SYSTEMATIC ENFORCEMENT OF STRESS BOUNDARY CONDITIONS IN THE ASSUMED STRESS HYBRID MODEL BASED ON THE DEFORMATION METHOD

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ABSTRACT

The favourable influence of the enforcing of stress boundary conditions in the assumed stress hybrid model is demonstrated with examples of singular behaviour. A systematic procedure of satisfying the stress boundary conditions exactly is described.

1. INTRODUCTION

In the assumed stress hybrid model, pioneered by Pian [1], developed further by himself and others [2 through 8], a stress distribution over an element satisfying microscopic equilibrium and independent boundary displacements are assumed. For convergence, the former does not have to satisfy the stress boundary conditions [6]. In general, if these are enforced, a favourable influence on the accuracy of the results is observed, although for certain rather well-behaved problems this effect diminishes as the number of elements increases [2]. In the next section two examples are presented, where the improvement remains as the number of degrees of freedom increases : a skew simply supported plate with a strong singularity, and a pinched cylinder are calculated using the hybrid stress model with and without satisfying stress boundary conditions for different subdivisions into elements. The results are compared to those of other finite elements. As a simple introductory example, a cantilever plate loaded in its plane is examined. In the third section, a consistent formulation for processing stress boundary conditions is presented. Because of the fact that every concentrated force at a cornerpoint is a linear combination of the stresses on the neighbouring sides, linear dependencies can arise. In this case the deformation corresponding to such a concentrated force depends on which stress boundary conditions have really been used in the analysis. Starting from the stress-distribution and all other cornerpoint-deformations of the element, such a deformation can be recalculated. This procedure is described in the fourth section. Apart from improving the accuracy, the enforcing of stress boundary conditions can also be used to introduce external loads and reaction forces into the structural system in special ways. A general finite-element analyser for plates and shells called STRIP [9 through 12], based on the stress hybrid model, and which places a lot of emphasis on enforcement of stress boundary conditions, has been developed. As an example, the calculation of a hyperbolic paraboloid is presented. The result of the finite-element analysis is compared with that of a model test in the last section.

2. ACCURACY WITH AND WITHOUT SATISFYING STRESS BOUNDARY CONDITIONS

The cantilever plate loaded in its plane of fig. 1 has been analysed for the specified loads with four finite-elements only. A quadratic stress distribution with 12 coefficients and a linear deformation along the boundary have been used [2 or 9, p. 172]. The analytical solution is given in fig. 1; depending on how certain boundary conditions in the built-in section are formulated, different theoretical results arise. Satisfying stress boundary conditions improves the deformations and the stresses considerably (fig. 1).

As a second example, a skew simply supported plate under a distributed load has been calculated (fig. 2). This example has singular points. The bending moments in the obtuse corner are infinite using Kirchhoff's theory. This singularity is strong; even based on the assumptions of Reissner, one of the moments is infinite if the same boundary conditions are introduced [13, p. 186]. No theoretical solution exists; Morley's solution using a series expansion with coefficients determined by the least-square method, is precise [14]. The effect of the singularity on the results of the assumed stress hybrid model with a quadratic stress distribution (17 coefficients) and cubic displacements and linear rotations on the boundary [2 or 9, p. 172] with and without satisfaction of stress boundary conditions for different subdivisions into elements is examined. For comparison a corresponding conforming displacement model CQ [15] with cubic deformations and an equilibrium model EQT [16] are used [13].

For the hybrid model, five different modelisations have been chosen, the shape of the element always being a parallelogram. When the stress boundary conditions are enforced, the bending moment m_n is set equal to zero along the boundary. The vertical deflection w and the two bending moments m_x and m_y in the centre of the plate are given as a function of the degree of freedom before introducing any deformation boundary conditions in the fig. 2 and 3. The results speak for themselves. If the stress boundary conditions are not satisfied, the hybrid model gives similar results as the conforming displacement model CQ (apart from the coarse meshes). For these two models, the influence of the singularity on the results in the centre of the plate is strong; even for a system of equations with more than a 1000 unknowns, the error is considerable. On the other hand the hybrid model with enforced stress boundary conditions gives good results, the accuracy of the moments is excellent! Fig. 4 shows how the singularity itself is represented. The variation of the principal moments m_x and m_y from the centre to the obtuse angle is plotted for the model CQ (14 x 14 elements, 1095 unknowns), EQT (6 x 6 elements, 481 unknowns) and for the hybrid models

(16 x 16 elements, 867 unknowns). The results of the hybrid model satisfying the stress boundary conditions are the best. The conforming displacement model CQ and the hybrid model without satisfying the stress boundary conditions cannot be used for the prediction of m_x; even the sign is wrong!

The last example consists of a simply supported cylinder, exposed to two concentrated loads which are in equilibrium (fig. 5). Flat elements are used for the assumed stress hybrid model. Because of symmetry, only an eighth had to be analysed. When applicable, the specified stress boundary conditions were used. To be able to apply the concentrated load in a reasonable way, no stress boundary conditions were introduced on the two sides of the loaded element. Five different subdivisions into elements were used for the hybrid models. The theoretical value of the displacement under the concentrated load is given in reference [17] . The same example has also been calculated by Key and Beisinger [18] , using their conforming displacement model, which is based on a curved middle surface, takes care of the rigid body motion and permits a general geometry. Fig. 5 represents the deformation under the load as a function of the mesh size and of the number of unknowns in the final system of equations before introducing the geometrical boundary conditions. Both hybrid models using flat elements give better results than the conforming displacement model with a curved middle surface, the comparison being based on the number of equations. Again, enforcing stress boundary conditions has a favourable influence on the accuracy of the results. As in the case of a skew plate, this improvement remains as the number of elements increases. The analysis using a 8 x 8 mesh satisfying stress boundary conditions involving 486 equations results in a value of 0,0121; if no stress boundary conditions are enforced, a mesh 20 x 20 with 2646 unknowns has to be chosen to give the same deflection (the latter is not shown in the fig. 5).

3. MATHEMATICAL FORMULATION

The mathematical formulation of systematically enforcing stress boundary conditions on an element basis is illustrated with a simple example. A quarter of the simply-supported square plate under evenly distributed load is represented in fig. 6. Triangular finite-elements are used. Starting from the assumed distribution of stresses in equilibrium and deformations along the boundaries, all coefficient matrices and the right-hand sides of eq. (2) and (3) can be determined [1, 5, 9] . This includes the stress-distribution of the "built-in" element, which is added to that calculated from the response of the total system, eq. (7). In the example, a second-degree polynomial with 17 coefficients β for the stresses is chosen. The particular solution indicates that the load is carried statically determinately with a beam in the x-direction.

$$
\begin{aligned}
m_x &= \beta_1 + \beta_2.x + \beta_3.y + \beta_4.x^2 + \beta_5.xy + \beta_6.y^2 - \tfrac{p}{2}\cdot x^2 \\
m_y &= \beta_7 + \beta_8.x + \beta_9.y + \beta_{10}.x^2 + \beta_{11}.xy + \beta_{12}.y^2 \\
m_{xy} &= \beta_{13} + \beta_{14}.x + \beta_{15}y + \beta_{16}x^2 - (\beta_4 + \beta_{12}).xy + \beta_{17}y^2
\end{aligned}
\tag{1}
$$

In addition, a cubic displacement and a linear rotation, leading to 3 degrees of freedom per cornerpoint, are chosen.

The structural behavior of an element is governed by the stress-displacement relations (2) and the equilibrium equations (3). The coefficients β take the place of the forces.

$$
\left[\begin{array}{c} H_1 \\ H \\ \hline H_2 \end{array}\right] \cdot \left[\begin{array}{c} \beta \end{array}\right] - \left[\begin{array}{c|c} T_1 & t_1 \\ T & t \\ \hline T_2 & t_2 \end{array}\right] \cdot \left[\begin{array}{c} q \\ \hline u \end{array}\right] = - \left[\begin{array}{c} \delta_0\beta_1 \\ \delta_0\beta \\ \hline \delta_0\beta_2 \end{array}\right]
\tag{2}
$$

$$
\left[\begin{array}{cc|c} T_1^t & & T_2^t \\ & T^t & \\ \hline t_1^t & t^t & t_2^t \end{array}\right] \cdot \left[\begin{array}{c} \beta_1 \\ \beta \\ \hline \beta_2 \end{array}\right] = \left[\begin{array}{c} P \\ \hline p \end{array}\right]
\tag{3}
$$

H = flexibility matrix

T^t = equilibrium matrix (concentrated forces $Q = T^t\beta$)

t^t = additional equilibrium equations (formulated stress boundary conditions)

β = (unknown) coefficients of the stress assumption

q = (unknown) additional deformations, corresponding to the formulated stress boundary conditions

$\delta_0\beta$ = initial strains (e.g. due to temperature effects)

P = concentrated external forces (determined by integrating the "built-in" stresses with the assumed deformations along the boundary)(s. fig. 6).

p = right-hand sides of the additional equilibrium equations (= specified boundary stress [normally zero] minus "built-in" stress)

For the triangular element at the bottom-left corner of the plate, the enforced stress boundary conditions are indicated in fig. 6. A moment (shear force) which varies quadratically (linearly) in the coordinate s measured along a side is determined by three (two) values e.g. the constant, linear and quadratic (constant and linear) terms. These are functions of the β's and represent the additional equilibrium equations t^t. In the example, 8 additional relations are formulated : 3 for m_n along side 3-2, 3 for m_{ns} and 2 for q_n along side 1-2 (s. fig. 7a). This procedure is well known in the statics of member systems where additional equilibrium equations can be formulated for hinges.

To be able to keep the size of the stiffness and load matrices the same as for the elements where no stress boundary conditions are enforced, the u's and the same number of β's ($=\beta_2$) are eliminated. The number of β's is reduced ($\beta_1 = \beta_r$).

Using the lower part of eq. (3) to eliminate β_2

$$t_1^{t}\cdot\beta_1 + t_2^{t}\cdot\beta_2 = p$$

$$\beta_2 = -(t_2^{t})^{-1}\cdot t_1^{t}\cdot\beta_1 + (t_2^{t})^{-1}\cdot p$$

$$\beta = \begin{bmatrix} \beta_1 \\ \hline \beta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ \hline -(t_2^{t})^{-1}\cdot t_1^{t} \end{bmatrix}}_{R_1} \cdot \beta_1 + \underbrace{\begin{bmatrix} O \\ \hline (t_2^{t})^{-1} \end{bmatrix}}_{R_2} p$$

$$\beta = R_1\cdot\beta_r + R_2\cdot p \tag{4}$$

Backsubstituting into the upper part of (3)

$$T^{t}\cdot R_1\cdot\beta_r = P - T^{t}\cdot R_2\cdot p$$

$$T_r^{t}\cdot\beta_r = P_r \tag{5}$$

with

$$T_r = R_1^{t}\cdot T$$

$$P_r = P - T^{t}\cdot R_2\cdot p$$

Using the lower part of eq. (2) to eliminate u

$$H_2\cdot\beta - T_2\cdot q - t_2\cdot u = -\delta_o\beta_2$$

$$u = -t_2^{-1}\cdot T_2\cdot q + t_2^{-1}\cdot H_2\cdot R_1\cdot\beta_r + t_2^{-1}\cdot\delta_o\beta_2 + t_2^{-1}\cdot H_2\cdot R_2\cdot p$$

Substituting into the upper part of eq. (2)

$$H_1\cdot\beta - T_1\cdot q - t_1\cdot u = -\delta_o\beta_1$$

$$(H_1\cdot R_1 - t_1\cdot t_2^{-1}\cdot H_2\cdot R_1)\cdot\beta_r - (T_1 - t_1\cdot t_2^{-1}\cdot T_2)\cdot q = -\delta_o\beta_1 + t_1\cdot t_2^{-1}\cdot\delta_o\beta_2 - H_1\cdot R_2\cdot p + t_1\cdot t_2^{-1}\cdot H_2\cdot R_2\cdot p$$

$$\underset{R_1^t}{\left[\begin{array}{c|c} I & -t_1\cdot(t_2)^{-1} \end{array}\right]}\cdot\left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right]\cdot R_1\cdot\beta_r - \underset{R_1^t}{\left[\begin{array}{c|c} I & -t_1\cdot(t_2)^{-1} \end{array}\right]}\cdot\left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right]\cdot q =$$

$$- \underset{R_1^t}{\left[\begin{array}{c|c} I & -t_1\cdot t_2^{-1} \end{array}\right]}\cdot\left[\begin{array}{c} \delta_0\beta_1 \\ \hline \delta_0\beta_2 \end{array}\right] - \underset{R_1^t}{\left[\begin{array}{c|c} I & -t_1\cdot t_2^{-1} \end{array}\right]}\cdot\left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right]\cdot R_2 p \qquad (6)$$

$$R_1^{t}\cdot H\cdot R_1\cdot\beta_r - R_1^{t}\cdot T\cdot q = - R_1^{t}\cdot\delta_0\beta - R_1^{t}\cdot H\cdot R_2\cdot p$$

$$H_r\cdot\beta_r - T_r\cdot q = -\delta_0\beta_r$$

with

$$H_r = R_1^{t}\cdot H\cdot R_1$$

$$\delta_0\beta_r = R_1^{t}\cdot\delta_0\beta + R_1^{t}\cdot H\cdot R_2\cdot p$$

Eq. (5) and (6) are "reduced" stress-displacement and equilibrium equations, respectively. Eliminating β_r, the stiffness matrix S is determined.

From eq. (6)

$$\beta_r = H_r^{-1}\cdot T_r\cdot q - H_r^{-1}\cdot\delta_0\beta_r \qquad (7)$$

Substituting into eq. (5)

$$T_r^{t}\cdot H_r^{-1}\cdot T_r\cdot q = P_r + T_r^{t}\cdot H_r^{-1}\cdot\delta_0\beta_r$$

$$S\cdot q = P_r + T_r^{t}\cdot H_r^{-1}\cdot\delta_0\beta_r \qquad (8)$$

with

$$S = T_r^{t}\cdot H_r^{-1}\cdot T_r$$

(s. fig. 6)

If in the additional equations t^t, all contributions of a (several) row (s) of T^t are formulated, a (several) linear dependency(ies) arises (arise). In the example, this is the case for the concentrated moment Q_{My} in the y-direction at cornerpoint 1 (see fig. 6 and 7a). All cases of linear dependencies for plates in bending are represented in fig. 8. In more complicated cases, linear dependencies among t^t can also be found (e. g. in point 2 of fig. 1). t_2^{-1} can be determined, by first exchanging the concentrated force(s) Q, corresponding to the row(s) of T^t, for a δ(or β's, fig. 7b), second by exchanging as many rows of t^t as possible (two-dimensional pivot search)(fig. 7c) and third by exchanging the concentrated force(s) Q back again (fig. 7d). The linear dependency (constant term of m_n along side 3-1) is thus automatically deleted. It can be shown that the resulting β distribution and the deformations q of those concentrated forces which have no linear dependancy do not depend on which extra equilibrium equations are actually used. If the constant term of m_n along side 3-1 is deleted, the rotation around the y-axes in cornerpoint 1 is calculated as 1232. By slightly modifying the two-dimensional pivot-search, any of the additional equations can be suppressed. For the linear term of m_n along side 3-1, the rotation is determined as 1210; for the quadratic : 1198. For the constant term of m_{ns} along side 1-2, it is determined as 1232, for the linear : 1156; and for the quadratic : 1158. For the constant term of q_n along side 1-2, it is determined as 1150; for the linear : 1147 (see appendix for proof).

4. DEFORMATIONS IN CORNERPOINTS WITH CONCENTRATED FORCES WHICH ARE DEPENDENT ON THE ENFORCED STRESS BOUNDARY CONDITIONS

Starting from the given stress state characterized by β and all other cornerpoint deformations q_A of the element, the deformation(s) q_B corresponding to the concentrated force(s) which is (are) dependent on the enforced stress boundary conditions is (are) determined in an independent second step. The columns of the T matrix are partitioned in the same way as q (T_A, T_B). Writing down the stress-displacement relations (9) and part of the equilibrium equations (10), we have :

$$H \cdot \beta - T_A \cdot q_A - T_B \cdot q_B = - \delta_0 \beta \tag{9}$$
$$T_B^{t} \cdot \beta = T_B^{t} \cdot \beta \tag{10}$$

Eliminating β from eq. (9) and substituting into the left-hand side of the eq. (10), we can determine q_B :

$$\beta = H^{-1} \cdot T_B \cdot q_B + H^{-1} \cdot T_A \cdot q_A - H^{-1} \cdot \delta_0 \beta$$
$$T_B^{t} \cdot H^{-1} \cdot T_B \cdot q_B = T_B^{t} \cdot \beta - T_B^{t} \cdot H^{-1} \cdot T_A \cdot q_A + T_B^{t} \cdot H^{-1} \cdot \delta_0 \beta$$
$$q_B = (T_B^{t} \cdot H^{-1} \cdot T_B)^{-1} \cdot \left[T_B^{t} \cdot \beta - T_B^{t} \cdot H^{-1} \cdot T_A q_A + T_B^{t} \cdot H^{-1} \cdot \delta_0 \beta \right] \tag{11}$$

Applying this procedure to the triangle of the plate-bending problem, (q_B = rotation at A, q_A = all other 8 deformations), q_B is determined as 1192. The analytic value, using Fourier series, is 1178. The value calculated by this procedure is much closer to the exact value than if it is determined by using the equations (8) of section 3 (1232 if the constant term of the moment m_n is not used in determining t_2^{-1}).

5. HYPERBOLIC PARABOLOID SHELL

As a practical example of a calculation performed with STRIP [9, 12], the roof of the new swimming-pool in Hamburg, a hypar shell, is analysed (consulting engineer : Leonhardt and Andrä, Stuttgart). The construction is described in detail in reference [19]. It consists of two hyperbolic paraboloids leaning against each other and supported in three points only (s. fig. 9). The edge beams are cantilevers and are thus not supported continuously. Their neutral axis lies in the middle surface of the shell. A continuous transition arises between the shell and the edge beams, which have variable cross-sections. As only symmetric loading cases were analysed, only half of the construction had to be calculated. The subdivision into elements is represented in fig. 10. The edge beams were also divided into finite elements (s. fig. 11). In the zone of transition from the full to the box cross-section, the upper and lower flange-plates of the hollow cross-section were connected eccentrically to the joints lying in the middle surface of the shell. 754 joints and 792 elements were used. The system of equations after introducing the geometrical boundary conditions has 4457 unknowns; the (half) bandwidth is 245. In fig. 11 the normal stress in the top and bottom fibers of the edgebeam BD for dead load is compared to the results of a model analysis performed at the Institute for Model Analysis of the University in Stuttgart (Prof. Dr. R. K. Müller). The scale was chosen as 1:26.67 [20]. In general, the agreement is good. In the centre of the shell (s. fig. 10) the principal normal stress resultant determined by the hybrid model is 387.0 kg/cm as compared to 394.7 kg/cm in the model test.

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APPENDIX : INFLUENCE OF THE LINEAR DEPENDENCY OF THE ENFORCED STRESS BOUNDARY CONDITIONS AND THE CONCENTRATED FORCES ON THE STRESS DISTRIBUTION AND THE DEFORMATIONS

The coefficient matrices of eq. (2) and (3) are partitioned in the following way :

$$
T^t_{tot_{ind.}}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.\!\!\!\!
\underbrace{\left[\begin{array}{c} T_A^{\ t} \\ \hdashline T_B^{\ t} \\ \hdashline t_A^{\ t} \\ \hdashline t_B^{\ t} \end{array}\right]}_{T^t_{tot}}
\qquad
q_{tot_{ind.}}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.\!\!\!\!
\underbrace{\left[\begin{array}{c} q_A \\ \hdashline q_B \\ \hdashline u_A \\ \hdashline u_B \end{array}\right]}_{q_{tot}}
\qquad
P_{tot_{ind.}}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.\!\!\!\!
\underbrace{\left[\begin{array}{c} P_A \\ \hdashline P_B \\ \hdashline p_A \\ \hdashline p_B \end{array}\right]}_{P_{tot}}
$$

$T_A^{\ t}$: independent concentrated forces (mostly coupled with those of other elements through the incidence matrix)

$T_B^{\ t}$: dependent concentrated forces (linear combination of t^t)

$t_A^{\ t}$: additional equilibrium relations, independent of $T_B^{\ t}$

$t_B^{\ t}$: additional equilibrium relations, dependent on $T_B^{\ t}$ and $t_A^{\ t}$

$T^t_{tot\,ind.}$: total independent equilibrium relations

Rows corresponding to a determinate support have been eliminated from $T_A^{\ t}$, so that the stiffness matrix, eq. (14), can be calculated. The deformations q and loads P are partitioned analogously.

The stress-displacement relationship (12) and the equilibrium equations (13) are solved for $q_{tot_{ind.}}$ and β. The dependent equations t^t_B (and unknowns u_B) are not used. For simplicity, the initial strain term $\delta_o\beta$ is dropped :

$$H \cdot \beta - T_{tot_{ind.}} \cdot q_{tot_{ind.}} = O \qquad (12)$$

$$T^t_{tot_{ind.}} \cdot \beta = P_{tot_{ind.}} \qquad (13)$$

$$q_{tot_{ind.}} = (T^t_{tot_{ind.}} \cdot H^{-1} \cdot T_{tot_{ind.}})^{-1} P_{tot_{ind.}} \qquad (14)$$

$$\beta = H^{-1} \cdot T_{tot_{ind.}} \cdot q_{tot_{ind.}} \qquad (15)$$

If other additional equations are assembled in the dependent part, the transformation (16) exists. Bars are used to denote the alternate values.

$$
\bar{T}^t_{tot_{ind.}} \left\{ \begin{bmatrix} \bar{T}_A{}^t = T_A{}^t \\ \hline \bar{T}_B{}^t = T_B{}^t \\ \hline \bar{t}_A{}^t \\ \hline \bar{t}_B{}^t \end{bmatrix} \right. = \begin{bmatrix} I & & \\ & I & \\ & & \; \end{bmatrix} \cdot \underset{T^t_{tot}}{\begin{bmatrix} T_A{}^t \\ \hline T_B{}^t \\ \hline t_A{}^t \\ \hline t_B{}^t \end{bmatrix}} \qquad (16)
$$

As $t_B{}^t$ is linearly dependent,

$$
t_B{}^t = L \cdot \begin{bmatrix} T_B{}^t \\ \hline t_A{}^t \end{bmatrix} \qquad (17)
$$

it can be eliminated. Eq. (16) then reads:

$$
\underset{\bar{T}^t_{tot_{ind.}}}{\begin{bmatrix} \bar{T}_A{}^t = T_A{}^t \\ \hline \bar{T}_B{}^t = T_B{}^t \\ \hline \bar{t}_A{}^t \end{bmatrix}} = \underset{R}{\begin{bmatrix} I & \\ & \; \end{bmatrix}} \cdot \underset{T^t_{tot_{ind.}}}{\begin{bmatrix} T_A{}^t \\ \hline T_B{}^t \\ \hline t_A{}^t \end{bmatrix}} \qquad (18)
$$

R is regular. Analogously, the loads are transformed :

$$
\bar{P}_{tot_{ind.}} = R \cdot P_{tot_{ind.}} \qquad (19)
$$

Writing down the stress-displacement and equilibrium equations for the alternate formulation, substituting eq. (18) and (19),

$$
H \cdot \bar{\beta} - T_{tot_{ind.}} \cdot R^t \cdot \bar{q}_{tot_{ind.}} = O \qquad (20)
$$

$$
R \cdot T^t_{tot_{ind.}} \cdot \bar{\beta} = R \cdot P_{tot_{ind.}} \qquad (21)
$$

and solving for $\bar{q}_{tot_{ind.}}$ and $\bar{\beta}$ and making use of eq. (14) and (15), we obtain

$$
\begin{aligned}
\bar{q}_{tot_{ind.}} &= (R \cdot T^t_{tot_{ind.}} \cdot H^{-1} \cdot T_{tot_{ind.}} \cdot R^t)^{-1} \cdot R \cdot P_{tot_{ind.}} \\
&= (R^t)^{-1} \cdot (T^t_{tot_{ind.}} \cdot H^{-1} \cdot T_{tot_{ind.}})^{-1} \cdot R^{-1} \cdot R \cdot P_{tot_{ind.}}
\end{aligned}
$$

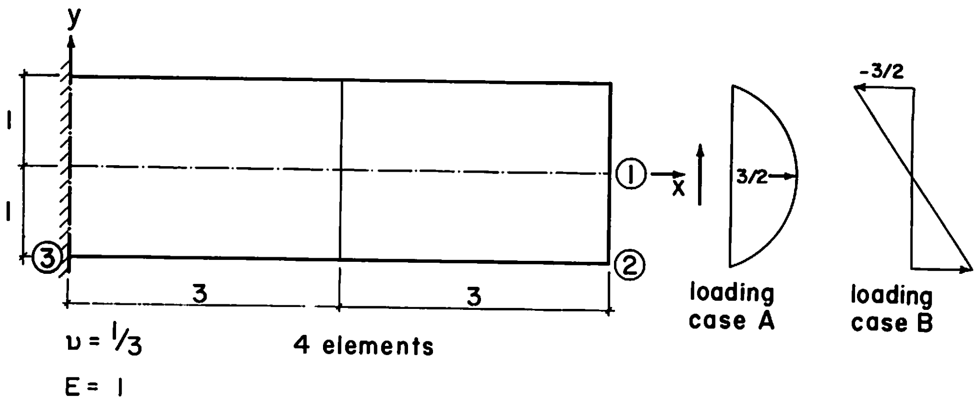
$$\bar{q}_{tot_{ind.}} = (R^t)^{-1} \cdot q_{tot_{ind.}} \tag{22}$$

$$\bar{\beta} = H^{-1} \cdot T_{tot_{ind.}} \cdot R^t \cdot \bar{q}_{tot_{ind.}}$$

$$\bar{\beta} = H^{-1} \cdot T_{tot_{ind.}} \, R^t \cdot (R^t)^{-1} \cdot q_{tot_{ind.}}$$

$$\bar{\beta} = \beta \tag{23}$$

Because of the unit matrix in the upper-left corner of the R-matrix (eq. 18) it follows from eq. (22) that q_A (but not q_B) and from eq. (23) that β are independent of the transformation matrix.



		theory of elasticity		stress hybrid model: stress boundary conditions not enforced		stress hybrid model: stress boundary conditions enforced	
		loading case		loading case		loading case	
		A	B	A	B	A	B
vertical deformation in point ①		2160÷3120	270.0	1291.8	151.4	2426.3	262.9
stresses in point ②	σ_x	0.	1.5	2.5	.8	0.	1.5
	σ_y	0.	0.	.2	-.1	0.	0.
	τ	0.	0.	-.3	-.4	0.	0.
stresses in point ③	σ_x	18.0	1.5	7.4	.8	19.6	1.6
	σ_y	0.	0.	.1	.0	0.	0.
	τ	0.	0.	4.9	.4	0.	0.

Fig.1

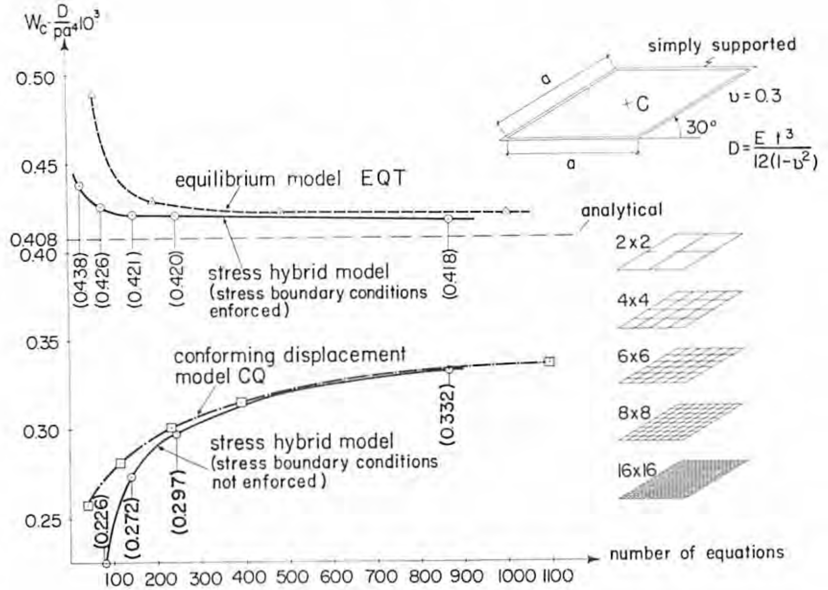

simply supported
a
C
υ = 0.3
30°
a
D = E t³ / 12(1-υ²)
equilibrium model EQT
analytical
stress hybrid model
(stress boundary conditions enforced)
conforming displacement model CQ
stress hybrid model
(stress boundary conditions not enforced)
2x2
4x4
6x6
8x8
16x16
number of equations
(0.438)
(0.426)
(0.421)
(0.420)
(0.418)
(0.226)
(0.272)
(0.297)
(0.332)

Fig. 2

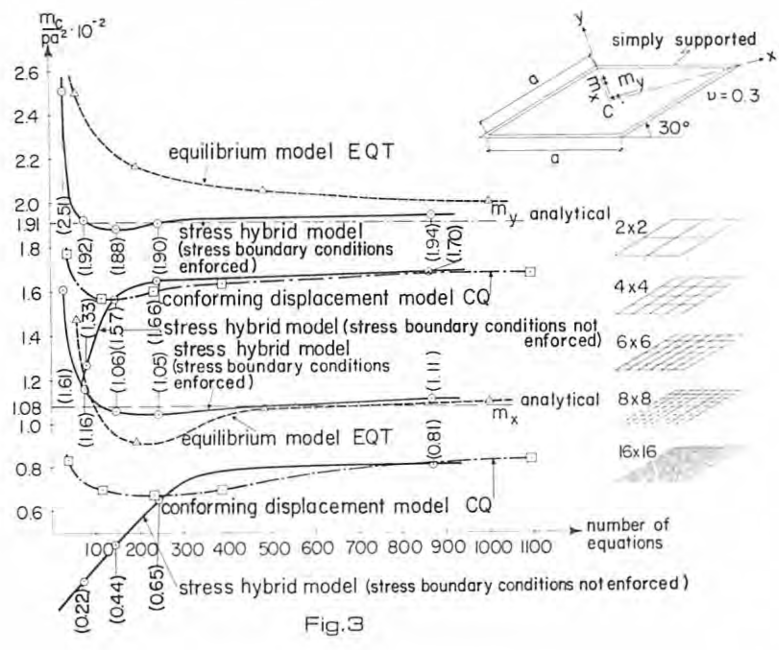
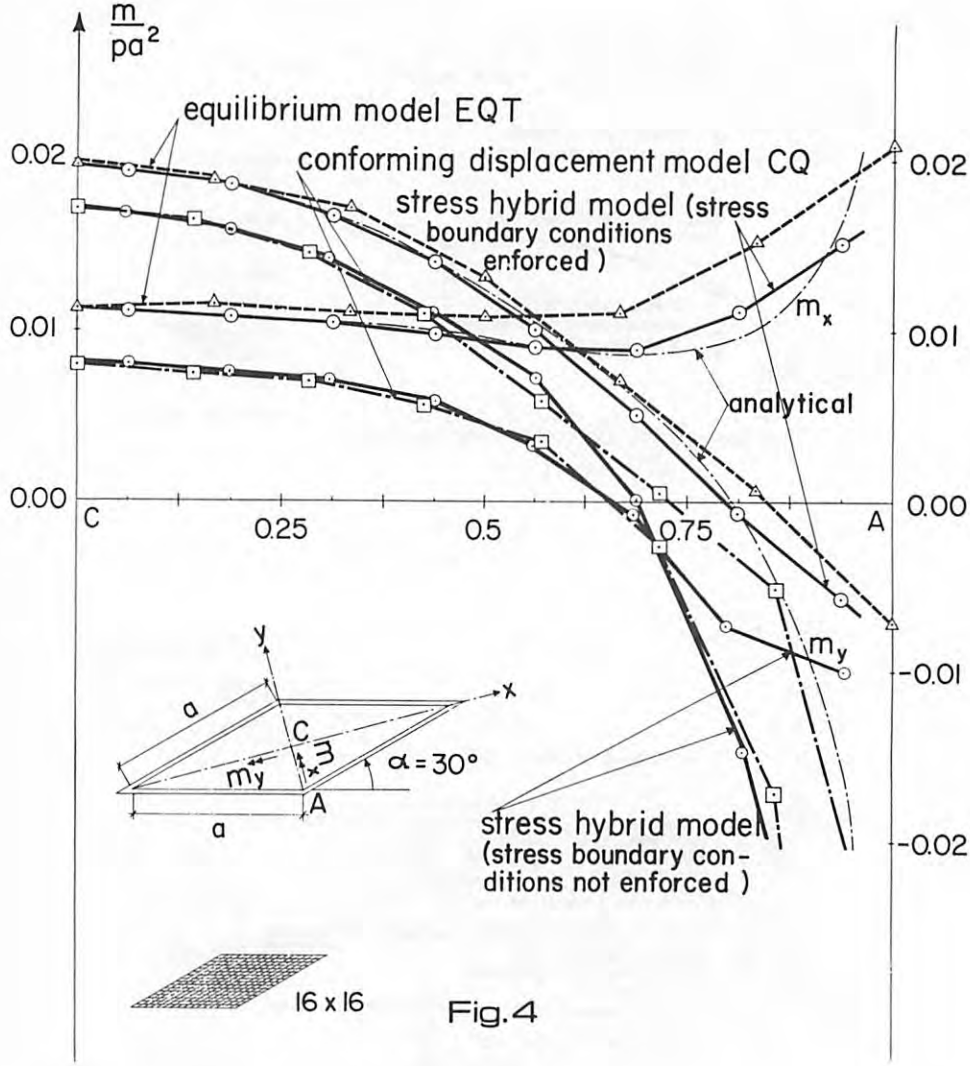

simply supported
a
υ = 0.3
30°
a
equilibrium model EQT
analytical
stress hybrid model
(stress boundary conditions enforced)
conforming displacement model CQ
stress hybrid model (stress boundary conditions not enforced)
stress hybrid model
(stress boundary conditions enforced)
analytical
equilibrium model EQT
conforming displacement model CQ
number of equations
stress hybrid model (stress boundary conditions not enforced)
2x2
4x4
6x6
8x8
16x16

Fig. 3



m / pa²
equilibrium model EQT
conforming displacement model CQ
stress hybrid model (stress boundary conditions enforced)
analytical
stress hybrid model (stress boundary conditions not enforced)
m_x
m_y
0.02
0.01
0.00
-0.01
-0.02
C
0.25
0.5
0.75
A
y
x
a
C
A
m_y
m_x
α = 30°
16 x 16

Fig.4

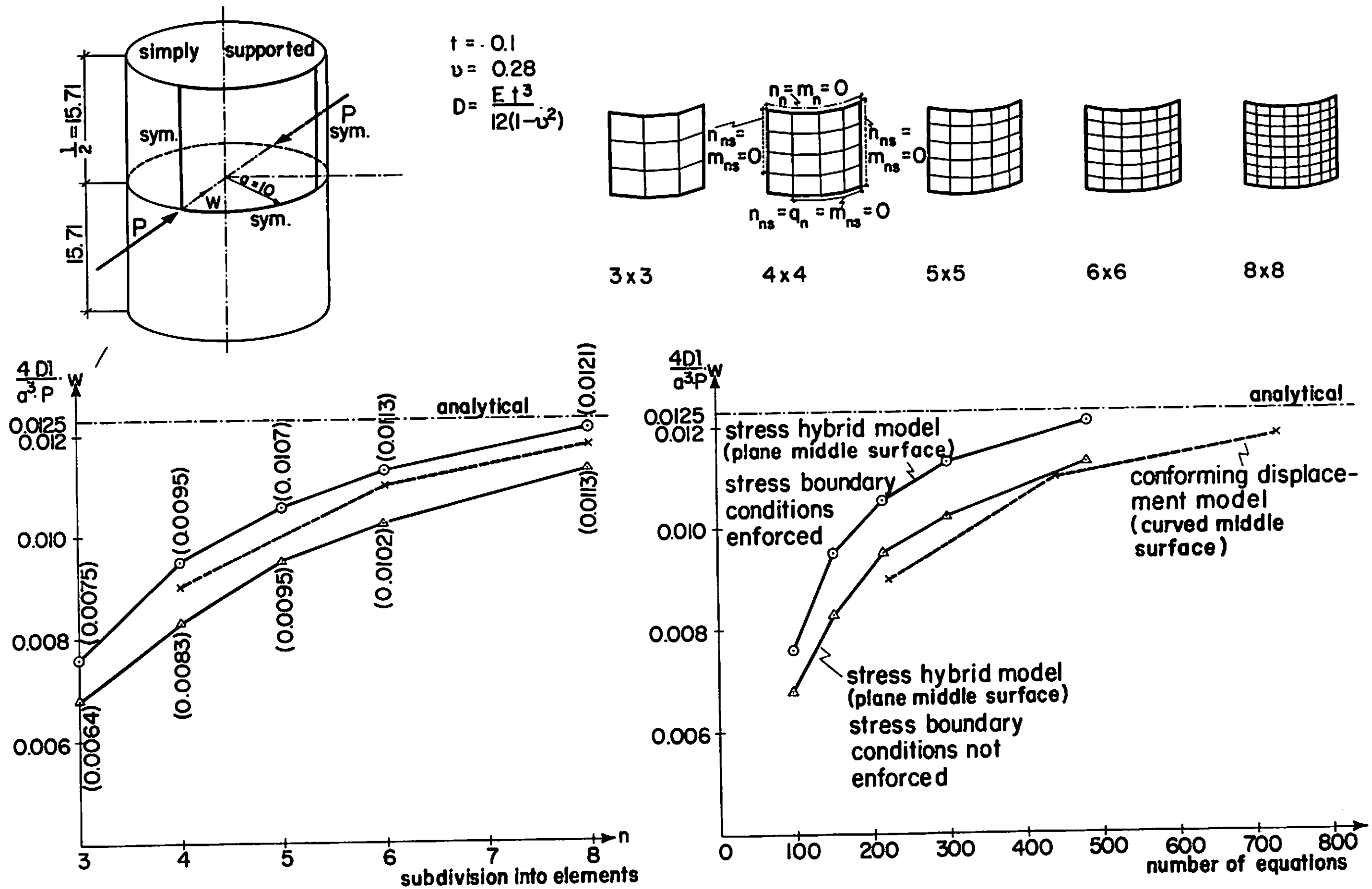

simply
supported
sym.
sym.
sym.
P
P
w
a=10
$\frac{l}{2}$=15.71
15.71
t = 0.1
ν = 0.28
D = $\frac{E t^3}{12(1-\nu^2)}$
$n_n = m_n = 0$
$n_{ns} = m_{ns} = 0$
$n_{ns} = q_n = m_{ns} = 0$
3x3
4x4
5x5
6x6
8x8
$\frac{4DI}{a^3 P} \cdot w$
analytical
(0.0075)
(0.0064)
(0.0095)
(0.0083)
(0.0107)
(0.0095)
(0.0113)
(0.0102)
(0.0121)
(0.0113)
subdivision into elements
n
stress hybrid model (plane middle surface)
stress boundary conditions enforced
conforming displacement model (curved middle surface)
stress hybrid model (plane middle surface)
stress boundary conditions not enforced
number of equations

Fig.5

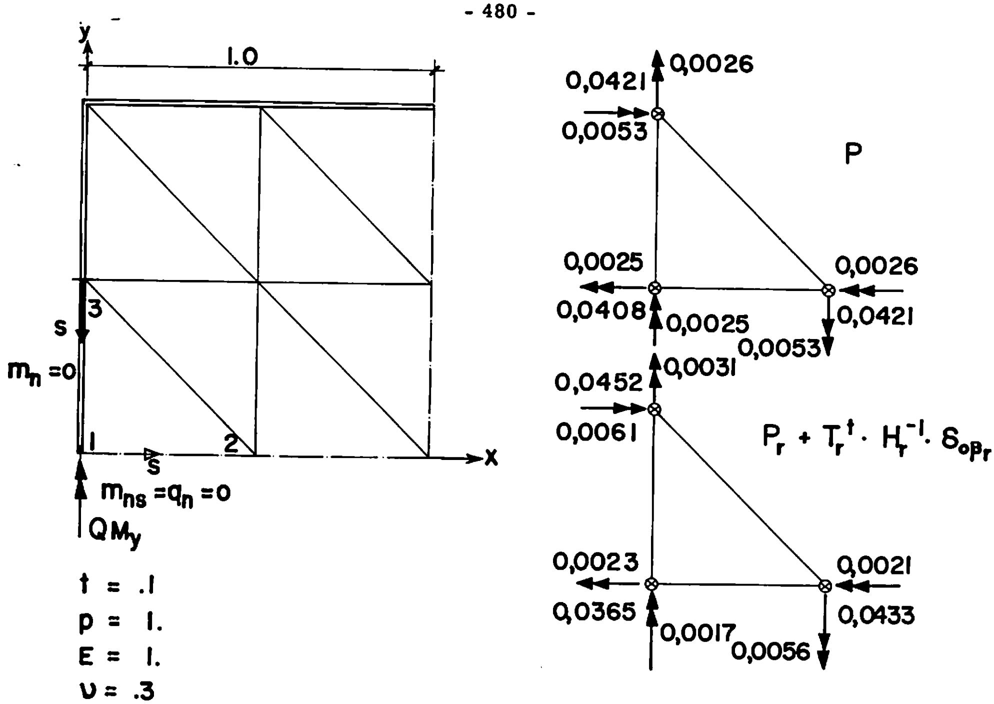


Fig.6

a)

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	β_{16}	β_{17}
QM_y	.25	-.00	.04	.00	-.00	.01	-.00	-.00	-.02	-.00	-.00	-.00	-.00	-.04	-.00	-.02	-.00
m_n c	1.00	.00	.50	.00	.00	.25	.00	.00	.00	.00	.00	.00	-.00	-.00	-.00	-.00	-.00
m_n l	.00	-.00	-1.00	-.00	-.00	-1.00	.00	-.00	-.00	-.00	-.00	-.00	.00	.00	.00	.00	.00
m_n q	.00	.00	.00	.00	.00	1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-.00	-.00
m_{ns} c	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00
m_{ns} l	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00
m_{ns} q	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00
q_n c	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	-1.00	.00	.00	.00
q_n l	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	-2.00	.00

b)

	QM_y	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	β_{16}	β_{17}
β_1	4.00	.00	-.17	-.00	.00	-.04	.00	.00	.08	.00	.02	.00	.00	.17	.00	.07	.00
m_n c	4.00	.00	.33	.00	.00	.21	.00	.00	.08	.00	.02	.00	.00	.17	-.00	.07	-.00
m_n l	.00	-.00	-1.00	-.00	-.00	-1.00	.00	-.00	-.00	-.00	-.00	-.00	.00	.00	.00	.00	.00
m_n q	.00	.00	.00	.00	.00	1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-.00	-.00
m_{ns} c	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00
m_{ns} l	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00
m_{ns} q	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00
q_n c	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	-1.00	.00	.00	.00
q_n l	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	-2.00	.00

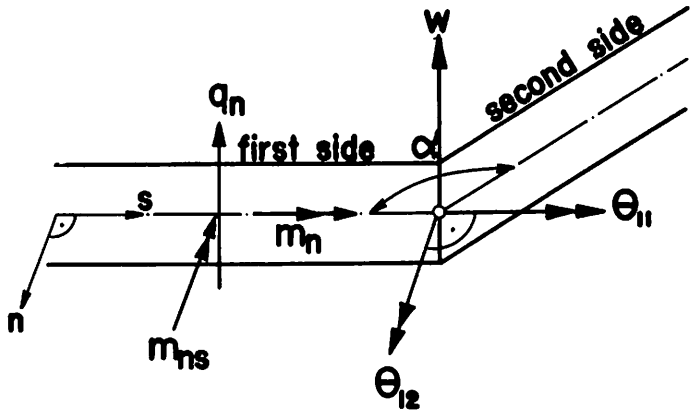
c)

	QM_y	q_n l	q_n c	m_n l	m_n q	m_{ns} c	m_{ns} l	m_{ns} q	β_7	β_{10}	β_8	β_{12}	β_5	β_4	β_{15}	β_2	β_{17}
β_1	4.00	-.02	-.08	.17	.12	-.00	-.08	-.03	.00	.00	.00	.00	.00	-.00	-.00	.00	-.00
β_{16}	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_{14}	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_3	.00	-.00	.00	-1.00	-1.00	.00	-.00	.00	.00	.00	-.00	.00	.00	.00	.00	-.00	.00
β_6	.00	.00	.00	.00	1.00	.00	.00	-.00	.00	-.00	.00	-.00	-.00	-.00	.00	.00	.00
β_{13}	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_9	.00	.00	-1.00	.00	.00	.00	1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_{11}	.00	-1.00	.00	.00	.00	.00	.00	2.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
m_n c	4.00	-.02	-.08	-.33	-.13	-.00	-.08	-.03	.00	.00	.00	.00	.00	.00	-.00	-.00	-.00

d)

	β_1	q_n l	q_n c	m_n l	m_n q	m_{ns} c	m_{ns} l	m_{ns} q	β_7	β_{10}	β_8	β_{12}	β_5	β_4	β_{15}	β_2	β_{17}
QM_y	.25	.00	.02	-.04	-.03	.00	.02	.01	-.00	-.00	-.00	-.00	.00	.00	.00	-.00	.00
β_{16}	.00	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_{14}	.00	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_5	.00	-.00	.00	-1.00	-1.00	.00	-.00	.00	.00	.00	-.00	.00	.00	.00	.00	-.00	.00
β_6	.00	.00	.00	.00	1.00	.00	.00	-.00	.00	-.00	.00	-.00	-.00	-.00	.00	.00	.00
β_{13}	.00	.00	.00	.00	.00	-1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_9	.00	.00	-1.00	.00	.00	.00	1.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
β_{11}	.00	-1.00	.00	.00	.00	.00	.00	2.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
m_n c	1.00	-.00	-.00	-.50	-.25	.00	-.00	-.00	.00	.00	.00	-.00	.00	.00	-.00	-.00	-.00

Fig.7



enforced stress boundary condition		restriction on α	effected deformation
first side	second side		
m_n	m_{ns} q_n	90°	Θ_{11}
m_n	m_n m_{ns} q_n	none	Θ_{11}
m_{ns} q_n	m_{ns} q_n	none	w
m_{ns} q_n	m_n m_{ns} q_n	none	w Θ_{12}
m_n m_{ns} q_n	m_n m_{ns} q_n	none	w Θ_{11} Θ_{12}

Fig. 8

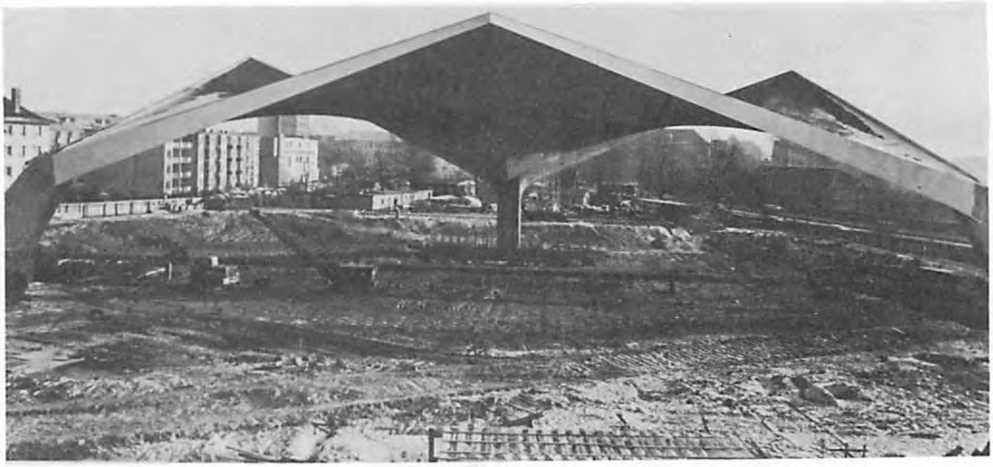


Fig. 9

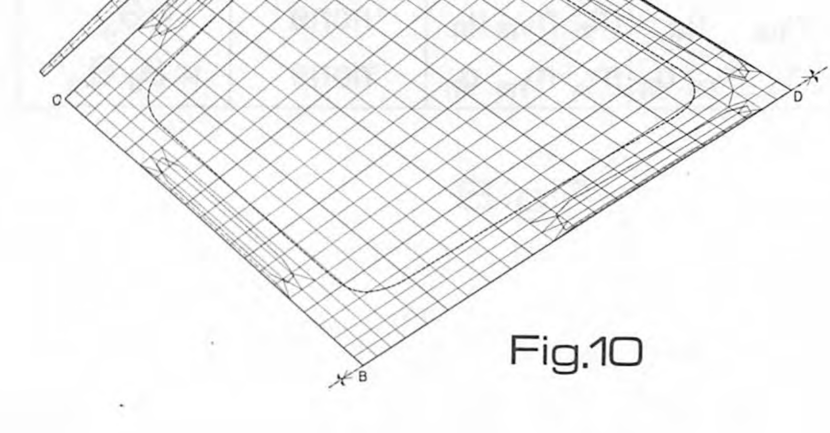


Fig.10

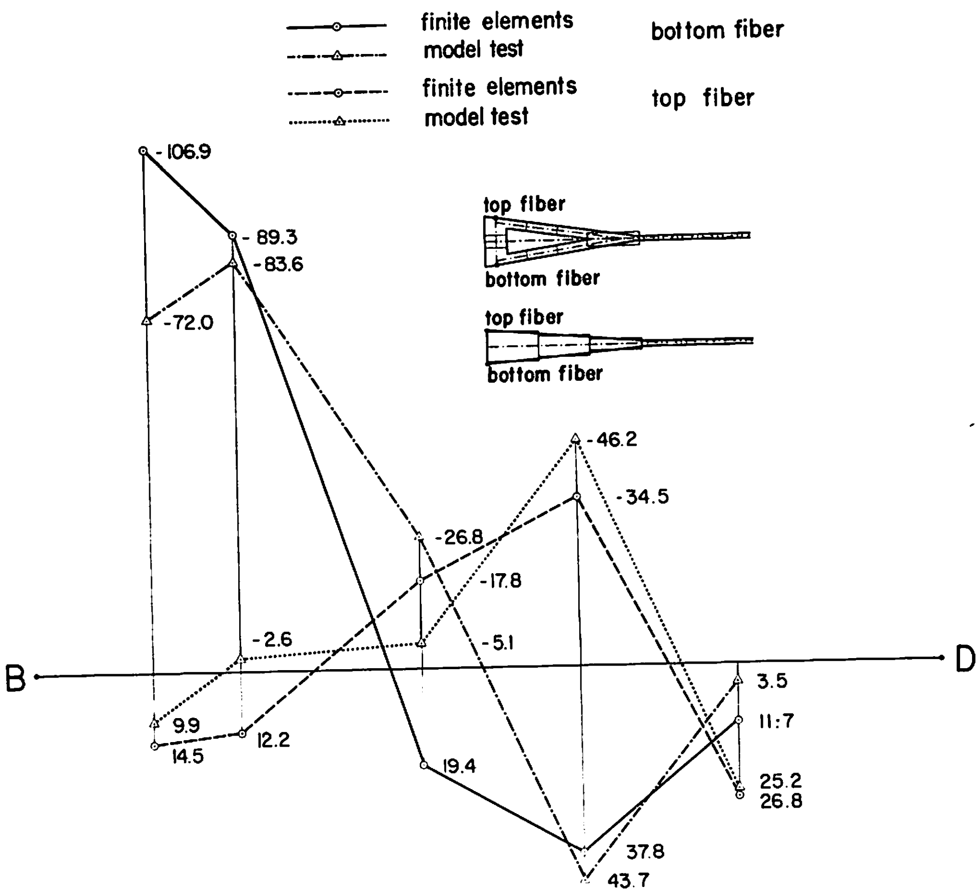

finite elements
model test
bottom fiber
finite elements
model test
top fiber
top fiber
bottom fiber
top fiber
bottom fiber
-106.9
-89.3
-83.6
-72.0
-46.2
-34.5
-26.8
-17.8
-2.6
-5.1
B
D
3.5
11:7
9.9
14.5
12.2
19.4
25.2
26.8
37.8
43.7

Fig.11

DISCUSSION

C . J. H. ARGYRIS, U. K. /Germany

In our experience pure displacement models, if correctly chosen, are at least as accurate as hybrid models and have in addition the advantage of clear convergence conditions. As far as shells are concerned, SHEBA elements (for arbitrary geometry) and SABA elements (for axisymmetrical shape) prove superior to anything in use to-day. They are so accurate that in the case of exact analytical results being available the difference cannot be discerned in a graph. Relatively few unknowns are necessary. In all cases a fifth order interpolation scheme is used. Both elements were developed in 1968 and published then. Some computational results were published in the J. R. A. S. and in the I. U. T. A. M. conference in Liège Sept. 1970. It is of interest that the SHEBA scheme when applied as a mapping procedure to plates may simulate singularities. For example, for oblique plates with obtuse corners one may introduce in this manner a singularity and obtain excellent agreement with Morley's analytical results.