ABSTRACT

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Having been able to realize highly entangled states on quantum processors that have fundamentally different geometries as hardware, poses an important question: Whether a state of a quantum spin system has any intrinsic geometry. Here, we show that for both the dynamics and the states of a spin system have a canonical, associated *coarse geometry*, in John Roe's sense, on the set of sites in the thermodynamic limit. For a state ϕ in the thermodynamic limit of a spin system we, we construct a unique coarse structure $\mathscr{E}_\phi$ such that ϕ has correlation decay with respect to any other coarse structure $\mathscr{E}$ if and only $\mathscr{E}_\phi \subseteq \mathscr{E}$. Further more, under reasonable assumptions we show that the coarsely connected completion, $(\mathscr{E}_\phi)_{con}$ is stable under quasi-local perturbations of ϕ. Alongside this, we develop a dynamical coarse structure for generic quantum channels with similar stability result. In addition to that we show that multiple order parameters depend on the coarse structure of some underlying metric. We also establish compatability between dynamical coarse structure of some circuit α and the coarse structure of a state $\psi \circ \alpha$ with ψ being a product state.

Universal Coarse Geometry of Spin Systems

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CHAPTER

1

INTRODUCTION

1.1 Motivation

Studying quantum spin systems is central when it comes to quantum information and condensed matter physics. Recently we have been able to experimentally "realize" highly entangled states on quantum processors. Specifically, demonstrated through highly entangled, topologically ordered ground states of gapped quantum spin liquids (Satzinger et al. 2021; Iqbal et al. 2024; Xu and other 2024; Andersen et al. 2023); these states have, in principle, the entanglement structure of the ground state of the theoretical spin liquid up to local perturbation. The properties of such spin liquid that depend on entanglement structure are not just simulated but arguably realized on the processor. Topological order is one important example of such property (Kitaev 2006; Zeng et al. 2019).

The geometry of the lattice plays a key role when it comes to such entanglement properties; that is, such properties depend on the geometry of the sites to define mathematically. In particular, for most experiments realizing topologically ordered states so far, the geometry of the hardware agrees with the geometry of the state being realized. Although, for most of the interesting states, there is an underlying geometry that is inherently different from that of the hardware (Iqbal et al. 2024).

To further emphasize the problem, suppose we are trying to encode a ground state of a 3D

spin liquid (Walker-Wang model (Walker and Wang 2012)) on some quantum processor with 2D hardware geometry. Specifically, we can make use of some bijective (necessarily non-local) encoding scheme to map sites of the 3D lattice to sites of the 2D lattice. We might think since we remember this encoding scheme, then the incompatibility between the 2 geometries is not an issue; after all, the universal properties that depend on the 2D metric are mathematically well defined as long as we use our encoding scheme of the 3D metric. Suppose now there was some local perturbation (manifesting in some kind of an error) that occurred in the 2D geometry; then this is necessarily non-local when mapped back to the 3D geometry. Here an important question is posed: what happens to the interesting entanglement properties that depended on the geometry? Another situation may occur: suppose we are just handed a state on the processor without knowing any encoding schemes; can we find a "correct" geometry on the set of sites that we can use to define and measure properties like topological order that is independent of the physical geometry of the local Hilbert spaces?

To this end we raise the following question as a starting point:

Question 1.1. *Is it possible to find a mathematically-precise "correct" geometry on the set of sites of a spin system such that we can define an entanglement-based order parameter with respect to it?*

To better explain the problem, we look at the various notions of equivalence of quantum phases. Normally, two gapped Hamiltonians have ground states in the same quantum phase if there's a path between the two keeping the gap open. The state-based definitions of equivalence of phases that do not reference the parent Hamiltonians have been motivated by quantum information perspective. Some of the important examples are finite-depth circuit equivalence, locality-preserving equivalence (Chen et al. 2010; Zeng et al. 2019) and and quasi-local automorphic equivalence (Bachmann et al. 2012). The key point is that these do not just depend on the state; they also depend a priori on the underlying geometry of the sites to be defined. Thus the "inherent" geometry that we are looking for should tell us which geometry we should use when characterizing quantum phases.

The framework of *coarse geometry* helps answer the question 1.1 positively in the thermodynamic limit.

1.2 Large Scale Geometry and Correlations

One crucial way that serves as the manifestation of the geometry of space is the notion of locality. Typically, locality is encoded in the Hamiltonian, where we require the interactions to appropriately decay as a function of distance (Bratteli and Robinson 1997; Zeng et al. 2019).

There is, however, one other way locality manifests, and that is through states (e.g., ground states or equilibrium/KMS states (Bratteli and Robinson 1997, Section 5.2)) through the decay of the two-point correlation functions as a function of distance. We do not mean the actual *rate* of decay of correlations; however, we refer to the mere *fact* of decay as a function of distance. This then motivates us to say that a state on a spin system with a given metric on the set of sites is *local* with respect to some metric function on the sites if the two-point correlation function decays as a function of the distance between the supports of localized observables.

We show in the current work that the notion of locality depends only on the *large-scale geometry* of the metric and is independent of the details at small-length scales (Elokl and Jones 2024). What we mean by that is that 2 metrics generate the "same" locality if they look the same when looked at from far away. Thinking from this perspective, we then have an answer for 1.1 using the framework of *coarse geometry*, which formalizes the notion of large-scale geometric structure. We make use of the equivalence relation on metric spaces known as *coarse equivalence*, which makes the idea of "looking the same" at large scales more formal. Take, for instance, the metric space $\mathbb{R}^n$, it is coarsely equivalent to any discrete n-dimensional lattice (e.g., $\mathbb{Z}^n$) despite obvious differences in topology, but $\mathbb{R}^n$ is coarsely equivalent to $\mathbb{R}^m$ if and only if $n = m$ (see chapter 2.1 for more details). Moreover, hyperbolic n-space $\mathbb{H}^n$ is *not* coarsely equivalent to $\mathbb{R}^n$, even though they are homeomorphic.

Recently, the coarse perspective has come to play a role in mathematical physics in the theory of disordered topological insulators (Ludewig 2023; Kubota 2017; Ewert and Meyer 2019; Sako 2020; Ludewig and Thiang 2021; Kubota et al. 2022; Ludewig and Thiang 2022). However, it came about originally in the context of geometric group theory and index theory of pseudo-elliptic operators on non-compact manifolds (Nowak and Yu 2012; Roe 1993; Willett and Yu 2020). Here, however, we will make use of Roe's framework of *abstract* coarse structures. This provides a metric-independent characterization of coarse geometry in the sense of (Roe 2003), to propose an answer to Question 1.1.

Given a set X, a coarse structure $\mathscr{E}$ is a set of subsets of $X \times X$, that is, $\mathscr{E} \subseteq \mathscr{P}(X \times X)$, satisfying several closure axioms (see Definition 2.3). We call the elements $E \in \mathscr{E}$ *controlled sets*. If (X, d) is a metric space, the the associated bounded coarse structure is defined by

$$\mathscr{E}^d := \{E \subseteq X \times X \mid \sup_{(x,y)\in E} d(x,y) < \infty\}.$$

Given a fixed set X, we will find it useful to think of the collection of all coarse structures on it as a partially ordered set with respect to containment. Now, two metrics on X give *the same* coarse structure exactly when they have the same controlled sets. Having the same coarse structure will be the notion that is more relevant to use than abstract coarse equivalence.

Now, we can have a more formal version of Question 1.1 it goes as follows: Given a state ϕ in the thermodynamic limit of a spin system, is there an inherent coarse structure on the sites with respect to which it is local? We are then naturally led to consider the collection of *all* coarse geometries on the set of sites with respect to which this state is local. Then, we can find the *smallest* coarse structure on X with respect to which ϕ is local. We show later that it is unique and that it satisfies a nice universal property.

To put it in mathematical terms, we start with an abstract (classical or quantum) spin system[1], with the sites indexed by some set X. Our system is disembodied; that is, we do not assume that any geometric structure X is given a priori. Let A_x be the C*-algebra of observables localized at site x. Commonly, it A_x is $M_d(\mathbb{C})$ in the quantum case and $\mathbb{C}^d$ in the classical case. Suppose A is the C*-algebra of quasi-local observables. This is simply the infinite tensor product $\otimes_{x\in X} A_x$. Let ϕ be a state on A, and define:

$$C_\phi(x,y) := \sup_{a\in A_x,\ b\in A_y} \frac{|\phi(ab)-\phi(a)\phi(b)|}{\|a\|\|b\|}.$$

$C_\phi(x,y)$ ss the maximal correlation between observables localized at points x and y respectively? ϕ is said to be *local*[2] with respect to a coarse structure $\mathscr{E}$ on X if C_ϕ has *controlled decay* with respect to $\mathscr{E}$, that is, for every $\epsilon > 0$, there exists some controlled set E such that for all $(x,y) \notin E$, with $C_\phi(x,y) < \epsilon$ (see Definition 3.1). Remember that a coarse structure is *connected* if every finite subset of $X \times X$ is controlled (see Definition 2.3). We now have one of the main results, which is a special case of the more general case 3.1.

Theorem 1.1. *Let ϕ be a state on an (abstract) spin system over the set X. Then, there is a unique coarse structure $\mathscr{E}_\phi$ on X such that ϕ is local with respect to the coarse structure $\mathscr{E}$ if and only if $\mathscr{E}_\phi \subseteq \mathscr{E}$.*

The previous theorem gives a complete characterization of coarse structures with respect to which ϕ is local. It also gives natural candidates for some choice of "universal coarse geometry." There are other natural variations of $\mathscr{E}_\phi$ that are similarly universal but with additional hypotheses. The main variation considered here is *higher-order* controlled decay, which gives rise to the universal coarse structure $\mathscr{E}_{\tilde{\phi}}$ (see Definition 3.3 and Theorem 3.3), where, instead of considering operators localized at points, we consider operators localized in uniformly bounded regions. For "reasonable, nice" states, we expect all these variations to agree. The reason for considering such variations here is that they make the statement of some theorems more convenient.

[1] here, abstract spin systems are characterized by discrete nets of C*-algebras; see Definition 2.15

[2] we will consider different variations of this definition, see Definition 3.3

1.3 Quasi-local perturbations

One of the main motivations was defining entaglement-based order parameters. To do so, we need the coarse geometry associated with a state to be stable under local perturbation. However, since we are working in the thermodynamic limit, the more natural constraint is to require that the geometry be stable under *quasi-local perturbation*. A quasi-local perturbation is defined to be a quantum channel Φ on the quasi-local algebra, which is a norm limit of quantum channels that are localized in finite regions (see Definition 4.1). For a pure state on a quantum spin system, stability under quasi-local perturbation is necessary and sufficient for a property to depend only on its quasi-equivalence class (Bratteli and Robinson 1987).

Recall that a coarse structure on a countable set X is *proper* if all bounded sets are finite, and monogenic if it can be generated by a single controlled set. The typical examples of proper, monogenic spaces are the coarse spaces associated to infinite, locally finite graphs, e.g., lattices $\mathbb{Z}^n$ and, more generally, Cayley graphs of groups. As it happens, the monogenic property of the universal coarse structure $\mathscr{E}_\phi$ is equivalent to the coarse-scale invariance of the correlations (see Proposition 3.1). One of our main results is the following theorem, which in the body of the paper is proved in a more general setting; see Theorem 4.1.

Theorem 1.2. *Suppose ϕ is a state in the thermodynamic limit of a spin system with $\mathscr{E}_\phi$ being proper and monogenic. Then for any quasi-local perturbation Ψ, $(\mathscr{E}_{\phi\circ\Psi})_{con} = (\mathscr{E}_\phi)_{con}$.*

We have two immediate consequences for a pure state ϕ in the thermodynamic limit of a spin system with $\mathscr{E}_\phi$ proper and monogenic.

1. (Corollary 4.1). Suppose ψ is any other pure state with $\|\phi - \psi\| < 2$, $(\mathscr{E}_\phi)_{con} = (\mathscr{E}_\psi)_{con}$.

2. If $(\mathscr{E}_\phi)_{con}$ is a non-trivial connected coarse structure, ϕ is not a quasi-local perturbation of a product state. Where $(\mathscr{E}_\phi)_{con}$ can be thought of as a "connected completion" of $\mathscr{E}_\phi$ which is generated by adding all finite subsets of $X \times X$ to $\mathscr{E}_\phi$.

Thus, at least under the reasonable assumption that $\mathscr{E}_\phi$ is proper and monogenic, $(\mathscr{E}_\phi)_{con}$ is robust under all local perturbations, making it a solid candidate for a universal order parameter.

1.4 Universal coarse geometry of circuits

Up till now we have focused on coarse geometry of states defined in terms of correlation decay; however, the mathematical formalism for constructing universal coarse structures on a set X is

very general. The input is basically any function $f : X \times X \to \mathbb{R}_+$ (a function from the product of power sets $\widetilde{f} : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+$ for the higher-order setting, see Chapter 3). In the previous discussion we utilized the two-point correlation function C_ϕ. However, there are many other interesting functions related to physical objects that we can consider, and we would expect them to decay with distance.

A particularly interesting example is coarse structures associated with dynamics of quantum spin systems. In the case of time evolution, a (discrete) time evolution on a quantum spin system is local with respect to some given geometry if its 'tails decay" with distance. To put it more formally, utilizing the same setup as before, for a given α, an automorphism of (or more generally, a unital completely positive (ucp) map) on the quasi-local C*-algebra A, then we use the tail function:

$$Q_\alpha(x, y) := \sup_{a \in A_x, b \in A_y} \frac{\|[\alpha(a), b]\|}{\|a\|\|b\|}.$$

The quantum dynamical coarse structure, $\mathscr{E}_\alpha$, is defined on X to be the universal coarse structure where Q_α has controlled decay. For dynamics arising from integrating gapped, local Hamitlonians or from quantum cellular automata, this construction generally recovers the original coarse geometry (see Section 3.4 for more details). Contrary to the case of the correlation coarse structure, which applies equally well to classical and quantum states, the dynamical coarse structure is only interesting for quantum systems for which localization of observables is detected by commutativity. However, we still explore the quantum dynamical structure side by side with the correlation coarse structure, since many results are completely analogous. Similarly to the correlation case, we also have a "higher order" version $\mathscr{E}_{\widetilde{\alpha}}$.

The quantum dynamical coarse structure gives, similar to the case of states, an interesting intrinsic coarse geometry for quantum circuits in the thermodynamic limit, and thus the discussion above motivating the study of correlation coarse structures for state realization works as well for universal properties of dynamics. Therefore, we have a similar stability result for the case of dynamics (see Theorem 4.2).

Theorem 1.3. *Let A be an (abstract) spin system over the set X, and $\alpha : A \to A$ be any ucp map such that $\mathscr{E}_\alpha$ is proper and monogenic. Then for any quasi-local perturbation $\Psi : A \to A$, $(\mathscr{E}_{\alpha \circ \Psi})_{con} = (\mathscr{E}_\alpha)_{con}$.*

And, again, there are two immediate consequences for unitary dynamics (or, when the channel α is an *automorphism of the quasi-local algebra*) on a *concrete* spin system A, given that $\mathscr{E}_\alpha$ is proper and monogenic.

1. (Corollary 4.2). Given $\beta \in \mathrm{Aut}(A)$ with $\|\alpha - \beta\| < 1$, then $(\mathscr{E}_\alpha)_{\mathrm{con}} = (\mathscr{E}_\beta)_{\mathrm{con}}$.

2. (see Proposition 4.1). For $(\mathscr{E}_\alpha)_{\mathrm{con}}$, a non-trivial connected coarse structure, α is not an inner automorphism.

1.5 Applications to state preparation

In Chapter 5, (higher-order) correlation and dynamical coarse structures are tied together by applying both the interaction of both to state preparation on quantum processors (Elokl and Jones 2024). This is done through exploring the behavior of coarse structures of states under locality-preserving channels. Consequently, we obtain the following theorem as a direct consequence of Corollary 5.2.

Theorem 1.4. *Let ϕ be a state, and suppose α is a circuit which prepares ϕ from a product state. Then $\mathscr{E}_{\widetilde{\phi}} \subseteq \mathscr{E}_{\widetilde{\alpha}}$*

We can interpret the coarse structures $\mathscr{E}_{\widetilde{\phi}}$ and $\mathscr{E}_{\widetilde{\alpha}}$ as a “coarse complexity class." With this in mind, we can see that the above result takes us beyond extrapolating information out of coarse structure by saying that any circuit used to prepare a state must be at least as coarsely complex as the state. Suppose, for instance, that we want to prepare a state whose universal (higher-order) coarse structure is equivalent to $\mathbb{Z}^3$ on a quantum processor with hardware that is coarsely equivalent to $\mathbb{Z}^2$, then we can not use a circuit which is (higher-order) local with respect to the hardware geometry.

1.6 Coarse order parameters

We finish up by considering entanglement-based order parameters of a state, such as topological order, and we note that they only depend explicitly on some kind of geometry to be defined. A family of candidates is proposed for canonical coarse geometries that need not be compatible with spatial geometries directly. However, we need to see to what extent do the order parameters of interest only depend on coarse geometries. This is the topic of Chapter 6.

Here there is an explicit metric d that we work with on the set of sites X, and aim at showing that various order parameters are invariant under coarse equivalence. We assume that the spaces are monogenic, which reduces coarse equivalence to quasi-equivalence (see Proposition 2.4). With this in mind, d is coarsely equivalent to another metric d' if and only if there exist constants $L > 0, C \geq 0$ such that for any $x, y \in X$,

$$L^{-1}d(x,y)-C \leq d'(x,y) \leq Ld(x,y)+C$$

We aim at exploring which definitions of various order parameters, depending a priori on a metric d, are invariant under this equivalence relation.

1.6.1 Superselection sectors and topological orders

The first order parameter we explore in Section 6.1 is the W*-category of cone-localized superselection sectors of a state ϕ defined on a spin system in $\mathbb{Z}^n$. In 2 dimensions, with some assumptions, we can upgrade this category to a braided tensor category, which characterizes the states' *topological order* (Naaijkens 2011; Ogata 2022). Given (X,d) with a finite asymptotic dimension (see Definition 2.9), there is a coarse variation of the definition of superselction sectors in terms of coarse cones, and it can be shown that this W*-category only depends on the coarse structure. Also, note that in the case of $\mathbb{Z}^n$ with the Euclidean metric, this agrees with the regular definition of superselction sectors in terms of localization in honest cones (see Proposition 6.1), thus we say that this W*-category is a coarse order parameter.

1.6.2 Correlation's Decay

The second important family of order parameters is related to the decay *rate* of correlation functions. In 6.2, we see how the decay rate depends only on the coarse structure of the metric. In general, not all decay rates are coarsely defined.

We start with the three most common classes of decay that are usually considered in the condensed matter literature: 0 correlation length (correlations are 0 when distance is large), finite correlation length (correlations have at most exponential decay), and finally infinite correlation length (correlations have super-exponential decay).

Proposition 1.1. *(See Proposition 6.2). Correlation length* 0*, finite correlation length, and infinite correlation length for a state ϕ depend only on the quasi-equivalence class of the metric. Therefore, if $\mathscr{E}_\phi$ is monogenic, then* 0*, finite, or infinite correlation length is a coarse order parameter.*

Having a finite correlation length is coarsely well defined. However, the exact value of the correlation length is not. We are then led to believe that generally the precise asymptotic "coarse rate of decay" for correlations is not something that would make sense for correlation functions.

Particularly, an important family of order parameters is *critical exponents*. They help determine the scaling limit CFT that is believed to characterize the universality class of critical

states of spin systems (refer to, for instance, (Stanley 1987)). These numbers are usually defined in terms of degrees of polynomials for parameters of the system with algebraic decay. Here, we, naturally, focus on algebraic decay of correlations. In section 6.2, we explore some theory that allows us to make sense of when a class of decay functions on $\mathbb{R}^+$ are asymptotically invariant under quasi-equivalent changes of the metric, introducing the notion of *coarsely stable*. Consequently, we are able to conclude the following.

Proposition 1.2. *(See Proposition 6.4). Algebraic decay of correlations and the critical exponent are both independent of the quasi-equivalence class of the given metric.*

The above result heavily suggests that what is usually thought of as a universal feature of the system, normally defined through properties of a scaling limit QFT, are sensible and could be defined at the coarse level instead. This maybe be more accessible from a mathematically rigorous viewpoint. Following this line of reasoning, it may be of interest to develop a notion of “coarse equivalence" of quantum field theories (for example, the framework of algebraic quantum field theories), and instead of defining a universality class in terms of renormalization group fixed points, we can define a universality class as a coarse equivalence class of quantum field theories obtained in the thermodynamic limit (possibly with some sort of stabilization).

1.7 Discussion

This line of thought that depends on asymptotic reasoning is inspired by similar ideas for extracting geometric structures from states and/or dynamics that have already appeared in the literature to describe *emergent spacetime* (for example, see (Qi 2013; Cao et al. 2017; Raasakka 2017)). They have different goals, mostly focusing on (small-scale) metric geometry, aiming to obtain something that looks like gravity on their emergent spatial structure. This contrasts with what’s presented here, which focuses on large-scale geometry emerging only in the thermodynamic limit of a spin system for the purpose of serving as a basis for defining universality classes. However, these ideas use essentially the same mathematical starting point of using quantum information as a basis for defining geometry. The work presented here is also related to (Qi and Ranard 2019), where the correlation functions are utilized with the aim of reconstructing Hamiltonians from states.

We also want to emphasize that the choice of the "sites" is taken to be given. From the perspective of state realization on a quantum processor, this is a natural starting point given that such machines have a built-in choice of sites and local observables. However, in the abstract case, one could wonder if we could derive a site structure itself from just a state on an algebra. This seems unlikely, since any two states in the thermodynamic limit of a spin system

are related by an automorphism. It would be very intriguing, however, to see if some natural additional structure, such as a Hamiltonian, could give rise to a decomposition into sites

Another interesting follow-up is whether interesting order parameters can be found that are valued in algebraic structures related to coarse spaces. For instance, coarse spaces have (co)-homology and K-theories. It would be very interesting to see if states have invariants valued in these groups, as in the case of disordered band insulators (Ewert and Meyer 2019). Another interesting issue is that of practically computing aspects of the emergent coarse structure, either theoretically or from a large but finite data set (see section 3.2). This problem has many similarities to what is done in topological data analysis (Chazal and Michel 2021) and, in particular, persistence homology.

Although the results above are stated for quantum spin systems, all the definitions and theorems are formulated in terms of abstract operator algebras, since proofs work at this level of generality. One possible advantage is that this framework applies equally well to classical systems with probability distribution states (that is, "macrostates") by assuming the algebras in question are commutative, as well as more complicated algebras arising from quantum field theory and topological order. We also note that the frame for extracting coarse structures on the set X is also very general and could be of independent interest. After all, it only requires as input a function $f : X \times X \to \mathbb{R}_+$. Therefore, many other objects relevant in physics (e.g., a Hamiltonian) or even in classical network theory could be used to cook up a function f to produce a coarse structure of interest.

The outline is as follows. In chapter 2 we review the C*-algebraic formulation of states, observables, and dynamics and their important and relevant results. We also introduce the framework of discrete nets of algebras used throughout the paper. Chapter 2 also gives a mostly self-contained review of the important basic concepts in coarse geometry and the tangents that may be relevant for us. Chapter 3 has the most weight. We introduce the formalism for extracting coarse geometries on some set X from a function $f : X \times X \to \mathbb{R}$, and discuss the main examples arising from correlation functions and dynamics. In Chapter 4, we have stability results for both correlation and dynamical structures, which solidify their importance. In Chapter 5, we explore how dynamical and correlation coarse structures interact. Finally, in Chapter 7, we discuss the dependence of the relevant order parameters, superselection sectors, and decay rates on the coarse structure.

CHAPTER

2

BACKGROUND AND LITERATURE

2.1 Coarse Geometry

Coarse geometry was originally developed by John Roe in the context of geometric group theory and index theory of pseudo-elliptic operators on non-compact manifolds, (Nowak and Yu 2012; Roe 1993; Willett and Yu 2020) so we refer the reader to (Roe 2003) for a detailed review. John Roe's book will be the main source for this part; we will also have some results based on (Elokl and Jones 2024). As mentioned before, the intuition behind *coarse geometry* is that it ignores the small details of the geometry (e.g., a metric space) while focusing more on what happens when we zoom out and look from far away. Take the integers with the taxicab metric and the real line equipped with the usual Euclidean metric. Think about what happens when we zoom out from the integers; the distance between each site diminishes until they look extremely close to each other, and eventually they look like they lie on the real line. This is the intuition behind large-scale geometry, and since metric spaces are a really intuitive example of coarse geometry, we will start by talking about them, which again, Roe goes into it in more detail.

Another illuminating analogy is that with topology and metric spaces. When defining continuity of a function on a metric space, we forget lots of details about the metric, since only small distances matter. Suppose d is a metric; then $d' = \min(d, 1)$ is also a metric. The topology of d' is exactly that of d even though d' erases all the information about d.

We want to give a precise sense of the *coarse* equivalence of $\mathbb{Z}$ and $\mathbb{R}$(Roe 2003). To this end, we start by defining

Definition 2.1. *Suppose X, Y are metric spaces with $f : X \to Y$ a map; then:*

- *f is (metrically) proper if the inverse image under f of every bounded subset of Y is a bounded subset of X*
- *f is (uniformly) bornologous if for every $R > 0$ there exists an $S > 0$ with:*

$$d(x, x') < R \implies d\big(f(x), f(x')\big) < S$$

- *if f is proper and bornologous then it is coarse*

Definition 2.2 (Large-Scale Lipschitz)**.** *Suppose X, Y are metric spaces. A map $f : X \to Y$ is large-scale Lipschitz if there exist constants c, A such that*

$$d\big(f(x), f(x')\big) \leq c\, d(x, x') + A$$

Note that large-scale Lipschitz implies bornologous.

Suppose X is a set Two maps f, f' from X into a metric space Y are said to be close if $d(f(x), f'(x))$ is bounded, uniformly in X. The metric spaces X and Y are coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ with $f \circ g$ and $g \circ f$ are close to the identity map on Y and the identity on, X respectively. In particular, if we take X, Y to be $\mathbb{Z}, \mathbb{R}$ respective, we can now see how they are coarsely equivalent. Using the inclusion $\mathbb{Z} \to \mathbb{R}$ as one of the maps and the inverse map up to closeness $\mathbb{R} \to \mathbb{Z}$ would be a map that takes x to the largest integer less than or equal to x (the floor map)(Roe 2003).

Here comes an important distinct scenario from the one above: what if we use the same base set and have 2 different metrics on it? Example 2.1 is an example of a comparison we will be doing throughouth this work.

Example 2.1 (Finitely Generated Groups)**.** *We say a group Γ is generated by $F \subseteq \Gamma$ if every $\gamma \in \Gamma$ can be written as word in elements of F and their inverses that is,*

$$\gamma = \gamma_1 \gamma_2 \cdots \gamma_n$$

such that γ_i is an element of F or an inverse of an element in F; if F isfinite, we say Γ is finitely generated. The smallest such n is called the word length of γ with respect to F denoted by $|\gamma|_F$. It is easy to see, then, that $d(x, y) = |x^{-1} y|$ defines a metric on Γ that is, by construction, left invariant

under the action of Γ *on itself. We then have the interesting fact that the coarse equivalence class of a word metric is independent of the choice of the generating set. Recall that a metric is proper if closed, bounded sets are compact.*

Proposition 2.1. *(Roe 2003, 1.15) Suppose* F *is a finite generating set for the group* Γ *and* d *is the associated word metric to* F*. Now, if* d' *is another metric on* Γ *that is translationally invariant and proper, then the identity* $(\Gamma, d) \to (\Gamma, d')$ *is a coarse equivalence. That is, any two word metrics on* Γ *are coarsely equivalent.*

Proof. Suppose $c = \max\{d'(e,\gamma) : \gamma \in F\}$. Then,

$$d'(x\gamma, x) \leq c \quad \forall x \in \Gamma$$

Now, using the triangle inequality and by induction, we have:

$$d'(x,e) \leq c\,|x|_F$$

then by translation invariance, $d'(x,y) \leq c\,d(x,y)$. Then the identity map $I_1 : (\Gamma, d) \to (\Gamma, d')$ is coarse. To prove the other way around, for $R > 0$, $B_{d'}(e,R)$ the $d'-$ball is finite. So, there exists $S > 0$ with $d'(e,x) \leq R \implies d(e,x) \leq S$. Again, by translation invariance, then the identity map, $I_2 : (\Gamma, d') \to (\Gamma, d)$ is coarse. □

We now move on to define what a *coarse structure* is. It will be our main tool to answer find our "correct" geometry inherent to the states.

Definition 2.3 (Coarse Structure.)**.** *Suppose* X *is a set. A coarse structure on* X*,* $\mathscr{E}$*, is a collection of subsets of* $X \times X$ *satisfying the following:*

- $\mathscr{E}$ *contains the diagonal, and is closed under taking finite unions and subsets.*
- *If* $E \in \mathscr{E}$*, then* $E^{-1} = \{(x,y) : (y,x) \in E\} \in \mathscr{E}$
- *If* $E, F \in \mathscr{E}$*, then* $E \circ F := \{(x,z) : (x,y) \in E, \text{and } (y,z) \in F \text{ for some } y\} \in \mathscr{E}$.

If $E \in \mathscr{E}$ then it is called *controlled.* A set $B \subseteq X$ is called *bounded* if $B \times B \in \mathscr{E}$. The following example motivates the terminology. When we think of a coarse structure we want to think of a structure that captures what happens when we look at the original structure from infinity (always keep in mind zooming out from the integers).

Example 2.2 (Bounded coarse structure of a metric space.)**.** *Suppose* X *is a set and* d *is a metric on* X*. Define a set* $E \subseteq X \times X$ *to be controlled if*

$$\sup_{(x,y)\in E} d(x,y) < \infty.$$

The result is coarse structure, denoted by $\mathscr{E}^d$ and is called the bounded coarse structure or simply the metric coarse structure of (X,d).

The idea behind the bounded coarse structure of a metric space is that it captures its large-scale behavior. It tells us which collection of pairs of points are a uniformly bounded distance apart as they go to infinity. This is in some sense a "dual" of the uniform structure of a metric space, which captures the small-scale structure of a metric.

Example 2.3 (Cayley Graph.)**.** *To expand more on finitely generated groups. Suppose Γ is a finitely generated group with a generating set S. We define the Cayley Graph, G, to be that with the elements of Γ and there is an edge between $x, g \in \Gamma$ if there exists $s \in S$ such that $sx = g$. Using the path metric on G makes into a discrete metric space. Any two choices of finite generating sets give rise to equivalent coarse structure.*

A coarse space $(X,\mathscr{E})$ is *metrizable* if there is some metric d on X whose bounded coarse structure is $\mathscr{E}$. A coarse structure is metrizable if and only if it is *countably generated*, (Roe 2003). All the coarse structures that occur naturally in this paper are metrizable.

We continue to introduce some useful definitions and terminology. Let $B \subseteq X$ and $E \in \mathscr{E}$ a controlled set, then the E-ball is defined as $E[B] := \{x \in E \ : (y,x) \in E \text{ for some y} \in B\}$. We call a (discrete) coarse structure *proper* if the E-ball of any bounded set is finite; this implies all bounded sets are finite. To generalize this to coarse structures on spaces with an underlying topology, we require E-balls of bounded sets to be compact, but here we will only be using the "discrete" version we described above.

We call a coarse space $(X,\mathscr{E})$ *coarsely connected* if it contains the collection of finite subsets $\mathscr{F}(X \times X)$. Equivalently, if every pair of points is contained in some controlled set then the coarse space is coarsely connected. In particular, for a metric space (X,d), $\mathscr{E}^d$ is coarsely connected if and only if any pair of points are separated by a finite distance.

We now define coarse functions and closeness of functions in a more general way; recall that those notions before were defined for metric spaces.

Definition 2.4 (Coarse functions)**.** *A function $\rho : X \to Y$ between coarse spaces $(X,\mathscr{E})$ and $(Y,\mathscr{F})$ coarse if the inverse image of bounded sets is bounded, and $(\rho \times \rho)(\mathscr{E}) \subseteq \mathscr{F}$ (the image of a controlled set is controlled).*

Definition 2.5 (Close functions)**.** *Two coarse functions $\rho, \gamma : (X,\mathscr{E}) \to (Y,\mathscr{F})$ are said to be close if $\{(\rho(x),\gamma(x)) \ : \ x \in X\} \in \mathscr{F}$.*

We see that closeness is an equivalence relation on the set of coarse functions. For the more categorically inclined, the *coarse category* consists of coarse spaces and coarse maps up to the equivalence relation of closeness, with the usual composition of functions. Two coarse spaces are *coarsely equivalent* if they are isomorphic in the coarse category.

Example 2.4. *This is an example to better clarify the change in cardinality. Suppose (X,d) is a metric space; let $R>0$, and $Y \subseteq X$ such that for every $x \in X$ there exists some $y \in Y$ with $d(x,y)<R$. Then Y is coarsely equivalent to X. In particular, any separable metric space (e.g., $\mathbb{R}^n$) is coarsely equivalent to a discrete metric space.*

As mentioned before example 2.1, we will be mostly considering coarse structures on the same set and if they are contained in each other. So, our main function comparing coarse structures will be the identity, and it may not be coarse in some situations that we might expect it to be. If $\mathscr{E}$ and $\mathscr{F}$ are two coarse structures on the set X with $\mathscr{E} \subseteq \mathscr{F}$, it need not be the case that the identity from $(X,\mathscr{E}) \to (X,\mathscr{F})$ is a coarse function , since $\mathscr{F}$-bounded sets are not necessarily $\mathscr{E}$-bounded.

Let $\mathscr{E}$ and $\mathscr{F}$ be coarse structures on the same set X with $\mathscr{E} \subseteq \mathscr{F}$. $\mathscr{E}$ is said to be*smaller* than $\mathscr{F}$ or that $\mathscr{F}$ is *larger* than $\mathscr{E}$. There is a smallest coarse structure called the *discrete* coarse structure. It consists only of all subsets of the diagonal. There is also a largest coarse structure, the *indiscrete* coarse structure, and it consists of all subsets of $X \times X$. In particular, the discrete coarse structure is the bounded coarse structure of a metric which gives an infinite value between any pair of distinct points. On the other hand, the indiscrete coarse structure is given by any metric which is globally bounded. The coarse structure $\mathscr{F}(X \times X)$ of finite subsets is the smallest connected coarse structure, and we will frequently refer to it as the trivial connected coarse structure.

The following proposition is useful when comparing coarse structure in metrizable coarse spaces.

Proposition 2.2. *(Elokl and Jones 2024, 3.5). Suppose d and d' are metrics on X. Then $\mathscr{E}^d \subseteq \mathscr{E}^{d'}$ if and only if there is some non-decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with*

$$d'(x,y) \leq \rho(d(x,y)),$$

where we interpret $\rho(\infty) = \infty$.

Proof. If $\mathscr{E}^d \subseteq \mathscr{E}^{d'}$. Then for every $R \geq 0$, $\{(x,y) \in X \times X \ : \ d(x,y) \leq R\} \in \mathscr{E}^{d'}$, we can pick some S_R with $d'(x,y) \leq S_R$. Since we are free to pick arbitrarily large S_R, we can choose S_R so that the function $\rho(R) := S_R$ is increasing. This satisfies the criterion. On the other hand,

given such a ρ, let $E \in \mathscr{E}^d$, and choose an $R \geq 0$ such that $d(x,y) \leq R$ for all $(x,y) \in E$. Then $d'(x,y) \leq \rho(d(x,y)) \leq \rho(R)$, thus $E \in \mathscr{E}^{d'}$. □

Example 2.5. *Consider the set $\mathbb{Z}^2$ with 2 different metrics, Taxicab metric and the Euclidean metric for example, then $\mathscr{E}^{d_{taxi}} = \mathscr{E}^{d_{euc}}$. Both metrics give the same coarse structure. To see this we note that: $\sqrt{x^2+y^2} \leq |x|+|y|$ and $|x|+|y| \leq x^2+y^2$.*

The intersection of a family of coarse structures is a coarse structure as well. Thus, if $\mathscr{S}$ is any family of subsets of $X \times X$, there is a smallest coarse structure containing $\mathscr{S}$, which we call the coarse structure generated by $\mathscr{S}$, denoted by $\langle \mathscr{S} \rangle$. We obtain this coarse structure by taking the intersection of all coarse structures containing it (it's non-empty since every family $\mathscr{S}$ is contained in the indiscrete coarse structure). We have the following proposition:

Proposition 2.3. *Let $\{\mathscr{E}_\epsilon\}_{\epsilon \in I}$ be a family of coarse structures on a set, X, then*

$$\langle \mathscr{E}_\epsilon \rangle_{\epsilon \in I} = \{E \subseteq X \times X \,:\, E \subseteq E_1 \circ \dots E_n \text{ where } E_i \in \mathscr{E}_{\epsilon_i},\ \epsilon_i \in I\}$$

Proof. It's clear that the collection $\mathscr{S} := \{E \subseteq X \times X \,:\, E \subseteq E_1 \circ \dots E_n \text{ where } E_i \in \mathscr{E}_{\epsilon_i},\ \epsilon_i \in I\}$ is a coarse structure containing $\mathscr{E}_\epsilon$, hence contains $\langle \mathscr{E}_\epsilon \rangle_{\epsilon \in I}$. Conversely, any coarse structure containing all $\mathscr{E}_\epsilon$ will necessarily contain $\mathscr{S}$, hence $\mathscr{S} \subseteq \langle \mathscr{E}_\epsilon \rangle_{\epsilon \in I}$. □

Example 2.6. *Suppose we have a coarse space $(X, \mathscr{E})$ with $\mathscr{E}$ that is not coarsely connected; we can always look at its connected quotient $\mathscr{E}_{con}$. We obtain it by staking the coarse structure generated by $\mathscr{E}$ and all finite subsets, that is, $\mathscr{E}_{con} := \langle \mathscr{E} \cup \mathscr{F}(X \times X) \rangle$.*

Example 2.7 (Locally-finite Connected Graph.)**.** *A coarse structure is said to be monogenic if it is generated by a single controlled set $E \subseteq X \times X$. If G is a graph, then the bounded coarse structure on the vertices $V(G)$ is monogenic with generator $E = \{(x,y) : \{x,y\} \in E(G)\}$. In fact, this graph presentation is characteristic of monogenic coarse structures. If $\mathscr{E} = \langle F \rangle$ is a monogenic coarse structure on X, take the graph G_F whose vertices are X and $\{x,y\} \in E(G_F)$ if either $(x,y) \in F$ or $(y,x) \in F$. Then it is clear that $\mathscr{E}$ is the bounded coarse structure induced by the path metric on G_F, denoted by d_F.*

In particular, monogenic coarse structures are precisely metrizable by path metrics on some graphs, and from this we can conclude that they are restrictions of geodesic metrics. More generally, one can show that a coarse space is coarsely geodesic if and only if it is monogenic (Roe 2003, 2.57).

Note that in the monogenic case, the coarse structure is coarsely connected if and only if the graph G_F is connected, and is proper if and only if G_F is locally finite.

A particular example that helps tie these concepts together is that of a locally-finite, connected graph. This will be our main example as we prove the main theorem in this paper. Such a graph

equipped with the path metric gives rise to a coarse structure that is proper, monogenic and connected.

Proposition 2.4. *(c.f. (Nowak and Yu 2023, 1.4.14)) Let $(X, \mathcal{E})$ be monogenic with generator F, and suppose d_F is the metric as in Example 2.7. Then for any different metric d', $\mathcal{E} \subseteq \mathcal{E}^{d'}$ if and only if there exists some $L > 0$ such that*

$$d'(x, y) \leq L d_F(x, y).$$

In particular, if G, F are 2 different generators for $\mathcal{E}$, then there exists a positive constant K such that

$$\frac{1}{K} d_G(x, y) \leq d_F(x, y) \leq K d_G(x, y).$$

Proof. It is clear that

$$d'(x, y) \leq L d_F(x, y)$$

implies $\mathcal{E} \subseteq \mathcal{E}^{d'}$ from Proposition 2.2. On the other hand, by Proposition 2.2 there is some non-decreasing $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d'(x, y) \leq \rho(d_F(x, y))$. Suppose $(x = x_0, x_1, \ldots, x_n = y)$ is a path in G_F so that $d_F(x, y) = n$. Then

$$d'(x, y) \leq \sum_i d'(x_i, x_{i+1}) \leq \sum_i \rho(d_F(x_i, x_{i+1})) = n\rho(1) = \rho(1) d_F(x, y)$$

□

So, for monogenic coarse structures, there is a canonical *Lipschitz equivalence class* of metric, which is actually stronger than coarse equivalence. In this paper though, we will be more interested in the coarse equivalence class of the metric. After all, it is the underlying coarse structure we will be studying.

The metric d_F arising from a monogenic coarse structure turns out to be an example of a *quasi-geodesic metric space*. For these, coarse equivalence reduces to a notion that's easier to work with, which is quasi-isometry. To make it more formal we have the following

Definition 2.6. *(Nowak and Yu 2023, 1.4.10) A metric space (X, d) is said to be quasi-geodesic if there is an $L, C > 0$ such that for any $x, y \in X$ with $d(x, y) = t < \infty$, there is a function $\gamma : [0, t] \to X$ such that $\gamma(0) = x$, $\gamma(t) = y$, and for all $0 \leq s, s' \leq t$*

$$L^{-1} d(\gamma(s), \gamma(s')) - C \leq |s' - s| \leq L d(\gamma(s), \gamma(s')) + C$$

It is obvious that if $\mathscr{E}$ is generated by F, then d_F is quasi-geodesic. That is, monogenic coarse structures can always be realized by quasi-geodesic metrics.

Proposition 2.5. *(Nowak and Yu 2023, 1.4.14) Suppose (X,d) is a quasi-geodesic metric space (and specifically if $\mathscr{E}^d$ is mongenic) and d' is any other coarsely equivalent metric, then there exist $L>0, C\geq 0$ such that*

$$L^{-1}d(x,y)-C\leq d'(x,y)\leq Ld(x,y)+C.$$

In general, two metrics d and d' that satisfy the above inequalities are said to be *quasi-isometric* or, equivalently, *large-scale Lipschitz equivalent*. In particular, proposition 2.5 says that for monogenic coarse spaces $(X,\mathscr{E})$, if we want to consider all metrics that generate this coarse structure, it suffices to consider the quasi-isometry class of any particular one.

At this point, the resemblance between coarse structures and topologies is obvious. We want to strengthen the resemblance in a way that will serve our purposes later. To this end we have the following section.

2.1.1 Coarse Cohomology

This subsection is meant to show how closely related coarse geometry is to topology, and the reader may move on to the following without losing context. We should note, however, that coarse cohomology would be one of the methods that we can make use of our results on real physical systems, as it bears lots of resemblance to how one uses regular cohomology theory in Topological Data Analysis (TDA).

Coarse algebraic topology has been known to be relevant to geometric group theory, particularly in cases where a group can carry non-trivial topology at infinity, with the ends of groups being one of the earliest examples. This part will discuss the definition of coarse cohomology and its relation to ordinary cohomology. For a detailed review of Coarse Cohomology we refer the reader to (Roe 2003, Ch 5). We assume basic knowledge of cohomology theory.

In defining coarse cohomology, we introduce the following notations. Given a coarse space X, we denote by X^{q+1} the Cartesian product of $q+1$ copies of X. The multi-diagonal is denoted by Δ and consists of points of the form $\{(x,\dots,x) : x\in X\}$. A subset E of X^{q+1} is said to be controlled if all $q+1$ coordinate projections $\pi_q : E\to X$ are close to a constant map. These definitions generalize naturally from the case $q=1$. When $q=0$, every subset is controlled, and the notion of boundedness retains its usual meaning.

Lemma 2.1. *(Roe 2003, 5.2) Let $f : X\to Y$ be a coarse map between coarse spaces. If E is a*

controlled subset of X^{q+1}, then its image

$$f_*(E) = \{(f(x_0), \ldots, f(x_q)) : (x_0, \ldots, x_q) \in E\}$$

is a controlled subset of Y^{q+1}. Moreover, for every bounded subset $B \subseteq Y^{q+1}$, the preimage

$$f_*^{-1}(B) = \{(x_0, \ldots, x_q) : (f(x_0), \ldots, f(x_q)) \in B\}$$

is a bounded subset of X^{q+1}.

Definition 2.7. *A subset $D \subseteq X^{q+1}$ is cocontrolled if, for every controlled set E, the intersection $D \cap E$ is bounded. When $q = 0$, cocontrolled sets are precisely the bounded ones.*

Lemma 2.2. *Let $f : X \to Y$ be a coarse map between coarse spaces. If D is a cocontrolled subset of Y^{q+1}, then $f_*(D)$ is cocontrolled in X^{q+1}.*

Proof. Consider $B = f_*(D) \cap E$, where $E \subseteq X^{q+1}$ is controlled. We have

$$f_*^{-1}(B) \subseteq D \cap f_*^{-1}(E)$$

D is cocontrolled and, by the first part of Lemma 2.1 $f_*^{-1}(E)$ is controlled, so $f_*^{-1}(B)$ is bounded. Then from the second part of Lemma 2.1 $B \subseteq f_*\left(f_*^{-1}(B)\right)$ is bounded also. Thus we have shown that $f_*(D) \cap E$ is bounded for every controlled E; so $f_*(D)$ is cocontrolled. □

Example 2.8. *(Roe 2003, 5.5) Consider the coarse space $X = \mathbb{R}$ with the metric coarse structure. Let $D \subset X^2$ be the union of the second and fourth quadrants of the Euclidean plane. Then D is co-controlled.*

Definition 2.8 (Coarse Cohomology)**.** *Let X be a coarse space, and let G be an abelian group. The coarse cochain complex $C^*(X;G)$ consists of functions $\varphi : X^{q+1} \to G$ that have controlled support. When the coefficient group G is $\mathbb{Z}$, it is omitted.*

The complex $C^*(X;G)$ is equipped with the usual coboundary map of Alexander-Spanier cohomology (Spanier 1995):

$$\partial\varphi(x_0, \ldots, x_{q+1}) = \sum (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1})$$

where the hat denotes omission of the specified term. We see that ∂ maps CX^q to CX^{q+1} and that $\partial^2 = 0$. Thus CX^* is a complex; the coarse cohomology $\mathrm{HX}^*(X;G)$ is defined to be the cohomology of this complex.

2.1.2 Asymptotic Dimension

A second tool that would be helpful when putting our theoretical framework to use in the physical world would be asymptotic dimension. Here, this will be an example of what we call an *invariant* of coarse spaces.

There are multiple equivalent definitions of the asymptotic dimension of a coarse space. Here we will focus on the one due to (Nowak and Yu 2023) which focuses on the case of metric spaces and aligns with our goals. For a more general treatment, see (Roe 2003).

Before we define the asymptotic dimension, we introduce some notation. Let $\mathscr{U} := \{U_i\}_{i\in I}$ be a cover for a metric space M. For $R > 0$, define the $R-$multiplicity of $\mathscr{U}$ to be the smallest integer n such that for every $x \in M$ the ball $B_R(x)$ intersects at most n elements of $\mathscr{U}$.

Definition 2.9. *(Nowak and Yu 2023) Suppose M is a metric space. The asymptotic dimension of M is defined to be the smallest integer $n \geq 0$ such that for every $R > 0$, there is a uniformly bounded cover $\mathscr{U}$ with R-multiplicity $n+1$. The asymptotic dimension of M is denoted by asdim M.*

The asymptotic dimension means to capture a notion of dimensionality of the space from a large-scale view, that is, in a way consistent with the idea of coarse geometry. It is, as we mentioned earlier, an invariant of coarse spaces (Roe 2003). We have $\text{asdim}(\mathbb{Z}^n) = n$ for any of the usual metrics (Nowak and Yu 2023). The above definition suggests that this number could be hard to compute in practice, and it is not clear that we could get a good estimate for this on a large, finite subset. One way we can attack this issue is by considering growth functions.

First we look again at the case of a discrete metric space (X, d) that has bounded geometry, that is, for each $r > 0$, $\sup_{x\in X} |B_r(x)| < \infty$. We then define the *growth function* of (X, d) to be

$$\gamma(r) = \sup_{x\in X} |B_r(x)|.$$

To give an idea, for any metric on $\mathbb{Z}^n$ where the identity is a coarse equivalence with $\mathbb{R}^n$, then $\gamma(r) \leq C\, r^n$, and hence these spaces are said to have *polynomial growth*. Another instance would be that of an infinite n-regular tree (a Cayley tree). We have $\gamma(r) \leq n(n-1)^{r-1}$, with equality when $r \in \mathbb{N}$, which is super-polynomial. However, having some intuition from the first example, we may expect that, for spaces with polynomial growth, we can extract the asymptotic dimension from the smallest such exponent.

Theorem 2.1. *(See(Papasoglu 2023)) Let (X, d) be a bounded geometry metric space arising from the path metric of a connected graph. If $\lim_{r\to\infty} \frac{\gamma(r)}{r^{k+1}} = 0$, then asdim $(X, d) \leq k$*

This theorem will be useful later when we talk about controlled-decay coarse structures.

We can also use the idea of growth functions to try to distinguish coarse spaces in such a way that will be useful in comparing coarse structures. For a more general theory of growth functions for bounded geometry coarse structures, check Roe (2003). However, we will restrict ourselves, for simplicity, to uniformly discrete, bounded geometry metric spaces. Here, we will frequently have the situation where we have a coarse structure $(X, \mathscr{E})$ and are able to conclude some other coarse structure $\mathscr{E}'$ on the same set X such that $\mathscr{E}' \subseteq \mathscr{E}$. In the general case, it is not the case that asymptotic dimension is decreasing with respect to inclusions of coarse structures in such sense. We then have the following question: how much information does this inclusion give us? To be specific, can we use this to rule out a large class of possibilities for the coarse equivalence class of $(X, \mathscr{E}')$ if we know the coarse equivalence class of $(X, \mathscr{E})$, and the other way around?

Proposition 2.6. *Suppose X and Y are uniformly discrete bounded geometry metric spaces, and let there be a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma_X(r) \leq f(r)$ and*

$$limsup_r \frac{\gamma_Y(r)}{f(r)} = \infty.$$

Then Y is not coarsely equivalent to $(X, \mathscr{E})$ for any coarse structure $\mathscr{E}$ on X that is contained in the metric coarse structure.

Proof. Suppose $\mathscr{F}$ denotes the metric coarse structure on Y. We proceed by proving the contrapositive of the claim.

Let (Y, d_Y) be coarsely equivalent to $(X, \mathscr{E})$ for some coarse structure $\mathscr{E}$ that is contained in the metric coarse structure of X. This implies that there are functions $\rho : Y \to X$ and $\pi : X \to Y$ such that $\rho \times \rho(\mathscr{F}) \subseteq \mathscr{E}$, $\pi \times \pi(\mathscr{E}) \subseteq \mathscr{F}$, and the sets $\{(y, \pi \circ \rho(y)) : y \in Y\} \in \mathscr{F}$ and $\{(x, \rho \circ \pi(y)) : x \in X\} \in \mathscr{E}$.

First, our claim is that $\sup_{x \in X} |\rho^{-1}(x)| < M$ for some finite M. Suppose, by way of contradiction, that this is not the case. Then, since (Y, d_Y) is a uniformly discrete, bounded geometry metric space, we have $\gamma(n) \to \infty$. Then there is a sequence $\{x_n\} \subseteq X$ where $|\rho^{-1}(x_n)| \geq \gamma_Y(n)$. This implies that for each n, there is $y_n, z_n \in \rho^{-1}(x_n)$ with $d_Y(y_n, z_n) \geq n$.

However, by assumption $F := \{(y_n, \pi \circ \rho(y_n))\}$, $G := \{(\pi \circ \rho(z_n), z_n)\} \in \mathscr{F}$. But also, $\pi \circ \rho(y_n)) = \pi(x_n) = \pi \circ \rho(z_n))$, hence

$$\{(y_n, z_n)\} \subseteq F \circ G \in \mathscr{F}$$

However, since $\mathscr{F}$ is the bounded coarse structure for (Y, d_Y), there exists N such that $d(y_n, z_n) \leq N$ for all n, which is a contradiction.

Now, suppose f is a function as in the hypothesis of the theorem. Then for any $y \in Y$ and $r > 0$, we have $|B_r(y)| \leq M|B_r(\rho(y))| \leq M f(r)$, hence

$$\frac{\gamma_Y(r)}{f(r)} \leq M$$

□

Example 2.9. *To see how we can use the above theorem, consider the metric spaces $\mathbb{Z}^n$ with the (coarse equivalence class of) the Euclidean metric. Then the literal size of balls in $\mathbb{Z}^n$ will depend on the details of the metric we choose, but it will asymptotically go like $C\,r^n$ for some constant C. Hence, for any $m > n$, by the above theorem, we cannot have $\mathbb{Z}^m$ (with the usual metric course structure) equivalent to a subcoarse structure of $\mathbb{Z}^n$ (with the usual metric coarse structure). Which could be seen, in some sense, as a manifestation of the fact that $\mathbb{Z}^m$ is coarsely equivalent to $\mathbb{Z}^n$ if and only if $m = n$.*

2.2 C*-Algebra Formulation

C*-algebras are our main framework for describing physical systems in this paper. In this section we review the basic notions and the ones that will aid us in our pursuit. We refer the reader to (Bratteli and Robinson 1987; Landsman 2017) for all definitions and results we mention in relation to C*-algebras.

We start with the most basic definitions and properties, and we move on to talk about how C*-algebras describe a physical theory.

Definition 2.10 (C*-algebra)**.** *A C*-algebra A is a Banach *-algebra (that is, a normed algebra that is complete with respect to the metric induced by the norm with $\|A^*\| = \|A\|$) satisfying the C* property:*

$$\|a^*a\| = \|a^*\|\|a\|$$

for all $a \in A$

If A has a unit (a (unique) element $1 \in A$ such that $a1 = 1a = 1$ for all $a \in A$) we call it unital, and it is the case that A may not have an identity; we will mostly be working with unital C*-algebras. The norm-property which marks a C^*-algebra is reminiscent of an underlying Hilbert space structure. This property combined with the product inequality gives $\|a^*\| = \|a\|$, since:

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\,\|a\|$$

and hence $\|a\| \leq \|a^*\|$. Switching A and A^* we see that $\|a\| = \|a^*\|$, for all $a \in A$.

Example 2.10. *A standard example of a (unital) C*-algebra would be that of all the bounded operators on a Hilbert space, $B(H)$. This, for an n—dimensional Hilbert space, is isomorphic to $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over the complex numbers, which will also be one of the main examples we have to which we can apply our methods. It is a non-commutative example which gives rise to quantum phenomena. We define products and sums of elements of $B(H)$ in the normal manner and equip this set with the operator norm*

$$\|a\| = \sup\{\|a\psi\|; \psi \in H, \|\psi\| = 1\}$$

The adjoint operation on the Hilbert space defines an involution on $B(H)$ and with respect to these operations and this norm, $B(H)$ is a C-algebra. Specifically, the C*-norm identity follows when making the following observation:*

$$\begin{aligned}
\|a\|^2 &= \sup\{\langle a\psi, a\psi\rangle; \psi \in H, \|\psi\| = 1\} \\
&= \sup\left\{\langle \psi, a^*a\psi\rangle; \psi \in H, \|\psi\| = 1\right\} \\
&\leq \sup\left\{\left\|a^*a\psi\right\|; \psi \in H, \|\psi\| = 1\right\} \\
&= \|a^*a\| \\
&\leq \|a^*\|\,\|a\| \\
&= \|a\|^2.
\end{aligned}$$

Example 2.11. *Suppose X is a locally compact space and $C_0(X)$ the continuous functions over X which go to zero at infinity. That is, for each $f \in C_0(X)$ and $\epsilon > 0$ there exists a compact $K \subseteq X$ such that $|f(x)| < \epsilon$ for all $x \in X \backslash K$, the complement of K in X. We define the algebraic operations as follows: $(f+g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$, $(fg)(x) = f(x)g(x)$, and an involution by $f^*(x) = \overline{f(x)}$. And the norm is introduced by:*

$$\|f\| = \sup\{|f(x)|; x \in X\}$$

$C_0(X)$ is an example of a commutative C-algebra. That is, the norm identity is satisfied, since:*

$$\left\|ff^*\right\| = \sup\left\{|f(x)|^2; x \in X\right\} = \|f\|^2.$$

We note that if μ is some measure on X and $H = L^2(X;\mu)$ is the Hilbert space of the μ-square integrable functions over X, we may then interpret $C_0(X)$ as an algebra of multiplication

operators on H. Hence, $C_0(X)$ is a sub- C-algebra of $B(H)$.*

As we mentioned before, a C*-algebra may not possess an identity. Even though the ones we work with here do have an identity, it's easy to resolve such an issue through the following proposition.

Proposition 2.7. *(Bratteli and Robinson 1987, 2.1.5) Suppose A is a C*-algebra without identity, and suppose $\widetilde{A}$ denote the algebra of pairs $\{(\alpha,a);\alpha\in\mathbb{C},a\in A\}$ with operations $(\alpha,a)+(\beta,b)=(\alpha+\beta,a+b),(\alpha,a)(\beta,b)=(\alpha\beta,\alpha b+\beta a+ab),(\alpha,a^*)=(\bar{\alpha},a^*)$. It follows then that the following*

$$\|(\alpha,a)\|=\sup\{\|\alpha b+ab\|,b\in A,\|B\|=1\}$$

give us a norm on $\widetilde{A}$ such that $\tilde{A}$ is a C-algebra with respect to it. We can identify A as the C*-subalgebra of $\widetilde{A}$ formed by the pairs $(0,a)$.*

Proof. To show that $\|(\alpha,a)\|$ is a norm, the triangle and product inequalities can be easily verified. Next, we show that $\|(\alpha,a)\|=0$ implies $\alpha=0,a=0$. However, $\|(0,a)\|=\|a\|$. Thus $\|(0,a)\|=0$ implies $a=0$. Therefore we can take $\alpha\neq 0$ and we can even make $\alpha=1$ by scalar multiplication. However,

$$\|b-ab\|\leq\|b\|\|(1,-a)\|$$

and hence $\|(1,-a)\|=0$ implies $b=ab$ for all $b\in A$. Through involution we then have $b=ba^*$ for all $b\in A$. Specifically, $a^*=aa^*=a$, and we get:

$$b=ab=ba$$

hence, a is an identity, which is a contradiction. The C*-norm property follows from:

$$\begin{aligned}\|(\alpha,a)\|^2&=\sup\left\{\|\alpha b+ab\|^2;b\in A,\|b\|=1\right\}\\&=\sup\{\|b^*(\bar{\alpha}\alpha b+\bar{\alpha}ab+\alpha a^*b+a^*ab)\|;b\in A,\|b\|=1\}\\&\leq\|(\alpha,a)^*(\alpha,a)\|\\&\leq\|(\alpha,a)^*\|\,\|(\alpha,a)\|.\end{aligned}$$

Thus,

$$\|(\alpha,a)\|\leq\|(\alpha,a)^*\|$$

□

Definition 2.11. *Suppose A is a C*-algebra without an identity. The C*-algebra $\widetilde{A}$ obtained by adjoining an identity $\mathbb{1}$ to A is defined as the algebra of pairs (α, a) as we described in Proposition 2.7. We denote $\alpha\mathbb{1} + a$ for the pair (α, a) and write $\widetilde{A} = \mathbb{C}1 + A$.*

Before we go on any further, we mention a couple of really nice theorems that make things intuitive.

Theorem 2.2. *(Bratteli and Robinson 1987, 2.1.10) Suppose A is a C*-algebra. Then A is isomorphic to a norm-closed, self-adjoint algebra of bounded operators on some Hilbert space.*

Theorem 2.3. *(Bratteli and Robinson 1987, 2.1.10) Suppose A is a commutative C*-algebra. Then A is isomorphic to the algebra $C_0(X)$ of continuous functions that go to infinity, over a locally compact Hausdorff space X.*

Definition 2.12. *A *-morphism between two *-algebras A, B is a map $\pi : A \to B$ that preserves the *-structure, that is:*

- $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,
- $\pi(AB) = \pi(A)\pi(B)$,
- $\pi(A^*) = \pi(A)^*$

for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$

Definition 2.13 (representation)**.** *A representation of a C*-algebra A is defined to be a pair, (H, π), of a complex Hilbert space H and a *-morphism, $\pi : A \to B(H)$, from A to the bounded operators over H. If π is a *-isomorphism between $A, \pi(A)$, then (H, π) is said to be faithful. That is it is faithful if and only if* $\ker\pi = \{0\}$

Now, it's time to start talking about states, and shortly, we will see an important example of using C*-algebras when it comes to physics.

Definition 2.14 (State)**.** *A linear functional over a C*-algebra A, $\omega : A \to \mathbb{C}$ is said to be positive if:*

$$\omega(aa^*) \geq 0$$

for all $a \in A$. ω is said to be a state if $\|\omega\| = 1$, where $\|\omega\| := \sup\{|\omega(a)|; \|a\| = 1\}$.

We can see the relevance of the notion of states by considering the following situation, (Bratteli and Robinson 1987). Suppose we have a representation, (H, π) of some C*-algebra A. Now let $\Omega \in H$ be any non-zero vector. Define ω_Ω through:

$$\omega_\Omega := \langle \Omega, \pi(a)\Omega \rangle$$

for all $a \in A$. It's obvious that ω_Ω is positive:

$$\omega_\Omega(a^*a) = \|\pi(a)\|^2 \geq 0$$

Furthermore, $\|\omega_\Omega\| = 1$ whenever $\|\Omega\| = 1$, so π is non-degenerate and ω_Ω is a state. Such states are called vector states of the representation (H, π). It can also be shown that every state is a vector state for some non-degenerate representation.(Bratteli and Robinson 1987).

At the heart of the connection between states and representations of a C*-algebra is the Gelfand–Naimark–Segal (GNS) construction. Given a state φ on $\mathscr{A}$, one defines a sesquilinear form via

$$\langle a, b \rangle = \varphi(b^*a) \quad \text{for all } a, b \in A.$$

The Cauchy–Schwarz inequality ensures that this form is positive semidefinite, and its null space

$$\mathscr{N}_\varphi = \{a \in A : \varphi(a^*a) = 0\}$$

is a closed left ideal. By forming the quotient $A/\mathscr{N}_\varphi$ and completing it with respect to the induced inner product, one obtains the Hilbert space H_φ. The representation $\pi_\varphi : A \to B(H_\varphi)$ is then given by

$$\pi_\varphi(a)[b] = [ab] \quad \text{for all } a, b \in A,$$

where $[b]$ is the class of b in $A/\mathscr{N}_\varphi$. And a distinguished cyclic vector $\Omega_\varphi = [1]$ satisfying

$$\varphi(a) = \langle \Omega_\varphi, \pi_\varphi(a)\Omega_\varphi \rangle \quad \text{for every } a \in A.$$

Different, non-unitarily equivalent GNS representations lead to distinct *superselection sectors*, each representing a different type of excitation or charge in the quantum system.

When a C*-algebra is represented on a Hilbert space and further closed in the weak (or strong) operator topology, it becomes a *von Neumann algebra* (or W*-algebra). For a more detailed review, see (Bratteli and Robinson 1987; Landsman 2017).

We now move on to see how C*-algebras lay the foundation of the physical theory. The mathematical structure of any dynamical system rests upon three main ingredients and the interaction between them. Observables (the sorts of measurements one can make), states

(accounting for possible configurations of the system), and dynamics (how the configurations or the observables evolve in time). With the C*-algebra formulation, these three ingredients are determined by a C*-algebra. The following are the postulates of an algebraic quantum theory:

1. **Observables** are self-adjoint (*Hermitian*) elements of some C*-algebra.
2. **States** are bounded linear functionals $\phi : A \to \mathbb{C}$ with $||\phi|| = 1$ and $\phi(x^*x) \geq 0$.
3. **Dynamics** are *-automorphisms[1] $\alpha \in \text{Aut}(A)$,

The collection of states $S(A)$ forms a convex set, and a key focus in many discussions is on *pure* states, which are the extreme points of $S(A)$. These states are often regarded as representing the "true" configurations of the system, as they encode the maximal possible information, whereas general states incorporate a degree of (classical) uncertainty. Given a state $\varphi \in S(A)$ and an observable $a \in A$, the quantity $\varphi(a)$ is interpreted as the expected value of the measurement outcome when the system is in state φ.

For an abelian C*-algebra $A \cong C(X)$, pure states correspond to evaluation functionals at points $x \in X$, while general states are associated with Radon measures on X. In the specific case of the C*-algebra of compact operators $K(H)$ on a Hilbert space H, the space of states can be characterized in terms of density matrices ρ, establishing the correspondence $\rho \longleftrightarrow \text{Tr}(\cdot\rho)$. This formulation aligns with the standard quantum mechanical framework, where states are represented by density matrices, and observables are evaluated via the trace operation.

A single automorphism $\alpha \in \text{Aut}(A)$ can be understood as representing a discrete time evolution. If a system is initially in the state φ at time $t = 0$, then after one time step, the state evolves to $\varphi \circ \alpha$. A continuous time evolution, on the other hand, is described by a homomorphism $\mathbb{R} \to \text{Aut}(A)$, given by $t \mapsto \alpha_t$. In this framework, if the system starts in the state φ at $t = 0$, then after time t, its state is given by $\varphi \circ \alpha_t$. When this homomorphism is continuous with respect to the norm topology, the family $\{\alpha_t\}$ possesses an infinitesimal generator δ, which is a (potentially unbounded) derivation of the C*-algebra A. We can identify this derivation as the Hamiltonian of the system, and the equation governing the time evolution which is generated by δ serves as a generalization of the Schrödinger equation (see (Bratteli and Robinson 1987)).

More broadly, one may consider *open dynamics*, which are governed by *quantum channels*. These are expressed in the Heisenberg picture as unital, completely positive maps $\alpha : A \to A$, where the evolution of a state is given by $\varphi \mapsto \varphi \circ \alpha$ (see (Paulsen 2002)). Notably, automorphisms, which we refer to as “unitary dynamics," provide examples of quantum channels.

[1] generally, unital completely positive (ucp) maps for open dynamics, (Paulsen 2002)

However, quantum channels generalize both states and dynamics: any state can also be viewed as a channel, defined by the mapping $a \mapsto \varphi(a)1_A$. We primarily consider unital completely positive maps as generalized dynamics (Elokl and Jones 2024).

Let us get to the core of the construction of our example, a spin system in the thermodynamic limit (Bratteli and Robinson 1997). We usually have a (countably infinite) set of sites (a lattice). We have a unital C*-algebra of localized observables, A_x at each site x. We also assume that each A_x is nuclear so we can ignore different types of tensor products. To make things more relatable, we usually think of $A_x \cong M_n(\mathbb{C})$, however, our results are more general.

Let $F \subseteq X$ be any finite subset; define the C*-algebra

$$A_F := \otimes_{x \in F} M_d(\mathbb{C}).$$

A_F is the tensor product algebra which has tensor factors indexed by the elements of F. For $F \subseteq G$, we get a natural inclusion $A_F \hookrightarrow A_G$, given by $a \mapsto 1^{\otimes G-F} \otimes a$. Define the C*-algebra:

$$A = \operatorname{colim}_{F \in \mathscr{F}(X)} A_F,$$

where $\mathscr{F}(X)$ is the partially ordered set of finite subsets, and the colimit is in the category of C*-algebras. To be less abstract, if we identify each A_F with its image in A_G for $F \subseteq G$, then we recognize this colimit as just the completion of the union algebra (see (Rørdam et al. 2000),for more technical details). Identifying each A_F with its image in A endows A with a distinguished family of subalgebras that are indexed by finite regions of X. If we take A with this family of subalgebras, we get a precise mathematical description of the local observables of the system. For two regions F, G with $F \cap G = \emptyset$, $[A_F, A_G] = 0$. This amounts to a connection between the algebraic relations of the observables and their "physical location in space".

The majority of the definitions work in great generality, even though spin systems constitute our main examples. We move on to abstract most of the features we introduced above through the following definition. This definition was recently used in the theory of topological order and symmetric quantum cellular automata (QCA) (Jones 2024), (Jones and Lim 2024; Jones et al. 2023; Tomba et al. 2023).It can be thought of as a discrete metric version of an algebraic quantum field theory (AQFT) in the sense of Haag and Kastler Haag and Kastler (1964); Haag (1996), but with no distinguished Hilbert space representation or a symmetry group.

Definition 2.15 (Discrete Nets of C*-algebras)**.** *(Elokl and Jones 2024)*

A discrete net of C-algebras comprises of a countable set X, a unital C*-algebra A called the quasi-local algebra, and associated to each finite subset $F \subseteq X$, a unital sub-algebra $A_F \subseteq A$ (called a local sub-algebra) such that*

1. *If $F \subseteq G$, then $A_F \subseteq A_G$.*

2. *IF* $F \cap G = \emptyset$*, then* $[A_F, A_G] = \emptyset$.

3. $\bigcup_{F \in \mathscr{F}(X)} A_F$ *is norm-dense in* A.

Now, for *any* subset $F \subseteq X$ (that is not necessarily finite), we are able to define A_F to be the C*-algebra generated by the collection of A_G where $G \subseteq F$ is finite, which allows us to make sense of the *quasi-local observables* localized in any arbitrary region F. Therefore, we have an assignment of a subalgebra of A to arbitrary subsets $\mathscr{P}(X)$ of X.

Next, we note that in practice, X typically has a metric space structure (e.g., the metric space $\mathbb{Z}^d \subseteq \mathbb{R}^d$), and we normally ask that their nets have additional assumptions that tie the algebra to the geometry (for instance, something like algebraic Haag duality in (Jones 2024)). However, the whole point here is that there is not a pre-existing geometry, and the purpose is to *derive* an intrinsic geometric structure from correlations or dynamics of the system (Elokl and Jones 2024).

Something else to keep in mind is that there is nothing fundamental about the unital assumption on A or the local algebras A_F, and we can use non-unital C*-algebras; however, we should likely require that the inclusions are non-degenerate in some sense. But the examples we bear in mind (spin systems and closely related examples) all possess local algebras A_F that are finite-dimensional, which are automatically unital. In the case of locally infinite-dimensional nets, it could be more useful to require the additional assumption that the local algebras are von Neumann algebras (which is automatically true in the locally finite-dimensional case), but this makes no difference in the discussion for our purposes.

We can think of a discrete net of C*-algebras as generalizing a spin system; however, the major difference (apart from the type of the local C*-algebras) crystallizes in *algebraic entanglement*. Let A be a general discrete net; for $F \cap G = \emptyset$, our inclusions give a natural map $A_F \otimes A_G \hookrightarrow A_{F \cup G}$. In a normal spin system, this is an isomorphism of algebras. However, in the case of a general net, this may fail to be injective or surjective. That is, products of operators localized in disjoint regions may be 0, and local operators may not factorize as sums of products of operators localized in F and G respectively. Physically, this last condition implies that we cannot determine local observables by observables localized at points. We discuss algebraic entanglement in more detail later on. Now we give some naturally occurring examples of discrete nets apart from ordinary spin systems.

1. **Symmetric spin systems**. Let $A := \otimes_{x \in X} B$ be a generalized spin system, and suppose we have an action of the group G on the local C*-algebra B. Then the diagonal action on A is called an *on-site global symmetry*. We can consider the C*-algebra A^G of G invariant operators, and the local operators A_F^G is just $A^G \cap A_F$. These systems generically exhibit algebraic entanglement. For a more concrete example, consider the following:

Example 2.12. *A real physical system that realizes symmetric spin systems with an on-site global symmetry is the Heisenberg model with global spin rotation symmetry (Sachdev 2011; Nielsen and Chuang 2000).*

Consider a system where $B = B(H)$ corresponds to the local observable algebra for a single spin, with $H = \mathbb{C}^{2S+1}$ for spin S. The total algebra $A = \bigotimes_{x \in X} B(H)$ describes the entire system. The on-site global symmetry corresponds to an action of a group G on each local spin space. A typical choice is $G = SU(2)$, which acts by rotating each spin independently but in the same manner across the entire system. The algebra of G-invariant observables, A^G, consists of operators that respect this symmetry, which include the exchange interactions of the Heisenberg Hamiltonian:

$$H = \sum_{\langle x,y \rangle} J_{xy} \mathbf{S}_x \cdot \mathbf{S}_y,$$

where J_{xy} are exchange coefficients and $\mathbf{S}_x$ are spin operators. The algebraic structure of A^G and its local restrictions A_F^G encode entanglement patterns due to the underlying symmetry, often leading to symmetry-protected topological order or algebraic entanglement.

2. **Categorical spin systems**. Considering higher categorical structures (unitary fusion categories, unitary braided fusion categories, etc.), we can construct nets of finite-dimensional C*-algebras. The following are the most studied of these *fusion spin chains* (Jones 2024). They show up not only as the local invariant operators under a (weak) Hopf or categorical symmetry, but also as the boundary algebras of 2+1D topological orders (Jones et al. 2023).

3. **AQFTs**. A (Euclidean) algebraic quantum field theory is defined through a net of C*-algebras B over a (complete) Riemannian manifold M, assigning C*-algebras to (bounded) open sets (Schlingemann 1999). Consider a geodesic triangulation S of M where the vertices are at least δ distance apart for some fixed $\delta > 0$. If we let X be the set of vertices of the triangulation, we can define a discrete net of C*-algebras A over X by defining, for any finite subset $F \subseteq X$, $A_F := B_{N_{\frac{\delta}{2}}(S(F))}$, where $S(F)$ is the sub-complex spanned by the vertices in F. This is a “discretization" of the original theory.

2.3 Relevant Experiments

In recent years, the pursuit of fault-tolerant quantum computation has led researchers to investigate phases of matter that exhibit topological order—phases characterized by long-range en-

tanglement and intrinsic robustness against local perturbations. Experimental breakthroughs have demonstrated that topologically ordered ground states, long the subject of theoretical study, can be physically realized on quantum processors. In particular, experiments have successfully prepared ground states that bear the same entanglement structure as those predicted by theoretical spin liquid models. At the core of these experiments is the preparation of highly entangled states that serve as analogues to the ground states of gapped quantum spin liquids. In these systems, the inherent protection offered by the non-local features of topological order makes them promising candidates for implementing error-resilient quantum information processing. These results have been reported in several studies, including those by (Satzinger et al. 2021; Iqbal et al. 2024; Xu and other 2024; Andersen et al. 2023).

Complementary to these results, in (Iqbal et al. 2024), the experimental advancement was extended by realizing non-abelian topological order on a trapped-ion processor. Their approach involved a carefully designed state-preparation protocol that encoded anyonic excitations within the processor's architecture. The experiment not only confirmed the existence of non-abelian anyons in a laboratory setting but also provided insight into the mechanisms by which topological features persist even when the hardware's spatial arrangement does not directly mimic the underlying theoretical lattice. This experimental success is particularly notable because it moves beyond mere simulation—by actual state preparation, the system intrinsically exhibits the same long-range order that characterizes its theoretical counterpart.

Another significant milestone was achieved in (Xu and other 2024), whose work on non-abelian braiding in a superconducting processor further emphasizes the experimental feasibility of harnessing topological order. In their study, they implemented a protocol that allowed for the manipulation and braiding of Fibonacci anyons—quasiparticles that serve as the basis for topologically protected quantum operations. This demonstration not only confirmed theoretical predictions regarding the braiding statistics of anyons but also underscored the potential for using such exotic quasiparticles in topological quantum computation.

In (Andersen et al. 2023), non-abelian braiding of graph vertices on a superconducting processor we realized. Their work emphasized that, even in systems where the physical architecture might impose certain geometric constraints, the intrinsic topological features of the prepared state can emerge robustly. The realization of such states on current-generation quantum hardware has profound implications for both quantum simulation and the practical realization of topologically protected quantum gates.

The experimental realizations described above are built on a solid theoretical foundation that has been established over decades of research into quantum many-body physics and topological phases. Central to these theoretical underpinnings is the concept of topological order, which was first rigorously formalized in (Kitaev 2006). In his seminal work, Kitaev introduced

exactly solved models that host anyonic excitations and described the nature of topological order in a framework that is robust against local perturbations (Kitaev 2006). These ideas have since been extended and refined by various studies, we refer the reader to (Zeng et al. 2019).

In contrast to traditional quantum simulation, where the goal is to emulate the properties of a many-body system on a classical computer, these experiments achieve state preparation on actual quantum hardware. This distinction is crucial because the process of preparing a state on a quantum processor inherently embeds the state within the dynamics and noise characteristics of the hardware (i.e., the characteristic geometry of the hardware). As such, the robustness of the topological features in these systems provides compelling evidence that the experimentally prepared states are genuinely reflective of an underlying geometry inherent to the states, which is the core motivation for our work here.

CHAPTER

3

CONTROLLED-DECAY COARSE STRUCTURES

Here we define the notion of controlled decay of a function $f : X \times X \to \mathbb{R}_+$ and how we can construct coarse structures defined on countable sets in terms of it. The main example will be to use the controlled decay of the correlation functions of a (classical or quantum) spin system, to which, for the moment, we give an in intuition. We start by taking observables *localized* at sites x and y and consider their covariance

$$\mathrm{cov}(a,b) := \frac{|\phi(ab) - \phi(a)\phi(b)|}{\|a\|\|b\|}$$

This quantity gives the statistical correlation of the observables a and b at different sites. If this happens to be 0, it means that the observables a and b are statistically independent, so we can think of the covariance as measuring how far these observables are from independence. This allows us to measure the maximal correlation between observables localized at sites x and y respectively taking the supremum of covariances between observables localized at different sites

$$C_\phi(x,y) := \sup_{a \in (A_x),\ b \in (A_y)} \frac{|\phi(ab) - \phi(a)\phi(b)|}{\|a\|\|b\|}.$$

3.1 Controlled Decay

The main idea here is that for states arising from *local* physical processes with respect to some metric d on X, localized observables should become decreasingly correlated as the distance between the sites increases. This is how we will operationally *define* the notion locality of a state with respect to a given metric d. As we noted before, this notion of locality of a state depends only on the bounded coarse structue of the metric and not on the local details of the metric, which is how coarse geometry comes into play. This gives rise to the following question: given a state on some discrete net of over X without any a-priori metric (i.e. geometry), what are all the coarse structures on X with respect to which a state is local?

We tackle such problem by abstracting away the physical details of states and correlations, and develop the theory for any given function $f : X \times X \to \mathbb{R}_+$. This way, we have the flexibility to apply our framework in other settings, more specifically, we also construct quantum dynamical coarse structures in Section **??**. Most of the following results are based on (Elokl and Jones 2024)

Definition 3.1. *Suppose $(X, \mathscr{E})$ is a coarse space and $f : X \times X \to \mathbb{R}_+$ is a function. Then f is said to have controlled decay with respect to $\mathscr{E}$ if for all $\epsilon > 0$, there exists a controlled set $E \in \mathscr{E}$ where $f(x, y) < \epsilon$ for all $(x, y) \notin E$.*

We should think of controlled decay as a way to formalize the notion that a function behaves well when we look from infinity, hence the name, controlled decay. The above definition allows us to describe a unique, "universal" coarse structure with respect to which a real-valued function f on $X \times X$ has controlled decay.

Theorem 3.1. *(Elokl and Jones 2024, 4.2) Consider any function $f : X \times X \to \mathbb{R}_+ \cup \{\infty\}$, then there is a coarse structure $\mathscr{E}_f$ with the property that any other coarse structure $\mathscr{E}$ on X, f has controlled decay with respect to $\mathscr{E}$ if and only if $\mathscr{E}_f \subseteq \mathscr{E}$.*

Proof. For a given $f : X \times X \to \mathbb{R}_+ \cup \{\infty\}$, and $\epsilon > 0$, define

$$E_{f,\epsilon} := \{(x, y) \in X \times X \ : \ f(x, y) \geq \epsilon\} \subseteq X \times X,$$

and define the monogenic coarse structure $\mathscr{E}_{f,\epsilon} := \langle E_{f,\epsilon} \rangle$. Then for $\epsilon \leq \epsilon'$, we get $\mathscr{E}_{\epsilon'} \subseteq \mathscr{E}_{\epsilon}$. Hence, we get an "order reversing" poset morphism from $\mathbb{R}_+$ to coarse structures on X.

Define the coarse structure $\mathscr{E}_f = \bigcup_{\epsilon > 0} \mathscr{E}_{f,\epsilon}$. We can see that this is a coarse structure, since the $\mathscr{E}_{f,\epsilon}$ are nested. Then, by construction, f will have controlled decay with respect to any coarse structure that contains $\mathscr{E}_f$.

Now, for the other direction, suppose f has controlled decay with respect to the coarse structure $\mathscr{E}$. Then, for each $\epsilon > 0$, there is a controlled set $E \in \mathscr{E}$ such that $E_{f,\epsilon} = \{(x, y) \ : f(x, y) \geq \epsilon\} \subseteq E$. Thus $\mathscr{E}_f = \bigcup_{\epsilon > 0} \mathscr{E}_{f,\epsilon} \subseteq \mathscr{E}$. □

Remark 3.1. *We note that the coarse structures $\mathscr{E}_{f,\epsilon}$ in the above proof will prove to be useful. For a given $\epsilon > 0$, we look at the graph $G_{f,\epsilon}$, whose vertices are elements of X and there exists an edge between $x, y \in X$ if $(x, y) \in E_{f,\epsilon}$. Endow $G_{f,\epsilon}$ with the path metric, if $G_{f,\epsilon}$ is connected, then it is is coarsely equivalent to $(X, \mathscr{E}_{f,\epsilon})$. We again note how important graphs play an important role on our coarse structures.*

Proposition 3.1. *$\mathscr{E}_f$ is monogenic if and only if $\mathscr{E}_f = \mathscr{E}_{f,\epsilon}$ for some $\epsilon > 0$.*

Proof. If $\mathscr{E}_f = \mathscr{E}_{f,\epsilon}$ then $\mathscr{E}_f$ is obviously monogenic. On the other hand, suppose that $\mathscr{E}_f$ is monogenic, with generator E. Then since $\mathscr{E}_f = \bigcup_\epsilon \mathscr{E}_{f,\epsilon}$, we have $E \in \langle \mathscr{E}_{f,\epsilon_0} \rangle$ for some $\epsilon_0 > 0$. Thus $\mathscr{E}_f = \langle E \rangle \subseteq \mathscr{E}_{f,\epsilon_0} \subseteq \mathscr{E}_f$, giving an equality $\mathscr{E}_{f,\epsilon_0} = \mathscr{E}_f$. □

Definition 3.2. *A function $f \times f \to \mathbb{R}_+$ is said to be coarsely stable if $\mathscr{E}_f$ is monogenic, or equivalently, for some $\epsilon > 0$, the bounded coarse structure of the graph $G_{f,\epsilon}$ is $\mathscr{E}_f$.*

Note that if ϵ is stable, then ϵ' is stable for any $\epsilon' < \epsilon$. We call the corresponding interval the *stable range*. We note that stability is a *property of the coarse structure $\mathscr{E}_f$*.

Suppose f is stable, then by choosing an ϵ we can recover an explicit metric model for $\mathscr{E}_f$ given by the path metric on $G_{f,\epsilon}$. Notice that the actual metric space is heavily dependent on ϵ in general, but the hope is that they will all be *coarsely equivalent* for a sufficiently small ϵ. Now, given a function $f : X \times X \to \mathbb{R}_+$, if $\mathscr{E}_f$ is monogenic, then the set $\{\epsilon > 0 : \mathscr{E}_f = \mathscr{E}_{f,\epsilon}\}$ is called the *stable range* for f. We also note that if $\mathscr{E}_f$ is monogenic, then it being proper is equivalent to $G_{f,\epsilon}$ being locally finite. It is evident, again, how understanding the coarse structures of graphs is at the heart of understanding our results. We follow with some instructive remarks.

Lower semi-continuity. Take the complete metric space $B_+(X \times X)$ of bounded, non-negative, real-valued functions on $X \times X$ taking the metric to be the sup norm, that is:

$$d(f, g) := \sup_{(x,y)\in X\times X} |f(x, y) - g(x, y)|.$$

We would like it if the the assignment $f \mapsto \mathscr{E}_f$ was continuous in some sense, for instance, if it is possible to extract information about $\mathscr{E}_f$ from coarse structures $\mathscr{E}_{f_\lambda}$, where $f_\lambda \to f$ is some net of functions converging to f. The following theorem allows us to interpret the map $B_+(X \times X) \to \mathrm{Coa}(X)$, as lower semi-continuity, where the latter denotes the set of coarse structures on X naturally viewed as a poset with inclusion.

Theorem 3.2. *(Elokl and Jones 2024) Suppose Λ is a directed set and let $f_\lambda \to f$ be a convergent net over Λ in $B_+(X \times X)$, then $\mathscr{E}_f \subset \langle \{\mathscr{E}_{f_\lambda}\}_{\lambda\in\Lambda} \rangle$.*

Proof. Suppose $\|f - g\| < \delta$. Note then that for any $\epsilon > \delta$ $E_{f,\epsilon} \subseteq E_{g,\epsilon-\delta}$ for all ϵ, therefore $\mathscr{E}_{f,\epsilon} \subseteq \mathscr{E}_g$. Now, for every $\epsilon > 0$, pick some λ_ϵ with $d(f, f_{\lambda_\epsilon}) < \epsilon$. Then we get,

$$\mathscr{E}_f \subseteq \langle \{\mathscr{E}_{f_{\lambda_\epsilon}}\}_\epsilon \rangle \subseteq \langle \{\mathscr{E}_{f_\lambda}\}_\lambda \rangle$$

□

Higher-order controlled decay. It is common in literature to find decay results for correlators that not only involve operators localized at a pair of sites, but localized at multiple sites, maybe even regions. These are called *higher-order correlators*. A good way to think about it would be the invariance of the coarse strucutre under "coarse graining". Consider a discrete net of C*-algebras over X, A. If (X,d) is a metric space, then for every $R>0$ we can "coarse-grain" the system. We do this by replacing the algebra A_x of operators of a point with $A_{N_R(x)}$, the algebra of operators localized in an R-ball around x. Then ϕ is said to have higher decay of correlations if for all $R>0$, the new rescaled correlation functions have controlled decay.

In most cases of interest, physical states satisfy this property with respect to the given metric. However, it is possible to have states which would be local with respect to the metric, but after coarse graining we obtain correlations that appear non-local. We are then lead to define a new type of coarse structure which is universal with respect to higher order decay of correlations. Again, we abstract away the physics and treat this at the a general level of functions.

Definition 3.3. *Let $\widetilde{f}:\mathscr{P}(X)\times\mathscr{P}(X)\to\mathbb{R}_+$ be a function, and suppose $\mathscr{E}$ is a coarse structure on X. Then $\widetilde{f}$ is said to have higher controlled decay with respect to $\mathscr{E}$ if for every $E\in\mathscr{E}$, the function $\widetilde{f}_E:X\times X\to\mathbb{R}_+$ defined by $\widetilde{f}_E(x,y):=\widetilde{f}(E[x],E[y])$ has controlled decay.*

Theorem 3.3. *There exists a unique coarse structure, $\mathscr{E}_{\widetilde{f}}$, with the property that $\widetilde{f}$ has higher-order controlled decay with respect to any other coarse structure $\mathscr{E}$ if and only if, $\mathscr{E}_{\widetilde{f}}\subseteq\mathscr{E}$.*

Proof. Let D be the diagonal of X, define $f:=\widetilde{f}_D:X\times X\to\mathbb{R}_+$. Then set $\mathscr{E}_0=\mathscr{E}_f$, and

$$\mathscr{E}_1 := \langle \bigcup_{E\in\mathscr{E}_0} \mathscr{E}_{\widetilde{f}_E} \rangle.$$

inductively, we define

$$\mathscr{E}_{n+1} := \langle \bigcup_{E\in\mathscr{E}_n} \mathscr{E}_{\widetilde{f}_E} \rangle.$$

Then $\mathscr{E}_f=\mathscr{E}_0\subseteq\mathscr{E}_1\subseteq\dots$ is an increasing sequence of coarse structures and we set

$$\mathscr{E}_{\widetilde{f}}=\bigcup_n \mathscr{E}_n$$

Note that $\widetilde{f}$ has higher-order controlled decay with respect to $\mathscr{E}_{\widetilde{f}}$. Indeed, for $E\in\mathscr{E}_{\widetilde{f}}$, there

exists some n such that $E \in \mathscr{E}_n$. By construction $\mathscr{E}_{\widetilde{f}_E} \in \mathscr{E}_{n+1} \subseteq \mathscr{E}_{\widetilde{f}}$, hence $\widetilde{f}_E$ has controlled decay with respect to $\mathscr{E}_{\widetilde{f}}$.

Now, suppose $\mathscr{E}$ is any coarse structure on X where $\widetilde{f}$ has higher-order controlled decay. Then using the diagonal controlled set in $\mathscr{E}_{\widetilde{f}}$, we see that f has controlled decay with respect to $\mathscr{E}_{\widetilde{f}}$ and thus $\mathscr{E}_0 = \mathscr{E}_f \subseteq \mathscr{E}$. And, for any $E \in \mathscr{E}_0$, $\widetilde{f}_E$ has controlled decay so $\mathscr{E}_1 \subseteq \mathscr{E}$. Continuing inductively we see $\mathscr{E}_{\widetilde{f}} \subseteq \mathscr{E}$ as desired. □

3.2 Asymptotic Reasoning

Here we aim to show how feasible is our construction when it comes to real, physical systems. Our definition of a coarse structure above for a function $f : X \times X$ works splendidly in theory. However, the question arises: what obstacles will we encounter trying to employ it and how can we tackle them?

First of all, we note that our formalism is based on asymptotic reasoning (Batterman 2002), which is common in condensed matter physics. The coarse structures we defined are mathematical idealizations, and are not a literal description of exact features of the physical systems we are modeling; these will normally have only finitely many degrees of freedom. Our coarse structures, on the other hand, are witnessed only when there are infinitely many degrees of freedom. However, such idealization is necessary to rigorously define concepts such as phase transitions.

We have implicitly assumed that the set X is infinite, as in the thermodynamic limit of a spin system, for us to get non-trivial coarse structures. However, real, physical systems to which we would like to apply our theory (e.g. states prepared on a quantum processor) will be finite dimensional. Specifically, they will have finite number of sites X. This, again, leads us to the question: how can we make since of our constructions in finite systems.

We can approach this question one way. We can look at coarse invariants and this brings us back to section 2.1.2. We can use Theorem 2.1 and attempt to estimate the asymptotic dimension of the coarse space in the thermodynamic limit on a large but finite system through finding the smallest k making this limit work “approximately". Now, let X be finite, and we possess some, empirically determined, function $f : X \times X \to \mathbb{R}_+$ via some measurements (f could simply be the correlations between observables localized at some sites). We choose some $\epsilon > 0$, and construct the graph $G_{f,\epsilon}$ where the vertices are the elements of X and (x, y) is an edge if $f(x, y) > \epsilon$. We then calculate the growth function $\gamma_\epsilon(r)$ for this graph and look at

$$\frac{\gamma_\epsilon(r)}{r^{k+1}}.$$

In the thermodynamic limit, we will find the values of k that make this ratio go to 0 as $r \to \infty$, then we would try to find the smallest such k. Taking such approach literally in the finite limit could lead to some problems. Suppose, for instance, that the smallest value of f is non-zero, once ϵ gets below this value, $G_{f,\epsilon}$ degenerates to a complete graph. Similarly, since each graph is finite, as $r \to \infty$ then $\gamma_\epsilon(r)$ becomes constant after r reaches the diameter of $G_{f,\epsilon,}$. Hence, $\frac{\gamma_\epsilon(r)}{r^{k+1}} \to 0$ for *any k*. Thus it will be necessary to look in some "intermediate range" of ϵ and r to see the most persistent features of these quantities, similar to one would do in persistent homology used in topological data analysis. So, we refer back to section 2.1.1 for a brief intro to coarse cohomology and to (Roe 2003, Ch 5) for a detailed review.

3.3 Coarse geometry for Correlation functions

We mentioned before that our constructions work for a general function $f : X \times X \to \mathbb{R}_+$. In this section we apply our results to correlation functions and see how coarse geometry answers our questions in the introduction.

Suppose A is a discrete net of C*-algebras over F, let B be any Banach algebra, then $\mathscr{B}_1(A \to B)$ denotes the space of linear maps from A to B with norm one. Then for $\phi \in \mathscr{B}_1(A \to B)$ define the 2-point correlation function,

$$C_\phi(x,y) = \sup_{a \in (A_x)_1, b \in (A_y)_1} \|\phi(ab) - \phi(a)\phi(b)\|.$$

and the higher order correlation function,

$$\widetilde{C}_\phi(F,G) = \sup_{a \in (A_F)_1, b \in (A_G)_1} \|\phi(ab) - \phi(a)\phi(b)\|.$$

The main case of interest will be when B is a C*-algebra and ϕ is a ucp map (quantum channel), even more specifically, when $B = \mathbb{C}$ such that ϕ is a state. In such case we interpret $C_\phi(x,y)$ to be the maximal correlation of observables localized at the sites x and y respectively.

Definition 3.4. *We define the correlation coarse structures by* $\mathscr{E}_\phi := \mathscr{E}_{C_\phi}$ *and* $\mathscr{E}_{\tilde{\phi}} := \mathscr{E}_{\widetilde{C}_\phi}$

We have the following inclusions:

$$\begin{array}{ccc} \mathscr{E}_\phi & \subseteq & \mathscr{E}_{\tilde{\phi}} \\ \cap\mathrel{|} & & \cap\mathrel{|} \\ (\mathscr{E}_\phi)_{\mathrm{con}} & \subseteq & (\mathscr{E}_{\tilde{\phi}})_{\mathrm{con}} \end{array}$$

where the bottom row is the connected quotients in the sense of Example 2.6.

The idea is that many states associated with a Hamiltonian, local with respect to some metric (X, d), would have (higher order) decay of correlations. We are being loose with the term "local" for Hamiltonian here, but we typically mean the interaction terms become smaller in norm as the support sets grow in diameter, usually with some specified bound. In the same sense, we will say that a state is *local* with respect to some coarse structure to mean the coarse structure contains either $\mathcal{E}_\phi$ or $\mathcal{E}_{\tilde{\phi}}$. Some of the motivating examples are listed below.

Example 3.1. ***Ground states of gapped Hamiltonains*** *For a local Hamiltonian of a quantum spin system on a discrete metric space, if H has a unique ground state then this state will generically have correlations with (higher order) controlled decay with respect to the metric (Hastings and Koma 2006; Nachtergaele and Sims 2006). The function C_ϕ will generically have exponential decay. A subclass of gapped Hamiltonains are those arising from commuting projector Hamiltonians with local topological order (Bravyi et al. 2010). The correlations of these ground states will typically have controlled support. To give a more concrete picture with more familiar terms. Consider:*

Matrix product states. *Matrix Product States (MPS) are canonical examples of states exhibiting exponential decay of correlations (Orús 2014), especially when associated with gapped, one-dimensional local Hamiltonians (Perez-Garcia et al. 2007; Schollwöck 2011). Consider a translation-invariant, infinite MPS with finite bond dimension D. Such a state φ is represented as*

$$|\varphi\rangle = \sum_{\{s_i\}} \mathrm{Tr}(A^{s_1} A^{s_2} \dots A^{s_N}) |s_1, s_2, \dots, s_N\rangle,$$

where the matrices $\{A^s\}_{s=1}^d$ are $D \times D$ matrices indexed by the local physical spin states $s = 1, \dots, d$, and d is the local Hilbert-space dimension (for spin-$\frac{1}{2}$, typically $d = 2$). The state is translation-invariant if the matrices A^s are identical at each site (Fannes et al. 1992).

Exponential Decay of Correlations

An essential feature of MPS with finite bond dimension and a spectral gap is their inherent exponential decay of correlations. Specifically, two-point correlation functions of local observables O_i, O_j at positions i, j satisfy the bound

$$|\langle O_i O_j \rangle_\varphi - \langle O_i \rangle_\varphi \langle O_j \rangle_\varphi| \leq M_\varphi e^{-|i-j|/\xi},$$

where M_φ is a constant dependent on the observables and the state, and ξ is the correlation length determined by the spectral gap of the MPS transfer operator (Perez-Garcia et al. 2007; Hastings 2007). The correlation length ξ explicitly characterizes the exponential decay.

MPS naturally arise as ground states of gapped, local, one-dimensional Hamiltonians, such as the Affleck-Kennedy-Lieb-Tasaki (AKLT) model or general gapped phases of quantum spin

chains (Affleck et al. 1988; Hastings 2007). A unique, gapped ground state of a local Hamiltonian is known to be efficiently approximable by an MPS, reflecting their intimate connection with states exhibiting exponential decay of correlations (Hastings 2007).

Concrete Example: AKLT Model

The AKLT state provides an explicit example of an exact MPS ground state, corresponding to the unique ground state of the following Hamiltonian defined on a spin-1 chain (Affleck et al. 1988):

$$H_{AKLT} = \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right].$$

This Hamiltonian has a finite spectral gap and a ground state represented exactly by an MPS with bond dimension $D = 2$. Consequently, the AKLT ground state explicitly exhibits exponentially decaying spin-spin correlations.

GHZ-state

The GHZ-state of N-1/2 is defined by:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N} \right)$$

where $|0\rangle$ and $|1\rangle$ are eigen states of the σ_z Pauli operator, for instance (Greenberger et al. 1989). This is a highly entangled quantum state of the N spins, which has some non-trivial entanglement properties (e.g. it violates certain N-partite Bell inequalities). Still, this state can be represented exactly by a MPS with bond dimension $D = 2$ and periodic boundary conditions.

Example 3.2. ***Equilibrium states****. In classical or quantum spin systems with a local Hamiltonian on a discrete metric space, "thermal equilibrium" states (Gibbs states or KMS states) have controlled (typically exponential) decay at low and high temperatures Araki, Gibbs states of a one dimensional quantum lattice. Near critical temperatures, however, the correlations of equilibrium states tend to decay super exponentially. The decay class is usually algebraic, and is indicative of scale-free behavior (Bluhm et al. 2022a; Fröhlich and Ueltschi 2015a; Kliesch et al. 2014).*

To give more concrete instances of equilibrium states we look at the following.

In classical and quantum spin systems with local Hamiltonians on discrete metric spaces, the behavior of correlation functions in thermal equilibrium states varies significantly with temperature. Below are specific examples illustrating these correlations at different temperature regimes:

- ***Two-Dimensional XY Model (Berezinskii–Kosterlitz–Thouless Transition):*** *In the two-dimensional XY model, near the Berezinskii–Kosterlitz–Thouless (BKT) transition temperature, correlations decay algebraically, indicating long-range order without spontaneous*

magnetization. Approaching the critical temperature, the correlation functions follow a power-law decay, reflecting scale-free behavior characteristic of critical phenomena (Fröhlich and Ueltschi 2015b; Bluhm et al. 2022b).

- ***Ising Model at Critical Temperature:*** *At the critical temperature of the two-dimensional Ising model, spin-spin correlation functions decay algebraically as*

$$C(r) \sim r^{-\eta},$$

where η is the critical exponent (Onsager 1944).

Example 3.3. ***States from locality preserving channels*** *This example highlights a key feature of our formalism. In the previous examples, the states were associated to some Hamiltonian. However, we can easily construct states local with respect to a coarse structure without reference to any Hamiltonian. Suppose A is a discrete net over X and let $\mathscr{E}$ be a fixed coarse structure. If $\Psi : A \to A$ is a locality preserving channel with respect to $\mathscr{E}$ (See Definion 5.2), then in [Corollary 5.2], we show that for any product state ϕ, $\mathscr{E}_{\widetilde{\phi\circ\Psi}} \subseteq \mathscr{E}$. Locality preserving channels are easy to construct (e.g. (quantum) cellular automata or even more concretely, finite depth circuits, see Example 3.7), hence, we can easily construct a variety of states whose coarse structures are (at least contained in) any desired coarse structure without reyling on any a Hamiltonian.*

Example 3.4. ***Spin systems on Cayley graphs****. Suppose we have a finitely-generated discrete group, Γ. Recall that Γ has a canonical coarse structure arising from a Cayley graph for the group. We choose a finite generating set S, and connect points x and y by an edge if there is a generator $g \in S$ such that $y = gx$. We know then that the coarse structure for the path metric on this graph is independent of the choice of finite generating set, and we denote it $\mathscr{E}_\Gamma$. So assume we have picked some S once and for all, and denote the associated metric by d_Γ.*

I this setting, there is a right action of Γ on Γ, that is defined by $R_g(h) := hg$. We note that R_g are isometries of (Γ, d_Γ)

Now, consider A, a discrete net of C-algebras over Γ, the latter is viewed as a set. An action $\Gamma \to Aut(A)$ is said to be covariant if $g(A_F) \subseteq A_{R_{g^{-1}}(F)}$. The main example is a generalized spin system $A := \otimes_{x\in\Gamma} A_x$, where A_x are all the same, and Γ acts by permuting the tensor factors on the right.*

We can use the following theorem to construct states that are local with respect to $\mathscr{E}_\Gamma$ by considering a state for a spin system on a Cayley graph invariant under translation symmetry.

Theorem 3.4. *(Elokl and Jones 2024) Suppose Γ is a finitely-generated, discrete group and let A be locally-finite dimensional discrete covariant net of C*-algebras over Γ. For any Γ-invariant*

factor state ϕ on A, $\mathcal{E}_{\widetilde{\phi}} \subseteq \mathcal{E}_\Gamma$. Specifically, given ϕ, the unique ground state for a translation invariant Hamiltonian on A, then $\mathcal{E}_{\widetilde{\phi}} \subseteq \mathcal{E}_\Gamma$.

Proof. For $R \geq 0$, define $E_R := \{(x,y) \in \Gamma \times \Gamma \ : \ d_\Gamma(x,y) \leq R\}$. It suffices to verify $(C_\phi)_{E_R}$ has controlled decay for all $R \geq 0$. Let $R, \epsilon > 0$. Suppose $e \in \Gamma$ is the identity, and pick a basis $\{a_1, \ldots, a_n\}$ of $A_{N_R(e)}$ where $\|a_i\| \leq 1$. Since all norms on a finite-dimensional C*-algebra are equivalent, there is some constant $D > 0$ such that if $a = \sum \lambda_i a_i$, $\sum_i |\lambda_i| \leq D\|a\|$.

Recall that a net of C*-algebras in our sense gives us a *quasi-local* C*-algebra in the sense of (Bratteli and Robinson 1987) where the poset in question is the set $\mathscr{P}(\Gamma)$ of all subsets of Γ, and the $\perp$ relation is disjointness. Then by (Bratteli and Robinson 1987, 2.6.5), since ϕ is a factor state, there are finite subsets $F_1, \ldots, F_n$ such that $\sup_{b \in (A_{F_i^c})_1} |\phi(a_i b) - \phi(a_i)\phi(b)| < \frac{\epsilon}{D}$.

Pick $S > 0$ sufficiently big such that $F_1, \ldots, F_n, N_R(e) \subseteq N_S(e)$. Then, for any $a = \sum \lambda_i a_i \in (A_{N_R(e)})_1$ and $b \in A_{N_S(e)^c}$, we get

$$|\phi(ab) - \phi(a)\phi(b)| \leq \left(\sum |\lambda_i|\right) \frac{\epsilon}{D} < \epsilon.$$

Thus, if we pick $T > R + S$, we get for any $x \in \Gamma$ with $d_\Gamma(e,x) > T$,

$$(C_\phi)_{E_R}(e,x) < \epsilon$$

For any $y, z \in \Gamma$ such that $d_\Gamma(y,z) > T$, then $d_\Gamma(e, zy^{-1}) > T$, and by translation invariance of ϕ we get

$$(C_\phi)_{E_R}(y,z) = (C_\phi)_{E_R}(e, zy^{-1}) < \epsilon.$$

For the last statement, if a ground state of a Hamiltonian H is unique, then it is pure, hence a factor state (Bratteli and Robinson 1987). Thus, if H is translation invariant, so is its unique ground state. □

Remark 3.2. *So far, in all the above cases, we have results that guarantee that ϕ is local with respect to some coarse structure $\mathscr{F}$. However, we can easily find sufficient conditions that yield the stronger statement $\mathscr{F} = \mathcal{E}_\phi$. For instance, if the state has some non-trivial uniform degree of entanglement between close sites x, y, then this should work. More Concretely, suppose we are working with coarse structure $\mathscr{F}$ corresponding to the quasi-geodesic metric space (X,d), and that there exists $R > 0$ and $c > 0$ such that if $d(x,y) \leq R$ then $C_\phi(x,y) \geq c$. Then we get that $\mathscr{F} \subseteq \mathcal{E}_\phi$. This condition will occur generically, for instance, in states built through finite depth quantum circuits as in Example 3.3.*

Close States. The space of bounded linear maps $\mathscr{B}_1(A \to B)$ with norm 1 has a natural topology,

with metric given by the norm

$$\|\phi-\psi\| = \sup_{a\in(A)_1} \|\phi(a)-\psi(a)\|.$$

We prove that the map $\mathscr{B}_1(A\to B)\to \mathrm{BdFun}(X\times X)$ is continuous, where the latter is the space of bounded real-valued functions on $X\times X$ with the sup norm $\|\cdot\|_\infty$. When taken together with Theorem 3.2 will give a lower semicontinuity of the map $\phi\mapsto\mathscr{E}_\phi$.

Proposition 3.2. *Suppose $\phi,\psi\in\mathscr{B}_1(A\to B)$ with $\|\phi-\psi\|\le\delta$. Then $\|C_\phi - C_\psi\|_\infty \le 3\delta$. Consequently,*

1. *$\phi\mapsto C_\phi$ is continuous and $\phi\mapsto\mathscr{E}_\phi$ is lower semicontinuous in the sense of Theorem 3.2.*
2. *If $\epsilon-3\delta>0$, then $\mathscr{E}_{\epsilon+3\delta,\phi}\subseteq\mathscr{E}_{\epsilon,\psi}\subseteq\mathscr{E}_{\epsilon-3\delta,\phi}$.*

Proof. For any $a\in(A_x)_1, b\in(A_y)_1$, we have

$$\begin{aligned}\|\phi(ab)-\phi(a)\phi(b)\| &\le \|\phi(ab)-\psi(ab)\|+\|\phi(a)\phi(b)-\psi(a)\psi(b)\|+\|\psi(ab)-\psi(a)\psi(b)\|\\ &\le 3\delta+\|\psi(ab)-\psi(a)\psi(b)\|\end{aligned}$$

Now, take the supremum over $a\in(A_x)_1, b\in(A_y)_1$, we get

$$C_\phi(x,y)\le C_\psi(x,y)+3\delta$$

By switching ϕ and ψ and putting these together, we obtain

$$\|C_\phi - C_\psi\|\le 3\delta.$$

If $(x,y)\in E_\epsilon^\phi=\{(x,y)\in X\times X\ : C_\phi(x,y)\ge\epsilon\}$, then $C_\phi(x,y)\ge C_\phi(x,y)-3\delta$, and thus $(x,y)\in E^\psi_{\epsilon-3\delta}$. Thus

$$\mathscr{E}_{\epsilon,\phi}=\langle E^\phi_\epsilon\rangle\subseteq\langle E^\psi_{\epsilon-3\delta}\rangle=\mathscr{E}_{\epsilon-3\delta,\psi}.$$

We can get the other inclusion by switching ϕ and ψ.

□

Corollary 3.1. *Suppose $\phi,\psi\in\mathscr{B}_1(A\to B)$ with $\mathscr{E}_\phi$ and $\mathscr{E}_\psi$ being monogenic, and let L be the length of the largest (open) interval contained in the intersection of the stable ranges of ϕ and ψ. If $\|\phi-\psi\|<\frac{L}{6}$, then $\mathscr{E}_\phi=\mathscr{E}_\psi$.*

Proof. Let $\delta = \|\phi - \psi\|$ and choose ϵ in the intersection of the two stable ranges such that $(\epsilon - 3\delta, \epsilon + 3\delta) \subseteq (0, L)$. Then from the above proposition, we see

$$\mathscr{E}_\phi = \mathscr{E}_{\epsilon+3\delta,\phi} \subseteq \mathscr{E}_{\epsilon,\psi} \subseteq \mathscr{E}_{\epsilon-3\delta,\phi} = \mathscr{E}_\phi,$$

hence $\mathscr{E}_\phi = \mathscr{E}_{\epsilon,\psi} = \mathscr{E}_\psi$. □

It would be nice if the states are sufficiently close their coarse structures agreed on the nose. However, as we saw earlier, the above proposition does not give us enough information to properly compare coarse structures. We will see that with some additional assumptions we can conclude such a result. For any state ϕ, let ϕ^u denote the state $\phi(u \cdot u^*)$.

Remark 3.3. ***Choice of tensor product decomposition for states on spin systems****. We note that the correlation coarse structure depends heavily on our choice of local sub-algebras. So far, we have taken the local algebras as a given. However, we can consider different tensor product decompositions of spin systems. In concrete contexts (e.g. simulating states on quantum processors) it is easy to make a natural choice for the tensor product decomposition. However, it is interesting to ask how different things can get with respect to a different tensor product decomposition. Suppose $\phi, \psi \in \mathscr{B}_1(A \to B)$, and suppose we have an automorphism $\alpha : A \to A$ with $\phi \circ \alpha = \psi$. Now, define a new point-structure over the same base set X by $\{A'_x = \alpha(A_x)\}_{x\in X}$. Note that this site structure is local (resp. generating) if and only if $\{A_x\}_{x\in X}$ is. With this new site structure, we compute the following,*

$$\begin{aligned} C'_\phi(x,y) &= \sup_{a'\in(A'_x)_1, b'\in(A'_y)_1} \|\phi(a'b') - \phi(a')\phi(b')\| \\ &= \sup_{a\in(A_x)_1, b\in(A_y)_1} \|\phi(\alpha(ab)) - \phi(\alpha(a))\phi(\alpha(b))\| \\ &= C_\psi(x,y) \end{aligned}$$

Hence, with respect to the new site structure $\{A'_x\}$, the two-point correlation function for ϕ looks exactly like the correlation function for ψ with respect to the original site structure.

Let us emphasize that two different site structures $\{A'_x\}_{x\in X}$ and $\{A_x\}_{x\in X}$ for the same state ϕ can result in massively different coarse structures and decay rates. To be specific, we could have a state which from one site structure looks like a 3D state with 0 correlation length, while with respect to another site structure, it looks like a 2D state with infinite correlation length (for formal definitions of correlation length, see 6.4)

The obvious way to put this is that the automorphism α taking ϕ to ψ is highly "non-local". That is, the point-like operators $a \in A'_x$ are expected to be wildly non-local with respect to

the original site structure. We might think that if we assume some sort of "locality-preserving" condition on α that maybe we would fare better in this regard. See chapter 5 for further discussion.

3.4 Coarse structures for dynamics

We have a built-in algebraic version of locality in our definition of discrete nets of algebras. This is a way to formalize the intuition that for observables localized in different regions of space, for *any* state of the system we expect to be able to form a joint probability distribution for the measurement outcomes (i.e. the elements of the algebra commute). Thus, for quantum systems, we expect spatial separation to be connected to the commutativity of localized operators.

The *Lieb-Robinson bound* for quantum spin systems, (Lieb and Robinson 1972; Hastings 2004; Nachtergaele and Sims 2006), is an important dynamical manifestation of this idea. Let H be a (for example, strictly) local Hamiltonian for a spin system on a square lattice. Suppose α_t is the automorphism of the quasi-local algebra that is obtained by evolving the system for time t. Then there exists positive constants C_t, A such that for $a \in A_x, b \in A_y$

$$\|[\alpha_t(a), b]\| \leq C_t e^{-Ad(x,y)}$$

This says that the "spread of information" has a speed limit in quantum spin systems, and has a big variety of important theoretical consequences in mathematical physics. If think of a channel $\alpha : A \to A$ as a time evolution, then we can define the following:

Definition 3.5. *Suppose A is a discrete net of algebras and $\alpha \in \mathscr{B}_1(A \to A)$. Define:*

$$Q_\alpha(x, y) := \sup_{a \in (A_x)_1, b \in (A_y)_1} \|[\alpha(a), b]\|.$$

Then the dynamical coarse structure of α is defined as $\mathscr{E}_\alpha := \mathscr{E}_{Q_\alpha}$.

We are often interested in unitary dynamics, where the channel $\alpha \in \mathrm{Aut}(A)$. In this case, $Q_\alpha(x, y) = Q_{\alpha^{-1}}(x, y)$, hence $\mathscr{E}_\alpha = \mathscr{E}_{\alpha^{-1}}$.

Example 3.5. ***Local time evolution on a spin system.*** *Suppose (X, d) is a discrete metric space, and suppose for each $x \in X$, $A_x \cong M_{n_x}(\mathbb{C})$ for some positive integer n_x. Define $A = \otimes_{x \in X} A_x$. Now, assume we have a local time evolution $t \mapsto \alpha_t$, which is derived from integrating a local Hamiltonian (for example, satisfying the hypothesis of (Nachtergaele et al. 2006)).*

Then, the Lieb-Robinson bounds (e.g. (Nachtergaele et al. 2006, Theorem 2.1)) show that for any time $t \geq 0$, we get the inclusion of coarse structures $\mathscr{E}_{\alpha_t} \subseteq \mathscr{E}_{(X,d)}$. That is, Q_{α_t} decays asymptotically with respect to the metric d. More concretely,

Consider a quantum spin system arranged on a discrete lattice Λ (e.g., a finite subset of $\mathbb{Z}^d$). Each site $x \in \Lambda$ is associated with a finite-dimensional Hilbert space, for instance, a spin-$\frac{1}{2}$ system described by the algebra of observables

$$A_x = M_2(\mathbb{C}),$$

the algebra of 2×2 complex matrices. The algebra of observables for the entire system is the tensor product algebra

$$A_\Lambda = \bigotimes_{x\in\Lambda} M_2(\mathbb{C}).$$

Given a local Hamiltonian defined as a finite-range interaction:

$$H_\Lambda = \sum_{X\subseteq\Lambda} \Phi(X),$$

where $\Phi(X)$ is an interaction term supported on subsets of sites, typically local, bounded, and self-adjoint. The quantum dynamics or time evolution on the observable algebra A_Λ is given by the one-parameter group of automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$, defined as

$$\alpha_t(a) = e^{itH_\Lambda} a e^{-itH_\Lambda}, \quad a \in A_\Lambda, \quad t \in \mathbb{R}.$$

This time evolution represents the solution of the ***Heisenberg equation****:*

$$\frac{d}{dt}\alpha_t(a) = i[H_\Lambda, \alpha_t(a)], \quad \alpha_0(a) = a, \quad a \in A_\Lambda,$$

where the Hamiltonian H_Λ is the generator of the automorphism group.

Explicitly, one can also write the automorphisms using an integral formula (exponential series expansion):

$$\alpha_t(a) = e^{itH_\Lambda} a e^{-itH_\Lambda} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} ad^n_{H_\Lambda}(a),$$

where the adjoint action is given by

$$ad_{H_\Lambda}(a) = [H_\Lambda, a], \quad \textit{and} \quad ad^{n+1}_{H_\Lambda}(a) = [H_\Lambda, ad^n_{H_\Lambda}(a)].$$

Example 3.6. ***Local time evolution on a spin sub-system****. Let L be a discrete metric space and suppose $N \subseteq L$ is a metric subspace. Choose a dimension d and take the spin systems with quasi-local algebras $B = \otimes_L M_d(\mathbb{C})$ and $A = \otimes_N M_d(\mathbb{C})$. Then we have a natural inclusion of quasi-local algebras $A \leq B$. Furthermore, there is a canonical condition expectation $E : B \to A$*

tracing out operators localized away from L.

Now, assume we have dynamics given by a quantum channel $\alpha : B \to B$, and consider the induced quantum channel $\phi : A \to A$ defined by $\phi(a) := E(\alpha(a \otimes 1))$. Then if $a \in A_x$ and $b \in A_y$ we compute the following

$$\begin{aligned} ||[\phi(a), b]|| &= ||E(\alpha(a \otimes 1))b - b\,E(\alpha(a \otimes 1)|| \\ &= ||E(\alpha(a \otimes 1)(b \otimes 1) - (b \otimes 1)(a \otimes 1)|| \\ &\leq ||\alpha(a \otimes 1)(b \otimes 1) - (b \otimes 1)(a \otimes 1)|| \\ &\leq Q_\alpha(x, y). \end{aligned}$$

Thus $Q_\phi(x, y) \leq Q_\alpha(x, y)$ for all $x, y \in N$. Therefore

$$\mathcal{E}_\phi \subseteq \mathcal{E}_\alpha|_N.$$

That is, the coarse structure of Q_ϕ is contained in the restriction to N of the coarse structure on L induced by α.

Example 3.7. ***Quantum Cellular Automata*** *. Suppose $A := \otimes_{x \in X} M_n(\mathbb{C})$ is a spin system and let (X, d) be, for simplicity, a metric arising from a locally-finite graph. A quantum cellular automata (QCA) is is defined as an automorphism $\alpha \in Aut(A)$ where Q_α has 0 length (in the sense of Definition 6.4 with respect to the metric coarse structure of (X, d). This is equivalent to the automorphism having bounded spread (Schumacher and Werner 2004; Arrighi 2019; Farrelly 2020).*

For sufficiently non-trivial QCA $\alpha \in Aut(A)$, $\mathcal{E}_\alpha$ will recover the metric coarse structure. Suppose we have an $a \in M_n(\mathbb{C})$ where the support of the operator $\alpha(a_x)$ contains the ball of radius 1 about x, $N_1(x)$. Here a_x is the identification of a in the factor $M_n(\mathbb{C})$ identified with the site x. We then have for any y adjacent to x, $[\alpha(a_x), A_y] \neq 0$, so that $Q_\alpha(x, y) > 0$, and we set $C_x = min\{Q_\alpha(x, y) : d(x, y) = 1\}$, then $C_x > 0$. If we assume some sort of sufficient regularity, translational symmetry for instance, then $inf_{x \in X}\{C_x\} = C > 0$. This implies that the controlled set $F = \{(x, y) : Q_\alpha(x, y) \geq C\} \in \mathcal{E}_\alpha$ has the adjacency relation $E := \{(x, y) : d(x, y) = 1\}$ as a subset, hence E is controlled in $\mathcal{E}_\alpha$. However, the coarse structure generated by E is exactly the coarse structure of the graph, hence $\mathcal{E}_d \subseteq \mathcal{E}_\alpha$. The other inclusion follows from the definition of a QCA.

A special class of QCA, which will provide a large class of easily constructible dynamics with accessible coarse structures, is finite-depth circuits. Say we have a spin system $A = \otimes_{x \in X} M_d(\mathbb{C})$, and suppose we have a metric structure on X arising from a discrete metric space. To define

a depth-one circuit, we partition of X into finite subsets F_i with $\sup_i diam(F_i) < R$ for some constant $R > 0$. Then pick a unitary $u_i \in A_{F_i}$. Note that since the F_i are mutually disjoint, then u_i mutually commute in A, so the map $\prod_i Ad(u_i)$ is well defined on the algebra A_{loc}, and extends to an automorphism α of A and is called a depth one circuit.

This allows us to define a finite-depth circuit. It is an automorphism $\alpha \in Aut(A)$ which can be written as a composition $\alpha_1 \circ \alpha_2 \circ \dots \alpha_n$ such that each α_i is a depth-one circuit.

***Close automorphisms**. As we have done with close states, we also investigate how close automorphisms and their coarse structures are related. The space $\mathscr{B}_1(A \to A)$ has multiple natural topologies, but the one we consider here is the norm topology, where*

$$\|\alpha\| = \sup_{a \in A} \frac{\|\alpha(a)\|}{\|a\|}.$$

Here, we have an analogue of Proposition 3.2 but for dynamical coarse structures.

Proposition 3.3. *Suppose $\alpha, \beta \in \mathscr{B}_1(A \to A)$ with $\|\alpha - \beta\| \leq \delta$. Then $\|Q_\alpha - Q_\beta\|_\infty \leq 2\delta$. Consequently,*

1. *$\alpha \mapsto Q_\alpha$ is continuous and $\alpha \mapsto \mathscr{E}_\alpha$ is lower semicontinuous in the sense of Theorem 3.2.*
2. *If $\epsilon - 2\delta > 0$, then $\mathscr{E}_{\epsilon+2\delta,\alpha} \subseteq \mathscr{E}_{\epsilon,\beta} \subseteq \mathscr{E}_{\epsilon-2\delta,\alpha}$.*

Proof. Consider any $a \in (A_x)_1, b \in (A_y)_1$,

$$\begin{aligned} \|\alpha(a)b - b\alpha(a)\| &\leq \|\alpha(a)b - \beta(a)b\| + \|b\alpha(a) - b\beta(a)\| + \|\beta(a)b - b\beta(a)\| \\ &\leq 2\delta + \|\beta(a) - b\beta(a)\| \end{aligned}$$

Now, take the supremum over $a \in (A_x)_1, b \in (A_y)_1$, we get

$$Q_\alpha(x,y) \leq Q_\beta(x,y) + 2\delta$$

By switching α and β and putting these together, we get

$$\|Q_\alpha - Q_\beta\| \leq 2\delta.$$

If $(x,y) \in E^\alpha_\epsilon$, then $Q_\alpha(x,y) \geq Q_\alpha(x,y) - 2\delta$, and thus $(x,y) \in E^\beta_{\epsilon-2\delta}$. Thus

$$\mathscr{E}_{\epsilon,\alpha} = \langle E^\alpha_\epsilon \rangle \subseteq \langle E^\beta_{\epsilon-2\delta} \rangle = \mathscr{E}_{\epsilon-2\delta,\beta}.$$

The other inclusion is achieved by switching the roles of α and β.

□

Dynamics open the door for more questions and interplay between coarse structures. Given two dynamics $\alpha, \beta : A \to A$, what could happen when we compose them? More specifically, what is $\mathscr{E}_{\alpha\circ\beta}$, in terms of $\mathscr{E}_\alpha$ and $\mathscr{E}_\beta$? We can see that the coarse structure can certainly get smaller. For instance, in the case of a spin system $A := \otimes_{x\in X} M_n(\mathbb{C})$, $\mathscr{E}_{\alpha\circ\alpha^{-1}} = \mathscr{E}_{Id_A}$, which reduces to the trivial diagonal coarse structure. Hence, we anticipate it would be difficult to precisely characterize $\mathscr{E}_{\alpha\circ\beta}$ without detailed knowledge of the channels.

Even so, if two channels are local with respect to a coarse structure $\mathscr{E}$, we would expect their composition to also be local with respect to $\mathscr{E}$. We can guess then that $\mathscr{E}_{\alpha\circ\beta} \subseteq \langle \mathscr{E}_\alpha \cup \mathscr{E}_\beta \rangle$.

Theorem 3.5. *Suppose $A = \otimes_{x\in X} M_n(\mathbb{C})$ is a spin system, and let $\alpha, \beta \in \mathscr{B}_1(A \to A)$. Then $\mathscr{E}_{\alpha\circ\beta} \subseteq \langle \mathscr{E}_{\tilde{\alpha}} \cup \mathscr{E}_\beta \rangle$. Particularly, if α has higher-order commutator decay, then $\mathscr{E}_{\alpha\circ\beta} \subseteq \langle \mathscr{E}_\alpha \cup \mathscr{E}_\beta \rangle$.*

Proof. Let $\epsilon > 0$. Pick $0 < \delta < \frac{\epsilon}{3}$. There is a controlled set $F \in \mathscr{E}_\beta$ such that for $x, y \notin F$, $\|[\beta(a), b]\| < \delta$ for $a \in (A_x)_1$ and $b \in (A_y)_1$. This implies by (Nachtergaele et al. 2013) that if we set $G = F[x]$,

$$\|\beta(a) - E_G(\beta(a))\| < 2\delta,$$

where $E_G : A \to A_G$ is the canonical conditional expectation, hence

$$\|[\alpha(\beta(a)), b]\| \leq \|[\alpha(E_G(\beta(a))), b]\| + \|[\alpha(\beta(a) - E_G(\beta(a))), b]\| \leq \|[\alpha(E_G(\beta(a))), b]\| + 2\delta$$

However, there is some controlled set $\widetilde{F} \in \mathscr{E}_{\tilde{\alpha}}$ with $\|[\alpha(E_G(\beta(a))), b]\| < \delta$ for $x, y \notin \widetilde{F}$. Therefore, for $x, y \notin F \cup \widetilde{F}$ we have

$$\|[\alpha(\beta(a)), b]\| \leq 3\delta < \epsilon$$

But $F \cup \widetilde{F} \in \langle \mathscr{E}_\alpha \cup \mathscr{E}_\beta \rangle$, thus, $\alpha \circ \beta$ is local with respect to the latter. □

Remark 3.4. ***Choice of tensor product decomposition for dynamics****. Here we do a similar analysis as in the case of states when we considered different choices of tensor product decomposition on the base set. We begin by noting that if $\phi, \psi \in \mathscr{B}_1(A \to A)$ with an automorphism $\alpha \in Aut(A)$ such that $\phi = \alpha^{-1} \circ \psi \circ \alpha$, and we define the new coarse structure $\{A'_x := \alpha(A_x)\}$, then $Q_\phi = Q'_\psi$. Which we should expect, since automorphisms are norm isometries,*

$$
\begin{aligned}
Q_\phi(x,y) &= sup_{a\in(A_x)_1, b\in(A_y)_1} ||\phi(a)b - b\phi(a)|| \\
&= sup_{a\in(A_x)_1, b\in(A_y)_1} ||\alpha^{-1}(\psi(\alpha(a)))b - b\alpha^{-1}(\psi(\alpha(a)))|| \\
&= sup_{a\in(A_x)_1, b\in(A_y)_1} ||\psi(\alpha(a))\alpha(b) - \alpha(b)\psi(\alpha(a))|| \\
&= sup_{a'\in(A'_x)_1, b'\in(A_y)_1} ||\psi(a')b' - b'\psi(a')|| \\
&= Q'_\psi(x,y).
\end{aligned}
$$

We obtain a similar result to the case of correlation structures discussed in 3.3. However, one key difference is that in the case of spin systems with $A = \otimes_{x\in X} M_d(\mathbb{C})$, as far as we know there is not a complete classification of automorphisms up to conjugacy. However, similar to the case of states, it is expected that in essence, any coarse structure can be realized via a given automorphism simply by picking an alternate site structure.

3.5 Algebraic entanglement

We noted before the existence of what's called the algebraic entanglement. Here we discuss this other type of coarse structure that is specifically interesting for nets of algebras that are not concrete spin systems.

For any concrete spin system, the local-observable algebras factorize as a product of algebras assigned to points, i.e. $A_F \cong \otimes_{x\in F} A_x$. This is not the case for most examples of nets of algebras though. Operators that are not products of operators localized in proper sub-regions can arise as the regions get larger. We call this situation the algebraic entanglement. Although, we would anticipate that for regions F and G that are sufficiently far apart, we should not have new emerging operators which are localized in $F \cup G$ that are not generated by operators localized in F and G respectively.

Consider an abstract net on X whose local algebras are finite dimensional, we can measure algebraic entanglement with a map, $X \times X \to \mathbb{N} \cup \{0\}$, defined by

$$
D(x,y) := dim(A_{x\cup y}) - dim(\langle A_x, A_y\rangle).
$$

Then, we can define the algebraic-entanglement coarse structure as the universal coarse structure

$$
\mathcal{E}_{ae}(A) = \mathcal{E}_D.
$$

in the sense of Theorem 3.1

Example 3.8. ***Fusion spin chains****: Fusion spin chains arise in the context of 2+1D topologically ordered spin systems and categorical dualities of spin chains and they are nets of algebras typically defined on* $\mathbb{Z}$*. Consider a fusion category* $\mathscr{C}$ *and a strongly tensor generating object* X*, there is an associated net of algebras* $\mathscr{A}(\mathscr{C}, X)$ *over* $\mathbb{Z}$*. For more details, see (Jones 2024; Jones and Lim 2024). It is clear that* $\mathscr{E}_{ae}(\mathscr{A}(\mathscr{C}, X))$ *is the standard metric coarse structure on* $\mathbb{Z}$*.*

Example 3.9. *Let* A *be a net of C*-algebras over* X*. Furthermore, suppose we have a metric* d *on* X*. Pick some* $t \geq 0$*. We can then define a refinement* $A^{d,t}$ *of* A *by setting*

$$A_F^{d,t} = \langle x \in A_F \ : \ x \in A_G, G \subseteq F \ , diam(G) \leq t \}$$

This assembles into a sub-net of A*, such that the algebraic entanglement coarse structure* $\mathscr{E}_{ae}(A^{d,t})$ *is contained in the metric coarse structure* $\mathscr{E}_{(X,d)}$*. Indeed, for* $d(x,y) > t$*, then* $A^{d,t}_{x\cup y} = \langle A^{d,t}_x, A^{d,t}_y \rangle$ *by definition hence* $D(x,y) = 0$*. Thus, the set* $E_k = \{(x,y) \in X \times X \ : D(x,y) \geq k\} \subseteq \{(x,y) \ : \ d(x,y) \leq t\}$*, hence, controlled in the metric coarse structure of* (X,d)*.*

CHAPTER

4

STABILITY FOR COARSE STRUCTURES

As we mentioned earlier, having a solid basis for defining entanglement-based order parameters in the context of our coarse-geometric framework relies heavily on the stability of such "intrinsic" coarse structure on "quasi-local" perturbations. Showing this, will be the purpose of this chapter.

We start by giving some intuition. Local perturbations, as the name suggests, are operations that are meant to change your system only in a small region. Quasi-local perturbations are defined to be norm limits of such operations. The stability of a property (e.g., of a state, Hamiltonian, or dynamics) on a spin system under quasi-local perturbation is an important criterion for the property to be considered macroscopic. Our specific aim will be to show that the universal correlation coarse structures defined for states and quantum dynamics are invariant under quasi-local perturbation under favorable circumstances.

Typically quasi-local perturbations are considered for Hamiltonians, so we start by giving a formal definition for states and dynamics.

Definition 4.1. *Suppose A is a discrete net of C*-algebras over a set X. A unital, completely positive map $\Psi : A \to A$ is said to be localized if there is some finite subset $F \subseteq X$ such that*

$$\Psi|_{A_{Fc}} = Id|_{A_{Fc}}.$$

*A channel Ψ is called a **quasi-local perturbation** if there is a sequence Ψ_n of localized channels*

such that $\|\Psi - \Psi_n\| \to 0$.

Note that the norm we use for channels is the uniform norm, i.e. $\|\Phi\| = \sup_{a \in (A)_1} \|\Phi(a)\|$.

Proposition 4.1. *Every inner automorphism of a discrete net A of C*-algebras is a quasi-local perturbation.*

Proof. Let $u \in U(A)$. Choose an increasing sequence of finite subsets $F_1 \subseteq F_2 \subseteq \ldots$ such that $\bigcup_n F_n = X$.

It is common knowledge (using standard C*-algebra arguments, for example see the proof of Lemma 3.1 in (Glimm 1960)) that there are unitaries $u_i \in A_{F_i}$ where $\|u_i - u\| \to 0$. But each $\mathrm{Ad}(u_i)$ is localized in the bounded set F_i. So, if we choose i sufficiently large such that $\|u - u_i\| < \epsilon$, then for $\|a\| \leq 1$, we have

$$\begin{aligned} \|u a u^* - u_i a u_i^*\| &= \|u_i a u^* + (u - u_i) a u^* - u_i a u_i^*\| \\ &\leq \epsilon + \|u_i a (u - u_i)\| \leq 2\epsilon. \end{aligned}$$

hence

$$\|\mathrm{Ad}(u_i) - \mathrm{Ad}(u)\| \to 0$$

□

The next result shows that for nice states, the universal 2-point correlation coarse structure is stable under quasi-local perturbation.

Theorem 4.1. *Suppose A is a discrete net of C*-algebras over the set X, and* $\phi : A \to B$ *be any ucp map with* $\mathscr{E}_\phi$ *is proper and monogenic. Then, given any quasi-local perturbation* $\Psi : A \to A$, $(\mathscr{E}_{\phi \circ \Psi})_{con} = (\mathscr{E}_\phi)_{con}$.

Proof. Pick some increasing sequence $F_1 \subseteq F_2 \ldots$ of finite subsets with Ψ_i localized in F_i and $\|\Psi_i - \Psi\| \to 0$. Note that $\|\phi \circ \Psi - \phi \circ \Psi_i\| \to 0$.

To start with, we claim that $(\mathscr{E}_\phi)_{con} = (\mathscr{E}_{\phi \circ \Psi_i})_{con}$. To see this, recall that if $\mathscr{E}$ is a proper coarse structure, then for any finite subset $C \subseteq X$, $\mathscr{E}_{con} = \langle \mathscr{E}|_{(X-C)\times(X-C)} \bigcup \mathscr{F}(X \times X) \rangle$, where, as before, $\mathscr{F}(X \times X)$ denotes the collection of finite subsets of $X \times X$, and $\mathscr{E}|_{(X-C)\times(X-C)} = \mathscr{E} \cap ((X - C) \times (X - C))$.

Now, since Φ_i is localized in F_i,

$$C_\phi|_{(X-F_i)\times(X-F_i)} = C_{\phi \circ \Psi_i}|_{(X-F_i)\times(X-F_i)}$$

Note that for any finite set $C \subseteq X$, a coarse structure $\mathscr{E}$ on X is proper if and only if the coarse structure $\mathscr{E}|_{(X-C)\times(X-C)}$ is proper. This is true since if $\mathscr{E}$ is proper, clearly it's restriction

is proper. For, suppose $\mathscr{E}$ is not proper, then there is an infinite, bounded subset $D \subseteq X$. But then $D \cap (X-C) \subseteq X-C$ is infinite since C is finite, but is bounded in $\mathscr{E}|_{(X-C)\times(X-C)}$. Thus $\mathscr{E}|_{(X-C)\times(X-C)}$ is not proper.

This implies

$$\mathscr{E}_\phi|_{(X-F_i)\times(X-F_i)} = \mathscr{E}_{\phi\circ\Psi_i}|_{(X-F_i)\times(X-F_i)},$$

hence $\mathscr{E}_{\phi\circ\Psi}$ is proper, and thus we have

$$(\mathscr{E}_\phi)_{con} = \langle (\mathscr{E}_\phi)|_{(X-F_i)\times(X-F_i)} \cup \mathscr{F}(X\times X)\rangle = \langle (\mathscr{E}_{\phi\circ\Psi_i})|_{(X-F_i)\times(X-F_i)} \cup \mathscr{F}(X\times X)\rangle = (\mathscr{E}_{\phi\circ\Psi_i})_{con}.$$

Furthermore, if for any state ψ the set $E^{F_i}_{\epsilon,\psi} = \{(x,y)\in(X-F_i)\times(X-F_i) \,:\, C_\psi(x,y)\geq\epsilon\}$, we have

$$E^{F_i}_{\epsilon,\phi} = E^{F_i}_{\epsilon,\phi\circ\Psi_i}$$

thus, if ϕ is stable, for ϵ in the stable range we have

$$(\mathscr{E}_{\phi\circ\Psi_i})_{con} = (\mathscr{E}_\phi)_{con} = \langle E^{F_i}_{\epsilon,\phi} \cup \mathscr{F}(X\times X)\rangle = \langle E^{F_i}_{\epsilon,\phi\circ\Psi_i} \cup \mathscr{F}(X\times X)\rangle.$$

Thus $\phi\circ\Psi_i$ is connectively stable, with the same stable range as ϕ.

Now, suppose $\epsilon > 0$ is in the interior of the stable range for ϕ. Pick i such that $\delta = 3||\phi\circ\Psi - \phi\circ\Psi_i||$ satisfies $\epsilon-\delta > 0$ and $\epsilon+\delta$ is in the stable range for ϕ. Then, by Proposition 3.2, $E_{\epsilon+\delta,\phi\circ\Psi_i} \subseteq E_{\epsilon,\phi\circ\Psi} \subseteq E_{\epsilon-\delta,\phi\circ\Psi_i}$

$$(\mathscr{E}_\phi)_{con} = \langle E_{\epsilon-\delta,\phi\circ\Psi_i} \bigcup \mathscr{F}(X\times X)\rangle \subseteq \langle E_{\epsilon,\phi\circ\Psi} \bigcup \mathscr{F}(X\times X)\rangle \subseteq \langle E_{\epsilon+\delta,\phi\circ\Psi_i} \bigcup \mathscr{F}(X\times X)\rangle \subset (\mathscr{E}_\phi)_{con}.$$

Since this is the case for all $\epsilon > 0$, we then get $(\mathscr{E}_\phi)_{con} = (\mathscr{E}_{\phi\circ\Psi})_{con}$,
giving us the desired result.

□

Remark 4.1. *We cannot remove the connectivity in the statement of the theorem. To see that, consider any product state ϕ on a quantum spin system $A = \otimes_{x\in X} M_d(\mathbb{C})$. Suppose we take an entangling unitary u localized in some finite region F. Then $\mathscr{E}_\phi$ is diagonal, while $\mathscr{E}_{\phi\circ Ad(u)}$ contains non-diagonal finite subsets.*

Corollary 4.1. *If ϕ is a pure state on A such that $\mathscr{E}_\phi$ is monogenic and proper, then for any pure state ψ with $d(\phi,\psi) < 2$, we have $(\mathscr{E}_\phi)_{con} = (\mathscr{E}_\psi)_{con}$.*

Proof. If two pure states are close, there exists some unitary $u \in A$ with $\psi = \phi \circ \mathrm{Ad}(u)$ (Glimm and Kadison 1960). The result then follows from 4.1 and 4.1. □

Remark 4.2. ***Closure properties for states****. Suppose $\mathscr{E}$ is monogenic and connected, and let $S(\mathscr{E})$ be the set of pure states whose connected correlation coarse structure is $\mathscr{E}$. The above theorem then shows that $S(\mathscr{E})$ is a closed subspace of all pure states in the norm topology on the state space. However, in general, $S(\mathscr{E})$ is not closed in the weak$*$-topology. For instance, consider a discrete net of C*-algebras whose quasi-local algebra A is simple (and norm separable). Then from (Kishimoto et al. 2003) then, for any two pure states ϕ and ψ, there is an approximately inner automorphism $\alpha \in Aut(A)$ such that $\phi \circ \alpha = \psi$. By approximately inner, we mean there is some sequence of unitaries $\{u_i \in A\}$ such that for all $a \in A$, $||u_i^* a\, u_i - \alpha(a)|| \to 0$.*

If $\phi \in S(\mathscr{E})$ and $\psi \notin S(\mathscr{E})$, let α and $\{u_i\}$ be as above. If $\mathscr{E}$ is monogenic, then $\phi^{u_i} \in S(\mathscr{E})$, however, the approximation condition entails that,

$$|\phi(u_i^* a\, u_i) - \phi(\alpha(a))| = |\phi(u_i^* a\, u_i) - \psi(a)| \to 0.$$

Hence, $\phi^{u_i} \to \psi$ in the weak-topology. But ψ was arbitrary, then the set $S(\mathscr{E})$ is dense in the pure state space. Consequently, the assignment $\phi \mapsto \mathscr{E}_\phi$ (or any variation, e.g. $(\mathscr{E}_\phi)_{con}$) is not expected to be continuous in any sense with respect to the weak$*$− topology, in contrast to the case with the norm metric of Corollary 4.1.*

Theorem 4.2. *Let A be a discrete net of C*-algebras over the set X, and $\alpha : A \to B$ be any ucp map such that $\mathscr{E}_\alpha$ is proper and monogenic. Then for any quasi-local perturbation $\Psi : A \to A$, $(\mathscr{E}_{\alpha\circ\Psi})_{con} = (\mathscr{E}_\alpha)_{con}$.*

Proof. The proof exactly follows the proof of 4.1 with Q_α in place of C_ϕ, and Proposition 3.3 in place of Proposition 3.2. □

As we saw previously, we established quite the similarity between states and dynamics when it comes to the intrinsic coarse structure. We state a similar result for stability of dynamical coarse structure under quasi-local perturbations.

Corollary 4.2. *Suppose A is simple and let $\mathscr{E}_\alpha$ be monogenic and proper for $\alpha \in Aut(A)$, then if $\beta \in Aut(A)$ such that $||\alpha - \beta|| < 1$, then $(\mathscr{E}_\alpha)_{con} = (\mathscr{E}_\beta)_{con}$.*

Proof. If $||\alpha - \beta|| < 1$, then $||\beta^{-1} \circ \alpha - \mathrm{Id}_A|| < 1$. Using the power series expansion, we have a derivation $\delta = \log(\beta^{-1} \circ \alpha)$. But since every derivation of a simple C*-algebra is inner, (Sakai 1968), there is some skew-adjoint $h \in A$ with $\delta(a) = [h, a]$. Particularly, $\beta^{-1} \circ \alpha(a) = \exp(h) a \exp(-h)$, hence $\alpha = \mathrm{Ad}(u) \circ \beta$, where $u = \beta(\exp(h))$. □

CHAPTER

5

STATE PREPARATION

In this chapter, we use X as a discrete set and let $\mathscr{E}$ be a fixed coarse structure on X. Here, we will study the collection of quantum channels, in some sense, preserving "coarse locality".

Definition 5.1. *A unital completely positive map $\Psi : A \to B$ is said to have controlled spread if for all $\epsilon > 0$, there is a controlled set E such that for any bounded subset $F \subseteq X$ and $a \in (A_F)_1$, there exists $b \in (B_{E[F]})_1$ such that $||\Psi(a) - b|| < \epsilon$.*

Notice that in practice, we are typically interested in connected, proper coarse spaces, so that bounded sets are precisely finite sets.

Example 5.1. ***Bounded-spread channels***. *Controlled spread generalizes the notion of bounded spread, which is simply the same previous condition but for $\epsilon = 0$. Hence, $\Psi : A \to A$ with controlled spread is a generalization of (quantum) cellular automata that are possibly stocashtic and, possibly, "have tails".*

Example 5.2. ***Higher-order commutator decay***. *Let $A := \otimes_{x \in X} M_n(\mathbb{C})$ be a quantum spin system, and $\Psi : A \to A$. If $\widetilde{Q}_\Psi$ has higher-order controlled decay with respect to $\mathscr{E}$, then Ψ has controlled spread. This follows immediately from (Nachtergaele et al. 2013). Particularly, time evolutions of local Hamiltonian have controlled spread.*

Definition 5.2. *A unital completely positive map $\Phi : A \to B$ is said to be locality preserving if satisfies the following*

1. *Φ has controlled spread.*

2. *$\widetilde{C}_\Phi$ has higher order controlled decay.*

If Φ is a homomorphism (e.g. unitary dynamics), then $\widetilde{C}_\Phi$ is identically 0, thus, the second condition is a redundancy. This condition is for quantum channels that are not multiplicative, but are asymptotically multiplicative.

Theorem 5.1. *Suppose A, B, C are discrete nets of C*-algebras over X. $\Psi : A \to B$ and $\Phi : B \to C$ are locality preserving, then so is $\Phi \circ \Psi$*

Proof. Suppose we have $\epsilon > 0$. Pick $E, E' \in \mathscr{E}$ implementing the controlled spread condition for $\frac{\epsilon}{2}$ for Ψ and Φ respectively. Define $E'' = E' \circ E$. Suppose $F \subseteq X$ is bounded. Then, for $a \in A_F$, there is $b \in B_{E[F]}$ with $\|\Psi(a) - b\| < \frac{\epsilon}{2}$ and $c \in C_{E'[E[F]]} = C_{(E' \circ E)[F]}$ with $\|\Phi(b) - c\| < \frac{\epsilon}{2}$. Hence,

$$\|\Phi \circ \Psi(a) - c\| \leq \|\Phi(b) - c\| + \frac{\epsilon}{2} < \epsilon.$$

Therefore, $\widetilde{C}_{\Phi\circ\Psi}$ has controlled spread.

Let $E \in \mathscr{E}$, and $a \in (A_{E[x]})_1, a' \in (A_{E[y]})_1$. Suppose $\epsilon > 0$ and pick δ such that $\delta^2 + 10\delta < \epsilon$. Then there exists a controlled set E' implementing the δ controlled spread condition for Ψ, therefore, there are $b \in B_{(E' \circ E)[x]}$ and $b' \in B_{(E' \circ E)[x]}$ such that $\|\Psi(a) - b\| < \delta$ and $\|\Psi(a') - b'\| < \delta$. Then, using the δ controlled spread condition for Φ as well, for some controlled set we have E'' $c \in C_{(E'' \circ E')[x]}$ and $c' \in C_{(E'' \circ E')[x]}$ with $\|\Phi(b) - c\| < \delta$ and $\|\Phi(b') - c'\| < \delta$. We also choose the higher order controlled decay bounds to be δ for both Ψ, Φ. Then,

$$\begin{aligned}
&\|\Phi \circ \Psi(aa') - \Phi \circ \Psi(a)\Phi \circ \Psi(a')\| \\
&\leq \|\Phi \circ \Psi(aa') - \Phi(b)\Phi(b')\| + 2\delta + \delta^2 \\
&= \|\Phi \circ \Psi(aa') - \left\{\Phi(b)[\Phi(b') - c'] + \Phi(b)c' + cc' - cc'\right\}\| + 2\delta + \delta^2 \\
&\leq \delta^2 + 4\delta + \|\Phi\big(\Psi(aa') - bb'\big) + \Phi(bb') - cc'\| \\
&\leq \delta^2 + 4\delta + \|\Phi\big(\Psi(aa') - bb'\big)\| + \|\Phi(bb') - cc'\| \\
&\leq \delta^2 + 4\delta + \|\Psi(aa') - bb'\| + \|\Phi(bb') - cc'\| \\
&\leq \delta^2 + 4\delta + \|\Psi(aa') - \Psi(a)\Psi(a') + \Psi(a)\Psi(a') - bb'\| + \|\Phi(bb') - cc'\| \\
&\leq \delta^2 + 4\delta + \|\Psi(aa') - \Psi(a)\Psi(a')\| + \|\Psi(a)(\Psi(a') - b')\| + \\
&\quad \|(\Psi(a) - b)b'\| + \|\Phi(bb') - cc'\| \\
&\leq \delta^2 + 7\delta + \|\Phi(bb') - \Phi(b)\Phi(b')\| + \|\Phi(b)(\Phi(b') - c')\| + \|(\Phi(b) - c)c'\| \\
&\leq \delta^2 + 10\delta < \epsilon
\end{aligned}$$

Hence, $\widetilde{C}_{\Phi\circ\Psi}$ has higher order controlled decay. □

Now, define the category $\mathscr{L}\mathscr{P}_{\mathscr{E}}$ defined as follows:

1. *Objects are discrete nets of C*-algebras over X.*
2. *Morphisms from A to B, denoted $\mathscr{L}\mathscr{P}_{\mathscr{E}}(A,B)$ are locality-preserving channels $\Psi : A \to B$*
3. *Composition is composition of channels.*

We can restrict our attention to the sub-category of invertible morphisms, which is composed of automorphisms with controlled spread, which is a groupoid we denote by $\mathscr{L}\mathscr{P}^0_{\mathscr{E}}$. This contains the groupoid Net_X studied in (Jones 2024).

The assignment $A \mapsto S(A)$ of the state space of A to it's quasi-local C-algebra can be though of as a functor $F : \mathscr{L}\mathscr{P}^{op}_{\mathscr{E}} \to Conv$, where Conv denotes the category of convex spaces, defined by*

$$F(\Psi)(\phi) = \phi \circ \Psi,$$

where $\Psi : A \to B$ is locality-preserving, and $\phi \in S(B)$.

Even though we do not continue this track here, a very intriguing question is the following: what are the orbits under the action of the groupoid $\mathscr{L}\mathscr{P}^0_{\mathscr{E}}$? Can they be characterized with some kind of invariant? This can be interpreted as a kind of state-based classification of "phases".

We show that the locality-preserving channels in fact preserve the locality of the states, as characterized by higher-order decay of correlations, which justifies our terminology.

Corollary 5.1. *Define $\mathscr{S}_{\mathscr{E}}(A) := \{\phi \in S(A) : \mathscr{E}_{\widetilde{\phi}} \subseteq \mathscr{E}\}$. Then, for any discrete nets A, B over X and $\Psi \in \mathscr{L}\mathscr{P}_{\mathscr{E}}(A,B)$, we have*

$$F(\Psi)S_{\mathscr{E}}(B) \subseteq S_{\mathscr{E}}(A)$$

Proof. Pick some $\phi \in S_{\mathscr{E}}(B)$ and $\Psi \in \mathscr{L}\mathscr{P}_{\mathscr{E}}(A,B)$. Suppose $\epsilon > 0$. Now, pick $\delta > 0$ such that $10\delta + \delta^2 < \epsilon$, and let $E \in \mathscr{E}$. Then there is some symmetric $H \in \mathscr{E}$ such that for $a \in (A_{E[x]})_1$ and $a' \in (A_{E[y]})_1$, there exist $b \in B_{H[E[x]]}$, $b' \in B_{H[E[y]]}$ where

$$\|\Psi(a) - b\| < \delta$$

with same bound for a' and b'. Since $(\widetilde{C}_{\phi})_{H\circ E}$ has controlled decay with respect to $\mathscr{E}$, there is some F (which we could be assumed to contain H by taking union) such that for $x, y \notin F$, $(\widetilde{C}_{\phi})_H(x,y) < \epsilon - (10\delta + \delta^2)$. Hence, for $(x,y) \notin F$, and by thinking of ϕ as a locality-preserving channel to the scalars (that is, thinking of the scalars as a trivial discrete net of algebras), then by Theorem 5.1 we get that

$$(\widetilde{C}_{\phi\circ\Psi})_F(x,y) < \epsilon.$$

Thus $\mathcal{E}_{\widetilde{\phi\circ\Psi}} \subseteq \mathcal{E}$. □

The next corollary is an important one as it highlights an important application of our framework.

Corollary 5.2. *(Elokl and Jones 2024) Suppose ϕ is a product state on a spin system $A = \otimes_{x\in X} M_d(\mathbb{C})$. Let $\alpha \in Aut(A)$. Then $\mathcal{E}_{\widetilde{\phi\circ\alpha}} \leq \mathcal{E}_{\widetilde{\alpha}}$. Particularly, if α is local with respect to $\mathbb{Z}_n$ then $\mathcal{E}_{\widetilde{\phi\circ\alpha}}$ is not equivalent to $\mathbb{Z}^m$ for any $m > n$.*

We can interpret the universal coarse structures of states and dynamics as measures of "coarse-geometric complexity". The above corollary then shows that to prepare a state with a certain geometric complexity, we need a circuit which is at least as geometrically complex. Particularly, if we want to prepare a state with d-dimensional correlations, we need a circuit which is local to some coarse structure containing $\mathbb{Z}^d$.

One way to see this is to consider the following application

CHAPTER

6

ORDER PARAMETERS AND COARSE GEOMETRY

This chapter explores to what extent entanglement-based order parameters, in fact, depend only on the coarse equivalence class of a metric rather than the specific details of the metric itself (Elokl and Jones 2024). An order parameter that only depends on the coarse equivalence class of a metric will be called a coarse order parameter. Our main focus will be the two examples: topological order and critical exponents. In both, it is not obvious that these features depend only on the coarse here, is to provide a rigorous proof of such fact.

6.1 Topological order: superselection sectors

We first look at topological order using the mathematical formulation for spin systems following (Naaijkens 2011, 2012; Fiedler and Naaijkens 2015; Cha et al. 2018; Ogata 2022). So we look at topological order as W-category of superselection sectors. In general this is only part of the structure of topological order. iHigher dimensions than 1, the category of cone-localized superselection sectors (under some assumptions on the state) can be equipped with the additional structure of a braided C*-tensor category (Ogata 2022). This braided category, in higher dimensions, is only a piece of the full structure of topological order, which should assemble into a higher braided*

category (Johnson-Freyd 2022). Nonetheless, the analysis shown here is expected to apply in more complicated situations. Therefore, topological order should depend only on the quasi-isometry class of the metric, rather than its isometry or Lipschitz class.

Consider a spin system with a quasi-local algebra, $A = \otimes_{x\in\mathbb{Z}^n} M_q(\mathbb{C})$, and suppose ϕ is a state on A. Let H_ϕ be the GNS representation of A associated to ϕ. We utilize the notion of quasi sub-equivalence of representations of two C-algebras. For H, K being two representations of a unital C*-algebra B we say $H \preceq K$ if H is unitarily equivalent to a summand of $K \otimes \ell^2(\mathbb{N})$.*

We recall the usual formulation of superselection sectors in terms of cones in $\mathbb{R}^n$ (or rather, a slight modification as in (Jones et al. 2023)). There are many possible variations of superselection sectors, but for the sake of staying close to Naaijkens (2011); Ogata (2022), a cone $\mathscr{C} \subseteq \mathbb{R}^n$ will be convex closure of a set $\{r_i\}_{i\in I}$ of geodesic rays emanating from a common point, whose interior has dimenion n. By a cone in the lattice $\mathbb{Z}^n$, we actually mean the intersection $\mathscr{D} = \mathscr{C} \cap \mathbb{Z}^n$, where $\mathscr{C}$ is a cone in $\mathbb{R}^n$.

Definition 6.1. *Suppose H is a Hilbert space representation of A. Then, for any subset $F \subseteq X$, H is said to be localizable in F if*

$$H|_{A_{F^c}} \preceq (H_\phi)|_{A_{F^c}}.$$

Definition 6.2. *A Hilbert space representation H of A is said to satisfy the superselection criterion if for any cone $\mathscr{D}$ in $\mathbb{Z}^n$, H is localizable in $\mathscr{D}$. Sec_ϕ denotes the W*-category of Hilbert space representations of A satisfying the superselection criterion.*

For many states (in fact, all ground states of gapped Hamiltonians) we can replace $\preceq$ in the above definition with unitary equivalence without loss of generality, and it recovers the usual definition. Sec_ϕ is a unitarily Cauchy complete W-category (Ghez et al. 1985).*

Our aim is to define a version of Sec_ϕ which makes sense for a spin system over any metric space (X, d) and that depends only on its quasi-isometry class.

So suppose (X, d) is a metric space that has a finite asymptotic dimension n, and let (X, d) be quasi-geodesic. Let $\mathscr{D}$ be a cone in $\mathbb{R}^n$. For $L > 0, C \geq 0$, define a coarse $L, C, \mathscr{D}$ cone to be a quasi-isometric embedding $f : \mathscr{D} \to (X, d)$, with Large-scale Lipschitz constants L, C. The asymptotic dimension assumption is just to make sure we are using the "correct" dimension of cone for the region, although, it is not strictly necessary below. Recall also that for a subset $E \subseteq X$, $E + R$ is the R-neighborhood of E it does not mean translation by R which does make sense in general.

Now take $A = \otimes_{x\in X} M_q(\mathbb{C})$ to be the quasi-local algebra of a spin system as above, and let ϕ a state on A.

Definition 6.3. *A Hilbert space representation of A is said to satisfy the coarse-superselection criterion on (X, d) if for any triple $L, C, \mathscr{D}$, there is some constant $R \geq 0$ where for any coarse*

$L, C, \mathscr{D}$ cone $f : \mathscr{D} \to X$, H is localizable in $f(\mathscr{D}) + R$. Sec^d_ϕ denotes the W-category of Hilbert space representations of A satisfying the coarse superselection criterion.*

We show that the coarse-superselection category as defined above agrees with the usual superselection category, and thus is an intrinsic property of a state on a quantum processor.

Proposition 6.1. *(Elokl and Jones 2024) Suppose (X, d) is a quasi-geodesic metric space with finite asymptotic dimension, and take $A = \otimes_{x \in X} M_q(\mathbb{C})$ to be a spin system over X and ϕ a state. Then,*

1. *For d', a metric on X that is quasi-isometric to d, $Sec^d_\phi = Sec^{d'}_\phi$.*
2. *If $(X, d) = \mathbb{Z}^n$ with the induced Euclidean metric, then $Sec^d_\phi = Sec_\phi$.*

Proof. To prove the first part, let there be constants $L > 0, C \geq 0$ with

$$L^{-1} d(x, y) - C \leq d'(x, y) \leq L d(x, y) + C,$$

for all $x, y \in X$. Then the identity $\mathrm{Id}_X : X \to X$, thought of as a map from (X, d) to (X, d'), is an L, C quasi-isometry.

Then any coarse $K, B, \mathscr{D}$ cone with respect to d is a coarse $LK, LB + C, \mathscr{D}$ cone with respect to d' and conversely, $K, B, \mathscr{D}$ cones with respect to (X, d') are $L^{-1}K, L^{-1}B - C, \mathscr{D}$ cones with respect to d. Now suppose $H \in \mathrm{Sec}^d_\phi$, and let $K, B, \mathscr{D}$ and $f : \mathscr{D} \to (X, d')$ be a coarse cone. Then there is some $R \geq 0$ with H being localizble in $f(\mathscr{D}) + R$. Then for any choice of $R' > LR + C$,

$d'(x, f(\mathscr{D})) > R'$ which implies $d(x, f(\mathscr{D})) \geq L^{-1}R' - C > R$. Thus

$$(f(\mathscr{D}) +_{d'} R')^c \subseteq (f(\mathscr{D}) +_d R)^c,$$

Thus, $A_{(\mathscr{D} +_{d'} R')^c}$ is contained in $A_{(\mathscr{D} +_d R)^c}$ and the superselction criterion restricts to

$$H|_{A_{(\mathscr{D} +_{d'} R')^c}} \preceq (H_\phi)|_{A_{(\mathscr{D} +_{d'} R')^c}}.$$

Therefore, $\mathrm{Sec}^d_\phi \subseteq \mathrm{Sec}^{d'}_\phi$. Since the quasi-isometry relation is symmetric, we can obtain the other inclusion by flipping the roles of d and d'.

Now for the second part, suppose $(X, d) = \mathbb{Z}^n$ with the restriction of the Euclidean metric. For any $K, B, \mathscr{D}$, there exists some $R \geq 0$ such that the R-neighborhood of the image of any $K, B, \mathscr{D}$ coarse cone contains an honest Euclidean cone (that is, the intersection of a cone in $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$). Now let $f : \mathscr{D} \to \mathbb{Z}^n$ be a coarse $K, B, \mathscr{D}$ cone. If $H \in \mathrm{Sec}_\phi$, then, since H is localizable in $\mathscr{D}'$, since $\mathscr{D}' \subseteq f(\mathscr{D}) + R$, H is also localizble in $f(\mathscr{D}) + R$. Since $K, B, \mathscr{D}$ and f were arbitrary, $H \in \mathrm{Sec}^d_\phi$.

Now let $H \in \mathrm{Sec}_\phi^d$, and $\mathscr{D}$ an honest cone. If $\mathscr{D} = \mathbb{Z}^n \cap \mathscr{D}'$ for $\mathscr{D}'$ a cone in $\mathbb{R}^n$, then $\mathscr{D}$ is $1, \sqrt{2}, \mathscr{D}'$ cones, where $f : \mathscr{D}' \to \mathscr{D}$ is the map that picks the closest element in $\mathscr{D}$. Note that for any vector $v \in \mathbb{Z}^n$, the honest cone $\mathscr{D}'' = \mathscr{D} + v$ is also a $1, \sqrt{2}, \mathscr{D}'$ coarse cone. Let R be the $1, \sqrt{2}, \mathscr{D}'$ localizing constant. Then by translating $\mathscr{D}$ into its interior, we can find an honest cone $\mathscr{D}''$ such that $\mathscr{D}'' + R \subseteq \mathscr{D}$. Then H is localized in $\mathscr{D}'' \subseteq \mathscr{D}$ hence also in $\mathscr{D}$, and thus $H \in \mathrm{Sec}_\phi$.

□

When (X, d) is coarsely equivalent to $\mathbb{Z}^2$, this category can be upgraded with the structure of a braided W-tensor category under a condition called approximate Haag duality (Ogata 2022), generalizing the familiar Haag duality from AQFT employed in earlier approaches (Naaijkens 2011, 2012; Fiedler and Naaijkens 2015; Cha et al. 2018). It is a bit tedious but not difficult to check using the style of argument like our previous proposition that this condition can be re-framed in terms of coarse cones, and that the construction of the tensor product and braiding can be carried out in the coarse setting. Alternatively, it can be shown that an appropriate poset of coarse cones satisfies the axioms in (Bhardwaj et al. 2024), which, from their main result would allow a direct definition of braided tensor category structure on Sec_ϕ^d.*

6.2 Decay rates in the coarse perspective

Decay rates of functions will depend on the choice of metric, however, there are certain "coarse" properties of the decay rate that only depend on the coarse equivalence class of the metric. Consider ϕ, a generic equilibrium state for a Hamiltonian with short-range interactions, we anticipate C_ϕ to decay at most exponentially as a function of distance, where the constant appearing in the exponent is the inverse correlation length. However, if a state is at a critical point in parameter space, it is expected that the function decay algebraically, in which case the correlation length is said to diverge. In this situation, there is a notion of critical exponent, which is usually used as a fundamental indicator of the states universality class (Stanley 1987). This is the motive for the next definitions (Elokl and Jones 2024).

Definition 6.4. *Let $f : X \times X \to \mathbb{R}_+$. Then we say f has length 0 if there exists a controlled set $E \in \mathscr{E}_f$ such that $f(x, y) = 0$ for all $(x, y) \notin E$.*

If $\mathscr{E}_f$ is monogenic, let d_ϵ be the path metric on the graph G_ϵ for some ϵ in the stable range. Then we say

1. *f is said to have finite length if there are positive constants A, B and a controlled set $E \in \mathscr{E}_f$ such that $f(x, y) \leq Ae^{-Bd_\epsilon(x,y)}$ for all $(x, y) \notin E$.*

2. *f has infinite length if for every exponential function $g(t) = Ae^{-Bt}$, there exists a controlled set E such that $f(x,y) > g(d_\epsilon(x,y))$ for all $(x,y) \notin E$.*

We can apply this to correlation functions, and replace the word "length" by "correlation length" to recover the familiar notions. Note that 0 *length implies finite length, and finite length implies not infinite length. Also, note that length* 0 *depends only on the coarse structure, however, so far, it is not clear that the notions of finite and infinite length are well-defined, since they depend a-priori on the choice of ϵ in the stable range. So we have the following proposition (Elokl and Jones 2024):*

Proposition 6.2. *Finite and infinite length are independent of the choice of ϵ in the stable range.*

Proof. Since f is stable, for any choices of ϵ, ϵ' in the stable range, there exists positive constants L, C such that $L^{-1}d_{\epsilon'}(x,y) - C \leq d_\epsilon(x,y) \leq L d_{\epsilon'}(x,y) + C$.

Now let there be constants A, B such that $f(x,y) \leq Ae^{-Bd_\epsilon(x,y)}$ for $(x,y) \notin E$ for some controlled set $E \in \mathscr{E}_f$. Then

$$f(x,y) \leq Ae^{-B(L^{-1}d_{\epsilon'}(x,y)-C)} = (Ae^{C})e^{-(BL^{-1})d_{\epsilon'}(x,y)},$$

as desired. Now let f have infinite length. Let $A, B > 0$. Then pick $R > 0$ so that $d_\epsilon(x,y) \geq R$ implies $f(x,y) > Ee^{-Dd_\epsilon(x,y)}$, where $E = Ae^{BCL^{-1}}$ and $D = BL^{-1}$. Since $d_{\epsilon'}(x,y) \geq L^{-1}d_\epsilon(x,y) - L^{-1}C$, and the exponential function is decreasing

$$Ae^{-Bd_{\epsilon'}(x,y)} \leq Ee^{-Dd_\epsilon(x,y)} < f(x,y).$$

□

Consequently, for any state on a spin system whose coarse structure is monogenic, the finiteness of the correlation length (or the length of the function C_ϕ) is intrinsic to the state.

Note that in the case of finite length, the actual constants A, B depend on the ϵ you choose, and so are not themselves invariant under the choice of coarse structure. Therefore, while we can make asymptotic boundedness statements as above (we have focused on the exponential case, one could use other classes of decaying functions), the exact character of asymptotic exponential decay seems to be not coarsely invariant. How, then, can we make sense of "asymptotic rate of decay" which depends only on the quasi-isometry class of the metric d? To this end, we introduce the following:

Definition 6.5. *Consider $g, h : \mathbb{R}_+ \to \mathbb{R}_+$. g and h are said to becoarsely commensurate, written $h \approxeq g$, if for any $L > 0$, $C \in \mathbb{R}$, there exists constants $D_1, D_2 > 0$ such that for all sufficiently large x,*

$$D_1 h(x) < g(Lx + C) < D_2 h(x).$$

h is said to be coarsely invariant if h is coarsely commensurate with itself.

In order for the inequality to make sense in the above definition, it is need at least that $x \geq -\frac{C}{L}$. Note that coarsely simple functions are closed under positive scaling, addition, and multiplication. Clearly, $\approxeq$ defines an equivalence relation on the collection of coarsely invariant functions.

Definition 6.6. *Take $f : X \times X \to \mathbb{R}_+$. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ to be a coarsely invariant, decreasing function. f is said to decay coarsely uniformly as g if there exist constants $C_1, C_2, R > 0$ such that if $d(x, y) \geq R$,*

$$C_1 g(d(x, y)) \leq f(x, y) \leq C_2 g(d(x, y)).$$

Proposition 6.3. *(Elokl and Jones 2024, 7.9) Suppose (X, d) is an unbounded quasi-geodesic metric space, and $f : X \times X \to \mathbb{R}_+$.*

1. *Take $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ to be coarsely invariant decreasing functions and $f : X \times X \to \mathbb{R}_+$ decays coarsely uniformly as g and h, then $g \approxeq h$. Therefore, coarse uniform decay gives a well-defined coarse commensurability class.*

2. *The coarse commensurability class of f depends only on the metric up to coarse equivalence.*

Proof. Suppose we have coarsely invariant functions $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ and constants $C_1, C_2, R > 0$ and $D_1, D_2, S > 0$ such that for $d(x, y) \geq R$,

$$C_1 g(d(x, y)) \leq f(x, y) \leq C_2 g(d(x, y)),$$

and for $d(x, y) \geq S$,

$$D_1 h(d(x, y)) \leq f(x, y) \leq D_2 h(d(x, y)).$$

Then, for $d(x, y) \geq R + S$, we get

$$C_1 g(d(x, y)) \leq f(x, y) \leq D_2 h(d(x, y))$$

so for any $t = d(x, y) \geq R + S$ for some $(x, y) \in X \times X$

$$g(t) \leq \frac{D_2}{C_1} h(t)$$

and ,similarly, for such a t,

$$h(t) \frac{D_1}{C_2} \leq g(t)$$

Since (X,d) is an unbounded, quasi-geodesic metric space there exists some $T \geq 0$ such that for all $t \geq R+S$, there exists $t_l \leq t \leq t_r$, with $t_l, t_r \in \{d(x,y) : (x,y) \in X \times X\}$ with $t - t_l, t_r - t \leq T$.

Now, using the coarse invariance of g, h, choose $E_1, E_2 > 0$ such that $h(t+T) \geq E_1 h(t)$ and $h(t-T) \leq E_2 h(t)$ for sufficiently large t. Then using the fact that g and h are decreasing, for sufficiently large t, we have

$$g(t) \geq g(t_r) \geq \frac{D_1}{C_2} h(t_r) = \frac{D_1}{C_2} h(t + (t_r - t)) \geq \frac{D_1}{C_2} h(t+T) \geq \frac{E_1 D_1}{C_2} h(t)$$

$$g(t) \leq g(t_l) \leq \frac{D_2}{C_1} h(t_l) = \frac{D_2}{C_1} h(t - (t - t_l)) \leq \frac{D_2}{C_1} h(t-T) \leq \frac{E_2 D_2}{C_1} h(t)$$

Therefore, g and h are coarsely commensurate. The second part follows from the coarse invariance assumption.

□

We thus have, from the above proposition, a well-defined meaning for "decay rate" for functions $f : X \times X \to \mathbb{R}_+$ that decay coarsely uniformly which depends only on the coarse structure of (X,d), at least in the case that (X,d) is quasi-geodesic. A main characteristic is that for this to be well-defined, we need the decaying test functions to be coarsely invariant.

As a counterexample, take the generic exponential function $g(t) = Aexp(-Bt)$ we consider when defining finite/infinite length. Then,

$$g(Lt + C) = Aexp(-BLt - BC) = Aexp(-BC)exp(-BLt),$$

which decays much faster than $g(t)$ at infinity if $L \neq 1$. Hence there is no constant D with $Dg(t) \leq g(Lt + C)$, so exponential functions are not coarsely stable. What is happening here is that while the class of exponential functions is stable under quasi-isometry, hence being bounded (above or below) by some exponential is well-defined coarsely. However, no exponential function is itself coarsely invariant in the sense we defined. Therefore, while finite length is well defined, the actual value of the length is not.

This raises the question: which classes of functions, in the context of spin systems, are coarsely simple, and can we characterize coarse commensurability of such classes?

Proposition 6.4. *(Elokl and Jones 2024, 7.10) The functions $\frac{1}{ln(t)}$ and $\frac{1}{t^\gamma}$ where $\gamma > 0$ are coarsely invariant. Moreover, $\frac{1}{t^\gamma} \approxeq \frac{1}{t^\delta}$ if and only if $\gamma = \delta$.*

Proof. For ln, we see that by exponentiation that for sufficiently large t, we get that for any $A > 0, B \in \mathbb{R}$

$$\frac{1}{2}\ln(t) \leq \ln(At + B) \leq 2\ln(t),$$

inverting and reversing the inequalities gives us the desired outcome.

Take $\gamma > 0$. Let C be positive then for any $K > L$, if $t \geq \frac{C}{K-L}$ then $Lt + C \leq Kt$, hence

$$L^\gamma t^\gamma \leq (Lt + C)^\gamma \leq K^\gamma t^\gamma,$$

and inverting gives the desired inequality. Similarly, if $C < 0$, then choosing $K < L$, for $t \geq \frac{-C}{L-K}$ we get

$$K^\gamma t^\gamma \leq (Lt + C)^\gamma \leq L^\gamma t^\gamma,$$

again, inverting gives the desired inequality.

□

It is very intriguing that the asymptotic algebraic and logarithmic decay of functions, which we typically notice in critical behavior and the correlations of observables in conformal field theories, is a coarse invariant. On the other hand, asymptotic exponential decay itself is not. One of the consequences of the above lemma is that for functions with algebraic coarse uniform decay, there is a well-defined analogue of the correlation critical exponent in statistical mechanics, however, it only depends on the coarse structure.

Definition 6.7. *If X is an unbounded, quasi-geodesic metric space, and if $f : X \times X \to \mathbb{R}_+$ decays coarsely uniformly as $t^{-\gamma}$, then γ is the coarse critical exponent.*

Thus, the assertion that the coarse critical exponent, more generally, the coarse commensurability class of a coarsely stable 2-point correlation function, is a universal property of the state.

CHAPTER

7

CONCLUSION

To wrap things up, we run briefly through what we have done here. We show that states in the thermodynamic limit on quantum spin systems can possess an intrinsic large-scale geometry—one that is independent of the physical layout of the underlying hardware. Motivated by recent developments in the realization of topologically ordered states on quantum processors, we introduced a framework using coarse geometry *to formalize this concept. And even though our main examples were states and dynamics on a quantum spin system our frame work could be applied more generally to functions $f : X \times X \to \mathbb{R}_+$ where X is the set of sites.*

Our central result is the construction of a universal coarse structure $\mathcal{E}_\phi$ *associated with a state ϕ in the thermodynamic limit of a quantum spin system. This structure captures the notion of locality in terms of correlation decay. Specifically, we showed that ϕ is local with respect to another coarse structure $\mathcal{E}$ if and only if $\mathcal{E}_\phi \subseteq \mathcal{E}$. This provides a canonical and intrinsic notion of geometry that depends only on the state itself, not on the choice of the hardware or the encoding scheme.*

We further established that under reasonable conditions—namely when $\mathcal{E}_\phi$ is proper and monogenic—the connected completion $(\mathcal{E}_\phi)_{\mathrm{con}}$ is stable under quasi-local perturbations. This is significant because it implies that $(\mathcal{E}_\phi)_{\mathrm{con}}$ serves as a robust basis for order parameters. In particular, this makes it a good candidate for characterizing quantum phases in the thermodynamic limit.

In addition to states, we developed a dynamical coarse structure $\mathcal{E}_\alpha$ associated with quantum channels or circuits. This structure is built from the decay of commutators and captures the locality of time evolution. Just like in the states case, we proved that $\mathcal{E}_\alpha$ is stable under quasi-local perturbations, reinforcing the idea that dynamics in quantum systems also carry an intrinsic large-scale geometry.

This connection between states and dynamical coarse structures led to an application in state preparation. We showed that any circuit used to prepare a state from a product state must be at least as coarsely complex as the state itself: formally, if $\phi = \psi \circ \alpha$, then $\mathcal{E}_{\tilde{\phi}} \subseteq \mathcal{E}_{\tilde{\alpha}}$. This offers a new way to think about the complexity of quantum circuits, not just in terms of gate counts but through their emergent geometry.

Furthermore, we analyzed how key physical properties—such as topological order and correlation decay—relate to coarse structures. We proved that topological order through superselection sectors and correlation length are invariant under coarse equivalence, meaning they are determined entirely by the large-scale geometry. This supports the idea that these properties are not sensitive to small-scale details and can be treated as coarse order parameters.

Together, these results point to a broader idea: coarse geometry provides a universal language for describing the structure and behavior of quantum spin systems. It captures locality, distinguishes phases, and constrains dynamics in a way that is mathematically precise and physically meaningful.

Several interesting directions ariseas a result. One natural question is whether coarse structures can be computed or approximated from finite experimental data. Another is whether this framework can be extended to systems without a clear tensor-product decomposition, such as certain quantum field theories. It would also be valuable to investigate what coarse equivalence might mean in the context of scaling limits or universality classes defined in terms of renormalization groups. Lastly, there may be potential in exploring.

By grounding these questions in the formalism of operator algebras and coarse geometry, this thesis lays the foundation for a new perspective on the large-scale behavior of quantum systems, opening pathways for further mathematical development and physical insight.

CHAPTER

8

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REFERENCES

Affleck, I., Kennedy, T., Lieb, E. H., and Tasaki, H. (1988). Valence bond ground states in isotropic quantum antiferromagnets. Comm. Math. Phys.*, 115(3):477–528.*

Andersen, T. I. et al. (2023). Non-abelian braiding of graph vertices in a superconducting processor. Nature*, 618(7964):264–269.*

Arrighi, P. (2019). An overview of quantum cellular automata. Natural Computing*, 18(4):885–899.*

Bachmann, S., Michalakis, S., Nachtergaele, B., and Sims, R. (2012). Automorphic equivalence within gapped phases of quantum lattice systems. Communications in Mathematical Physics*, 309(3):835–871.*

Batterman, R. W. (2002). The devil in the details. *Oxford Studies in Philosophy of Science. Oxford University Press, New York. Asymptotic reasoning in explanation, reduction, and emergence.*

Bhardwaj, A., Brisky, T., Chuah, C. Y., Kawagoe, K., Keslin, J., Penneys, D., and Wallick, D. (2024). Superselection sectors for posets of von neumann algebras.

Bluhm, A., Capel, Á., and Pérez-Hernández, A. (2022a). Exponential decay of mutual information for Gibbs states of local Hamiltonians. Quantum*, 6:650.*

Bluhm, A., Ángela Capel, and Pérez-Hernández, A. (2022b). Exponential decay of mutual information for Gibbs states of local Hamiltonians. Quantum*, 6:650.*

Bratteli, O. and Robinson, D. W. (1987). Operator algebras and quantum statistical mechanics. 1. *Texts and Monographs in Physics. Springer-Verlag, New York, second edition. C^*- and W^*-algebras, symmetry groups, decomposition of states.*

Bratteli, O. and Robinson, D. W. (1997). Operator algebras and quantum statistical mechanics. 2. *Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition. Equilibrium states. Models in quantum statistical mechanics.*

Bravyi, S., Hastings, M. B., and Michalakis, S. (2010). Topological quantum order: Stability under local perturbations. Journal of Mathematical Physics*, 51(9):093512.*

Cao, C., Carroll, S. M., and Michalakis, S. (2017). Space from Hilbert space: recovering geometry from bulk entanglement. Phys. Rev. D*, 95(2):024031, 20.*

Cha, M., Naaijkens, P., and Nachtergaele, B. (2018). The complete set of infinite volume ground states for Kitaev's abelian quantum double models. Comm. Math. Phys.*, 357(1):125–157.*

Chazal, F. and Michel, B. (2021). An introduction to topological data analysis: Fundamental and practical aspects for data scientists. Frontiers in Artificial Intelligence*, 4.*

Chen, X., Gu, Z.-C., and Wen, X.-G. (2010). Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. Phys. Rev. B*, 82:155138.*

Elokl, A. and Jones, C. (2024). Universal coarse geometry of spin systems.

Ewert, E. E. and Meyer, R. (2019). Coarse geometry and topological phases. Communications in Mathematical Physics, *366(3):1069–1098.*

Fannes, M., Nachtergaele, B., and Werner, R. F. (1992). Finitely correlated states on quantum spin chains. Comm. Math. Phys., *144(3):443–490.*

Farrelly, T. (2020). A review of Quantum Cellular Automata. Quantum, *4:368.*

Fiedler, L. and Naaijkens, P. (2015). Haag duality for Kitaev's quantum double model for abelian groups. Rev. Math. Phys., *27(9):1550021, 43.*

Fröhlich, J. and Ueltschi, D. (2015a). Some properties of correlations of quantum lattice systems in thermal equilibrium. Journal of Mathematical Physics, *56(5):053302.*

Fröhlich, J. and Ueltschi, D. (2015b). Some properties of correlations of quantum lattice systems in thermal equilibrium. Journal of Mathematical Physics, *56(5):053302.*

Ghez, P., Lima, R., and Roberts, J. E. (1985). W^*-categories. Pacific J. Math., *120(1):79–109.*

Glimm, J. G. (1960). On a certain class of operator algebras. Trans. Amer. Math. Soc., *95:318–340.*

Glimm, J. G. and Kadison, R. V. (1960). Unitary operators in C^*-algebras. Pacific J. Math., *10:547–556.*

Greenberger, D. M., Horne, M. A., and Zeilinger, A. (1989). Going Beyond Bell's Theorem, *pages 69–72. Springer Netherlands, Dordrecht.*

Haag, R. (1996). Local quantum physics. *Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition. Fields, particles, algebras.*

Haag, R. and Kastler, D. (1964). An algebraic approach to quantum field theory. J. Mathematical Phys., *5:848–861.*

Hastings, M. B. (2004). Lieb-schultz-mattis in higher dimensions. Phys. Rev. B, *69:104431.*

Hastings, M. B. (2007). An area law for one-dimensional quantum systems. J. Stat. Mech. Theory Exp., *(8):P08024, 14.*

Hastings, M. B. and Koma, T. (2006). Spectral gap and exponential decay of correlations. Commun. Math. Phys., *265:781–804.*

Iqbal, M. et al. (2024). Non-abelian topological order and anyons on a trapped-ion processor. Nature, *626(7999):505–511.*

Johnson-Freyd, T. (2022). On the classification of topological orders. Comm. Math. Phys., *393(2):989–1033.*

Jones, C. (2024). DHR bimodules of quasi-local algebras and symmetric quantum cellular automata. Quantum Topology.

Jones, C. and Lim, J. (2024). An index for quantum cellular automata on fusion spin chains. Annales Henri Poincaré, *25(10):4399–4422.*

Jones, C., Naaijkens, P., Penneys, D., and Wallick, D. (2023). Local topological order and boundary algebras.

Kishimoto, A., Ozawa, N., and Sakai, S. (2003). Homogeneity of the pure state space of a separable C^*-algebra. Canad. Math. Bull., *46(3):365–372.*

Kitaev, A. (2006). Anyons in an exactly solved model and beyond. Annals of Physics, *321(1):2–111. January Special Issue.*

Kliesch, M., Gogolin, C., Kastoryano, M. J., Riera, A., and Eisert, J. (2014). Locality of temperature. Phys. Rev. X, *4:031019.*

Kubota, Y. (2017). Controlled topological phases and bulk-edge correspondence. Comm. Math. Phys., *349(2):493–525.*

Kubota, Y., Ludewig, M., and Thiang, G. C. (2022). Delocalized spectra of Landau operators on helical surfaces. Comm. Math. Phys., *395(3):1211–1242.*

Landsman, K. (2017). Foundations of quantum theory, *volume 188 of* Fundamental Theories of Physics. *Springer, Cham. From classical concepts to operator algebras.*

Lieb, E. H. and Robinson, D. W. (1972). The finite group velocity of quantum spin systems. Comm. Math. Phys., *28:251–257.*

Ludewig, M. (2023). Coarse geometry and its applications in solid state physics.

Ludewig, M. and Thiang, G. C. (2021). Gaplessness of Landau Hamiltonians on hyperbolic half-planes via coarse geometry. Comm. Math. Phys., *386(1):87–106.*

Ludewig, M. and Thiang, G. C. (2022). Large-scale geometry obstructs localization. J. Math. Phys., *63(9):Paper No. 091902, 8.*

Naaijkens, P. (2011). Localized endomorphisms in Kitaev's toric code on the plane. Rev. Math. Phys., *23(4):347–373.*

Naaijkens, P. (2012). Haag duality and the distal split property for cones in the toric code. Lett. Math. Phys., *101(3):341–354.*

Nachtergaele, B., Ogata, Y., and Sims, R. (2006). Propagation of correlations in quantum lattice systems. J. Stat. Phys., *124(1):1–13.*

Nachtergaele, B., Scholz, V. B., and Werner, R. F. (2013). Local approximation of observables and commutator bounds. In Operator methods in mathematical physics, *volume 227 of* Oper. Theory Adv. Appl., *pages 143–149. Birkhäuser/Springer Basel AG, Basel.*

Nachtergaele, B. and Sims, R. (2006). Lieb-Robinson bounds and the exponential clustering theorem. Comm. Math. Phys., *265(1):119–130.*

Nielsen, M. A. and Chuang, I. L. (2000). Quantum computation and quantum information. *Cambridge University Press, Cambridge.*

Nowak, P. W. and Yu, G. (2012). Large scale geometry. *EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich.*

Nowak, P. W. and Yu, G. ([2023] ©2023). Large scale geometry. *EMS Textbooks in Mathematics. EMS Press, Berlin, second edition.*

Ogata, Y. (2022). A derivation of braided C^*-tensor categories from gapped ground states satisfying the approximate Haag duality. J. Math. Phys.*, 63(1):Paper No. 011902, 48.*

Onsager, L. (1944). Crystal statistics. I. A two-dimensional model with an order-disorder transition. Physical Review*, 65(3-4):117–149.*

Orús, R. (2014). A practical introduction to tensor networks: matrix product states and projected entangled pair states. Ann. Physics*, 349:117–158.*

Papasoglu, P. (2023). Polynomial growth and asymptotic dimension. Israel J. Math.*, 255(2):985–1000.*

Paulsen, V. (2002). Completely bounded maps and operator algebras*, volume 78 of* Cambridge Studies in Advanced Mathematics. *Cambridge University Press, Cambridge.*

Perez-Garcia, D., Verstraete, F., Wolf, M. M., and Cirac, J. I. (2007). Matrix product state representations. Quantum Inf. Comput.*, 7(5-6):401–430.*

Qi, X.-L. (2013). Exact holographic mapping and emergent space-time geometry.

Qi, X.-L. and Ranard, D. (2019). Determining a local Hamiltonian from a single eigenstate. Quantum*, 3:159.*

Raasakka, M. (2017). Spacetime-free approach to quantum theory and effective spacetime structure. SIGMA Symmetry Integrability Geom. Methods Appl.*, 13:Paper No. 006, 33.*

Roe, J. (1993). Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc.*, 104(497):x+90.*

Roe, J. (2003). Lectures on coarse geometry*, volume 31 of* University Lecture Series. *American Mathematical Society, Providence, RI.*

Rørdam, M., Larsen, F., and Laustsen, N. (2000). An introduction to K-theory for C^*-algebras*, volume 49 of* London Mathematical Society Student Texts. *Cambridge University Press, Cambridge.*

Sachdev, S. (2011). Quantum phase transitions. *Cambridge University Press, Cambridge, second edition.*

Sakai, S. (1968). Derivations of simple C^*-algebras. J. Functional Analysis*, 2:202–206.*

Sako, H. (2020). Finite-dimensional approximation properties for uniform Roe algebras. J. Lond. Math. Soc. (2), *102(2):623–644.*

Satzinger, K. J., , et al. (2021). Realizing topologically ordered states on a quantum processor. Science, *374(6572):1237–1241.*

Schlingemann, D. (1999). From euclidean field theory to quantum field theory. Reviews in Mathematical Physics, *11(09):1151–1178.*

Schollwöck, U. (2011). The density-matrix renormalization group in the age of matrix product states. Ann. Physics, *326(1):96–192.*

Schumacher, B. and Werner, R. F. (2004). Reversible quantum cellular automata.

Spanier, E. H. ([1995?]). Algebraic topology. *Springer-Verlag, New York. Corrected reprint of the 1966 original.*

Stanley, H. (1987). Introduction to Phase Transitions and Critical Phenomena. *International series of monographs on physics. Oxford University Press.*

Tomba, M., Wei, S., Hungar, B., Wallick, D., Kawagoe, K., Chuah, C. Y., and Penneys, D. (2023). Boundary algebras of the kitaev quantum double model.

Walker, K. and Wang, Z. (2012). (3+1)-tqfts and topological insulators. Frontiers of Physics, *7(2):150–159.*

Willett, R. and Yu, G. (2020). Higher index theory, *volume 189 of* Cambridge Studies in Advanced Mathematics. *Cambridge University Press, Cambridge.*

Xu, S. and other (2024). Non-abelian braiding of fibonacci anyons with a superconducting processor. Nature Physics, *20(9):1469–1475.*

Zeng, B., Chen, X., Zhou, D.-L., and Wen, X.-G. (2019). Quantum information meets quantum matter. *Quantum Science and Technology. Springer, New York. From quantum entanglement to topological phases of many-body systems, With a foreword by John Preskill.*

APPENDIX

APPENDIX

A

ACRONYMS

A summary of all acronyms is documented in Table A.1.

Table A.1: A summary of acronyms used in alphabetical order.

Acronym	*Abbreviation*
Unital Completely Positive	*ucp*
Quantum Field Theory	*QFT*
Conformal Field Theory	*CFT*