \frac{3\gamma_G}{4C_G}\boldsymbol{\chi}_G^T\boldsymbol{\chi}_G$ being the element of the vector $\boldsymbol{\phi}^D$.

The evolution rules for the global plastic strain vector and internal thermodynamical forces can be described by means of plastic potential vector, respectively

$$\dot{\boldsymbol{\epsilon}}^p = \frac{\partial\boldsymbol{\phi}^T}{\partial\sigma}\dot{\boldsymbol{\lambda}} = \frac{\partial\boldsymbol{\mathcal{F}}^T}{\partial\sigma}\dot{\boldsymbol{\lambda}} \qquad \dot{\boldsymbol{\eta}} = \frac{\partial\boldsymbol{\phi}^T}{\partial\boldsymbol{\chi}}\dot{\boldsymbol{\lambda}} = \frac{\partial\boldsymbol{\mathcal{F}}^T}{\partial\boldsymbol{\sigma}}\dot{\boldsymbol{\lambda}} - \frac{\partial\boldsymbol{\phi}^{D^T}}{\partial\boldsymbol{\chi}}\dot{\boldsymbol{\lambda}} \tag{9}$$

Then, the global back stress vector has to satisfy the following differential equation

$$\dot{\boldsymbol{\chi}} = \frac{2}{3}\mathbf{C}\dot{\boldsymbol{\eta}} = \frac{2}{3}\mathbf{C}\dot{\boldsymbol{\epsilon}}^p - \frac{2}{3}\mathbf{C}\frac{\partial\boldsymbol{\phi}^{D^T}}{\partial\boldsymbol{\chi}}\dot{\boldsymbol{\lambda}} \tag{10}$$

From the consistency condition

$$\delta\boldsymbol{\mathcal{F}} = \frac{\partial\boldsymbol{\mathcal{F}}}{\partial\sigma}\delta\boldsymbol{\sigma} - \mathbf{H}\delta\boldsymbol{\lambda} \tag{11}$$

the global matrix of plastic hardening moduli can be found

$$\boldsymbol{H} = \frac{2}{3}\left(\frac{\partial\boldsymbol{\mathcal{F}}}{\partial\sigma}\mathbf{C}\frac{\partial\boldsymbol{\mathcal{F}}^T}{\partial\sigma} - \frac{\partial\boldsymbol{\mathcal{F}}}{\partial\sigma}\mathbf{C}\frac{\partial\boldsymbol{\phi}^{D^T}}{\partial\boldsymbol{\chi}}\right) \tag{12}$$

For practical use, the matrices presented above should be constructed by the assemblation of all quantities in Gauss points.

4. SHAKEDOWN CONCEPT FOR NONLINEAR HARDENING MATERIALS

4.1 *Bounds on the shakedown multiplier*

Materials exhibiting nonlinear kinematic hardening constitute a relatively new part in the shakedown theory. They can be treated in a similar way as typical nonassociated elastic-perfectly plastic materials, although the plastic strain rate is governed by the associated plastic flow rule. The reason for such a treatment is that the nonassociativity occurs in the augmented space of stresses $\boldsymbol{\sigma}$ and backstresses $\boldsymbol{\chi}$ for the kinematic variables $\dot{\boldsymbol{\epsilon}}^p$ and $\dot{\boldsymbol{\eta}}$, respectively. A direct consequence of such constitutive modeling is a possibility of ratcheting description on the material point level.

The static approach leading to a shakedown criterion which in associated plasticity is both sufficient and necessary, splits now into two separate criteria: the lower and the upper bounds on shakedown load multiplier μ ([2]). The optimization problems resulting from these criteria are presented below:

1. The lower bound on the shakedown load multiplier

$$\mu^L = \max_{\mu, \boldsymbol{\sigma}^{res}, \boldsymbol{\chi}} \mu \tag{13}$$

subject to

$$\boldsymbol{\mathcal{F}}\left(\mu\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{res} - \boldsymbol{\chi}\right) + \boldsymbol{\phi}^D(\boldsymbol{\chi}) \leq \mathbf{0} \tag{14}$$

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\chi}) - \mathbf{g} \leq \mathbf{0}, \qquad \boldsymbol{\sigma}^{res} = \mathbf{Z}\boldsymbol{\epsilon}^p, \qquad \boldsymbol{\sigma}^E \in \partial\Omega \tag{15}$$

where the vector g depends on the material constants $g_G = C_G/\gamma_G$.

2. The upper bound on the shakedown load multiplier

$$\mu^U = \max_{\mu, \boldsymbol{\sigma}^{res}, \boldsymbol{\chi}} \mu \tag{16}$$

subject to

$$\boldsymbol{\mathcal{F}}\left(\mu\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{res} - \boldsymbol{\chi}\right) \leq \mathbf{0} \tag{17}$$

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\chi}) - \mathbf{g} \leq \mathbf{0}, \qquad \boldsymbol{\sigma}^{res} = \mathbf{Z}\boldsymbol{\epsilon}^p, \qquad \boldsymbol{\sigma}^E \in \partial\Omega \tag{18}$$

In both cases the lower and the upper bound load multipliers are obtained by optimizing the respective problems with respect to the same variables, i.e. the load multiplier μ, the residual stress vector $\boldsymbol{\sigma}^{res}$ and the vector of backstresses $\boldsymbol{\chi}$. The only differences existing between these two approaches can be clearly seen for inequalities (14) and (17). The former inequality represents reduced augmented yield domain (RAYD) whereas the latter one is referred to as augmented yield domain (AYD) (see [2]). The other inequalities $(15)_1$ and $(18)_1$ impose some limitations on backstresses $\boldsymbol{\chi}$ what means that there exists a saturation state resulting from evolution rules.

4.2 *Numerical formulation of shakedown problem*

Following [8] and [9] a new effective incremental approach can provided to solve the optimization problems (13)–(15) and (16)–(18) . For the simplicity of considerations and due to the clarity of relevance with the mentioned papers our focus will be limited only to the solution of the second problem, i.e. the upper bound on the shakedown load multiplier. The first problem concerning the lower bound approach to shakedown multiplier can be proceeded in the analogous way.

In the sequel the concept of incremental shakedown analysis presented in [8] and [9] will be extended to the more complex material model with nonlinear kinematic hardening. For this reason consider a discretized structure with volume V subjected to variable repeated loading represented by vertices $\mathbf{P}^k$, $k = 1, \ldots, n_k$. For the given residual stresses $\boldsymbol{\sigma}^{res}$ and backstresses $\boldsymbol{\chi}$ the structure can be divided into separate parts V_l, $l = 1, \ldots, n_k$, $V \equiv \cup V_l$, in the following way.

First, at each point $\mathbf{x}_G$ the dominant load vertex $\mathbf{P}^l$ is sought

$$\mathbf{P}^l: \quad \begin{array}{l} \mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G) \geq \mathcal{F}(\boldsymbol{\sigma}^k_G - \boldsymbol{\chi}_G) \\ k = 1, \ldots, n_k, \quad \mathbf{x}_G \in V \end{array} \tag{19}$$

with $\boldsymbol{\sigma}^k_G$ and $\boldsymbol{\sigma}^l_G$ defined as the sum of vectors of elastic and residual stresses

$$\boldsymbol{\sigma}^k_G(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p) = \boldsymbol{\sigma}^E_G(\mathbf{x}_G, \mu\mathbf{P}^k) + \boldsymbol{\sigma}^{res}_G(\mathbf{x}_G, \boldsymbol{\epsilon}^p) \tag{20}$$

The above decomposition is arbitrary chosen but uniquely relates the elastic stresses $\boldsymbol{\sigma}^E$ to external loading $\mathbf{P}^k$ and residual stresses $\boldsymbol{\sigma}^{res}$ to plastic strains $\boldsymbol{\epsilon}^p$. Then the volume V_l can be simply identified by all points $\mathbf{x}_G$ for which the vertex P^l is a dominant load vertex, i.e.

$$V_l: \quad \mathbf{x}_G \in V_l \quad \Leftrightarrow \mathbf{P}^l \text{ is the dominant load vertex in } \mathbf{x}_G \tag{21}$$

Having defined the volumes V_l and the associated vertices $\mathbf{P}^l$ the upper bound shakedown criterion can be specified for the choosen vertex l

$$\mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G) \leq 0, \quad \mathbf{x}_G \in V_l, \quad l = 1, \ldots, n_k \tag{22}$$

For numerical purposes it is convenient to define plastic and elastic parts V_p and V_e ($V_p \cup V_e = V$), respectively. It can be simply done by noting that in the plastic part the stresses should be situated on the yield surface

$$V_p: \quad \mathbf{x_G} \in V_p \Leftrightarrow \mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G) = 0 \tag{23}$$

Then the shakedown criterion (22) can be splitted into equality and strict inequality

$$\begin{array}{ll} \mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G) = 0, & \mathbf{x}_G \in V_l \cap V_p, \\ \mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G) < 0, & \mathbf{x}_G \in V_l \cap V_e, \end{array} \quad l = 1, \ldots, n_k \tag{24}$$

Expressing the yield function by variables μ and $\boldsymbol{\epsilon}^p$ instead of $\boldsymbol{\sigma}$ the equivalent criterion reads

$$\begin{array}{ll} \mathcal{G}^l(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p, \boldsymbol{\chi}_G) = 0, & \mathbf{x}_G \in V_l \cap V_p, \\ \mathcal{G}^l(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p, \boldsymbol{\chi}_G) < 0, & \mathbf{x}_G \in V_l \cap V_e, \end{array} \quad l = 1, \ldots, n_k \tag{25}$$

where $\mathcal{G}^l(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p, \boldsymbol{\chi}_G) \equiv \mathcal{F}(\boldsymbol{\sigma}^l_G - \boldsymbol{\chi}_G)$

Assembling all the yield functions $\mathcal{G}^l(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p, \boldsymbol{\chi}_G)$, $\mathbf{x}_G \in V_p$ into the vector $\boldsymbol{\mathcal{G}}_p$ and $\mathcal{G}^l(\mathbf{x}_G, \mu, \boldsymbol{\epsilon}^p, \boldsymbol{\chi}_G)$, $\mathbf{x}_G \in V_e$ into $\boldsymbol{\mathcal{G}}_e$ the inequalities (25) can be rewritten to the more comprehensive form

$$\begin{array}{ll} \boldsymbol{\mathcal{G}}_p(\mu, \boldsymbol{\epsilon}^p_p, \boldsymbol{\chi}_p) = \mathbf{0}, & \text{in } \mathrm{V_p} \\ \boldsymbol{\mathcal{G}}_e(\mu, \boldsymbol{\epsilon}^p_p, \boldsymbol{\chi}_e) < \mathbf{0}, & \text{in } \mathrm{V_e} \end{array} \tag{26}$$

4.3 *Prdeictor-corrector methods in shakedown analysis of hardened structures*

As long as the part of the volume remains elastic V_e the plastic strains and backstresses are not varying during the loading process. They can be assumed to be constant and equal to $\hat{\boldsymbol{\epsilon}}_e^p$ and $\hat{\boldsymbol{\chi}}_e$, respectively. Thus the set of inequalities (26) can be replaced by the equivalent set of equations

$$\begin{aligned} \boldsymbol{\mathcal{G}}_p\left(\mu, \boldsymbol{\epsilon}_p^p, \boldsymbol{\chi}_p\right) &= \mathbf{0}, \qquad && \text{in } V_p \\ \boldsymbol{\epsilon}_e^p - \hat{\boldsymbol{\epsilon}}_e^p &= \mathbf{0}, && \text{in } V_e \\ \boldsymbol{\chi}_e - \hat{\boldsymbol{\chi}}_e &= \mathbf{0}, && \text{in } V_e \end{aligned} \tag{27}$$

Denote by $\mathbf{r}$ a vector consisting of the load multiplier, plastic strains and backstresses in the plastic part of the volume V_p

$$\mathbf{r}^T = \left(\mu, \boldsymbol{\epsilon}_p^p, \boldsymbol{\chi}_p\right) \tag{28}$$

Assume that a set of the solution vectors $\mathbf{r}_i$ (for $i = 1, \ldots, \omega$) to the problem (27) has been determined. The next solution point $\mathbf{r}_{\omega+1}$, for which the load multiplier $\mu_{\omega+1}$ is greater or at least equal to μ_ω, can be found using incremental-iterative procedure

$$\mathbf{r}_{\omega+1} = \mathbf{r}_\omega + \Delta\mathbf{r}_\omega \tag{29}$$

The increments $\Delta\mathbf{r}_\omega$ are determined using a series of approximating points $\mathbf{r}^{\nu+1}$ (ν denotes the number of approximation) as follows

$$\mathbf{r}^{\nu+1} = \mathbf{r}^\nu + \delta\mathbf{r}^\nu \tag{30}$$

The first approximation $\mathbf{r}^1$ is established by an initial sub-increment $\delta\mathbf{r}^0$, which is calculated at $\mathbf{r}^0 \equiv \boldsymbol{r}_\omega$ in a predictor step. Then, the solution is improved using the subsequent sub-increments $\delta\mathbf{r}^\nu$, which are calculated at $\mathbf{r}^\nu$ (for $\nu > 0$) during a corrector step. The correction proceeds until the specified numerical accuracy ε is achieved

$$\max_{\mathbf{x}_G} \mathcal{G}_p\left(\mathbf{x}_G, \mathbf{r}_G^{\nu+1}\right) \leq \varepsilon \qquad \mathbf{x}_G \in V_p^{\nu+1} \tag{31}$$

It can be noted that the plastic volumes V_p^ν may vary during the iteration process for any ν starting from $\nu := 1$ to $\nu := \nu + 1$, and the yield conditions may be violated at different points $\mathbf{x}_G \in V_p^\nu$

$$\mathcal{G}(\mathbf{x}_G, \mathbf{r}_G^\nu) > 0 \tag{32}$$

The crucial problem in continuation algorithms is, how to determine the sub-increments $\delta\mathbf{r}^\nu$. The incremental system can be conveniently derived taking into account only equations $(26)_1$, which are formulated in the plastic part of the structure. In the vicinity of the last approximation $\mathbf{r}^\nu$ a truncated Taylor series can be written as follows

$$\boldsymbol{\mathcal{G}}_p\left(\mathbf{r}^{\nu+1}\right) = \boldsymbol{\mathcal{G}}_p\left(\mathbf{r}^\nu\right) + \left.\frac{\partial \boldsymbol{\mathcal{G}}_p}{\partial \mathbf{r}}\right|_{\mathbf{r}^\nu} \delta\mathbf{r}^\nu + \boldsymbol{\Theta} \tag{33}$$

Herein, $\boldsymbol{\Theta}$ denotes the sum of higher order terms. The linearized set of equations is obtained assuming $\boldsymbol{\Theta} \cong \mathbf{0}$ and $\boldsymbol{G}_p(\mathbf{r}^{\nu+1}) \cong \mathbf{0}$.

It is assumed that the class of plastic strain increments and backstress increments adopted to the analysis are continuous with respect to the increment of the load multipliers. Therefore, the plastic flow rule (9) and evolution rule for kinematic hardening variable (10) can be utilized in the incremental form

$$\delta \boldsymbol{\epsilon}^p = \frac{\partial \boldsymbol{\mathcal{F}}_p^T}{\partial \boldsymbol{\sigma}^l}\,\delta \boldsymbol{\lambda} \qquad \delta \boldsymbol{\chi} = \frac{2}{3}\mathbf{C}\delta \boldsymbol{\epsilon}^p - \frac{2}{3}\mathbf{C}\frac{\partial \boldsymbol{\phi}^{D^T}}{\partial \boldsymbol{\chi}}\delta \boldsymbol{\lambda} \tag{34}$$

Finally the set of equations (33) can be rewritten in view of $\delta \boldsymbol{\lambda}^\nu$ as follows

$$\frac{\partial \boldsymbol{\mathcal{G}}_p}{\partial \mu^\nu}\delta \mu^\nu + (\mathbf{D}^\nu - \mathbf{H}^\nu)\,\delta \boldsymbol{\lambda}^\nu = -\boldsymbol{\mathcal{G}}_p^\nu \tag{35}$$

where matrices $\mathbf{D}^\nu$ and $\mathbf{H}^\nu$ denote

$$\mathbf{D}^\nu = \frac{\partial \boldsymbol{\mathcal{F}}_p}{\partial \boldsymbol{\sigma}}\frac{\partial \boldsymbol{\sigma}_p}{\partial \boldsymbol{\epsilon}^p}\frac{\partial \boldsymbol{\mathcal{F}}_p^T}{\partial \boldsymbol{\sigma}}\delta \boldsymbol{\lambda} \qquad \mathbf{H}^\nu = \frac{2}{3}\left(\frac{\partial \boldsymbol{\mathcal{F}}_p}{\partial \boldsymbol{\sigma}}\mathbf{C}\frac{\partial \boldsymbol{\mathcal{F}}_p^T}{\partial \boldsymbol{\sigma}} - \frac{\partial \boldsymbol{\mathcal{F}}_p}{\partial \boldsymbol{\sigma}}\mathbf{C}\frac{\partial \boldsymbol{\phi}_p^{D^T}}{\partial \boldsymbol{\chi}}\right) \tag{36}$$

The nonlinear system of equations (33) can be solved using predictor-corrector methods. In the predictor step ($\nu = 0$) $\boldsymbol{\mathcal{G}}_p^0$ is equal to zero and the increment $\delta\mu^0 = \delta\hat{\mu}$ is assumed to be known. In the corrector step the equation (35) has to be augmented by the so called constrain equation

$$h_\mu \delta\mu^\nu + \boldsymbol{h}_\lambda \delta \boldsymbol{\lambda}^\nu = \gamma \tag{37}$$

The quantities h_μ, $\boldsymbol{h}_\lambda$ and γ denote assumed coefficients, which characterize a specific path-following technique. It seems that the most efficient versions of iterative procedures, that can be employed for the shakedown analysis, are strain control scheme and orthogonal plane scheme as it was shown in the paper [9]. In general, the predictor-corrector methods for hardening structures can be presented in the following form

$$\begin{array}{ll} \text{1. Predictor step, } \nu = 0 & \text{2. Corrector steps, } \nu = 1, 2, \ldots \\ (\mathbf{D}^0 - \mathbf{H}^0)\,\delta \boldsymbol{\lambda}^0 = -\dfrac{\partial \boldsymbol{\mathcal{G}}_p}{\partial \mu}\delta\mu^0 & \dfrac{\partial \boldsymbol{\mathcal{G}}_p}{\partial \mu^\nu}\delta \mu^\nu + (\mathbf{D}^\nu - \mathbf{H}^\nu)\,\delta \boldsymbol{\lambda}^\nu = -\boldsymbol{\mathcal{G}}_p^\nu \\ \delta\mu^0 = \delta\hat{\mu} & h_\mu \delta\mu^\nu + \boldsymbol{h}_\lambda \delta \boldsymbol{\lambda}^\nu = \gamma \end{array} \tag{38}$$

Let us notice that the matrix $\mathbf{H}^\nu$ is a diagonal matrix of plastic hardening moduli (12) obtained in Section 3. For a particular case of elastic-perfectly plastic material this matrix vanishes and the system is simply reduced to the one obtained in the paper [9]. Hence, it is believed, that the high effectiveness of the shakedown method for elastic-perfectly plastic material model will be projected to the shakedown with kinematic hardening material model.

5. CONCLUSIONS

The paper presents the numerical procedure for the solution of the upper bound shakedown load multiplier by a static approach. The structure is made of nonlinear hardening material exhibiting ratcheting on the material level. The shakedown problem is presented as a set of nonlinear equations which can be solved using classical predictor-corrector iterative schemes.

The new formulation constitute a generalization of the approach proposed in recent papers [8] and [9] to nonlinear hardening material. The effective solution of the nonlinear problem is restricted to the area where plastic deformations can develop. In each such a Gauss point belonging to the plastic zone there is only one unknown, i.e. plastic multiplier that occurs in the associated plastic flow rule. Therefore, the size of the considered problem is limited to the number of Gauss points beeing in yielding during variable process of deformation.

The second advantage of the problem is that the solution phase doesn't depend on the number of load cycles that have to be performed to get elastic stabilization. In fact, the set of nonlinear algebraic equations combines the load multiplier with the plastic multipliers assuming that the obtained solution has to satisfy the shakedown asymptotic properties. In this manner the time required for the solution is limited only to the number of operations related to the solution of iterative scheme.

Promising results were obtained in [9] for a particular case of the elastic-perfectly plastic material. One storey frame was subjected to cyclic loading represented by 8 load vertices. In the consequence the shakedown solution was achieved after 23 to 56 iterations depending on the iterative scheme employed and the size of the incremental step. On the other hand the solution obtained for the classical step by step incremental analysis resulted in thousands of iterations. Therefore, it seems that the general approach outlined in the present paper for nonlinear hardening material will result in effectiveness similar to that presented in [9].

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