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SOME SEQUENTIAL BAYES PROCEDURES FOR COMPARING TWO BINOMIAL PARAMETERS WHEN OBSERVATIONS ARE TAKEN IN PAIRS

by

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CHAPTER I

INTRODUCTION

1.0 Summary

In section 1.1 the basic problem of the thesis is stated, namely that of comparing the parameters of two Bernoulli processes from paired samples.

Section 1.2 contains a brief survey of existing methods of solving the problem, most of which are based on the Neyman-Pearson theory.

In section 1.3 some shortcomings of the existing methods are discussed.

In section 1.4 a decision-theoretic formulation of the basic problem along with some realistic specifications of loss and cost is proposed.

In section 1.5 the original problem of comparing two binomial probabilities is reduced to that of comparing two cell probabilities in a certain trinomial problem.

Section 1.6 contains a corresponding formulation for a single Bernoulli process.

Section 1.7 contains a brief review of some results that are relevant for the analysis of the various decision problems formulated.

Finally, a brief chapter by chapter summary of the thesis is presented in section 1.8, which is concluded with a schematic table of contents.

1.1 Statement of the problem.

Consider two independent data-generating processes. Each process yields a sequence of independent observations of dichotomous type, say, successes and failures. Suppose also that the observations are obtained in pairs from the two processes. Let p_1 and p_2 be the unknown probabilities of a success for the two processes. Now consider the problem of selecting the process with the larger success probability on the basis of a sequence of pairs of observations. The method of sampling being sequential, our primary problem is that of obtaining a good sampling rule that tells us when to stop sampling.

The statistical formulation as stated above is typical of many practical problems arising in various fields of application. In clinical trials, for example, one may wish to compare two alternative treatments or drugs, say T_1 and T_2. Observations may be obtained in pairs, one member of each pair for each treatment. Suppose they are categorized into successes and failures according as the patients are cured or not, or say, according as the blood pressure readings of the patients are below a certain level or not. Here p_1 and p_2 are the unknown probabilities of a success with the treatments T_1 and T_2 respectively.

Similar situations arise in industrial production processes where the effectiveness of two alternative production methods P_1 and P_2, say, are to be compared. The effectiveness of a process is measured in terms of the proportion of effective units produced, all units being classified simply as effective or defective. Here $p_1(p_2)$ is the unknown probability of a unit being effective when it is produced using method $P_1(P_2)$.

For the sake of a descriptive terminology, but without any loss of generality, we shall sometimes refer to the two situations described above.

1.2 Existing methods.

The formulation stated in the previous section is, as yet, incomplete. The specification of further details including the ultimate objective to which the results of the investigation are to be put makes all the difference in the particular approach to be used in analysing such data. In almost all of the existing methods of analysis, this problem of double dichotomies[1] has been posed as a hypothesis testing problem in the framework of the Neyman-Pearson theory. A very brief review of the important existing methods is given below.

(a) Fixed sample size procedures.

Perhaps the most common method is the classical one of taking samples of fixed size and analysing the sample proportions of successes as if they were normally distributed, using a one-sided or a two-sided test of the difference between the two means. The two-sided test is, in effect, a chi-square test.

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[1]We have used the term 'double dichotomies' in the general sense as used for example in Wald [39]* and not in the restricted sense of Barnard [7]* .

*The numbers in square brackets refer to the bibliography.

Another common non-sequential method is to apply Fisher's exact test, as mentioned briefly below. The data is put in a 2 x 2 table as follows.

	Process I	Process II	Total
Success	a_1	a_2	$a_1 + a_2$
Failure	$n - a_1$	$n - a_2$	$2n - a_1 - a_2$
Total	n	n	$2n$

The exact probability of such a configuration, conditional on the marginals being fixed and the null hypothesis $p_1 = p_2$ being true, is given by the hypergeometric term

$$\binom{n}{a_1}\binom{n}{a_2}\Big/\binom{2n}{a_1+a_2} .$$

Probabilities of "more extreme" (depending on whether a one-sided or a two-sided test is wanted[2]) configurations are obtained similarly. The observed result is judged to be insignificant or not at a fixed level of significance, according as the total probability of the observed configuration along with more extreme configurations exceeds the fixed level, or not.

The hypergeometric distribution being discrete, the size of Fisher's exact test may not attain the traditional levels of significance. The corresponding randomized version of the test [27, p. 142] is a simi-

[2] Fisher's exact test as described in [18, p. 96] is one-sided.

lar test of Neyman structure and is essentially a uniformly most powerful unbiased test of the hypothesis $u \leq 1$ against $u > 1$ for the one-sided case, or of the hypothesis $u = 1$ against $u \neq 1$ for the two-sided case, where

$$u = \frac{p_1}{1 - p_1} \Big/ \frac{p_2}{1 - p_2}, \qquad 0 < u < \infty ,$$

is called the ratio of odds and $u \gtreqless 1$ according as $p_1 \gtreqless p_2$.

Fisher's exact test is a conditional one in the sense that in the proper "frame of reference" the marginal totals of the above 2×2 table are kept fixed. A different sort of conditioning may be applied to the data to obtain another exact, though not so efficient, test[3] that is particularly relevant when only successes (or equivalently failures) are observable. Keeping the total number of successes fixed, the number of successes with treatment T_1 is distributed binomially with the parameter

$$\omega = p_1/(p_1 + p_2), \qquad 0 < \omega < 1 ,$$

$\omega \gtreqless 1/2$ according as $p_1 \gtreqless p_2$. The problem of testing the difference between the two binomial parameters p_1 and p_2 is thus reduced to that of testing the departure of the single binomial parameter ω from $1/2$. One-sided or two-sided exact tests of the hypothesis $\omega = 1/2$ may be constructed in the usual way.

The methods described so far do not take account of the natural pairing of the observations present in our formulation of the problem in section 1.1. Each pair of observations coming from the two

[3]This test is reminiscent of a test for the equality of two Poisson parameters [27, p. 141], and, along with its sequential analog, was brought to our attention by Professor W. J. Hall.

processes is one of the four types SS, FF, SF, and FS. We shall use the term <u>tied pairs</u> for those recorded as SS or FF and <u>untied pairs</u> for those recorded as SF or FS. The problem of testing the equality of p_1 and p_2 may be reduced as before to that of testing the departure from the value 1/2 of another binomial parameter v defined as

$$v = \frac{p_1(1 - p_2)}{p_1(1-p_2)+ p_2(1-p_1)} = \frac{u}{1+u} , \qquad 0 < v < 1 ,$$

the probability that an untied pair is an SF. It is easily verified that $v \gtreqless 1/2$ according as $p_1 \gtreqless p_2$. Wald [39, p. 107] proposed an exact non-sequential test for the double dichotomies using the above reduction. The test of course depends on a given pairing of the observations from the two processes.

It is to be noted that any test, sequential or non-sequential, for the double dichotomies when based on the first type of reduction, depends only on the successes (or only on the failures) irrespective of any pairing, while when based on the second type of reduction, depends only on the untied pairs for a given pairing. Inasmuch as these tests fail to utilize all the relevant data they cannot be fully efficient.

In constructing a test for the double dichotomies, it is, however, quite a standard practice to take account of any natural pairing of observations for some well-understood reasons also explained by Wald [39, p. 108]. It would be interesting to know the relative merits of the two types of reduction and to make a comparison between the two associated tests as to their relative performances vis a vis other tests, viz., the classical test and Fisher's exact test for the non-sequential case.

(b) Sequential procedures.

In the absence of any natural pairing of observations the problem of testing $p_1 = p_2$ sequentially may be reduced to that of testing $\omega = 1/2$ sequentially. Since by the very formulation of our problem we assume the existence of such a pairing and since most of the existing sequential procedures make use of this fact by basing the tests on the untied pairs we shall no longer consider any test based on successes alone. It is to be noted, however, that corresponding to each of the tests based on untied pairs to be described below we could have a test based on successes alone.

The existing sequential procedures may be considered in two broad groups, according as the number of hypotheses considered being two or three giving rise to the so-called two-decision or three-decision approaches.

(i) Two-decision approaches.

Wald's sequential procedure [39, p. 109] for analyzing double dichotomies consists in selecting two parameter points v_o and v_1 ($v_o < 1/2 < v_1$) and two error probabilities α and β ($\alpha, \beta > 0$, $\alpha + \beta < 1$) and then to set up a sequential probability ratio test (SPRT) of the hypothesis H_2: $v = v_o$ against H_1: $v = v_1$ in the usual way.

Since the v-parameter is not intuitively very meaningful Wald describes the test and its operating characteristic (OC) and average sample number (ASN) function in terms of the parameter $u = v/(1-v)$, the ratio of odds, considered by Girshick [19] as an appropriate measure of divergence between the two parameters p_1 and p_2. It may be possible for the experimenter to choose u_o and u_1 $(0 < u_o < 1 < u_1 < \infty)$ such that a divergence measure $\leq u_o$ or $\geq u_1$ between p_1 and p_2 are worth detecting. Then $v_i = u_i/(1 + u_i)$, $i = 0, 1$.

Wald has also discussed the considerations that may govern the choice of u_o and u_1. Because of the symmetrical footing in which the two treatments stand with respect to each other in our problem, attention might be restricted to procedures with $u_1 = 1/u_o$; i.e. with $|v_i - 1/2| = (1 - u_o)/2(1 + u_o) \equiv \delta_o$ (say), $i = 0, 1$. It may be verified that the resulting procedure is then the same as that of Girshick [19].

If, moreover, the two error probabilities α, β are taken to be equal we arrive at <u>symmetrical</u> procedures. For this class of procedures, Hall [22] has proposed a modification to effect a reduction in the ASN when p_1 and p_2 are approximately equal.

(ii) <u>Three-decision approaches</u>.

Besides the two decisions of judging one of the two treatments to be better than the other, the third one of making no such judgment and proclaiming them to be the same for all practical purposes, is now introduced. This is the approach of the common two-sided non-sequential tests. Most of the procedures considered by

Armitage also have this feature.

The three decisions a_o, a_1, a_2 corresponding to $p_1 = p_2$, $p_1 > p_2$, $p_1 < p_2$ respectively correspond in terms of v to $v = 1/2$, $v > 1/2$, $v < 1/2$ respectively. Armitage [3] has suggested a procedure obtained by running simultaneously two one-sided SPRT's, viz.,

$$T^+: \text{ Testing } H_o: v = 1/2 \text{ against } H_1: v = 1/2 + \delta_o \ ,$$

$$T^-: \text{ Testing } H_2: v = 1/2 - \delta_o \text{ against } H_o: v = 1/2 \ .$$

The OC and ASN of this procedure might be obtained from the results of Boer [10] who developed the theory for such procedures for the binomial parameter, just as Sobel and Wald [35] did for the normal mean problem.

All the sequential procedures described so far (except Hall's [22]) are <u>open</u> in the sense that they are not truncated (although they terminate with probability one). Armitage [4] has considered a class of <u>restricted</u> (truncated) procedures in which sampling is continued until one of three specified boundaries is reached leading to the three possible decisions about v and hence about p_1 and p_2. He considers various types of boundaries and finds the performance of the resulting procedures in terms of the usual criteria of the Neyman-Pearson theoretic formulation.

The sequential procedures described so far were concerned with the derived problem in terms of the single binomial parameter v and hence are based solely on the untied observations. Darwin [16] has considered within the three-decision approach a symmetrical truncated procedure which utilizes the tied pairs without discriminating between the two types of tied pairs. If (X_{1i}, X_{2i}) is the sequence of pairs of observations where

$$X_{ki} = \begin{cases} 1 & \text{for a success} \\ 0 & \text{for a failure} \end{cases}$$

$i = 1, 2, \ldots, M$. $k = 1, 2$, and

$$Z_i = X_{1i} - X_{2i}, \; i = 1, 2, \ldots, M ,$$

$$S_n = \sum_{i=1}^{n} Z_i , \quad n = 1, 2, \ldots, M ,$$

then, Darwin's procedure accepts the three decisions a_1, a_2, a_o according as S_n crosses the boundaries $S_n = h$, $S_n = -h$ and $n = M$, where h and M are certain positive integers. He considers the problem of choosing h and M such that the procedure has a high probability of leading to the correct decision. We shall also propose in section 1.5 procedures based on Z_n's.

1.3 Criticism of the existing procedures.

As mentioned and explained by Armitage [5, p. 53], the first difficulty "may be caused by the fact that the measurements in which one is primarily interested are the probabilities of success p_1 and p_2, whereas the sequential design is specified in terms of the derived quantity v". This difficulty becomes much more relevant when one considers a decision-theoretic formulation of the problem. Once it is agreed that v is the primary parameter of interest rather than p_1 and p_2, all the usual decision-theoretic specifications regarding loss, cost, prior distribution, etc. might well center around v, a binomial parameter as considered in section 1.6.

In all the sequential procedures mentioned in the previous section where the original problem is reduced to one about a single binomial parameter v, the ASN function as well as the terminologies 'truncation' or 'termination', etc., refer only to the untied portion of the sequence of pairs of all observations. The total amount of expected sampling can be obtained in each case by dividing the ASN by

$$p_1(1 - p_2) + p_2(1 - p_1) = 1 - \pi_o \text{ , say,}$$

where π_o is the probability of a pair being tied. Thus the total amount of sampling before reaching a decision might be quite high if π_o is sufficiently large. The motivation behind introducing the class of restricted procedures is that unlike those of Wald and Girshick they have a relatively more attractive feature of possessing a known upper bound, but only on the number of untied pairs and not on the total number of pairs. Thus these procedures of Armitage (as well as of Hall) are really not truncated in the sense that Darwin's procedure is truncated.

It is quite reasonable that the tied observations do not intuitively contain much, if any, information about the difference $p_1 - p_2$ and hence may not seem relevant for any test of $p_1 - p_2$. They cannot be discarded by sufficiency or invariance considerations, however, but they certainly involve some cost. It might happen in all the sequential procedures mentioned above except Darwin's that the available resources for continuing sampling are exhausted before a sufficient number of untied observations are obtained to reach a decision. Also in Darwin's approach, the choice of a particular procedure with desirable power properties depends on the unknown probability π_o.

It is true, of course, that in practice, the experimenter has a rough knowledge of the magnitude of π_o and may utilize this, as shown by Armitage [5, p. 54] in choosing a suitable sequential plan. But this is exactly what a Bayesian analysis attempts to do in a more formal way. We may mention here that informal Bayesian analyses of double dichotomies are to some extent available. By assigning a prior distribution for the two parameters, one may calculate the posterior distribution after any amount of sampling, and discontinue the experiment when the posterior distribution seems adequately concentrated to permit a decision. Novick [31] has discussed such approaches in the context of clinical trials.

Finally, Anscombe [2] has questioned the appropriateness for our problem of the very foundations on which the existing procedures of the previous section rest, viz., the operating-characteristic concepts of the Neyman-Pearson theory of tests. His basic thesis in [2] is that the existing procedures might be appropriate in devising a routine decision procedure that will work sufficiently well in the long run, but not so in a problem where we have to analyze, interpret and make some recommendations on the basis of just one series of observations. Whatever happened is relevant for our present action and not whatever could have happened but did not.

Anscombe also criticizes the consideration of the third decision of taking no positive action. In some applications one has to recommend only one of the two treatments or one of the two production processes to be used in the future.

1.4 A decision-theoretic formulation.

Anscombe [2] suggests a way in which the problem should be tackled and illustrates to some extent with the corresponding normal distribution problem. He suggests a decision-theoretic formulation of the problem in terms of loss and cost functions, which is adapted and elaborated for our problem given below.

(a) actions: The action $a_1(a_2)$ corresponds to the recommendation that treatment $T_1(T_2)$ is to be used with future patients. Action $a_1(a_2)$ is preferable if $p_1 - p_2 \equiv \theta > 0$ $(\theta \leq 0)$.

(b) costs per observation: We shall consider only linear cost functions, i.e., the cost of n observations will be proportional to n, the constant of proportionality, denoted by C, being termed the cost per observation. When interpreted as the usual cost of performing an experiment, C may be regarded as a constant. There arise, however, certain situations where C might depend on the unknown parameter θ. We illustrate this possibility below.

Suppose in the industrial process example that the effective items produced during the period of trial (to determine which one is the better process) are marketable. Suppose the monetary utility per effective unit (viz, profit) is, say k in some monetary unit. Had we known which one was the better process and allocated the two units to it, the expected utility would have been $2k \max(p_1, p_2)$. Instead we expect only $k(p_1 + p_2)$ by allocating one unit to each process. Each pair of observations thus costs (on the average)

$$2k \max(p_1, p_2) - k(p_1 + p_2) = k|p_1 - p_2| \equiv k|\theta|$$

in unrealized profit.

In the drug-testing situation, apart from the monetary costs of experimentation, we incur a different type of cost peculiar to this situation. For each pair of patients allocated to the two treatments, one patient is subjected to the inferior treatment giving rise to a so-called ethical cost, considered earlier by Anscombe [2]. Although it is difficult, if not impossible, to assess this cost in terms of any monetary unit, it may not be too unreasonable to assume that the ethical cost, thus involved, is greater, the greater the difference in the efficacy of the two treatments. By similar considerations as in the previous situation, it turns out that the ethical cost per pair of observations is increasing in $|\theta|$. Suppose, for simplicity it is $k\,|\theta|$, where k is any arbitrary positive constant. We shall ignore in this situation any monetary cost that might be incurred in obtaining a pair of observations, as it cannot easily be combined with the ethical cost.

It is to be noted that the symbol k arising above in different connections is generic and stands for any arbitrary positive constant. For reasons to be explained in the next subsection we may take $k = 1$ Thus taking a pair of observations as the new unit of observation, we shall be concerned with the two following special types of cost per observation, viz.,

Case (i): $C(\theta) = 1$, for all θ, called constant cost

Case (ii): $C(\theta) = |\theta|$, called absolute deviation cost.

(c) losses due to wrong action: It is well-known that a testing of hypothesis approach on which some of the current procedures are based, when formulated as a decision-theoretic problem, involves only simple loss functions. We shall consider below two different types of loss structure that may be more realistic.

I. Linear loss structure: The losses for the respective actions are

$$(1.1)\quad L(\theta, a_1) = \begin{cases} 2A\,|\theta| , & \text{if } \theta \leq 0; \\ 0 , & \text{if } \theta > 0 . \end{cases}$$

$$L(\theta, a_2) = \begin{cases} 0 , & \text{if } \theta \leq 0 ; \\ 2A\,\theta , & \text{if } \theta > 0 . \end{cases}$$

It is instructive to write them as regret functions, viz.,

$$(1.2)\qquad L(p_1, p_2; a_i) = 2A \max (p_1, p_2) - 2A\, p_i , \quad i = 1, 2.$$

The constant of proportionality 2A (the factor 2 is introduced as a notational convenience to be understood subsequently) may be interpreted as some sort of index of the future use envisaged for the recommended drug or process (Anscombe [2]). Since it is the relative magnitude of the loss vis a vis the cost per observation that matters in the ultimate analysis, the constant k in $C(\theta)$, as mentioned earlier, can always be (and henceforth is assumed to be) standardized to unity, by adjusting the factor 2A. After such adjustment, the constant 2A may be interpreted as the number of future patients who will be the potential users of the recommended drug in the drug-testing example, or the number of future items to be produced by the recommended process in the industrial process example.

Perhaps, in practice, the experimenter will be able to assess this constant, just as in the Neyman-Pearson-theoretic formulation he is required to assess the least difference in the parameters considered to be worth detecting. It will follow from the asymptotic theory to be developed in Chapter IV that the sensitivity of the optimum procedures to it diminishes as it increases. Anscombe [2] discusses how to make some guess about this constant in the context of the clinical trial situation. Anyhow, this type of loss structure has been considered by so many authors [11, 26, 29, 33, 2] to name only a few that it seems to be well-established in this branch of the statistical literature.

II. Modified loss structure: We shall have occasion to consider a slightly altered version of the linear loss structure that is especially relevant in the clinical trial situation, as envisaged by Anscombe [2]. For the problem where $\theta = \theta_1 - \theta_2$ is the difference between the unknown means of two normal populations with known variances, and where y_n stands for the cumulative sum of the differences of n successive pairs of observations, Anscombe introduces this type of loss structure as follows:

"Perhaps $2A$ should be assessed, not as a constant, but as an increasing function of $|y_n|/n$, since the more striking the treatment difference indicated the more likely it is that the experiment will be noticed and so will affect the treatment given to the future patients. One way of introducing such a dependence of $2A$ on $|y_n|/n$ is to assess $2A + 2n$ as a constant, as though the sum of the number of patients in the trial and of the number of later patients directly affected by the trial were fixed."

If we denote this latter constant as 2A (which should not be confused with the earlier 2A in the definition of the linear loss structure), and if an action is taken after performing tests on n pairs of patients, then the numberof prospective patients to use the recommended drug is reduced to $2A - 2n$. The constant of proportionality in the proposed loss function will thus depend on n, the stage of sampling. Explicitly then, we define the <u>modified loss structure</u> as follows:

$$(1.3) \quad L(\theta, a_1; n) = \begin{cases} 2(A-n)|\theta| , & \text{if } \theta \leq 0, \ n < A ; \\ 0 , & \text{if } \theta \leq 0, \ n \geq A ; \\ 0 , & \text{if } \theta > 0, \ n = 0,1,2,\ldots \end{cases}$$

$$L(\theta, a_2; n) = \begin{cases} 0 , & \text{if } \theta \leq 0, \ n = 0,1,2,\ldots ; \\ 0 , & \text{if } \theta > 0, \ n \geq A ; \\ 2(A-n)\theta , & \text{if } \theta > 0, \ n < A . \end{cases}$$

The considerations that led to the formulation of the modified loss function suggest that the absolute deviation cost combines more naturally with this type of loss structure. Hence the modified loss function will be considered in Chapter III only in conjunction with $C(\theta) = |\theta|$ and not with constant cost.

Remark: It seems worthwhile at this stage to point out an interesting connection between the decision-theoretic formulation of our problem, with the modified loss and absolute deviation cost, and the classical two-armed bandit problem with a certain restricted class of strategies as considered by Vogel [38]. The original two-armed bandit problem is concerned with devising a sampling plan (i.e. prescribing a policy of selecting one of the two possible binomial experiments with unknown probabilities of a success, p_1 and p_2, at each of a fixed total of $2A$ experiments) in order to maximize the total return. Vogel considers, and also gives some justification for doing so, a class of strategies as follows. In the first $2N$ steps each of the two kinds of experiments, I and II, say, is performed an equal number, N, of times. The remaining $2A - 2N$ steps are performed either with I or II alone selected on the basis of the results of the first N pairs of observations. Now the decision whether to continue taking one more pair of observations or to discontinue taking pairs and performing the remaining steps with I or II, is made with the help of a sequential procedure, δ, say. N is thus a random variable bounded above by A.

If $E(N)$ denotes the expected number of pairs of steps before deciding to continue with a single type of experiment, if $\gamma [1 - \gamma]$ denotes the probability of continuing with I [II] when the procedure δ is used, then the total expected return following this plan is given by

$$W(p_1,p_2;\delta) = (p_1+p_2)E(N) + p_1 \gamma [2A - 2E(N)] + p_2(1-\gamma)[2A - 2E(N)],$$

while the best possible expected reutnr would have been

$$2A \max(p_1, p_2) = \max_{\delta} W(p_1, p_2; \delta).$$

Thus the regret in following δ is given by

$$\begin{aligned} r(p_1, p_2; \delta) &\equiv \max_{\delta} W(p_1, p_2; \delta) - W(p_1, p_2; \delta) \\ &= E(N)|p_1 - p_2| + \gamma [2A - 2E(N)] \max [0, p_2 - p_1] \\ &\quad + (1-\gamma)[2A - 2E(N)] \max [0, p_1 - p_2], \end{aligned}$$

which may easily be verified to be the same as the risk function (defined as the expected loss plus expected cost) associated with the procedure δ for our problem with modified loss and absolute deviation cost.

(d) a priori distributions: Although it is theoretically possible to carry out a Bayesian analysis for any a priori distribution, it has become quite customary for the sake of mathematical convenience to deal with only those a priori distributions that are closed under sampling [40], which have alternatively been christened as natural conjugate priors (NCP) by Raiffa and Schlaifer [33]. In most circumstances prior knowledge of the experimenter, which is to be reflected in the prior distribution, is rough and can adequately be quantified with the help of a particular member of any reasonably large family of a priori distributions.

The natural conjugate prior distribution of

of p_1 and p_2 is a product of two beta distributions. This specification may be satisfactory when a priori it is reasonable to consider p_1 and p_2 to be independent. If, however, the experimenter has some prior reasons to believe that the curative potentialities of the two drugs go together (which might be due to the fact that the ingredients of the two drugs are to some extent similar), then one should select some joint distribution of p_1 and p_2 with a positive correlation between them. As this will frequently be the case, a natural conjugate prior is ruled out for the model as it stands. We shall instead convert the problem to another one for which it is possible to specify a natural conjugate family of prior distributions with correlations between p_1 and p_2. The a priori distribution with which we shall be mostly concerned in our analysis in Chapter II and III will thus be specified at the end of the next subsection in which we describe the intended reduction of the original problem.

1.5 Reduction of the original problem.

In so far as we are interested only in the difference $p_1 - p_2$ θ and not in the actual values of p_1 and p_2 (the specification of loss and cost in the previous section involving p_1 and p_2 only through θ) the problem would have been a lot easier, if, like the corresponding normal problem, there existed a sufficient statistic Y whose distribution involved only θ. We can, however, represent θ as

$$\theta \equiv p_1 - p_2 = p_1(1-p_2) - p_2(1-p_1) \equiv \pi_1 - \pi_2 \text{ (say)},$$

the difference of the probabilities of the two types of untied observations. Moreover, due to the symmetrical footing on which the two drugs stand to each other, it would seem that the two types of tied observations contain little or no discriminatory information about $\theta = p_1-p_2$, and, therefore, may be coalesced into a single group with little loss of efficiency. The probability of a tied pair is thus

$$p_1 p_2 + (1 - p_1)(1 - p_2) = \pi_o \quad \text{(say)}$$

We are thus led to consider the statistic

$$Z = X_1 - X_2 \ ,$$

the difference of the two Bernoulli random variables. Z has the following trinomial distribution:

$$(1.4) \qquad \begin{aligned} P[Z = 0] &= \pi_o \\ P[Z = 1] &= \pi_1 \\ P[Z = -1] &= \pi_2 \end{aligned} \qquad \begin{aligned} &\pi_i \geq 0, \ i = 0,1,2 \\ &\pi_o + \pi_1 + \pi_2 = 1 \ . \end{aligned}$$

The specifications about the actions, losses and costs per observation described in the previous section remain the same for this modified problem if one interprets θ not as $p_1 - p_2$ but as $\pi_1 - \pi_2$. We have thus reduced the original problem of comparing two binomial parameters into a problem of comparing two cell probabilities in a certain trinomial model.

The fact that for the corresponding problem of comparing two normal means it is sufficient to restrict attention to the difference $Y = X_1 - X_2$ of the two observations by appealing to the principle of invariance and sufficiency [21], and the fact that the binomial problem is approximated by the normal problem at least for large samples, lend some additional justification for the present reduction. The main argument for this modification is, however, the resulting mathematical simplicity engendered out of the existence of a convenient family of a priori distributions that is NCP for the trinomial model. This is defined below as:

a priori distribution:

$$(1.5)\quad d\xi(a,b,c) \equiv d\xi(\pi_1,\pi_2|a,b,c) = \frac{\Gamma(a,b,c)}{\Gamma a\,\Gamma b\,\Gamma c}\,\pi_1^{a-1}\pi_2^{b-1}(1-\pi_1-\pi_2)^{c-1}\times$$

$$\times\ d\pi_1\,d\pi_2,\quad \pi_i \geq 0,\ i = 0, 1, 2;\ \pi_o + \pi_1 + \pi_2 = 1\ ;$$

$$a,b,c \text{ positive integers}$$

This has been called the (bivariate) Dirichlet distribution [41, p. 177]. This family possesses certain properties that make it as tractable mathematically as the beta family. Some of these properties are recorded in the subsection below for future reference.

1.5.1 Some Properties of the Dirichlet distribution.

(a) The marginal distributions of the Dirichlet distribution are beta distributions. Explicitly,

$$d\,\xi(\pi_1) = \frac{\Gamma(a+b+c)}{\Gamma a\ \Gamma(b+c)}\ \pi_1^{a-1}\,(1-\pi_1)^{b+c-1}\,d\,\pi_1,\quad 0 \leq \pi_1 \leq 1\ ,$$

$$d\,\xi(\pi_2) = \frac{\Gamma(a+b+c)}{\Gamma b\ \Gamma(a+c)}\ \pi_2^{b-1}\,(1-\pi_2)^{a+c-1}\,d\,\pi_2,\quad 0 \leq \pi_2 \leq 1\ .$$

(b) The means, variances and covariance of π_1 and π_2 are given by

$$(1.5a) \qquad E(\pi_1) = \frac{a}{a+b+c}, \qquad V(\pi_1) = \frac{a(b+c)}{(a+b+c)^2(a+b+c+1)};$$

$$(1.5b) \qquad E(\pi_2) = \frac{b}{a+b+c}, \qquad V(\pi_2) = \frac{b(a+c)}{(a+b+c)^2(a+b+c+1)};$$

$$\text{Cov}(\pi_1, \pi_2) = -\frac{ab}{(a+b+c)^2(a+b+c+1)}.$$

From these it follows that

$$E(\pi_1 - \pi_2) = \frac{a-b}{a+b+c},$$

$$(1.6) \qquad V(\pi_1 - \pi_2) = \frac{4\,ab + ac + bc}{(a+b+c)^2(a+b+c+1)}.$$

(c) The family of Dirichlet distributions form a natural conjugate family to the trinomial distribution, so that the posterior distribution of π_1, π_2 after observing n independent observations each following a trinomial distribution, is another Dirichlet distribution with new parameters. Specifically, if the a priori distribution is $d\,\xi(\pi_1, \pi_2 | a_o, b_o, c_o)$ and if out of the n observations $a, b, c (a+b+c = n)$ belong to the three cells with probability $\pi_1, \pi_2, 1 - \pi_1 - \pi_2$ respectively, then the a posteriori distribution is $d\,\xi(\pi_1, \pi_2 | a_o + a, b_o + b, c_o + c)$.

1.5.2 Reduction of prior information.

The modification of the original two-binomial problem into the trinomial problem gives rise to the technical problem of converting the prior information about p_1 and p_2 into that about π_1 and π_2.

Although a precise analytical transformation of this information is inconvenient mathematically (in fact, the transformation from p_1 p_2 to π_1, π_2 is not one to one, but one to two at most), any such sophisticated treatment may not be necessary since a priori information is usually of a rough nature. A few lower order moments of π_1 and π_2 are sufficient to fit a particular Dirichlet distribution. Now it might be easier on the part of the experimenter to form some ideas about the moments of p_1 and p_2 rather than about π_1 and π_2 from previous experience. In such situations the following relations might be helpful; these relations give the means and variances of π_1 and π_2 in terms of those of p_1 and p_2 assuming independence of p_1 and p_2.

$$E(\pi_i) = E(p_i)[1 - E(p_{3-i})] ,$$

(1.7)

$$V(\pi_i) = V(p_i)[1 - E(p_{3-i})]^2 + V(p_{3-i})E^2(p_i) + V(p_i)V(p_{3-i});$$

$$i = 1, 2 .$$

It may be interesting to know, in particular, what a uniform distribution of p_1 and p_2, reflecting little or no a priori knowledge about p_1, p_2, means in terms of π_1, π_2 and vice versa. A rough calculation using the above formulae shows that a uniform prior distribution of p_1, p_2 over the square $[0, 1] \times [0, 1]$ (i.e., assuming p_1 and p_2 independent and uniform) corresponds approximately to the Dirichlet distribution $d \xi$ (2,2,5).

If the experimenter feels that p_1 and p_2 are not independent, then the following formulae might be useful.

$$E(\pi_i) = E(p_i)[1 - E(p_{3-i})] - \rho[V(p_i)\, V(p_{3-i})]^{1/2},$$

$$V(\pi_i) \approx V(p_i)[1 - E(p_{3-i})]^2 + V(p_{3-i})\, E^2(p_i)$$

$$- 2\, E(p_i)[1 - E(p_{3-i})]\ \rho[V(p_i)\, V(p_{3-i})]^{1/2}\ ; \ i=1,2.$$

where ρ is the correlation coefficient between p_1 and p_2.

Using the above formulae it may be verified that a joint distribution of p_1 and p_2 with uniform marginals and a correlation $\rho = 1/2$ corresponds approximately to the Dirichlet distribution $d\,\xi(4,4,11)$.

1.6 A decision-theoretic formulation for the binomial problem.

In this section we write down, for the sake of completeness, a corresponding decision-theoretic formulation for the problem involving a single binomial parameter, p. The binomial problem has an independent interest of its own. In fact, the problem with linear loss structure and constant cost has already been considered by Moriguti and Robbins [30]. Moreover, this might serve as a statistical formulation for the original problem when it is reduced, as shown in page 6 to the problem about a single binomial parameter, neglecting the ties. This model also arises with continuous responses, such as temperature or blood pressure reading, to the two treatments in the following way. Instead of discretizing the continuous responses as lying above or below a certain level as mentioned in page 2, we consider the original responses as such. Suppose specifically that X_1 and X_2 are distributed with absolutely continuous distribution functions F_1 and F_2 respectively. Suppose also that the experimenter does not wish to make any assumption about any specific parametric form, say normal, exponential etc. for F_1 and F_2, and wants to test whether X_1 is larger than X_2 in some probability sense (e.g., stochastically larger [27, p. 73]); then the present formulation can be applied to the following binomial parameter

$$p = P[X_1 > X_2] = \int F_2 \, dF_1 \quad .$$

The various elements of a decision-theoretic formulation are specified below:

<u>model</u>: $P[X = 1] = p, \quad P[X = 0] = 1-p, \quad 0 \le p \le 1$.

<u>actions</u>: a_1 preferable if $p > 1/2$,

a_2 preferable if $p \le 1/2$.

<u>costs per observation</u>: (i) $C(p) = 1$, all p : constant cost

(ii) $C(p) = |p-1/2|$: absolute deviation cost.

<u>losses</u>: I. linear loss structure:

$$L(p,a_1) = \begin{cases} 2A|p-1/2| , & \text{if } p \le 1/2 , \\ 0 , & \text{if } p > 1/2 . \end{cases}$$

(1.9)

$$L(p,a_2) = \begin{cases} 0 , & \text{if } p \le 1/2 , \\ 2A(p-1/2), & \text{if } p > 1/2 . \end{cases}$$

II. modified loss structure:

$$(1.10)\quad L(p,a_1;n) = \begin{cases} 2(A-n)|p-1/2|, & \text{if } p \leq 1/2,\ n < A, \\ 0 & , \text{if } p \leq 1/2,\ n \geq A, \\ 0 & , \text{if } p > 1/2,\ n = 0,1,2\ldots \end{cases}$$

$$L(p,a_2;n) = \begin{cases} 0 & , \text{if } p \leq 1/2,\ n = 0,1,2,\ldots, \\ 0 & , \text{if } p > 1/2,\ n \geq A, \\ 2(A-n)(p-1/2), & \text{if } p > 1/2,\ n < A. \end{cases}$$

a priori distribution:

$$(1.11)\quad d\,\xi\,(a,b) \equiv d\,\xi(p|a,b) = \frac{\Gamma(a+b)}{\Gamma a\ \Gamma b}\ p^{a-1}(1-p)^{b-1}dp,$$

$$0 \leq p \leq 1\ ,$$

where a,b are positive integers.

We shall use the term binomial problem to denote the decision problem formulated above. In Chapter II we shall consider a slightly more general situation, viz., testing $p \leq p_o$ against $p > p_o$, where p_o is an arbitrary number $0 < p_o < 1$. The above formulation corresponds to the special case $p_o = 1/2$.

The corresponding decision problem concerning the trinomial model will be called the trinomial problem. Another important reason for introducing the binomial problem in this thesis is the fact that, being mathematically simpler to solve it constitutes a first step towards the solution of the trinomial problem. In fact the analyses for the two problems are so analogous that in the remaining chapters only the binomial problem is treated in details. The results for the trinomial problem are just stated omitting many of the details.

1.7 Review of some earlier relevant work.

We have already mentioned Anscombe [2] which contributes rather heavily towards the formulation of our basic problem as stated in section 1.4. Anscombe considers the corresponding normal problem and gives an easily computable outer bound on the optimum boundary. We are, however, concerned with the exact optimum boundary for our problems.

We have also mentioned Moriguti and Robbins [30] in section 1.6 in connection with the binomial problem. Specifically, they consider only linear loss, constant cost and $p_o = 1/2$. From rather elementary considerations they show the existence of the Bayes procedure for their problem and describe it constructively. They have also given the numerical values of the optimum boundary for certain values of A. Next, they give a heuristic treatment (which we shall adapt for our generalizations) of the limiting behavior of the optimal solution under two different types of limiting tendencies. For the second type of limiting tendency that turned out to fit the observed numerical results better than the first, they reduced the problem of finding the optimal solution to that of solving a free boundary value problem, thus arriving at the same mathematical problem as Chernoff [14] and Lindley [28] arrived at for the normal problem. But they gave for the first time a formal series expansion solution of the free boundary value problem (in the permissible range) which has later been shown to be an asymptotic expansion by Breakwell and Chernoff [12]. Moriguti and Robbins also considered some alternative ad-hoc sampling rules and compared them with the optimal one at least in some restricted sense. Finally, they presented some charts showing the OC curves and ASN curves for the optimal sampling rule for some representative values of the relevant parameters involved.

It will be seen later that the limiting behavior of the optimal solution for the binomial problem reduces to that of the optimal solution for the normal problem under appropriate limiting tendencies. For this reason, results connected with the latter problem are relevant for our purpose. Towards this we mention an important contribution of Bather [8] in outlining some methods of obtaining certain bounds on the optimal solution. These bounds, of course, have nothing to do with the outer boundary of Anscombe, mentioned earlier in this section, for the normal problem, and the possible inner boundary that might be obtained from the modified Bayes rule developed by Amster [1].

The works reviewed above are directly relevant for our specific problem. It is not intended here to review any of the rapidly growing volume of works connected with the general theory of sequential analysis. An excellent summary of the current state of affairs in this field is given in [25] by Johnson.

1.8 Chapter by chapter summary.

Chapter I. The basic problem of double dichotomies is stated. Existing methods of analysing such data within the framework of Neyman-Pearson theory are described briefly along with their main defects and their inappropriateness in the present context. Following Anscombe [2], the problem is then formulated as a decision-theoretic one within the framework of Wald theory involving more than one realistic loss and cost function. The problem thus formulated is then modified, for reasons that are explained, to one involving the difference of two cell-probabilities in a certain trinomial distribution, called the trinomial problem. The corresponding binomial problem is also formulated.

Chapter II. The general theory of statistical decision as given in Blackwell and Girshick [9] is applied to show that the Bayes procedures for the various decision problems (with linear loss) are truncated with probability one. It is also described here how to obtain the Bayes sampling rule and hence the optimum (Bayes) boundary constructively, for both the binomial and trinomial problems (linear loss structure and both constant cost and absolute deviation cost).

Chapter III. The usual characterization of a Bayes procedure in the general set up considered in section 2.1 is extended in section 3.1 to take account of any "stage-dependent" loss function. These results are then specialized to obtain some simple recursion relations characterizing the optimal procedures for the various decision problems with modified loss and absolute deviation cost. These procedures are also shown to be necessarily truncated at a known stage.

Chapter IV. The limiting behavior in large samples of the optimum boundaries for the decision problems studied in Chapter II is considered. Extending Moriguti and Robbins' [30] approach the problem of finding the normalized optimal boundaries with appropriate normalizations is reduced to that of solving certain partial differential equations free boundary value problems. A series expansion for a certain portion of the optimum boundary is then obtained for each problem. Expansions which enable the computation of the average risk are also given.

Chapter V. An indirect verification of the series expansions of Chapter IV is offered with the help of certain upper and lower bounds on the optimum boundary obtained by applying some methods of Bather [8].

A SCHEMATIC CLASSIFICATION OF THE RESULTS BY SECTIONS

Model	Optimum boundary						
	Exact			Asymptotic		Approximate asymptotic	
	linear loss		modified loss	linear loss		linear loss	
	constant cost	absolute deviation cost	absolute deviation cost	constant cost	absolute deviation cost	constant cost	absolute deviation cost
binomial	2.2-2.6 Moriguti and Robbins	2.2-2.6	3.3	4.1-4.2 Moriguti and Robbins	4.1-4.2	Bather	5.1
trinomial	2.7-2.11	2.7-2.11	3.4	4.3-4.4	4.3-4.4	5.2	5.2

CHAPTER II

BAYES PROCEDURES WITH LINEAR LOSS STRUCTURE

2.0 Summary.

In section 2.1 we collect together some known results of the general theory of statistical decisions that are applied with increasing specializations in the subsequent sections. In section 2.2 a binomial model with linear loss and a general p_o is introduced. Sufficient conditions are obtained for a Bayes procedure to be truncated. In section 2.3, a beta prior distribution is introduced and the sufficient condition of section 2.2 is shown to be satisfied. In section 2.4 the point of truncation of a Bayes procedure is defined; it is also shown how to determine it. Section 2.5 gives a description of how to obtain the optimal boundary from the point of truncation by working backwards step by step. Section 2.6 contains some points useful for computation of the optimal boundary as described in section 2.5. Sections 2.7-2.11 treat the trinomial problem and correspond in the numerical order with the sections 2.2-2.6.

2.1 Some well-known results about Bayes sequential procedures with a general model.

In this section, we collect together some known results [9] that will be useful in obtaining the Bayes sequential procedures for the binomial and the trinomial model with linear loss structure.

Notations:

$\{X_i\}$: sequence of independent and identically distributed
$i=1,2,\ldots$ random variables.

$f_\theta(x_1)$: probability density function of the random variable X_1 with respect to some measure μ.

$\Theta = \{\theta\}$: space of states of nature.

a_1, a_2 : two possible actions or terminal decisions.

$L(\theta, a_i)$, $i=1,2$: losses.

$C(\theta)$: cost per observation.

ξ : a priori distribution (on a σ-field of subsets of Θ).

$$C(\xi) \equiv \int_\Theta C(\theta)\, d\,\xi(\theta) .$$

$$L(\xi, a_i) \equiv \int_\Theta L(\theta, a_i)\, d\,\xi(\theta), \quad i = 1, 2 .$$

$$\rho_0(\xi) \equiv \min_{i=1,2} L(\xi, a_i) .$$

$$f_\xi(x) \equiv \int_\Theta f_\theta(x)\, d\,\xi(\theta) .$$

$$\underline{x}_j \equiv (x_1, x_2, \ldots, x_j), \quad j = 1, 2, \ldots .$$

$$f_{\theta,j} \equiv f_{\theta,j}(\underline{x}_j) \equiv \prod_{i=1}^{j} f_\theta(x_i), \quad j = 1, 2, \ldots .$$

$$f_{\xi,j} \equiv f_{\xi,j}(\underline{x}_j) \equiv \int_\Theta f_{\theta,j}\, d\,\xi(\theta), \quad j = 1, 2, \ldots .$$

$$d\,\xi_j(\theta) \equiv f_{\theta,j}\, d\,\xi(\theta)/f_{\xi,j}, \qquad j = 1, 2, \ldots .$$

$$d\,\xi_o(\Theta) \equiv d\,\xi(\Theta) \quad .$$

$$d\,\xi_1^{(y)}(\Theta) \equiv f_\Theta(y)\; d\,\xi(\Theta)/f_\xi(y) \quad .$$

N : sample size (random variable) using a specified decision function .

$\psi \equiv \{\psi_j\}$, $j = 0, 1, \ldots$: a sampling rule, where

$\psi_j \equiv \psi_j(\underline{x}_j)$, $j = 0, 1, \ldots$ is the probability that $N = j$, that is, the probability of termination after observing $x_1,\ldots,x_j$.

$\phi \equiv \{\phi_j\}$, $j = 0, 1, \ldots$: a terminal decision rule, where

$\phi_j \equiv \phi_j(\underline{x}_j)$, $j = 0, 1, \ldots$ is the probability of taking action a_2 after observing $x_1,\ldots,x_j$ conditional upon $N = j$.

$\delta \equiv (\psi, \phi)$: a sequential decision function .

Δ_n, $n = 0,1,2,\ldots$: class of all sequential decision functions truncated at $N = n$, i.e., with $\psi_o + \psi_1 + \ldots + \psi_n = 1$ identically in $\underline{x}_n$.

$\Delta^o \equiv \bigcup_{n=1}^{\infty} \Delta_n$: class of all truncated sequential decision functions .

Δ^1 : class of all sequential decision functions which terminate almost certainly, i.e. for which

$$\sum_{j=o}^{\infty} \psi_j = 1, \quad \text{a.e. for all } \Theta \in \Theta \; .$$

Δ^2 : class of all sequential decision functions subject to usual [9] measurability conditions.

$\delta_\xi \equiv (\psi_\xi, \phi_\xi)$: Bayes decision function in a specified class Δ .

$r(\xi, \delta)$: average risk associated with δ .

$\rho_n(\xi) \equiv \min_{\delta \in \Delta_n} r(\xi, \delta)$, $n = 0,1,2,\ldots$: Bayes risk in Δ_n .
(this notation is consistent with earlier one, $\rho_o(\xi)$) .

$\rho(\xi) \equiv \min_{\delta \in \Delta^2} r(\xi, \delta)$: Bayes risk in Δ^2 .

We state the results in the form of the following lemmas.

Lemma 2.1. A Bayes terminal decision rule for $\delta \in \Delta$, where Δ is Δ_n for any n or Δ^k for any $r = 0,1$ or 2, satisfies

$$(2.1) \qquad \phi_{\xi,j} = \begin{cases} 0 & \text{if } \rho_o(\xi_j) = L(\xi_j, a_1) < L(\xi_j, a_2) , \\ u & \text{if } \quad '' \quad = \quad '' \quad = \quad '' , \\ 1 & \text{if } \quad '' \quad = L(\xi_j, a_2) < L(\xi_j, a_1) , \end{cases}$$

$j = 0,1,2,\ldots n$, if $\Delta = \Delta_n$, $n = 0,1,2,\ldots$,

or $j = 0,1,2,\ldots$ if $\Delta = \Delta^k$, $k = 0,1$ or 2;

where u is arbitrary $(0 \le u \le 1)$.

Lemma 2.2. If $\delta_\xi = (\psi_\xi, \phi_\xi)$ is a Bayes decision function in Δ^1 , then

$$(2.2) \qquad \rho(\xi) = \min [\rho_o(\xi), \quad \rho(\xi_1^{(y)}) f_\xi(y) \, d\mu(y) + C(\xi)],$$

and, after observing $x_1, \ldots, x_j$, sampling is terminated if and only if

$$\rho(\xi_j) = \rho_o(\xi_j) .$$

Lemma 2.3. If

(2.3) $$\xi\left\{C(\Theta) > 0\right\} = 1,$$

then

$$\lim_{n \to \infty} \rho_n(\xi) = \rho(\xi).$$

Notations.

$n \geq 1$

(2.3a) $$u_m \equiv [\rho_0(\xi_m) + m\, C(\xi_m)] f_{\xi,m}, \qquad m = 0,1,2,\ldots,n$$

v_m, w_m defined recursively for $m = m, n-1,\ldots,1$ by

(2.4) $$\begin{cases} v_n = u_n ; \\ w_m = \min(u_m, v_m), & m = n, n-1,\ldots,1; \\ v_m = \int w_{m+1}(\underline{x}_m, x_{m+1})\, d\mu(x_{m+1}), & m=n-1,n-2,\ldots,1 . \end{cases}$$

Lemma 2.4. If there exists a least positive integer n_o such that for every $n \geq n_o$,

(2.5) $$u_{n-1} \leq v_{n-1}, \quad \text{for all } \underline{x}_{n-1},$$

then

(2.6) $$\rho_{n_o} = \rho_{n_o+1} = \cdots$$

Notations.

(2.7) $$\begin{cases} \Gamma_i(\xi) \equiv \left\{\xi \mid L(\xi, a_i) \leq L(\xi, a_{3-i})\right\}, & i = 1,2 . \\ \Gamma_o(\xi) \equiv \left\{\xi \mid L(\xi, a_1) = L(\xi, a_2)\right\} = \Gamma_1 \cap \Gamma_2 . \end{cases}$$

(2.8) $\quad \Lambda_i(\xi) \equiv \{\xi \mid \rho(\xi) = \rho_o(\xi) = L(\xi, a_i)\}, \quad i = 1,2 .$

$\Lambda \equiv \Lambda_1 \cup \Lambda_2$: stopping set .

(2.9) $\quad \bar{\Lambda} \equiv$ complement of Λ : continuation set .

Lemma 2.5. Λ_1 and Λ_2 are convex, i.e., if ξ_1 and ξ_2 belong to Λ_i, then any convex linear combination ξ of ξ_1 and ξ_2, namely

$$\lambda \xi_1 + (1 - \lambda) \xi_2 \quad , \quad 0 \leq \lambda \leq 1$$

belongs to Λ_i, $i = 1, 2$.

Note. Lemma 2.3 follows from Theorem 1 of [23]. All other lemmas are essentially contained in Blackwell and Girshick's book [9].

2.2 Binomial model.

In this section we apply the results of the previous section with the following specializations and call the decision problem simply the binomial problem.

(2.10) $\quad f_p(1) = p, \quad f_p(0) = 1-p; \quad 0 \leq p \leq 1 .$

a_1 is preferable if $p > p_o$,

a_2 is preferable if $p < p_o$.

where p_o is an arbitrary constant, $0 < p_o < 1$.

(2.11)
$$L(p,a_1) = \begin{cases} 2A(p_o - p) & , \text{ if } p \leq p_o , \\ 0 & , \text{ if } p > p_o . \end{cases}$$

$$L(p,a_2) = \begin{cases} 0 & , \text{ if } p \leq p_o , \\ 2A(p - p_o) & , \text{ if } p > p_o . \end{cases}$$

It follows from the above specializations that

$$
(2.12)\quad \begin{cases} \Gamma_1(\xi) = \{\xi \mid E_\xi p \geq p_o\} , \\ \Gamma_2(\xi) = \{\xi \mid E_\xi p \leq p_o\} , \\ \Gamma_o(\xi) = \{\xi \mid E_\xi p = p_o\} ; \end{cases}
$$

and

$$
(2.13)\qquad \rho_o(\xi) = \begin{cases} A\, E_\xi |p-p_o| - A\, E_\xi(p-p_o) , & \text{for } \xi \in \Gamma_1 , \\ A\, E_\xi |p-p_o| - A\, E_\xi(p_o-p) , & \text{for } \xi \in \Gamma_2 , \\ A\, E_\xi |p-p_o| , & \text{for } \xi \in \Gamma_o ; \end{cases}
$$

which may be written compactly as

$$
(2.14)\qquad \rho_o(\xi) = A\, E_\xi |p-p_o| - A|E_\xi(p-p_o)| .
$$

$\phi_{\xi,j}$ of lemma 2.1 and $\rho(\xi)$ of lemma 2.2 (equations (2.1) and (2.2)) reduce to

$$
(2.15)\qquad \phi_{\xi,j} = \begin{cases} 0 , & \text{if } E_{\xi_j} p > p_o , \\ u , & \text{if } \quad " \quad = p_o , \\ 1 , & \text{if } \quad " \quad < p_o , \end{cases}
$$

$$
j = 0,1,2,\ldots,n; \quad n = 0,1,2,\ldots .
$$

$$
(2.16)\qquad \rho(\xi) = \min [\, \rho_o(\xi),\ f_\xi(1)\, \rho(\xi_1^{(1)}) + f_\xi(0)\rho(\xi_1^{(0)}) + c(\xi) \,]
$$

where

$$
(2.17)\qquad f_\xi(i) = \int_0^1 p^i\, (1-p)^{1-i}\, d\,\xi(p) , \quad i = 0, 1 .
$$

and $\rho_o(\xi)$ is given by (2.14) .

We develop below a sufficient condition for a Bayes procedure for the binomial problem to be truncated with probability one, i.e., δ_ξ in Δ^o coincides with δ_ξ in Δ^2. Under the assumption of ξ being absolutely continuous, Sobel [36] proves that the Bayes sequential procedure δ_ξ in Δ^2 is truncated with probability one for a general set up that includes ours with constant cost. His method of proof did not allow extension for a cost depending on the parameter; hence the necessity of the following theorem.

Theorem 2.1. A sufficient condition for a Bayes sequential procedure δ_ξ for the binomial problem to be truncated with probability one is:

(2.18) (a) $\xi \{C(p) > 0\} = 1$

and

(2.19) (b) $\lim_{n \to \infty} \text{Var}(p \mid \xi_n)/ C(\xi_n) = 0$ a.e.

uniformly in n.

Moreover, δ_ξ takes at most n_o observations, where n_o is the smallest positive integer such that for every $n \geq n_o - 1$

(2.20) $\text{Var}(p \mid \xi_n)/ C(\xi_n) \leq 1/2A$ a.e.

Note. Condition (a) is introduced in order that lemma 2.3 holds. It is trivially satisfied for constant cost for any ξ. For absolute deviation cost, viz., $C(p) = |p-p_o|$, it is satisfied for any absolutely continuous ξ and hence in particular when ξ is a beta distribution (1.11).

Proof: Because of lemma 2.3 and lemma 2.4, it is sufficient to show that there exists a least positive integer n_o such that for every $n \geq n_o$,

$$(2.21) \qquad u_{n-1} \leq v_{n-1} \ , \quad \text{for all } \underline{x}_{n-1} \ .$$

Writing explicitly the definitions of u_{n-1} and v_{n-1}, (2.3a) and (2.4), (2.21) reduces to

$$(2.22) \qquad [\rho_o(\xi_{n-1}) + (n-1)\, C(\xi_{n-1})]\, f_{\xi,n-1} \leq \int [\rho_o(\xi_n) + n\, C(\xi_n)]\, f_{\xi,n}\, d\mu\, (x_n), \text{ for all } \underline{x}_{n-1} \ ,$$

i.e., for all ξ_{n-1} for a fixed ξ .

Writing

$$(2.23) \qquad \lambda(\xi) \equiv \rho_o(\xi) - f_\xi(1)\, \rho_o(\xi_1^{(1)}) - f_\xi(0)\, \rho_o(\xi_1^{(0)}) \ ,$$

for convenience, (2.22) reduces under the present set up, to

$$(2.24) \qquad \lambda(\xi_{n-1}) \leq C(\xi_{n-1}) \quad \text{for all } \xi_{n-1} \ .$$

Now from the definitions

$$(2.25) \qquad d\, \xi = f_\xi(1)\, d\, \xi_1^{(1)} + f_\xi(0)\, d\, \xi_1^{(0)} \ .$$

Also, expectation being a linear operation, it follows from (2.25) that for any integrable $g(p)$,

$$(2.26) \qquad E_\xi\, g(p) = f_\xi(1)\, E_{\xi_1^{(1)}}\, g(p) + f_\xi(0)\, E_{\xi_1^{(0)}}\, g(p) \ .$$

It follows from (2.13) and (2.26) by identifying ξ with ξ_{n-1}, that $\lambda(\xi_{n-1})$ is identically zero for all those ξ_{n-1} such that ξ_{n-1}, $(\xi_{n-1})_1^{(0)}$, $(\xi_{n-1})_1^{(0)}$ all belong to the same Γ_i, $i = 1$ or 2. (2.24) is then satisfied for all these ξ_{n-1} and for any non-negative cost $C(p)$.

Each of the remaining ξ_{n-1}'s belong to one and only one of the following three mutually disjoint subsets:

$$(2.27) \quad \begin{cases} \Omega_0 \equiv \left\{ \xi \mid E_\xi\, p = p_0 \right\} , \\ \Omega_1 \equiv \left\{ \xi \mid E_{\xi_1^{(0)}}\, p < p_0 < E_\xi\, p \right\} \subset \Gamma_1 , \\ \Omega_2 \equiv \left\{ \xi \mid E_\xi\, p < p_0 < E_{\xi_1^{(1)}} p \right\} \subset \Gamma_2 . \end{cases}$$

If $\xi \in \Omega_0$, then $\xi_1^{(1)} \in \Gamma_1$ and $\xi_1^{(0)} \in \Gamma_2$, so that it follows from (2.23), (2.13) and (2.26) that

$$\begin{aligned} \lambda(\xi) &= A\, f_\xi(1)\, E_{\xi_1^{(1)}} (p-p_0) + A\, f_\xi(0)\, E_{\xi_1^{(0)}} (p_0-p) \\ (2.28) \qquad &= A \int_0^1 (p-p_0) p\, d\,\xi(p) + A \int_0^1 (p_0-p)(1-p)\, d\,\xi(p) \\ &= 2A \operatorname{Var}(p \mid \xi) \end{aligned}$$

remembering that

$$p_0 = E_\xi p \quad \text{for} \quad \xi \in \Omega_0 .$$

If $\xi \in \Omega_1$, then $\xi_1^{(1)} \in \Gamma_1$ and $\xi_1^{(0)} \in \Gamma_2$, so it again follows from (2.23), (2.13) and (2.26) that

$$
\begin{aligned}
\lambda(\xi) &= 2A\, f_{\xi}(0)\; E_{\xi_1(0)}(p_o - p) \\
&= 2A \int_0^1 (p_o - p)(1-p)\, d\,\xi\,(p) \\
(2.29) \qquad &= 2A\, [E_{\xi}\,(1-p)^2 - (1-p_o)\, E_{\xi}(1-p)] \\
&< 2A\, [E_{\xi}(1-p)^2 - E^2_{\xi}\,(1-p)]
\end{aligned}
$$

since $E_{\xi} p > p_o$, for $\xi \in \Omega_1$, and this

$$= 2A \text{ Var } (p|\xi) .$$

It can be shown similarly that for $\xi \in \Omega_2$,

$$
\begin{aligned}
(2.30) \qquad \lambda(\xi) &= 2A\, [E_{\xi}\, p^2 - p_o\, E_{\xi}\, p] \\
&< 2A \text{ Var } (p|\xi) .
\end{aligned}
$$

Hence (2.21) is satisfied if there exists an n_o such that for every $n \geq n_o$,

$$(2.31) \qquad \frac{\text{Var } (p|\xi_{n-1})}{C(\xi_{n-1})} \leq \frac{1}{2A} \quad \text{a.e.}$$

for which a sufficient condition is (2.19) .

The second part of the theorem follows immediately from (2.31). The proof of the theorem is thus complete.

Remarks: (1) It is to be noted that n_o defined in (2.20) depends on the particular a priori distribution ξ , while the sets Ω_i , i = 0,1,2 (2.27) are defined independently of it.

(2) It follows from the proof above that for the conclusion of the theorem to remain valid it is sufficient that (2.19) hold good only for those ξ_n which belong to $\Gamma \equiv \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.

2.3 Beta a priori distribution.

From now onwards we shall consider an a priori distribution given by

$$(2.32)\quad d\,\xi(a_o,b_o) \equiv d\,\xi(p|a_o,b_o) = \frac{\Gamma(a_o+b_o)}{\Gamma(a_o)\Gamma(b_o)}\; p^{a_o-1}(1-p)^{b_o-1}dp\,,$$

$$0 \leq p \leq 1\,,$$

where a_o, b_o are positive integers.

It is well-known [40] that the beta family of distributions is closed under sampling. In the terminology of Raiffa and Schlaifer [33], (2.32) forms a natural conjugate prior for the model binomial distribution. In other words, if a_n and b_n denote the number of successes and failures respectively in a sample of size n, the a posteriori distribution is given by

$$(2.33)\qquad d\,\xi_n(p) = d\,\xi(a_o + a_n,\ b_o + b_n),\quad n = 1,2,\ldots,$$

Thus the set of all possible a posteriori distributions that can arise from the a priori distribution (2.32) can be represented by the set of all lattice points in a positive quadrant with origin at (a_o,b_o).

It is well-known [9] that the Bayes decision-function can be completely characterized by the sets Λ_1, Λ_2, $\bar{\Lambda}$ defined in (2.8), (2.9). In this special case, these sets that are associated with (2.32) can be represented as certain sets in the space of lattice points mentioned above. Without any risk of confusion we shall denote these various (a,b) sets by the same notations as the corresponding ξ sets. The optimum boundary (for the decision problem) that will be mentioned henceforth quite often, is formally defined (unlike the terminology in Moriguti and Robbins [30]) as the set of points (a,b) in the stopping set Λ such that (a-1,b) or (a,b-1) is in the continuation set $\bar{\Lambda}$.

Due to the closedness property of the beta family mentioned above, it is not necessary to determine the optimal boundary along with the various sets Λ_1, Λ_2, $\bar{\Lambda}$ for each a priori distribution separately. It is possible to describe them in terms of a fixed origin say (0,0), and to obtain them for a particular a priori distribution, like (2.32), we need only to shift the origin to (a_o, b_o). It amounts to considering the a priori distribution (2.32) as being arrived at with a sample of $n_o-2 \equiv a_o + b_o-2$ observations with a_o-1 successes and b_o-1 failures, starting with a uniform prior.

In the following analysis the terms optimum boundary, stopping sets, continuation sets, etc. are defined with respect to this fixed frame of reference. Accordingly, by $d\,\xi_n(p)$, unlike previously, we shall mean $d\,\xi(a,b)$ where $a + b = n$. It is obvious that any sequential procedures which is Bayes for $d\,\xi(a_o, b_o)$ and is truncated at n_o (in the sense that it takes at most n_o observations before it tells one to stop), is also truncated at $n_o + a_o + b_o$ (in the above sense) when referred to this fixed frame of reference. Since any a priori beta distribution can be represented by $d\,\xi(a_o, b_o)$ where a_o, b_o are finite, the condition (2.19) for a Bayes procedure to be truncated remains valid when ξ_n is given the new interpretation above. We apply this condition in this new light to arrive at the following corollary to Theorem 2.1.

Corollary 2.1. A Bayes sequential procedure for the binomial problem with a beta prior distribution is truncated with probability one.

Proof: As noted before, condition (a) of Theorem 2.1 is satisfied for the particular cases we are concerned with here.

Now

$$(2.34) \qquad \mathrm{Var}\,(p|a,b) = \frac{ab}{(a+b)^2\,(a+b+1)} \quad .$$

Case (i). Constant cost, i.e. $C(a,b) = 1$ for all (a,b) :

$$(2.35) \qquad \frac{\mathrm{Var}(p|a,b)}{C(a,b)} = \frac{ab}{(a+b)^2(a+b+1)} < \frac{1}{a+b+1} \longrightarrow 0$$

uniformly as $a + b \longrightarrow \infty$.

Condition (2.19) is thus ensured in this case.

Case (ii). Absolute deviation cost, i.e., $C(a,b) = \int_0^1 |p-p_o|\,d\xi(a,b)$:

As noted in remark (2) after Theorem 2.1, we need only verify the condition (2.19) for those ξ belonging to Ω_i, $i = 0,1,2$ which may be written in this special case as follows:

$$(2.36) \qquad \Omega_o = \left\{ (a,b) \;\middle|\; \frac{a}{a+b} = p_o \right\} \quad ,$$

$$(2.37) \qquad \Omega_1 = \left\{ (a,b) \;\middle|\; \frac{a}{a+b+1} < p_o < \frac{a}{a+b} \right\} ,$$

$$(2.38) \qquad \Gamma_2 = \left\{ (a,b) \;\middle|\; \frac{a}{a+b} < p_0 \frac{a+1}{a+b+1} \right\} .$$

Also, it is sufficient to show that (2.19) holds only asymptotically as $(a+b) \longrightarrow \infty$. Now

$$(2.39) \qquad C(a,b) = \frac{1}{B(a,b)} \int_0^1 |p-p_0|\, p^{a-1} (1-p)^{b-1}\, dp$$

$$\sim \frac{1}{B(p_0\, \overline{a+b},\, \overline{1-p_0}\, \overline{a+b})} \int_0^1 |p-p_0|\, p^{p_0(a+b)-1} (1-p)^{\overline{1-p_0}(a+b)-1}\, dp$$

for $(a,b) \in \Gamma_i$, $i = 0,1,2$, where $B(a,b)$ is the well-known beta function. Also, the right hand side of (2.39) is asymptotically [15, p. 252] equal to

$$(2.40) \qquad [2p_0(1-p_0)/\pi]^{1/2} \; (a+b)^{-1/2} \qquad .$$

Thus

$$(2.41) \qquad Var(p|a,b)/C(a,b) \sim [\pi\, p_0(1-p_0)/2]^{1/2} \; (a+b)^{-1/2} \longrightarrow 0$$

uniformly as $a + b \longrightarrow \infty$,

concluding the proof of the corollary.

2.4 Determination of the point of truncation in the binomial problem.

The set Γ_0 has been called, for obvious reasons, the *neutral boundary* by Weatherill [40]. We shall call Γ the *extended neutral* boundary for reasons to be explained below. The *point of truncation* is then formally defined to be *that point of* Γ, *say* (a^0,b^0) *such that* $a^0 + b^0$ *is smallest and no point* (a,b) *of* Γ *with* $a+b \geq a^0 + b^0$ *is* a continuation point. The point of truncation, thus defined, may or may not belong to the optimum boundary defined earlier. It does not belong to the optimum boundary if it is in Γ_o. In the symmetric case, i.e., when $p_o = 1/2$, the point of truncation may alternatively be defined as the first stopping point on Γ (which is the same as Γ_o when $p_o = 1/2$). Here by "first" we mean "with smallest sum of co-ordinates".

The determination of the point of truncation is important in the sense that, once it is known, it is possible to obtain the optimum boundary, as will be described afterwards, by working backwards from this point with the help of the recursion formula (2.16).

Before we proceed further, we note the following facts about the sets Γ_i , $i = 0,1,2$.

(1) Γ_o is empty if p_o is irrational. If p_o is rational, in which case it can be written as a pure fraction, say,

$$(2.42) \qquad p_o = r_a/(r_a + r_b)$$

where r_a and r_b are mutually prime to each other, then the sequence of points $\{r_a k, r_b k\} \equiv \{\xi_k\}$, say, belong to Γ_o . It is obvious that the farther the ratio r_a/r_b is from unity, the fewer the points in Γ_o. In the special case $p_o = 1/2$, Γ_o consists of all the points (a,a) on the diagonal $a=b$.

(2) Γ_1 and Γ_2 are empty when $p_o = 1/2$. They have been primarily introduced for the situation $p_o \neq 1/2$. They represent points near Γ_o but not exactly on it, on the two sides of it,

Γ_i being on the side of Λ_i, $i = 1, 2$. In fact, a point in $\Gamma_1(\Gamma_2)$, though belonging to $\Lambda_1(\Lambda_2)$, is so near to the neutral boundary, Γ_o, that a failure (success) at the point leads the resulting posterior distribution to $\Lambda_2(\Lambda_1)$.

(3) Let $L_n \equiv \{(a,b) \mid a+b = n\}$ denote the set of lattice points on the line $a + b = n$. Then $\Gamma \cap L_n$ is empty if and only if $\Gamma \cap L_{n+1}$ is not empty and $\Gamma \cap L_{n+1} \subset \Gamma_o$. In all other situations, Γ intersects L_n in exactly one point. In the special case $p_o = 1/2$, since $\Gamma = \Gamma_o$, it follows that $L_n \cap \Gamma$ is empty if n is odd while $L_n \cap \Gamma$ is the point $(n/2, n/2)$ if n is even.

(4) If p_o is given by (2.42), then there are exactly r_a-1 points of Γ_1 interwoven with exactly r_b-1 points of Γ_2 in Γ between any two points, say, $(r_a k, r_b k)$ and $(r_a \overline{k+1}, r_b \overline{k+1})$ of Γ_o.

The results of the previous section ensure that a Bayes sequential procedure is truncated. This means that there exists an integer n^o sufficiently large such that the sampling region (or the continuation set) lies below the straight line L_{n^o}. Using the above facts, the region of sampling can be narrowed down further as stated in the following lemma.

<u>Lemma 2.6. If the Bayes sequential procedure for the binomial problem with a beta prior distribution is known to be truncated at n^o (i.e., every point (a,b) with $a+b > n^o$ is a stopping point), then the continuation set $\bar{\Lambda}$ is contained within a rectangle (with a corner removed in the first two cases below)</u>:

(i) *If the line* $a+b = n^{\circ}$ *intersects* Γ *at a point, say* (a°,b°) *of* Γ_1, *then* $\bar{\Lambda}$ *is bounded by* $a = a^{\circ}$, $b = b^{\circ}+1$, *and the diagonal joining* (a°, b°) *with* $(a^{\circ}-1, b^{\circ}+1)$.

(ii) *If the line* $a + b = n^{\circ}$ *intersects* Γ *at a point, say* (a°,b°) *of* Γ_2 , *then* $\bar{\Lambda}$ *is bounded by* $a = a^{\circ}+1$, $b = b^{\circ}$ *and the diagonal joining* (a°,b°) *with* $(a^{\circ}+1, b^{\circ}-1)$.

(iii) *If the line* $a+b = n^{\circ}$ *intersects* Γ *at a point, say,* (a°,b°) *of* Γ_0, *then* $\bar{\Lambda}$ *is bounded by*

$$a = a^{\circ} , \ b = b^{\circ} .$$

(iv) *If the line* $a+b = n^{\circ}$ *does not intersect* Γ, *which case arises when the line* $a+b = n^{\circ}+1$ *intersects* Γ *at a point, say* (a°,b°) *of* Γ_0 , *then* $\bar{\Lambda}$ *is bounded by*

$$a = a^{\circ} , \quad b = b^{\circ} .$$

The proof of this long lemma is simple and follows readily from lemma 2.5 and the above facts (1), (2) and (3).

The above lemma is useful in locating the point of truncation. If it is known that all the points on a certain line L_n are stopping points, then so are all those on the line L_{n-1} except perhaps the one in Γ, if any. In other words, if $L_{n-1} \cap \Gamma$ is empty, we move on to the next line L_{n-2}; if, on the other hand, $L_{n-1} \cap \Gamma$ is not empty, in which case it consists of just one point belonging to any of the Γ_i's, $i = 0,1,2$, according to (3) above, then it needs to be verified whether it is a stopping point or a continuation point. If it is a stopping point, we move on to the next line, and proceed analogously till we reach the first continuation point on Γ. The next theorem gives an easily applicable criterion by which it is possible to verify whether the point $\xi(a,b)$ in $L_{n-1} \cap \Gamma$ is a stopping point or a continuation point.

<u>Theorem 2.2. If $\xi(a,b)$ is a point on Γ such that $\xi_1^{(1)}$, i.e., $(a+1,b)$, and $\xi_1^{(0)}$, i.e., $(a,b+1)$, are stopping points, then $\xi(a,b)$ is a stopping point or a continuation point according as</u>

$$(2.43) \qquad \lambda(\xi) \leq C(\xi) \quad \text{or} \quad \lambda(\xi) > C(\xi) ,$$

<u>where</u>

$$(2.44) \qquad \lambda(\xi) = \begin{cases} 2A \ \mathrm{Var}(p \mid \xi) & , \text{ if } \xi \in \Gamma_0 \\ 2A \left[E_\xi (1-p)^2 - (1-p_0) E_\xi (1-p) \right] & , \text{ if } \xi \in \Gamma_1 , \\ 2A \left[E_\xi p^2 - p_0 E_\xi p \right] & , \text{ if } \xi \in \Gamma_2 . \end{cases}$$

<u>Proof.</u> Since the Bayes procedure is known to be truncated, the hypothesis of lemma 2.2 is satisfied so that (2.16) holds. Subtracting $\rho_0(\xi)$ from both sides of (2.16), we get

$$(2.45) \quad \rho(\xi)-\rho_0(\xi) = \min\left[0, f_\xi(1) \left\{\rho(\xi_1^{(1)})-\rho_0(\xi_1^{(0)})\right\} + f_\xi(0) \left\{\rho(\xi_1^{(0)}) - \rho_0(\xi_1^{(0)})\right\} + C(\xi)-\lambda(\xi)\right]$$

where $\lambda(\xi)$ is defined as in (2.23).

Since $\xi_1^{(1)}$ and $\xi_1^{(0)}$ are stopping points, by hypothesis,

$$\rho(\xi_1^{(i)}) = \rho_0(\xi_1^{(i)}), \quad i = 0, 1 .$$

Hence, (2.45) reduces to

$$(2.46) \qquad \rho(\xi) - \rho_0(\xi) = \min [0, C(\xi) - \lambda(\xi)]$$

Now ξ is a stopping point or a continuation point according as

$$\rho(\xi) = \rho_0(\xi) \qquad \text{or} \qquad \rho(\xi) < \rho_0(\xi) ,$$

i.e., by (2.46), according as

$$\lambda(\xi) \leq C(\xi) \qquad \text{or} \qquad \lambda(\xi) > C(\xi) .$$

The expressions for $\lambda(\xi)$ given in (2.44) were already obtained in (2.28), (2.29) and (2.30). The proof of the theorem is complete.

We now derive bounds on $\lambda(\xi) = \lambda(a,b)$ for $\xi = \xi(a,b)$.

Lemma 2.7.

$$(2.47) \qquad 0 \leq \lambda(a,b) < 2A\, p_0(1-p_0)/(a+b) .$$

We are now in a position to determine the exact location of the point of truncation. For simplicity we consider the symmetric case first.

Special case: $p_0 = 1/2$.

As noted earlier, Γ_1 and Γ_2 are empty in this case; Γ_0 consists of points (a,a). It follows from the procedure indicated above and Theorem 2.2 that the point of truncation is given by (a^0, a^0) where a^0 is the smallest positive integer such that

$$(2.51) \qquad \lambda(a^0, a^0) \leq C(a^0, a^0) .$$

Now for constant cost, $C = 1$, while for absolute deviation cost

$$C(a,a) = \int_0^1 \left| p - \frac{1}{2} \right| d\xi\ (a,a)$$

$$= \frac{1}{2} \frac{(2a)!}{a!\ a!} \frac{1}{2^{2a}} \equiv \frac{1}{2} b_i\ (a,a), \text{ say}$$

where $b_i(a,a)$ denotes the central term of the symmetrical binomial distribution with parameters $(2a, \frac{1}{2})$.

Thus it follows from (2.51), (2.52) that the truncation point is (a^o, a^o) where a^o is the smallest positive integer such that

$$(2.53) \qquad a^o \geq \frac{1}{2}\left(\frac{A}{2} - 1\right),$$

for the case of constant cost; and such that

$$(2.54) \qquad (2a^o + 1)\ b_i(a^o, a^o) \geq A$$

for the case of absolute deviation cost. Standard binomial tables may be used to obtain a^o from (2.54). For large values of A, we may use instead the asymptotic relation

$$(2.55) \qquad a^o \sim \pi A^2/4.$$

Corollary 2.2. If (a^o, a^o) is the point of truncation, then the points (a,a) with $a < a^o$ are continuation points.

This follows readily from the definition of the point of truncation, the monotonically decreasing property of $\lambda(a,a)/C(a,a)$ with respect to a, and Theorem 2.2, especially the equation (2.45) remembering that $\rho \leq \rho_o$ for any point (a,b).

General case. $0 < p_o < 1$

It follows from (2.43) and (2.47) that any point (a,b) with

$$(2.56) \qquad a + b \geq \frac{2A\, p_o(1-p_o)}{C(a,b)}$$

is a stopping point. Thus, if (a', b') be the point of $(\bar{\ })$ with smallest $a+b$ satisfying (2.56) and if (a^o,b^o) be the point of truncation, then

$$(2.57) \qquad a^o + b^o < a' + b'$$

Consider the points of $(\bar{\ })$ in a sequence with decreasing $a+b$, starting from (a',b'). With the help of the criterion developed in Theorem 2.2 we may then search for (a^o,b^o), as described already, along this sequence.

special case: $\left\{ \begin{array}{l} \text{constant cost} \\ p_o = \dfrac{r_a}{r_a + r_b}, \quad \text{as defined in (2.42)} \end{array} \right.$

In this special situation we do not have to go beyond the point (a_1,b_1) say, along this sequence, where

$$(a_1,b_1) = (r_a k^o + 1,\ r_b k^o + 1)$$

k^o being the largest positive integer k for which

$$(2.58) \qquad \lambda(r_a k,\ r_b k) \equiv \frac{p_o(1-p_o)}{(r_a+r_b)\, k+1} > \frac{1}{2A} .$$

In fact, it may be shown in this special situation that the point of truncation, (a^o,b^o) is one of the $(r_a + r_b - 1)$ points of

Γ between the two successive points $(r_a(k^o+1)\ ,\ r_b(k^o+1))$ and $(r_a\,k^o,\ r_b k^o)$ of Γ_o including the first. This follows since from the definition (2.58) of k^o, $(r_a\,k^o,\ r_b\,k^o)$ is a continuation point while $(r_a(k^o+1),\ r_b(k^o+1))$ is a stopping point. Moreover, all points (a,b) with

$$(2.59)\qquad a + b \geq (r_a + r_b)(k^o + 1) + 1$$

are stopping points. This follows from (2.43) since for any such point (a,b),

$$\lambda(a,b) < \frac{2A\ p_o(1-p_o)}{a + b} \qquad \text{by lemma 2. .}$$

$$\leq \frac{2A\ p_o(1-p_o)}{(r_a+r_b)(k^o+1)+1} \qquad \text{by (2.59)}$$

$$\leq 1$$

by (2.43), $(r_a(k^o+1),\ r_b(k^o+1))$ being a stopping point.

Finally, it may be shown just like in corollary 2.2 that the points $(r_a\,k,\ r_b\,k)$ with $k \leq k^o$ are all continuation points. We cannot assert, however, that all the points of Γ_1 and Γ_2 for which $a+b < a^o + b^o$ are also continuation points, since $\lambda(a,b)$ is not necessarily monotonically decreasing in $a+b$ for $(a,b) \in \Gamma_i$, $i = 1,2$. This is the reason for the particular definition of the point of truncation that we chose in the beginning of this section. We expect, however, that most of them will be continuation points because by Lemma 2.7 $\lambda(a,b)$ is closely bounded above by a function that is monotonically decreasing in $a+b$.

2.5 Procedure for obtaining the optimum boundary in the binomial problem.

In this section, we describe how to obtain the optimum boundary constructively once the point of truncation is known. For simplicity we first consider the case $p_o = 1/2$.

Special case: $p_o = 1/2$.

The neutral boundary, Γ_o, consists of points (a,a) in this case. Because of the symmetry of the problem, the optimum boundary will be symmetric with respect to the line $a = b$. Hence, without any loss of generality, we consider only $a \geq b$.

Let (a^o, a^o) be the point of truncation obtained as indicated in the previous section. By Lemma 2.6, all the points on $a = a^o$ are stopping points. Also, from Corollary 2.2, all the points (a,a) with $a < a^o$ are continuation points. Next, the recursion relation (2.16) reduces to

$$(2.60)\quad \rho(a,b) = \min \left[\rho_o(a,b),\ \frac{a}{a+b}\,\rho(a+1,b) + \frac{b}{a+b}\,\rho(a,b+1) + C(a,b)\right]$$

where

$$(2.61)\quad \rho_o(a,b) = A\,E\left[\,|p - \tfrac{1}{2}|\,\middle|\,a,b\right] - A\,\frac{|a-b|}{2(a+b)}$$

and

$$(2.62)\quad C(a,b) = \begin{cases} 1, \text{ all } (a,b) & \text{, for constant cost;} \\ E\left[\,|p - \tfrac{1}{2}|\,\middle|\,a,b\right] & \text{, for absolute deviation cost.} \end{cases}$$

For stopping points $\rho = \rho_o$, a known function, and for continuation points $\rho < \rho_o$, as follows from Lemma 2.2. Thus a point may be classified as a stopping point or not according as $\rho = \rho_o$ or $\rho < \rho_o$ at this point.

Consider the points on the line $a = a^o - 1$, starting from (a^o-1, a^o-1) which is a continuation point. $\rho(a^o-1,a^o-1)$ may be obtained with the help of (2.60) since (a^o, a^o-1) and (a^o-1,a^o) are stopping points as noted earlier and hence have known ρ-values. Using this $\rho(a^o-1,a^o-1)$, and the known ρ-values of the stopping points on the line $a = a^o$, the ρ-values of the points on the line $a = a^o - 1$ may be calculated with the help of (2.60) successively with increasing a co-ordinates. Thus, these points may be examined for a stopping point or a continuation point one after another. We need, however, only continue on this line till a stopping point is reached for the first time. This is due to the fact that all further points on this line are stopping points as can easily be shown using lemma 2.5.

The procedure described in the previous paragraph determines the boundary point (i.e., the first stopping point encountered) on the line $a = a^o - 1$. We then consider finding this point on the next line $a = a^o - 2$, in the same way, by examining the points on it for a stopping point till we reach one. This time we start with the point $(a^o - 2, a^o - 2)$ whose ρ-values can be obtained from (2.60) by those of (a^o-1, a^o-2) and (a^o-2, a^o-1) already obtained. This process is then repeated to find the boundary points on the lines $a = a^o - k$, successively for $k = 3,4$, etc.. The boundary points on the part $a \leq b$ are obtained from those on $a \geq b$ by symmetry.

General case: $0 < p_o < 1$

As before for each point (a,b) we associate the Bayes risk $\rho(a,b)$ which satisfies the same recursion relation as (2.60) where now (2.61) is replaced by

$$(2.63) \qquad \rho_o(a,b) = A\, E\,[\,|p-p_o|\,|a,b\,] - A\,\left[\,\left|\frac{a}{a+b} - p_o\right|\,\right]$$

and (2.62) by

$$(2.64) \qquad C(a,b) = \begin{cases} 1\ , & \text{all } a,b, \quad \text{for constant cost;} \\ E[\,|p-p_o|\,|a,b\,]\ , & \text{for absolute deviation cost.} \end{cases}$$

In the general case, the optimum boundary may not be symmetric with respect to Γ_o: $a = b\,p_o/(1-p_o)$. We have to consider the boundary both above and below this line. Let (a^o,b^o) be the point of truncation on Γ, obtained as indicated in the previous section. Three different cases arise according as (a^o, b^o) belongs to Γ_o, Γ_1 or Γ_2.

(1) : $(a^o, b^o) \in \Gamma_o$

By Lemma 2.6 the points on the lines $a = a^o$ and $b = b^o$ are stopping points, hence their ρ-values are known. The ρ-value of the next point on Γ, i.e., (a^o-1, b^o-1) is obtained from those of (a^o, b^o-1) and (a^o-1, b^o). Using this $\rho(a^o-1,b^o-1)$ and the ρ-values of the points on the line $a = a^o$, in the equation (2.60), those of the points on the line $a = a^o-1$ may be obtained successively from the point (a^o-1, b^o-1) till we reach the first stopping point on this line. Similarly using $\rho(a^o-1, b^o-1)$ and the ρ-values of the points on the line $b = b^o$, those of the points on the line $b = b^o - 1$ may be obtained successively. Consider the next point on Γ which may be either on the line $a = a^o - 2$ or on the line $b = b^o - 2$. In either case, it should be sufficiently apparent by now how to proceed.

(2): $(a^o, b^o) \in \Gamma_1$.

By Lemma 2.6, the points on the lines $a = a^o$ and $b = b^o + 1$ are stopping points in this case; hence their ρ-values are known. Proceeding as indicated above, the boundary point on the line $b = b^o$ is obtained first. Next the boundary points on the lines $a = a^o - 1$, $b = b^o - 1$ and so on.

(3): $(a^o, b^o) \in \Gamma_2$.

By Lemma 2.6, again, the points on the line $a = a^o + 1$, and $b = b^o$ are stopping points. The boundary point on the line $a = a^o$ is obtained first. Next the process is repeated with lines $a = a^o - k$, $b = b^o - k$, successively with increasing k.

2.6 <u>Some points useful for computation of the optimal boundary</u>.

Since the optimum (Bayes) boundary does not depend on the absolute magnitudes of the Bayes risk ρ, but rather on the relative magnitude of the two quantities on the right hand side of (2.60), we may, for convenience of computation, consider the following translated quantity

$$(2.65) \qquad V(a,b) = A\,E\,[\,|p-p_o|\,|a,b] - \rho(a,b)$$

which may be called the <u>gain function</u> associated with (a,b). Using the fact that

$$(2.66) \quad E[\,|p-p_o|\,|a,b] = \frac{a}{a+b} E[\,|p-p_o|\,|a+1,b] + \frac{b}{a+b} E[\,|p-p_o|\,|a,b+1]$$

the equation (2.60) may be written in terms of (2.65) as

$$(2.67) \quad V(a,b) = \max\left[V_o(a,b),\ \frac{a}{a+b} V(a+1,b) + \frac{b}{a+b} V(a,b+1) - C(a,b)\right]$$

where

$$(2.68) \qquad V_o(a,b) = A \left| \frac{a}{a+b} - p_o \right|$$

Following Moriguti and Robbins [30], for actual computation, we may still consider another quantity defined as

$$(2.69) \qquad M(a,b) = V(a,b) - V_o(a,b)$$

which may be called the <u>net gain function</u> associated with (a,b). It can be shown from (2.67) and (2.68) that the optimum boundary may be characterized in terms of M(a,b) as follows,

$$(2.70) \quad M(a,b) = \max\left[0,\ \frac{a}{a+b} M(a+1,b) + \frac{b}{a+b} M(a,b+1) + \lambda(a,b) - C(a,b)\right]$$

where

$$\lambda(a,b) = \begin{cases} 2A \cdot ab/[(a+b)^2(a+b+1)], & \text{if } (a,b) \in \Gamma_o, \\ 2A \cdot b/(a+b) \cdot [p_o - a/(a+b+1)], & \text{if } (a,b) \in \Gamma_1, \\ 2A \cdot a/(a+b) \cdot [(a+1)/(a+b+1) - p_o], & \text{if } (a,b) \in \Gamma_2, \\ 0, & \text{otherwise}. \end{cases}$$

We give below a closed form expression for C(a,b) for the absolute deviation cost in terms of tabulated functions:

(2.71)
$$C(a,b) \equiv E[\,|p-p_o|\,|a,b] = \int_0^1 |p-p_o| \, d\,\xi_{a,b}(p)$$
$$= p_o\,[2\,I_{p_o}(a,b)-1] - \frac{a}{a+b}\,[2\,I_{p_o}(a+1,b) - 1]$$

where $I_x(p,q)$ is the incomplete beta integral

(2.72)
$$\frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt\,, \qquad p,q > 0$$

tabulated by K. Pearson [32].

For the special case $p_o = 1/2$, we give below in the form of a lemma another expression for C(a,b) that might be more convenient for computation:

Lemma 2.8

(2.73)
$$C(a,b) \equiv E[\,|p-\tfrac{1}{2}|\,|a,b] = \frac{\max(a,b)}{a+b}\; g(a,b)$$

where

(2.74)
$$g(a,b) = bi(a,b;\tfrac{1}{2}) \sum_{j=o}^{\left[\frac{|a-b|}{2}\right]} \frac{\binom{|a-b|}{2j}}{\binom{\min(a,b)+j}{j}}$$

and

(2.75)
$$bi(a,b;\tfrac{1}{2}) = \binom{a+b}{a}\frac{1}{2^{a+b}}$$

and $[x]$ denotes the largest integer $\leq x$.

2.7 Trinomial model.

In this section we write down without much elaboration the analogous results for the trinomial problem stated below. These results can be obtained similarly as in the binomial problem with $p_o = 1/2$.

(2.76) $f_{\pi_1,\pi_2}(i) = \pi_i$, $i = 0,1,2$; $0 \leq \pi_i \leq 1$, $\pi_o + \pi_1 + \pi_2 = 1.$

a_1 is preferable if $\pi_1 > \pi_2$,

a_2 if preferable if $\pi_1 < \pi_2$.

(2.77)
$$\begin{cases} L(\pi_1,\pi_2;\, a_1) = \begin{cases} 2A(\pi_2 - \pi_1), & \text{if } \pi_1 \leq \pi_2 , \\ 0 & \text{, if } \pi_1 > \pi_2 . \end{cases} \\ L(\pi_1,\pi_2;\, a_2) = \begin{cases} 0 & \text{, if } \pi_1 \leq \pi_2 , \\ 2A(\pi_1 - \pi_2) & \text{, if } \pi_1 > \pi_2 . \end{cases} \end{cases}$$

It follows from the above specializations that

(2.78)
$$\begin{cases} \Gamma_1 = \left\{ \xi \mid E_\xi \pi_1 \geq E_\xi \pi_2 \right\} , \\ \Gamma_2 = \left\{ \xi \mid E_\xi \pi_1 \leq E_\xi \pi_2 \right\} , \\ \Gamma_o = \left\{ \xi \mid E_\xi \pi_1 = E_\xi \pi_2 \right\} \end{cases}$$

and

(2.79)
$$\rho_o(\xi) = A\, E_\xi |\pi_1 - \pi_2| - A \mid E_\xi(\pi_1 - \pi_2) \mid .$$

Also, (2.1) reduces to

(2.80)
$$\phi_{\xi,j} = \begin{cases} 0 & \text{, if } E_{\xi_j} \pi_1 > E_{\xi_j} \pi_2 , \\ u & \text{, if } \quad " = \quad " , \\ 1 & \text{, if } \quad " < \quad " ; \end{cases}$$

$j = 0,1,2,\ldots,n$; $n = 0,1,2,\ldots$.

and (2.2) reduces to

$$(2.81)\quad \rho(\xi) = \min \left[\rho_o(\xi),\ \sum_{o}^{2} f_\xi(i)\, \rho(\xi_1^{(i)}) + C(\xi) \right]$$

where

$$(2.82)\qquad f_\xi(i) = E_\xi\ \pi_i\ ,\quad i = 0,1,2$$

and $\rho_o(\xi)$ is given by (2.79).

Corresponding to Theorem 2.1 we have in this case the following theorem which can be proved analogously.

<u>Theorem 2.3.</u> <u>A sufficient condition for a Bayes sequential procedure δ_ξ for the trinomial problem to be truncated with probability one is:</u>

$$(2.83)\quad (a)\ \ \xi \left\{ C(\pi_1,\pi_2) > 0 \right\} = 1$$

and

$$(2.84)\quad (b)\ \lim_{n \to \infty} \mathrm{Var}\,(\pi_1 - \pi_2 \mid \xi_n)/\ C(\xi_n) = 0 \ \text{ a.e.,}$$

uniformly in n .

Moreover, δ_ξ takes at most n_o observations, where n_o is the smallest positive integer such that for every $n \geq n_o - 1$,

$$(2.85)\quad \mathrm{Var}\,(\pi_1 - \pi_2 \mid \xi_n)/C(\xi_n) \leq 1/A\ ,\ \text{a.e.}$$

It is possible to show as in the remark (2) after Theorem 2.1 that it is only necessary to show that condition (b) holds for those $\xi_n \in \Gamma_o$.

2.8 Dirichlet a priori distribution.

From now onwards we consider an a priori distribution belonging to the bivariate Dirichlet distribution family (1.5) some of whose properties have already been considered in subsection 1.5.1. From the property of closedness of this family under sampling, all the considerations of the binomial problem hold equally well. The only change in this case is that now we have to consider all the lattice points of the positive octant in the three-dimensional space. The various sets Λ_1, Λ_2, $\bar{\Lambda}$, Γ_1, Γ_2, Γ_0 are considered as before, with respect to the fixed frame of reference with (0,0,0) as the origin in the 3-space. From (1.5 a,b) and (2.) we may write

$$(2.86)\quad \Gamma_1 = \{a,b,c \mid a \geq b\}, \quad \Gamma_2 = \{a,b,c \mid a \leq b\},$$

$$\Gamma_0 = \{a,b,c \mid a = b\} \text{ i.e., } = \{a,a,c\} : \text{ neutral boundary.}$$

The optimum boundary may be defined in this case as the set of stopping points (a,b,c) such that (a-1,b,c), (a,b-1,c) or (a,b,c-1) is a continuation point.

As before, by $d\xi_n(\pi_1,\pi_2)$ we shall mean $d\xi(a,b,c)$ where $a + b + c = n$.

Analogous to corollary 2.1, we have the following corollary:

Corollary 2.3. A Bayes sequential procedure for the trinomial problem with a Dirichlet prior distribution is truncated with probability one.

Outline of proof: Condition (a) of Theorem 2.3 may be shown to be satisfied for the special cases of cost and a priori distributions we are concerned with here.

Now, for $\xi \in \Gamma_0$, it follows from (1.6), putting $a = b$

(2.87) $$\mathrm{Var}\,(\pi_1 - \pi_2 | \xi(a,a,c)) = 2a/(2a+c)(2a+c+1) .$$

Also, it can be easily verified that

(2.88) $$C(a,a,c) = E[C(\pi_1,\pi_2)|\xi(a,a,c)]$$

(2.89) $$= \begin{cases} 1 , & \text{for constant cost: case (i)} \\ E[|\pi_1 - \pi_2|\,|a,a,c] , & \text{for absolute deviation cost: case (ii)} \end{cases}$$

(2.90) $$= 4a/[(2a+c)bi(a,a)]$$

where $bi(a,a)$ is the central term in the binomial distribution with parameters $(2a, 1/2)$. From these results, (2. 4) follows immediately for constant cost. For the other cost, we have

(2.91) $$\mathrm{Var}(\pi_1 - \pi_2|a,a,c)/C(a,a,c) = [2(2a+c+1)bi(a,a)]^{-1}$$

$$\sim (\pi a)^{1/2}/2(2a+c+1) \longrightarrow 0$$

$$\text{as } 2a+c \longrightarrow \infty .$$

2.9 Determination of the points of truncation in the trinomial problem.

The results of the previous section ensure that a Bayes procedure for the trinomial problem is truncated. This means that for a sufficiently large value of n, all the lattice points of the plane, say L_n: $a+b+c=n$ are stopping points. In other words the continuation set,

$\bar{\Lambda}$, lies below this plane. Just like Lemma 2.6, the lemma below narrows down $\bar{\Lambda}$ to some subset of $a+b+c < n$.

Let the two planes L_n and Γ_0 intersect each other (which they do perpendicularly) along the line, say, ℓ_n: $a = (n-c)/2$. Consider a plane P_i^n in Γ_i passing through ℓ_n at 45° to the plane Γ_0 , $i = 1,2$. Obviously, P_1^n and P_2^n meet each other perpendicularly at ℓ_n .

<u>Lemma 2.9.</u> <u>If a Bayes procedure for the trinomial problem is truncated at n, then the continuation set $\bar{\Lambda}$ is contained within the region R^n bounded above by the two planes P_1^n and P_2^n .</u>

<u>Proof</u>: The lemma asserts that for each c^o-plane, where c^o is such that $a^o = (n-c^o)/2$ is an integer (positive), the continuation set, $\bar{\Lambda}$ (c_o), in that plane is contained within the square bounded above by $a = a^o$ and bounded on the right by $b = a^o$. Consider the point (a^o, a^o-1, c^o) in Γ_1. It can be represented as a convex linear combination of the points (a^o, a^o, c^o), (a^o+1, a^o-1, c^o) and (a^o, a^o-1, c^o+1) all in Λ_1 (being in L_n) and hence by Lemma 2.5 (a^o, a^o-1, c^o) is also in Λ_1. It can similarly be shown that (a^o-1,a^o,c^o) belongs to Λ_2. (a^o, a^o-1,c^o) and (a^o-1,a^o,c^o) are two points of the plane L_{n-1} on the c_o-plane. It can be shown successively that all the points of the plane L_{n-1} except on Γ_0 , if any, are stopping points. Utilizing this fact again, all the points on the plane L_{n-2} and on or outside R^n may be shown to be stopping points. Continuing in this way successively we arrive at the above lemma.

Formal definition of a point of truncation:

For a fixed c, we may formally define the point of truncation in the c-plane just as in section 2.4, as that point (a^o, a^o, c) of $\overline{(\ )}$, such that a^o is the smallest integer such that no point (a,a,c) in $\overline{(\ )}_o$ on the c-plane with $a \geq a^o$ is a continuation point. Because of the symmetry of the problem, it will turn out that just as in the symmetric binomial problem (i.e., with $p_o = 1/2$) the point of truncation on the c-plane might alternatively be defined as the "first" stopping point (a^o, a^o, c) in $\overline{(\ )}_o$ on this c-plane, "first" here meaning with smallest a^o. It may be noted here that unlike the symmetric binomial case, the points of truncation defined as such do belong to the optimum boundary as defined in the previous section. In fact, it may easily be seen from the above lemma that they cannot be reached from any point in the same c-plane, but may be reached from the next lower c-plane.

We now state how to determine these points of truncation. Just as in the binomial problem, the knowledge of their exact location enables us to obtain the optimum boundary, as will be described in section 2.10, by working backwards from these points with the help of the recursion formula (2.81).

To find the points of truncation for a Bayes procedure known to be truncated at n, say, we need, because of Lemma 2.9, only the set of points $L_{n-1} \cap \overline{(\ )}_o$ to determine whether they are stopping points or not. A point belonging to $L_{n-1} \cap \overline{(\ )}_o$ is of the form (a,a,c) where $2a + c = n-1$. Notice that these points are on c-planes with odd (even) c if n is even (odd). A further observation at this point leads to either $(a+1,a,c)$, $(a,a+1,c)$ or $(a,a,c+1)$, each of which is a stopping point. The next theorem gives an easily applicable criterion to determine whether or not such a point (a,a,c) is a stopping point.

Theorem 2.4. *If* (a,a,c) *is a point on the neutral boundary* Γ_0 *such that* $(a+1,a,c)$, $(a,a+1,c)$ *and* $(a,a,c+1)$ *are stopping points, then* (a,a,c) *is a stopping point or a continuation point, according as*

$$(2.92)\quad \lambda(a,a,c) \leq C(a,a,c) \quad \text{or} \quad \lambda(a,a,c) > C(a,a,c)$$

where

$$(2.93)\quad \lambda(a,a,c) = A\,\mathrm{Var}(\pi_1 - \pi_2|a,a,c)\ ,$$

and

$$(2.94)\quad C(a,a,c) = E(C(\pi_1,\pi_2)|a,a,c)\ .$$

The proof of this theorem is analogous to that of Theorem 2.2 and hence omitted.

If all the points of $L_{n-1} \cap \Gamma_0$ are thus found to be stopping points, we then restrict our attention to the next smaller region R^{n-1}. By Lemma 2.9, again, we need examine only the points of $L_{n-2} \cap \Gamma_0$ for stopping or continuation points, and this is done with the help of the above theorem as before. We proceed in the same way till we reach a stage when not all the points of, say, $L_{n-k} \cap \Gamma_0$ are stopping points. It turns out that some will be stopping and some will be continuation points depending on the value of c. Suppose for a fixed value c^o of c, the point $((n-k-c^o)/2, (n-k-c^o)/2, c^o)$ is found to be a continuation point. Now, using the monotonically decreasing property of $\lambda(a,a,c^o)/C(o,a,c^o)$ with respect to a for the two types of cost C that we are considering, it is possible to show just as in Corollary 2.2, that all the points (a,a,c^o) with $a < (n-k-c^o)/2$ are continuation points. Hence, by definition, the point

$$(2.95) \qquad (\tfrac{1}{2}(n-k-c^o) + 1, \ \tfrac{1}{2}(n-k-c^o) + 1, \ c^o)$$

is the point of truncation on the c^o-plane. If, for other values of c, all the points of $L_{n-k} \cap \Omega_o$ are still stopping points, we move on the next set of points $L_{n-k-1} \cap \Omega_o$ and so on. It is now evident how the points of truncation may be obtained for any value of c.

Using (2.92) and (2.87) we are thus led to conclude for the case of constant cost that the set of boundary points for the Bayes procedure consists of the lattice points of a curve, having the following parametric representation:

$$(2.96) \qquad (a^o(c), \ a^o(c), \ c)$$

where $a^o(c)$ is the smallest positive integer such that

$$(2.97) \qquad \frac{2a}{(2a+c)\ (2a+c+1)} \leq \frac{1}{A} ,$$

for a fixed c. Obviously a^o depends on c and hence the particular notation $a^o(c)$. In fact, it is easily seen from (2.97) that $a^o(c)$ is decreasing in c. This means that the point of truncation moves to the side of the origin as c increases.

The set of truncation points for the case of absolute deviation cost is also given by the same type of curve as (2.96) where (using (2.87) and (2.90)) $a^o(c)$ is now defined to be the smallest positive integer for which

$$(2.98) \qquad \frac{1}{2\,(2a + c + 1)} \cdot \frac{1}{bi(a,a)} \leq \frac{1}{A}$$

for a fixed c. For large A, $a^o(c)$ is given approximately by

$$(2.99) \qquad \frac{\sqrt{\pi}\;a^o}{2(2a^o + c + 1)} \backsim \frac{1}{A}\,.$$

As a numerical illustration, we give below for $A = 10$, some of the points of truncation (2.96).

<u>constant cost:</u>

c	1	2
$a^o(c)$	4	1

absolute deviation cost:

c	1	2	3	4	5	6
$a^o(c)$	20	17	16	15	14	13

Finally, we note that if the point (2.95) is a truncation point, then the point $((n-k-c^o)/2,\ (n-k-c^o)/2,\ c^o + 1)$ belongs to L_{n-k+1} and hence, is a stopping point from the very definition of L_{n-k}. We may formally put this observation in the form of the following lemma which may otherwise be proved directly from the definitions of $a^o(c)$ from (2.97) and (2.98).

<u>Lemma 2.10.</u> <u>If</u> $(a^o(c), a^o(c), c)$ <u>represents a truncation point, then</u> $(a^o(c)-1, a^o(c)-1, c+1)$ <u>belongs to the stopping set</u>.

We shall use this lemma in the next section.

2.10. Procedure for obtaining the optimum boundary in the trinomial problem.

We describe in this section how to obtain the optimum boundary systematically, once the points of truncation are known, using some recursion relations as well as some simple facts about the optimum stopping and continuation sets as embodied in Lemmas 2.5, 2.9, and 2.10. Because of the symmetry of the problem, the optimum boundary is symmetric with respect to the plane $a=b$. Without any loss of generality, we may thus consider only $a \geq b$.

Suppose the points of truncation $(a^o(c), a^o(c), c)$ are obtained as indicated in the previous section. Consider any particular c-plane, in which, omitting the fixed co-ordinate for the moment, $(a^o(c), a^o(c))$ is the point of truncation. By Lemma 2.9, all the points on the line $a = a^o$ are stopping points. Moreover, as noted earlier in the previous section all the points on the diagonal $a=b$ with $a < a^o$ are continuation points. Finally, the recursion relation (2. 1) reduces to

$$(2.100) \quad \rho(a,b,c) = \min \left[\rho_o(a,b,c), \frac{a}{a+b+c} \rho(a+1,b,c) + \frac{b}{a+b+c} \rho(a,b+1,c) + \frac{c}{a+b+c} \rho(a,b,c+1) + C(a,b,c) \right]$$

where

$$(2.101) \quad \rho_o(a,b,c) = A\, E\left[|\pi_1 - \pi_2| \,\big|\, a,b,c \right] - A \frac{|a-b|}{a+b+c} ,$$

and

$$(2.102) \qquad C(a,b,c) = \begin{cases} 1 & \text{, all } (a,b,c), \text{ for constant cost} \\ E[\,|\pi_1 - \pi_2|\,|a,b,\underline{c}] & \text{, for absolute deviation cost.} \end{cases}$$

For stopping points $\rho = \rho_o$, a known function and for continuation points $\rho < \rho_o$, as follows from Lemma 2.2. Thus a point may be examined as a stopping point or not, according as $\rho = \rho_o$ or $\rho < \rho_o$ at this point.

Consider the points on the line $a = a^o - 1$, starting from (a^o-1, a^o-1). (a^o, a^o-1, c), (a^o-1, a^o, c) are stopping points, as noted earlier, by Lemma 2.9. So is $(a^o-1, a^o-1, c+1)$ by Lemma 2.10. Hence, $\rho(a^o-1, a^o-1, c)$ may be obtained from (2.100). Also, since $(a^o-1, a^o-1, c+1)$ is a stopping point, by Lemma 2.9 again, all the points on the line $a = a^o-1$ on the $(c+1)$-plane are stopping points. Using this fact, the ρ-values associated with the other points on the line $a = a^o-1$ on the c-plane may be calculated with the help of (2.100) successively from (a^o-1, a^o-1, c) onwards and hence may be determined to be stopping points or not. One needs to proceed till one encounters a stopping point on this line $a = a^o - 1$ for the first time. All further points on this line can easily be shown to be stopping points with the help of Lemma 2.5.

The procedure described in the above paragraph determines the boundary point (the first stopping point encountered) on the line $a = a^o(c) - 1$ on the c-plane. This is now carried out for all possible values of c, before one considers finding out the boundary point on the next line $a = a^o(c) - 2$. For a fixed c, we start from the point (a^o-2, a^o-2, c) and proceed to the left on the same c-plane as before. It is to be noticed that the ρ-values of the points $(a^{o}-2,\ a^{o}-2-k,\ c+1)$, $k \geq 0$, that are needed in this connection, must have already been obtained in the procedure described so far (if they are not already stopping points, of course). This is now carried out for all possible values of c. In this way the boundary point on the lines $a = a^{o}(c) - k$ are obtained for $k = 3, 4$, etc., successively.

The boundary points on the part $a \leq b$ are obtained from those on $a \geq b$, by symmetry. The description of the procedure is thus complete.

2.11 Some points useful for computation of the optimum boundary.

For convenience of computation we may replace (2.100), as in the binomial problem, by an analogous relation in terms of the so-called gain function $V(a,b,c)$ defined as

$$(2.103)\qquad V(a,b,c) = A\, E[\,|\pi_1-\pi_2|\,|a,b,c] - \rho(a,b,c) \quad .$$

Using the relation

$$(2.104)\quad E[\,|\pi_1-\pi_2|\,|a,b,c] = \frac{a}{a+b+c}\, E[\,|\pi_1-\pi_2|\,|a+1,b,c] + \frac{b}{a+b+c}\, E[\,|\pi_1-\pi_2|\,|a,b+1,c] + \frac{c}{a+b+c}\, E[\,|\pi_1-\pi_2|\,|a,b,c+1]$$

(2.100) becomes in terms of V:

$$(2.105) \qquad V(a,b,c) = \max\Big[V_o(a,b,c), \frac{a}{a+b+c} V(a+1,b,c) + \frac{b}{a+b+c} V(a,b+1,c) + \frac{c}{a+b+c} V(a,b,c+1) - C(a,b,c)\Big]$$

where

$$(2.106) \qquad V_o(a,b,c) = A \frac{|a-b|}{a+b+c} \quad .$$

Alternatively, (2.105) may be put in terms of the so-called net gain function, $M(a,b,c)$ defined as

$$(2.107) \qquad M(a,b,c) = V(a,b,c) - V_o(a,b,c) \; ,$$

as follows:

$$(2.108) \quad M(a,b,c) = \max\Big[0, \frac{a}{a+b+c} M(a+1,b,c) + \frac{b}{a+b+c} M(a,b+1,c) + \frac{c}{a+b+c} M(a,b,c+1) - C(a,b,c)\Big]$$

for $a \neq b$. For $a = b$,

$$(2.109) \quad M(a,a,c) = V(a,a,c) = \max \Big[0, \lambda(a,a,c) - C(a,a,c) + \frac{2a}{2a+c} M(a+1,a,c)\Big]$$

where $\lambda(a,a,c)$ is defined in (2.93).

We give an expression for $C(a,b,c)$ for the absolute deviation cost that might be useful for computation:

Lemma 2.11

$$(2.110) \qquad C(a,b,c) \quad E\left[\,|\pi_1-\pi_2|\,\middle|\,a,b,c\right] = \frac{4 \max(a,b)}{a+b+c} \cdot g(a,b)$$

where $g(a,b)$ is defined in (2.74).

CHAPTER III

BAYES PROCEDURES WITH MODIFIED LOSS STRUCTURE

3.0 Introduction and summary

The modified loss structures as defined in (1.3) and (1.10) are only some special cases of the general situation when loss incurred at any stage depends on the observed outcome. One important feature of these special cases is, however, that any optimum sequential procedure for a decision problem involving this type of loss must be truncated at A. This follows intuitively from the fact that any terminal decision after A observations must be correct in the sense that no loss is involved and the situation cannot be improved upon by taking further observations whereas any further observation is going to cost us more for any type of positive cost function. This argument, may, however, be put in formal mathematical terminologies by developing an analogous version of lemma 2.4 for the modified loss structure.

It is well-known [6, 9] how to obtain the Bayes sequential procedure within the class of procedures truncated at a fixed stage, by the working backwards technique, for loss and cost depending on the observed outcome in general. The existence of the Bayes procedures for the various decision problems involving the modified loss structure are thus assured. The problem is then to give a simpler characterization of the Bayes procedures that might be applicable readily in actual computation. Section 3.1 contains such a characterization for the general model that is analogous to (although not exactly the same as) that given in [9] for the traditional loss, constant cost, and independent and identically distributed sequence of observations.

A further simplification in the characterization results as shown in section 3.2 when one considers the appropriate absolute deviation type of cost corresponding to a modified loss. As noted already in section 1.4, it follows from physical considerations that the modified loss structure fits more naturally with the absolute deviation cost.

The results of the first two sections are then applied in section 3.3 and 3.4 to the binomial and the trinomial model respectively. It is in section 3.3 that a rather undesirable feature of the Bayes procedures with the modified loss structure compared to those with linear loss structure is illustrated in details with the binomial model.

3.1 Characterization of the Bayes procedure with a general model.

Extending the results in [24] in the desired direction we arrive at the intended characterization stated as a theorem below. Apart from the assumptions of independent observations and constant (but may depend on the unknown parameter) cost per observation, we introduce the assumptions regarding the particular structure of the loss in the appropriate places. We shall retain the notation of section 2.1 as far as possible changing only the relevant ones.

Notations: Let

$$L_j(\Theta,a_i) \equiv L(\Theta,\ a_i,\ \underline{x}_j),\ j = 1,2,\ldots,n;\ i = 1,2;\ \Theta \in \textcircled{H},$$

be the loss incurred due to action a_i after observing $x_1,\ldots,x_j$, when Θ is the true parameter. For consistency of notation we denote by

$$L_o(\Theta,a_i)\ , \qquad i = 1,2;\ \ \Theta \in \textcircled{H}$$

the loss incurred when a_i is taken without taking any observation and when Θ is true. We assume the usual [9] measurability and integrability conditions for $L_j(\Theta,\ a_i)$. Let

$$L_j(\xi,a_i) = \int_{\textcircled{H}} L_j(\Theta,\ a_i)\ d\ \xi(\Theta),\ j = 0,1,2,\ldots,n;\ i = 1,2.$$

(3.0) $$\lambda_j(\xi) = \min_{i=1,2} L_j(\xi,a_i),\ \ j = 0,1,2,\ldots,n.$$

Let $\delta \equiv (\psi,\ \emptyset)$ be an arbitrary (but measurable) sequential decision procedure truncated at n, i.e.,

$$\psi_o + \psi_1 + \ldots + \psi_n \equiv 1 \quad \text{a.e. for all } \Theta \in \textcircled{H}\ .$$

Let $\delta_\xi \equiv (\psi_\xi,\ \emptyset_\xi)$ denote the Bayes procedure for ξ . Also let

$$\pi_{\xi,j} = \frac{\psi_{\xi,j}}{\psi_{\xi,j} + \psi_{\xi,j+1} \ldots + \psi_{\xi,n}}\ ,\ \ j = 0,1,2,\ldots,\ n$$

be the conditional probability of stopping with j observations given that j or more observations are taken. $\psi_{\xi,j}$'s may be obtained uniquely from $\pi_{\xi,j}$'s and vice versa.

With these added notations to those of section 2.1, we may write the following results.

The risk $r(\Theta, \delta)$ associated with δ when Θ is true is given by

$$r(\Theta,\delta) = \text{Expected loss} + \text{Expected cost}$$

$$(3.1) \qquad = \sum_{j=0}^{n} \int_{\mathfrak{X}^j} \psi_j \left[\left\{ L_j(\Theta,a_2)\phi_j + L_j(\Theta,a_1)(1-\phi_j) \right\} + j\, C(\Theta) \right] p_{\Theta,j}\, d\mu^j \ .$$

The average risk associated with δ is given by

$$(3.2) \qquad r(\xi,\delta) = \int_{\Theta} r(\Theta,\delta)\, d\xi(\Theta) \ .$$

Substituting (3.1) in (3.2) and changing the order of integration

$$(3.3) \quad r(\xi,\delta) = \sum_{j=0}^{n} \int_{\mathfrak{X}^j} \psi_j \left[\left\{ L_j(\Theta,a_1)(1-\phi_j) + L_j(\Theta, a_2)\phi_j + j\, C(\Theta) \right] d\xi_j(\Theta) \cdot f_{\xi,j}\, d\mu^j$$

$$(3.4) \qquad = \sum_{j=0}^{n} \int_{\mathfrak{X}^j} \psi_j \left[\left\{ L_j(\xi_j,a_2)\phi_j + L_j(\xi_j,a_1)(1-\phi_j) \right\} + j\, C(\xi_j) \right] f_{\xi,j}\, d\mu^j$$

$$(3.5) \qquad \geq \sum_{j=0}^{n} \int_{\mathfrak{X}^j} \psi_j \left[\min\ L_j(\xi_j,a_2), L_j(\xi_j,a_1) + j\, C(\xi_j) \right] f_{\xi,j}\, d\mu^j$$

$$(3.6) \qquad = \sum_{j=0}^{n} \int_{\mathfrak{X}^j} \psi_j \left[\lambda_j(\xi_j) + j\, C(\xi_j) \right] f_{\xi,j}\, d\mu^j \ .$$

The equality is obtained (3.5) when $\phi = \phi_\xi$ defined as

$$(3.\psi) \quad \phi_{\xi,j} = \begin{cases} 0 & , \text{ if } \lambda_j(\xi_j) = L_j(\xi_j,a_1) < L_j(\xi_j,a_2) , \\ u & , \text{ if } \quad " \quad = \quad " \quad = \quad " , \\ 1 & , \text{ if } \quad " \quad = L_j(\xi_j,a_2) < L_j(\xi_j,a_1) , \end{cases}$$

$j = 0,1,2,\ldots,n$.

where u is any arbitrary number, $0 \leq u \leq 1$.

Assumption (L_1): The loss depends on the observations in a rather special way in the sense that it does not depend on the actual magnitude of the observations. Specifically,

$$(3.8) \quad L_j(\Theta, a_i) = (n-j)\, \overline{L}(\Theta,a_i),\ j = 0,1,\ldots,n;\ i=1,2; \Theta \in \Theta .$$

<u>Further notations:</u>

$$(3.9) \quad \overline{L}(\xi,a_i) \equiv \int_{\Theta} \overline{L}(\Theta,a_i)\, d\xi(\Theta), \quad i = 1,2.$$

$$\psi^*_{j-1} \equiv \frac{\psi_j}{1-\psi_o} , \quad j = 1,2,\ldots,n;$$

$$\psi^*_o + \ldots + \psi^*_{n-1} = 1$$

$$\phi^*_{j-1} \equiv \phi_j , \ j = 1,2,\ldots,n;$$

$$\delta^* = \left\{ \psi^*_o , \ldots, \psi^*_{n-1}; \phi^*_o , \ldots, \phi^*_{n-1} \right\} .$$

We shall show below that with the restriction (3.8), we may write

$$(3.10) \quad r(\xi,\delta) = \psi_o [L_o(\xi,a_2)\, \phi_o + L_o(\xi,a_1)(1-\phi_o)]$$
$$+ (1 - \psi_o) [\int_{\mathcal{X}} r(\xi_1^{(x_1)}, \delta^*)\, f_\xi(x_1)\, d\mu(x_1) + C(\xi)] .$$

Writing the first term in the summation in (3.3) separately and using (3.8) we have

$$r(\xi,\delta) = \psi_o . n[\ \bar{L}(\xi,a_2)\ \phi_o + \ \bar{L}(\xi,a_1)(1-\phi_o)]$$

(3.11)
$$+ \sum_{j=1}^{n} \int_{\mathcal{X}^j} \psi_j \int_{\Theta} [(n-j)\{\bar{L}(\theta,a_2)\phi_j + \bar{L}(\theta,a_1)(1-\phi_j)\} + j\,c(\theta)]d\xi_j(\theta)\ f_{\xi,j}d\mu^j .$$

$$= \psi_o \cdot n[\ \bar{L}(\xi,a_2)\phi_o + \bar{L}(\xi,a_1)(1-\phi_o)]$$

$$+ (1-\psi_o)[A_1 + c(\xi)]$$

(3.12)
$$+ (1-\psi_o) \sum_{j=2}^{n} \int_{\mathcal{X}^j} \psi^*_{j-1} \int_{\Theta} [(n-1 - \overline{j-1})\{\bar{L}(\theta,a_2)\ \phi^*_{j-1} + L(\theta,a_1)(1-\phi^*_{j-1})\} + (j-1)c(\theta)]d\xi_j(\theta)\ f_{\xi,j}d\mu^j ,$$

where

(3.13)
$$(1-\psi_o)A_1 = \int_{\mathcal{X}} \psi_1 \int_{\Theta} (n-1)\ \{\bar{L}(\theta,a_2)\phi_1 + \bar{L}(\theta,a_1)(1-\phi_1)\}\, d\xi_1^{(x_1)}(\theta)\ f_{\xi,1}\ d\mu^1 .$$

i.e.,

(3.14)
$$A_1 = \int_{\mathcal{X}} \psi^*_o(n-1)\ [\ \bar{L}(\xi_1^{(x_1)},\ a_2)\phi^*_o + \ \bar{L}(\xi_1^{(x_1)},\ a_1)(1-\phi^*_o)]\ f_\xi(x_1)d\mu(x_1)$$

Now we may write

$$(3.15)\quad d\xi_j(\Theta)\, f_{\xi,j}\, d\mu^j = d\,\xi_{1,j-1}^{(x_1)}(\Theta)\, f_{\xi_1^{(x_1)},\, j-1}\, d\mu^{j-1}(x_2,\ldots,x_j)$$

$$f_\xi(x_1)\ d\mu(x_1)\ .$$

Substituting (3.15) in the sum, A_2 say, in (3.12), we have

$$(3.16)\quad A_1+A_2 = \int_{\mathcal{X}} f_\xi(x_1)d\mu(x_1) \Big\{ \psi_o^*(n-1)[\ \bar{L}(\xi_1^{(x_1)},a_2)\phi_o^* + \bar{L}(\xi_1^{(x_1)},a_1)(1-\phi_o^*)]$$

$$+ \sum_{j=1}^{n-1} \int_{\mathcal{X}^j} \psi_j^* \int_{\Theta} [(n-1 - j)\left\{\bar{L}(\Theta,a_2)\phi_j^* + \bar{L}(\Theta,a_1)(1-\phi_j^*)\right\} + j\,c\,(\Theta)]\ d\,\xi_{1,j}^{(x_1)}(\Theta)\ f_{\xi_{1,j}^{(x_1)}}\,d\mu^j \Big\}.$$

Identifying (3.16) properly with (3.11), we may write

$$(3.17)\quad A_1 + A_2 = \int_{\mathcal{X}} f_\xi(x_1) \cdot d\mu(x_1) \cdot r(\xi_1^{(x_1)}, \delta^*)\ ,$$

thus arriving at (3.10). Now remembering the notation

$$\rho_n(\xi) = \inf_{\delta\, \epsilon\, \Delta_n} r(\xi, \delta)$$

we have

$$(3.18)\quad \int_{\mathcal{X}} r(\xi_1^{(x_1)}, \delta^*)\, f_\xi(x_1)d\mu(x_1) \geq \rho_{n-1}(\xi_1^{(x_1)})f_\xi(x_1)d\mu(x_1)\ .$$

Hence it follows from (3.7), and (3.10), that

$$(3.19)\quad \inf_{\substack{\delta \in \Delta_n \\ \psi_o \text{ fixed}}} r(\xi,\delta) = \psi_o \lambda_o(\xi) + (1-\psi_o)\left[\int_{\mathscr{X}} \rho_{n-1}(\xi_1^{(x_1)}) f_\xi(x_1) d\mu(x_1) + c(\xi)\right]$$

Thus

$$(3.20)\quad \rho_n(\xi) = \min\left[\lambda_o(\xi), \int_{\mathscr{X}} \rho_{n-1}(\xi_1^{(x_1)}) f_\xi(x_1) d\mu(x_1) + c(\xi)\right]$$

and hence

$$(3.21)\quad \psi_{\xi,o} \equiv \pi_{\xi,o} = \begin{cases} 0, & \text{if } \rho_n(\xi) < \lambda_o(\xi), \\ 1, & \text{if } \quad '' \quad = \quad '' \quad . \end{cases}$$

It may similarly be shown that

$$(3.22)\quad \rho_{n-j}(\xi) = \min\left[\lambda_j(\xi), \int_{\mathscr{X}} \rho_{n-j-1}(\xi_1^y) f_\xi(y) d\mu(y) + c(\xi)\right], \quad j = 1,2,\ldots,n-1 ;$$

and hence

$$(3.23)\quad \pi_{\xi,j} = \begin{cases} 0, & \text{if } \rho_{n-j}(\xi_j) < \lambda_j(\xi_j), \\ 1, & \text{if } \quad '' \quad = \quad '' \quad . \end{cases} \quad j = 1,2,\ldots,n-1 ;$$

where by (3.0) and (3.8)

$$(3.24)\quad \lambda_j(\xi) = (n-1) \min_{i=1,2} \bar{L}(\xi, a_i),$$

$$(3.25) \qquad \equiv (n-j) \quad \overline{\lambda}(\xi) \quad , \text{ say} .$$

and $\rho_o(\xi)$ is defined to be identically 0 for all ξ. We may thus state the results in the form of the following theorem:

Theorem 3.1.

Under the assumptions of identically and independently distributed sequence of observations and with constant cost per observation, the Bayes terminal decision rule in the class Δ_n is given by (3.7). If, moreover, the loss function satisfies the assumption (L_1), i.e., (3.8), then the Bayes sampling rule in the class Δ_n can be characterized as:

$$(3.26) \quad \pi_{\xi,j} = \begin{cases} 0 \quad , \text{ if } \rho_{n-j}(\xi_j) < \lambda_j(\xi_j) , \\ 1 \quad , \text{ if } \rho_{n-j}(\xi_j) = \lambda_j(\xi_j); \end{cases} \quad j = 0,1,\ldots,n-1 ,$$

where

$$(3.27) \quad \rho_{n-j}(\xi) = \min \left[\lambda_j(\xi), \int_{\mathcal{X}} \rho_{n-j-1}(\xi_1^{(y)}) f_\xi(y) d\mu(y) + C(\xi) \right] \qquad j = 0,1,\ldots,n-1 ;$$

$\lambda_j(\xi)$ being given by (3.25).

We may express the Bayes sampling rule in words as follows: After any stage of sampling, j, say, $j = 0,1,\ldots,n-1$, we stop or take another observation according as $\rho_{n-j}(\xi_j) = (n-j)\overline{\lambda}(\xi)$ or $\rho_{n-j}(\xi_j) < (n-j)\, \overline{\lambda}(\xi)$.

We may alternatively define the Bayes sampling rule in terms of certain sets in $\mathcal{X}^n$. For this, let us denote, following Blackwell and Girshick [9],

$$(3.28)\quad \Xi'_{n-j} = \left\{ \xi:\ \rho_{n-j} = (n-j)\,\bar{\lambda}\,(\xi) \right\}, \qquad j = 0,1,\ldots,n-1 .$$

It may be verified that

$$(3.29)\quad \Xi'_0 \supset \Xi'_1 \supset \cdots \quad \cdots \supset \Xi'_n ,$$

where Ξ'_0 is the set of all ξ's. Now let

$$(3.30)\qquad S^{*\prime}_{jn} = \left\{ \underline{x}_n:\ \xi_r \notin \Xi_{n-r} \ \text{ for } \ r < j\ ,\ \xi_j \in \Xi_{n-j} \right\} .$$

$j = 0,1,\ldots,n.$

Then,

$$(3.31)\qquad S^{*\prime}_{\underline{n}} \equiv (S^{*\prime}_{on}\ ,\ S^{*\prime}_{1n},\ \ldots,\ S^{*\prime}_{nn}\ )$$

forms the required partition of the sample space $\mathscr{X}^n$ such that the Bayes procedure takes j observations if and only if $\underline{x}_n \in S^{*\prime}_{jn}$.

It may also be verified that

$$(3.32)\qquad S^{*\prime}_{jn} \subset S^{*}_{jn}\ ,\quad j = 0,1,\ldots,n$$

where $S^*_n \equiv (S^*_{on}\ ,\ \ldots,\ S^*_{n,n})$ is the partition of the sample space $\mathscr{X}^n$ corresponding to the truncated Bayes procedure for the decision problem with the traditional loss function

$$(3.33)\qquad L(\theta,a_i) = n\,\bar{L}\,(\theta,a_i), \qquad i = 1,2, \qquad \theta \in \Theta$$

This means in other words that if the truncated Bayes procedure with the loss (3.33) tells one to stop then so will also be told by the Bayes procedure with the modified loss (3.8), which is rather obvious from physical considerations.

3.2 Modified loss structure with absolute deviation cost.

We now consider the following special situations:

I. Assumption (L_2):

$$\bar{L}(\theta, a_1) = \begin{cases} 2\theta & , \text{ if } \theta \le 0 , \\ 0 & , \text{ if } \theta > 0 ; \end{cases}$$

(3.34)

$$\bar{L}(\theta, a_2) = \begin{cases} 0 & , \text{ if } \theta \le 0 , \\ 2\theta & , \text{ if } \theta > 0 . \end{cases}$$

Then,

(3.35) $$\bar{\lambda}(\xi) = E_\xi |\theta| - |E_\xi \theta|$$

From (3.35), it follows that the various sets Γ_0, Γ_1, Γ_2 corresponding to (2.7) with this modified loss may be defined analogously as in (2.12) or (2.78) in terms $E_\xi \theta$.

II. Assumption (C):

$$C(\theta) = k|\theta|$$

where k is a positive integer. It may easily be seen from (3.35), (3.25), (3.27) and (3.26) that with these specializations the Bayes procedure is truncated at $n-k$. With $k=1$, i.e., for the absolute deviation type of cost we may thus state the result in the form of the following corollary:

Corollary 3.1. The Bayes procedure for the decision problem with the modified loss structure, viz.,

$$L(\theta, a_1; j) = \begin{cases} 2(n-j)|\theta| & , \text{ if } \theta \le 0, j \le n , \\ 0 & , \text{ if } \theta \le 0, j \ge n, \\ 0 & , \text{ if } \theta > 0, j = 0,1,2,\ldots. \end{cases}$$

(3.36)

$$L(\theta, a_2; j) = \begin{cases} 0 & , \text{ if } \theta \le 0, j = 0,1,2,\ldots. \\ 0 & , \text{ if } \theta > 0, j \ge n , \\ 2(n-j)\theta & , \text{ if } \theta > 0, j \le n , \end{cases}$$

and with the absolute deviation cost, viz.,

$$(3.37) \qquad C(\theta) = |\theta|$$

is truncated at $n-1$.

From now on we shall only consider these two specializations, (3.36) and (3.37) which lead to the simpler characterization of the Bayes sampling rule in terms of the following quantity,

$$(3.38) \qquad V_j(\xi) = \begin{cases} (n-j)\, E_\xi |\theta| - \rho_{n-j}(\xi), & j = 0,1,2\ldots, n, \\ 0 & , j \ge n \quad ; \end{cases}$$

which may be called the gain function. It follows from (3.27), (3.38), (3.25) and (3.35), that

$$(3.39) \qquad V_j(\xi) = \max \Big[(n-j)\, E_\xi |\theta| - (n-j) \left\{ E_\xi |\theta| - |E_\xi \theta| \right\} , \\ (n-j)\, E_\xi |\theta| - \int_{\mathcal{X}} \rho_{n-j-1}(\xi_1^{(y)}) f_\xi(y)\, d\mu(y) \\ - E_\xi |\theta| \Big], \; j = 0,1,\ldots,n-1.$$

Now, using the fact that

$$(3.40) \qquad E_\xi |\theta| = \int_{\mathcal{X}} E_{\xi_1^{(y)}} |\theta| \quad f_\xi(y)\, d\mu(y) \quad ,$$

it foloows from (3.39) that

$$(3.41)\qquad V_j(\xi) = \max\left[(n-j)\ |E_\xi\theta|\ ,\ \int_{\mathcal{X}} V_{j+1}(\xi_1^{(y)}) f_\xi(y)\, d\mu(y)\right],$$

$$j = 0,1,\ldots,n-1.$$

The Bayes sampling rule may thus be stated as follows: after any stage of sampling, say, $j = 0,1,\ldots,n-1$, we stop or take another observation according as

$$(3.42)\qquad V_j(\xi_j) = (n-j)\ |E_{\xi_j}\theta| \quad \text{or} \quad V_j(\xi_j) < (n-j)|E_{\xi_j}\theta| \ .$$

It may be noted that it follows from the definition (3.38) of V_j that

$$(3.43)\qquad V_{n-1}(\xi) = \max\left[|E_\xi\theta|,\ 0\right] = |E_\xi\theta| \quad \text{for all } \xi$$

thus corroborating with the corollary 3.1 that the Bayes procedure is truncated at n-1.

3.3 Binomial model.

To obtain the optimum (Bayes) sequential procedure for the special problems we have been considering, the results of the previous section are specialized in the usual way, replacing the n there by A, and j there by n, θ there by $p-p_o$. Γ_o is defined as in (2.36). Γ_1 and Γ_2 are defined in an obvious way. Equation (3.41) reduces in this case to ,

$$(3.44)\qquad V_n(a,b) = \max\left[V_{n,o}(a,b),\ \frac{a}{a+b} V_{n+1}(a+1,b) + \frac{b}{a+b} V_{n+1}(a,b+1)\right]$$

$$n = 0,1,\ldots,A-1\ ;$$

where

$$(3.45)\quad V_{n,o}(a,b) = (A-n)\left|\frac{a}{a+b} - p_o\right| , \quad n = 0,1,\ldots,A-1 .$$

From (3.43) it follows that

$$(3.46)\quad V_{A-1}(a,b) = V_{A-1,o}(a,b) = \left|\frac{a}{a+b} - p_o\right| \quad \text{for all } (a,b).$$

With the help of (3.44) it is possible, at least in principle, to calculate $V_n(a,b)$ for all possible values of (a,b) and n starting from the known values (3.46) for $n = A-1$. Once these values are known, the optimum sampling rule becomes specified.

We now compare this procedure with that obtained in the previous chapter with linear loss structure. We first note that the subscript n in $V_n(a,b)$ denotes the number of observations already obtained. If we start with an a priori distribution $d\xi(a_o,b_o)$, then the relevant points (a,b) for which $V_n(a,b)$ has to be defined are those for which

$$a+b = (a_o + b_o) + n .$$

The dependence of V on the stage of sampling, n, is thus, essentially, a dependence on the particular a priori distribution $d\xi(a_o, b_o)$ through $a_o + b_o$. The prior distribution being fixed for a particular decision problem, we may, alternatively, write (3.44), omitting the dependence on n, as thus ,

$$(3.4\ )\quad V(a,b) = \max\left[V_o(a,b), \frac{a}{a+b} V(a+1,b) + \frac{b}{a+b} V(a,b+1)\right] ,$$

where

$$(3.48) \quad V_o(a,b) = (A + a_o + b_o - a - b) \quad \left|\frac{a}{a+b} - p_o\right|$$

(which is not to be confused with the earlier $V_o(a,b)$ in (3.44)). The difference of (3.47) and (2.67) lies in the difference between the two different definitions (3.47) and (2.67) of $V_o(a,b)$. It may be noted that the nature of a point (a,b) as to whether it is a stopping point or a continuation point as determined with the help of (2.67) did not depend on the particular a priori distribution one starts with, whereas this is the situation in the present case with (3.47). This is an undesirable feature of the procedures considered in this chapter in comparison to those of Chapter II in the sense that one has to compute the optimal boundary separately for each prior distribution, whereas, as noted earlier in section 2, for the procedures of that section, we need compute the optimal boundary only for the particular a priori distribution $d\xi(1,1)$ and the optimal boundary for any other a priori distribution may be obtained by a simple change in the frame of reference.

It is possible to prove the convexity of the two stopping regions in this case (for the general model) just as in the proof of lemma 2.5 for the traditional loss. This result may analogously be used along with the fact that all the points (a,b) with $a+b = a_o + b_o + A - 1$ are stopping points, as in the traditional case, to arrive at a result corresponding to lemma 2.6, delimiting the region of sampling. We have, however, preferred to show this directly using (3.44). For simplicity we only consider the symmetric case, i.e., with $p_o = 1/2$.

Lemma 3.1

$$(3.49) \quad (a) \qquad V_n(a,a) > V_{n,o}(a,a) = 0 , \quad n < A-1, \text{ all } a ,$$

$$(3.50) \quad (b) \qquad V_n(a,b) = V_{n,o}(a,b) \quad \text{for } |a-b| \geq A-n, \; n \leq A-1 .$$

Proof:

(a) Putting $a = b$ in (3.44) and (3.45) with $p_o = 1/2$,

$$(3.51) \qquad V_n(a,a) = \max[0, \tfrac{1}{2} (V_{n+1}(a+1,a) + \tfrac{1}{2} V_{n+1}(a,a+1)]$$

Now, because of symmetry of the problem,

$$V_n(a,b) = V_n(b,a) \quad \text{all } (a,b), \text{ all } n;$$

and hence in particular,

$$(3.53) \qquad V_{n+1}(a+1,a) = V_{n+1}(a,a+1) .$$

We, thus, have from (3.51)

$$(3.54) \qquad V_n(a,a) = \max [0, V_{n+1}(a+1,a)]$$

Now from the definition (3.44) of V, it follows that

$$V_{n+1}(a+1,a) \geq V_{n+1,0}(a+1,a)$$

$$= \frac{A - n - 1}{2(2a + 1)} \quad \text{by } (3.45)$$

$$> 0 \qquad \text{for } n < A-1.$$

Hence

$$(3.55) \quad V_n(a,a) = V_{n+1}(a+1,a) > 0 = V_{n,o}(a,a) \text{ for all } a, \; n < A-1 .$$

(b) Because of symmetry, it is sufficient to prove (3.51) for $a > b$ only, i.e., to show

$$V_n(a+k,a) = V_{n,o}(a+k,a), \text{ for } k \geq A-n,\ n \leq A-1 . \tag{3.56}$$

We prove this (3.56) by induction on n. Suppose (3.56) is true for $n = m$, we show that (3.56) is also true $n = m-1$.

From (3.44) and (3.45) it follows that

$$V_{m-1}(a+k,a) = \max\left[-\frac{(A-m+1)k}{2(2a+k)},\ \frac{a+k}{2a+k} V_m(a+k+1,a) + \right. \tag{3.57}$$

$$\left. + \frac{a}{2a+k} V_m(a+k,a+1)\right]$$

From induction hypothesis, we get

$$V_m(a+k+1,a) = V_{m,o}(a+k+1,a) = \frac{(A-m)(k+1)}{2(2a+k+1)},\ k \geq A-m-1 , \tag{3.58}$$

and

$$V_m(a+k,a+1) = V_{m,o}(a+k,a+1) = \frac{(A-m)(k-1)}{2(2a+k+1)} ,\ k \geq A-m+1 . \tag{3.59}$$

Hence, (3.57) reduces to

$$V_{m-1}(a+k,a) = \max\left[-\frac{(A-m+1)k}{2(2a+k)} ,\ \frac{(A-m)k}{2(2a+k)}\right],\quad k \geq A-\overline{m-1} \tag{3.60}$$

$$= V_{m-1,o}(a+k,a) ,\qquad k \geq A-\overline{m-1} .$$

Thus (3.56) is true for $n = m-1$. But (3.56) is true for $n = A-1$, by (3.46). Hence (3.56) follows by induction and the proof of the lemma is complete.

When interpreted in words, part (a) of the above lemma means that the points (a,a) on the neutral boundary that may be reached at any stage of sampling up to A-2 are continuation points. We stop anyhow after stage A-1, whatever the point (a,b) is reached. Hence the point of truncation, may be defined as the point (a^o, a^o) where $a^o = (a_o+b_o+A-1)/2$, in case a_o+b_o+A-1 is even or otherwise $a^o = (a_o+b_o+A-1)/2 + 1/2$ in case a_o+b_o+A-1 is odd. Part (b) of the above lemma then asserts that the optimal region of sampling lies within the square bounded above by $a = a^o$ and on the right by $b = a^o$.

A possible method for obtaining the optimum boundary numerically may be described analogously as in section 2.5 with the help of (3.47), starting from the point of truncation and working backwards.

3.4 Trinomial model.

For this specialization, we need put $\Theta = \pi_1 - \pi_2$ in the results of section 3.1. Γ_0, Γ_1, Γ_2 are defined in the same way as in (2.86). Equation (3.41) reduces in this case to

$$(3.61)\quad V_n(a,b,c) = \max \left[V_{n,o}(a,b,c), \frac{a}{a+b+c} V_{n+1}(a+1,b,c) + \frac{b}{a+b+c} V_{n+1}(a,b+1,c) + \frac{c}{a+b+c} V_{n+1}(a,b,c+1) \right]$$

where

$$(3.62)\qquad V_{n,o}(a,b,c) = (A-n) \frac{|a-b|}{a+b+c} \ .$$

Again, it follows from (3.43) that

$$(3.63)\quad V_{A-1}(a,b,c) = V_{A-1,o}(a,b,c) = \frac{|a-b|}{a+b+c}, \text{ for all } (a,b,c) \ .$$

With the help of (3.61) it is possible, at least in principle, to calculate $V_n(a,b,c)$ for all possible values of (a,b,c) and n starting from the known values (3.63) for $n = A-1$. Once these values are known, the optimum sampling rule becomes specified.

As in the binomial problem, we may alternatively write (3.61) as

$$(3.64)\quad V(a,b,c) = \max\left[V_o(a,b,c),\ \frac{a}{a+b+c}V(a+1,b,c) + \frac{b}{a+b+c}V(a,b+1,c) + \frac{c}{a+b+c}V(a,b,c+1)\right],$$

where

$$(3.65)\quad V_o(a,b,c) = (A + a_o + b_o + c_o - a - b - c)\ \frac{|a-b|}{a+b+c}$$

and (a_o, b_o, c_o) represents the starting point. The same remarks as in the binomial case apply in comparing (3.64) with (2.105).

We state below without proof a lemma corresponding to Lemma 3.1 for the binomial problem.

<u>Lemma 3.2</u>

(a) $V_n(a,a,c) > V_{n,o}(a,a,c) = 0,\ n < A-1,$ all a,c .

(b) $V_n(a,b,c) = V_{n,o}(a,b,c)$, for $|a-b| \geq A-n,\ n \leq A-1,$ all c.

The proof is analogous to that of Lemma 3.1. Just like Lemma 2.9, this one also delimits the region of sampling within a certain region defined as in Lemma 2.9, where n is now replaced by $a_o + b_o + c_o + A-1$.

Using the lemma above and the recursion relation (3.64), a possible method for obtaining the optimum boundary systematically may be described analogously as in section 2.10 starting from the points of truncation $(a^o(c), a^o(c), c)$ where $a^o(c)$ is the smallest positive integer satisfying $2a^o(c)+c \geq a_o + b_o + c_o + A-1, \qquad c \geq c_o$.

It follows from the definition (3.66) of $a^{o}(c)$ that $a^{o}(c)$ is non increasing in c.

Suppose we start with a uniform prior i.e., with

$$a_o = b_o = c_o = 1$$

Then for $A = 10$, the points of truncation are obtained from (3.66) as

C	1	2	3	4	5	6	7
$a^{o}(C)$	6	5	5	4	4	3	3

These may be compared with those values obtained in page 70.

CHAPTER IV

ASYMPTOTIC BEHAVIOUR OF OPTIMAL SOLUTION

4.0 Introduction and summary

In this chapter we study the limiting behaviour of the optimum boundaries of the various sequential decision problems considered in Chapter II under limiting conditions which tend to require large samples. As sample size is not one of the given parameters of a sequential problem, a natural approach to the large sample theory is to make the cost of an observation tend to zero (when the loss is held fixed) as considered by Wald [39a]. Since it is the relative magnitude of the two types of cost, namely, the cost of an observation and the cost of making a wrong decision, that matters in obtaining the optimum boundaries, an equivalent approach is to make the loss tend to infinity keeping the cost fixed. We consider this alternative approach as it is particularly suited for our sequential decision problems. The limiting condition suggested can then be obtained by letting the parameter A in the various loss functions approach infinity.

From physical considerations it is apparent that as A becomes larger, more and more sampling is needed to reach the boundary. Thus, in the limiting case, any point with finite co-ordinates (a,b) or (a,b,c), whichever the case may be, will be within the continuation set. The boundary will consist of points with their co-ordinates also approaching infinity. The limiting behvaiour of the boundary, if any, has to be specified only in terms of suitably normalized co-ordinates.

The first point to be resolved in this study, then, is what constitutes a suitable normalization of the co-ordinates in each case, in order to yield non-degenerate limits. It turns out that the appropriate normalization for a given case depends on the cost function.

Throughout this chapter the binomial problem is considered for simplicity with $p_o = 1/2$. The asymptotic behaviour of the optimal solution has been studied by Moriguti and Robbins [30] for this problem with constant cost. Following them closely the problem of finding the asymptotic optimal solution for the binomial problem with absolute deviation cost is reduced in section 4.1 to that of finding the solution of a partial differential equation free boundary value problem. In section 4.2 certain series representations of the latter solution are obtained and compared with those given by Moriguti and Robbins for the corresponding case with constant cost.

Sections 4.3 and 4.4 correspond to sections 4.1 and 4.2 respectively for the trinomial problem. The ~~series~~ representation of the optimal solution is obtained in this case for both types of cost.

The formal expansions of the asymptotic optimal boundaries for the various decision problems, obtained in this chapter, are useful in giving us some idea of the asymptotic shapes of the Bayes sequential testing regions, a concept developed by Schwarz [34] in a somewhat similar context. We hope that the exact boundary for large A can be computed from the asymptotic results involving no appreciable error.

4.1 Binomial problem.

As shown in Chapter II, (2.67), the optimum boundary can be characterized in terms of the gain function $V(a,b)$ which assumes, on the boundary and outside it, the value

$$(4.1) \qquad V(a,b) = V_o(a,b) \equiv A|a-b|/2(a+b)$$

and satisfies in the go-region (continuation set) the following recursion relation:

$$(4.2) \qquad V(a,b) = \frac{a}{a+b} V(a+1,b) + \frac{b}{a+b} V(a,b+1) - C(a,b)$$

where

$$(4.3) \qquad C(a,b) = \begin{cases} 1 & \text{, for constant cost,} \\ E[\,|p-1/2|\,|a,b] & \text{, for absolute deviation cost.} \end{cases}$$

It is possible to give, as in (2.70), a similar (with obvious modifications) characterization of the optimum boundary in terms of the net gain function,

$$(4.4) \qquad M(a,b) = V(a,b) - V_o(a,b) .$$

We now consider what kind of limiting conditions are appropriate if the boundary is to be non-degenerate. Without going into details we state that so long as p_o is a fixed positive number bounded away from 0 and 1, a and b must approach infinity at the same rate in order that the asymptotic boundary be non-degenerate. In this limiting condition, the problem of testing $p \leq p_o$ against $p > p_o$ reduces after proper standardization to that of testing $\mu \leq \mu_o$

against $\mu > \mu_o$, where μ is the mean rate of increase in a standard Gaussian process. With $p_o = 1/2$, $\mu_o = 0$. In this symmetric case, it is the limiting tendencies of the two pertinent parameters $a+b \equiv u$ and $a-b \equiv v$ that determines whether or not the optimal boundary is non-degenerate asymptotically. It is possible to show from Chernoff [14] that the appropriate normalization of u, v and V or M for a non-degenerate boundary depends on the type of cost and can, in fact, be determined as given below.

The appropriate normalization consists of:

Case (i): constant cost:

$$(4.5)\quad \begin{cases} x = u/A^{2/3} = (a+b)/A^{2/3}, \\ y = v/A^{1/3} = (a-b)/A^{1/3}, \\ \bar{V}(x,y) = V(a,b)/A^{2/3}; \end{cases}$$

Case (ii): absolute deviation cost:

$$(4.6)\quad \begin{cases} x = u/A = (a+b)/A, \\ y = v/A^{1/2} = (a-b)/A^{1/2}, \\ \bar{V}(x,y) = V(a,b)/A^{1/2}. \end{cases}$$

Moriguti and Robbins [30] studied the limiting behaviour of the optimum boundary under normalization (4.5). We shall do the same in this chapter under normalization (4.6). For convenience we denote by L the limiting tendency in which $a,b \to \infty$ along with A such that x,y in (4.6) remain finite.

Lemma 4.1

$$A^{1/2}C(a,b) \xrightarrow{L} \frac{1}{\sqrt{x}}\left\{ \phi\left(\frac{y}{\sqrt{x}}\right) + \frac{|y|}{\sqrt{x}}\left[\frac{1}{2} - \Phi\left(-\frac{|y|}{\sqrt{x}}\right)\right]\right\} \equiv \bar{C}(x,y),$$

say ,

where ϕ and Φ are the standard normal density and distribution function respectively.

Proof: It is well-known [15, p.252] that the beta density of p converges to the normal density under L. The standardized variable is

$$t = \frac{p - E(p)}{\sqrt{V(p)}} \quad ,$$

where

$$E(p) = \frac{a}{a+b} = \frac{1}{2} + \frac{1}{A^{1/2}} \cdot \frac{y}{2x}$$

and

$$V(p) = \frac{ab}{(a+b)^2(a+b+1)} = \frac{\frac{1}{4} - \frac{y^2}{4x^2} \cdot \frac{1}{A}}{A(x + \frac{1}{A})} \quad .$$

Hence,

$$p = E(p) + t\sqrt{V(p)}$$

$$= \frac{1}{2} + \frac{y}{2x} \cdot \frac{1}{A^{1/2}} + \frac{t}{2(xA)^{1/2}} \left(\frac{1 - \frac{y^2}{x^2 A}}{1 + \frac{1}{xA}} \right)^{1/2}$$

so that

$$(4.6) \qquad A^{1/2}(p - \tfrac{1}{2}) = \frac{y}{2x} + \frac{1}{2\sqrt{x}}\, t + o\left(\frac{|t|}{A}\right) .$$

Now if t' is $N(0,1)$, it follows from Theorem 2 of [13] that for fixed a' and b',

$$E|a' + b't| \xrightarrow{L} E|a' + b' t'|$$

which implies that $E|t|$ is bounded and hence the last term in (4.6) converges to zero in mean as $A \to \infty$. Thus

$$A^{1/2} C(a,b) \equiv A^{1/2} E[\,|p - \tfrac{1}{2}|\,|a,b] \xrightarrow{L} E\left|\frac{y}{2x} + \frac{1}{2\sqrt{x}} t'\right|$$

$$= \frac{1}{\sqrt{x}}\left\{\phi\left(\frac{y}{\sqrt{x}}\right) + \frac{|y|}{\sqrt{x}}\left[\frac{1}{2} - \Phi\left(-\frac{|y|}{\sqrt{x}}\right)\right]\right\}.$$

The following development parallels very closely the corresponding development sketched very briefly by Moriguti and Robbins [30] for the case of constant cost.

It follows from (4.2) that within the go-region $\bar{V}(x,y)$ satisfies the following relation:

$$(4.7)\quad A^{1/2}[\bar{V}(x,y) - \bar{V}(x+A^{-1},y)] = \tfrac{1}{2} A^{1/2}[\bar{V}(x+A^{-1},y+A^{-1/2}) - 2\bar{V}(x+A^{-1},y) + \bar{V}(x+A^{-1},y-A^{-1/2})]$$

$$+ (y/2x)A^{-1/2} A^{1/2}[\bar{V}(x+A^{-1},y+A^{-1/2}) - \bar{V}(x+A^{-1},y-A^{-1/2})]$$

$$- C(\tfrac{1}{2}Ax + \tfrac{1}{2}A^{1/2}y,\ \tfrac{1}{2}Ax - \tfrac{1}{2}A^{1/2}y).$$

Multiplying both sides of (4.7) by $A^{1/2}$, letting $A \to \infty$ and using lemma 4.1, we see that $\bar{V}$ satisfies, under L, the following partial differential equation:

$$(4.8) \qquad -\frac{\partial \bar{V}}{\partial x} = \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial y^2} + \frac{y}{x}\frac{\partial \bar{V}}{\partial y} - \bar{C}(x,y) .$$

On or outside the boundary, $\bar{V}(x,y)$ is given by

$$(4.9) \qquad \bar{V}(x,y) = \bar{V}_o(x,y) ,$$

where

$$(4.10) \qquad \bar{V}_o(x,y) = V_o(a,b)/A^{1/2} = V_o(1/2 \cdot Ax + 1/2 \cdot A^{1/2}y, 1/2 \cdot Ax - 1/2 \cdot A^{1/2}y)/A^{1/2} = |y|/(2x).$$

If follows from (4.10) that $\bar{V}_o$ satisfies, in the region $y > 0$, the following homogeneous partial differential equation:

$$(4.11) \qquad \frac{1}{2}\frac{\partial^2 \bar{V}_o}{\partial y^2} + \frac{y}{x}\frac{\partial \bar{V}_o}{\partial y} + \frac{\partial V_o}{\partial x} = 0 .$$

Thus, the standardized net gain function,

$$(4.12) \qquad W(x,y) \equiv M(a,b)/A^{1/2} = [V(a,b) - V_o(a,b)]/A^{1/2} = \bar{V}(x,y) - \bar{V}_o(x,y)$$

satisfies, under L, the partial differential equation,

$$(4.13) \qquad \frac{1}{2}\frac{\partial^2 W}{\partial y^2} + \frac{y}{x}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial x} = \bar{C}(x,y)$$

in the region between the straight line $y=0$ and the boundary curve

$$(4.14) \qquad y = y_1(x) ,$$

say. At the boundary, $y = 0$, symmetry implies the condition:

$$(4.15) \qquad \left.\frac{\partial \bar{V}}{\partial y}\right|_{y=0} = 0$$

and this, together with

$$(4.16) \qquad \left.\frac{\partial \bar{V}_o}{\partial y}\right|_{y=0} = \frac{1}{2x} ,$$

yields the boundary condition

$$(4.17) \qquad \left.\frac{\partial W}{\partial y}\right|_{y=0} = -\frac{1}{2x} .$$

At $y = y_1(x)$, a continuity requirement yields the boundary condition

$$(4.18) \qquad W\Big|_{y=y_1(x)} = 0$$

Another boundary condition is obtained by the same arguments as in Moriguti and Robbins. Suppose that the point (u,v) is on the boundary, $(u+1,v+1)$ being outside , $(u+1,v-1)$ being inside the go-region. Then

$$V'(u,v) = V_o'(u,v) ,$$

and

$$V'(u+1,v+1) = V_o'(u+1,v+1);$$

where

$$V'(u,v) = V((u+v)/2, (u-v)/2) ,$$

and

$$V_o'(u,v) = V_o((u+v)/2, (u-v)/2) .$$

Let us further assume that $v-1 > 0$. Then substitution of $V'(u,v) = V_o'(u,v) = Av/2u$, and $V'(u+1,v+1) = V_o'(u+1,v+1) = A(v+1)/2(u+1)$ into

$$V'(u,v) = \frac{1}{2}[V'(u+1,v+1) + V'(u+1,v-1)]$$

$$+ \frac{v}{2u}[V'(u+1,v+1) - V'(u+1,v-1)] - C(\frac{u+v}{2}, \frac{u-v}{2})$$

obtained from (4.2), yields

$$\frac{Av}{2u} = \frac{1}{2}\left[\frac{A(v+1)}{2(u+1)} + V'(u+1,v-1)\right]$$

$$+ \frac{v}{2u}\left[\frac{A(v+1)}{2(u+1)} - V'(u+1,v-1)\right] - C\left(\frac{u+v}{2}, \frac{u-v}{2}\right) ,$$

i.e.,

$$V'(u+1,v-1) = \frac{A(v-1)}{2(u+1)} + \frac{2u}{u-v} C\left(\frac{u+v}{2}, \frac{u-v}{2}\right) .$$

But

$$V_o'(u+1,v-1) = \frac{A(v-1)}{2(u+1)} .$$

Thus

$$V'(u+1,v-1) - V_o'(u+1,v-1) = \frac{2u}{u-v} C\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

i.e., $W(x+A^{-1},y-A^{-1/2}) = \dfrac{2x}{A^{1/2}x-y} C\left(\frac{1}{2} Ax + \frac{1}{2}A^{1/2}y, \frac{1}{2}Ax - \frac{1}{2} A^{1/2} y\right)$.

Expanding the left hand side and using Lemma 4.1, we get

$$(4.19) \qquad W(x,y) - A^{-1/2} \frac{\partial W}{\partial y} + O(A^{-1}) = O(A^{-1}) .$$

But $W(x,y) = 0$, since (x,y) is on the boundary. Hence, (4.19) implies that

$$(4.20) \qquad \frac{\partial W}{\partial y} = O(A^{-1/2}) .$$

Therefore, in the limit, we should expect that

$$(4.21) \qquad \left.\frac{\partial W}{\partial y}\right|_{y=y_1(x)} = 0$$

The problem of finding the optimum boundary along with the optimum net gain function thus reduces to the following

FREE-BOUNDARY PROBLEM:

To solve for the unknown $W(x,y)$ and the unknown boundary $y_1(x)$, given that W satisfies the partial differential equation

$$(4.22) \qquad \frac{1}{2}\frac{\partial^2 W}{\partial y^2} + \frac{y}{x}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial x} = \overline{C}(x,y)$$

where

$$(4.23) \qquad \overline{C}(x,y) = \frac{y}{2x} + \frac{1}{\sqrt{x}}\left\{ \phi\left(\frac{y}{\sqrt{x}}\right) - \frac{y}{\sqrt{x}} \quad \left(-\frac{y}{\sqrt{x}}\right)\right\} .$$

in the region

$$(4.24) \qquad 0 < y \leq y_1(x) \ , \quad 0 < x < \infty \ ,$$

and where $W(x,y)$ satisfies the boundary condition

$$(4.25) \qquad \left.\frac{\partial W}{\partial y}\right|_{y=0} = -\frac{1}{2x} \ , \quad \text{for } 0 < x < \infty \ ,$$

and the free boundary conditions

$$(4.26) \qquad W\Big|_{y = y_1(x)} = 0 \ , \quad \text{for } 0 < x < \infty \ ,$$

$$(4.27) \qquad \left.\frac{\partial W}{\partial y}\right|_{y = y_1(x)} = 0 \ , \quad \text{for } 0 < x < \infty \ .$$

$y_1(x)$ gives the upper boundary. The problem being symmetrical with respect to y, and $W(x,y) = W(x,-y)$, the lower boundary is given simply by $-y_1(x)$. Thus it is sufficient to consider only $y > 0$.

We now make a transformation to reduce (4.22) to the heat equation; we conclude, omitting the proof:

Lemma 4.2. The non-homogeneous partial differential equation

$$(4.28) \qquad \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + \frac{y}{x}\frac{\partial F}{\partial y} + \frac{\partial F}{\partial x} = f(x,y) \quad ,$$

where $f(x,y)$ satisfies the corresponding homogeneous partial differential equation, is reduced to the heat equation, viz.,

$$(4.29) \qquad \frac{\partial^2 F_1}{\partial z^2} - \frac{\partial F_1}{\partial t} = 0 \quad ,$$

by the following transformation :

$$(4.30) \qquad \begin{cases} t = \dfrac{1}{2x} \quad , \\ z = \dfrac{y}{x} \quad , \\ F_1(t,z) = F(x,y) - x\, f(x,y) \; . \end{cases}$$

It is easily verified that $\overline{C}(x,y)$ satisfies the homogeneous partial differential equation:

$$(4.31) \qquad \frac{1}{2}\frac{\partial^2 \overline{C}}{\partial y^2} + \frac{y}{x}\frac{\partial \overline{C}}{\partial y} + \frac{\partial \overline{C}}{\partial x} = 0 \quad .$$

Hence, by the lemma above, if we define

$$(4.32) \qquad W_1(t,z) = W(x,y) - x\,\overline{C}(x,y)$$

where t and z are defined by (4.30), then the free-boundary problem stated above reduces to the following

TRANSFORMED FREE-BOUNDARY PROBLEM:

To solve for the unknown $W_1(t,z)$ and the unknown boundary $z_1(t)$, given that W_1 satisfies the following partial differential equation:

$$(4.33)\qquad \frac{\partial^2 W_1}{\partial z^2} - \frac{\partial W_1}{\partial t} = 0 \text{ in } \begin{cases} 0 < z \leq z_1(t) \\ 0 < t < \infty \end{cases} ,$$

and satisfies the following boundary conditions:

$$(4.34)\qquad \left.\frac{\partial W_1}{\partial z}\right|_{z=0} = -\frac{1}{2} - \left.\frac{\partial \bar{C}_1}{\partial z}\right|_{z=0} = -\frac{1}{2} ,$$

$$(4.35)\qquad W_1\Big|_{z=z_1(t)} = -\bar{C}_1(t, z_1(t)) ,$$

$$(4.36)\qquad \left.\frac{\partial W_1}{\partial z}\right|_{z=z_1(t)} = -\left.\frac{\partial \bar{C}_1}{\partial z}\right|_{z=z_1(t)} ;$$

where

$$(4.37)\qquad \bar{C}_1(t,z) = \frac{1}{2t}\,\bar{C}\left(\frac{1}{2t}, \frac{z}{2t}\right) .$$

4.2 Series expansion for the optimal solution for large x (binomial problem)

As it is very difficult to find the solution of the transformed free-boundary problem, our objective in the following analysis is limited to finding a formal series expansion of its solution for small t (as in Moriguti and Robbins [30]). Thus, assuming a series expansion for the unknown boundary as

$$(4.38)\qquad z_1(t) = \sum_m C_m t^m$$

for small t, where the range of the summation index m will be settled presently, we are led to consider solutions of the heat equation (4.33) involving some powers of t. It may be verified that the function

$$(4.39) \qquad F_1(t,z) = A_n t^n \, {}_1F_1(-n, 1/2; -z^2/4t) ,$$

where

$$(4.40) \qquad {}_1F_1(\alpha,\beta;x) = 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{x^2}{2!} + \ldots.$$

denotes the confluent hypergeometric function and n, A_n are constants, satisfies the heat equation (4.33).

The most general form of the solution of (4.33) which belongs to this family and satisfies (4.34) is, thus, given by

$$(4.41) \qquad W_2(t,z) = \sum_n A_n t^n \, {}_1F_1(-n, 1/2; -z^2/4t) - z/2 ,$$

where the range of the summation index n is determined along with that of m in (4.38) from the two free boundary conditions (4.35) and (4.36). It will be seen below that these two conditions are sufficient to determine the unknown constants C_m and A_n .

From (4.41) and (4.40), (4.35) and (4.36) reduce to:

$$(4.42) \quad \sum_n A_n t^n \, {}_1F_1(-n, 1/2; -z_1^2(t)/4t) = \frac{1}{2} z_1(t) - \bar{C}_1(t, z_1(t))$$

$$= \frac{1}{2} z_1(t) - \frac{1}{\sqrt{2t}} \left\{ \phi\left(\frac{z_1(t)}{\sqrt{2t}}\right) + \right.$$

$$\left. + \frac{z_1(t)}{\sqrt{2t}} \left[\frac{1}{2} - \phi\left(-\frac{z_1(t)}{\sqrt{2t}}\right) \right] \right\},$$

$$(4.43) \quad \sum_n 2n A_n t^n \frac{z_1(t)}{2t} \, {}_1F_1(-n+1, 3/2; -z_1^2(t)/4t) = \frac{1}{2} - \left. \frac{\partial \bar{C}_1}{\partial z} \right|_{z=z_1(t)}$$

$$= \frac{1}{2} - \frac{1}{2t} \left\{ \frac{1}{2} - \phi\left(-\frac{z_1(t)}{\sqrt{2t}}\right) \right\} .$$

To facilitate the later analysis we express the ϕ and Φ functions in terms of confluent hypergeometric functions. To this end we note the following results (Erdelyi [17])

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\ {}_1F_1(\alpha,\ \alpha;\ -x^2/2)\ ,\quad -\infty < x < \infty\ ,\quad \alpha \neq 0\ .$$

(4.44) $$\frac{1}{2} - \Phi(-x) = \frac{x}{\sqrt{2\pi}}\ {}_1F_1(1/2,\ 3/2;\ -x^2/2),\ -\infty < x < \infty\ .$$

$$y\ {}_1F_1(\alpha,\beta + 1;y) = \beta[\ {}_1F_1(\alpha,\ \beta;\ y) - {}_1F_1(\alpha-1,\ \beta;\ y)]$$

and in particular

$$2y\ {}_1F_1(1/2,\ 3/2;\ y) = {}_1F_1(1/2,\ 1/2;\ y) - {}_1F_1(-1/2,\ 1/2;\ y).$$

Therefore, putting $y = -x^2/2$,

(4.45) $$\begin{aligned}\phi(x) + x[\ \tfrac{1}{2} - \Phi(-x)] &= \frac{1}{\sqrt{2\pi}}\left\{{}_1F_1(1/2,1/2;-x^2/2)+x^2\,{}_1F_1(1/2,3/2,-x^2/2)\right\}\\ &= \frac{1}{\sqrt{2\pi}}\ {}_1F_1(-1/2,1/2;-x^2/2).\end{aligned}$$

Utilizing (4.44) and (4.45), (4.42) and (4.43) can be expressed as:

$$\Sigma\ A_n t^n\ {}_1F_1(-n,1/2;-z_1^2(t)/4t) = \frac{1}{2}\ z_1(t) - \frac{1}{\sqrt{2t}}\ \frac{1}{\sqrt{2\pi}}\ {}_1F_1(-1/2,1/2;-z_1^2(t)/4t)$$

$$\Sigma\ nA_n t^n\ \frac{z_1(t)}{t}\ {}_1F_1(-n+1,3/2;-z_1^2(t)/4t) = \frac{1}{2} - \frac{1}{\sqrt{2t}}\ \frac{1}{\sqrt{2\pi}}\ \frac{z_1(t)}{2t}\ {}_1F_1(1/2,3/2;-z_1^2(t)/4t)$$

We now consider the range of m and n in (4.38) and (4.41) respectively. We first note that in (4.38) m must be non-negative, otherwise the series might diverge for sufficiently small t. Next, it has been shown in Chernoff [14] that $y_1(x)/\sqrt{x}$ is decreasing in

x for the case of constant cost. His argument remains valid for absolute deviation cost too. This means, in terms of the transformed variables, that $z_1(t)/\sqrt{2t}$ is increasing in t, i.e.,

$$(4.46) \qquad \sum_m \frac{c_m}{\sqrt{2}} \left(m - \frac{1}{2}\right) t^{m-3/2} > 0 \,.$$

This rules out the possibility that $m < 1/2$, because, otherwise, for sufficiently small t, the term involving a negative coefficient in (4.46) might be dominant making (4.46) nonpositive contrary to hypothesis. Thus if

$$(4.47) \qquad z_1(t) = \sum_{m \geq 1/2} c_m t^m \,,$$

then

$$(4.48) \qquad \frac{z_1^2(t)}{4t} = \sum_{m \geq 0} d_m t^m \,,$$

say. Substituting (4.48) in (4.42), we notice that in the right hand side of (4.42) the minimum power of t is $-1/2$. Hence the series on the left hand side should start with $n = -1/2$. Now by what steps n should increase in the left hand side of (4.42) depends, of course, by what steps m increases in (4.38). For simplicity, we take the step in (4.38) as $1/2$, any smaller steps only increasing the number of unknown coefficients to be determined. We are thus finally led to consider both m and n as half-integers, m starting from $1/2$ and n starting from $-1/2$ in (4.38) and (4.41) respectively.

If we now substitute

$$(4.49) \qquad z_1(t) = C_1\sqrt{t} + C_2(\sqrt{t})^2 + C_3(\sqrt{t})^3 + \dots .$$

in (4.42) and (4.43) and equate the coefficients of equal powers of t from both sides of the equations we are led to the following results.

Equating coefficients of $t^{-1/2}$ from both sides of (4.42), we get

$$(4.50) \quad -\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{2\pi}} \; {}_1F_1(-1/2,\ 1/2;-C_1^2/4) = A_{-1/2}\ {}_1F_1(1/2,1/2;-C_1^2/4)\ ,$$

while equating coefficients of t^{-1} from both sides of (4.43), we get

$$(4.51) \quad -\ \frac{1}{2}\ \frac{1}{\sqrt{2\pi}} \ \frac{C_1}{\sqrt{2}}\ {}_1F_1(1/2,3/2;-C_1^2/4) = -\ \frac{1}{2}\ A_{-1/2}C_1\ {}_1F_1(3/2,3/2;-C_1^2/4).$$

If $C_1 \neq 0$, using (4.45), we get from (4.50)

$$A_{-1/2}\ e^{-C_1^2/4} = -\left\{ \phi(\frac{C_1}{\sqrt{2}}) + \frac{C_1}{\sqrt{2}} \left[\phi(\frac{C_1}{\sqrt{2}}) - \frac{1}{2} \right] \right\} < 0 \quad \text{for all}$$

C_1, while from (4.51),

$$A_{-1/2}\ e^{-C_1^2/4} = \frac{1}{C_1} \left[\phi\ (\frac{C_1}{\sqrt{2}}) - \frac{1}{2} \right] > 0 \quad \text{for all} \quad C_1\ ,$$

a contradiction. Hence $C_1 = 0$ and hence from (4.51), $A_{-1/2} = -1/2\sqrt{\pi}$.

Substituting these values for the constants C_1, $A_{-1/2}$ in the subsequent equations, the next two coefficients C_2, A_o are determined. These are then substituted in the subsequent equations. This process is repeated, giving us the following result:

$$C_n' \equiv C_{4n-1} \neq 0\ , \quad n = 1,2,3,\dots\ ,$$

$$A_n' \equiv A_{(4n-1)/2} \neq 0\ ,\ n = 0,1,2,\dots\ ,$$

and all other coefficients are zero. The first few values are found to be:

$$C_1' = \pi^{1/2} \,, \qquad A_o' = -\frac{1}{2}\pi^{-1/2} \,,$$

$$C_2' = \frac{7}{12}\pi^{3/2} \,, \qquad A_1' = \frac{1}{4}\pi^{1/2} \,,$$

$$C_3' = \frac{197}{160}\pi^{5/2} \,, \qquad A_2' = \frac{1}{6}\pi^{3/2} \,,$$

$$A_3' = \frac{529}{1440}\pi^{5/2} \,.$$

We thus arrive at the following series expansion of the solution of the transformed free boundary problem:

$$(4.52) \quad z_1(t) = t^{3/2}[C_1' + C_2' t^2 + C_3' t^4 + \ldots]$$

$$= t^{3/2}[\pi^{1/2} - \frac{7}{12}\pi^{3/2} t^2 + \frac{197}{160}\pi^{5/2} t^4 - \ldots]$$

and

$$(4.53) \quad W_1(t,z) = -\frac{3}{2} - \frac{1}{2\sqrt{\pi}} t^{-1/2} \, {}_1F_1(-1/2,1/2;-z^2/4t)$$

$$+ \frac{1}{4}\pi^{1/2} t^{3/2} \, {}_1F_1(-3/2,1/2;-z^2/4t)$$

$$- \frac{1}{6}\pi^{3/2} t^{7/2} \, {}_1F_1(-7/2,1/2;-z^2/4t)$$

$$+ \frac{529}{1440}\pi^{5/2} t^{11/2} \, {}_1F_1(-11/2,1/2;-z^2/4t)$$

$$- \ldots \,.$$

In terms of the original variables x and y (see (4.6)) we get the following series expansions, for large x, of the <u>optimum boundary</u> as

$$y_1(x) = x\, z_1\left(\frac{1}{2x}\right)$$

$$= \frac{\sqrt{2\pi}}{4} \cdot \frac{1}{\sqrt{x}} \left[1 - \frac{7\pi}{48} \cdot \frac{1}{x^2} + \frac{197\,\pi^2}{160 \cdot 16} \cdot \frac{1}{x^4} - \ldots \right]$$

and of the <u>optimum</u> <u>net</u> <u>gain</u> <u>function</u> (4.12) as

$$W(x,y) = x\, \bar{C}(x,y) + W_1\left(\frac{1}{2x}, \frac{y}{x}\right)$$

$$= -\frac{y}{2x} + y\left[\frac{1}{2} - \Phi\left(-\frac{y}{\sqrt{x}}\right)\right] + \frac{\sqrt{2\pi}}{4^2}\, \frac{1}{x^{3/2}}\, {}_1F_1(-3/2, 1/2; -y^2/2x)$$

$$- \frac{(2\pi)^{3/2}}{3 \cdot 4^3} \cdot \frac{1}{x^{7/2}}\, {}_1F_1(-7/2, 1/2; -y^2/2x)$$

$$+ \frac{(2\pi)^{5/2} \cdot 529}{1440 \cdot 4^4} \cdot \frac{1}{x^{11/2}}\, {}_1F_1(-11/2, 1/2; -y^2/2x)$$

$$- \ldots .$$

In particular,

$$(4.56)\quad W(x,0) = \frac{(2\pi)^{1/2}}{4^2} \cdot \frac{1}{x^{3/2}} - \frac{(2\pi)^{3/2}}{3 \cdot 4^3} \cdot \frac{1}{x^{7/2}} + \frac{(2\pi)^{5/2} \cdot 529}{1440 \cdot 4^4} \cdot \frac{1}{x^{11/2}} - \ldots$$

$$= \frac{\sqrt{2\pi}}{16} \cdot \frac{1}{x^{3/2}} \left[1 - \frac{\pi}{6} \cdot \frac{1}{x^2} + \frac{\pi^2 \cdot 529}{5760}\, \frac{1}{x^4} - \ldots \right] .$$

For the case of constant cost, Moriguti and Robbins [30] found the following:

$$y_1(x) = \frac{1}{4x} - \frac{1}{48x^4} + \frac{7}{960\, x^7} - \ldots .$$

$$W(x,y) = x - \frac{y}{2x} - x\, {}_1F_1\left(1, \frac{1}{2}; -\frac{y^2}{2x}\right)$$

$$+ \frac{1}{16x^2} \; {}_1F_1(-2, \tfrac{1}{2}; - \tfrac{y^2}{2x})$$

$$- \frac{5}{3\cdot 16x^5} \; {}_1F_1(-5, \tfrac{1}{2}; - \tfrac{y^2}{2x})$$

$$+ \frac{437}{5\cdot 3^2\cdot 16^3 x^8} \; {}_1F_1(-8, \tfrac{1}{2}; - \tfrac{y^2}{2x})$$

$$- \cdots ,$$

$$W(x,0) = \frac{1}{16x^2} - \frac{5}{3\cdot 16^2 x^5} + \frac{437}{5\cdot 3^2\cdot 16^3 x^8} - \cdots ,$$

where x and y are defined in (4.5) and

$$W(x,y) = M(a,b)/A^{2/3} .$$

4.3 Trinomial problem.

As shown in Chapter II, (2.105), the optimum boundary can be characterized in terms of the gain function $V(a,b,c)$ which assumes, on and outside the boundary the value

$$V(a,b,c) = V_o(a,b,c) = A \cdot \frac{|a-b|}{a+b+c} , \tag{4.57}$$

and satisfies in the go-region the recursion relation

$$V(a,b,c) = \frac{a}{a+b+c} V(a+1,b,c) + \frac{b}{a+b+c} V(a,b+1,c) + \frac{c}{a+b+c} V(a,b,c+1) - C(a,b,c) \tag{4.58}$$

where

$$C(a,b,c) = \begin{cases} 1 , & \text{for constant cost ,} \\ E[\,|\pi_1 - \pi_2|\,|a,b,c] , & \text{for absolute deviation cost} \end{cases} \tag{4.59}$$

It is possible to give, as in (2.108), a similar (with obvious modifications) characterization of the optimum boundary in terms of the <u>net gain function</u> ,

$$M(a,b,c) = V(a,b,c) - V_0(a,b,c) \tag{4.60}$$

As in the binomial problem, the appropriate limiting conditions for the asymptotic boundary to be non-degenerate depend on the type of cost per observation. In this problem, apart from the two pertinent parameters $a+b = u$ and $a-b = v$, we have another, c. It is clear that if the order of magnitude of c is less than that of $a+b$ as these parameters tend to infinity, the trinomial problem essentially reduces asymptotically to the binomial problem considered in the last two sections. To retain the essential feature of the trinomial problem, then, we keep the order of c the same as that of $a+b$. The two different kinds of approaches of u and v to infinity in this problem for the boundaries to be non-degenerate are the same as those in the binomial problem for the corresponding two types of cost function. Specifically they are as follows:

The appropriate normalization consists of:

case (i): <u>constant cost</u>:

$$(4.61) \qquad \begin{aligned} x &= \frac{a+b}{A^{2/3}} , \qquad x > 0 ; \\ y &= \frac{a-b}{A^{1/3}} ; \\ z &= \frac{a+b+c}{A^{2/3}} , \quad z \geq x ; \\ \bar{V}(x,y,z) &= \frac{V(a,b,c)}{A^{2/3}} . \end{aligned}$$

case (ii): absolute deviation cost:

$$(4.62)\quad \begin{cases} x = \dfrac{a+b}{A}, & x > 0\,; \\ y = \dfrac{a-b}{A^{1/2}}\,; & \\ z = \dfrac{a+b+c}{A}, & z \geq x\,; \\ \overline{V}(x,y,z) = \dfrac{V(a,b,c)}{A^{1/2}}\,. & \end{cases}$$

We denote, for convenience, the two types of limiting tendencies specified in the above two normalizations by L_1 and L_2 respectively. We note that if the order of magnitude of c is less than that of a+b, then the corresponding solutions can be obtained as a special case with $z = x$.

We now treat the two cases separately.

case (i). It follows from (4.58) that

$$(4.63)\quad A^{2/3}[\,\overline{V}(x,y,z) - \overline{V}(x,y,z + A^{-2/3})\,]$$

$$= \frac{x}{2z} A^{2/3}[\,\overline{V}(x+A^{-2/3},y+A^{-1/3},z+A^{-2/3}) - 2\overline{V}(x+A^{-2/3},y,z+A^{-2/3}) + \overline{V}(x+A^{-2/3},y-A^{-1/3},z+A^{-2/3})\,]$$

$$+ \frac{x}{z} A^{2/3}[\,\overline{V}(x+A^{-2/3},y,z+A^{-2/3}) - \overline{V}(x,y,z+A^{-2/3})\,]$$

$$+ \frac{y}{2z} A^{-1/3}A^{2/3}[\,\overline{V}(x+A^{-2/3},y+A^{-1/3},z+A^{-2/3}) - \overline{V}(x+A^{-2/3},y-A^{-1/3},z+A^{-2/3})\,]$$

$- 1 .$

Letting $A \to \infty$ we see that $\bar{V}$ satisfies, under L_1, the following partial differential equation:

$$(4.64) \qquad -\frac{\partial \bar{V}}{\partial z} = \frac{x}{2z}\frac{\partial^2 \bar{V}}{\partial y^2} + \frac{x}{z}\frac{\partial \bar{V}}{\partial x} + \frac{y}{z}\frac{\partial \bar{V}}{\partial y} - 1 .$$

On or outside the boundary , $\bar{V}(x,y,z)$ is given by

$$(4.65) \qquad \bar{V}(x,y,z) = \bar{V}_o(x,y,z)$$

where

$$(4.66) \qquad V_o(x,y,z) = V_o(a,b,c)/A^{2/3} = V_o(\tfrac{1}{2}A^{2/3}x + \tfrac{1}{2}A^{1/3}y, \tfrac{1}{2}A^{2/3}x - \tfrac{1}{2}A^{1/3}y, A^{2/3}z - A^{2/3}x)/A^{2/3}$$

$$= |y| /z ,$$

which satisfies, in the region $y > 0$, the homogeneous partial differential equation:

$$(4.67) \qquad \frac{x}{2z}\frac{\partial^2 \bar{V}_o}{\partial y^2} + \frac{x}{z}\frac{\partial \bar{V}_o}{\partial x} + \frac{y}{z}\frac{\partial \bar{V}_o}{\partial y} + \frac{\partial \bar{V}_o}{\partial z} = 0 .$$

Thus the normalized net gain function ,

$$(4.68) \qquad W(x,y,z) = M(a,b,c)/A^{2/3} = [V(a,b,c) - V_o(a,b,c)]A^{2/3} =$$

$$= \bar{V}(x,y,z) - \bar{V}_o(x,y,z)$$

satisfies, under L_1, the partial differential equation:

$$(4.69) \qquad \frac{x}{2z}\frac{\partial^2 W}{\partial y^2} + \frac{x}{z}\frac{\partial W}{\partial x} + \frac{y}{z}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial y} = 1 .$$

We state below a lemma, without proof, that will be used in the second case as well as in Chapter V.

Lemma 4.3

If the parameters a,b,c of the Dirichlet distribution $\xi(a,b,c)$ tend to infinity along with A in such a way that

$$(4.70)\quad \begin{cases} \dfrac{a+b}{A^{2\alpha}} = x \\[2ex] \dfrac{a-b}{A^{\alpha}} = y \quad , \qquad \alpha > 0 \\[2ex] \dfrac{a+b+c}{A^{2\alpha}} = z \end{cases}$$

remain finite, then

$$(4.71)\quad \lim_{a \to \infty} A^{\alpha}\, E\left[\, |\pi_1 - \pi_2| \,\middle|\, \xi(a,b,c) \right] = \frac{2\sqrt{x}}{z} \left\{ \phi\left(\frac{y}{\sqrt{x}}\right) + \frac{|y|}{\sqrt{x}} \left[\frac{1}{2} - \Phi\left(-\frac{|y|}{\sqrt{x}}\right) \right] \right\}$$

$$\equiv \overline{C}(x,y,z), \text{ say .}$$

case (ii): Just as in case (i), it follows from (4.58) in this case

$$(4.72)\quad A^{1/2}\left[\overline{V}(x,y,z) - \overline{V}(x,y,z+A^{-1}) \right]$$

$$= \frac{x}{2z} A^{1/2}\left[\overline{V}(x+A^{-1},y+A^{-1/2},z+A^{-1}) - 2\overline{V}(x+A^{-1},y,z+A^{-1}) + \overline{V}(x+A^{-1},y-A^{-1/2},z+A^{-1}) \right]$$

$$+ \frac{x}{z} A^{1/2}\left[\overline{V}(x+A^{-1},y,z+A^{-1}) - \overline{V}(x,y,z+A^{-1}) \right]$$

$$+ \frac{y}{2z} \quad A^{-1/2} \cdot A^{1/2}\left[\overline{V}(x+A^{-1},y+A^{-1/2},z+A^{-1}) - \overline{V}(x+A^{-1},y-A^{-1/2},z+A^{-1}) \right]$$

$$- C\left(\tfrac{1}{2} Ax + \tfrac{1}{2}A^{1/2}y, \tfrac{1}{2} Ax - \tfrac{1}{2} A^{1/2}y,\ Az - Ax\right) .$$

Multiplying both sides of (4.72) by $A^{1/2}$, letting $A \rightarrow \infty$, and using Lemma 4.3 with $\alpha = 1/2$, we see that $\overline{V}$ satisfies, under L_2, the following partial differential equation:

$$(4.73) \qquad -\frac{\partial \overline{V}}{\partial z} = \frac{x}{2z} \frac{\partial^2 \overline{V}}{\partial y^2} + \frac{x}{z}\frac{\partial \overline{V}}{\partial x} + \frac{y}{z}\frac{\partial \overline{V}}{\partial y} - \overline{C}(x,y,z) .$$

It turns out that $\overline{V}_o(x,y,z) = V_o(a,b,c)/A^{1/2} = |y|/z$, as in case (i) and hence satisfies the equation (4.67). Hence the normalized net gain function,

$$(4.73a) \qquad W(x,y,z) = M(a,b,c)/A^{1/2} ,$$

satisfies, under L_2, the same differential equation (4.73).

Combining the two cases, we may state that under the appropriate limiting conditions W satisfies for both types of costs per observation, the partial differential equation:

$$(4.74) \qquad \frac{x}{2z}\frac{\partial^2 W}{\partial y^2} + \frac{x}{z}\frac{\partial W}{\partial x} + \frac{y}{z}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = \overline{C}(x,y,z) ,$$

where

$$(4.75) \qquad \overline{C}(x,y,z) = \begin{cases} 1 & \text{, for constant cost} \\ \frac{2\sqrt{x}}{z}\left\{\phi\left(\frac{y}{\sqrt{x}}\right) + \frac{|y|}{\sqrt{x}}\left[\frac{1}{2} - \Phi\left(-\frac{|y|}{\sqrt{x}}\right)\right]\right\}, & \text{for absolute deviation cost} \end{cases}$$

in the three dimensional region between the plane $y=0$ and the boundary surface $y = y_1(x,z)$, say. At the boundary surface $y=0$, symmetry gives the condition $\partial \overline{V}/\partial y|_{y=o} = 0$ in both cases. This, together with $\partial \overline{V}_o/\partial y|_{y=o} = 1/z$, gives the boundary condition satisfied by W, in both cases, as

$$(4.76) \qquad \left.\frac{\partial W}{\partial y}\right|_{y=0} = -\frac{1}{z} .$$

As before, from continuity, we have

$$(4.77) \qquad W\Big|_{y=y_1(x,z)} = 0 ,$$

for both cases. The other free boundary condition can also be derived as in the binomial problem; it turns out to be

$$(4.78) \qquad \left.\frac{\partial W}{\partial y}\right|_{y=y_1(x,z)} = 0 \begin{cases} \text{up to } O(A^{-1/3}) \text{ for case (i)} \\ \text{up to } O(A^{-1/2}) \text{ for case (ii)} \end{cases}$$

Again, due to symmetry, $W(x,y,z) = W(x,-y,z)$ and the optimum boundary $y_1(x,z)$ is symmetric with respect to the plane $y=0$. Thus it is sufficient to consider only $y > 0$.

The problem of finding the optimum boundary

along with the optimum net gain function is thus reduced to a free-boundary problem which can be stated as follows after the following transformation of variables:

$$r = \frac{z}{x} , \quad 1 \leq r < \infty ,$$

$$s = \frac{y}{x} , \quad 0 < s < \infty ,$$

$$t = \frac{1}{2x} , \quad 0 < t < \infty ,$$

$$W_1(r,s,t) = W(x,y,z) - x\,\bar{C}(x,y,z)$$

$$= W\left(\frac{1}{2t}, \frac{s}{2t}, \frac{r}{2t}\right) - \bar{C}_1(r,s,t), \text{ say} .$$

TRANSFORMED FREE-BOUNDARY PROBLEM:

To solve for the unknown boundary $s_1(r,t)$ and the unknown $W_1(r,s,t)$ where W_1 satisfies the partial differential equation:

$$(4.79)\qquad \frac{\partial^2 W_1}{\partial s^2} - \frac{\partial W_1}{\partial t} = 0 \qquad \text{in} \begin{cases} 0 < s \leq s_1(r,t) , \\ 1 \leq r < \infty , \\ 0 < t < \infty . \end{cases}$$

with the following boundary conditions:

$$(4.80)\qquad \left.\frac{\partial W_1}{\partial s}\right|_{s=0} = -\frac{1}{r} - \left.\frac{\partial \bar{C}_1}{\partial s}\right|_{s=0} ,$$

$$(4.81)\qquad \left.W_1\right|_{s=s_1(r,t)} = -\bar{C}_1(r,s_1(r,t),t) ,$$

$$(4.82)\qquad \left.\frac{\partial W_1}{\partial s}\right|_{s=s_1(r,t)} = -\left.\frac{\partial \bar{C}_1}{\partial s}\right|_{s=s_1(r,t)} .$$

Now

$$(4.83)\qquad \bar{C}_1(r,s,t) = \begin{cases} \dfrac{r}{2t} & \text{, for case (i)} \\ \dfrac{1}{t}\cdot\sqrt{2t}\left[\phi\left(\dfrac{s}{\sqrt{2t}}\right) + \dfrac{s}{\sqrt{2t}}\left\{\dfrac{1}{2} - \Phi\left(-\dfrac{s}{\sqrt{2t}}\right)\right\}\right] & \text{, for case (ii)} \end{cases}$$

the latter formula reducing to

$$\frac{2}{t}\cdot\frac{1}{\sqrt{2\pi}}\ {}_1F_1(-1/2,1/2;-s^2/4t) ,$$

using (4.45). Thus

$$(4.84)\qquad \frac{\partial \bar{C}_1}{\partial s} = \begin{cases} 0 . & \text{, for case (i)} \\ \dfrac{2}{t}\cdot\dfrac{1}{\sqrt{2\pi}}\ \dfrac{s}{2t}\ {}_1F_1(1/2,3/2;-s^2/4t) & \text{, for case (ii).} \end{cases}$$

Hence $\partial \bar{C}_1/\partial s|_{s=o}$ is 0 for both cases.

4.4 Series expansion for the optimal solution for large x (trinomial problem)

Let

$$(4.85) \qquad W_2(r,s,t) = W_1(r,s,t) + \frac{s}{r} .$$

We now consider the two cases separately.

case (i): constant constant:

To solve for the unknown $s_1(r,t)$ and $W_2(r,s,t)$ where W_2 satisfies the partial differential equation:

$$(4.86) \qquad \frac{\partial^2 W_2}{\partial s^2} - \frac{\partial W_2}{\partial t} = 0 \text{ in } 0 < s \leq s_1(r,t), 1 \leq r < \infty, 0 < t < \infty$$

with the boundary conditions:

$$(4.87) \qquad \left.\frac{\partial W_2}{\partial s}\right|_{s=o} = 0 ,$$

$$(4.88) \qquad W_2|_{s=s_1(r,t)} = -\frac{r}{2t} + \frac{s_1(r,t)}{r} ,$$

$$(4.89) \qquad \left.\frac{\partial W_2}{\partial s}\right|_{s=s_1(r,t)} = \frac{1}{r} .$$

As we are interested in solving this free boundary problem only for small t, we may assume, as in the free-boundary problem of Moriguti and Robbins [30], a series of expansion in t of the unknown boundary surface as

$$(4.90) \qquad s_1(r,t) = C_o(r) + C_1(r)t + C_2(r)\, t^2 + \ldots$$

and, hence, a solution of (4.86) with boundary condition (4.87) of the form

(4.91) $$W_2(r,s,t) = \sum_n A_n(r)\, t^n \, {}_1F_1(-n,1/2;-s^2/4t) \,,$$

where, unlike the previous situation, the coefficients $C_m(r)$ and $A_n(r)$ may now depend on r. It is possible to show as in Moriguti and Robbins' problem, that the summation in (4.91) is over integral values of n starting with $n = -1$. Moreover, the two free boundary conditions (4.88) and (4.89) are sufficient to determine the arbitrary functions $A_n(r)$ and $C_m(r)$ completely. In fact, they may be obtained successively just as before by equating the coefficients of equal powers of t from both sides of each of the following equations obtained by substituting (4.90) and (4.91) in (4.88) and (4.89):

(4.92) $$\sum_{n=-1}^{\infty} A_n(r)t^n \, {}_1F_1\Big(-n,1/2;-\Big[\sum_{m=0}^{\infty} C_m(r)t^m\Big]^2/4t\Big) =$$

$$= -\frac{r}{2t} + \frac{1}{r}\Big[\sum_{m=0}^{\infty} C_m(r)\, t^m\Big]$$

(4.93) $$\sum_{n=-1}^{\infty} 2nA_n(r)t^n \cdot \frac{1}{2t}\Big[\sum_{m=0}^{\infty} C_m(r)\, t^m\Big] {}_1F_1\Big(-n+1,3/2;-\Big[\sum_{m=0}^{\infty} C_m(r)t^m\Big]^2/4t\Big)$$

$$= \frac{1}{r} \,.$$

The results are:

$$C'_m(r) \equiv C_{3m-1}(r) \neq 0 \,, \quad m = 1,2,3,\ldots \,.$$

$$A'_n(r) \equiv A_{3n-1}(r) \neq 0 \,, \quad n = 0,1,2,\ldots \,.$$

and all other C's and A's vanish. The first few coefficients are:

$$C_1'(r) = 2r^{-2} \;, \qquad A_0'(r) = -\tfrac{1}{2}r \;,$$

$$C_2'(r) = -\frac{16}{3}\, r^{-6} \;, \qquad A_1'(r) = r^{-3}$$

$$C_3'(r) = \frac{896}{15}\, r^{-10} \;; \qquad A_2'(r) = -\frac{10}{3}\, r^{-7} \;,$$

$$\vdots \qquad\qquad A_3'(r) = \frac{148}{45}\, r^{-11} \;.$$

$$\vdots$$

Thus a series expansion in t of the optimum boundary $s_1(r,t)$ for small t may be written as:

$$(4.94) \qquad s_1(r,t) = 2\left(\frac{t}{r}\right)^2 - \frac{16}{3r}\left(\frac{t}{r}\right)^5 + \frac{896}{15r^2}\left(\frac{t}{r}\right)^8 - \dots \;.$$

which may be put in terms of the original variables (see (4.61)) as:

$$(4.95) \qquad y_1(x,z) = x\, s_1\left(\frac{z}{x}, \frac{1}{2x}\right) = \frac{1}{2z}\left(\frac{x}{z}\right)\left[1 - \frac{1}{3z^3}\left(\frac{x}{z}\right) + \frac{7}{15z^6}\left(\frac{x}{z}\right)^2 - \dots\right]$$

$$= \frac{1}{2x}\left(\frac{x}{z}\right)^2\left[1 - \frac{1}{3x^3}\left(\frac{x}{z}\right)^4 + \frac{7}{15x^6}\left(\frac{x}{z}\right)^8 - \dots\right]$$

whichever form is convenient. As $z \to x$, both, however, reduce to

$$(4.96) \qquad y_1(x,x) = \frac{1}{2x} - \frac{1}{6x^4} + \frac{7}{30\,x^7} - \dots$$

A more physically meaningful reparametrization of the boundary surface may be given in terms of the parameter

$$(4.97) \qquad \varpi = \frac{1}{r} = \frac{x}{z} \;, \quad 0 < \varpi \leq 1$$

(the proportion of untied observations, in terms of the original double dichotomies problem) as:

$$(4.98)\quad y_1^*(z,\varpi) = \frac{\varpi}{2z} - \frac{\varpi^2}{6z^4} + \frac{7\varpi^3}{30\,z^7} - \dots \text{ (for large } z\text{)}$$

z denoting the total amount of sampling.

A series expansion for the net gain function (4.68) for large x may be written as

$$(4.99)\quad W(x,y,z) = z - \frac{y}{z} - z\ {}_1F_1(1,1/2;-y^2/2x) + \frac{x}{4z^3}\ {}_1F_1(2,1/2;-y^2/2x)$$
$$- \frac{5x^2}{48z^7}\ {}_1F_1(5,1/2;-y^2/2x) + \frac{37x^3}{2880z^{11}}\ {}_1F_1(8,1/2;-y^2/2x) - \dots .$$

Thus

$$(4.100)\quad W(x,0,z) = \frac{x}{4z^3} - \frac{5x^2}{48z^7} + \frac{37\,x^3}{2880z^{11}} - \dots .$$

In the special case $z = x$,

$$(4.101)\quad W(x,0,x) = \frac{1}{4x^2} - \frac{5}{48x^5} + \frac{37}{2880x^8} - \dots .$$

case (ii): absolute deviation cost:

In this case W_2 satisfies (4.86) with the boundary condition (4.87) same as in case (i). Only the free-boundary conditions (4.88) and (4.89) are changed to:

$$(4.102)\quad W_2\Big|_{s=s_1(r,t)} = -\frac{2}{t}\cdot\frac{1}{\sqrt{2t}}\ {}_1F_1(-1/2,1/2;-s_1^2(r,t)/4t)+s_1(r,t)/r,$$

$$(4.103)\quad \frac{\partial W_2}{\partial s}\Big|_{s=s_1(r,t)} = -\frac{2}{t}\cdot\frac{1}{\sqrt{2t}}\,\frac{s_1(r,t)}{2t}\ {}_1F_1(1/2,3/2;-s_1^2(r,t)/4t) + 1/r\ ,$$

respectively. For small t, a series expansion in t of the solution may be taken as

$$(4.104) \qquad s_1(r,t) = \sum_m C_m(r)\, t^m$$

$$(4.105) \qquad W_2(r,s,t) = \sum_n A_n(r)\, t^n \; {}_1F_1(-n,1/2;-s^2/4t)$$

where m and n are half integers and the summations starting from $m=3/2$ in (4.104) and $n = -1/2$ in (4.105), just as in the corresponding binomial situation. Substituting (4.104) and (4.105) in (4.102) and (4.103), and equating coefficients of equal powers of t from both sides of the resulting equations, the unknown coefficients $C_m(r)$'s and $A_n(r)$'s may be determined successively. It is found that

$$C'_m(r) \equiv C_{(4m-1)/2}(r) \neq 0 \;, \; m = 1,2,3,\ldots \;,$$

$$A'_n(r) \equiv A_{(4n-1)/2}(r) \neq 0 \;, \; n = 0,1,2,3,\ldots,$$

and all other C_n's and A_n's are zero. The first few non-zero coefficients are:

$$C'_1(r) = \frac{\sqrt{\pi}}{r} \;, \qquad A'_0(r) = -\frac{1}{\sqrt{\pi}} \;,$$

$$C'_2(r) = -\frac{7}{12}\left(\frac{\sqrt{\pi}}{r}\right)^3 , \qquad A'_1(r) = \frac{1}{2\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{r}\right)^2 ,$$

$$C'_3(r) = \frac{197}{160}\left(\frac{\sqrt{\pi}}{r}\right)^5 . \qquad A'_2(r) = -\frac{1}{3\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{r}\right)^4 ,$$

$$A'_3(r) = \frac{1058}{1440\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{r}\right)^6 .$$

Thus a series expansion in t of the optimum boundary $s_1(r,t)$ for small t may be written as:

$$(4.106) \quad s_1(r,t) = t^{3/2}\left[\frac{\sqrt{\pi}}{r} - \frac{7}{12}\left(\frac{\sqrt{\pi}}{r}\right)^3 t^2 + \frac{197}{160}\left(\frac{\sqrt{\pi}}{r}\right)^5 t^4 + \ldots\right],$$

which may be put in terms of the original variables (see (4.62)) as:

$$(4.107)\quad y_1(x,z) = \left(\frac{x}{z}\right)^{1/2} \cdot \frac{\sqrt{2\pi}}{4}\ \frac{1}{\sqrt{z}}\left[1 - \frac{7\pi}{48}\ \frac{1}{z^2} + \frac{197\pi^2}{160\cdot 16}\ \frac{1}{z^4} - \ldots\right],$$

or, alternatively, in terms of ϖ, the proportion of untied observations, and z, the total amount of sampling, as

$$(4.108)\quad y_1^*(z,\varpi) = \frac{\sqrt{2\pi\varpi}}{4}\ \frac{1}{\sqrt{z}}\left[1 - \frac{7\pi}{48}\ \frac{1}{z^2} + \frac{197\pi^2}{160\cdot 16}\ \frac{1}{z^4} - \ldots\right],$$

or, still, alternatively, in terms of ϖ and x, as

$$(4.109)\quad y_1^o(x,\varpi) = \frac{\sqrt{2\pi}}{4}\ \frac{\varpi}{\sqrt{x}}\left[1 - \frac{7\pi}{48}\ \frac{\varpi^2}{x^2} + \frac{197\pi^2}{160\cdot 16}\ \frac{\varpi^4}{x^4} - \ldots\right].$$

All three forms coincide as $z \to x$ i.e., as $\varpi \to 1$ to

$$(4.110)\quad y_1(x,x) = y_1^*(x,1) = y_1^o(x,1) = \frac{\sqrt{2\pi}}{4}\ \frac{1}{\sqrt{x}}\left[1 - \frac{7\pi}{48}\ \frac{1}{x^2} + \frac{197\pi^2}{160\cdot 16}\ \frac{1}{x^4} - \ldots\right]$$

The net gain function turns out as

$$\begin{aligned}(4.111)\quad W(x,y,z) = &-\frac{y}{z} + 2y\left[\frac{1}{2} - \Phi\left(-\frac{y}{\sqrt{x}}\right)\right] \\ &+ \frac{2\sqrt{2\pi}}{4^2}\left(\frac{x}{z}\right)^2 \frac{1}{x^{3/2}}\ {}_1F_1(-3/2,1/2;-y^2/2x) \\ &- \frac{2(2\pi)^{3/2}}{3\cdot 4^3}\left(\frac{x}{z}\right)^4 \frac{1}{x^{7/2}}\ {}_1F_1(-7/2,1/2;-y^2/2x) \\ &+ \frac{2(2\pi)^{5/2}\,529}{1440\cdot 4^4}\left(\frac{x}{z}\right)^6 \frac{1}{x^{11/2}}\ {}_1F_1(-11/2,1/2;-y^2/2x) \\ &- \ldots\ .\end{aligned}$$

Thus

$$(4.112)\quad W(x,0,z) = \frac{2\sqrt{2\pi}}{4^2}\left(\frac{x}{z}\right)^2 \frac{1}{x^{3/2}} - \frac{2(2\pi)^{3/2}}{3 \cdot 4^3}\left(\frac{x}{z}\right)^4 \frac{1}{x^{7/2}} + \frac{2(2\pi)^{5/2}\,529}{1440 \cdot 4^4}\left(\frac{x}{z}\right)^6 \frac{1}{x^{11/2}} - \cdots$$

In the special case, $z = x$,

$$(4.113)\quad W(x,0,x) = \frac{2\sqrt{2\pi}}{4^2}\,\frac{1}{x^{3/2}} - \frac{2(2\pi)^{3/2}}{3 \cdot 4^3}\,\frac{1}{x^{7/2}} + \frac{2(2\pi)^{5/2}\,529}{1440 \cdot 4^4}\,\frac{1}{x^{11/2}} - \cdots .$$

It may be noted that (4.110) is the same as (4.54), the expansion for the boundary $y_1(x)$ in the binomial problem with absolute deviation cost. Also, $W(x,y,x)$ and $W(x,0,x)$ in (4.113) are found to be double the corresponding expressions (4.55) and (4.56) for the later situation.

CHAPTER V

BOUNDS ON ASYMPTOTIC SOLUTIONS

5.0 Introduction and summary.

In Chapter IV we obtained a formal expansion in negative powers of x (for large x) of the (normalized) optimal (Bayes) boundary, y_1 and that of the corresponding (normalized) net gain function W as $A \to \infty$ for the various sequential decision problems considered in Chapter II. It is, however, very difficult to justify these formal expansions. The expansion obtained by Moriguti and Robbins [30] for their problem (involving binomial model with constant cost) has been shown by Breakwell and Chernoff [12] to be an asymptotic one as $x \to \infty$ in the sense that the error is small compared to the last term. Their proof is rather involved. We shall consider in this chapter a less ambitious objective of obtaining some bounds on the true boundary using Bather's [8] method. These bounds are valid for all x, large and small. In fact, they converge to the true boundary when $x \to \infty$ as well as when $x \to 0$. In as much as these bounds provide even approximately the true boundary for intermediate values of x, they are useful in themselves. Besides, in so far as the expansions of these bounds for large x are consistent with the expansions obtained in Chapter IV, they provide an independent check on them.

As we are going to follow Bather's methods rather closely, we try to keep matters simpler by retaining his notations as far as possible consistent with our previous ones. Bather's results, like Chernoff's [14], are concerned with a Gaussian process and deal with the risk function rather than the gain function approach of Moriguti and Robbins [30] to characterize the optimum (Bayes) sampling rule. Since the optimum boundary remains the same for either approach, our choice in the various studies has been prompted by the relative advantages one offers over the other. In Chapters II and III we concerned ourselves with the risk function as far as the general theory developed with respect to it [9, 24] could take us and then changed to the gain function for some obvious reasons. The recursion relations developed in Chapter III could only be obtained in terms of the gain function. In Chapter II, the computations of the optimum boundary are easier with the recursion relations involving the gain function rather than with those involving the risk function. In Chapter IV, we retained the gain function in deriving the partial differential equation and the various boundary conditions that it satisfies just to be consistent with Moriguti and Robbins, but this portion could equally well have been carried out in terms of the risk function with slight modifications of the resulting partial differential equations (viz., changing the sign of $\bar{C}$, etc.) and the boundary conditions. For the asymptotic theory the two approaches are thus reversible. In order to adopt Bather's technique, it is, however, imperative that one should consider the risk function rather than the gain function, because some of his arguments hinge critically on the particular functional form that the risk function takes on the boundary. In the following analysis, therefore, we adapt the derivation of the various differential equations along with the boundary conditions in terms of the normalized risk function. These could, however, be easily anticipated for the binomial problem with absolute deviation cost, from Chernoff's results for the Gaussian process problem with constant cost. Under the particular type of limiting tendencies that we are considering, the trinomial problem would have reduced to a certain (rather specialized) bivariate Gaussian process problem, just as the binomial one reduces to the univariate Gaussian one. We have, however, no special reasons for considering this bivariate Gaussian process problem in order to derive the partial differential equations and the boundary conditions satisfied by the standardized risk function for the trinomial problem. They are simply obtained by modifying, as in the binomial problem, the corresponding expressions in terms of the gain function that are derived in Chapter IV.

In section 5.1, we first consider the binomial problem with absolute deviation cost. The associated free-boundary problem (FBP) is stated in terms of the normalized risk, $\bar{\rho}(x,y)$[4]. Transforming the variables, the FBP is put in the appropriate form. The upper bound $\bar{\lambda}(t)$ for the true boundary $\bar{\sigma}(t)$ and its expansion for large t are obtained. A similar development is given for the lower bound $\bar{\rho}(t)$[4]. Section 5.2 is concerned with the corresponding developments for the trinomial problem with both types of cost.

[4] The ambiguous use of the same symbol $\bar{\rho}$ for two different things may be resolved by referring to the arguments.

5.1 Binomial problem with absolute deviation cost.

We have seen in Chapter II, (2.60) that the optimum boundary can be characterized by the Bayes risk $\rho(a,b)$ which satisfies in the go-region the recursion relation

$$\rho(a,b) = \frac{a}{a+b}\rho(a+1,b) + \frac{b}{a+b}\rho(a,b+1) + C(a,b)$$

where

$$C(a,b) = E[\,|p - 1/2|\;|a,b\,] \;;$$

and assumes on and outside the boundary the value

$$\rho_o(a,b) = A\,E[\,|p - 1/2|\,|a,b\,] - A\,\frac{|a-b|}{2(a+b)}$$

(5.1) $$= A\,C(a,b) - A\,\frac{|a-b|}{2(a+b)} \quad .$$

Using the appropriate normalization (4.62) for this case, namely,

(5.2) $$x = \frac{a+b}{A} \quad ,$$

(5.3) $$y = \frac{a-b}{A^{1/2}} \quad ,$$

$$\bar{\rho}(x,y) = \frac{\rho(a,b)}{A^{1/2}} \quad ;$$

and modifying the results of section 4.1, we see that, under L, the normalized risk $\bar{\rho}(x,y)$ satisfies the partial differential equation

$$\frac{1}{2}\frac{\partial^2 \bar{\rho}}{\partial y^2} + \frac{y}{x}\frac{\partial \bar{\rho}}{\partial y} + \frac{\partial \bar{\rho}}{\partial x} = -\bar{C}(x,y) \quad ,$$

in the symmetric region

$$-y_1(x) < y < y_1(x) \quad , \qquad 0 < x < \infty \quad ,$$

where $y_1(x)$ is the unknown boundary on which bounds are desired; the boundary conditions are

$$\frac{\partial \bar{\rho}}{\partial y} = 0 \qquad \text{on} \quad y = 0$$

$$\left.\begin{aligned} \bar{\rho} &= \bar{\rho}_o \qquad (5.4) \\ \frac{\partial \bar{\rho}}{\partial y} &= \frac{\partial \bar{\rho}_o}{\partial y} \qquad (5.5) \end{aligned}\right\} \quad \text{on the unknown boundary}$$

where

$$\bar{\rho}_o(x,y) = \rho_o(a,b)/A^{1/2} \quad . \tag{5.6}$$

Using (5.1), (5.2), (5.3) , (5.4) and Lemma 4.1, we may write

$$\bar{\rho}_o(x,y) = \bar{C}(x,y) - |y|/2x$$

$$= \frac{1}{\sqrt{x}} \left\{ \phi\left(\frac{y}{\sqrt{x}}\right) - \frac{|y|}{\sqrt{x}} \, \Phi\left(- \frac{|y|}{\sqrt{x}} \right) \right\} .$$

Considering only the upper boundary, (5.4) and (5.5) become

$$\bar{\rho}(x,y_1(x)) = \bar{\rho}_o(x,y_1(x)) \quad ,$$

$$\bar{\rho}'(x,y_1(x)) = \bar{\rho}_o'(x,y_1(x)) \quad ;$$

where the prime denotes differentiation with respect to the second argument.

Let us now put the above facts in Bather's notation. Let

$$s = \frac{y}{\sqrt{x}} \quad ,$$

$$t = x \quad ,$$

$$f(s,t) = \bar{\rho}(x,y) \quad ,$$

$$K(s,t) = \bar{\rho}_o(x,y) \quad ,$$

$$c(s,t) = \bar{C}(x,y) ,$$

$$\bar{\sigma}(t) = y_1(x)/\sqrt{x} .$$

Then $f(s,t)$ satisfies the partial differential equation:

$$(5.7) \qquad \frac{\partial^2 f}{\partial s^2} + s\frac{\partial f}{\partial s} + 2t \left[c(s,t) + \frac{\partial f}{\partial t} \right] = 0$$

in

$$|s| < \bar{\sigma}(t)$$

with the following boundary conditions:

$$(5.8) \qquad \frac{\partial f}{\partial s} = 0 \quad \text{on} \quad s = 0$$

due to symmetry ,

$$f(s,t) = K(s,t)$$

$$(5.9) \qquad = \frac{1}{\sqrt{t}}\left\{ \phi(s) - |s|\,\Phi(-|s|) \right\} \equiv t^{-1/2}L(s), \text{ say}$$

whenever

$$|s| \geq \bar{\sigma}(t) , \qquad 0 < t < \infty ,$$

and finally

$$\frac{\partial f}{\partial s}(\bar{\sigma}(t),t) = \frac{\partial K}{\partial s}(\bar{\sigma}(t),t)$$

$$(5.10) \qquad = \frac{1}{\sqrt{t}} \cdot -\Phi(-\bar{\sigma}(t)) .$$

Due to symmetry again, we need consider only $s \geq 0$.

Upper bound:

Following Bather, we consider the following family of separable solutions of (5.7):

$$f(s,t) = t^r g(s) , \quad r \text{ any real constant.}$$

It follows from (5.9) that $r = -1/2$. Hence $g(s)$ must satisfy the ordinary differential equation

$$g''(s) + s\,g'(s) - g(s) = -2t^{3/2}\,c(s,t)$$

(5.11)

$$= -2t\left\{\phi(s) + s\left[\tfrac{1}{2} - \Phi(-s)\right]\right\}$$

in order that $t^{-1/2}g(s)$ is a solution of (5.7). Now (5.11) does not make sense as the right hand side involves t. In order to make sense we consider for the present an auxiliary problem where everything remains the same as in the original problem except that the cost function is now given by

$$c_1(s,t) = \frac{a}{t}\,c(s,t)$$

where a is some positive constant. The optimal continuation region for this auxiliary problem then is bounded by straight lines, viz.,

$$\mathcal{C}_1 = \left\{(s,t)\ ;\ -\bar{\lambda} < s < \bar{\lambda}\right\}.$$

The unknown constant $\bar{\lambda}$ and $g(s)$ are found from the following facts:

$g(s)$ satisfies the differential equation

(5.12) $$g'' + sg' - g = -2a\left\{\phi(s) + s\left[\tfrac{1}{2} - \Phi(-s)\right]\right\}$$

in $\mathcal{C}_1$ with the boundary conditions:

(5.13) $$g'(0) = 0$$

(5.14) $$g(\bar{\lambda}) = L(\bar{\lambda}),$$

(5.15) $$g'(\bar{\lambda}) = L'(\bar{\lambda}).$$

Now the general symmetrical solution of the homogeneous equation

$$g'' + sg' - g = 0$$

may be written as

$$g_o(s) = \gamma \left[\phi(s) + s\left\{\frac{1}{2} - \Phi(-s)\right\} \right]$$

where γ is an arbitrary constant. A particular solution of (5.12) with (5.13) is given by

$$g_1(s) = a\,\phi(s) - 2a\,\psi_1(s) \quad ,$$

where

(5.16) $$\psi_1(s) = s\int_0^s \frac{1}{u^2}\, e^{-u^2/2}\, du \int_0^u v^2\, e^{v^2/2} dv \int_0^v \phi(w)\, dw \quad .$$

Let

$$h(s) = g_o(s) + g_1(s)$$

be the general solution. Then the two constants γ and $\bar{\lambda}$ are determined from (5.14) and (5.15), i.e., from

(5.17) $$h(\bar{\lambda}) = L(\bar{\lambda}) \quad ,$$

(5.18) $$h'(\bar{\lambda}) = L'(\bar{\lambda}) \quad .$$

Writing (5.17) and (5.18) explicitly, we have

(5.19) $$\gamma\left[\phi(\bar{\lambda}) + \bar{\lambda}\left\{\frac{1}{2} - \Phi(-\bar{\lambda})\right\}\right] + a\,\phi(\bar{\lambda}) - 2a\,\psi_1(\bar{\lambda}) = \phi(\bar{\lambda}) - \bar{\lambda}\,\Phi(-\bar{\lambda})$$

and

(5.20) $$\gamma\left[\frac{1}{2} - \Phi(-\bar{\lambda})\right] - a\,\bar{\lambda}\,\phi(\bar{\lambda}) - 2a\,\psi_1'(\bar{\lambda}) = -\Phi(-\bar{\lambda})$$

Eliminating γ from (5.19) and (5.20), we get

(5.21) $$\bar{\lambda}\,\phi(\bar{\lambda}) + 2\,\psi_1'(\bar{\lambda}) + \left[\frac{1}{2} - \Phi(-\bar{\lambda})\right]\left\{1 + \bar{\lambda}^2 + \frac{2}{\phi(\bar{\lambda})}\left[\bar{\lambda}\,\psi_1'(\bar{\lambda}) - \psi_1(\bar{\lambda})\right]\right\} = \frac{1}{2a}$$

which determines the critical level $\bar{\lambda}$ for each $a > 0$ uniquely. Uniqueness follows from the positivity of the left hand side of (5.21) which is a consequence of the definitions of the various functions involved therein.

If we now replace the constant a by the running variable t, then the resulting solution $\bar{\lambda}(t)$ will constitute, following Bather, an upper bound on $\bar{\sigma}(t)$. It may be verified, moreover, that $\bar{\lambda}(t)$ is decreasing in t. To show this it is sufficient to show that the left hand side of (5.21) is increasing in $\bar{\lambda}$. This can be proved in its turn by direct differentiation and by utilization of the differential equation that $\psi_1(s)$ satisfies, namely,

$$\psi_1''(s) + s\,\psi_1'(s) - \psi_1(s) = s\left[\tfrac{1}{2} - \Phi(-s)\right] .$$

It aan also be seen from (5.21) that $\bar{\lambda}(t) \longrightarrow \infty$ and $t \longrightarrow 0$ and $\bar{\lambda}(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

It is thus possible, in principle, to obtain the actual values of $\bar{\lambda}(t)$ by solving (5.21) for $\bar{\lambda}$ after substituting $a = t$. In practice, however, difficulty arises because of the presence of terms involving the untabulated functions ψ_1 and ψ_1'. Now it follows directly from the definition (5.16) of ψ_1 that

$$\psi_1'(\bar{\lambda}) \geq 0 ,$$

and

$$\bar{\lambda}\,\psi_1'(\bar{\lambda}) - \psi_1(\bar{\lambda}) \geq 0 .$$

It can be seen from (5.21) that if some positive terms are omitted from its left hand side, the resulting solution, $\breve{\lambda}(t)$, say, can only exceed $\bar{\lambda}(t)$. Thus an upper bound to the upper bound $\bar{\lambda}(t)$ may be obtained by solving the equation:

$$(5.22)\quad \tilde{\lambda}(t)\,\phi(\tilde{\lambda}(t)) + [\tfrac{1}{2} - \phi(-\tilde{\lambda}(t))][1 + \tilde{\lambda}^2(t)] = \frac{1}{2t}\ .$$

$\tilde{\lambda}(t)$, unlike $\bar{\lambda}(t)$, is amenable to computation using the normal integral tables. It may also be verified from (5.22) that $\tilde{\lambda}(t)$ is decreasing in t and that $\tilde{\lambda}(t) \longrightarrow \infty$ as $t \longrightarrow 0$ and $\tilde{\lambda}(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Asymptotic form of upper bound for large t.

Lemma 5.1

$$\psi_1(s) = O(s^4)$$

This follows readily from the definition (5.16) of $\psi_1(s)$ using the following transformation:

$$w' = \frac{w}{v}\ ,\quad v' = \frac{v}{u}\ ,\quad u' = \frac{u}{s}\ ,\quad 0 \le w',\ v',\ u' \le 1$$

In fact, one finds that

$$(5.23)\quad \sqrt{2\pi}\ \psi_1(s) = s^4 \int_0^1 \int_0^1 \int_0^1 u'^2\, v'^3\, e^{-s^2/2[u'^2\{1-v'^2(1-w'^2)\}]}\, dw'\, dv'\, du' .$$

$$= M_4\, s^4 + M_6\, s^6 + \dots\ .$$

$$= O(s^4)\ ,$$

where the coefficients M_4, M_6, ... may be determined by expanding the integrand in (5.23) and integrating term by term.

Using Lemma 5.1, it can be shown that for large t,

$$(5.24)\quad \bar{\lambda}(t) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{t} - \frac{1}{2}\left(\frac{\sqrt{2\pi}}{4}\right)^3 \frac{1}{t^3} + o\left(\frac{1}{t^3}\right)\ ,$$

and

$$\check{\lambda}(t) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{t} - \frac{1}{6}\left(\frac{\sqrt{2\pi}}{4}\right)^3 \frac{1}{t^3} + o\left(\frac{1}{t^3}\right) .$$

For large t, we do not lose much by using the computationally more convenient form $\check{\lambda}(t)$ instead of $\bar{\lambda}(t)$.

Lower bound:

Without going into the details of the arguments that lead to the inner bound, for which we refer to Bather, we just indicate the pertinent steps relevant for our case only. Following Bather, we consider here the following family of solutions satisfying (5.7) and (5.8) :

$$(5.25) \qquad F_\alpha(s,t) = -\sqrt{t}\, s\left[\tfrac{1}{2} - \Phi(-s)\right] + \sqrt{t}\,\alpha\,\phi(s) .$$

Eliminating the parameter α from the two equations below obtained from the boundary conditions (5.9) and (5.10) ,

$$F_\alpha(\bar{\rho}(t),t) = K(\bar{\rho}(t),t) ,$$

$$F'_\alpha(\bar{\rho}(t),t) = K'(\bar{\rho}(t),t) ,$$

the inner bound, $\bar{\rho}(t)$ is given by the equation:

$$(5.26) \qquad \frac{\bar{\rho}(t)\phi(\bar{\rho}(t)) + \left[\tfrac{1}{2} - \Phi(-\bar{\rho}(t))\right]\left[1 + \bar{\rho}^2(t)\right]}{2\left[1 + \bar{\rho}^2(t)\right]\phi(\bar{\rho}(t))\left[\dfrac{\Phi(-\bar{\rho}(t))}{\phi(\bar{\rho}(t))} - \dfrac{\bar{\rho}(t)}{1+\bar{\rho}^2(t)}\right]} = \frac{1}{2t} .$$

The existence and positivity of $\bar{\rho}(t)$ follows from the positivity of the left hand side of (5.26) that is obtained by using Gordon's inequality [20] on Mill's ratio, viz.,

$$\frac{\Phi(-x)}{\phi(x)} \geq \frac{x}{1+x^2} \quad , \quad x \geq 0$$

equality obtaining as $x \to \infty$.

It may be verified from (5.26) that $\bar{\rho}(t)$ is decreasing in t by showing that the numerator and the denominator of the left hand side of (5.26) are, respectively, increasing and decreasing in $\bar{\rho}$. It may also be verified that $\bar{\rho}(t) \to \infty$ as $t \to 0$ and $\bar{\rho}(t) \to 0$ as $t \to \infty$. From (5.26) again the asymptotic form of $\bar{\rho}(t)$ for large t turns out as follows:

$$(5.27) \quad \bar{\rho}(t) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{t} - \frac{\sqrt{2\pi}}{4}\,\frac{1}{t^2} + o\left(\frac{1}{t^3}\right) .$$

Now writing (5.24) and (5.27) in terms of the original variables x and y given by (5.2) and (5.3) respectively, the bounds on the optimum upper boundary for large x may be written as

$$\bar{y}_1(x) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{\sqrt{x}}\left[1 - \frac{\pi}{16}\,\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right] ,$$

and

$$\underline{y}_1(x) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{\sqrt{x}}\left[1 - \frac{1}{x} + o\left(\frac{1}{x^2}\right)\right]$$

whereas the exact upper boundary (4.54) from Chapter IV is

$$y_1(x) = \frac{\sqrt{2\pi}}{4}\,\frac{1}{\sqrt{x}}\left[1 - \frac{7\pi}{48}\,\frac{1}{x^2} + o\left(\frac{1}{x^4}\right)\right]$$

It is to be noted that the first order term is the same in all three expansions above and that $y_1(x)$ lies between $\bar{y}_1(x)$ and $\underline{y}_1(x)$ for sufficiently large x.

5.2 Trinomial problem

As already noted in Chapter II, (2.100), the optimum boundary can be characterized with the help of the Bayes risk $\rho(a,b,c)$ which satisfies in the go-region the recursion relation

$$\rho(a,b,c) = \frac{a}{a+b+c}\,\rho(a+1,b,c) + \frac{b}{a+b+c}\,\rho(a,b+1,c) + \frac{c}{a+b+c}\,\rho(a,b,c+1)$$

$$+ \; C(a,b,c) \; ,$$

where

$$C(a,b,c) = \begin{cases} 1 \; , & \text{for constant cost ,} \\ E[\,|\pi_1-\pi_2|\,|a,b,c] \; , & \text{for absolute deviation cost ;} \end{cases}$$

and assumes on and outside the boundary the value

$$\rho_o(a,b,c) = A\, E[\,|\pi_1-\pi_2|\,|a,b,c] - A\,\frac{|a-b|}{a+b+c} \; .$$

Considering the two different types of normalizations for the two types of cost, viz.,

$$x = (a+b)/A^{2/3} \; ,$$
$$y = (a-b)/A^{1/3} \; ,$$
$$z = (a+b+c)/A^{2/3} \; ;$$

$$\bar{\rho}(x,y,z) = \rho(a,b,c)/A^{2/3} \; ,$$
$$\bar{\rho}_o(x,y,z) = \rho_o(a.b.c)/A^{2/3} \; ;$$

for (i) constant cost, and

$$x = (a+b)/A ,$$
$$y = (a-b)/A^{1/2} ,$$
$$z = (a+b+c)/A ;$$
$$\bar{\rho}(x,y,z) = \rho(a,b,c)/A^{1/2} ,$$
$$\bar{\rho}_o(x,y,z) = \bar{\rho}_o(a,b,c)/A^{1/2} ;$$

for (ii) absolute deviation cost, using Lemma 4.3 with $\alpha = 1/3$ for case (i) and with $\alpha = 1/2$ for case (ii), and re-expressing the results of section 4.3 in terms of the normalized Bayes risk $\bar{\rho}(x,y,z)$, we have the following:

Under the appropriate limiting tendencies for the two types of cost, $\bar{\rho}(x,y,z)$ satisfies the partial differentiation equation

$$(5.28) \quad \frac{x}{2z} \frac{\partial^2 \bar{\rho}}{\partial y^2} + \frac{y}{z} \frac{\partial \bar{\rho}}{\partial y} + \frac{x}{z} \frac{\partial \bar{\rho}}{\partial x} + \frac{\partial \bar{\rho}}{\partial z} = - \bar{C}(x,y,z)$$

in the region (symmetric with respect to $y=o$):

$$- y_1(x,z) < y < y_1(x,z) , \qquad 0 < x \leq z < \infty \quad ;$$

$y_1(x,z)$ being the unknown upper boundary surface on which bounds are desired; and where

$$\bar{C}(x,y,z) = \begin{cases} 1 & \text{for case (i)} , \\ \frac{|y|}{z} + \frac{2\sqrt{x}}{z} \left\{ \phi\left(\frac{|y|}{\sqrt{x}}\right) - \frac{|y|}{\sqrt{x}} \, \Phi\left(- \frac{|y|}{\sqrt{x}}\right) \right\} & \text{for case (ii)} . \end{cases}$$

Further, $\bar{\rho}(x,y,z)$ satisfies the following boundary conditions:

(5.29) $$\frac{\partial \bar{\rho}}{\partial y} = 0 \quad \text{on } y=0 \text{ due to symmetry ,}$$

(5.30) $$\bar{\rho} = \bar{\rho}_o$$

(5.31) $$\frac{\partial \bar{\rho}}{\partial y} = \frac{\partial \bar{\rho}_o}{\partial y}$$

on the unknown boundary surface ;

where

$$\bar{\rho}_o(x,y,z) = \frac{2\sqrt{x}}{z} \left\{ \phi\left(\frac{y}{\sqrt{x}}\right) - \frac{|y|}{\sqrt{x}} \Phi\left(-\frac{|y|}{\sqrt{x}}\right) \right\} .$$

Considering only the upper boundary $y_1(x,z)$, (5.30) and (5.31) may be written explicitly as

(5.32) $$\bar{\rho}(x,y_1(x,z),z) = \bar{\rho}_o(x,y_1(x,z),z)$$

(5.33) $$\bar{\rho}'(x,y_1(x,z),z) = \bar{\rho}_o'(x,y_1(x,z),z)$$

where the prime denotes differentiation with respect to the second argument. We now make the following transformation

$$r = \frac{z}{x} \quad , \quad 1 \leq r < \infty$$

$$s = \frac{y}{\sqrt{x}} \quad , \quad -\infty < s < \infty$$

$$t = x \quad , \quad 0 < t < \infty$$

$$f(r,s,t) \equiv \bar{\rho}(x,y,z)$$

(5.34) $$K(r,s,t) \equiv \bar{\rho}_o(x,y,z) = \frac{2}{r\sqrt{t}} \left\{ \phi(s) - |s| \Phi(-|s|) \right\}$$

$$c(r,s,t) \equiv \frac{z}{x} \bar{C}(x,y,z)$$

$$(5.35) \quad = \begin{cases} r & \text{, for case (i)} \\ \frac{2}{\sqrt{t}} \left\{ \phi(s) + |s| \left[\frac{1}{2} - \Phi(-|s|) \right] \right\} & \text{, for case (ii)} \end{cases}$$

$$\bar{\sigma}(r,t) \equiv \frac{y_1(x,z)}{\sqrt{x}} \quad .$$

It follows from (5.28), (5.29), (5.30) and (5.31) that $f(r,s,t)$ satisfies the partial differential equation:

$$(5.36) \qquad \frac{\partial^2 f}{\partial s^2} + s \frac{\partial f}{\partial s} + 2t \left[c(r,s,t) + \frac{\partial f}{\partial t} \right] = 0$$

in

$$|s| < \bar{\sigma}(r,t) \,, \quad 1 \leq r < \infty \,, \quad 0 < t < \infty \,,$$

with the following boundary conditions:

$$(5.37) \qquad \frac{\partial f}{\partial s} = 0 \quad \text{on } s = 0 \text{ due to symmetry} \,,$$

$$f(r,s,t) = K(r,s,t) \text{ whenever } |s| \geq \bar{\sigma}(r,t) \,,$$

and

$$\frac{\partial f}{\partial s}(r, \bar{\sigma}(r,t), t) = \frac{\partial K}{\partial s}(r, \bar{\sigma}(r,t), t)$$

$$= \frac{2}{r\sqrt{t}} \cdot - \Phi(-\bar{\sigma}(r,t)) \text{ by (5.34)} \,.$$

Again, due to symmetry, $f(r,s,t) = f(r,-s,t)$, and it is sufficient to consider only $s \geq 0$.

case (i): constant cost:

Upper bound:

Because of the particular separable form (5.34) of $K(r,s,t)$ we seek solutions of the form

$$(5.38) \qquad f(r,s,t) = \frac{1}{r\sqrt{t}} \; g(s)$$

Substituting (5.38) in (5.36), we see that $g(s)$ should satisfy the ordinary differential equation

$$(5.39) \qquad g'' + sg' - g = -\,2rt^{3/2} \; c(r,s,t)$$

Remembering that

$$c(r,s,t) = r$$

for this case, we are led to consider the auxiliary problem with the new cost function as

$$c_1(r,s,t) = \frac{a}{r^2 \, t^{3/2}}$$

where a is an arbitrary positive constant. Now proceeding as in the binomial problem, an upper bound $\bar{\lambda}(r,t)$ on $\bar{\sigma}(r,t)$ is obtained from the following equation:

$$(5.40) \qquad e^{\bar{\lambda}^2(r,t)} \int_0^{\bar{\lambda}(r,t)} e^{-y^2/2} \, dy = \frac{1}{2r^2 \, t^{3/2}} \; .$$

From (5.40) one can easily deduce the following facts:

(5.41) 1. $\bar{\lambda}$ is decreasing in r as well as in t.

(5.42) 2. For a fixed $r_o \geq 1$,

$$\bar{\lambda}(r_o,t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \; ,$$

$$\bar{\lambda}(r_o,t) \longrightarrow \infty \text{ as } t \longrightarrow 0 \; .$$

(5.43) 3. For any two fixed $1 \leq r_1 \leq r_2$,

$$\bar{\lambda}(r_1,t) \geq \bar{\lambda}(r_2,t) , \quad 0 < t < \infty ,$$

and hence

$$\bar{\lambda}(r,t) \leq \bar{\lambda}(1, t) , \quad r \geq 1, \ 0 < t < \infty .$$

4. For large tr, an asymptotic form of $\bar{\lambda}$ is given by

$$\text{(5.44)} \quad \bar{\lambda}(r,t) = \frac{1}{2r^2 t^{3/2}} \left[1 - \frac{1}{12} \left(\frac{1}{r^2 t^{3/2}} \right)^2 + O\left(\frac{1}{r^2 t^{3/2}} \right)^4 \right] .$$

Lower bound:

We consider the following family of curves satisfying (5.36) and (5.37):

$$\text{(5.45)} \quad F_\alpha(r,s,t) = - rt\, \psi(s) + \alpha(r) \sqrt{t}\, e^{-s^2/2} ,$$

where the parameter α is now allowed to depend on r, and

$$\psi(s) = s\, e^{-s^2/2} \int_0^s e^{x^2/2}\, dx .$$

The boundary conditions (5.32) and (5.33) for the family (5.45) involve the inner bound $\bar{\rho}(r,t)$ as can be shown similarly to the binomial problem, using the arguments of Bather, as follows:

$$\text{(5.46)} \quad F_\alpha(r, \bar{\rho}(r,t),t) = K(r, \bar{\rho}(r,t), t) ,$$

$$\text{(5.47)} \quad F'_\alpha(r, \bar{\rho}(r,t),t) = K'(r,\bar{\rho}(r,t),t) .$$

Eliminating the parameter $\alpha(r)$ from (5.46) and (5.47), using (5.34) and the relation

$$\psi(s) - s\,\psi'(s) = s^2[\psi(s) - 1],$$

$\bar{\rho}(r,t)$ is given by the following equation

$$(5.48)\qquad \frac{(1+\bar{\rho}^2)\,\Phi(-\bar{\rho}) - \bar{\rho}\,\phi(\bar{\rho})}{\bar{\rho} + \psi(\bar{\rho})/\bar{\rho}} = \frac{1}{2}\, r^2\, t^{3/2}\ .$$

The facts (5.41), (5.42), (5.43) stated above for $\bar{\lambda}$ also hold for $\bar{\rho}$, as can be verified from (5.48) utilizing some facts about ψ given by Bather. Also, using the asymptotic form of $\psi(s)$ given by Bather, an asymptotic expansion of $\bar{\rho}$ for large rt is given by

$$(5.49)\qquad \bar{\rho}(r,t) = \frac{1}{2r^2\, t^{3/2}}\left[1 - \frac{2}{\sqrt{2\pi}}\left(\frac{1}{r^2 t^{3/2}}\right) + O\left(\frac{1}{r^2 t^{3/2}}\right)^2\right]\ .$$

The asymptotic expansion of the exact boundary $y_1(x,z)$ obtained in (4.95) for large x and hence for lage z, when written in the notation of this chapter, is given by

$$(5.50)\qquad \bar{\sigma}(r,t) = \frac{1}{2r^2\, t^{3/2}}\left[1 - \frac{1}{3}\left(\frac{1}{r^2\, t^{3/2}}\right) + \frac{7}{15}\left(\frac{1}{r^2\, t^{3/2}}\right)^4 - \dots\right]$$

A comparison of (5.50) with (5.49) and (5.44) gives a verification of (4.95).

case (ii): <u>absolute deviation cost:</u>

<u>Upper bound:</u> Equations (5.38) and (5.39) are obtained as in case (i). Remembering from (5.35) the expression for $c(r,s,t)$ in this case, we are led, as before, to consider the auxiliary problem with the new cost function as

$$c_1(r,s,t) = \frac{a}{rt}\left\{\phi(s) + |s|\left[\frac{1}{2} - \Phi(-|s|)\right]\right\}\ .$$

Proceeding as usual we arrive at the same equation (5.21) defining $\bar{\lambda}$ for a fixed a in the auxiliary problem. Replacing a in (5.21) by rt, the resulting equation gives $\bar{\lambda}(r,t)$, an upper bound on $\bar{\sigma}(r,t)$. We may, as in the binomial problem, obtain an upper bound $\tilde{\lambda}(r,t)$ to $\bar{\lambda}(r,t)$ defined by an equation like (5.22) with the t in its right hand side replaced by rt. The same three facts (5.47), (5.42) and (5.43) also hold for either $\bar{\lambda}(r,t)$ or $\tilde{\lambda}(r,t)$ in this case. Finally, for large rt, asymptotic expansions for them may be obtained as follows:

$$(5.51) \quad \bar{\lambda}(r,t) = \frac{\sqrt{2\pi}}{4} \frac{1}{rt} \left[1 - \frac{1}{2} \left(\frac{\sqrt{2\pi}}{4}\right)^2 \left(\frac{1}{rt}\right)^2 + O\left(\frac{1}{rt}\right)^3\right],$$

$$\tilde{\lambda}(r,t) = \frac{\sqrt{2\pi}}{4} \frac{1}{rt} \left[1 - \frac{1}{6} \left(\frac{\sqrt{2\pi}}{4}\right)^2 \left(\frac{1}{rt}\right)^2 + O\left(\frac{1}{rt}\right)^3\right].$$

<u>Lower bound</u>: As in the binomial problem (Sec (5.25)), we consider the following family of solutions to (5.36) and (5.37),

$$F_\alpha(r,s,t) = -2\sqrt{t} \quad s \left[\frac{1}{2} - \Phi(-s)\right] + \alpha(r)\sqrt{t} \quad \phi(s)$$

where the parameter α may now depend, as in case (i), on r.

Eliminating $\alpha(r)$ from the following two equations, resulting from the boundary conditions (5.32) and (5.33),

$$F_\alpha(r, \bar{\rho}(r,t),t) = K(r, \bar{\rho}(r,t),t) ,$$

$$F_\alpha'(r, \bar{\rho}(r,t),t) = K'(r, \bar{\rho}(r,t),t) ,$$

the inner bound $\bar{\rho}(r,t)$ is given by the same equation as (5.26) with its right hand side replaced by $2(1 + \frac{1}{rt})$. The same three facts (5.41), (5.42) and (5.43) also hold for $\bar{\rho}(r,t)$. An asymptotic expansion of $\bar{\rho}(r,t)$ for large tr is given by

$$(5.52) \qquad \bar{\rho}(r,t) = \frac{\sqrt{2\pi}}{4} \frac{1}{rt} \left[1 - \left(\frac{1}{rt}\right)^2 + O\left(\frac{1}{rt}\right)^3 \right].$$

Expressing the boundary $y_1(x,z)$, obtained for this case in (4.107) of Chapter IV, in the notations of this chapter gives

$$(5.53) \qquad \bar{\sigma}(r,t) = \frac{\sqrt{2\pi}}{4} \frac{1}{rt} \left[1 - \frac{7\pi}{48} \left(\frac{1}{rt}\right)^2 + O\left(\frac{1}{rt}\right)^4 \right].$$

Comparing (5.53) with (5.51) and (5.52) gives an independent check on (4.107).

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