# A GRADIENT--REGRESSION SEARCH PROCEDURE FOR SIMULATION EXPERIMENTATION

William Ernest Biles

University of Notre Dame

## ABSTRACT

This paper examines a gradient search procedure for simulation experimentation with constrained systems. This procedure combines gradient search with curvilinear regression in moving toward a constrained optimal solution for a system involving n controllable variables. In a direction-determining block, at least n+1 simulation trials are performed around a current base point to establish an improving direction. Then in a step determining block, t simulation trials are performed along the improving direction to establish the most favorable step in moving to the next base point. This sequential block process, in which each block is executed in one input to the computer, is repeated until an approximate solution is found which satisfies all system constraints.

## I. INTRODUCTION

Although simulation is most often used as a descriptive technique, much attention has been given to the problem of using optimization in conjunction with computer simulation. This entails performing experiments with a simulation model of the system under study to determine the optimum response y* for some probabilistic function of unknown form,

$$y = F(X) + \epsilon , \qquad (1)$$

where y is some measure of system effectiveness, X is an n-dimensional vector of controllable variables ($x_i$, i=1, ···, n), and $\epsilon$ is an error component usually assumed to be normally distributed with mean zero and standard deviation $\sigma_\epsilon$. Simulation experimentation is typically conducted by controlling the levels of the system variables X at p distinct sets of values ($X^k$, k=1, ···, p), observing the p simulated responses ($y^k$, k=1, ···, p) and selecting X* so as to achieve the most beneficial expected response y*.

Mihram [8, 9], Schmidt and Taylor [13], and Smith [15] have presented excellent treatments of multivariable optimization with a single simulated response. Most realistic systems, however, require simultaneous consideration of several responses ($y_j$, j=0, 1, ···, m). For instance, in a simple inventory system one might have to consider safety stock level as well as total system cost. An expedient approach to multiple-response simulation experimentation is that of constrained optimization. In this approach one response, $y_o$, is designated a primary or objective response, while the remaining responses ($y_j$, j=1, ···, m) become constraints by placing restrictions on the values they may assume. The mathematical statement of this problem is as follows:

$$\text{Maximize (or minimize) } y_o = F(X) + \epsilon_o \qquad (2)$$

subject to

$$a_i \le x_i \le c_i, \; i = 1, \cdots, n \qquad (3)$$

$$y_j = G_j(X) + \epsilon_j \; \{\le, =, \ge\} \; d_j, \; j=1, \cdots m \qquad (4)$$

where

- X = n-dimensional vector of controllable variables ($x_i$, i=1, ···, n);
- $x_i$ = value of the i-th controllable variable;
- $a_i$ = lower bound on the value of the i-th controllable variable $x_i$;
- $c_i$ = upper bound on the value of the i-th controllable variable $x_i$;
- F = objective function, a probabilistic function of unknown form;
- $y_o$ = objective response variable;
- $\epsilon_o$ = error component in the objective system response;
- $G_j$ = j-th system response function, often a probabilistic function of unknown form;
- $y_j$ = j-th system response variable;
- $\epsilon_j$ = error component in the j-th system response;
- $d_j$ = specification level or bound on the j-th system response $y_j$;
- n = number of controllable variables for the system;
- m = number of system responses treated as constraining conditions.

Biles [1] has examined the multiple-response problem in connection with discrete-event simulation and demonstrated several techniques that can be applied to solve the problem in a succession of sequential experimental blocks. Sequential-block experimentation consists of performing sequential sets or "blocks" of experiments with the simulation model of the system under study. Each "block" consists of several "trials" performed in a single computer run. A "trial" is an execution of the simulation model at a specified set of values for the controllable variables ($x_i$, $i=1, \cdots, n$). After each block, the experimenter examines available results and decides where to place the next block of experiments; hence, sequential-block search is an interactive technique in that it permits the experimenter to use his judgment in determining the sequence of experiments and when to terminate the search.

A gradient approach to sequential-block simulation experimentation would consist of several iterations, with each iteration consisting of a gradient-determining block followed by a step-determining block. Biles [1] illustrated this procedure in a rather informal fashion with a GASP-II simulation model of a periodic review inventory system. The present paper develops a more formal algorithm for handling constrained problems. This algorithm employs a gradient direction when the search is interior to the feasible region given by (3) and (4), and a gradient projection direction when the search has progressed to a point $X^k$ which lies at a boundary of the feasible region. This gradient projection direction is the same as that developed by Rosen [12, 13] for solving nonlinear programming problems. Rosen's gradient projection method, however, was developed for optimization problems involving functions of known algebraic form without statistical variation and is a computational procedure rather than an experimental one. Although neither of these assumptions is applicable to computer simulation, it is possible to develop modifications to Rosen's method that enable it to be used with simulation optimization. This paper examines these modifications.

## II. GRADIENT SEARCH

Given the problem of finding the maximum of a known function $F(X)$, which has no statistical variation, gradient search proceeds from a current point $X^k$ to a new point $X^{k+1}$ according to the relation

$$X^{k+1} = X^k + \lambda^k[\nabla F(X^k)] \quad (5)$$

where

$$\nabla F(X^k) = [\partial F/\partial x_1, \partial F/\partial x_2, \cdots, \partial F/\partial x_n] \quad (6)$$

That is, $\nabla F(X^k)$ is the n-component vector of first partial derivatives of the function $F(X)$, evaluated at the point $X^k$. This "gradient vector" describes the optimal improving direction away from the point $X^k$. Since the gradient direction is a local property of the function $F(X)$, there is some point along this direction for which the function $F(X)$ obtains its maximum; that is, there is a step $\lambda^k$ for which

$$F(X^k+\lambda^k[\nabla F(X^k)]) = \max_{\lambda} F(X^k+\lambda[\nabla F(X^k)]) \quad (7)$$

Hence, gradient search consists of alternately determining a gradient direction $\nabla F(X^k)$ and an optimal step $\lambda^k$ along this gradient direction. This process is repeated until an optimal or near-optimal solution is found.

Box and Wilson [3] developed an experimentation procedure which invokes the principles of gradient search. Called "response surface methodology", or RSM for short, their procedure employs a first-order designed experiment around a current base point $X^k$ to develop an estimate of the gradient direction. Using a $2^n$ factorial design or an n-dimensional simplex design [2], both illustrated below, it is possible to estimate the linear equation

$$\hat{y} = b_o + \sum_{i=1}^{n} b_i x_i \quad (8)$$

where $\hat{y}$ = an estimate of the system response under study;
$x_i$ = value of the i-th controllable variable;
$b_i$ = regression coefficient.

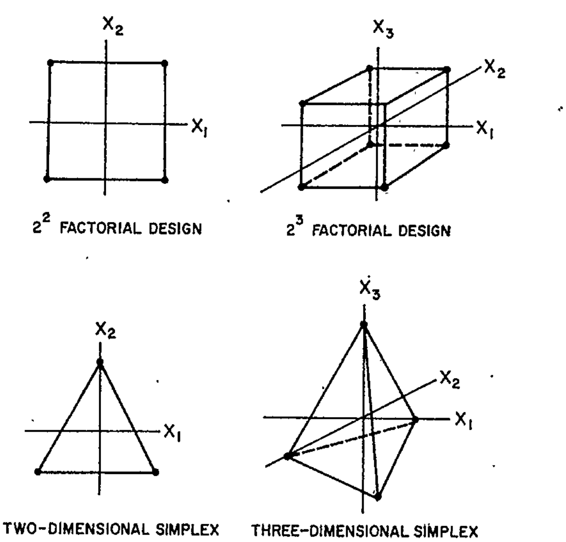


Figure 1. First-Order Experimental Designs

Multiple linear regression is used to determine the n+1 estimates ($b_i$, $i=0, 1, \cdots, n$). Therefore, any experimental design employed for this purpose must have at least n+1 design points. In terms of computer simulation, this block of n+1 or more design points is performed in a single computer run consisting of a simulation trial at each design point. The regression coefficients $b_1, \cdots, b_n$ provide the estimate of the gradient direction $\nabla F(X^k)$.

RSM then proceeds by experimentally determining the optimum step $\lambda^k$ away from the current base point $X^k$. In effect, this is a univariable search with a new controllable variable $\lambda$. This search can be performed in a single experimental block by employing t simulation trials in a single computer run. These t trials are performed at points $\lambda_1, \lambda_2, \cdots, \lambda_t$ along the gradient direction. The values of the controllable variables $X_j$ corresponding to these points ($\lambda_j$, j=1, ···, t) are given by

$$X_j = X^k + \lambda_j[\nabla F(x^k)] \, , \tag{9}$$

which for each variable ($x_i$, i=1, ···, n) is

$$x_{ij} = x_i^k + \lambda_j b_i \tag{10}$$

Now the univariate search points ($\lambda_j$, j=1, ···, t) must be chosen so that the t simulation trials span the region of interest. The t responses ($y_j$, j=1, ···, t) are recorded and curvilinear regression is used to fit the most statistically significant polynomial equation of the form

$$\hat{y} = a_o + \sum_{k=1}^{r} a_k \lambda^k \tag{11}$$

where $r \leq t-1$. For instance, if a second degree equation best fits the data, the polynomial is

$$\hat{y} = a_o + a_1 \lambda + a_2 \lambda^2$$

The appropriate polynomial expression is solved for the value $\lambda^*$ which maximizes $\hat{y}$. Then the new base point in the exploration is given by

$$X^{k+1} = X^k + \lambda^*[\nabla F(X^k)] \tag{12}$$

Hence, two experimental blocks, each consisting of a single computer run, are used in moving from point $X^k$ to an improved point $X^{k+1}$. This process is iterated in moving from a starting point $X^o$ to a solution $X^p$. Termination at $X^p$ usually takes place according to a criterion such as

$$|F(X^p) - F(X^{p-1})| \leq \delta \tag{13}$$

where $\delta$ is an arbitrarily small increment. A hypothetical gradient search performed in sequential blocks is illustrated below.

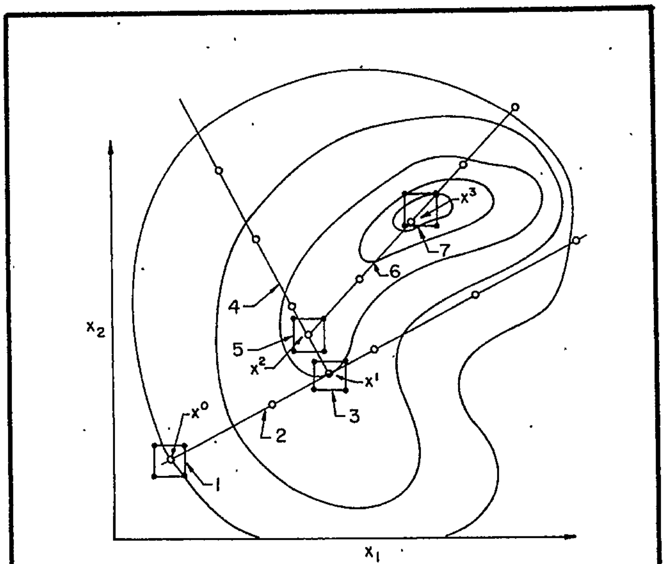


Figure 2. Unconstrained Search

There is little thus far to distinguish the proposed procedure from the RSM method by Box and Wilson [3], except that the simulation trials are executed in experimental blocks with each block requiring a single computer run. However, we have not yet considered the constrained problem in any formal fashion.

## III. GRADIENT PROJECTION METHOD

Given the constrained optimization problem represented by relations (2)-(4), but without the error components ($\epsilon_j$, j=0, 1, ···, m), Rosen's gradient projection method [12, 13] is one of gradient search coupled with orthogonal projection of the negative gradient onto a linear sub-manifold of any binding constraints. The algorithm begins at a feasible point $X^k$. A feasible direction $S^k$ is defined, and a step of length $\lambda^k$ is taken in this feasible direction that maximizes F(X) and yields a new point

$$X^{k+1} = X^k + \lambda^k S^k \tag{14}$$

that is still feasible. As long as $X^k$ is in the interior of the feasible region enclosed within (3) and (4), the direction $S^k$ is the gradient direction given by (6). As with gradient search, a step $\lambda^k$ is sought so as to maximize the function $F(X^k + \lambda S^k)$. If $\lambda^k$ causes one or more of the constraints (3) and (4) to be violated, it is necessary to determine a quantity $\rho^k$, $0 < \rho^k < 1$, such that

$$X^{k+1} = X^k + \rho^k \lambda^k S^k \, , \tag{15}$$

where $X^{k+1}$ lies on a boundary of the feasible region. That is, at least one of the constraints (3) and (4) is satisfied at the equality and is said to be "active".

At point $X^{k+1}$ the gradient direction $\nabla F(X^{k+1})$ is determined. If this direction points back into the feasible region, the standard gradient search procedure is again invoked. If the gradient direction points away from the feasible region, however, it is necessary to proceed in a direction which lies along the intersection of the linearized forms of the "active" constraints. An appropriate search direction $S^{k+1}$ is "projected" onto this intersection of linearized constraints, which is called the "linear sub-manifold".

Suppose that at point $X^{k+1}$, q constraints are satisfied as equalities. Let $B_q$ be the q x n matrix of first partial derivatives of these q active constraints ($G_j(X)$, j=1, ···, q) evaluated at $X^{k+1}$. That is, $B_q$ consists of the q gradient vectors ($\nabla G_j(X^{k+1})$, j=1, ···, q), or

$$B_q = \begin{bmatrix} \partial G_1/\partial x_1, \cdots, \partial G_q/\partial x_1 \\ \partial G_1/\partial x_2, \cdots, \partial G_q/\partial x_2 \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \partial G_1/\partial x_n, \cdots, \partial G_q/\partial x_n \end{bmatrix} \tag{16}$$

Note that we have arbitrarily designated the active constraints as being the first q constraints. We can reorder the constraints (3) and (4) as we like. There are 2n constraints in (3) and m constraints in (4). At most n of the 2n constraints in (3) can be active; hence, $q \leq m+n$.

If $\nabla F(X^{k+1})$ is the gradient direction for the objective function given by (2), then the gradient projection direction from point $X^{k+1}$ is

$$S^{k+1} = [\nabla F(X^{k+1})] - B_q(B_q' B_q)^{-1} B_q' [\nabla F(X^{k+1})] \quad (17)$$

where $B_q'$ is the transpose of $B_q$, and $(B_q' B_q)^{-1}$ is the inverse of the product of $B_q'$ and $B_q$. Hence, $S^{k+1}$ is an n-component vector. The search continues until the termination criterion given by (13) is satisfied.

The extension of Rosen's gradient projection method to the experimental realm is accomplished in much the same fashion as that for unconstrained gradient search. At an interior point $X^k$, a designed experiment consisting of n+1 or more simulation trails is conducted to develop the estimate of the gradient direction for the objective function, given by

$$\hat{y}_o = b_{00} + \sum_{i=1}^{n} b_{0i} x_i \quad (18)$$

The $2^n$ factorial and n-dimensional simplex designs are also used here. The coefficients $b_{01}, \cdots, b_{0n}$ provide the estimate of the gradient direction, but now all m+1 curvilinear regressions are computed to yield

$$\hat{y}_j = a_{j0} + \sum_{k=1}^{n} a_{jk} \lambda^k, \; j = 0, 1, \cdots, m \quad (19)$$

These m+1 equations need not be of the same degree. The polynomial function for the objective system response,

$$\hat{y}_o = a_{00} + \sum_{k=1}^{r_o} a_{0k} \lambda^k, \quad (20)$$

is solved for $\lambda^*$ which maximizes $\hat{y}_o$. Then this $\lambda^*$ is used to check the remaining m functions given in (19) to ascertain whether constraint violation has occurred. If not, (12) holds and the gradient search procedure is again invoked. If constraint violation occurs, then $\rho^k$ is determined so that (3) and (4) hold; hence, the new base point $X^{k+1}$ is determined from (15). A gradient projection direction is then found by performing a designed experiment about the new base point $X^{k+1}$, employing multilinear regression to find the m+1 expressions given by

$$\hat{y}_j = b_{j0} + \sum_{i=1}^{n} b_{ji} x_i, \; j = 0, 1, \cdots, m, \quad (21)$$

and solving for $S^{k+1}$ in equation (17) where

$$\nabla F(X^{k+1}) = (b_{01}, \ldots, b_{0n})' \quad (22)$$

and

$$\nabla G_j(X^{k+1}) = (b_{j1}, \ldots, b_{jn})', \; j = 1, \ldots, m \quad (23)$$

Once this gradient projection direction is established, t simulation trials are executed along this direction just as with the gradient search procedure. Illustrated below is a hypothetical gradient projection search for a constrained system involving two controllable variables and two nonlinear constraints. The progress of this search is described in the following example.

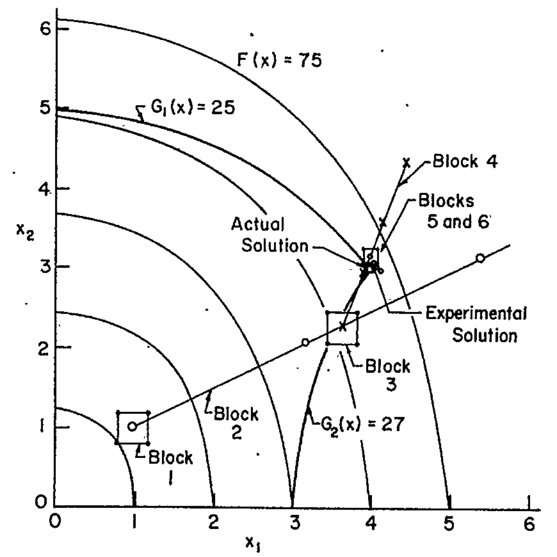


Figure 3. Constrained Search

## IV. EXAMPLE PROBLEM

Consider a simple two-variable problem in which the objective system response $y_o$ is restricted by two other system responses $y_1$ and $y_2$. Suppose the problem is one in which these system responses follow the equations

$$y_0 = F(X) = 3x_1^2 + 2x_2^2 + \epsilon_o(0, 0.5) \quad (24)$$

$$y_1 = G_1(X) = x_1^2 + x_2^2 + \epsilon_1(0, 0.5) \quad (25)$$

$$y_2 = G_2(X) = 9x_1 - x_2^2 + \epsilon_2(0, 0.5) \quad (26)$$

The term $\epsilon_j(0, 0.5)$ indicates that the experimental error is normally distributed with mean zero and standard deviation 0.5. This constrained optimization problem is as follows:

$$\max y_o = F(X) \quad (27)$$

$$\text{subject to} \quad x_1, x_2 \geq 0 \quad (28)$$

$$y_1 = G_1(X) \le 25 \quad (29)$$

$$y_2 = G_2(X) \le 27 \quad (30)$$

Suppose that the search is initiated at the point $X^o = (1, 1)$.

The first step in solving this problem is to perform a designed experiment about the point $X^o$ to estimate the search direction away from $X^o$. Table 1 below gives the design points and responses for a $2^2$ factorial experiment about the point $X^o$.

TABLE 1

First Direction-Determining Experimental Block

| Design Point | $x_1$ | $x_2$ | $y_o$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|
| 1 | 0.8 | 0.8 | 3.73 | 1.03 | 5.56 |
| 2 | 0.8 | 1.2 | 4.58 | 2.67 | 5.19 |
| 3 | 1.2 | 0.8 | 7.00 | 3.09 | 9.50 |
| 4 | 1.2 | 1.2 | 5.71 | 2.19 | 10.30 |

The values of the responses $y_1$ and $y_2$ clearly indicate that we are in the interior of the feasible region given by (28) - (30). Therefore, multiple linear regression will be used to estimate the gradient direction $\nabla F(X^o)$ away from $X^o$. The regression equation, statistically significant at a 95 per cent confidence level, is

$$\hat{y}_0 = -2.92 + 5.5\, x_1 + 2.675\, x_2 \quad (31)$$

Hence, the estimated gradient direction is

$$\nabla F(X^o) = (5.5,\ 2.675)'$$

Thus, the first direction-determining block is completed.

The first step-determining block involves performing several trials, in this case four, along the estimated gradient direction. Since we could reasonably ascertain from the first block results that constraint violation would occur if $x_1$ goes much beyond 5 and $x_2$ beyond 3, the range for $\lambda$ is chosen to be $0 \le \lambda \le 1.2$. Table 2 below gives the design points and responses for the four trials along the gradient direction, as computed by equation (10).

TABLE 2

First Step-Determining Experimental Block

| Design Point | $\lambda$ | $x_1$ | $x_2$ | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1.00 | 1.00 | 5.53 | 1.75 | 7.00 |
| 2 | 0.4 | 3.20 | 2.07 | 39.07 | 15.11 | 23.94 |
| 3 | 0.8 | 5.40 | 3.14 | 107.00 | 39.23 | 38.88 |
| 4 | 1.2 | 7.60 | 4.21 | 208.84 | 75.59 | 50.81 |

The curvilinear regression equations computed from these results, all statistically significant at a 95 per cent confidence level, are as follows:

$$\hat{y}_0 = 5.51 + 41.4\lambda + 106.72\lambda^2 \quad (32)$$

$$\hat{y}_1 = 1.82 + 18.29\lambda + 35.94\lambda^2 \quad (33)$$

$$\hat{y}_2 = 6.95 + 46.00\lambda - 7.83\lambda^2 \quad (34)$$

Equation (32) suggests that $\lambda^o$ should be made as large as possible; however, $\lambda^o$ must be chosen so that the constraints (29) and (30) are not violated. Equating the right side of (33) to the boundary value 25 and solving for $\lambda^o$ yields the solution $\lambda^o = 0.588$. Equating the right side of (34) to the boundary value 27 in (30) and solving yields $\lambda^o = 0.472$. Hence, the smaller value must be chosen to prevent constraint violation.. Thus, $\lambda^o = 0.472$ and

$$X^1 = X^o + \lambda^o[\nabla F(X^o)]$$

$$X^1 = (1,\ 1)' + 0.472\ (5.5,\ 2.675)'$$

$$X^1 = (3.60,\ 2.26)'$$

The point $X^1$ lies on the estimate of the boundary $G_2(X)$. This does not mean that (3.60, 2.26)' necessarily lies on the actual boundary $G_2(X)$.

Performing a second direction-determining block, a second $2^2$ factorial experiment is employed about the point $X^1$ to estimate a search direction. Table 3 gives the design points and responses for this block of experiments.

TABLE 3

Second Direction-Determining Experimental Block

| Design Point | $x_1$ | $x_2$ | $y_o$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|
| 1 | 3.40 | 2.06 | 43.79 | 15.69 | 27.29 |
| 2 | 3.40 | 2.46 | 46.54 | 16.48 | 24.80 |
| 3 | 3.80 | 2.06 | 51.73 | 18.56 | 29.06 |
| 4 | 3.80 | 2.46 | 55.54 | 20.41 | 29.14 |

The multiple linear regression equation representing the objective function $F(X)$ is

$$\hat{y}_0 = -45.79 + 21.25\, x_1 + 8.25\, x_2 \quad (35)$$

Then the gradient direction is

$$\nabla F(X^1) = (21.25,\ 8.25)'$$

As this gradient direction points away from the feasible region given by (28)-(30), it is necessary to estimate the gradient projection direction $S^1$ given by equation (17). As the matrix $B_q$ is simply the gradient vector $\nabla G_2(X^1)$ here, the only additional information needed is the multiple linear regression equation representing $G_2(X^1)$. From the data in Table 3, this equation is

$$\hat{y}_2 = 6.57 + 7.625\, x_1 - 2.875\, x_2 \quad (36)$$

Therefore,

$$\nabla G_2(X^1) = (7.625,\ -2.875)'$$

From equation (17),

$$S^1 = \begin{bmatrix} 21.25 \\ 8.25 \end{bmatrix} -$$

$$\begin{bmatrix} 7.625 \\ -2.875 \end{bmatrix} \left[ \begin{pmatrix} 7.625 \\ -2.875 \end{pmatrix}' \begin{pmatrix} 7.625 \\ -2.875 \end{pmatrix} \right]^{-1} \begin{pmatrix} 7.625 \\ -2.875 \end{pmatrix}' \begin{bmatrix} 21.25 \\ 8.25 \end{bmatrix}$$

$$S^1 = \begin{bmatrix} 5.35 \\ 14.25 \end{bmatrix}$$

Then in the second step-determining block, four design points are conducted along this gradient projection direction in the range $0 \leq \lambda \leq 0.15$. Table 4 gives the design points and responses for this fourth experimental block.

TABLE 4

Second Step-Determining Experimental Block

| Design Point | $\lambda$ | $x_1$ | $x_2$ | $y_0$ | $y_2$ | $y_2$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 3.60 | 2.26 | 49.06 | 18.25 | 27.66 |
| 2 | 0.05 | 3.87 | 2.97 | 62.10 | 22.95 | 26.76 |
| 3 | 0.10 | 4.14 | 3.68 | 78.82 | 30.03 | 23.57 |
| 4 | 0.15 | 4.41 | 4.39 | 97.21 | 39.48 | 20.09 |

The curvilinear regression equations computed from these results are as follows:

$$\hat{y}_0 = 48.96 + 242\lambda + 535\lambda^2 \tag{37}$$

$$\hat{y}_1 = 18.25 + 70.3\lambda + 475\lambda^2 \tag{38}$$

$$\hat{y}_2 = 27.76 - 13.1\lambda - 258\lambda^2 \tag{39}$$

Equating (38) and (39) to 25 and 27, respectively, and solving for $\lambda^1$ yields $\lambda^1 = 0.0345$ with $G_2(X)$ as the binding constraint. The estimated solution at this point is

$$X^2 = (3.79,\ 2.75)';\ y_0 = 58;\ y_1 = 21.2;\ y_2 = 27.$$

However, if $\lambda^1 = 0.066$ with $G_1(X)$ binding, a better solution is obtained without violating the $G_2(X)$ constraint. This estimated solution is

$$X^2 = (3.95,\ 3.20)';\ y_0 = 67.2;\ y_1 = 25;\ y_2 = 25.8$$

Performing a third direction-determining block around point $X^2$ using a $2^2$ factorial experimental design, the results given in Table 5 are obtained. Because the search has neared the intersection of constraints, the design points are placed closer together.

TABLE 5

Third Direction-Determining Experimental Block

| Design Point | $x_1$ | $x_2$ | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|
| 1 | 3.85 | 3.10 | 64.05 | 24.26 | 25.37 |
| 2 | 3.85 | 3.30 | 66.10 | 25.63 | 23.77 |
| 3 | 4.05 | 3.10 | 69.06 | 26.30 | 26.31 |
| 4 | 4.05 | 3.30 | 71.17 | 27.23 | 24.77 |

The regressson equation for the objective function (FX) is

$$\hat{y}_0 = -65.2 + 25.2x_1 + 10.4x_2 \tag{40}$$

The gradient direction $(25.2,\ 10.4)'$ points away from the feasible region. Hence, it is necessary to compute the gradient projection direction $S^2$ with $G_1(X)$ as the binding constraint. From the data in Table 5, the regression equation for $G_1(X)$ is

$$\hat{y}_1 = -28.5 + 9.1x_1 + 5.75x_2 \tag{41}$$

with the estimate of $\nabla G_1(X^2)$ as $(9.1,\ 5.75)'$. Hence, $S^2 = (2.6,\ -3.9)'$ from equation (17).

Performing the third step-determining block along this gradient projection direction, the results in Table 6 are obtained.

TABLE 6

Third Step-Determining Experimental Block

| Design Point | $\lambda$ | $x_1$ | $x_2$ | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 3.95 | 3.20 | 67.05 | 25.71 | 24.63 |
| 2 | 0.02 | 4.00 | 3.12 | 67.08 | 24.98 | 26.99 |
| 3 | 0.04 | 4.05 | 3.04 | 67.23 | 25.35 | 26.93 |
| 4 | 0.06 | 4.10 | 2.96 | 68.03 | 25.85 | 28.95 |

The curvilinear regression equations resulting from these experiments are not significant at a 95 per cent confidence level, owing to the close spacing of the four design points. Since the second design point yields estimated values for $y_1$ and $y_2$ that lie approximately at the intersection of constraints $G_1(X)$ and $G_2(X)$, this point is taken as a solution and the search is terminated. Therefore, the experimental solution is

$$X^3 = (4.00,\ 3.12)';\ y_0 = 67;\ y_1 = 25;\ y_2 = 27$$

The actual system responses at $X^3 = (4.00,\ 3.12)'$ are

$$y_0 = 67.46; \quad y_1 = 25.73; \quad y_2 = 26.26$$

The known solution to this test problem is

$X^* = (4, 3);\quad y_0^* = 66;\quad y_1^* = 25;\quad y_2^* = 27$

Thus, we have used six experimental blocks of four trials each to determine a solution which is within 2.5% of the known solution for this test problem. The progress of this search is illustrated in Figure 3.

## V. CONCLUSIONS

This paper has demonstrated a novel and efficient search technique for simulation experimentation with constrained systems. Although the use of Rosen's gradient projection method [6, 12, 13] with simulation experimentation has not been attempted before, it differs only slightly in application from the Box and Wilson response surface method [3] which is widely accepted. It employs first-order experimental designs which have been previously shown to be useful in simulation methodology [2, 4, 8, 15].

This study did not seek to compare the effectiveness of various experimental designs for use with multiple-response experimentation. Brooks and Mickey [4] concluded that a design with exactly n+1 design points, such as the simplex design, is just as effective as factorial designs in estimating the gradient direction in dealing with a single system response. There needs to be a study of the effectiveness of various designs in estimating the gradient projection direction. Moreover, there needs to be a study which examines the effect of the magnitude of the random errors ($\varepsilon_j$, j=0, 1, $\cdots$, m) on the effectiveness of gradient projection search along binding constraints.

Finally, there are other search methods which could also be studied for use in simulation experimentation. Zoutendijk's method of feasible directions [17] is one possibility. Simplex methods [11, 16] also hold promise, but it would be necessary to develop effective strategies to be employed when the search has progressed to a binding constraint.

## BIBLIOGRAPHY


1. Biles, W.E., "Constrained Sequential-Block Search in Simulation Experimentation", Proceedings of the 1973 Winter Simulation Conference, San Francisco, January 1973.
2. Box, G.E.P., "Multi-factor Designs of First Order", Biometrika, Vol. 39, No. 1, 1952.
3. Box, G.E.P. and K.P. Wilson, "On the Experimental Attainment of Optimum Conditions", Journal of the Royal Statistical Society, Series B, Vol. 13, No. 1, 1951.
4. Brooks, S.H., and M.R. Mickey, "Optimum Estimation of Gradient Direction in Steepest Ascent Experiments", Biometrics, Vol. 17, No. 1, 1961.
5. Griffith, R.E., and R.A. Stewart, "A Nonlinear Programming Technique for the Optimization of Continuous Processing Systems", Management Science, Vol. 7, 1961.
6. Himmelblau, D.M., Applied Nonlinear Programming, McGraw-Hill Book Company, New York, 1972.
7. Hooke, R., and T.A. Jeeves, "Direct Search Solution of Numerical and Statistical Problems", Journal of the Association of Computing Machinery, Vol. 8, No. 2, April 1961.
8. Mihram, G.A., "An Efficient Procedure for Locating the Optimal Simular Response", Proceedings of the Fourth Conference on the Applications of Simulation, New York, December 1970.
9. Mihram, G.A., Simulation: Statistical Foundations and Methodology, Academic Press, New York, 1972.
10. Myers, R.L., Response Surface Methodology, Allyn and Bacon, Boston, Massachusetts, 1971.
11. Nelder, J.A., and R. Mead, "A Simplex Method for Function Minimization", Computer Journal, Vol. 7, 1965.
12. Rosen, J.B., "The Gradient Projection Method for Nonlinear Programming, Part I - Linear Constraints", Journal of the Society of Industrial and Applied Mathematics, Vol. 8, 1961.
13. Rosen, J.B., "The Gradient Projection Method for Nonlinear Programming, Part II - Nonlinear Constraints", Journal of the Society of Industrial and Applied Mathematics, Vol. 9, 1961.
14. Schmidt, J.W., and R.E. Taylor, Simulation and Analysis of Industrial Systems, Richard D. Irwin, Homewood, Illinois, 1970.
15. Smith, D.E., "Requirements of an 'Optimizer' for Computer Simulations", Naval Research Logistics Quarterly, Vol. 20, No. 1, March, 1973.
16. Spendley, W., G.R. Hext, and R.F. Himsworth, "Sequential Application of Simplex Designs in Optimization and Evolutionary Operation", Technometrics, Vol. 4, November 1962.
17. Zoutendijk, G., Methods of Feasible Directions, Elsevier Press, Amsterdam, 1960.