Stress and Displacement of a Laminated Thin Circular Plate Subjected to a Core Thermal Distribution

Russell C. Hibbeler
University of Louisiana at Lafayette
Lafayette, Louisiana 70504

ABSTRACT

This paper provides an analysis of the stress and deflection of a laminated thin plate, which is subjected to an axisymetric temperature distribution, caused by heat input along its inner boundary. The solution is based on a thermoelastic analysis of the plate, which is made from two isotropic materials that are firmly bonded together. The results can be applied to laminated stiffeners used on piping that carries hot fluids.

INTRODUCTION

The analysis of thermal stresses in plates or thin disks has been investigated in numerous papers. Some of the principal treatments of the subjected are listed in several sources. [1] Through the years problems related to the thermal analysis of laminated plates have also been solved. [2,3,4,5] In this investigation classical thermoelasticity is used to solve for the displacement and stress within a double laminated plate for which a high constant temperature is imposed on the inner boundary of a hole or pipe passing through the center of the plate. Here we require each lamina to be bonded with rigid insulating material. Each lamina is assumed to be thin and have unique material properties and a constant thickness, with a common inner and outer radius a and b, respectively.

ANALYSIS

A cylindrical coordinate system will be considered having its origin at the mean thickness of each lamina. The equations of equilibrium for this axisymmetric case are

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$
$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\sigma_{rz} = 0 \qquad (1)$$

For isotropic materials, the strain relations are

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha T(r,z)$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \alpha T(r,z) \qquad (2)$$
$$\varepsilon_{rz} = \frac{1}{2G}\sigma_{rz}$$

And finally, the strain displacement relationships are

$$\varepsilon_r = \frac{\partial u_r}{\partial r}; \quad \varepsilon_\theta = \frac{u_r}{r}$$
$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z}\right) \qquad (3)$$

In these equations, σ is the stress, ε is the strain, u_r is the radial displacement, w is the deflection in the z direction, E is the modulus of elasticity, G is the shear modulus, ν is Poisson's ratio, α is the coefficient of thermal expansion, and $T(r, z)$ is the temperature difference above a reference temperature.

If we define the stress resultants and couples for each plate A and B as

$$N_r = \int_{-h/2}^{h/2} \sigma_r dz, \quad N_\theta = \int_{-h/2}^{h/2} \sigma_\theta dz, \quad Q = \int_{-h/2}^{h/2} \sigma_{rz} dz$$
$$M_r = \int_{-h/2}^{h/2} z\sigma_r dz, \quad M_\theta = \int_{-h/2}^{h/2} z\sigma_\theta dz \tag{4}$$

then the equilibrium equations for each plate can be expressed in terms of these resultants using the boundary conditions

$$\sigma_z\big|_{-t_A/2} = \sigma_z\big|_{t_B/2} = \sigma_0, \quad \sigma_{rz_A}\big|_{-t_A/2} = \sigma_{rz_B}\big|_{t_B/2} = \tau_0$$
$$\sigma_{rz_A} z_A\big|_{-t_A/2} = -\frac{t_A\tau_0}{2}, \quad \sigma_{rz_B} z_B\big|_{t_B/2} = \frac{t_B\tau_0}{2} \tag{5}$$

Here σ_0 and τ_0 are the normal and shear stress acting at the bonded joint.
After integration, the plate equilibrium equations become

$$\frac{dN_{r_A}}{dr} + \frac{N_{r_A} - N_{\theta_A}}{r} - \tau_0 = 0 \tag{6}$$

$$\frac{dM_{r_A}}{dr} + \frac{M_{r_A} - M_{\theta_A}}{r} - \frac{h_A\tau_0}{2} - Q_A = 0 \tag{7}$$

$$\frac{dQ_A}{dr} + \frac{Q_A}{r} - \sigma_0 = 0 \tag{8}$$

$$\frac{dN_{r_B}}{dr} + \frac{N_{r_B} - N_{\theta_B}}{r} + \tau_0 = 0 \tag{9}$$

$$\frac{dM_{r_B}}{dr} + \frac{M_{r_B} - M_{\theta_B}}{r} + \frac{h_B\tau_0}{2} - Q_B = 0 \tag{10}$$

$$\frac{dQ_B}{dr} + \frac{Q_B}{r} + \sigma_0 = 0 \tag{11}$$

For each plate, we will assume the displacements are of the form

$$w = w(r)$$
$$u_r(r,z) = u_0(r) + z\,f(r) \tag{12}$$

Here u_0 is the in-plane displacement of the middle surface of each plate. Neglecting the shearing strain, ε_{rz}, then the last of Eqs. (3) becomes

$$\frac{\partial w}{\partial r} = -\frac{\partial u_r}{\partial z}$$

Substituting this into Eq. (12), $f(r) = -dw/dr = -w'$ the displacements become

$$w = w(r)$$
$$u_r = u_0 - zw' \tag{13}$$

The strain-displacement relations for each plate are

$$\varepsilon_r = u_0' - zw''$$
$$\varepsilon_\theta = \frac{u_0}{r} - \frac{z}{r}w' \tag{14}$$

The stress resultants and couples for each plate can be expressed in terms of w and u_0 by substituting Eq.(14) into Eqs.(2) and integrating, using Eq.(4). The results are

$$N_r = K(u_0' + \frac{\nu}{r}u_0) - \frac{1}{1-\nu}T^* \tag{15}$$
$$N_\theta = K(\nu\ u_0' + \frac{u_0}{r}) - \frac{1}{1-\nu}T^* \tag{16}$$
$$M_r = -D(w'' + \frac{\nu}{r}w') - \frac{1}{1-\nu}M^* \tag{17}$$
$$M_\theta = -D(\frac{w'}{r} + \nu\ w'') - \frac{1}{1-\nu}M^* \tag{18}$$

Here $K = Et/(1-\nu^2)$ and $D = Eh^3/12\ (1-\nu^2)$ and the thermal stress resultant and thermal stress couples are

$$T^* = \int_{-t/2}^{t/2} E\alpha T\, dz \tag{19}$$
$$M^* = \int_{-t/2}^{t/2} E\alpha T z\, dz \tag{20}$$

The problem now involves six equilibrium equations, [Eqs.(8) through (11)], and eight equations derived from the stress-strain and strain displacement relations [Eqs.(15) - (18)]. To obtain the sixteen unknowns, two more equations are needed. Since transverse normal deformations are neglected,

$$w = w_A = w_B \tag{21}$$

Also, since no slippage occurs at the joint, from Eq. (13),

$$u_{0A} = u_{0B} - \frac{t_A + t_B}{2}w' \tag{22}$$

To solve it is necessary to obtain the lateral deflection and the displacement u_0. To do this it is reasonable to assume $\nu_1=\nu_2=\nu$. Solving for Q_1 and Q_2 in Eqs. (7) and (10), and substituting them into Eqs. (8) and (11), using Eqs. (17) and (18), yields two equations, which can be added and subtracted, giving

$$(D_A + D_B)\nabla^4 w = -\frac{1}{1-\nu}\left[\nabla^2 M_A{}^* + \nabla^2 M_B{}^*\right] + \frac{t_A + t_B}{2}(\tau_0' + \frac{\tau_0}{r}) \tag{23}$$

$$(D_A - D_B)\nabla^4 w = -\frac{1}{1-\nu}\left[\nabla^2 M_A^* - \nabla^2 M_B^*\right] + \frac{t_A - t_B}{2}\left(\tau_0' + \frac{\tau_0}{r}\right) - 2\sigma_0 \tag{24}$$

Here the Laplacian operators are defined as

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right), \quad \nabla^4 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\nabla^2\right)$$

In order to eliminate the stress from the above equations, we can add and subtract Eqs.(6) and (9), and substitute Eqs. (15), (16), and (22) into the result. After simplification,

$$(K_A + K_B)\frac{d}{dr}\left(u_{0A}' + \frac{u_{0A}}{r}\right) + K_B\frac{t_A + t_B}{2}\frac{d}{dr}(\nabla^2 w) - \frac{1}{1-\nu}(T_A^{*\prime} + T_B^{*\prime}) = 0 \tag{25}$$

$$\tau_0 = \frac{-K_A K_B (t_A + t_B)}{2(K_A + K_B)}\frac{d}{dr}(\nabla^2 w) - \frac{K_B T_A^{*\prime} - K_A T_B^{*\prime}}{(1-\nu)(K_A + K_B)} \tag{26}$$

Finally, substituting Eq. (26) and its derivative into Eq. (23), gives

$$D_e \nabla^4 w = -\frac{1}{(1-\nu)}\left(C_0\nabla^2 T_A^* + C_1\nabla^2 T_B^* + \nabla^2 M_A^* + \nabla^2 M_B^*\right) \tag{27}$$

Here,

$$\begin{aligned} C_0 &= \frac{(t_A + t_B)K_B}{2(K_A + K_B)}, \quad C_1 = \frac{(t_A + t_B)K_A}{2(K_A + K_B)} \\ D_e &= D_A + D_B + \frac{K_A K_B (t_A + t_B)^2}{4(K_A + K_B)} \end{aligned} \tag{28}$$

By comparison with a single plate, here the rigidity D_e is increased by the last term due to the bonding of the two plates. Integrating this equation twice, yields

$$\nabla^2 w = -\frac{1}{D_e(1-\nu)}\left(C_0 T_A^* + C_1 T_B^* + M_A^* + M_B^*\right) + C_4 \ln r + C_5 \tag{29}$$

The normal force between the plates can be determined by substituting Eq.(26) and its derivative into Eq.(24). We get

$$\begin{aligned} \sigma_0 = {} & \frac{1}{2D_e(1-\nu)}\left(D_A - D_B + \frac{K_A K_B (t_A^2 - t_B^2)}{4(K_A + K_B)}\right)\left(C_0\nabla^2 T_A^* + C_1\nabla^2 T_B^* + \nabla^2 M_A^* + \nabla^2 M_B^*\right) \\ & - \frac{1}{2(1-\nu)}\left(C_2\nabla^2 T_A^* - C_3\nabla^2 T_B^* + \nabla^2 M_A^* + \nabla^2 M_B^*\right) \end{aligned} \tag{30}$$

where

$$C_2 = \frac{(t_A - t_B)K_B}{2(K_A + K_B)} \quad \text{and} \quad C_3 = \frac{(t_A - t_B)K_A}{2(K_A + K_B)}$$

Equation (27) may be solved for the lateral displacement of the plate provided the temperature distribution within the plate is specified. As assumed in classical thermoleasticity, the temperature distribution is unaffected by the stress or displacement fields of the plate.

As a special case we will consider a pipe of outer radius a which passes through the center hole of the plate. A hot fluid flows through the pipe, which causes its outer surface to be subjected to a constant temperature T_a. In order to obtain the temperature distribution through each plate will assume the bonding material between the plates provides enough insulation so that no heat transfer between the plates takes place. Furthermore, because the plates are thin, the temperature in the z direction is assumed to be constant. Heat is conducted in the radial direction and heat flux is lost from the top surface of the top plate and the bottom surface of the bottom plate.

If a thermal energy balance is made for this case, it results in Bessel's zero-order differential equation, namely

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}-N\,T=0, \quad \text{where } N=\sqrt{h/kT}$$

Here h is the unit surface conductance between the plate and air and k is the thermal conductivity. The general solution to this equation is of the form,

$$T=AJ_0(iNr)+BY_0(iNr)$$

If we apply the boundary conditions of $T = T_a$ at $r = a$ and $dT/dr = 0$ at $r = b$, then the temperature distribution within each plate is represented by

$$T=\frac{I_0(Nr)K_1(Nb)+I_1(Nb)K_0(Nr)}{I_0(Na)K_1(Nb)+I_1(Nb)K_0(Na)} \tag{31}$$

where I_0, K_0, and I_1, K_1, are the modified Bessel functions of the first kind, of zero and first order, respectively. Substituting this equation into Eq. (29) and integrating twice, yields

$$w'=-\left(\frac{T_A(C_0E_A\alpha_At_A+C_1E_B\alpha_Bt_B)}{D_e(1-\nu)N}\right)\left(\frac{I_1(Nr)K_2(Nb)-I_1(Nb)K_1(Nr)}{I_0(Na)K_1(Nb)-I_1(Nb)K_0(Na)}\right) + C_4\frac{r}{2}(\ln r-\frac{1}{2})+\frac{C_6}{r} \tag{32}$$

$$w=-\left(\frac{T_A(C_0E_A\alpha_At_A+C_1E_B\alpha_Bt_B)}{D_e(1-\nu)N^2}\right)\left(\frac{I_0(Nr)K_1(Nb)-I_1(Nb)K_0(Nr)}{I_0(Na)K_1(Nb)-I_1(Nb)K_0(Na)}\right) + \frac{C_4}{4}r^4(\ln r-1)+\frac{C_5}{4}r^2+C_6\ln r+C_7 \tag{33}$$

Finally, the displacement u_{0A} can be determined from Eqs.(25) by first integrating, then solving the differential equation. Using Eq. (32), the result is

$$u_{0A}=-C_0w'+\frac{T_a(E_A\alpha_At_A+E_B\alpha_Bt_B)r}{N(1-\nu)(K_A+K_B)}\left(\frac{I_1(Nr)K_1(Nb)-I_1(Nb)K_1(Nr)}{I_0(Na)K_1(Nb)-I_1(Nb)K_0(Na)}\right)+\frac{C_8}{2}r+\frac{C_9}{r} \tag{34}$$

Since both w and u_{0A} are known, the constants of integration can be evaluated using the boundary conditions for the plate, which depends, upon how it is supported at its inside and outside boundaries. The stress resultant and stress couples can then be determined by back substitutions. In particular, the radial and circumferential stress distributions are determined from

$$\sigma_r = \frac{N_r}{t} + \frac{M_r z}{t^2/12} \tag{35}$$

$$\sigma_\theta = \frac{N_\theta}{t} + \frac{M_\theta z}{t^2/12} \tag{36}$$

In summary, this analysis can be used to solve for the displacements and stresses in a laminated thin plated when the plate is subjected to radial thermal loading caused by high temperature applied to its inner radius. The stress and deflection caused by various lateral loadings can be superimposed provided the material remains elastic and the deflection remains small.

Applications for this analysis include modeling laminated plates used as stiffeners on cylinders or laminated plates used as cooling fins on pipes. Furthermore, if a single plate is thick, and the thermal gradient is so severe as to alter the material properties, the plate may be approximated by laminae, whereby each segment is assigned average material properties. Lastly, if the temperature is transient, then this analysis provides a quasi-static statement of the stress and deflection at specific times during the temperature input.

REFERENCES

B.A. Boley, J.H. Weiner, *Theory of Thermal Stresses*, John Wiley and Sons, Inc.

J.R. Vinson, "Thermal Stress in Laminated Circular Plates", *Proceedings of the Midwestern Conference on Solid Mechanics*, 1959, p467-471.

J.M. Witney, et.al., "Thermo-Stress Analysis of a Plated-Through-Hole in a Composite Printed Wiring Board", *Proceedings of the 7th Technology Conference of the American Society of Composites*, 1972, p.191-197.

K.N. Cho, "Thermal Stress Analysis of Laminate Using Higher Order Theory in Each Layer", *Journal of Thermal Stresses*, v12, n3, 1989 p.321-332.

Y.A. Zeijarnik, et.al. "Thermal Deformations of External Surfaces of Multilayered Circular Plates, Slabs, and Cylinders, *Prob. Prochn*, v 7, July, 1987,p.86-88.