Exact Analytical Solutions in Elastoplasticity

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ABSTRACT

Numerical calculations of elastoplastic structures are often very expensive and the precision of the results remain unknown. Exact solutions are necessary to experiment various methods in the aim to reduce the cost of computations or to estimate the error. The well-known analytical solutions are not representative of complex structures. We present a method to build various exact analytical solutions in elastoplasticity by an inverse method. We give different ways to use them in the control of a computer code, in the training of an analyst and in the adaptive meshing of a structure.

INTRODUCTION

Actually, most structures are numerically computed assuming an elastoplastic constitutive law. The cost of these computations is very high and often difficulties of convergence appear. Indeed, the reliability of such analysis can be discussed since there is no method to estimate the real error between the exact solution of the problem and its numerical solution (here we do not consider another source of error with the real data identification or the validity of the constitutive law); the parameters of the numerical computation can be badly chosen, or worst, the computer code can be false or inaccurate.

Moreover, to reduce the cost of such calculations, no theoretical or empirical solution exists to optimize the parameters of a numerical computation i.e. the mesh, the time discretization or the type of algorithm.

In order to estimate the cost of numerical elastoplastic analysis, many parameters must be taken into account. They are :

• *Number of degrees of freedom (DOF) and number of elements*

The cost of an elastoplastic numerical computation increases with the number of DOF. The global matrix must be factorized at least once. The number of elements occurs in the computation of local stiffness matrices and especially, after each external iteration, in the computation of out-of-balance forces. This is made by integrating on the Gauss points of each element of the difference between the trial stress and the plastically admissible stress. If the number of external iterations is high, the cost of these computations may be prohibitive.

• *Time discretization*

The time step used during the loading has a great influence on the cost. A time step who gives a too high load step increases the number of external iterations and may drive a divergence of the algorithms.

• *Convergence tolerance*

The numerical algorithm stops iterations when the out-of-balance force is smaller than a fraction of the second member. If the size of this remainder is small, some plastic iterations are necessary and the cost increases; but, we do not know if the error decreases at the same speed.

• *Type of the algorithm*

Various algorithms can be used such as initial stiffness, tangent stiffness or secant stiffness. The most efficient one depends of the problem (existence of hardening, alternate loading path...).

In order to estimate errors, a lot of works have been made for the evaluation *a priori* or *a posteriori* of the error between the exact solution of a mechanical problem and the numerical solution computed by a code. We recall that the notion of error must be clearly defined. An "error" is a mesure of the difference between an exact quantity and the approximate associated quantity. An "error estimator" is an approximation of the real error, generally obtained without the exact solution. An "error indicator" is a quantity which cannot be associated to an error but has the same properties (its value is zero only when the numerical solution is exact).

• A priori *estimation of the error*

In elasticity, mathematical estimations may be found in the form

$$\| \mathbf{u}_{ex} - \mathbf{u}_h \|_{m,\Omega} \leq C_m \, h^{p+1-m} \, | \mathbf{u}_{ex} |_{p+1,\Omega}$$

where p is the maximum degree of the shape functions of the element, h the size of the element, $\mathbf{u}_{ex}$ and $\mathbf{u}_h$ exact and approximated displacement fields and $\| \cdot \|_{m,\Omega}$ and $| \cdot |_{m,\Omega}$ the norm and semi-norm of order m associated to the Sobolev spaces.

Such rules are generally not practical but they can give rates of convergence.

• A posteriori *estimation of the error*

A numerical solution is available. Some authors propose in the elastic case error estimators and error indicators computed from the approximate stress field. They use generally the jumps of stress between elements and the remainders of equilibrium equations. For example, Coffignal (1987) proposed the definition of the error in constitutive law by using an associated statically admissible stress field; this field is locally computed from the approximate stress field and does not verify the constitutive law. The method was extended to elastoplastic structures.

Another problem is the search of optimal meshes. Some authors proposed meshing rules from the error indicators or from the results of low-cost computations on a coarse mesh to find a better mesh. They do not know the associated error. For example, Turcke & Mc Neice (1974) proposed to dispose the nodes on iso-SED lines (SED is strain energy density or $\epsilon.L.\epsilon$).

As underlined, many approaches of error evaluation and optimal discretization exist in the elastic case but only a few in the elastoplastic case. The reliability of these rules is limited to their trial on known exact solutions which exist in sufficient number in elasticity but not in elastoplasticity.

For that reason, in this paper, our main purpose will be to propose the construction of analytical exact solutions in elastoplasticity. These solutions will allow the test of various methods to reduce the cost of numerical calculations.

PRINCIPLES OF EXACT SOLUTIONS

An exact solution for the response of a continuous structure is the set of a displacement field, a strain field exactly computed from the previous displacement field, a stress field exactly computed by a material constitutive law which must satisfy the loading and geometry data.

In elasticity, several analytical solutions of various problems are available. Various kinds of geometries, loadings, boundary conditions can be found even for contact (Hertz problems), infinite and semi-infinite bodies (Boussinesq), singular solutions of cracks... In elastoplasticity, only some solutions are known such as the simple traction or torsion of bars, thick cylinders or spheres under pressure (Hill, Mandel). Usually these examples have only one spatial variable and they are not representative of mechanical problems that engineers have indeed to solve.

Pseudo-Exact Solutions Obtained by Extrapolation

For many elastic problems, it can be shown that the exact solution may be reached for infinitely fine meshes (if we assume that there is no troncature error during the numerical calculations). A pseudo-exact solution can be obtained by extrapolation of the results of meshes when each mesh contains the nodes of the previous one. However, for some problems, as in classical shell theories, the computation may diverge when the refinement is too high. In elastoplasticity, no such proof is available!

Exact Solutions by an Inverse Method

Since the solutions given by the extrapolation method are not rigorously exact and are expensive to build, we propose the use of an inverse method.

At first, we shall explain the procedure in the elastic case.

Exact fields

Exact fields are evaluated with the following scheme :

• *Displacements*

We define a displacement field $u_i = f_u(x_j,t)$ with some analytical functions f_u.

• *Strains*

In the case of infinitesimal strains, strains are computed from displacements by analytical derivation :

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (1)$$

• *Stresses*

For an elastic body, the HOOKE law gives the stresses from the strains by $\sigma = \mathbf{L}.\epsilon$.

• *Body forces*

Since this stress field must satisfy equilibrium equations, we find the necessary associated body forces

$$X_i = -\frac{\partial \sigma_{ij}}{\partial x_j} \text{ (statical case)} \quad \text{or} \quad -\frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial^2 u_i}{\partial t^2} \text{ (dynamical case).} \qquad \text{(2a and b)}$$

So, we have defined the 4 exact fields $\mathbf{u}$, ϵ, σ and $\mathbf{X}$.

Boundary value problem

Now, with always these fields, we may define an infinite number of boundary value problems :

• *Geometry*

We choose one domain Ω of arbitrary shape and its boundary $\partial\Omega$. This boundary may be splitted in any $\partial_u\Omega$, where the displacements are taken from the previous displacement field and the complementary part $\partial_F\Omega$.

• *Loading*

In Ω, the loading is taken from the previous body forces field.

On $\partial_F\Omega$, we compute the surface forces associated to the stress field by

$F_s = \sigma_{ij}n_j$ where n_j is the external normal of $\partial_F\Omega$ at the given point. (3)

• *Material constitutive law*

Of course, it is the elastic law used for the computation of the stress field.

This boundary values problem is well defined mathematically but, of course, we know its exact solution.

We can obtain its numerical solution with any finite element code. Now, the error may be exactly computed by the difference between the exact field and the corresponding approximated field.

Practical usage of the inverse method

• *Hints for the choice of the functions f_u*

These functions must have second derivatives versus space variables; in the dynamical case, second derivatives versus time must be finite too.

When only the purely mathematic aspect of the problem is considered, special functions with high or infinite strains can be used, such as during fracture mechanics.

The variations of the strain field may be strong at some places and weak at some others; a "good" displacement function must offer a convenient mixture of low and high gradients of strain.

In elasticity, functions which will give zero body forces can be found; the loading of all boundary values problems based on these functions is limited at surface forces and assumed displacements.

• *Equivalent nodal forces*

After having defined the boundary values problem, we have to compute it with a numerical code. A discretization into finite elements is classically made for the geometry. We have to give to the code a discretization of the loading. If the displacements functions f_u are complex, the body forces field may have a complex shape, with fast variations. A finite element model uses a discretized model of a continuous structure. The unknown terms are the nodal displacements and the second member contains nodal values of loading. If the code accepts distributed loads in its data, usually, it converts them into nodal values. Generally distributed loads are limited to constant or linear values in the volume of each element. The variation of the loading can be too complex to allow an exact evaluation of the nodal forces by the code. However, an exact evaluation of these forces is necessary. Otherwise the numerical problem will not correspond to the exact problem. For isoparametric elements, the expression of the nodal force F_i at node i, equivalent to the body force X on the volume V_e of an element, has the value

$$F_i = \int_{V_e} N_i(x,y)\, X(x,y)\, dx\, dy \qquad (4)$$

or on the parent element : $$F_i = \int_{V_{parent}} N_i(s,t)\, X(x(s,t),y(s,t))\, J_\phi(s,t)\, ds\, dt, \qquad (5)$$

where x and y are the real coordinates, s and t the local coordinates, J_ϕ the jacobian and N_i the shape functions.

• *Evaluation of the errors*

An "absolute" error between numerical and exact solutions is computed by a measure of the difference between the exact field (stress, strain, displacement...) and the corresponding approximate field. The "relative" error will be the absolute error divided by the norm of the exact field.

• *Numerical integration*

The previous computation of nodal forces needs integration of complex functions which can have jumps (the second derivatives of the displacement functions f_u must exist but must not be continuous). The high-precision integration can need more than 10 Gauss points by space direction, i.e. 100 points for a quadrilateral element or 1000 for a brick.

For the computation of the norm of the error, the fields are more regular and the integration has not to be so precise.

EXACT SOLUTIONS IN ELASTOPLASTICITY

Hypotheses

General hypotheses

In this paper, we limit the study to the case of small strains and purely mechanical problems without thermal or other effects.

Mechanical hypotheses

The materials are homogeneous and isotropic. The elastoplastic constitutive law is the standard plasticity. Indeed, we shall only consider the Mises criterion and perfect plasticity or linear positive kinematic hardening. The initial state of the material is the natural virgin state.

The mechanical properties of materials are thus described by :

- E and ν, (or λ and μ) the elastic coefficients
- C the hardening modulus
- σ_0 the yield limit in uniaxial tension with $y_0 \equiv \sigma_0^2/3$
- ρ the specific mass.

The HOOKE law has the form $\boldsymbol{\sigma} = \mathbf{L}.\boldsymbol{\epsilon}$ or $\boldsymbol{\epsilon} = \mathbf{M}.\boldsymbol{\sigma}$ (6 a and b)

and the Mises criterion is written $f(\sigma, \epsilon^p) = 1/2(\mathbf{S} - C\epsilon^p) : (\mathbf{S} - C\epsilon^p) - y_0 \leq 0$ (7)

where **L** and **M** are the elastic coefficients matrices, $\boldsymbol{\sigma}$ is the stress tensor, ϵ the strain tensor, ϵ^p the plastic strain tensor, $\mathbf{S} = \mathrm{dev}(\sigma)$ = deviatoric part of the stress tensor $\boldsymbol{\sigma}$ and $\mathbf{a}{:}\mathbf{b} \equiv$ the inner product of **a** and **b**.

General equations of standard plasticity are

- stress/strain relation $\boldsymbol{\sigma} = \mathbf{L}\ (\epsilon - \epsilon^p)$ (8)

or for the deviatoric parts $\mathbf{S} = 2\mu\ (\epsilon' - \epsilon^p)$ with $\epsilon' = \mathrm{dev}\ (\epsilon)$ (9)

- flow rule $\dot{\epsilon}^p = \lambda_p \dfrac{\partial f}{\partial \sigma} = \lambda_p \dfrac{\partial f}{\partial \mathbf{S}}$. (10)

Stress and Strain Increments in Elastoplasticity

At a point P, at the time t_k, we have ϵ_k strain tensor, ϵ_k^p plastic strain tensor, σ_k stress tensor; σ_k and ϵ_k^p verify the criterion (7). During the time increment Δt, we add a strain increment $\Delta\epsilon$ and we search the stress tensor σ_{k+1} which verifies (7) at time $t_{k+1} = t_k + \Delta t$.

The trial stress is $\boldsymbol{\sigma}^* = \sigma_k + \Delta\sigma^{el}$; the trial stress increment is $\Delta\sigma^{el} = \mathbf{L}.\Delta\epsilon$.

If $f\left(\boldsymbol{\sigma}^*, \epsilon_k^p\right) \leq 0$ ==> we write $\sigma_{k+1} = \boldsymbol{\sigma}^*$; the problem is solved.

If $f\left(\boldsymbol{\sigma}^*, \epsilon_k^p\right) > 0$ ==> we have to find $\Delta\epsilon^p$ which verifies $f\left(\sigma_k + \mathbf{L}.(\Delta\epsilon - \Delta\epsilon^p), \epsilon_k^p + \Delta\epsilon^p\right) = 0$ (11)

Two methods may be used to solve these equations.

First method or radial projection

It uses a first-order development of the flow rule for the computation of $\Delta\epsilon^p$:

$$\Delta\epsilon^p = \Delta\lambda_p.\left(\frac{\partial f}{\partial \sigma}\right)_{\sigma=\sigma_{k+1}} \tag{12}$$

In the deviatoric stress space, we project normally the point $\boldsymbol{\sigma}^*$ on the convex set of plasticity at time t_{k+1} (implicit algorithm), in order to verify the criterion (11) ((Nguyen, 1973) or (Zarka, 1978)).

For the Mises criterion, the spheric part of the stress tensor stays unchanged :

$$\sigma_{k+1} - S_{k+1} = \boldsymbol{\sigma}^* - \mathbf{S}^*$$

and the value of the deviatoric stress increment ΔS^{ex} is

$$\Delta S^{ex} = \frac{1}{C_2}\left(\Delta S^{el}\left(\frac{C}{2\mu} + \eta\right) + \left(S_k - C\epsilon_k^p\right)(\ \eta - 1\ )\right) \tag{13}$$

with $C_2 = 1+\frac{C}{2\mu}$, $C_1 = \frac{C_2}{2\mu}$, $\eta = \sqrt{\frac{2y_0}{\xi}}$, $\xi = \left(S^* - C\epsilon^P_k\right) : \left(S^* - C\epsilon^P_k\right)$ and $S_{k+1} = S_k + \Delta S^{ex}$.

This is an approximated first-order method and, for a numerical use with a reasonable accuracy, the strain increments $\Delta\epsilon$ must be "not too far" from the convex set of plasticity (except during a radial loading). A solution computed with radial projection is not rigorously "exact" but can be as much precise as wished by dividing the strain increments.

Second method or analytical integration

For the Mises criterion and for a finite strain increment $\Delta\epsilon$ with the form $\Delta\epsilon = \dot{\epsilon}\,\Delta t$, the analytical integration of the evolution equations is possible. Thus the exact value of the plastic strain and stress increments can be computed (Krieg & Krieg, 1977), (Yoder & Whirley, 1984).

The start point P_c (usually called contact point) is taken on the boundary of the convex of plasticity; it verifies $f\left(\sigma_c, \epsilon^P_c\right) = 0$. (see Fig. 1)

The flow rule is written for ψ, the angle between the strain increment direction $\dot{\epsilon}$ and the normal to the convex at the given point. During a plastic loading, the analytical integration gives the evolution of ψ; the exact variation $\Delta\sigma^P$ of σ is computed from the variation of ψ.

At the contact point P_c, we have ϵ_c, ϵ^P_c, σ_c known tensors. We add $\Delta\epsilon_c$; the angle ψ_c is given by

$$\cos\psi_c = \frac{\frac{3}{2}\left(\epsilon'_c - C_2\,\epsilon^P_c\right) : \Delta\epsilon'}{R\sqrt{\frac{3}{2}\Delta\epsilon' : \Delta\epsilon'}} \quad \text{with } R = \frac{\sigma_0}{2\mu}. \tag{14}$$

The flow rule becomes
$$\mathrm{tg}\,\frac{\psi}{2} = \mathrm{tg}\,\frac{\psi_c}{2}\,\exp\left(-\frac{\sqrt{\frac{3}{2}\Delta\epsilon':\Delta\epsilon'}}{R}\right) = K\ \mathrm{tg}\,\frac{\psi_c}{2} \tag{15}$$

and the value of the deviatoric part of the stress increment is

$$\Delta S^{ex} = \frac{\alpha - 1}{C_1}\,\epsilon'_c + \Delta\epsilon'\left(2\mu - \frac{1-\alpha\beta}{C_1}\right) + 2\mu\,(1-\alpha)\,\epsilon^P_c \tag{16}$$

with the terms $\alpha = \frac{\sin\psi}{\sin\psi_c} = K\,\frac{1 + \mathrm{tg}^2\,\frac{\psi_c}{2}}{1 + K^2\ \mathrm{tg}^2\,\frac{\psi_c}{2}}$ and $\beta = \frac{R}{\sqrt{\frac{3}{2}\Delta\epsilon':\Delta\epsilon'}}\left(\frac{\cos\psi}{\alpha} - \cos\psi_c\right)$.

Body Forces in Elastoplasticity

The computation of the body forces needs the derivatives of the stresses with respect to spatial coordinates; of course, the plasticity will modify their values. Two formulations can be presented as for the computation of stress increments.

Radial projection

The derivative of (13) with respect to space variables is

$$\frac{\partial(\Delta S^{ex})}{\partial x} = \frac{1}{C_2}\left[\left(\frac{C}{2\mu} + \eta\right)\frac{\partial(\Delta S^{el})}{\partial x} + (\eta - 1)\frac{\partial S_1}{\partial x} + S_2\,\frac{\partial\eta}{\partial x}\right] \tag{17}$$

with $\sigma_1 = \sigma_k - C\,\epsilon^P_k$, $\sigma_2 = \sigma^* - C\,\epsilon^P_k$ and S_1 and S_2 their deviatoric parts.

This derivative presents a jump at the boundary between the elastic and plastic zones; the body forces will have a jump too.

Let us assume that the point σ_k is a contact point for which $f\left(\sigma_k, \epsilon^P_k\right) = 0^-$ (the point reaches the boundary of the convex set, without crossing it). We add a strain increment $\Delta\epsilon$ such that $f\left(\sigma_k + L.\,\Delta\epsilon, \epsilon^P_k\right) > 0$ and such that its norm $||\ \Delta\epsilon\ ||$ tends towards zero.

The expression (17) becomes $\dfrac{\partial(\Delta S^{ex})}{\partial x} = -\dfrac{1}{2y_0C_2}\left(S_1 : \dfrac{\partial S_1}{\partial x}\right) S_1$. (18)

Generally, it does not tend towards zero with $||\ \Delta\epsilon\ ||$. This residual value gives the jump of the body forces, at the crossing of the boundary between elastic and plastic zones.

Analytical integration

The start point must be a contact point. The derivative of (16) versus space variables is not sufficient to obtain the value of the body force increment since the derivative of the contact relation versus space variables $\dfrac{\partial f(\sigma_c\ ,\ \epsilon_c^p\ )}{\partial x} = 0$ is not verified if the jump (18), computed by radial projection, is not added to the derivative of (16).

The complete formulation of this method can be found in (Hablot, 1989).

Cases of Axisymmetric and Plane Stress Structures

In the axisymmetric case, the use of cylindrical coordinates modifies the expression of the strains and the body forces. The formulation of the elastoplastic problem stays unchanged.

In the plane stress case, some difficulties appear due to the specific hypotheses of plane stress problems. The unknown non-zero z-displacement prevents from building solutions in the dynamical case. Moreover, the projection rules are very complicated. Akel (1987) showed that we can compute ϵ_{zz} which satisfies the Mises criterion and the plane stress conditions $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, by solving numerically an implicit equation. This method is also a first-order method and no analytical integration is available. We refer to (Hablot, 1989) for the complete formulation of statical plane stress problems.

APPLICATIONS

Optimization of Elastoplastic Computations

Test of semi-empirical rules of remeshing

After examining the meshing rules given by some authors in the elastic case, we propose a generalized rule :

$$\frac{h}{r_0} = \frac{k}{\left(\dfrac{r}{r_0}\right)^{\alpha} (r_0\ ||\ \mathrm{grad}(\phi^{\beta})\ ||)^{\gamma}} \qquad (19)$$

where h is the size of an element, r the radius (in the case of a singularity), r_0 a characteristical length of the structure, ϕ a quantity computed with a coarse mesh of the structure, k a constant and α, β and γ unknown exposants. This rule gives an element size field and we use an automatic mesh generator which is able to satisfy this field.

The Turcke & Mc Neice rule is a particular case of the previous rule with $\alpha = 0$, $\beta = 1$, $\gamma = 1$ and ϕ = SED (or SED/E to become adimensional). In elastoplasticity with the Mises criterion, ϕ can be taken as the equivalent stress : $\phi = \sqrt{(3/2S : S)}/\sigma_0$. We test this rule in elastoplasticity with different values of the parameters but where the value ϕ is computed on the structure **assumed elastic**. In this case, the cost of the evaluation of ϕ is low in comparison with the cost of the elastoplastic computation.

Some authors (Coffignal, 1987) have assumed that a mesh is optimal if the contribution of each element to the global error is the same. The exact solutions allow the eventual verification of this rule.

Optimal values of various parameters of numerical elastoplastic computations

Miscellaneous parameters such as the convergence tolerance, the maximum number of plastic iterations or the type of algorithm have also an influence on the cost and the reliability of the computation. The influence of these parameters may be strictly evaluated.

Benchmarks and Training of Users

Exact solutions can be used to test the quality of computer codes, to find formulation or programmation errors during their development or to compare the performances of some algorithms.

Didactic problems can be defined to new users of an industrial code for their training. The possible computation of the error allows them to find step by step the optimal choices of the parameters of elastoplastic computations.

EXAMPLES

As illustration of the preceding paragraphs we give this elementary following example. It consists of a statical two-dimensional plane strain model with the material constants E = 216000 MPa, $\nu = 0.2$, σ_0 = 400 MPa and C = 7200 MPa. At first we build a class of exact solutions; then, we verify the applications of some empirical rules.

Exact solution

We shall take the displacement functions in the polynomial form

$$u = [\ -0.032x^2y + 0.16xy\ ]\ \zeta(t)$$
$$v = [\ 0.032xy^2 - 0.08y^2\ ]\ \zeta(t)$$

where $\zeta(t)$ is a piecewise linear function with $\zeta(t = 0) = 0$, $\zeta(t = 20) = 0.001$, $\zeta(t = 60) = 0.04$, $\zeta(t = 100) = -0.004$. We note that u and v $\equiv$ 0 when y = 0 and that u $\equiv$ 0 when x = 0.

With the special constitutive constants, these displacement functions may give an elastic or elastoplastic loading between times 0 and 60, then an elastic unloading and at last, an elastoplastic reloading between times 60 and 100. The exact stress tensor is purely deviatoric in all the space.

Particular solution

We shall now take a special geometry $\partial\Omega$ such shown in Fig. 2 and we shall consider the boundary conditions in displacement on y = 0 and x = 0 where the exact displacements are equal to zero. The boundary conditions in surface forces are defined on $\partial_F\Omega$ as previously indicated.

For this special Ω, the exact stress is elastic until time t = 20.

Numerical models and Errors

We used the computer code **Dynaflow** for a support of this example. Various meshes were taken which have all the common features :

- the initial stiffness method with only one factorization of the global matrix
- the time step is equal to 5 for all computations
- the elements are 3-node triangles with one stress point.
- the global relative errors are computed from the field $\mathbf{S} - C\epsilon^p$.

Applications

Meshing rules

The convergence tolerance was taken equal to 0.01 for all the meshes. The Fig. 3 shows the errors for five meshing rules. The most efficient rule seems to be the rule with "constant local contribution of each element to the global error". But this error can only be computed when the exact solution is known. (We did not test the direct error in constitutive law such as given by Coffignal, 1987.)

The Fig. 4 shows the variation during the loading of the exact and approximated values of the equivalent stress at the point (x = 5, y = 5) of the structure. (An error is due to the distance between this point and the center of the element which contains the point).

The Fig. 5 shows the variation of the equivalent plastic strain at the same point.

Convergence tolerance

We study the effect of different values of the convergence tolerance for the same mesh and time step. This parameter has a high influence on the cost of computations. When the maximum number of plastic iterations is unlimited, the code continues to compute until it reaches the given tolerance. A maximum value of the tolerance equal to 0.75 was used for the comparison.

The Fig. 6 shows the cost (CPU time) versus the number of iterations.

The Fig. 7 shows the number of iterations versus the tolerance (semi-logarithmic scale).

The Fig. 8 gives the error versus the tolerance; the error increases fast with the tolerance after a value close to 0.02.

At last the Fig. 9 and 10 indicate the evolution of equivalent stress and equivalent plastic strain versus time for different values of the tolerance. High tolerance gives entirely wrong results. The gap between the exact solution and the best approximate solutions (for a tolerance close to 0.001) is due to the time step (here 5) and not to the tolerance.

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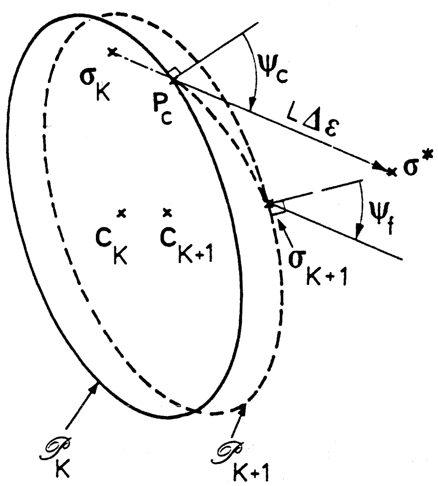


Fig. 1 : Stress increments

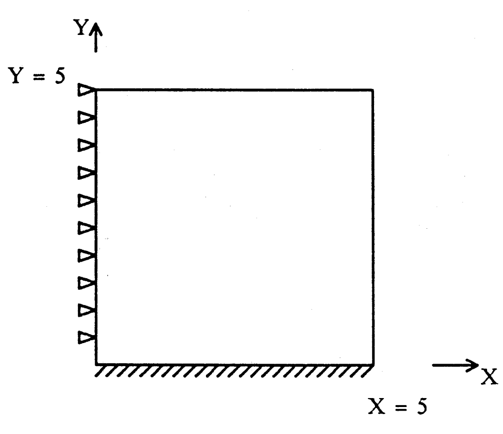
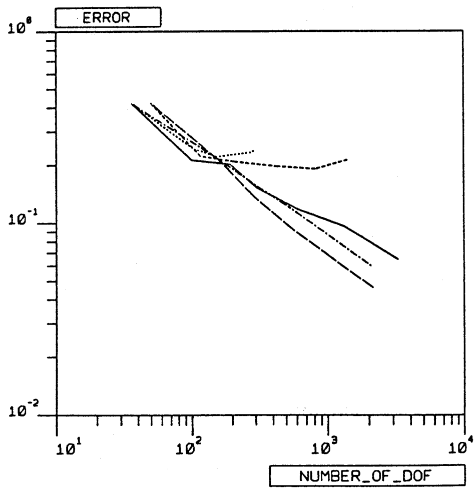


Fig. 2 : Geometry and constrained nodes



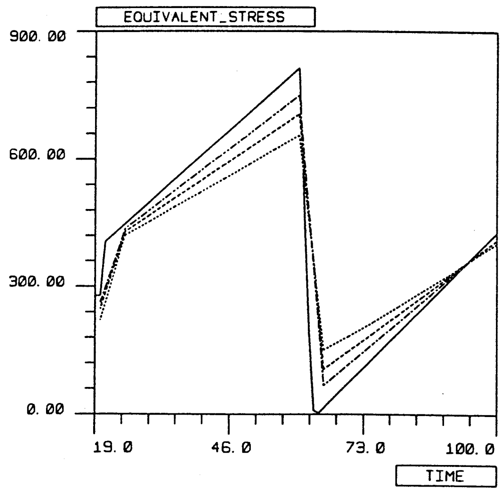
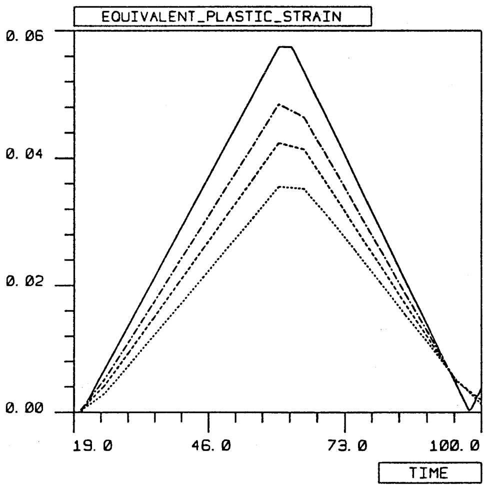


Fig. 3 : Error versus number of DOF with the rules :
1. rule (19) with α=0, β=1, γ=1 and ϕ=Mises equivalent stress
2. idem but γ=3
3. idem but γ=2
4. uniform triangular meshes
5. local error constant for each element

Fig. 4 : Equivalent stress versus time
1. exact solution
2. uniform mesh 128 elements
3. uniform mesh 512 elements
4. uniform mesh 2048 elements



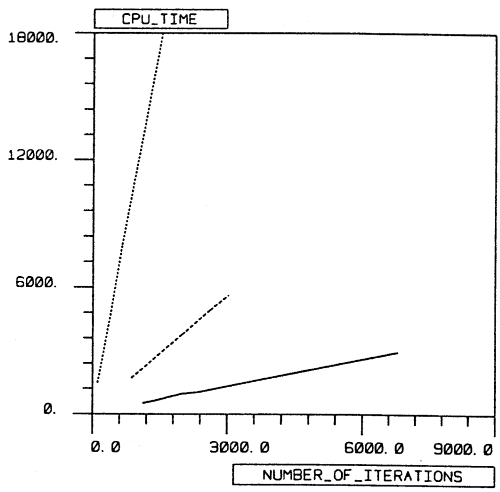
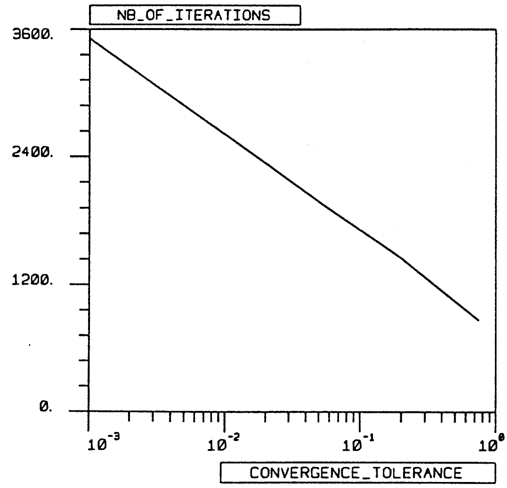


Fig. 5 : Equivalent plastic strain versus time
1. exact solution
2. uniform mesh 128 elements
3. uniform mesh 512 elements
4. uniform mesh 2048 elements

Fig. 6 : Cost versus number of plastic iterations
1. 57 elements mesh
2. 3282 elements mesh
3. 498 elements mesh



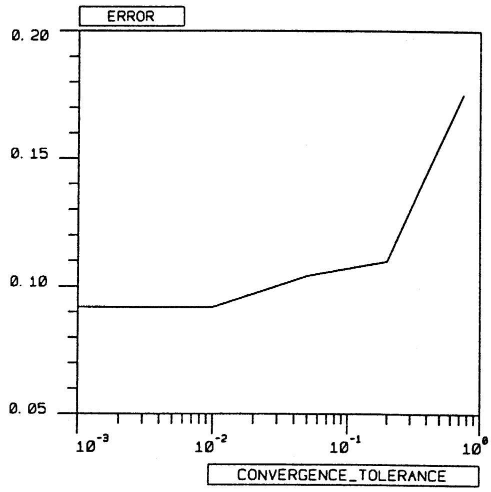


Fig. 7 : Number of iterations versus tolerance

Fig. 8 : Global error versus tolerance

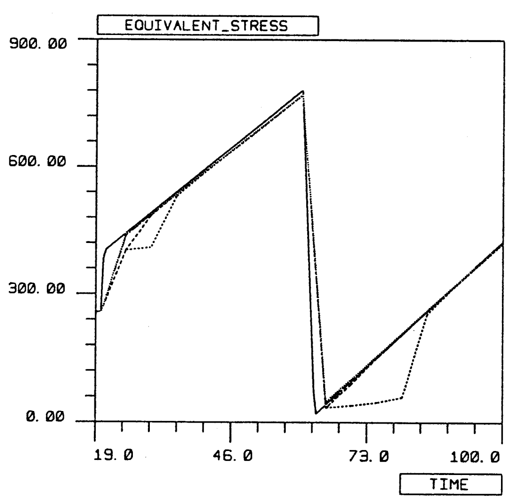


Fig. 9 : Equivalent stress versus time
- 1. exact solution
- 2. convergence tolerance = 0.75
- 3. convergence tolerance = 0.2
- 4. convergence tolerance = 0.05
- 5. convergence tolerance = 0.01
- 6. convergence tolerance = 0.001

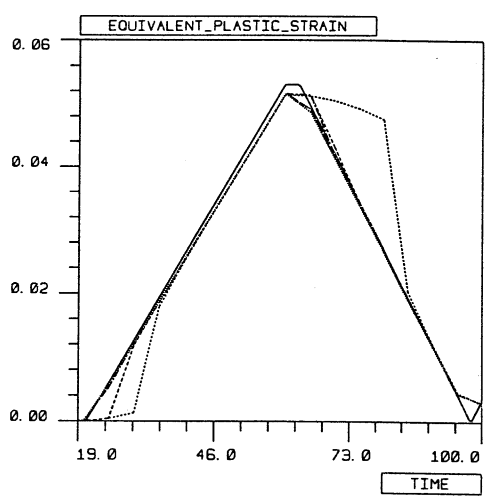


Fig. 10 : Equivalent plastic strain versus time
- 1. exact solution
- 2. convergence tolerance = 0.75
- 3. convergence tolerance = 0.2
- 4. convergence tolerance = 0.05
- 5. convergence tolerance = 0.01
- 6. convergence tolerance = 0.001