

INSTITUTE OF STATISTICS  
BOX 5457  
STATE COLLEGE STATION  
RALEIGH, NORTH CAROLINA

UNIVERSITY OF NORTH CAROLINA  
Department of Statistics  
Chapel Hill, N. C.

A STUDY OF SOME SINGLE-COUNTER QUEUEING PROCESSES

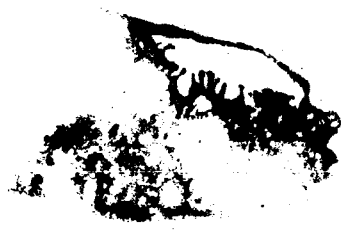
by

Meckinley Scott

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## INTRODUCTION

This thesis deals with the study of a class of single-counter queueing processes. In Chapters I, II, and III we assume that those in charge of the system are capable of controlling "to some extent" the arrival and/or service intensity. The purpose of this is, in effect, to reduce the mean number of units waiting for service to a desirable length. In Chapter IV we assume that the arrival intensity is neither constant nor controllable but rather a random process of a Markovian type.

Chapter I constitutes a special case of, and is introductory to, the more general case dealt with in Chapter II. Furthermore, we are able to get explicit forms for most formulas in the special case. A brief description of the model for the special case is as follows: The population of units that demand service from the station from time to time is classified into two categories, say 0 and 1. When the number of units in the system ( $l_s$ ) reaches a certain prescribed number  $R$ , the server forbids any category 0 unit from joining the queue and invites any such unit again for service, if and only if,  $l_s$  reduces to a size  $r \geq 0$ ,  $r < R$ . The exclusion rule does not, however, affect those who are already waiting in line. The service time is assumed to have an exponential distribution with parameter  $\mu$  (= reciprocal of the average service time) for all units irrespective of the categories to which they belong.

The first part of Chapter I deals with the determination of the stationary probability distribution of  $l_s$  and the problems arising therefrom, for example, evaluation of  $\text{Prob} [r \leq l_s \leq R]$ , certain conditional probabilities, etc. For the existence of steady state (stationary) conditions we have to impose the condition  $\rho_1 = \lambda_1/\mu < 1$ . It is intuitively obvious that if  $\rho_1 \geq 1$ , the server may not have any idle time and that a divergent queue may ensue (cf. Cox and Smith [2], page 41). There is no such condition imposed on  $\rho_0 = \lambda_0/\mu$ . The expected (mean) number of units in the system ( $L_s$ ) is obtained directly from the stationary distribution but for the purpose of calculating higher moments it is easier to deal with the probability generating function which we give in Section 1.4. The expected waiting time in the system ( $W_s$ ) is obtained from the formula  $L_s = \bar{\lambda} W_s$ , where  $\bar{\lambda}$  is the average arrival intensity (cf. Little [4]).

In Sections 1.5 and 1.6 we are concerned with some distribution problems. In the case of the distribution of the busy period we obtain its Laplace transform and its expected length. For this, we consider a modified process which ceases as soon as the number of units in the system falls to zero (cf. Bailey [1]). Similar is the case of distribution of time the system spends with  $\lambda_0$  (or  $\lambda_1$ ) as the arrival intensity before it changes to  $\lambda_1$  (or  $\lambda_0$ ) for the first time. Using the same mathematical technique as the above mentioned distribution problems, we also obtain the generating function of the Laplace transform with respect to time, of the time-dependent probability of the number of units in the system. It appears that it is

very difficult to obtain the inverse transform.

In Chapter II we also assume that the service time is exponentially distributed with parameter  $\mu$  but the population of units demanding service is comprised of  $(N + 1)$  categories, say  $0, 1, 2, \dots, N$ . Consequently, the arrival intensity  $\lambda$  assumes  $(N + 1)$  values  $\lambda_0, \lambda_1, \dots, \lambda_N$  and associated with these values of  $\lambda$ , we have  $N$  pairs of integers  $(r_1, R_1), (r_2, R_2), \dots, (r_N, R_N)$  satisfying  $r_1 \geq 0, r_i < R_i (i = 1, 2, \dots, N); r_i < r_{i+1}, R_i < R_{i+1} (i = 1, 2, \dots, N - 1)$ . When  $\ell_s$  reaches a size  $R_i + 1 (i = 0, 1, \dots, N-1)$ , the server forbids any unit belonging to categories  $0, 1, \dots, i$  from joining the queue and any category  $i$  unit is invited again for service, if and only if,  $\ell_s$  reduces to a size  $r_{i+1}$ . The exclusion rule does not, however, affect those who are already waiting in line.

The problems we deal with in Chapter II are exactly the same as those of Chapter I, except with  $(N + 1)$  categories instead of only 2. We are able to obtain explicit expressions for both the stationary distribution and the probability generating function of  $\ell_s$ . For the busy period distribution we simply show how to obtain its Laplace transform from a set of  $(2N + 1)$  equations involving  $(2N + 1)$  unknowns. The distribution problems concerning first passage times for the parameters  $\lambda_0$  and  $\lambda_N$  are exactly the same as those concerning the parameters  $\lambda_0$  and  $\lambda_1$  respectively of Chapter I. For the parameter  $\lambda_i$  where  $i \neq 0, N$ , the problem is solved by considering a set of difference equations satisfied by  $f_m(t)$ 's, where  $f_m(t)$  is the probability density function (p.d.f.) of the required distribution starting

initially from a position  $m$ . This is discussed in detail in Section 2.6. We have not given the expression for the generating function of the Laplace transform with respect to time of time-dependent probabilities, though we are able to do so as in Chapter I, since we are unable to obtain its inverse transform.

Lastly, in Chapter II mention must be made that the existence of stationary conditions is achieved only when  $\rho_N = \lambda_N / \mu < 1$ . There are no restrictions whatsoever on the values of  $\rho_i = \lambda_i / \mu$  ( $i=0,1,\dots,N-1$ ). Also when  $N$  is infinite we will still have a stationary distribution provided a certain series converges (See Section 2.7).

A study of the above queueing process suggests another problem which is mathematically analogous to the problems discussed before but whose physical interpretation is quite different. This is considered in Chapter III. It is the case where the arrival intensity is constant but the service rate  $\mu$  changes according to a rule very similar to the change of  $\lambda$  in Chapters I and II. The  $(N+1)$  values of  $\mu$  are denoted by  $\mu_0, \mu_1, \dots, \mu_N$  and, associated with these values of  $\mu$ , are  $N$  pairs of integers  $(r_1, R_1), (r_2, R_2), \dots, (r_N, R_N)$  satisfying the same requirements as in Chapter II. The problems we deal with in Chapter III are exactly the same as in Chapters I and II and the corresponding results are very similar in form. Thus, for example, the probability of the server being idle in the case when  $N=1$  is given by the reciprocal of

$$\frac{1}{1 - \rho_0} - (R - r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \frac{\rho_0 - \rho_1}{1 - \rho_1}, \quad \rho_0 \neq 1$$

in the case where  $\lambda$  is changing and  $\mu$  fixed (Chapter I), and, by the reciprocal of

$$\frac{1}{1 - b_0} - (R - r) \frac{b_0^{R+r-1}}{b_0^R - b_0^R} = \frac{b_0 - b_1}{1 - b_1}, \quad b_0 \neq 1$$

where  $b_i = \lambda/\mu_i$  ( $i = 0, 1$ ), in the case where  $\mu$  is changing and  $\lambda$  fixed (Chapter III).

In Chapter III we have not considered the above special case ( $N = 1$ ), but all appropriate formulas can be obtained from the general case by proper identification of the parameters. In the special case, it may be noted that if we put  $\mu_0 = 0$ ,  $r = 0$  we have what is known as the  $(0, R)$  doctrine introduced by Yadin and Naor [7].

The title of Chapter III is "A single-counter queue with changeable service rate." The reason why we call it a single-counter instead of a single-server is because, in effect, we have only one counter but we bring in or drop out helpers or auxiliary servers depending upon whether the number of units in the System increases or decreases within a certain specified range.

In Chapter IV we consider a queueing process under the assumptions that the service time has an exponential distribution with parameter  $\mu$  and the arrival intensity  $\lambda$  is neither constant nor controllable, but rather a random process of a Markovian type. We have considered a very special case where  $\lambda$  assumes only two values  $\lambda_0$  and  $\lambda_1$ , though in principle the methods developed are general. We assume that the probability of arrival in a small interval of time

of length  $\delta t$  is either  $\lambda_0 \delta t$  or  $\lambda_1 \delta t$ . If at any instant of time the arrival rate is  $\lambda_0$ , then in the next interval  $\delta t$  the probability that  $\lambda_0$  will change to  $\lambda_1$  is  $\kappa_0 \delta t$ . Similarly when the arrival rate is  $\lambda_1$ , the probability that it will change to  $\lambda_0$  in the next interval  $\delta t$  is  $\kappa_1 \delta t$ .

A considerable part of Chapter IV is devoted to problems arising from the arrival process described above. Specifically, these problems are (i) the distribution of the inter-arrival time, (ii) the joint distribution of two successive inter-arrival times, (iii) the probability generating function of the number of arrivals in an interval of length  $t$ , and (iv) the joint probability generating function of the number of arrivals in two successive non-overlapping intervals of lengths  $t_1$  and  $t_2$ . As expected, when we put  $\lambda_0 = \lambda_1 (= \lambda \text{ say})$ , all results reduce to those of Poisson arrivals with parameter  $\lambda$ . A result which is intuitively obvious is that the mean number of arrivals in an interval of length  $t$  is given by  $(\pi_0 \lambda_0 + \pi_1 \lambda_1) t$ , where  $\pi_i$  is the proportion of time in which the arrival rate is  $\lambda_i$  ( $i=0,1$ ).

As far as the queueing part is concerned, we can determine only the fraction of time in which the server is idle or busy. We are able to write down the queueing equations completely; but the solution, at present, seems intractable. The probability of the server being idle is obtained by the method of generating function, and this is

$$P(0) = P(0, \lambda_0) + P(0, \lambda_1) = 1 - (\pi_0 \rho_0 + \pi_1 \rho_1), \quad \rho_i = \lambda_i / \mu \quad (i=0,1).$$

This result is intuitively obvious. We are not able to determine

$P(0, \lambda_0)$  and  $P(0, \lambda_1)$  separately, where  $P(0, \lambda_i)$  is the probability that the server is idle and that the arrival rate is  $\lambda_i$ .

If, with the same arrival process, we consider the queueing process with a limited waiting room, of size  $N$  say, it is possible to obtain an explicit expression for the stationary distribution for small values of  $N$ . For large values of  $N$ , the solution can be expressed in a determinantal form.

## CHAPTER I

### A SINGLE-SERVER QUEUE WITH CONTROLLABLE

#### ARRIVAL RATE - A SPECIAL CASE

1.1 Introduction. In this chapter we are considering a single-server queueing process under the following assumptions:-

- (a) The service time is exponentially distributed with mean duration  $1/\mu$ .
- (b) The arrival process is what may be called "controllable Poisson arrival process with two arrival rates".

#### Meaning of "controllable Poisson arrival process with two arrival rates":

In any queueing process we have a cycle consisting of two phases called the idle and busy periods respectively. Suppose at some instant of time the system is idle, that is, there are no units (customers) in the system. Units then start coming in a Poisson stream with intensity  $\lambda = \lambda_0$ . The arrival rate  $\lambda$  is equal to  $\lambda_0$  as long as the number of units in the system ( $l_s$ ) is less than some prescribed positive integer  $R$ . As soon as  $l_s$  reaches a value  $R$ , the arrival rate  $\lambda$  changes from  $\lambda_0$  to  $\lambda_1$  instantaneously and it continues to be  $\lambda_1$  as long as  $l_s$  is strictly greater than some other prescribed integer  $r$ , ( $r \geq 0$ ,  $r < R$ ). When  $l_s$  reaches the value  $r$ ,  $\lambda$  changes back from  $\lambda_1$  to  $\lambda_0$  and the same process is repeated; that is, the arrival rate will continue to be  $\lambda_0$  until  $l_s$  takes a value  $R$  again.

With the above assumptions, we shall determine the stationary probability distribution of the number of units in the system (and also give the expression for its generating function) and its expected value, the expected waiting time in the system, the Laplace transform of the distribution of the busy period and its expected value, etc. In the case of the number of units in the system we shall also give an expression for the generating function of the Laplace transform with respect to time of the time dependent probabilities.

1.2 The distribution of  $l_s$ , the number of units in the system:

Let  $(n, \lambda_i)$  denote the state that  $l_s = n$  and  $\lambda_i$  is the rate of arrival where  $n = 0, 1, 2, \dots$  and  $i = 0, 1$ . By virtue of our assumptions, clearly, the states  $(n, \lambda_0)$  for  $n \geq R$  and the states  $(n, \lambda_1)$  for  $n \leq r$  are inadmissible. The diagram associated with the process is the following:-

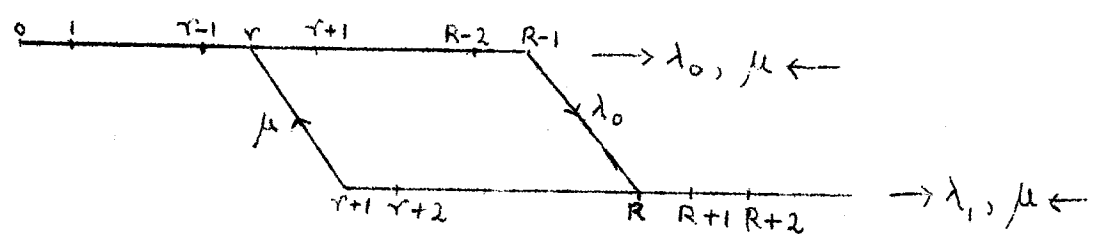


Diagram 1.1

Let  $P(t, n, \lambda_i)$  denote the probability, at time  $t$ , of the state  $(n, \lambda_i)$  and denote by  $P(n, \lambda_i)$  the corresponding stationary probability. By employing the usual ' $\delta t$ ' technique, we have the following differential-difference equations for the process:

$$(1.2.1) \quad P'(t, 0, \lambda_0) = -\lambda_0 P(t, 0, \lambda_0) + \mu P(t, 1, \lambda_0)$$

$$(1.2.2) \quad P'(t, n, \lambda_0) = -(\mu + \lambda_0)P(t, n, \lambda_0) + \mu P(t, n+1, \lambda_0) \\ + \lambda_0 P(t, n-1, \lambda_0) \quad (n = 1, 2, \dots, r-1)$$

$$(1.2.3) \quad P'(t, r, \lambda_0) = -(\mu + \lambda_0)P(t, r, \lambda_0) + \mu P(t, r+1, \lambda_0) \\ + \lambda_0 P(t, r-1, \lambda_0) + \mu P(t, r+1, \lambda_1)$$

$$(1.2.4) \quad P'(t, n, \lambda_0) = -(\mu + \lambda_0)P(t, n, \lambda_0) + \mu P(t, n+1, \lambda_0) \\ + \lambda_0 P(t, n-1, \lambda_0) \quad (n=r+1, r+2, \dots, R-2)$$

$$(1.2.5) \quad P'(t, R-1, \lambda_0) = -(\mu + \lambda_0)P(t, R-1, \lambda_0) + \lambda_0 P(t, R-2, \lambda_0)$$

$$(1.2.6) \quad P'(t, r+1, \lambda_1) = -(\mu + \lambda_1)P(t, r+1, \lambda_1) + \mu P(t, r+2, \lambda_1)$$

$$(1.2.7) \quad P'(t, n, \lambda_1) = -(\mu + \lambda_1)P(t, n, \lambda_1) + \mu P(t, n+1, \lambda_1) \\ + \lambda_1 P(t, n-1, \lambda_1) \quad (n=r+2, r+3, \dots, R-1)$$

$$(1.2.8) \quad P'(t, R, \lambda_1) = -(\mu + \lambda_1)P(t, R, \lambda_1) + \mu P(t, R+1, \lambda_1) \\ + \lambda_1 P(t, R-1, \lambda_1) + \lambda_0 P(t, R-1, \lambda_0)$$

$$(1.2.9) \quad P'(t, n, \lambda_1) = -(\mu + \lambda_1)P(t, n, \lambda_1) + \mu P(t, n+1, \lambda_1) \\ + \lambda_1 P(t, n-1, \lambda_1) \quad (n \geq R+1)$$

The following difference equations which the stationary probabilities satisfy are obtained from the above differential-difference equations by merely putting  $P'(t, n, \lambda_i) = 0$  and suppressing the argument  $t$  from  $P(t, n, \lambda_i)$  for all  $n$  and  $i$ . They can also be obtained directly from the fact that the intensity of leaving a certain state is equal to the intensity of coming into the state (cf. Morse [5], page 16).

$$(1.2.10) \quad \mu P(1, \lambda_0) - \lambda_0 P(0, \lambda_0) = 0$$

$$(1.2.11) \quad \mu P(n+2, \lambda_0) - (\mu + \lambda_0)P(n+1, \lambda_0) + \lambda_0 P(n, \lambda_0) = 0$$

$$(n = 0, 1, 2, \dots, r-2)$$

$$(1.2.12) \quad \mu P(r+1, \lambda_1) + \mu P(r+1, \lambda_0) - (\mu + \lambda_0)P(r, \lambda_0) + \lambda_0 P(r-1, \lambda_0) = 0$$

$$(1.2.13) \quad \mu P(n+2, \lambda_0) - (\mu + \lambda_0)P(n+1, \lambda_0) + \lambda_0 P(n, \lambda_0) = 0$$

$$(n = r, r+1, \dots, R-3)$$

$$(1.2.14) \quad -(\mu + \lambda_0)P(R-1, \lambda_0) + \lambda_0 P(R-2, \lambda_0) = 0$$

$$(1.2.15) \quad \mu P(r+2, \lambda_1) - (\mu + \lambda_1)P(r+1, \lambda_1) = 0$$

$$(1.2.16) \quad \mu P(n+2, \lambda_1) - (\mu + \lambda_1)P(n+1, \lambda_1) + \lambda_1 P(n, \lambda_1) = 0$$

$$(n = r+1, r+2, \dots, R-2)$$

$$(1.2.17) \quad \mu P(R+1, \lambda_1) - (\mu + \lambda_1)P(R, \lambda_1) + \lambda_1 P(R-1, \lambda_1) + \lambda_0 P(R-1, \lambda_0) = 0$$

$$(1.2.18) \quad \mu P(n+2, \lambda_1) - (\mu + \lambda_1)P(n+1, \lambda_1) + \lambda_1 P(n, \lambda_1) = 0$$

$$(n \geq R)$$

We shall solve the above difference equations and express all  $P(n, \lambda_1)$ 's in terms of  $P(0, \lambda_0)$ , where  $P(0, \lambda_0)$  will be determined later from the normalizing equation.

Notice that (1.2.11) is a homogeneous difference equation of the second order with (1.2.10) giving the initial condition. We shall first assume that  $\lambda_0 \neq \mu$ ; then the solution to (1.2.10) and (1.2.11) is

$$(1.2.19) \quad P(n, \lambda_0) = \rho_0^n P(0, \lambda_0) \quad (n = 0, 1, 2, \dots, r) \quad \text{and} \\ \rho_0 = \lambda_0 / \mu .$$

Similarly the solution to (1.2.13) with (1.2.14) as the initial condition is given by

$$(1.2.20) \quad P(n, \lambda_0) = \frac{\rho_0^r}{\rho_0^r - \rho_0^R} (\rho_0^n - \rho_0^R) P(0, \lambda_0) \\ (n = r, r+1, \dots, R-1) .$$

we can write equation (1.2.12) as

$$P(r+1, \lambda_1) = (1 + \rho_0) P(r, \lambda_0) - \rho_0 P(r-1, \lambda_0) - P(r+1, \lambda_0)$$

and substituting the values of  $P(r, \lambda_0)$ ,  $P(r-1, \lambda_0)$  and  $P(r+1, \lambda_0)$  from (1.2.13) and (1.2.14) we have the relation

$$(1.2.21) \quad P(r+1, \lambda_1) = \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (1 - \rho_0) P(0, \lambda_0) .$$

Again from (1.2.19) we have

$$(1.2.22) \quad P(R-1, \lambda_0) = \frac{\rho_0^{R+r-1}}{\rho_0^r - \rho_0^R} (1 - \rho_0) P(0, \lambda_0) .$$

Thus we have

$$(1.2.23) \quad P(r+1, \lambda_1) = \rho_0 P(r-1, \lambda_0) = \frac{\rho_0^{R+r}}{r - \rho_0^R} (1 - \rho_0) P(0, \lambda_0).$$

The case where  $\lambda_0 = \mu$  or  $\rho_0 = 1$  is obtained by solving the equations directly or by taking the limits of (1.2.19) and (1.2.20) as  $\rho_0$  tends to 1. The following are the solutions

$$(1.2.24) \quad P(n, \lambda_0) = P(0, \lambda_0) \quad (n = 0, 1, \dots, r),$$

$$(1.2.25) \quad P(n, \lambda_0) = \frac{R-n}{R-r} P(0, \lambda_0) \quad (n = r, r+1, \dots, R-1) \text{ and}$$

$$(1.2.26) \quad P(r+1, \lambda_1) = P(R-1, \lambda_0) = \frac{P(0, \lambda_0)}{(R-r)}.$$

In solving the remaining equations we shall assume that  $\rho_1 = \lambda_1/\mu < 1$ . The fact that we need the restriction  $\rho_1 < 1$  will become obvious later when we impose the normalizing condition on the probabilities.

The solution to (1.2.16) with (1.2.15) giving the initial condition is

$$(1.2.27) \quad P(n, \lambda_1) = \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r}).$$

$$(n = r+1, r+2, \dots, R)$$

where  $P(r+1, \lambda_1)$  is given by (1.2.21) or (1.2.26) depending upon the value of  $\rho_0$ .

Next, equation (1.2.17) can be written as

$$P(R+1, \lambda_1) = (1 + \rho_1) P(R, \lambda_1) - \rho_1 P(R-1, \lambda_1) - \rho_0 P(R-1, \lambda_0) = 0.$$

Making use of (1.2.27) and (1.2.23) or (1.2.26) depending upon the value of  $\rho_0$ , and after simplification we obtain

$$(1.1.28) \quad P(R+1, \lambda_1) = \rho_1 \frac{P(r+1, \lambda_1)}{(1 - \lambda_1)} (1 - \rho_1^{R-r})$$

$$= \rho_1 P(r, \lambda_1) .$$

Lastly, the solution to the difference equation (1.2.18) with (1.1.28) giving the initial condition is

$$(1.2.29) \quad P(n, \lambda_1) = P(R, \lambda_1) \rho_1^{n-R}$$

$$= \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{R-r}) \rho_1^{n-R}$$

$$(n \geq R) .$$

On normalization we obtain

$$(1.2.30) \quad \sum_{n=0}^{\infty} [P(n, \lambda_0) + P(n, \lambda_1)] = 1 .$$

$$\text{Now} \quad \sum_{n=0}^{\infty} P(n, \lambda_0) = \sum_{n=0}^{R-1} P(n, \lambda_0) = \sum_{n=0}^r P(n, \lambda_0) + \sum_{n=r+1}^{R-1} P(n, \lambda_0) .$$

Case (i)  $\rho_0 \neq 1$ , we have

$$(1.2.31) \quad \sum_{n=0}^r P(n, \lambda_0) = \sum_{n=0}^r \rho_0^n P(0, \lambda_0) = P(0, \lambda_0) \frac{1 - \rho_0^{r+1}}{1 - \rho_0}$$

and

$$(1.2.32) \quad \sum_{n=r+1}^{R-1} P(n, \lambda_0) = \sum_{n=r+1}^{R-1} \frac{\rho_0^r}{\rho_0^r - \rho_0^R} (\rho_0^n - \rho_0^R) P(0, \lambda_0)$$

$$= \frac{\rho_0^r P(0, \lambda_0)}{\rho_0^r - \rho_0^R} \left[ \frac{\rho_0^{r+1} - \rho_0^R}{(1 - \rho_0)} - (R-r-1) \rho_0^R \right].$$

Adding (1.2.31) and (1.2.32) and simplifying we obtain

$$\begin{aligned} (1.2.33) \quad \sum_{n=0}^{R-1} P(n, \lambda_0) &= \frac{P(0, \lambda_0)}{1 - \rho_0} - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} P(0, \lambda_0) \\ &= \frac{P(0, \lambda_0)}{1 - \rho_0} - (R-r) \frac{P(r+1, \lambda_1)}{1 - \rho_0} \end{aligned}$$

on using the relation (1.2.21).

Case (ii)  $\rho_0 = 1$ , we have

$$(1.2.34) \quad \sum_{n=0}^r P(n, \lambda_0) = \sum_{n=0}^r P(0, \lambda_0) = (r+1)P(0, \lambda_0)$$

and

$$\begin{aligned} (1.2.35) \quad \sum_{n=r+1}^{R-1} P(n, \lambda_0) &= \sum_{n=r+1}^{R-1} \frac{R-n}{R-r} P(0, \lambda_0) \\ &= \frac{P(0, \lambda_0)}{R-r} [R(R-r-1) - \frac{1}{2}(R+r)(R-r-1)] \\ &= \frac{1}{2} (R-r-1)P(0, \lambda_0). \end{aligned}$$

Adding (1.2.34) and (1.2.35) we have when  $\rho_0 = 1$

$$(1.2.36) \quad \sum_{n=0}^{R-1} P(n, \lambda_0) = \frac{1}{2} (R+r+1)P(0, \lambda_0)$$

Note that (1.2.36) can also be obtained by taking the limit of (1.2.33) as  $\rho_0$  tends to 1.

$$\text{Next } \sum_{n=0}^{\infty} P(n, \lambda_1) = \sum_{n=r+1}^{\infty} P(n, \lambda_1) \quad \text{and}$$

$$(1.2.37) \quad \sum_{n=r+1}^{R-1} P(n, \lambda_1) = \sum_{n=r+1}^{R-1} \frac{P(r+1, \lambda_1)}{(1 - \rho_1)^{n-r}} (1 - \rho_1^{n-r})$$

$$= \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} \left[ (R-r-1) - \frac{\rho_1 - \rho_1^{R-r}}{(1 - \rho_1)} \right]$$

$$(1.2.38) \quad \sum_{n=R}^{\infty} P(n, \lambda_1) = \sum_{n=R}^{\infty} P(R, \lambda_1) \rho_1^{n-R}$$

$$= \frac{P(R, \lambda_1)}{(1 - \rho_1)}$$

$$= \frac{P(r+1, \lambda_1)}{(1 - \rho_1)^2} (1 - \rho_1^{R-r}) ,$$

provided  $\rho_1 < 1$ .

Note that if  $\rho_1 \geq 1$ , the series  $\sum_{n=R}^{\infty} P(n, \lambda_1)$  would not converge. That is, we have a stationary distribution if and only if  $\rho_1 < 1$ . This condition is intuitively obvious.

Summing (1.2.37) and (1.2.38) we obtain

$$(1.2.39) \quad \sum_{n=r+1}^{\infty} P(n, \lambda_1) = (R-r) \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} .$$

Substituting (1.2.33) and (1.2.39) into (1.2.30) we have for the case  $\rho_0 \neq 1$

$$(1.2.40) \quad \frac{P(0, \lambda_0)}{(1 - \rho_0)} - (R-r) \frac{(\rho_0 - \rho_1)}{(1 - \rho_0)(1 - \rho_1)} P(r+1, \lambda_1) = 1$$

or on using (1.2.21) we obtain

$$(1.2.41) \quad P(0, \lambda_0) = \frac{1}{\frac{1}{1 - \rho_0} - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \frac{\rho_0 - \rho_1}{1 - \rho_1}}, \quad \rho_0 \neq 1$$

and substituting (1.2.36) and (1.2.39) into (1.2.30) we have for the case  $\rho_0 = 1$

$$(1.2.42) \quad \frac{1}{2} (R+r+1) P(0, \lambda_0) + (R-r) \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} = 1$$

or on using (1.2.26) we obtain

$$(1.2.43) \quad P(0, \lambda_0) = \frac{1}{\frac{1}{2} (R+r+1) + \frac{1}{1 - \rho_1}}$$

It has been verified that (1.2.43) can also be obtained as a limiting case of (1.2.41).

Summarizing all the results obtained so far, we have for the distribution of the number of units in the system, subject to the restriction that  $\rho_1 < 1$ ,

$$P(n, \lambda_0) = \begin{cases} \rho_0^n P(0, \lambda_0) & , \quad \rho_0 \neq 1 \\ & (n=0, 1, \dots, r) \\ P(0, \lambda_0) & , \quad \rho_0 = 1 \\ \frac{\rho_0^r P(0, \lambda_0)}{\rho_0^r - \rho_0^R} (\rho_0^n - \rho_0^R), & \rho_0 \neq 1 \\ & (n=r, r+1, \dots, R-1) \\ \frac{R-n}{R-r} P(0, \lambda_0) & , \quad \rho_0 = 1 . \end{cases}$$

$$P(n, \lambda_1) = \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r}) \quad (n=r+1, \dots, R) ,$$

$$(1.2.44) \quad P(n, \lambda_1) = \frac{P(r+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{R-r}) \rho_1^{n-r} \quad (n \geq R)$$

where

$$P(r+1, \lambda_1) = \begin{cases} \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (1 - \rho_0) P(0, \lambda_0), & \rho_0 \neq 1 \\ \frac{P(0, \lambda_0)}{(R-r)} & , \quad \rho_0 = 1 \end{cases}$$

and

$$P(0, \lambda_0) = \begin{cases} \frac{1}{\frac{1}{1-\rho_0} - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \frac{\rho_0 - \rho_1}{1 - \rho_1}} & , \quad \rho_0 \neq 1 \\ \frac{1}{\frac{1}{2} (R+r+1) + \frac{1}{1 - \rho_1}} & , \quad \rho_0 = 1 . \end{cases}$$

The probability of finding the system where  $\lambda_i$  is the rate of arrival is given by

$$P(\lambda_i) = \sum_{n=0}^{\infty} P(n, \lambda_i) \quad (i = 0, 1) .$$

From (1.2.33) and (1.2.36) we have

$$(1.2.45) \quad P(\lambda_0) = \begin{cases} \left[ \frac{1}{1 - \rho_0} - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \right] P(0, \lambda_0), \rho_0 \neq 1 \\ \frac{1}{2} (R+r+1) P(0, \lambda_0) , & \rho_0 = 1 \end{cases}$$

and (1.2.39) gives

$$(1.2.46) \quad P(\lambda_1) = (R-r) \frac{P(r+1, \lambda_1)}{(1 - \rho_1)}$$

where  $P(0, \lambda_0)$  and  $P(r+1, \lambda_1)$  are given by (1.2.44).

By means of straightforward algebra, it can be shown that the probability that the number of units in the system ( $l_s$ ) lies between  $r$  and  $R$  is given by

$$(1.2.47) \quad P[r \leq l_s \leq R] = \begin{cases} \rho_0^r \left[ \frac{1}{1 - \rho_0} - \frac{\rho_0^R}{\rho_0^r - \rho_0^R} \left\{ (R-r) \frac{(\rho_0 - \rho_1)}{(1 - \rho_1)} + \frac{(1 - \rho_0)}{(1 - \rho_1)} \sum_{n=1}^{R-r} \rho_1^n \right\} \right] P(0, \lambda_0) (\rho_0 \neq 1) \\ \left[ \frac{1}{2} (R-r+1) + \frac{(R-r) - \sum_{n=1}^{R-r} \rho_1^n}{(R-r)(1 - \rho_1)} \right] P(0, \lambda_0) & (\rho_0 = 1) . \end{cases}$$

Similarly, the expressions for the conditional probabilities that the arrival rates are  $\lambda_0$  and  $\lambda_1$  given that  $r \leq l_s \leq R$  are, respectively, given by

$$(1.2.48) \quad P[\lambda_0/r \leq l_s \leq R] = \left\{ \begin{array}{l} \frac{\frac{1}{1-\rho_0} - (R-r) \frac{\rho_0^R}{\rho_0^r - \rho_0^R}}{\frac{1}{1-\rho_0} - \frac{\rho_0^R}{\rho_0^r - \rho_0^R} \left\{ (R-r) \frac{(\rho_0 - \rho_1)}{(1-\rho_1)} + \frac{(1-\rho_0) R-r}{(1-\rho_1) \sum_{n=1}^R \rho_1^n} \right\}}, \\ \frac{\frac{1}{2} (R-r+1)}{(R-r) - \sum_{n=1}^{R-r} \rho_1^n}, \\ \frac{1}{2} (R-r+1) + \frac{(R-r) - \sum_{n=1}^{R-r} \rho_1^n}{(R-r)(1-\rho_1)} \end{array} \right. ,$$

and

$$(1.2.49) \quad P[\lambda_1/r \leq l_s \leq R] = \left\{ \begin{array}{l} \frac{\frac{\rho_0^R}{r - \rho_0^R} \frac{1-\rho_0}{1-\rho_1} \left\{ (R-r) - \sum_{n=1}^{R-r} \rho_1^n \right\}}{\frac{1}{1-\rho_0} - \frac{\rho_0^R}{\rho_0^r - \rho_0^R} \left\{ (R-r) \frac{(\rho_0 - \rho_1)}{(1-\rho_1)} + \frac{(1-\rho_0) R-r}{(1-\rho_1) \sum_{n=1}^R \rho_1^n} \right\}}, \\ \frac{(R-r) - \sum_{n=1}^{R-r} \rho_1^n}{(R-r)(1-\rho_1)}, \\ \frac{1}{2} (R-r+1) + \frac{(R-r) - \sum_{n=1}^{R-r} \rho_1^n}{(R-r)(1-\rho_1)} \end{array} \right. ,$$

$\rho_0 = 1 .$

1.3. The expected number of units in the system ( $L_s$ ) and the corresponding waiting time ( $W_s$ ): The contribution to  $L_s$  from the states  $(n, \lambda_0)$  ( $n = 0, 1, \dots, R-1$ ) is given by

$$L_{s_0} = \sum_{n=0}^{R-1} n P(n, \lambda_0) = \sum_{n=0}^r n P(n, \lambda_0) + \sum_{n=r+1}^{R-1} n P(n, \lambda_0).$$

Case (i)  $\rho_0 \neq 1$

Now  $\sum_{n=0}^r n P(n, \lambda_0) = \sum_{n=0}^r n \rho_0^n P(0, \lambda_0)$  and after simplification we obtain

$$(1.3.1) \quad \sum_{n=0}^r n P(n, \lambda_0) = P(0, \lambda_0) \frac{\rho_0}{(1-\rho_0)^2} [(1-\rho_0^r) - r \rho_0^r (1-\rho_0)].$$

Similarly, it can be shown that

$$(1.3.2) \quad \sum_{n=r+1}^{R-1} n P(n, \lambda_0) = \sum_{n=r+1}^{R-1} \frac{n \rho_0^r P(0, \lambda_0)}{(\rho_0^r - \rho_0^R)} (\rho_0^n - \rho_0^R)$$

$$= P(0, \lambda_0) \frac{\rho_0^r}{(\rho_0^r - \rho_0^R)} \left[ \frac{\rho_0}{(1-\rho_0)^2} \left\{ r \rho_0^r (1-\rho_0) - R \rho_0^{R-1} (1-\rho_0) + \rho_0^r - \rho_0^R \right\} - \frac{1}{2} (R+r)(R-r-1) \rho_0^R \right].$$

Adding (1.3.1) and (1.3.2) we obtain

$$(1.3.3) \quad L_{s_0} = \left[ \frac{\rho_0}{(1-\rho_0)^2} - \frac{(R-\rho_0 r)}{1-\rho_0} \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} - \frac{1}{2} (R+r)(R-r-1) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \right] P(0, \lambda_0)$$

$$= \frac{\rho_0}{(1-\rho_0)^2} P(0, \lambda_0) - (R-\rho_0 r) \frac{P(r+1, \lambda_1)}{(1-\rho_0)^2} - \frac{1}{2} (R+r)(R-r-1) \frac{P(r+1, \lambda_1)}{(1-\rho_0)}$$

on using (1.2.21),  $\rho_0 \neq 1$ .

Case (ii)  $\rho_0 = 1$

$$(1.3.4) \quad \sum_{n=0}^r n P(n, \lambda_0) = \sum_{n=0}^r n P(0, \lambda_0) = P(0, \lambda_0) \frac{r(r+1)}{2}$$

and it can be shown that

$$(1.3.5) \quad \sum_{n=r+1}^{R-1} n P(n, \lambda_0) = \sum_{n=r+1}^{R-1} n \frac{R-n}{R-r} P(0, \lambda_0) \\ = \frac{P(0, \lambda_0)}{R-r} \left[ \frac{R}{2} \{(R-1)R-r(r+1)\} \right. \\ \left. - \frac{1}{6} \{(R-1)R(2R-1)-r(r+1)(2r+1)\} \right].$$

Combining (1.3.4) and (1.3.5) and simplifying we obtain

$$(1.3.6) \quad L_{s_0} = \frac{1}{6} (R^2 + Rr + r^2 - 1) P(0, \lambda_0) \\ = \frac{1}{6} (R-r)(R^2 + Rr + r^2 - 1) P(r+1, \lambda_1)$$

on using (1.2.26),  $\rho_0 = 1$ .

The contribution to  $L_s$  from the states  $(n, \lambda_1)$  ( $n \geq r+1$ ) is given by

$$L_{s_1} = \sum_{n=r+1}^{\infty} n P(n, \lambda_1) = \sum_{n=r+1}^{R-1} n P(n, \lambda_1) + \sum_{n=R}^{\infty} n P(n, \lambda_1).$$

It can be shown that

$$(1.3.7) \quad \sum_{n=r+1}^{R-1} n P(n, \lambda_1) = \sum_{n=r+1}^{R-1} n \frac{P(r+1, \lambda_1)}{(1-\rho_1)} (1-\rho_1)^{n-r} \\ = \frac{P(r+1, \lambda_1)}{(1-\rho_1)} \left[ \frac{1}{2} (R+r)(R-r-1) - \frac{1}{\rho_1^r} \frac{\rho_1}{(1-\rho_1)^2} \times \right. \\ \left. \times \{r \rho_1^r (1-\rho_1) - R \rho_1^{R-1} (1-\rho_1) + \rho_1^r - \rho_1^R\} \right]$$

$$\begin{aligned}
 (1.3.8) \quad \sum_{n=R}^{\infty} nP(n, \lambda_1) &= \sum_{n=R}^{\infty} n \frac{P(r+1, \lambda_1)}{(1-\rho_1)} (1 - \rho_1^{R-r}) \rho_1^{n-R} \\
 &= \frac{P(r+1, \lambda_1)}{(1-\rho_1)} (1 - \rho_1^{R-r}) \frac{\rho_1}{(1-\rho_1)^2} \left( \frac{R}{\rho_1} - R + 1 \right)
 \end{aligned}$$

and addition of (1.3.7) and (1.3.8) leads to

$$(1.3.9) \quad L_{s_1} = \frac{P(r+1, \lambda_1)}{(1-\rho_1)} \left[ \frac{1}{2}(R+r)(R-r-1) + \frac{(R-\rho_1 r)}{(1-\rho_1)} \right].$$

Adding (1.3.3) and (1.3.9) we obtain

$$\begin{aligned}
 (1.3.10) \quad L_s &= P(0, \lambda_0) \frac{\rho_0}{(1-\rho_0)^2} - P(r+1, \lambda_1) \left[ \frac{(R-\rho_0 r)}{(1-\rho_0)^2} \right. \\
 &\quad \left. + \frac{1}{2}(R+r)(R-r-1) \frac{(\rho_0 - \rho_1)}{(1-\rho_0)(1-\rho_1)} - \frac{(R-\rho_1 r)}{(1-\rho_1)^2} \right],
 \end{aligned}$$

$$\rho_0 \neq 1$$

where  $P(0, \lambda_0)$  and  $P(r+1, \lambda_1)$  are, respectively, given by (1.2.41) and (1.2.21).

Similarly, adding (1.3.6) and (1.3.9) we obtain

$$\begin{aligned}
 (1.3.11) \quad L_s &= P(r+1, \lambda_1) \left[ \frac{1}{6}(R-r)(R^2+Rr+r^2-1) + \frac{1}{2} \frac{(R+r)(R-r-1)}{(1-\rho_1)} \right. \\
 &\quad \left. + \frac{(R-\rho_1 r)}{(1-\rho_1)^2} \right], \quad \rho_0 = 1
 \end{aligned}$$

where  $P(r+1, \lambda_1) = \frac{P(0, \lambda_0)}{(R-r)}$  and  $P(0, \lambda_0)$  is given by (1.2.43).

The expected waiting time in the system, by making use of the well-known formula of Little [4], is given by

$$(1.3.12) \quad W_s = \frac{1}{\bar{\lambda}} L_s$$

where  $\bar{\lambda} (= \lambda_0 P(\lambda_0) + \lambda_1 P(\lambda_1))$  is the actual expected arrival rate under the operation of the management doctrine, and  $P(\lambda_0)$  and  $P(\lambda_1)$  are, respectively, given by (1.2.45) and (1.2.46).

1.4 The generating function of the probabilities of the number of units in the system: Define the partial generating functions

$$(1.4.1) \quad \begin{cases} h(z, \lambda_0) = \sum_{n=0}^{R-1} z^n P(n, \lambda_0) \\ h(z, \lambda_1) = \sum_{n=r+1}^{\infty} z^n P(n, \lambda_1) \end{cases} .$$

Denote the probability of finding  $n$  units in the system by  $P(n)$ ; then clearly the generating function of the  $P(n)$ 's is

$$(1.4.2) \quad h(z) = h(z, \lambda_0) + h(z, \lambda_1) .$$

With  $h(z, \lambda_0)$  given by (1.4.1) equations (1.2.10) through (1.2.14) lead to

$$\begin{aligned} & \frac{\mu}{z} [h(z, \lambda_0) - P(0, \lambda_0)] - (\mu + \lambda_0)h(z, \lambda_0) + \mu P(0, \lambda_0) \\ & + \lambda_0 z [h(z, \lambda_0) - z^{R-1}P(R-1, \lambda_0)] + \mu z^r P(r+1, \lambda_1) = 0 \end{aligned}$$

which on simplification gives

$$(1.4.3) \quad h(z, \lambda_0) = \frac{\mu(1-z)P(0, \lambda_0) + \lambda_0 z^{R+1}P(R-1, \lambda_0) - \mu z^{r+1}P(r+1, \lambda_1)}{\lambda_0 z^2 - (\mu + \lambda_0)z + \mu} .$$

Similarly with  $h(z, \lambda_1)$  given by (1.4.1) considering equations (1.2.15) through (1.2.18) we obtain

$$(1.4.4) \quad h(z, \lambda_1) = \frac{\mu z^{r+1}P(r+1, \lambda_1) - \lambda_0 z^{R+1}P(R-1, \lambda_0)}{\lambda_1 z^2 - (\mu + \lambda_1)z + \mu} .$$

Case (i)  $\rho_0 \neq 1$ : Clearly,  $h(z, \lambda_0)$  converges for all values of  $z$ . The denominator on the right hand side of (1.4.3) has two zeros occurring at  $z = 1$  and  $z = \mu / \lambda_0 = 1/\rho_0$  and therefore the numerator must vanish for these two values of  $z$ . That is, we have the following two equations

$$\lambda_0 P(R-1, \lambda_0) - \mu P(r+1, \lambda_1) = 0$$

$$\mu(1-1/\rho_0)P(0, \lambda_0) + \lambda_0(1/\rho_0)^{R+1}P(R-1, \lambda_0) - \mu(1/\rho_0)^{r+1}P(r+1, \lambda_1) = 0 .$$

Solving these equations we have

$$(1.4.5) \quad P(r+1, \lambda_1) = \rho_0 P(R-1, \lambda_0) \\ = \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (1-\rho_0)P(0, \lambda_0) .$$

As expected (1.4.5) is the same as (1.2.21).

Notice that  $h(z, \lambda_1)$  converges for  $|z| \leq 1$  and that the denominator on the right hand side of (1.4.4) has two zeros occurring at  $z=1$  and  $z = 1/\rho_1$ . As has been remarked before, for the existence of

equilibrium conditions we require  $\rho_1 < 1$ , that is,  $1/\rho_1 > 1$ , otherwise the server will be unable to deal with the customers as fast as they arrive. We, therefore, disregard the zero  $z = 1/\rho_1$ . As before the numerator on the right hand side of (1.4.4) must vanish at  $z = 1$  and this leads to the relation

$$P(r+1, \lambda_1) = \rho_0 P(R-1, \lambda_0)$$

which is the same as (1.4.5) and on using this relation we can write

$$(1.4.6) \quad h(z, \lambda_0) = \frac{\mu(1-z)P(0, \lambda_0) + \lambda_0 P(R-1, \lambda_0)(z^{R+1} - z^{r+1})}{\lambda_0 z^2 - (\mu + \lambda_0)z + \mu}$$

$$(1.4.7) \quad h(z, \lambda_1) = \frac{\lambda_0 P(R-1, \lambda_0)(z^{r+1} - z^{R+1})}{\lambda_1 z^2 - (\mu + \lambda_1)z + \mu}$$

where

$$P(R-1, \lambda_0) = \frac{\rho_0^{R+r-1}}{\rho_0^r - \rho_0^R} (1 - \rho_0) P(0, \lambda_0) .$$

On using the normalizing condition

$$1 = \lim_{z \rightarrow 1} h(z) = \lim_{z \rightarrow 1} [h(z, \lambda_0) + h(z, \lambda_1)]$$

it has been verified that  $P(0, \lambda_0)$  is given by (1.2.41). Thus we have determined the generating function  $h(z)$  completely when  $\rho_0 \neq 1$ .

Case (ii)  $\rho_0 = 1$ : When  $\rho_0 = 1$  (1.4.3) gives

$$h(z, \lambda_0) = \frac{(1-z)P(0, \lambda_0) + z^{R+1}P(R-1, \lambda_0) - z^{r+1}P(r+1, \lambda_1)}{(1-z)^2} .$$

Using similar arguments as before we obtain

$$(1.4.8) \quad P(r+1, \lambda_1) = P(R-1, \lambda_0) \\ = \frac{P(0, \lambda_0)}{(R-r)},$$

and

$$(1.4.9) \quad h(z, \lambda_0) = \frac{P(0, \lambda_0)}{(1-z)} \left[ 1 - \frac{1}{(R-r)} \sum_{n=r+1}^R z^n \right].$$

The expression for  $h(z, \lambda_1)$  remains the same as before, that is, it is given by (1.4.7) except that in this case  $P(R-1, \lambda_0)$  is given by (1.4.8).

Again on using the normalizing condition we arrive at

$$P(0, \lambda_0) = \frac{1}{\frac{1}{2}(R+r+1) + \frac{1}{(1-\rho_1)}}$$

which is the same as (1.2.43).

Having determined  $h(z)$ , we can find out the moments of the distribution of the number of units in the system. As a check for the expected number of units in the system we calculated  $h'(z)/z=1$  and got the same expressions as those given in section 1.3.

1.5 The Laplace transform of the distribution of the busy period and its expected length: To obtain this we follow Bailey's method for finding the distribution of the busy period in case of a simple queue [1]. That is, we consider the modified process which ceases as soon as the number of units in the system falls to zero. In other words,

the state  $(0, \lambda_0)$  may be considered as an absorbing state with the condition that at the beginning of the period the system is in a state  $(1, \lambda_0)$ .

With this understanding, we denote by  $Q(t, n, \lambda_1)$  the probability of the state  $(n, \lambda_1)$  at time  $t$ . Note that  $P(t, n, \lambda_1)$  introduced in section 1.2 has the same meaning as  $Q(t, n, \lambda_1)$  but for a different process. The differential-difference equations which the  $Q(t, n, \lambda_1)$ 's satisfy are:

$$(1.5.1) \quad Q'(t, 0, \lambda_0) = \mu Q(t, 1, \lambda_0)$$

$$(1.5.2) \quad \left\{ \begin{array}{l} Q'(t, 1, \lambda_0) = -(\mu + \lambda_0)Q(t, 1, \lambda_0) + \mu Q(t, 2, \lambda_0) \\ Q'(t, n, \lambda_0) = -(\mu + \lambda_0)Q(t, n, \lambda_0) + \mu Q(t, n+1, \lambda_0) + \lambda_0 Q(t, n-1, \lambda_0) \\ \qquad \qquad \qquad [n = 2, 3, \dots, r-1] \\ Q'(t, r, \lambda_0) = -(\mu + \lambda_0)Q(t, r, \lambda_0) + \mu Q(t, r+1, \lambda_0) \\ \qquad \qquad \qquad + \lambda_0 Q(t, r-1, \lambda_0) + \mu Q(t, r+1, \lambda_1) \\ Q'(t, n, \lambda_0) = -(\mu + \lambda_0)Q(t, n, \lambda_0) + \mu Q(t, n+1, \lambda_0) + \lambda_0 Q(t, n-1, \lambda_0) \\ \qquad \qquad \qquad [n=r+1, r+2, \dots, R-2] \\ Q'(t, R-1, \lambda_0) = -(\mu + \lambda_0)Q(t, R-1, \lambda_0) + \lambda_0 Q(t, R-2, \lambda_0) \end{array} \right.$$

$$\begin{aligned}
 (1.5.3) \quad & \left\{ \begin{aligned}
 Q'(t, r+1, \lambda_1) &= -(\mu + \lambda_1)Q(t, r+1, \lambda_1) + \mu Q(t, r+2, \lambda_1) \\
 Q'(t, n, \lambda_1) &= -(\mu + \lambda_1)Q(t, n, \lambda_1) + \mu Q(t, n+1, \lambda_1) + \lambda_1 Q(t, n-1, \lambda_1) \\
 & \quad [n = r+2, r+3, \dots, R-1] \\
 Q'(t, R, \lambda_1) &= -(\mu + \lambda_1)Q(t, R, \lambda_1) + \mu Q(t, R+1, \lambda_1) \\
 & \quad + \lambda_1 Q(t, R-1, \lambda_1) + \lambda_0 Q(t, R-1, \lambda_0) \\
 Q'(t, n, \lambda_1) &= -(\mu + \lambda_1)Q(t, n, \lambda_1) + \mu Q(t, n+1, \lambda_1) + \lambda_1 Q(t, n-1, \lambda_1) \\
 & \quad [n \geq R+1] .
 \end{aligned} \right.
 \end{aligned}$$

Define the partial generating functions

$$(1.5.4) \quad \left\{ \begin{aligned}
 q(t, z, \lambda_0) &= \sum_{n=1}^{R-1} z^n Q(t, n, \lambda_0) \\
 q(t, z, \lambda_1) &= \sum_{n=r+1}^{\infty} z^n Q(t, n, \lambda_1)
 \end{aligned} \right.$$

and because of the initial condition  $Q(0, n, \lambda_i) = 1 \quad n = 1, \quad i = 0$   
 $= 0$  otherwise

we have

$$(1.5.5) \quad \left\{ \begin{aligned}
 q(0, z, \lambda_0) &= z \\
 q(0, z, \lambda_1) &= 0
 \end{aligned} \right. .$$

Making use of (1.5.4), the set of equations (1.5.2) leads to

$$(1.5.6) \quad \frac{z \partial q(t, z, \lambda_0)}{\partial t} = [\lambda_0 z^2 - (\mu + \lambda_0)z + \mu]q(t, z, \lambda_0) - \mu z Q(t, 1, \lambda_0) \\ - \lambda_0 z^{R+1} Q(t, R-1, \lambda_0) + \mu z^{r+1} Q(t, r+1, \lambda_1).$$

Let us denote the Laplace transform of any function  $\varphi(t)$  by  $\varphi^*(s)$  where

$$(1.5.7) \quad \varphi^*(s) = \int_0^{\infty} e^{-st} \varphi(t) dt.$$

Taking the Laplace transform with respect to time of both sides of

(1.5.6) and using (1.5.5) we get

$$z[s q^*(s, z, \lambda_0) - z] = [\lambda_0 z^2 - (\mu + \lambda_0)z + \mu]q^*(s, z, \lambda_0) - \mu z Q^*(s, 1, \lambda_0) \\ - \lambda_0 z^{R+1} Q^*(s, R-1, \lambda_0) + \mu z^{r+1} Q^*(s, r+1, \lambda_1)$$

that is

$$(1.5.8) \quad q^*(s, z, \lambda_0) = \frac{\mu z Q^*(s, 1, \lambda_0) + \lambda_0 z^{R+1} Q^*(s, R-1, \lambda_0) - \mu z^{r+1} Q^*(s, r+1, \lambda_1) - z^2}{\lambda_0 z^2 - (s + \mu + \lambda_0)z + \mu}.$$

Since  $q^*(s, z, \lambda_0)$  converges for all values of  $z$  provided  $\text{Re}(s) > 0$ , the zeros of the denominator in (1.5.8) must coincide with those of the numerator. Let us denote the zeros of the quadratic expression  $\lambda_1 z^2 - (s + \mu + \lambda_1)z + \mu$  by  $\alpha_i(s)$  and  $\beta_i(s)$  where

$$(1.5.9) \quad \begin{cases} \alpha_i(s) = \frac{1}{2\lambda_i} \{(s + \mu + \lambda_i) + [(s + \mu + \lambda_i)^2 - 4\mu\lambda_i]^{1/2}\} \\ \beta_i(s) = \frac{1}{2\lambda_i} \{(s + \mu + \lambda_i) - [(s + \mu + \lambda_i)^2 - 4\mu\lambda_i]^{1/2}\} \end{cases} \\ (i = 0, 1)$$

so that  $\alpha_i + \beta_i = \frac{1}{\lambda_i} (s + \mu + \lambda_i)$ ,  $\alpha_i \beta_i = \frac{\mu}{\lambda_i}$  and in (1.5.9) we choose that value of the square root for which the real part is positive. Setting the numerator of (1.5.8) equal to zero and substituting the two zeros  $\alpha_0$  and  $\beta_0$  (at each of which the numerator must vanish), we have

$$(1.5.10) \begin{cases} \mu Q^*(s, 1, \lambda_0) + \lambda_0 \alpha_0^R Q^*(s, R-1, \lambda_0) - \mu \alpha_0^r Q^*(s, r+1, \lambda_1) = \alpha_0 \\ \mu Q^*(s, 1, \lambda_0) + \lambda_0 \beta_0^R Q^*(s, R-1, \lambda_0) - \mu \beta_0^r Q^*(s, r+1, \lambda_1) = \beta_0 \end{cases}$$

Similarly considering equations (1.5.3) and on making use of (1.5.4), (1.5.7) and (1.5.5) we obtain

$$(1.5.11) \quad q^*(s, z, \lambda_1) = \frac{\mu z^{r+1} Q^*(s, r+1, \lambda_1) - \lambda_0 z^{R+1} Q^*(s, R-1, \lambda_0)}{\lambda_1 z^2 - (s + \mu + \lambda_1)z + \mu}$$

Since  $q^*(s, z, \lambda_1)$  converges inside and on the unit circle for  $\text{Re}(s) > 0$ , the zeros of the denominator of (1.5.11) inside and on  $|z| = 1$  must coincide with the corresponding zeros of the numerator. Using Rouché's theorem, it can be shown that (see Saaty [6], page 89) the denominator of (1.5.11) has only one zero inside the unit circle and this zero is  $\beta_1$  as given by (1.5.9), and there is no zero on  $|z| = 1$ . Setting the numerator of (1.5.11) equal to zero at  $z = \beta_1$  we obtain

$$(1.5.12) \quad \mu Q^*(s, r+1, \lambda_1) = \lambda_0 \beta_1^{R-r} Q^*(s, R-1, \lambda_0)$$

Solving for  $Q^*(s, 1, \lambda_0)$  from equations (1.5.10) and (1.5.12) we obtain after lengthy but simple algebra and noting that  $\alpha_i \beta_i = \mu / \lambda_i$

$$(1.5.13) \quad Q^*(s, 1, \lambda_0) = \frac{(\alpha_0^{R-1} - \beta_0^{R-1})}{\lambda_0 (\alpha_0^R - \beta_0^R)} + \left(\frac{\mu}{\lambda_0}\right)^{r-1} \frac{(\alpha_0^{R-r} - \beta_0^{R-r})}{\lambda_0 (\alpha_0^R - \beta_0^R)} \frac{\beta_1^{R-r} (\alpha_0 - \beta_0)}{\{(\alpha_0^R - \beta_0^R) - \beta_1^{R-r} (\alpha_0^r - \beta_0^r)\}}.$$

Now the p.d.f. of the length of the busy period  $f_b(t)$  is clearly given by

$$(1.5.14) \quad f_b(t) = Q'(t, 0, \lambda_0) \\ = \mu Q(t, 1, \lambda_0).$$

Taking the Laplace transform of both sides of (1.5.14) we have

$$(1.5.15) \quad f_b^*(s) = \mu Q^*(s, 1, \lambda_0) \\ = \frac{(\mu/\lambda_0)}{(\alpha_0^R - \beta_0^R)} \left[ (\alpha_0^{R-1} - \beta_0^{R-1}) + \right. \\ \left. + (\mu/\lambda_0)^{r-1} \beta_1^{R-r} \frac{(\alpha_0 - \beta_0)(\alpha_0^{R-r} - \beta_0^{R-r})}{\{(\alpha_0^R - \beta_0^R) - \beta_1^{R-r} (\alpha_0^r - \beta_0^r)\}} \right].$$

Recall that  $\alpha_i$  and  $\beta_i$  ( $i = 0, 1$ ) are functions of  $s$  and are given by (1.5.9).

Thus we have determined the Laplace transform of the distribution of the busy period. On differentiating  $f_b^*(s)$  and setting  $s$  equal to zero we obtain the expected length of the busy period given by

$$(1.5.16) \quad E_b(t) = \begin{cases} \frac{1}{\mu - \lambda_0} + \frac{R-r}{\mu - \lambda_1} \frac{\lambda_1 - \lambda_0}{\lambda_0} \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R}, & \rho_0 \neq 1 \\ \frac{1}{\mu} \left[ \frac{1}{2}(R+r+1) + \frac{\rho_1}{1 - \rho_1} \right], & \rho_0 = 1. \end{cases}$$

1.6 The Laplace transform of the distribution of time the system spends in the subset of states  $\{(n, \lambda_1)\}$  before it goes out of the subset for the first time and its expected value,  $(i = 0, 1)$ : The distribution of time the system spends in the subset of states  $\{(n, \lambda_0)\}$  ( $n = 0, 1, \dots, R-1$ ) before it goes out of the subset for the first time is clearly the same thing as the distribution of time, in a simple queue, that elapses before the number of units in the system grows from 0 to R. The latter is known (see Saaty [6], Exercise no. 17, page 129) and we shall simply write down the results: Let  $f_0(t)$  denote the p.d.f. of the required distribution and let  $f_0^*(s)$  denote its Laplace transform, then

$$(1.6.1) \quad f_0^*(s) = \frac{\lambda_0(\alpha_0 - \beta_0)}{\lambda_0(\alpha_0^{R+1} - \beta_0^{R+1}) - \mu(\alpha_0^R - \beta_0^R)}$$

where  $\alpha_0$  and  $\beta_0$  are given by (1.5.9). The expected length of the period is given by

$$(1.6.2) \quad E_0(t) = \begin{cases} \frac{R}{\lambda_0 - \mu} + \frac{\mu}{(\lambda_0 - \mu)^2} [(\mu/\lambda_0)^R - 1], & \lambda_0 \neq \mu \\ \frac{R(R+1)}{2\mu}, & \lambda_0 = \mu. \end{cases}$$

Similarly, the distribution  $f_1(t)$  of the time the system spends in the subset of states  $\{(n, \lambda_1)\}$  ( $n = r+1, r+2, \dots$ ) before it goes out of the subset for the first time is the same thing as the distribution of time, in a simple queue, that elapses before the number of units in the system reaches  $r < R$ , where  $R$  is the number of units at time  $t = 0$ . The Laplace transform of  $f_1(t)$  is given by

$$(1.6.3) \quad f_1^*(s) = \beta_1^{R-r},$$

where again  $\beta_1$  is given by (1.5.9) and the expected value

$$(1.6.4) \quad E_1(t) = \frac{R-r}{\mu - \lambda_1}.$$

Note: Saaty has also found the inverse transform of  $f_1^*(s)$ .

1.7 The generating function of the Laplace transform with respect to time of the time-dependent probabilities of the number of units in the system: Let us denote by  $P(t, n)$  the probability at time  $t$  that there are  $n$  units in the system. Then clearly,

$$(1.7.1) \quad P(t, n) = P(t, n, \lambda_0) + P(t, n, \lambda_1)$$

Define the generating functions

$$(1.7.2) \quad \left\{ \begin{array}{l} h(t, z, \lambda_0) = \sum_{n=0}^{R-1} z^n P(t, n, \lambda_0) \\ h(t, z, \lambda_1) = \sum_{n=r+1}^{\infty} z^n P(t, n, \lambda_1) \\ h(t, z) = h(t, z, \lambda_0) + h(t, z, \lambda_1). \end{array} \right.$$

We assume that at time  $t = 0$  there are no units in the system and this leads to the initial condition

$$(1.7.3) \quad h(0, z, \lambda_1) = \begin{cases} 1, & i = 0 \\ 0, & i = 1 \end{cases} .$$

By considering equations (1.2.1) through (1.2.5) and making use of (1.7.2) we obtain

$$(1.7.4) \quad \frac{z \partial h(t, z, \lambda_0)}{\partial t} = [\lambda_0 z^2 - (\mu + \lambda_0)z + \mu]h(t, z, \lambda_0) + \mu(z-1)P(t, 0, \lambda_0) \\ - \lambda_0 z^{R+1}P(t, R-1, \lambda_0) + \mu z^{r+1}P(t, r+1, \lambda_1)$$

Taking the Laplace transform with respect to  $t$  of both sides of (1.7.4) and making use of (1.7.3) we obtain after a little simplification

$$(1.7.5) \quad h^*(s, z, \lambda_0) = \frac{\mu(1-z)P^*(s, 0, \lambda_0) + \lambda_0 z^{R+1}P^*(s, R-1, \lambda_0) - \mu z^{r+1}P^*(s, r+1, \lambda_1) - z}{\lambda_0 z^2 - (s + \mu + \lambda_0)z + \mu} .$$

Making the same kind of argument as in section (1.5) we arrive at the following equations

$$(1.7.6) \quad \mu(1-\alpha_0)P^*(s, 0, \lambda_0) + \lambda_0 \alpha_0^{R+1}P^*(s, R-1, \lambda_0) - \mu \alpha_0^{r+1}P^*(s, r+1, \lambda_1) = \alpha_0$$

$$\mu(1-\beta_0)P^*(s, 0, \lambda_0) + \lambda_0 \beta_0^{R+1}P^*(s, R-1, \lambda_0) - \mu \beta_0^{r+1}P^*(s, r+1, \lambda_1) = \beta_0$$

where  $\alpha_0$  and  $\beta_0$  are given by (1.5.9).

Similarly considering equations (1.2.6) through (1.2.9) we obtain as before

$$(1.7.7) \quad h^*(s, z, \lambda_1) = \frac{\mu z^{r+1} P^*(s, r+1, \lambda_1) - \lambda_0 z^{R+1} P^*(s, R-1, \lambda_0)}{\lambda_1 z^2 - (s + \mu + \lambda_1)z + \mu}$$

where

$$(1.7.8) \quad \mu P^*(s, r+1, \lambda_1) = \lambda_0 \beta_1^{R-r} P^*(s, R-1, \lambda_0)$$

and again  $\beta_1$  is given by (1.5.9).

Now (1.7.6) and (1.7.8) are three linear equations in three unknowns and hence all the unknowns can be determined. That is, both  $h^*(s, z, \lambda_0)$  and  $h^*(s, z, \lambda_1)$  can be determined. The generating function of the Laplace transform with respect to time of the  $P(t, n)$ 's is obviously given by

$$(1.7.9) \quad h^*(s, z) = h^*(s, z, \lambda_0) + h^*(s, z, \lambda_1) .$$

Note: Because of the fact that

$$(1.7.10) \quad \lim_{s \rightarrow 0} s P^*(s, n) = \lim_{t \rightarrow \infty} P(t, n)$$

we can get (1.4.3) and (1.4.4) from (1.7.5) and (1.7.6) respectively by means of (1.7.10). As a check we see that if we use the same operation on (1.7.8) we obtain (1.4.5).

1.8 Remarks: On putting  $\lambda_0 = \lambda_1$  all the results or formulas for a simple queue, subject to the usual restriction  $\rho_0 < 1$  for the existence of equilibrium conditions, can be obtained from this chapter.

It is possible to superimpose a cost structure on the problem and on defining a cost function, optimization may be carried out. The parameters  $R$ ,  $r$  and  $\lambda_1$  are assumed to be under the control of management and appendix (a) summarizes an approach to the selection of their optimal values.

## CHAPTER II

### A SINGLE-SERVER QUEUE WITH CONTROLLABLE ARRIVAL

#### RATE - THE GENERAL CASE

2.1 Introduction. In this chapter the assumptions for the distribution of the service time is the same as in Chapter I. The assumption regarding the arrival process is more general in the sense that we allow the arrival rate  $\lambda$  to take  $(N+1)$  values, where  $N$  is an integer  $\geq 1$ .

Let the  $(N+1)$  values of  $\lambda$  be denoted by  $\lambda_0, \lambda_1, \dots, \lambda_N$  respectively and let  $(r_1, R_1), (r_2, R_2), \dots, (r_N, R_N)$  be  $N$  pairs of integers satisfying  $r_1 \geq 0, r_1 < r_2 < \dots < r_N;$   
 $R_1 < R_2 < \dots < R_N$  and  $r_1 < R_1, r_2 < R_2, \dots, r_N < R_N$ .  
Suppose at some instant of time there are no units in the system. Units then start coming in a Poisson stream with intensity  $\lambda = \lambda_0$ . This arrival rate  $\lambda_0$  remains unchanged as long as the number of units in the system ( $l_s$ ) is strictly less than  $R_1$ . When  $l_s$  assumes a value  $R_1$  the arrival rate changes instantaneously from  $\lambda_0$  to  $\lambda_1$ . After the change takes place, the arrival rate will continue to be  $\lambda_1$  as long as  $r_1+1 \leq l_s \leq R_2-1$ . If  $l_s$  goes down to  $r_1$  then changes back from  $\lambda_1$  to  $\lambda_0$  and in order that  $\lambda_0$  change again to  $\lambda_1, l_s$  has to grow to size  $R_1$  and the same process is repeated. With  $\lambda$  assuming a value  $\lambda_1, l_s$  should grow to a size  $R_2$

the arrival rate changes instantaneously from  $\lambda_1$  to  $\lambda_2$  and it continues to be  $\lambda_2$  as long as  $r_2 + 1 \leq l_s \leq R_3 - 1$ . If  $l_s$  goes down to  $r_2$  then  $\lambda$  changes back to  $\lambda_1$  whereas if  $l_s$  reaches a value  $R_3$  the arrival rate changes instantaneously from  $\lambda_2$  to  $\lambda_3$ . When  $\lambda$  changes from  $\lambda_2$  to  $\lambda_3$ , the arrival rate will continue to be  $\lambda_3$  as long as  $r_3 + 1 \leq l_s \leq R_4 - 1$  and will change back to  $\lambda_2$  if and only if  $l_s$  goes down to  $r_3$  and will assume a value  $\lambda_4$  should  $l_s$  reach a size  $R_4$  and so on. When  $\lambda$  assumes a value  $\lambda_{N-1}$  we have  $r_{N-1} + 1 \leq l_s \leq R_N - 1$ . If  $l_s$  goes down to a value  $r_{N-1}$ ,  $\lambda$  changes from  $\lambda_{N-1}$  to  $\lambda_{N-2}$  whereas if  $l_s$  grows to size  $R_N$   $\lambda$  will change instantaneously to  $\lambda_N$  and at this last stage it will continue to be  $\lambda_N$  unless  $l_s$  goes down to  $r_N$  when  $\lambda$  changes back to  $\lambda_{N-1}$ ; that is,  $\lambda = \lambda_N$  for all  $l_s \geq r_N + 1$ .

In this chapter we shall restrict ourselves strictly to equilibrium conditions and we need (as in Chapter I) the restriction

$\lambda_N < \mu$  for the existence of stationary probabilities. When we draw the state diagram of this process we see that there are  $N$  loops which may or may not overlap depending upon the values of the  $r_i$ 's and  $R_i$ 's ( $i = 1, 2, \dots, N$ ). In every case we have  $(N+1)$  sets of equations corresponding to the  $(N+1)$  values of  $\lambda$ . We shall again denote by  $(n, \lambda_i)$  the state that  $l_s = n$  and  $\lambda = \lambda_i$  ( $n = 0, 1, 2, \dots$ ;  $i = 0, 1, 2, \dots, N$ ).

2.2 The distribution of  $l_s$ , the number of units in the system:

(I) The case of  $N$  non-overlapping loops: This is the case where  $(0 \leq r_1 < R_1 < r_2 < R_2 < \dots < R_{N-1} < r_N < R_N)$ . The diagram of the process for this case is as follows

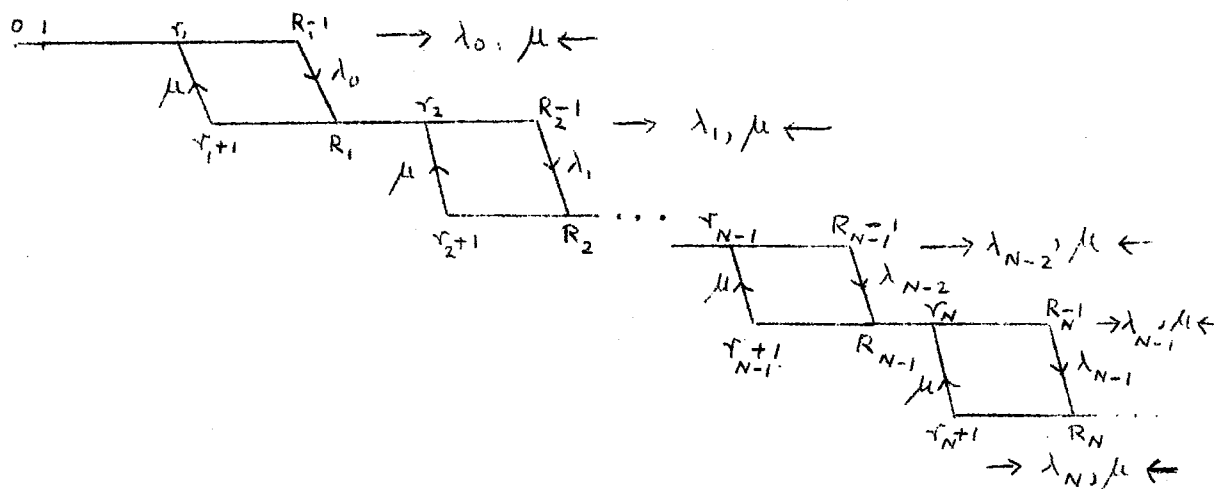


Diagram 2.1

The stationary probabilities  $P(n, \lambda_i)$ 's satisfy the following  $(N+1)$  sets of difference equations. We shall refer to the set corresponding to  $\lambda_i$  as the  $i$ -th set and denote it by (i) ( $i = 0, 1, \dots, N$ ).

Set (0)

$$(2.2.1) \quad \mu P(1, \lambda_0) - \lambda_0 P(0, \lambda_0) = 0$$

$$(2.2.2) \quad \mu P(n+2, \lambda_0) - (\mu + \lambda_0) P(n+1, \lambda_0) + \lambda_0 P(n, \lambda_0) = 0$$

$$(n = 0, 1, \dots, r_1 - 2)$$

$$(2.2.3) \quad \mu P(r_1+1, \lambda_1) + \mu P(r_1+1, \lambda_0) - (\mu + \lambda_0) P(r_1, \lambda_0) + \lambda_0 P(r_1-1, \lambda_0) = 0$$

$$(2.2.4) \quad \mu P(n+2, \lambda_0) - (\mu + \lambda_0)P(n+1, \lambda_0) + \lambda_0 P(n, \lambda_0) = 0$$

$$(n = r_1, r_1+1, \dots, R_1-3)$$

$$(2.2.5) \quad - (\mu + \lambda_0)P(R_1-1, \lambda_0) + \lambda_0 P(R_1-2, \lambda_0) = 0.$$

Set (i)  $i = 1, 2, \dots, N-1$

$$(2.2.6) \quad \mu P(r_i+2, \lambda_i) - (\mu + \lambda_i)P(r_i+1, \lambda_i) = 0$$

$$(2.2.7) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i)P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = r_i+1, r_i+2, \dots, R_i-2)$$

$$(2.2.8) \quad \mu P(R_i+1, \lambda_i) - (\mu + \lambda_i)P(R_i, \lambda_i) + \lambda_i P(R_i-1, \lambda_i) + \lambda_{i-1} P(R_i-1, \lambda_{i-1})$$

$$= 0$$

$$(2.2.9) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i)P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = R_i, R_i+1, \dots, r_{i+1}-2)$$

$$(2.2.10) \quad \mu P(r_{i+1}, \lambda_{i+1}) + \mu P(r_{i+1}, \lambda_i) - (\mu + \lambda_i)P(r_{i+1}, \lambda_i) + \lambda_i P(r_{i+1}, \lambda_i) = 0$$

$$(2.2.11) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i)P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = r_{i+1}, r_{i+1}+1, \dots, R-3)$$

$$(2.2.12) \quad - (\mu + \lambda_i)P(R_{i+1}-1, \lambda_i) + \lambda_i P(R_{i+1}-2, \lambda_i) = 0.$$

Set (N)

$$(2.2.13) \quad \mu P(r_N+2, \lambda_N) - (\mu + \lambda_N)P(r_N+1, \lambda_N) = 0$$

$$(2.2.14) \quad \mu P(n+2, \lambda_N) - (\mu + \lambda_N)P(n+1, \lambda_N) + \lambda_N P(n, \lambda_N) = 0$$

$$(n = r_N+1, r_N+2, \dots, R_N-2)$$

$$(2.2.15) \quad \mu P(R_N+1, \lambda_N) - (\mu + \lambda_N)P(R_N, \lambda_N) + \lambda_N P(R_N-1, \lambda_N) + \lambda_{N-1} P(R_N-1, \lambda_{N-1}) = 0$$

$$(2.2.16) \quad \mu P(n+2, \lambda_N) - (\mu + \lambda_N)P(n+1, \lambda_N) + \lambda_N P(n, \lambda_N) = 0 \quad (n \geq R_N)$$

In solving these  $(N+1)$  sets of equations we shall deal only with the case where  $\lambda_i \neq \mu$  ( $i = 0, 1, 2, \dots, N-1$ ). The results for some or all  $\lambda_i = \mu$  can be obtained as limiting cases.

Notice that when we replace  $r_1$  and  $R_1$  by  $r$  and  $R$  respectively equations of the set (0) are the same as equations (1.2.10) through (1.2.14) and hence the solutions to equations of the set (0) are given by

$$(2.2.17) \quad \begin{cases} P(n, \lambda_0) = \rho_0^n P(0, \lambda_0) & (n = 0, 1, \dots, r_1), \\ P(n, \lambda_0) = \frac{\rho_0^{r_1}}{\rho_0^{r_1} - \rho_0^{R_1}} (\rho_0^n - \rho_0^{R_1}) P(0, \lambda_0) & (n = r_1, r_1+1, \dots, R_1-1) \end{cases}$$

and

$$(2.2.18) \quad \begin{aligned} P(r_1+1, \lambda_1) &= \rho_0 P(R_1-1, \lambda_0) \\ &= \frac{\rho_0^{R_1+r_1}}{\rho_0^{r_1} - \rho_0^{R_1}} (1 - \rho_0) P(0, \lambda_0) \end{aligned}$$

where  $\rho_i = \lambda_i / \mu$  ( $i = 0, 1, 2, \dots, N$ ).

Next we shall solve equations of the set

(i) for  $i = 1$ . Equations (2.2.6), (2.2.7) and (2.2.8) are the same as (1.2.15), (1.2.16) and (1.2.17) when we replace  $r_1$  and  $R_1$  by

$r$  and  $R$  respectively and therefore we have

$$(2.2.19) \quad P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r_1})$$

$$(n=r_1+1, r_1+2, \dots, R_1)$$

and

$$(2.2.20) \quad P(R_1+1, \lambda_1) = \rho_1 P(R_1, \lambda_1) .$$

The solution to (2.2.9) which is an ordinary homogeneous difference equation of the second degree with (2.2.20) as the initial condition is

$$(2.2.21) \quad P(n, \lambda_1) = P(R_1, \lambda_1) \rho_1^{n-R_1}$$

$$= \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{R_1-r_1}) \rho_1^{n-R_1}$$

$$(n = R_1, R_1+1, \dots, r_2) .$$

Similarly the solution to (2.2.11) with (2.2.12) as the initial condition is given by

$$(2.2.22) \quad P(n, \lambda_1) = \frac{P(r_2, \lambda_1)}{\rho_1^{r_2} - \rho_1^{R_2}} (\rho_1^n - \rho_1^{R_2})$$

$$(n = r_2+1, r_2+2, \dots, R_2-1)$$

where  $P(r_2, \lambda_1)$  is given by (2.2.21).

Notice that we can write equation (2.2.10) as

$$P(r_2+1, \lambda_2) = (1 + \rho_1)P(r_2, \lambda_1) - \rho_1 P(r_2-1, \lambda_1) - P(r_2+1, \lambda_1)$$

and substituting the values of  $P(r_2, \lambda_1)$ ,  $P(r_2-1, \lambda_1)$  and  $P(r_2+1, \lambda_1)$  from (2.2.21) and (2.2.22) we arrive at the relation

$$\begin{aligned} (2.2.23) \quad P(r_2+1, \lambda_2) &= \frac{P(r_2, \lambda_1)}{\rho_1 - \rho_1} \rho_1^{R_2} (1 - \rho_1) \\ &= \rho_1 P(R_2-1, \lambda_1) \quad , \end{aligned}$$

on using (2.2.22) for  $n = R_2 - 1$ .

Also on using (2.2.21) for  $n = r_2$  we can express  $P(r_2+1, \lambda_2)$  in terms of  $P(r_1+1, \lambda_1)$ . The relation is given by

$$(2.2.24) \quad P(r_2+1, \lambda_2) = \frac{\rho_1^{R_2+r_2}}{\rho_1 - \rho_1} \frac{\rho_1^{r_1} - \rho_1^{R_1}}{\rho_1^{R_1+r_1}} P(r_1+1, \lambda_1)$$

and recall that  $P(r_1+1, \lambda_1)$  is given by (2.2.18).

On using (2.2.23) we obtain an expression alternative to (2.2.22) for  $P(n, \lambda_1)$  and this is

$$(2.2.25) \quad P(n, \lambda_1) = \frac{P(r_2+1, \lambda_2)}{(1 - \rho_1)} (\rho_1^{n-R_2} - 1)$$

$$(n = r_2, r_2+1, r_2+2, \dots, R_2-1) .$$

Let us now summarize the solutions to equations of the set (1)

$$(2.2.26) \quad \left\{ \begin{array}{l} P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r_1}) \\ \qquad \qquad \qquad (n=r_1+1, r_1+2, \dots, R_1) \ , \\ P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{R_1-r_1}) \rho_1^{n-R_1} \\ \qquad \qquad \qquad (n=R_1, R_1+1, \dots, r_2) \ , \\ P(n, \lambda_1) = \frac{P(r_2+1, \lambda_2)}{(1 - \rho_1)} (\rho_1^{n-R_2} - 1) \\ \qquad \qquad \qquad (n=r_2, r_2+1, \dots, R_2-1) \end{array} \right.$$

where

$$\begin{aligned} P(r_2+1, \lambda_2) &= \frac{\rho_1^{R_2+r_2}}{\rho_1^{r_2} - \rho_1^{R_2}} \frac{\rho_1^{r_1} - \rho_1^{R_1}}{\rho_1^{R_1+r_1}} P(r_1+1, \lambda_1) \\ &= \rho_1 P(R_2-1, \lambda_1) \end{aligned}$$

and

$$\begin{aligned} P(r_1+1, \lambda_1) &= \rho_0 P(R_1-1, \lambda_0) \\ &= \frac{\rho_0^{R_1+r_1}}{\rho_0^{r_1} - \rho_0^{R_1}} (1 - \rho_0) P(0, \lambda_0) \ . \end{aligned}$$

Recall that this last relation is given by the set (0).

Notice that the sets of equations (i) ( $i = 1, 2, \dots, N-1$ ) are exactly the same except for the differences in the subscripts of the parameters. In solving the set (1) we made use of the information  $P(r_1+1, \lambda_1) = \rho_0 P(R_1-1, \lambda_0)$  given by the set (0). Similarly set (1) gives the information  $P(r_2+1, \lambda_2) = \rho_1 P(R_2-1, \lambda_1)$  for

solving set (2). In view of the remark made at the beginning of this paragraph, clearly the solution to set (2) can be obtained from the solution to set (1) by merely replacing  $\lambda_1, \lambda_2, \rho_1, r_1, R_1, r_2$  and  $R_2$  respectively by  $\lambda_2, \lambda_3, \rho_2, r_2, R_2, r_3$  and  $R_3$ . The information supplied by set (2) for solving set (3) is

$P(r_3+1, \lambda_3) = \rho_2 P(R_3-1, \lambda_2)$ . In this way we solve equations of the sets (1), (2), ..., (N-1) successively. Furthermore, we notice that equations of the set (N) are the same as equations (1.2.15) through (1.2.18) when we replace  $\lambda_0, \lambda_1, r$  and  $R$  respectively by  $\lambda_{N-1}, \lambda_N, r_N$  and  $R_N$  and hence the solution to equations of the set (N) can be obtained from the solution to equations (1.2.15) through (1.2.18).

We now write in condensed notation the solutions of the equations of the sets (0), (1), ..., (N) under the restrictions  $\rho_i \neq 1$  ( $i = 0, 1, \dots, N-1$ ) and, as has been remarked before, for the existence of the stationary probabilities we need  $\rho_N < 1$ .

$$(2.2.27) \quad \left\{ \begin{array}{l} P(n, \lambda_0) = \rho_0^n P(0, \lambda_0) \quad (n=0, 1, \dots, r_1) , \\ P(n, \lambda_0) = \frac{\rho_0^{r_1}}{r_1 - \rho_0} \left( \rho_0^n - \rho_0^{R_1} \right) P(0, \lambda_0) \\ \quad (n=r_1, r_1+1, \dots, R_1-1) . \end{array} \right.$$

For  $i = 1, 2, \dots, N-1$ , we have

$$(2.2.28) \begin{cases} P(n, \lambda_i) = \frac{P(r_i+1, \lambda_i)}{(1 - \rho_i)} (1 - \rho_i)^{n-r_i} (n=r_i+1, r_i+2, \dots, R_i) , \\ P(n, \lambda_i) = \frac{P(r_i+1, \lambda_i)}{(1 - \rho_i)} (1 - \rho_i)^{R_i-r_i} \rho_i^{n-R_i} (n=R_i, R_i+1, \dots, r_{i+1}), \\ P(n, \lambda_i) = \frac{P(r_{i+1}+1, \lambda_{i+1})}{(1 - \rho_i)} (\rho_i^{n-R_{i+1}-1}) \\ \quad (n=r_{i+1}, r_{i+1}+1, \dots, R_{i+1}). \end{cases}$$

Finally, we have

$$(2.2.29) \begin{cases} P(n, \lambda_N) = \frac{P(r_N+1, \lambda_N)}{(1 - \rho_N)} (1 - \rho_N)^{n-r_N} \\ \quad (n=r_N+1, r_N+2, \dots, R_N) , \\ P(n, \lambda_N) = \frac{P(r_N+1, \lambda_N)}{(1 - \rho_N)} (1 - \rho_N)^{R_N-r_N} \rho_N^{n-R_N} (n \geq R_N) \end{cases}$$

where

$$(2.2.30) \begin{cases} P(r_1+1, \lambda_1) = \rho_0 P(R_1-1, \lambda_0) \\ \quad = \frac{\rho_0^{R_1+r_1}}{r_1 - \rho_0} (1 - \rho_0) P(0, \lambda_0), \quad \text{and} \\ P(r_{i+1}+1, \lambda_{i+1}) = \rho_i P(R_{i+1}-1, \lambda_i) \end{cases}$$

$$(2.2.30) \quad = \frac{\rho_i^{R_{i+1}+r_{i+1}}}{r_{i+1} - \rho_i^{R_{i+1}}} \frac{\rho_i^{r_i} - \rho_i^{R_i}}{R_{i+1} + r_i} P(r_{i+1}, \lambda_i)$$

$$(i = 1, 2, \dots, N-1) .$$

The only unknown involved now is  $P(0, \lambda_0)$  which will be determined later from the normalizing equation. We postpone this for the time being.

(II) The case of  $N$  overlapping loops: That is, the case where besides the general restrictions on the  $r_i$ 's and  $R_i$ 's given in section (2.1) we have an additional restriction  $r_{i+1} < R_i$  ( $i = 1, 2, \dots, N-1$ ). The diagram of the process for this case is given below

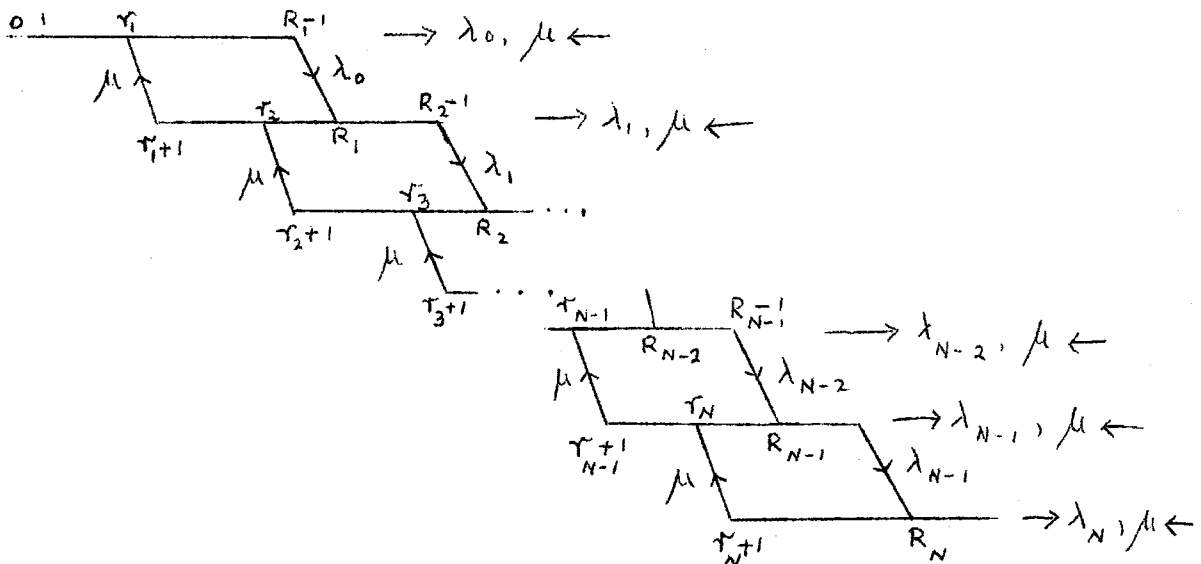


Diagram 2.2

The sets (O) and (N) of equations which the stationary probabilities  $P(n, \lambda_i)$ 's satisfy are exactly the same as in case (I). The  $i$ -th set is given by

Set (i)  $i = 1, 2, \dots, N-1$

$$(2.2.31) \quad \mu P(r_i+2, \lambda_i) - (\mu + \lambda_i) P(r_i+1, \lambda_i) = 0$$

$$(2.2.32) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i) P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = r_i+1, r_i+2, \dots, r_{i+1} - 2)$$

$$(2.2.33) \quad \mu P(r_{i+1}+1, \lambda_{i+1}) + \mu P(r_{i+1}, \lambda_{i+1}) - (\mu + \lambda_i) P(r_{i+1}, \lambda_i) + \lambda_i P(r_{i+1}-1, \lambda_i) = 0$$

$$(2.2.34) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i) P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = r_{i+1}, r_{i+1}+1, \dots, R_{i+1}-2)$$

$$(2.2.35) \quad \mu P(R_{i+1}, \lambda_i) - (\mu + \lambda_i) P(R_{i+1}, \lambda_i) + \lambda_i P(R_{i+1}-1, \lambda_i) + \lambda_{i-1} P(R_{i+1}-1, \lambda_{i-1}) = 0$$

$$(2.2.36) \quad \mu P(n+2, \lambda_i) - (\mu + \lambda_i) P(n+1, \lambda_i) + \lambda_i P(n, \lambda_i) = 0$$

$$(n = R_i, R_i+1, \dots, R_{i+1} - 3)$$

$$(2.2.37) \quad - (\mu + \lambda_i) P(R_{i+1}-1, \lambda_i) + \lambda_i P(R_{i+1}-2, \lambda_i) = 0.$$

Again we first solve equations of the set (i) for  $i = 1$ . Notice that (2.2.31) and (2.2.32) are exactly the same as (2.2.6) and (2.2.7) respectively when we replace  $r_{i+1}$  by  $R_i$  and hence we have

$$(2.2.38) \quad P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r_1})$$

$$(n = r_1+1, r_1+2, \dots, r_2)$$

where  $P(r_1+1, \lambda_1)$  is given by (2.2.18).

The equation (2.2.33) can be written as

$$P(r_2+1, \lambda_1) = (1 + \rho_1)P(r_2, \lambda_1) - \rho_1 P(r_2-1, \lambda_1) - P(r_2+1, \lambda_2).$$

Substituting the values of  $P(r_2, \lambda_1)$  and  $P(r_2-1, \lambda_1)$  from (2.2.38) in the above equation after some algebraical steps we arrive at

$$(2.2.39) \quad P(r_2+1, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{r_2-r_1+1}) - P(r_2+1, \lambda_2)$$

where  $P(r_2+1, \lambda_2)$  is not yet known.

The solution to the difference equation (2.2.34) with (2.2.38) for  $n = r_2$  and (2.2.39) as initial conditions is given by

$$(2.2.40) \quad P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} (1 - \rho_1^{n-r_1}) - \frac{P(r_2+1, \lambda_2)}{(1 - \rho_1)} (1 - \rho_1^{n-r_2})$$

$$(n = r_2, r_2+1, \dots, R_1).$$

Next, equation (2.2.35) can be written as

$$P(R_1+1, \lambda_1) = (1 + \rho_1)P(R_1, \lambda_1) - \rho_1 P(R_1-1, \lambda_1) - \rho_0 P(R_1-1, \lambda_0) .$$

Making use of (2.2.40) for  $n = R_1$  and  $n = R_1-1$  and also the relation  $P(r_1+1, \lambda_1) = \rho_0 P(R_1-1, \lambda_0)$  we arrive at

$$(2.2.41) \quad P(R_1+1, \lambda_1) = \rho_1 P(R_1, \lambda_1) - P(r_2+1, \lambda_2) .$$

The solution to (2.2.36) with (2.2.40) for  $n = R_1$  and (2.2.41) as initial conditions is given by

$$(2.2.42) \quad P(n, \lambda_1) = \frac{P(r_1+1, \lambda_1)}{(1 - \rho_1)} \frac{\rho_1^{r_1} - \rho_1^{R_1}}{\rho_1^{R_1+r_1}} \rho_1^n + \frac{P(r_2+1, \lambda_2)}{(1 - \rho_1)} (\rho_1^{n-r_2} - 1) \\ (n = R_1, R_1+1, \dots, R_2-1) .$$

Lastly, equation (2.2.37) leads to

$$(1 + \rho_1)P(R_2-1, \lambda_1) = \rho_1 P(R_2-2, \lambda_1) .$$

Substituting the values of  $P(R_2-1, \lambda_1)$  and  $P(R_2-2, \lambda_1)$  from (2.2.42) into the above equation and after some algebraical steps we obtain

$$(2.2.43) \quad P(r_2+1, \lambda_2) = \frac{\rho_1^{R_2+r_2}}{\rho_2 - \rho_1} \frac{\rho_1^{r_1} - \rho_1^{R_1}}{\rho_1^{R_1+r_1}} P(r_1+1, \lambda_1) .$$

On substituting the value of  $P(r_1+1, \lambda_1)$  from (2.2.43) into (2.2.42), it can be shown that

$$(2.2.43) \quad P(n, \lambda_1) = \frac{P(r_2+1, \lambda_2)}{(1 - \rho_1)} \left( \rho_1^{n-R_2} - 1 \right) \\ (n = R_1, R_1+1, \dots, R_2-1),$$

from which we also have the relation

$$(2.2.44) \quad P(r_2+1, \lambda_2) = \rho_1 P(R_2-1, \lambda_1) .$$

Thus we have completely solved equations of the set (1) and note that for solving set (1) the only information we need from the set (0) is the relation  $P(r_1+1, \lambda_1) = \rho_0 P(R_1-1, \lambda_0)$ . Similarly, we use the relation  $P(r_2+1, \lambda_2) = \rho_1 P(R_2-1, \lambda_1)$  for solving set (2). Making exactly the same argument as in case (I) we give below the solutions of the sets (i) ( $i = 0, 1, \dots, N$ ),

$$(2.2.45) \quad \begin{cases} P(n, \lambda_0) = \rho_0^n P(0, \lambda_0) & (n = 0, 1, \dots, r_1) , \\ P(n, \lambda_0) = \frac{\rho_0^{r_1}}{\rho_0^{r_1} - \rho_0^{R_1}} (\rho_0^n - \rho_0^{R_1}) P(0, \lambda_0) & (n = r_1, r_1+1, \dots, R_1-1) . \end{cases}$$

For  $i = 1, 2, \dots, N-1$  we have

$$\begin{aligned}
 (2.2.46) \quad & P(n, \lambda_i) = \frac{P(r_{i+1}, \lambda_i)}{(1-\rho_i)} (1 - \rho_i^{n-r_i}) \\
 & \quad (n=r_i+1, r_i+2, \dots, r_{i+1}) \quad , \\
 & P(n, \lambda_i) = \frac{P(r_{i+1}, \lambda_i)}{(1-\rho_i)} (1 - \rho_i^{n-r_i}) - \\
 & \quad - \frac{P(r+1, \lambda_{i+1})}{(1-\rho_i)} (1 - \rho_i^{n-r_{i+1}}) \\
 & \quad (n=r_{i+1}, r+1, \dots, R_i) \quad , \\
 & P(n, \lambda_i) = \frac{P(r+1, \lambda_{i+1})}{(1-\rho_i)} (\rho_i^{n-R_{i+1}} - 1) \\
 & \quad (n = R_i, R_i+1, \dots, R_{i+1}) \quad .
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 (2.2.47) \quad & P(n, \lambda_N) = \frac{P(r_N+1, \lambda_N)}{(1-\rho_N)} (1 - \rho_N^{n-r_N}) \\
 & \quad (n = r_N+1, r_N+2, \dots, R_N) \quad , \\
 & P(n, \lambda_N) = \frac{P(r_N+1, \lambda_N)}{(1-\rho_N)} (1 - \rho_N^{R_N-r_N}) \rho_N^{n-R_N} \quad (n \geq R_N)
 \end{aligned}$$

where

$$\begin{aligned}
 (2.2.48) \quad P(r_1+1, \lambda_1) &= \rho_0 P(R_1-1, \lambda_0) \\
 &= \frac{\rho_0}{r_1 - \rho_0} \frac{R_1+r_1}{R_1} (1 - \rho_0) P(0, \lambda_0), \quad \text{and} \\
 P(r_{i+1}+1, \lambda_{i+1}) &= \rho_i P(R_i-1, \lambda_i) \\
 &= \frac{\rho_i}{r_{i+1} - \rho_i} \frac{R_{i+1}+r_{i+1}}{R_{i+1}} \frac{\rho_i - \rho_i}{R_i+r_i} P(r_i+1, \lambda_i) \\
 &\quad (i = 1, 2, \dots, N-1) .
 \end{aligned}$$

Again  $P(0, \lambda_0)$  will be determined later from the normalizing equation.

(III) The case of  $N$  trivially overlapping loops: This is the case where  $r_{i+1} = R_i$  ( $i = 1, 2, \dots, N-1$ ) besides the restrictions on the  $r_i$ 's and  $R_i$ 's as given in section (2.1).

The diagram of the process for this case is as follows

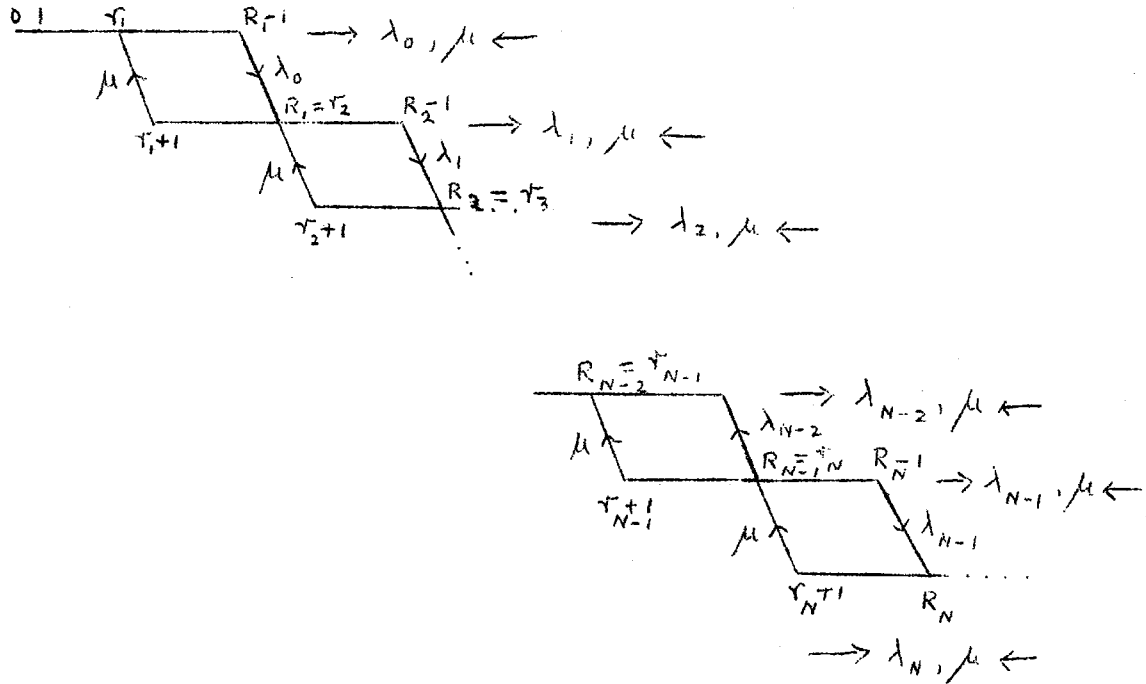


Diagram 2.3

This case is really a special case of both (I) and (II) with the understanding that  $r_{i+1} = R_i$  ( $i = 1, 2, \dots, N-1$ ). The difference equations satisfied by the stationary probabilities are the same as in cases (I) and (II), except that instead of equations (2.2.8), (2.2.9), (2.2.10) of case (I) or equations (2.2.33), (2.2.34), (2.2.35) of case (II) we only have the single equation

$$(2.2.49) \quad \mu P_{i+1}(r+1, \lambda_{i+1}) + \mu P_{i+1}(r+1, \lambda_i) - (\mu + \lambda_i) P_{i+1}(r, \lambda_i) \\ + \lambda_i P_{i+1}(r-1, \lambda_i) + \lambda_{i-1} P_{i+1}(R_i-1, \lambda_{i-1}) = 0.$$

The solutions for the  $P(n, \lambda_i)$ 's are the same as in case (I) except that in (2.2.28) the case where  $n = R_i+1, \dots, r_{i+1}$  does not arise, or are the same as in case (II) except that in (2.2.45) the case where  $n = r+1, \dots, R_i$  does not arise.

Having obtained the solutions for the above three basic cases it is easy to see how to obtain the solutions for other cases that we have not considered.

Let  $P(\lambda_i)$  ( $i = 0, 1, \dots, N$ ) denote the probability of finding the system in a state where the arrival rate is  $\lambda_i$ . Clearly  $P(\lambda_i) = \sum_{n=0}^{\infty} P(n, \lambda_i)$ . It can be shown, on making use of relations between  $P(r+1, \lambda_{i+1})$ ,  $P(r_i+1, \lambda_i)$  and  $P(0, \lambda_0)$  as given by (2.2.30) or (2.2.48), that in every case

$$(2.2.50) \quad P(\lambda_0) = \sum_{n=0}^{R_1-1} P(n, \lambda_0) = \frac{P(0, \lambda_0)}{(1 - \rho_0)} - (R_1 - r_1) \frac{P(r_1+1, \lambda_1)}{(1 - \rho_0)},$$

$$(2.2.51) \quad P(\lambda_i) = \sum_{n=r_i+1}^{R-1} P(n, \lambda_i) = (R_i - r_i) \frac{P(r_i+1, \lambda_i)}{(1 - \rho_i)} - (R_{i+1} - r_{i+1}) \frac{P(r+1, \lambda_{i+1})}{(1 - \rho_i)}$$

$$(i = 1, 2, \dots, N-1),$$

$$(2.2.52) \quad P(\lambda_N) = \sum_{n=r_N+1}^{\infty} P(n, \lambda_N) = (R_N - r_N) \frac{P(r_N+1, \lambda_N)}{(1 - \rho_N)}.$$

The normalizing condition requires  $1 = \sum_{i=0}^N P(\lambda_i)$  which on using (2.2.50), (2.2.51) and (2.2.52) leads to

$$(2.2.53) \quad 1 = \frac{P(0, \lambda_0)}{(1 - \rho_0)} - \sum_{i=1}^N (R_i - r_i) \frac{(\rho_{i-1} - \rho_i)}{(1 - \rho_{i-1})(1 - \rho_i)} P(r_i + 1, \lambda_i)$$

where the  $P(r_i + 1, \lambda_i)$ 's are expressed in terms of  $P(0, \lambda_0)$  by means of relations (2.2.30). Thus we can determine  $P(0, \lambda_0)$  from (2.2.53). We have therefore determined the probability distribution of the number of units in the system for all cases considered.

2.3 The expected number of units in the system ( $L_s$ ) and the corresponding waiting time ( $W_s$ ): We discovered above that the expressions for the  $P(\lambda_i)$ 's for all cases considered are exactly the same. The same is true in this case too and this will be obvious from the next section when we deal with the generating function.

Denote by  $L_{s_i}$  the contribution to  $L_s$  from the states  $(n, \lambda_i)$ , where

$$L_{s_i} = \sum_{n=0}^{\infty} n P(n, \lambda_i) \quad (i = 0, 1, 2, \dots, N).$$

It can be shown by means of simple but lengthy algebra, wherein we make use of the relations (2.2.30), that for all cases

$$(2.3.1) \quad L_{s_0} = \sum_{n=0}^{R_1-1} n P(n, \lambda_0) = \frac{\rho_0}{(1 - \rho_0)^2} P(0, \lambda_0) - (R_1 - \rho_0 r_1) \frac{P(r_1 + 1, \lambda_1)}{(1 - \rho_0)^2} - \frac{1}{2} (R_1 + r_1)(R_1 - r_1 - 1) \frac{P(r_1 + 1, \lambda_1)}{(1 - \rho_0)}.$$

For  $i = 1, 2, \dots, N-1$

$$(2.3.2) L_{s_i} = \sum_{n=r_i+1}^{R-1} n P(n, \lambda_i) = \frac{1}{2}(R_i+r_i)(R_i-r_i-1) \frac{P(r_i+1, \lambda_i)}{(1-\rho_i)}$$

$$+ (R_i - \rho_i r_i) \frac{P(r_i+1, \lambda_i)}{(1-\rho_i)^2} - (R_{i+1} - \rho_i r_{i+1}) \frac{P(r_i+1, \lambda_{i+1})}{(1-\rho_i)^2}$$

$$- \frac{1}{2}(R_{i+1}+r_{i+1})(R_{i+1}-r_{i+1}-1) \frac{P(r_i+1, \lambda_{i+1})}{(1-\rho_i)}$$

$$(2.3.3) L_{s_N} = \sum_{n=r_N+1}^{\infty} n P(n, \lambda_N) = \frac{1}{2}(R_N+r_N)(R_N-r_N-1) \frac{P(r_N+1, \lambda_N)}{(1-\rho_N)}$$

$$+ (R_N - \rho_N r_N) \frac{P(r_N+1, \lambda_N)}{(1-\rho_N)^2}$$

Combining (2.3.1), (2.3.2) and (2.3.3) we obtain

$$(2.3.4) L_s = \sum_{i=0}^N L_{s_i} = \frac{\rho_0}{(1-\rho_0)^2} P(0, \lambda_0)$$

$$- \sum_{i=1}^N \left[ \frac{(R_i - \rho_{i-1} r_i)}{(1-\rho_{i-1})^2} + \frac{1}{2} (R_i+r_i)(R_i-r_i-1) \frac{(\rho_{i-1} - \rho_i)}{(1-\rho_{i-1})(1-\rho_i)} \right. \\ \left. - \frac{(R_i - \rho_i r_i)}{(1-\rho_i)^2} \right] P(r_i+1, \lambda_i)$$

where the  $P(r_i+1, \lambda_i)$ 's and  $P(0, \lambda_0)$  are known from (2.2.30) and (2.2.53).

The expecting waiting time in the system ( $W_s$ ) is, on using Little's formula, given by



$$(2.4.3) \quad h(z, \lambda_0) = \frac{\mu(1-z)P(0, \lambda_0) + \lambda_0 z^{R_1+1} P(R_1-1, \lambda_0) - \mu z^{r_1+1} P(r_1+1, \lambda_1)}{\lambda_0 z^2 - (\mu + \lambda_0)z + \mu}.$$

This is so because equations of the set (0) for all cases are the same as equations (1.2.10) through (1.2.14) when we replace  $r_1$  and  $R_1$  by  $r$  and  $R$  respectively.

Also it is easy to see that in every case equations of the set (i) ( $i = 1, 2, \dots, N-1$ ) lead to

$$\begin{aligned} & \frac{\mu}{z} [h(z, \lambda_i) - z^{r_i+1} P(r_i+1, \lambda_i)] - (\mu + \lambda_i)h(z, \lambda_i) \\ & + \lambda_i z [h(z, \lambda_i) - z^{R-1} P(R-1, \lambda_i)] \\ & + \lambda_{i-1} z^{R_i} P(R_i-1, \lambda_{i-1}) + \mu z^{r_i+1} P(r_i+1, \lambda_{i+1}) = 0, \end{aligned}$$

which on simplification gives

$$(2.4.4) \quad h(z, \lambda_i) = \frac{[\mu z^{r_i+1} P(r_i+1, \lambda_i) - \lambda_{i-1} z^{R_i} P(R_i-1, \lambda_{i-1}) + \lambda_i z^{R+1} P(R-1, \lambda_i) - z^{r_i+1} P(r_i+1, \lambda_{i+1})]}{[\lambda_i z^2 - (\mu + \lambda_i)z + \mu]}.$$

Lastly, the set (N) of equations for all cases is the same as equations (1.2.14) through (1.2.18) when we replace  $r_N$  and  $R_N$  by  $r$  and  $R$  respectively and hence

$$(2.4.5) \quad h(z, \lambda_N) = \frac{\mu z^{r_N+1} P(r_N+1, \lambda_N) - \lambda_{N-1} z^{R_N+1} P(R_N-1, \lambda_{N-1})}{\lambda_N z^2 - (\mu + \lambda_N)z + \mu}.$$

Except for the subscript 1 in  $r_1$  and  $R_1$  equation (2.4.3) is the same as (1.4.3) and hence as in Chapter I we arrive at the following relation

$$(2.4.6) \quad P(r_1+1, \lambda_1) = \rho_0 P(R_1-1, \lambda_0)$$

$$= \frac{\rho_0^{R_1+r_1}}{\rho_0^{r_1} - \rho_0^{R_1}} (1 - \rho_0) P(0, \lambda_0) .$$

Let us first consider (2.4.4) for  $i = 1$ . Making use of (2.4.6) we can write

$$(2.4.7) \quad h(z, \lambda_1) = \frac{[\mu P(r_1+1, \lambda_1)(z^{r_1+1} - z^{R_1+1}) + \lambda_1 z^{R_2+1} P(R_2-1, \lambda_1) - \mu z^{r_2+1} P(r_2+1, \lambda_2)]}{[\lambda_1 z^2 - (\mu + \lambda_1)z + \mu]} .$$

Since  $h(z, \lambda_1)$  converges for all values of  $z$ , the zeros of the denominator of (2.4.7) must coincide with the zeros of the numerator.

But clearly the zeros of the denominator occur at  $z=1$  and  $z = 1/\rho_1$  and hence we arrive at the following equations

$$\lambda_1 P(R_2-1, \lambda_1) - \mu P(r_2+1, \lambda_2) = 0$$

$$\mu P(r_1+1, \lambda_1) [(1/\rho_1)^{r_1+1} - (1/\rho_1)^{R_1+1}] + \lambda_1 (1/\rho_1)^{R_2+1} P(R_2-1, \lambda_1) - \mu (1/\rho_1)^{r_2+1} P(r_2+1, \lambda_2) = 0 .$$

Solving these two equations we obtain

$$\begin{aligned}
 (2.4.8) \quad P(r_2+1, \lambda_2) &= \rho_1 P(R_2-1, \lambda_1) \\
 &= \frac{\rho_1^{R_2+r_2}}{r_2 - \rho_1} \frac{\rho_1^{r_1} - \rho_1^{R_1}}{\rho_1^{R_1+r_1}} P(r_1+1, \lambda_1) .
 \end{aligned}$$

Similarly considering (2.4.4) for  $i=2$  and making use of (2.4.6) and making exactly the same argument as above we get

$$\begin{aligned}
 P(r_3+1, \lambda_3) &= \rho_2 P(R_3-1, \lambda_2) \\
 &= \frac{\rho_2^{R_3+r_3}}{r_3 - \rho_2} \frac{\rho_2^{r_2} - \rho_2^{R_2}}{\rho_2^{R_2+r_2}} P(r_2+1, \lambda_2) .
 \end{aligned}$$

In this way we continue considering (2.4.4) successively for  $i = 2, 3, \dots, N-1$  and obtain

$$\begin{aligned}
 (2.4.9) \quad P(r_{i+1}, \lambda_{i+1}) &= \rho_i P(R_{i+1}-1, \lambda_i) \\
 &= \frac{\rho_i^{R_{i+1}+r_{i+1}}}{r_{i+1} - \rho_i} \frac{\rho_i^{r_i} - \rho_i^{R_i}}{\rho_i^{R_i+r_i}} P(r_i+1, \lambda_i)
 \end{aligned}$$

$$(i = 1, 2, \dots, N-1) .$$

Lastly, notice that (2.4.5) is the same as (1.4.4) when we replace  $N$  by  $1$  and hence we consider only the zero  $z=1$  which occurs in the denominator of (2.4.5). We disregard the zero  $z = 1/\rho_N$

since for the existence of equilibrium conditions we require  $1/\rho_N > 1$  and  $h(z, \lambda_N)$  may not converge for  $|z| > 1$ .

Setting the numerator of (2.4.5) equal to zero and substituting  $z=1$  (at which the numerator must vanish) we obtain the relation

$$P(r_N + 1, \lambda_N) = \rho_{N-1} P(R_N - 1, \lambda_{N-1})$$

which is the same as (2.4.9) for  $i = N - 1$ .

Using (2.4.6) and (2.4.9) we can now write

$$(2.4.10) \quad h(z, \lambda_0) = \frac{\mu (1-z)P(0, \lambda_0) + \mu P(r_1+1, \lambda_1)(z^{R_1+1} - z^{r_1+1})}{\lambda_0 z^2 - (\mu + \lambda_0)z + \mu}$$

$$= \frac{(1-z)P(0, \lambda_0) + (z^{R_1+1} - z^{r_1+1})P(r_1+1, \lambda_1)}{\rho_0 z^2 - (1 + \rho_0)z + 1}$$

$$(2.4.11) \quad h(z, \lambda_i) = \frac{(z^{r_i+1} - z^{R_i+1})P(r_i+1, \lambda_i) + (z^{R_{i+1}+1} - z^{r_{i+1}+1})P(r_{i+1}+1, \lambda_{i+1})}{\rho_i z^2 - (1 + \rho_i)z + 1}$$

$$(i = 1, 2, \dots, N-1)$$

$$(2.4.12) \quad h(z, \lambda_N) = \frac{(z^{r_N+1} - z^{R_N+1})P(r_N+1, \lambda_N)}{\rho_N z^2 - (1 + \rho_N)z + 1}$$

Taking the limits as  $z \rightarrow 1$  of (2.4.10), (2.4.11) and (2.4.12) and using l'Hospital's rule we obtain the expressions for the  $P(\lambda_i)$ 's since

$$P(\lambda_i) = \lim_{z \rightarrow 1} h(z, \lambda_i)$$

and as expected these expressions are the same as (2.2.49), (2.2.50) and (2.2.51). Thus we have determined the generating function  $h(z) = \sum_{i=0}^N h(z, \lambda_i)$ , where the  $h(z, \lambda_i)$ 's are given by (2.4.10), (2.4.11) and (2.4.12). The relation between the  $P(r_i+1, \lambda_i)$ 's and  $P(0, \lambda_0)$  is obtained from (2.4.6) and (2.4.9) and  $P(0, \lambda_0)$  is determined from the normalizing condition (2.2.52).

It may be noted that (2.4.10) and (2.4.11) are true even if some or all  $\rho_i$ 's ( $i = 0, 1, \dots, N-1$ ) are equal to unity whereas (2.4.6) and (2.4.9) will have to be modified by taking limits.

Having determined  $h(z)$ , we can find the moments of the distribution of the number of units in the system by usual methods.

2.5 The Laplace transform of the distribution of the busy period. As in the case of the generating function, we get the same expression for the Laplace transform of the busy period distribution for all cases considered in Section 2.2. We shall therefore deal only with case (I), that is, the case of  $N$  non-overlapping loops. As in Section (1.5) we consider the modified process which ceases as soon as the number of units in the system falls to zero. The initial condition is that at the beginning of the period the system is in a state  $(1, \lambda_0)$ . Again we denote by  $Q(t, n, \lambda_i)$  the probability of the state  $(n, \lambda_i)$  at time  $t$ . The  $(N+1)$

sets of differential-difference equations which this process satisfies are;

Set (0)

$$(2.5.1) \quad Q'(t, 0, \lambda_0) = \mu Q(t, 1, \lambda_0)$$

$$Q'(t, 1, \lambda_0) = -(\mu + \lambda_0)Q(t, 1, \lambda_0) + \mu Q(t, 2, \lambda_0)$$

$$Q'(t, n+1, \lambda_0) = -(\mu + \lambda_0)Q(t, n+1, \lambda_0) + \mu Q(t, n+2, \lambda_0) + \lambda_0 Q(t, n, \lambda_0)$$

$$(n = 1, 2, \dots, r_1 - 2)$$

$$(2.5.2) \quad Q'(t, r_1, \lambda_0) = -(\mu + \lambda_0)Q(t, r_1, \lambda_0) + \mu Q(t, r_1 + 1, \lambda_0) + \lambda_0 Q(t, r_1 - 1, \lambda_0)$$

$$+ \mu Q(t, r_1 + 1, \lambda_1)$$

$$Q'(t, n+1, \lambda_0) = -(\mu + \lambda_0)Q(t, n+1, \lambda_0) + \mu Q(t, n+2, \lambda_0) + \lambda_0 Q(t, n, \lambda_0)$$

$$(n = r_1, r_1 + 1, \dots, R_1 - 3)$$

$$Q'(t, R_1 - 1, \lambda_0) = -(\mu + \lambda_0)Q(t, R_1 - 1, \lambda_0) + \lambda_0 Q(t, R_1 - 2, \lambda_0),$$

Set (i)  $i = 1, 2, \dots, N-1$

$$Q'(t, r_i + 1, \lambda_i) = -(\mu + \lambda_i)Q(t, r_i + 1, \lambda_i) + \mu Q(t, r_i + 2, \lambda_i)$$

$$Q'(t, n+1, \lambda_i) = -(\mu + \lambda_i)Q(t, n+1, \lambda_i) + \mu Q(t, n+2, \lambda_i) + \lambda_i Q(t, n, \lambda_i)$$

$$(n = r_i + 1, r_i + 2, \dots, R_i - 2)$$

$$Q'(t, R_i, \lambda_i) = -(\mu + \lambda_i)Q(t, R_i, \lambda_i) + \mu Q(t, R_i + 1, \lambda_i) + \lambda_i Q(t, R_i - 1, \lambda_i)$$

$$+ \lambda_{i-1} Q(t, R_i - 1, \lambda_{i-1})$$

$$Q'(t, n+1, \lambda_i) = -(\mu + \lambda_i)Q(t, n+1, \lambda_i) + \mu Q(t, n+2, \lambda_i) + \lambda_i Q(t, n, \lambda_i)$$

$$(n = R_i, R_i+1, \dots, r_{i+1}-2)$$

$$Q'(t, r_{i+1}, \lambda_i) = -(\mu + \lambda_i)Q(t, r_{i+1}, \lambda_i) + \mu Q(t, r_{i+1}+1, \lambda_i)$$

$$+ \lambda_i Q(t, r_{i+1}-1, \lambda_i) + \mu Q(t, r_{i+1}+1, \lambda_{i+1})$$

$$Q'(t, n+1, \lambda_i) = -(\mu + \lambda_i)Q(t, n+1, \lambda_i) + \mu Q(t, n+2, \lambda_i) + \lambda_i Q(t, n, \lambda_i)$$

$$(n = r_{i+1}, r_{i+1}+1, \dots, R_{i+1}-3)$$

$$Q'(t, R_{i+1}-1, \lambda_i) = -(\mu + \lambda_i)Q(t, R_{i+1}-1, \lambda_i) + \lambda_i Q(t, R_{i+1}-2, \lambda_i),$$

Set (N)

$$Q'(t, r_N+1, \lambda_N) = -(\mu + \lambda_N)Q(t, r_N+1, \lambda_N) + \mu Q(t, r_N+2, \lambda_N)$$

$$Q'(t, n+1, \lambda_N) = -(\mu + \lambda_N)Q(t, n+1, \lambda_N) + \mu Q(t, n+2, \lambda_N) + \lambda_N Q(t, n, \lambda_N)$$

$$(2.5.4) \quad (n = r_N+1, r_N+2, \dots, R_N-2)$$

$$Q'(t, R_N, \lambda_N) = -(\mu + \lambda_N)Q(t, R_N, \lambda_N) + \mu Q(t, R_N+1, \lambda_N) + \lambda_N Q(t, R_N-1, \lambda_N)$$

$$+ \lambda_{N-1} Q(t, R_N-1, \lambda_{N-1})$$

$$Q'(t, n+1, \lambda_N) = -(\mu + \lambda_N)Q(t, n+1, \lambda_N) + \mu Q(t, n+2, \lambda_N) + \lambda_N Q(t, n, \lambda_N)$$

$$(n \geq R_N) \quad .$$

Introduce the partial generating functions

$$\begin{aligned}
 q(t, z, \lambda_0) &= \sum_{n=1}^{R_1-1} z^n Q(t, n, \lambda_0) \\
 (2.5.5) \quad q(t, z, \lambda_i) &= \sum_{n=r_i+1}^{R_{i+1}-1} z^n Q(t, n, \lambda_i) \\
 &\quad (i = 1, 2, \dots, N-1) \\
 q(t, z, \lambda_N) &= \sum_{n=r_N+1}^{\infty} z^n Q(t, n, \lambda_N)
 \end{aligned}$$

and because of the initial condition mentioned above we have

$$\begin{aligned}
 (2.5.6) \quad q(0, z, \lambda_0) &= z \\
 q(0, z, \lambda_i) &= 0, \quad i \neq 0.
 \end{aligned}$$

Applying (2.5.5) to all equations of the  $(N+1)$  sets above except the first equation of the set  $(0)$ , that is, Equation (2.5.1), and then taking Laplace transforms with respect to time  $t$  and making use of (2.5.6), we obtain as in Section 1.5

$$(2.5.7) \quad q^*(s, z, \lambda_0) = \frac{\mu z Q^*(s, 1, \lambda_0) + \lambda_0 z^{R_1+1} Q^*(s, R_1-1, \lambda_0) - \mu z^{r_1+1} Q^*(s, r_1+1, \lambda_1) - z^2}{\lambda_0 z^2 - (s + \mu + \lambda_0)z + \mu}$$

$$(2.5.8) \quad q^*(s, z, \lambda_i) = [\mu z^{r_i+1} Q^*(s, r_i+1, \lambda_i) - \lambda_{i-1} z^{R_i+1} Q^*(s, R_i-1, \lambda_{i-1}) \\ + \lambda_i z^{R_{i+1}+1} Q^*(s, R_{i+1}-1, \lambda_i) \\ - \mu z^{r_{i+1}+1} Q^*(s, r_{i+1}+1, \lambda_{i+1})] / [\lambda_i z^2 - (s+\mu+\lambda_i)z + \mu]$$

$$(i = 1, 2, \dots, N-1)$$

$$(2.5.9) \quad q^*(s, z, \lambda_N) = \frac{\mu z^{r_N+1} Q^*(s, r_N+1, \lambda_N) - \lambda_{N-1} z^{R_N+1} Q^*(s, R_N-1, \lambda_{N-1})}{\lambda_N z^2 - (s+\mu+\lambda_N)z + \mu}$$

Let the zeros of the quadratic  $\lambda_i z^2 - (s+\mu+\lambda_i)z + \mu$  be denoted by  $\alpha_i(s)$  and  $\beta_i(s)$ :

$$(2.5.10) \quad \alpha_i(s) = \frac{1}{2\lambda_i} \{ (s+\mu+\lambda_i) + [(s+\mu+\lambda_i)^2 - 4\mu\lambda_i]^{1/2} \}$$

$$\beta_i(s) = \frac{1}{2\lambda_i} \{ (s+\mu+\lambda_i) - [(s+\mu+\lambda_i)^2 - 4\mu\lambda_i]^{1/2} \}$$

$$(i = 0, 1, 2, \dots, N)$$

where in (2.5.10) we choose that value of the square root for which the real part is positive.

Making exactly the same argument as in Section 1.5, that is, that the zeros of the denominator must coincide with the zeros of the numerator in (2.5.7), (2.5.8) and (2.5.9) and further that we consider only those zeros that lie inside the circle of convergence we arrive at the

following equations

$$\mu Q^*(s, 1, \lambda_0) + \lambda_0 \alpha_0^{R_1} Q^*(s, R_1 - 1, \lambda_0) - \mu \alpha_0^{r_1} Q^*(s, r_1 + 1, \lambda_1) = \alpha_0$$

$$\mu Q^*(s, 1, \lambda_0) + \lambda_0 \beta_0^{R_1} Q^*(s, R_1 - 1, \lambda_0) - \mu \beta_0^{r_1} Q^*(s, r_1 + 1, \lambda_1) = \beta_0$$

$$\mu \alpha_i^{r_i} Q^*(s, r_i + 1, \lambda_i) - \lambda_{i-1} \alpha_{i-1}^{R_i} Q^*(s, R_i - 1, \lambda_{i-1}) + \lambda_i \alpha_i^{R_{i+1}} Q^*(s, R_{i+1} - 1, \lambda_i)$$

$$(2.5.11) \quad -\mu \alpha_i^{r_{i+1}} Q^*(s, r_{i+1} + 1, \lambda_{i+1}) = 0$$

$$\mu \beta_i^{r_i} Q^*(s, r_i + 1, \lambda_i) - \lambda_{i-1} \beta_{i-1}^{R_i} Q^*(s, R_i - 1, \lambda_{i-1}) + \lambda_i \beta_i^{R_{i+1}} Q^*(s, R_{i+1} - 1, \lambda_i)$$

$$-\mu \beta_i^{r_{i+1}} Q^*(s, r_{i+1} + 1, \lambda_{i+1}) = 0$$

$$(i = 1, 2, \dots, N-1)$$

$$\mu \beta_N^{r_N} Q^*(s, r_N + 1, \lambda_N) - \lambda_{N-1} \beta_{N-1}^{R_N} Q^*(s, R_N - 1, \lambda_{N-1}) = 0$$

(2.5.11) is a set of  $(2N+1)$  linear equations in  $(2N+1)$  unknowns and hence  $Q^*(s, 1, \lambda_0)$  can be determined.

But the p.d.f. of the length of the busy period  $f_b(t)$  is clearly given by

$$f_b(t) = Q'(t, 0, \lambda_0)$$

$$= \mu Q(t, 1, \lambda_0), \text{ on using (2.5.1).}$$

That is

$$(2.5.12) \quad f_b^*(s) = \mu Q^*(s, 1, \lambda_0)$$

on taking Laplace transform, where  $Q^*(s, l, \lambda_0)$  is determined from (2.5.11). Thus we have shown how in principle to obtain the Laplace transform of the distribution of the busy period. In practice it may be very complicated.

2.6 The Laplace transform of the distribution of time the system spends in the subset of states  $\{(n, \lambda_1)\}$ , ( $i = 0, 1, \dots, N$ ) before it goes out of the subset for the first time. Clearly the problem for the two cases  $i = 0$  and  $i = N$  is the same as in Section 1.6. More specifically, for  $i = 0$  the result is given by (1.6.1) with  $r$  and  $R$  replaced by  $r_1$  and  $R_1$  respectively and for  $i = N$  the result is given by (1.6.3) with  $\beta_1, r$  and  $R$  replaced by  $\beta_N, r_N$  and  $R_N$  respectively.

We shall now deal with the case where  $i \neq 0, N$ . Without any loss of generality let us consider the case  $i = 1$ . That is, we are interested in the distribution of time the system spends in the subset of states  $\{(n, \lambda_1)\}$  before it goes out of the subset for the first time either to the subset  $\{(n, \lambda_2)\}$  or the subset  $\{(n, \lambda_0)\}$ . Recall that for the subset  $\{(n, \lambda_1)\}$ ,  $n$  goes from  $r_1 + 1$  to  $R_2 - 1$ .

Let us denote by  $f_m(t)$  the p.d.f. of the required distribution given that initially it started from the state  $(m, \lambda_1)$ . On using the notation  $e(\mu + \lambda_1)$  for the exponential distribution with parameter  $\mu + \lambda_1$  it is easy to see that the following equations are true:

$$(2.6.1) \quad f_{r_1+1} = \frac{\mu}{\mu + \lambda_1} e(\mu + \lambda_1) + \frac{\lambda_1}{\mu + \lambda_1} e(\mu + \lambda_1) \otimes f_{r_1+2}$$

$$(2.6.2) \quad f_{n+1} = \frac{\mu}{\mu+\lambda_1} e^{(\mu+\lambda_1)t} \otimes f_n + \frac{\lambda_1}{\mu+\lambda_1} e^{(\mu+\lambda_1)t} \otimes f_{n+2}$$

$$(n = r_1+1, r_1+2, \dots, R_2-3)$$

$$(2.6.3) \quad f_{R_2-1} = \frac{\mu}{\mu+\lambda_1} e^{(\mu+\lambda_1)t} \otimes f_{R_2-2} + \frac{\lambda_1}{\mu+\lambda_1} e^{(\mu+\lambda_1)t}$$

where  $\otimes$  stands for the convolution and we have suppressed the argument  $t$  in the  $f_n(t)$ 's for convenience.

On taking the Laplace transform with respect to  $t$  of (2.6.1), (2.6.2) and (2.6.3) we have

$$(2.6.4) \quad f_{r_1+1}^*(s) = \frac{\mu}{\mu+\lambda_1+s} + \frac{\lambda_1}{(\mu+\lambda_1+s)} f_{r_1+2}^*(s)$$

$$(2.6.5) \quad f_{n+1}^*(s) = \frac{\mu}{\mu+\lambda_1+s} f_n^*(s) + \frac{\lambda_1}{\mu+\lambda_1+s} f_{n+2}^*(s)$$

$$(n = r_1+1, r_1+2, \dots, R_2-3)$$

$$(2.6.6) \quad f_{R_2-1}^*(s) = \frac{\mu}{\mu+\lambda_1+s} f_{R_2-2}^*(s) + \frac{\lambda_1}{\mu+\lambda_1+s}$$

Notice that (2.6.5) is a homogeneous difference equation of the second order and (2.6.4) and (2.6.6) give the initial conditions. The solution to (2.6.5) is

$$(2.6.7) \quad f_m^*(s) = c_1 [\alpha(s)]^m + c_2 [\beta(s)]^m$$

$$(m = r_1+1, r_1+2, \dots, R_2-1)$$

where

$$\alpha(s) = \frac{1}{2\lambda_1} [(\mu + \lambda_1 + s) + \{(\mu + \lambda_1 + s)^2 - 4\mu\lambda_1\}^{1/2}]$$

$$\beta(s) = \frac{1}{2\lambda_1} [(\mu + \lambda_1 + s) - \{(\mu + \lambda_1 + s)^2 - 4\mu\lambda_1\}^{1/2}]$$

$$c_1 = \rho_1^{r_1+1} [\mu \beta^{R_2-2} \{(\mu + \lambda_1 + s) \beta - \mu\} - \lambda_1 \beta^{r_1+1} \{(\mu + \lambda_1 + s) - \lambda_1 \beta\}] / [\beta^{R_2-r_1-3} \{(\mu + \lambda_1 + s) - \lambda_1 \alpha\} \{(\mu + \lambda_1 + s) \beta - \mu\} - \alpha^{R_2-r_1-3} \{(\mu + \lambda_1 + s) - \lambda_1 \beta\} \{(\mu + \lambda_1 + s) \alpha - \mu\}]$$

and

$$c_2 = \rho_1^{r_1+1} [\lambda_1 \alpha^{r_1+1} \{(\mu + \lambda_1 + s) - \lambda_1 \alpha\} - \mu \alpha^{R_2-2} \{(\mu + \lambda_1 + s) \alpha - \mu\}] / [\beta^{R_2-r_1-3} \{(\mu + \lambda_1 + s) - \lambda_1 \alpha\} \{(\mu + \lambda_1 + s) \beta - \mu\} - \alpha^{R_2-r_1-3} \{(\mu + \lambda_1 + s) - \lambda_1 \beta\} \{(\mu + \lambda_1 + s) \alpha - \mu\}]$$

Thus, we have obtained the Laplace transform of the p.d.f. of the required distribution.

2.7 The case when N is infinite. When N is infinite it is clear that the existence of equilibrium conditions depends upon the convergence of the series [see (2.2.52)]

$$(2.7.1) \quad \sum_{i=1}^{\infty} (R_i - r_i) \frac{(\rho_{i-1} - \rho_i)}{(1 - \rho_{i-1})(1 - \rho_i)} \frac{P(r_i + 1, \lambda_i)}{P(0, \lambda_0)}$$

and recall that because of (2.2.30) the ratio  $P(r_i+1, \lambda_i)/P(0, \lambda_0)$  involves only the  $\rho_i$ 's,  $r_i$ 's and  $R_i$ 's. If the above series converges then all the appropriate formulas developed in this chapter will still hold when the number of loops is infinite.

2.8 Remarks. On putting  $\lambda_1 = \lambda_2 = \dots = \lambda_N = \lambda_0$ , all the corresponding results or formulas for a simple queue, subject to the usual restriction  $\rho_0 < 1$  for the existence of equilibrium conditions, can be obtained from this chapter.

It is also possible to superimpose a cost structure on the problem and define a suitable cost function. The parameters involved in the cost function are clearly  $r_i$ ,  $R_i$  and  $\rho_i$  ( $i = 1, 2, \dots, N$ ). We shall not attempt to pursue this problem further as even in the case of a single loop we are not able to get explicit expressions for the optimal values of the parameters.

## CHAPTER III

### A SINGLE-COUNTER QUEUE WITH CHANGEABLE SERVICE RATE

3.1 Introduction. In this chapter we are considering another queueing process under assumptions which are in some sense the opposite of those considered in the first two chapters. The following assumptions are made:

- (a) Arrivals constitute a Poisson process with fixed parameter  $\lambda$ .
- (b) Service time is distributed exponentially, but the parameter of the distribution,  $\mu$ , is subject to changes depending on realizations of queue size.

Let  $(r_1, R_1), (r_2, R_2), \dots, (r_N, R_N)$  be  $N$  pairs of integers satisfying  $r_1 \geq 0, r_1 \leq 0, r_1 < R_1, r_2 < R_2, \dots, r_N < R_N; r_1 < r_2 < \dots < r_N; R_1 < R_2 < \dots < R_N$ . Beginning from the moment the system is idle, the service rate  $\mu$  assumes a value  $\mu_0$  as long as the number of units in the system ( $l_s$ ) is strictly less than  $R_1$ . When  $l_s$  assumes a value  $R_1$ , the service rate changes instantaneously from  $\mu_0$  to  $\mu_1$  and as long as  $l_s$  satisfies  $r_1 + 1 \leq l_s \leq R_2 - 1$  the service rate will remain constant at  $\mu_1$ . Should  $l_s$  go down to  $r_1$ ,  $\mu$  will change back from  $\mu_1$  to  $\mu_0$  and in order that  $\mu_0$  changes to  $\mu_1$  again  $l_s$  has to grow to size  $R_1$  and the same process is repeated. With  $\mu$  assuming a value  $\mu_1$  should  $l_s$  reach a size  $R_2$

$\mu$  will change instantaneously from  $\mu_1$  to  $\mu_2$  where the service rate will stay constant as long as  $\lambda_s$  satisfies  $r_2 + 1 \leq \lambda_s \leq R_3 - 1$ . It will change back to  $\mu_1$  when  $\lambda_s$  drops to a size  $r_2$  but will take a value  $\mu_3$  when  $\lambda_s$  attains a size  $R_3$ . The case here is similar;  $\mu = \mu_3$  as long as  $r_3 + 1 \leq \lambda_s \leq R_4 - 1$  and  $\mu$  will change to  $\mu_2$  when  $\lambda_s = r_3$  but will assume a value  $\mu_4$  when  $\lambda_s$  reaches a value  $R_4$  and so on. When  $\mu = \mu_{N-1}$  we have  $r_{N-1} + 1 \leq \lambda_s \leq R_N - 1$ . If  $\lambda_s$  drops down to a size  $r_{N-1}$ ,  $\mu$  will change from  $\mu_{N-1}$  to  $\mu_{N-2}$  whereas if  $\lambda_s$  grows to a size  $R_N$ ,  $\mu$  will change instantaneously to  $\mu_N$  and at this last stage there will be no further change in  $\mu$  unless  $\lambda_s$  goes down to a size  $r_N$  when  $\mu$  changes back to  $\mu_{N-1}$ . That is,  $\mu = \mu_N$  for all  $\lambda_s \geq r_N + 1$ .

Under the above model, we are interested in finding the stationary distribution of the number of units in the system (and also the generating function) and its expected value, the expected waiting time in the system, the Laplace transform of the distribution of the busy period, etc. It will be seen later that for the existence of steady state conditions we need the sole restriction  $\lambda < \mu_N$ .

As in Chapter II we shall restrict ourselves to three cases only, that is, the case of  $N$  non-overlapping loops, the case of  $N$  overlapping loops and the case of  $N$  trivially overlapping loops. The solution to other cases can easily be inferred from the solutions of the above three cases. We shall deal with the first case in detail whereas for the second and the third we shall simply write the solutions directly as the mathematical techniques are very

similar to those in Chapter II. We denote by  $(n, \mu_i)$  the state that  $\ell_s = n$  and  $\mu = \mu_i$  ( $n = 0, 1, 2, \dots; i = 0, 1, 2, \dots, N$ ). By virtue of our assumptions the following states are clearly inadmissible

$$(n, \mu_0) \text{ for } n > R_1 - 1$$

$$(n, \mu_i) \text{ for } n < r_i + 1, n > R_{i+1} - 1 \\ (i = 1, 2, \dots, N-1)$$

$$(n, \mu_N) \text{ for } n < r_N + 1$$

### 3.2 The distribution of $\ell_s$ , the number of units in the system.

(I) The case of  $N$  non-overlapping loops, that is, the case where  $(0 \leq r_1 < R_1 < r_2 < R_2 < \dots < R_{N-1} < r_N < R_N)$ . The diagram of the process for this case is given below:

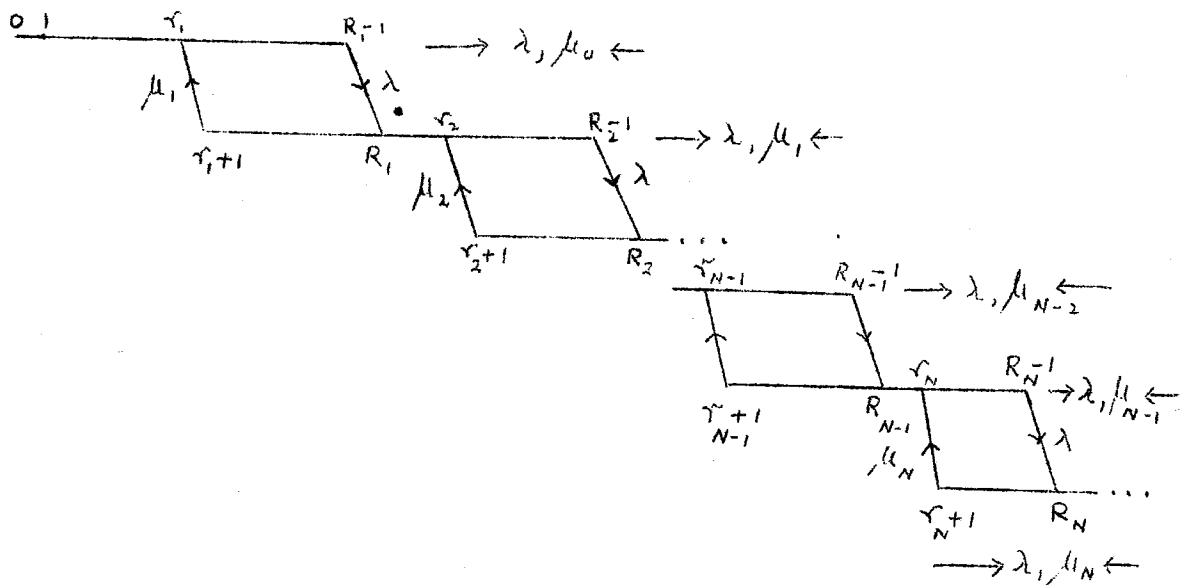


Diagram 3.1

Denoting by  $P(n, \mu_i)$  the stationary probability of the state  $(n, \mu_i)$ , the system satisfies the following  $(N + 1)$  sets of difference equations. We will denote by (i)  $(i = 0, 1, \dots, N)$  the set corresponding to  $\mu_i$ .

Set (0)

$$(3.2.1) \quad \mu_0 P(1, \mu_0) - \lambda P(0, \mu_0) = 0$$

$$(3.2.2) \quad \mu_0 P(n+2, \mu_0) - (\mu_0 + \lambda) P(n+1, \mu_0) + \lambda P(n, \mu_0) = 0$$

$$(n = 0, 1, \dots, r_1 - 2)$$

$$(3.2.3) \quad \mu_0 P(r_1 + 1, \mu_0) - (\mu_0 + \lambda) P(r_1, \mu_0) + \lambda P(r_1 - 1, \mu_0)$$

$$+ \mu_1 P(r_1 + 1, \mu_1) = 0$$

$$(3.2.4) \quad \mu_0 P(n+2, \mu_0) - (\mu_0 + \lambda) P(n+1, \mu_0) + \lambda P(n, \mu_0) = 0$$

$$(n = r_1, r_1 + 1, \dots, R_1 - 3)$$

$$(3.2.5) \quad - (\mu_0 + \lambda) P(R_1 - 1, \mu_0) + \lambda P(R_1 - 2, \mu_0) = 0$$

Set (i)  $i = 1, 2, \dots, N-1$

$$(3.2.6) \quad \mu_i P(r_i + 2, \mu_i) - (\mu_i + \lambda) P(r_i + 1, \mu_i) = 0$$

$$(3.2.7) \quad \mu_i P(n+2, \mu_i) - (\mu_i + \lambda) P(n+1, \mu_i) + \lambda P(n, \mu_i) = 0$$

$$(n = r_i + 1, r_i + 2, \dots, R_i - 2)$$

$$(3.2.8) \quad \mu_i P(R_i + 1, \mu_i) - (\mu_i + \lambda) P(R_i, \mu_i) + \lambda P(R_i - 1, \mu_i)$$

$$+ \lambda P(R_i - 1, \mu_{i-1}) = 0$$

$$(3.2.9) \quad \mu_i P(n+2, \mu_i) - (\mu_i + \lambda) P(n+1, \mu_i) + \lambda P(n, \mu_i) = 0$$

$$(n = R_i, R_i+1, \dots, r_{i-1}-2)$$

$$(3.2.10) \quad \mu_i P(r_{i+1}+1, \mu_i) - (\mu_i + \lambda) P(r_{i+1}, \mu_i) + \lambda P(r_{i+1}-1, \mu_i)$$

$$+ \mu_{i+1} P(r_{i+1}+1, \mu_{i+1}) = 0$$

$$(3.2.11) \quad \mu_i P(n+2, \mu_i) - (\mu_i + \lambda) P(n+1, \mu_i) + \lambda P(n, \mu_i) = 0$$

$$(n = r_{i+1}, r_{i+1}+1, \dots, R_{i+1}-3)$$

$$(3.2.12) \quad - (\mu_i + \lambda) P(R_{i+1}-1, \mu_i) + \lambda P(R_{i+1}-2, \mu_i) = 0$$

Set (N)

$$(3.2.13) \quad \mu_N P(r_N+2, \mu_N) - (\mu_N + \lambda) P(r_N+1, \mu_N) = 0$$

$$(3.2.14) \quad \mu_N P(n+2, \mu_N) - (\mu_N + \lambda) P(n+1, \mu_N) + \lambda P(n, \mu_N) = 0$$

$$(n = r_N+1, r_N+2, \dots, R_N-2)$$

$$(3.2.15) \quad \mu_N P(R_N+1, \mu_N) - (\mu_N + \lambda) P(R_N, \mu_N) + \lambda P(R_N-1, \mu_N)$$

$$+ \lambda P(R_N-1, \mu_{N-1}) = 0$$

$$(3.2.16) \quad \mu_N P(n+2, \mu_N) - (\mu_N + \lambda) P(n+1, \mu_N) + \lambda P(n, \mu_N) = 0$$

$$(n \geq R_N) .$$

We shall solve the above (N+1) sets of difference equations and express all  $P(n, \mu_i)$ 's in terms of  $P(0, \mu_0)$  which will be determined later from the normalizing equation. Furthermore we shall only deal with the case  $\lambda \neq \mu_i$  ( $i = 0, 1, 2, \dots, N-1$ ). The

results for some or all  $\lambda = \mu_1$  can be obtained as limiting cases.

Equations (3.2.2) of the set (0) is a homogeneous difference equation of the second order with (3.2.1) giving the initial condition. The solution is given by

$$(3.2.17) \quad P(n, \mu_0) = b_0^n P(0, \mu_0) \quad (n = 0, 1, \dots, r_1)$$

where  $b_i = \lambda/\mu_i, i = 0, 1, 2, \dots, N.$

(Note that  $b_i$  and  $\rho_i$  introduced in Chapter II have the same meaning, but for different processes.)

Similarly the solution to (3.2.4) with (3.2.5) as the initial condition is

$$(3.2.18) \quad P(n, \mu_0) = \frac{b_0^{r_1}}{b_0^{r_1} - b_0^{R_1}} (b_0^n - b_0^{R_1}) P(0, \mu_0) \quad (n=r_1, r_1+1, \dots, R_1-1).$$

Substitution of  $P(r_1+1, \mu_0)$ ,  $P(r_1, \mu_0)$  and  $P(r_1-1, \mu_0)$  from (3.2.18) and (3.2.17) into (3.2.3) leads to

$$(3.2.19) \quad P(r_1+1, \mu_1) = b_1 P(R_1-1, \mu_0)$$

$$= \frac{b_1 b_0^{R_1+r_1-1}}{b_0^{r_1} - b_0^{R_1}} (1 - b_0) P(0, \mu_0).$$

This formula may be compared with formula (2.2.18).

Next we shall solve equations of the Set (i) for  $i = 1$ . The solution to the difference equation (3.2.7) with (3.2.6) as the initial condition is

$$(3.2.20) \quad P(n, \mu_1) = \frac{P(r_1+1, \mu_1)}{(1 - b_1)} (1 - b_1^{n-r_1}) (n = r_1+1, r_1+2, \dots, R_1).$$

Substituting the values of  $P(R_1, \mu_1)$  and  $P(R_1-1, \mu_1)$  from (3.2.20) into (3.2.8) and making use of (3.2.19), we arrive at the relation

$$(3.2.21) \quad P(R_1+1, \mu_1) = b_1 \frac{P(r_1+1, \mu_1)}{(1 - b_1)} (1 - b_1^{R_1-r_1})$$

$$= b_1 P(R_1, \mu_1) .$$

Now (3.2.21) serves as an initial condition for equations (3.2.9), the solution of which is

$$(3.2.22) \quad P(n, \mu_1) = \frac{P(r_1+1, \mu_1)}{(1 - b_1)} (1 - b_1^{R_1-r_1}) b_1^{n-R_1} (n=R_1, R_1+1, \dots, r_2).$$

The solution to (3.2.11) with (3.2.12) as the initial condition is

$$(3.2.23) \quad P(n, \mu_1) = \frac{P(r_2, \mu_1)}{b_1^{r_2} - b_1^{R_2}} (b_1^n - b_1^{R_2}) (n=r_2, r_2+1, \dots, R_2-1).$$

Substituting the values of  $P(r_2+1, \mu_1)$ ,  $P(r_2, \mu_1)$  and  $P(r_2-1, \mu_1)$  into (3.2.10) we obtain on using (3.2.22)

$$\begin{aligned}
 (3.2.24) \quad P(r_2+1, \mu_2) &= b_2 P(R_2-1, \mu_1) = \frac{b_2 b_1^{R_2-1} P(r_2, \mu_1)}{r_2^{R_2} b_1^{R_2} - b_1^{R_2}} (1 - b_1) \\
 &= \frac{b_2 b_1^{R_2+r_2-1}}{r_2^{R_2} b_1^{R_2} - b_1^{R_2}} \cdot \frac{b_1^{r_1} - b_1^{R_1}}{b_1^{R_1+r_1}} P(r_1+1, \mu_1)
 \end{aligned}$$

where  $P(r_1+1, \mu_1)$  is given by (3.2.19).

Making use of (3.2.24) we can write (3.2.23) in an alternative form

$$(3.2.25) \quad P(n, \mu_1) = \frac{b_1}{b_2} \frac{P(r_2+1, \mu_2)}{(1 - b_1)} (b_1^{n-R_2-1}) (n = r_2, r_2+1, \dots, R_2-1).$$

Thus we have solved equations of the set (1) completely. Notice that the set of equations (1), ( $i = 1, 2, \dots, N-1$ ) are exactly the same except for the differences in the subscripts of the parameters. In solving the set (1) we made use of the information  $P(r_1+1, \mu_1) = b_1 P(R_1-1, \mu_0)$  given by the set (0). Similarly set (1) gives the information  $P(r_2+1, \mu_2) = b_2 P(R_2-1, \mu_1)$  for solving set (2). Clearly the solution to the set (2) can be obtained from the solution to the set (1) by merely replacing  $\mu_1, \mu_2, b_1, r_1, R_1, r_2,$  and  $R_2$  by  $\mu_2, \mu_3, b_2, r_2, R_2, r_3,$  and  $R_3$  respectively. In solving the set (3) we make use of the information  $P(r_3+1, \mu_3) = b_3 P(R_3-1, \mu_2)$  and so on. In this way we solve all equations of the sets (1), (2),  $\dots$ , ( $N-1$ ) successively. The solution to the ( $N-1$ )<sup>th</sup> set can be obtained from that of the first set by merely replacing  $\mu_1, \mu_2, b_1, r_1, R_1, r_2,$  and  $R_2$  by  $\mu_{N-1}, \mu_N, b_{N-1}, r_{N-1}, R_{N-1}, r_N,$  and  $R_N$  respectively. Furthermore, we have the information  $P(r_N+1, \mu_N) = b_N P(R_N-1, \mu_{N-1})$

for solving the  $(N)^{\text{th}}$  set. With this information the solution of the  $(N)^{\text{th}}$  set is given by

$$(3.2.26) \quad P(n, \mu_N) = \frac{P(r_N+1, \mu_N)}{(1 - b_N)} (1 - b_N^{n-r_N})_{(n=r_N+1, r_N+2), \dots, R_N}$$

$$(3.2.27) \quad P(n, \mu_N) = \frac{P(r_N+1, \mu_N)}{(1 - b_N)} (1 - b_N^{R_N-r_N}) b_N^{n-R_N} \quad (n \geq R_N) .$$

Summarizing the results, we have for the solutions of the equations of the sets (i) ( $i = 0, 1, \dots, N$ )

$$(3.2.28) \quad \left\{ \begin{array}{l} P(n, \mu_0) = b_0^n P(0, \mu_0) \quad (n = 0, 1, \dots, r_1) \\ P(n, \mu_0) = \frac{b_0^{r_1}}{b_0^{r_1} - b_0^{R_1}} (b_0^n - b_0^{R_1}) P(0, \mu_0) \quad (n = r_1, r_1+1, \dots, R_1-1) \end{array} \right.$$

For  $i = 1, 2, \dots, N-1$  we have

$$(3.2.29) \quad \left\{ \begin{array}{l} P(n, \mu_i) = \frac{P(r_i+1, \mu_i)}{(1 - b_i)} (1 - b_i^{n-r_i}) \quad (n = r_i+1, r_i+2, \dots, R_i) \\ P(n, \mu_i) = \frac{P(r_i+1, \mu_i)}{(1 - b_i)} (1 - b_i^{R_i-r_i}) b_i^{n-R_i} \quad (n = R_i, R_i+1, \dots, r_{i+1}) \\ P(n, \mu_i) = \frac{b_i}{b_{i+1}} \frac{P(r_{i+1}+1, \mu_{i+1})}{(1 - b_i)} (b_i^{n-R_{i+1}-1})_{(n=r_{i+1}, r_{i+1}+1), \dots, R_{i+1}-1)} \end{array} \right.$$

$$(3.2.30) \left\{ \begin{aligned} P(n, \mu_N) &= \frac{P(r_N+1, \mu_N)}{(1 - b_N)} (1 - b_N)^{n-r_N} \quad (n = r_N+1, r_N+2) \dots, R_N) \\ P(n, \mu_N) &= \frac{P(r_N+1, \mu_N)}{(1 - b_N)} (1 - b_N)^{R_N-r_N} b_N^{n-R_N} \quad (n \geq R_N) \end{aligned} \right.$$

where

$$(3.2.31) \left\{ \begin{aligned} P(r_1+1, \mu_1) &= b_1 P(R_1-1, \mu_0) \\ &= \frac{b_1 b_0^{R_1+r_1-1}}{r_1 - b_0} (1 - b_0) P(0, \mu_0) \text{ and} \end{aligned} \right.$$

$$P(r_{i+1}+1, \mu_{i+1}) = b_{i+1} P(R_{i+1}-1, \mu_i)$$

$$= \frac{b_{i+1} b_i^{R_{i+1}+r_{i+1}-1}}{r_{i+1} - b_i} \frac{b_i - b_i^{R_i}}{R_i + r_i} P(r_i+1, \mu_i)$$

$$(i = 1, 2, \dots, N-1) .$$

The only unknown involved now is  $P(0, \mu_0)$  which will be determined later.

(II) The case of  $N$  overlapping loops: The diagram of the process for this case is as given below

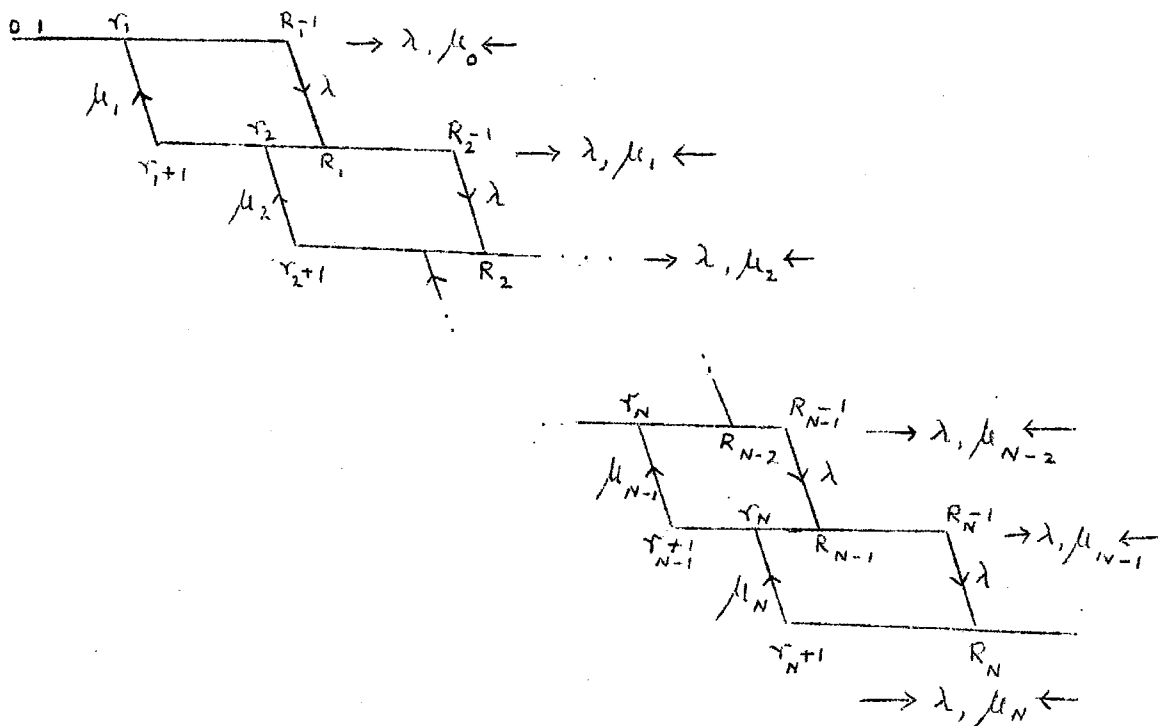


Diagram 3.2

The solutions to the  $(N+1)$  sets of difference equations corresponding to this case are as follows

$$(3.2.32) \quad \begin{cases} P(n, \mu_0) = b_0^n P(0, \mu_0) & (n = 0, 1, \dots, r_1) \\ P(n, \mu_0) = \frac{b_0^{r_1}}{b_0^{r_1} - b_0^{R_1}} (b_0^n - b_0^{R_1}) P(0, \mu_0) & (n = r_1, r_1 + 1, \dots, R_1 - 1). \end{cases}$$

For  $i = 1, 2, \dots, N-1$  we have

$$(3.2.33) \quad \left\{ \begin{aligned} P(n, \mu_i) &= \frac{P(r_i+1, \mu_i)}{(1-b_i)} (1-b_i)^{n-r_i} (n=r_i+1, r_i+2, \dots, r_{i+1}), \\ P(n, \mu_i) &= \frac{P(r_i+1, \mu_i)}{(i-b_i)} (i-b_i)^{n-r_i} \frac{b_i}{b_{i+1}} \frac{P(r_{i+1}+1, \mu_{i+1})}{(1-b_i)} (1-b_i)^{n-r_{i+1}} \\ &\quad (n = r_{i+1}, r_{i+1}+1, \dots, R_i), \\ P(n, \mu_i) &= \frac{b_i}{b_{i+1}} \frac{P(r_{i+1}+1, \mu_{i+1})}{(i-b_i)} (b_i)^{n-R_{i+1}-1} (n=R_i, R_i+1, \dots, R_{i+1}-1). \end{aligned} \right.$$

Finally, we have

$$(3.2.34) \quad \left\{ \begin{aligned} P(n, \mu_N) &= \frac{P(r_N+1, \mu_N)}{(1-b_N)} (1-b_N)^{n-r_N} (n=r_N+1, r_N+2, \dots, R_N), \\ P(n, \mu_N) &= \frac{P(r_N+1, \mu_N)}{(1-b_N)} (1-b_N)^{R_N-r_N} b_N^{n-R_N} (n \geq R_N) \end{aligned} \right.$$

where the relations between  $P(r_{i+1}+1, \mu_{i+1})$  and  $P(r_i+1, \mu_i)$  and between  $P(r_{i+1}+1, \mu_{i+1})$  and  $P(r_i+1, \mu_i)$  ( $i = 1, 2, \dots, N-1$ ) are the same as given by (3.2.31).

(III) The case of  $N$  trivially overlapping loops: The diagram of the process for this case is as shown below

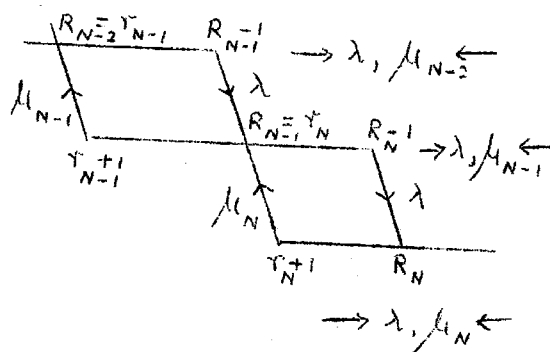
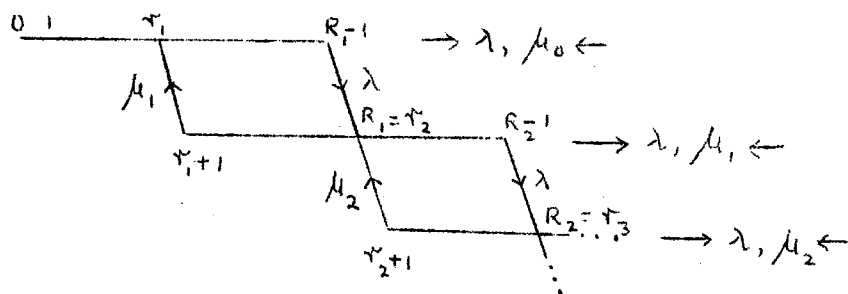


Diagram 3.3

This is really a special case of both (I) and (II) with the understanding that  $r_{i+1} = R_i$  ( $i = 1, 2, \dots, N-1$ ). The solutions for the  $P(n, \mu_i)$ 's are the same as in case (I) except that in (3.2.29) the case where  $n = R_i + 1, \dots, r_{i+1}$  does not arise, or are the same as in case (II) except that in (3.2.33) the case where  $n = r_{i+1} + 1, \dots, R_i$  does not arise.

The probability  $P(\mu_i)$  ( $i = 0, 1, \dots, N$ ) of finding the system in a state where the service rate is  $\mu_i$  is clearly given by  $P(\mu_i) = \sum_{n=0}^{\infty} P(n, \mu_i)$ . On making use of (3.2.31), it can be shown that in every case we have

$$(3.2.35) \quad \left\{ \begin{aligned} P(\mu_0) &= \frac{P(0, \mu_0)}{(1 - b_0)} - (R_1 - r_1) \frac{b_0}{b_1} \frac{P(r_1 + 1, \mu_1)}{(1 - b_0)} \\ P(\mu_i) &= (R_i - r_i) \frac{P(r_i + 1, \mu_i)}{(1 - b_i)} - (R_{i+1} - r_{i+1}) \frac{b_i}{b_{i+1}} \frac{P(r_{i+1} + 1, \mu_{i+1})}{(1 - b_i)} \\ &\quad (i = 1, 2, \dots, N-1) \\ P(\mu_N) &= (R_N - r_N) \frac{P(r_N + 1, \mu_N)}{(1 - b_N)} \quad \text{provided } b_N < 1. \end{aligned} \right.$$

If  $b_N \geq 1$  the series  $\sum_{n=0}^{\infty} P(n, \mu_N)$  does not converge and there will be no stationary distribution.

The normalizing condition requires  $1 = \sum_{i=0}^N P(\mu_i)$  which, on making use of (3.2.25), leads to

$$(3.2.36) \quad 1 = \frac{P(0, \mu_0)}{(1 - b_0)} - \sum_{i=1}^N (R_i - r_i) \frac{(b_{i-1} - b_i)}{b_i (1 - b_{i-1}) (1 - b_i)} P(r_i + 1, \mu_i)$$

where the  $P(r_i + 1, \mu_i)$ 's are expressed in terms of  $P(0, \mu_0)$  by means of relations (3.2.31).

Thus we can determine  $P(0, \mu_0)$  from (3.2.36). That is, we have completely determined the probability distribution of the number of units in the system.

3.3 The expected number of units in the system ( $L_S$ ) and the corresponding waiting time ( $W_S$ ). Just as in Section 2.3 all the three cases considered lead to the same expressions for  $L_S$  and  $W_S$ .

Let  $L_{S_i}$  denote the contribution to  $L_S$  from the states  $(n, \mu_i)$  ( $i = 0, 1, \dots, N$ ). Clearly

$$L_{S_i} = \sum_{n=0}^{\infty} n P(n, \mu_i).$$

On making use of relations (3.2.31) it can be shown that

$$(3.3.1) \quad L_{S_0} = \frac{b_0}{(1-b_0)^2} P(0, \mu_0) - (R_1 - b_0 r_1) \frac{b_0}{b_1} \frac{P(r_1+1, \mu_1)}{(1-b_0)^2} \\ - \frac{1}{2} (R_1 + r_1)(R_1 - r_1 - 1) \frac{b_0}{b_1} \frac{P(r_1+1, \mu_1)}{(1-b_0)}.$$

For  $i = 1, 2, \dots, N-1$

$$(3.3.2) \quad L_{S_i} = \frac{1}{2} (R_i + r_i)(R_i - r_i - 1) \frac{P(r_i+1, \mu_i)}{(1-b_i)} + (R_i - b_i r_i) \frac{P(r_i+1, \mu_i)}{(1-b_i)^2} \\ - (R_{i+1} - b_i r_{i+1}) \frac{b_i}{b_{i+1}} \frac{P(r_{i+1}+1, \mu_{i+1})}{(1-b_i)^2} \\ - \frac{1}{2} (R_{i+1} + r_{i+1})(R_{i+1} - r_{i+1} - 1) \frac{b_i}{b_{i+1}} \frac{P(r_{i+1}+1, \mu_{i+1})}{(1-b_i)},$$

$$(3.3.3) \quad L_{S_N} = \frac{1}{2} (R_N + r_N)(R_N - r_N - 1) \frac{P(r_N + 1, \mu_N)}{(1 - b_N)} + (R_N - b_N r_N) \frac{P(r_N + 1, \mu_N)}{(1 - b_N)^2}.$$

Combining (3.3.1), (3.3.2) and (3.3.3) we obtain

$$(3.3.4) \quad L_S = \sum_{i=0}^N L_{S_i} = \frac{b_0}{(1 - b_0)^2} P(0, \mu_0) \\ - \sum_{i=1}^N \left[ \frac{(R_i - b_{i-1} r_i)}{(1 - b_{i-1})^2} \frac{b_{i-1}}{b_i} + \frac{1}{2} (R_i + r_i)(R_i - r_i - 1) \right. \\ \left. \frac{(b_{i-1} - b_i)}{b_i (1 - b_{i-1})(1 - b_i)} - \frac{(R_i - b_i r_i)}{(1 - b_i)^2} \right] P(r_i + 1, \mu_i)$$

where  $P(0, \mu_0)$  and  $P(r_i + 1, \mu_i)$  are given by (3.2.31).

The expected waiting time in the system is obtained from the formula (3.3.5)  $W_S = L_S / \lambda$ .

3.4. The generating function of the probabilities of the number of units in the system. The probability  $P(n)$  of finding  $n$  units in the system is

$$P(n) = \sum_{i=0}^N P(n, \mu_i).$$

Define

$$(3.4.1) \quad h(z, \mu_i) = \sum_{n=0}^{\infty} z^n P(n, \mu_i) \quad (i = 0, 1, \dots, N)$$

then clearly the generating function of the  $P(n)$ 's is given by

$$(3.4.2) \quad h(z) = \sum_{n=0}^{\infty} z^n P(n) = \sum_{i=0}^N h(z, \mu_i) .$$

Using the definition (3.4.1) by considering the  $(N+1)$  sets of equations of case (I), that is, equations (3.2.1) through (3.2.16) or the  $(N+1)$  sets of equations corresponding to case (II) or Case (III), we arrive as in Section 2.4 at the following expressions for the  $h(z, \mu_i)$ 's

$$(3.4.3) \quad h(z, \mu_0) = \frac{\mu_0 (1-z) P(0, \mu_0) + \lambda z^{R_1+1} P(R_1-1, \mu_0) - \mu_1 z^{r_1+1} P(r_1+1, \mu_1)}{\lambda z^2 - (\mu_0 + \lambda)z + \mu_0}$$

For  $i = 1, 2, \dots, N-1$

$$(3.4.4) \quad h(z, \mu_i) = [\mu_i z^{r_i+1} P(r_i+1, \mu_i) - \lambda z^{R_i+1} P(R_i-1, \mu_{i-1}) + \lambda z^{R_{i+1}+1} P(R_{i+1}-1, \mu_i) - \mu_{i+1} z^{r_{i+1}+1} P(r_{i+1}+1, \mu_{i+1})] / [\lambda z^2 - (\mu_i + \lambda)z + \mu_i]$$

$$(3.4.5) \quad h(z, \mu_N) = \frac{\mu_N z^{r_N+1} P(r_N+1, \mu_N) - \lambda z^{R_N+1} P(R_N-1, \mu_{N-1})}{\lambda z^2 - (\mu_N + \lambda)z + \mu_N} .$$

Clearly the zeros of the quadratic  $\lambda z^2 - (\mu_i + \lambda)z + \mu_i$  are given by  $z = 1$  and  $z = 1/b_i$  ( $i = 0, 1, \dots, N$ ). Using exactly the same kind of argument as in Section 2.4, that is, that the zeros of the denominators must coincide with the corresponding zeros of the

numerators for which the  $h(z, \mu_i)$ 's converge, we finally obtain

$$(3.4.6) \quad h(z, \mu_0) = \frac{\mu_0 (1-z)P(0, \mu_0) + \mu_1 P(r_1+1, \mu_1) (z^{R_1-1} z^{r_1+1})}{\lambda z^2 - (\mu_0 + \lambda)z + \mu_0} .$$

For  $i = 1, 2, \dots, N-1$

$$(3.4.7) \quad h(z, \mu_i) = [\mu_i P(r_i+1, \mu_i) (z^{r_i+1} - z^{R_i+1}) + \mu_{i+1} P(r_{i+1}+1, \mu_{i+1}) (z^{R_{i+1}+1} - z^{r_{i+1}+1})] / [\lambda z^2 - (\mu_i + \lambda)z + \mu_i]$$

$$(3.4.8) \quad h(z, \mu_N) = \frac{\mu_N P(r_N+1, \mu_N) (z^{r_N+1} - z^{R_N+1})}{\lambda z^2 - (\mu_N + \lambda)z + \mu_N} , \quad b_N < 1$$

where

$$(3.4.9) \quad P(r_1+1, \mu_1) = \frac{b_1 b_0^{R_1+r_1-1}}{b_0^{r_1} - b_0^{R_1}} (1-b_0) P(0, \mu_0)$$

$$P(r_{i+1}+1, \mu_{i+1}) = \frac{b_{i+1} b_i^{R_{i+1}+r_{i+1}-1}}{b_i^{r_{i+1}+1} - b_i^{R_{i+1}+1}} \frac{b_i^{r_i} - b_i^{R_i}}{b_i^{R_i+r_i}} P(r_i+1, \mu_i)$$

( $i = 1, 2, \dots, N-1$ ) .

Taking the limits as  $z \rightarrow 1$  of (3.4.6), (3.4.7) and (3.4.8) and on using l'Hospital's rule we obtain the expressions for the  $P(\mu_i)$ 's since  $P(\mu_i) = \lim_{z \rightarrow 1} h(z, \mu_i)$ . As expected these expressions

are the same as those given by (3.2.25) and the only unknown quantity,  $P(0, \mu_0)$ , involved in the  $h(z, \mu_i)$ 's is determined from (3.2.26).

Having determined  $h(z, \mu_i)$  ( $i = 0, 1, \dots, N$ ) the generating function  $h(z)$  of the  $P(n)$ 's can be obtained from (3.4.2) and hence the moments can be found in the usual manner.

It may be noted that (3.4.3) and (3.4.4) are true even if some or all  $b_i$ 's ( $i = 0, 1, \dots, N-1$ ) are equal to unity whereas the relations given by (3.4.9) have to be modified on taking limits.

### 3.5 The Laplace transform of the distribution of the busy period.

As in the case of the generating function all the three cases considered in Section 3.2 lead to the same result. We shall therefore deal only with case (I). Again we consider the modified process which ceases as soon as the number of units in the system falls to zero. The initial condition is that at the start of the period the system is in state  $(1, \mu_0)$ . Denoting by  $Q(t, n, \mu_1)$  the probability of the state  $(n, \mu_1)$  at time  $t$ , the modified process satisfies the following  $(N+1)$  sets of differential-difference equations:

Set (0)

$$(3.5.1) \quad Q'(t, 0, \mu_0) = \mu_0 Q(t, 1, \mu_0)$$

$$Q'(t, 1, \mu_0) = -(\mu_0 + \lambda)Q(t, 1, \mu_0) + \mu_0 Q(t, 2, \mu_0)$$

$$Q'(t, n+1, \mu_0) = -(\mu_0 + \lambda)Q(t, n+1, \mu_0) + \mu_0 Q(t, n+2, \mu_0) + \lambda Q(t, n, \mu_0)$$

(3.5.2)

$$(n = 1, 2, \dots, r_1 - 2)$$

$$Q'(t, r_1, \mu_0) = -(\mu_0 + \lambda)Q(t, r_1, \mu_0) + \mu_0 Q(t, r_1 + 1, \mu_0) + \lambda Q(t, r_1 - 1, \mu_0) \\ + \mu_1 Q(t, r_1 + 1, \mu_1)$$

$$Q'(t, n+1, \mu_0) = -(\mu_0 + \lambda)Q(t, n+1, \mu_0) + \mu_0 Q(t, n+2, \mu_0) + \lambda Q(t, n, \mu_0)$$

$$(n = r_1, r_1 + 1, \dots, R_1 - 3)$$

$$Q'(t, R_1 - 1, \mu_0) = -(\mu_0 + \lambda)Q(t, R_1 - 1, \mu_0) + \lambda Q(t, R_1 - 2, \mu_0)$$

Set (i) ,  $i = 1, 2, \dots, N-1$ 

$$Q'(t, r_i + 1, \mu_i) = -(\mu_i + \lambda)Q(t, r_i + 1, \mu_i) + \mu_i Q(t, r_i + 2, \mu_i)$$

$$Q'(t, n+1, \mu_i) = -(\mu_i + \lambda)Q(t, n+1, \mu_i) + \mu_i Q(t, n+2, \mu_i) + \lambda Q(t, n, \mu_i)$$

$$(n = r_i + 1, r_i + 2, \dots, R_i - 2)$$

$$Q'(t, R_i, \mu_i) = -(\mu_i + \lambda)Q(t, R_i, \mu_i) + \mu_i Q(t, R_i + 1, \mu_i) + \lambda Q(t, R_i - 1, \mu_i)$$

$$+ \lambda Q(t, R_i - 1, \mu_{i-1})$$

$$(3.5.3) \quad Q'(t, n+1, \mu_i) = -(\mu_i + \lambda)Q(t, n+1, \mu_i) + \mu_i Q(t, n+2, \mu_i) + \lambda Q(t, n, \mu_i)$$

$$(n = R_i, R_i + 1, \dots, r_{i+1} - 2)$$

$$Q'(t, r_{i+1}, \mu_i) = -(\mu_i + \lambda)Q(t, r_{i+1}, \mu_i) + \mu_i Q(t, r_{i+1} + 1, \mu_i) \\ + \lambda Q(t, r_{i+1} - 1, \mu_i) + \mu_{i+1} Q(t, r_{i+1} + 1, \mu_{i+1})$$

$$Q'(t, n+1, \mu_i) = -(\mu_i + \lambda)Q(t, n+1, \mu_i) + \mu_i Q(t, n+2, \mu_i) + \lambda Q(t, n, \mu_i) \\ (n = r_{i+1}, r_{i+1} + 1, \dots, R_{i+1} - 3)$$

$$Q'(t, R_{i+1} - 1, \mu_i) = -(\mu_i + \lambda)Q(t, R_{i+1} - 1, \mu_i) + \lambda Q(t, R_{i+1} - 2, \mu_i)$$

Set (N)

$$Q'(t, r_N + 1, \mu_N) = -(\mu_N + \lambda)Q(t, r_N + 1, \mu_N) + \mu_N Q(t, r_N + 2, \mu_N)$$

$$(3.5.4) \quad Q'(t, n+1, \mu_N) = -(\mu_N + \lambda)Q(t, n+1, \mu_N) + \mu_N Q(t, n+2, \mu_N) + \lambda Q(t, n, \mu_N) \\ (n = r_N + 1, r_N + 2, \dots, R_N - 2)$$

$$Q'(t, R_N, \mu_N) = -(\mu_N + \lambda)Q(t, R_N, \mu_N) + \mu_N Q(t, R_N + 1, \mu_N) \\ + \lambda Q(t, R_N - 1, \mu_N) + \lambda Q(t, R_N - 1, \mu_{N-1})$$

$$Q'(t, n+1, \mu_N) = -(\mu_N + \lambda)Q(t, n+1, \mu_N) + \mu_N Q(t, n+2, \mu_N) + \lambda Q(t, n, \mu_N) \\ (n \geq R_N) .$$

Introduce the partial generating functions

$$(3.5.5) \quad \begin{cases} q(t, z, \mu_0) = \sum_{n=1}^{\infty} z^n Q(t, n, \mu_0) \\ q(t, z, \mu_i) = \sum_{n=0}^{\infty} z^n Q(t, n, \mu_i) \quad (i = 1, 2, \dots, N) . \end{cases}$$

The initial conditions are given by

$$(3.5.6) \quad q(0, z, \mu_i) = \begin{cases} z & , i = 0 \\ 0 & , i \neq 0 \end{cases}$$

Applying (3.5.5) to all equations of the (N+1) sets above excepting equation (3.5.1) of the set (0) and then taking Laplace transforms with respect to time t, and making use of (3.5.6), we obtain as in Sections 1.5, 2.5

$$(3.5.7) \quad q^*(s, z, \mu_0) = \frac{\mu_0 z Q^*(s, 1, \mu_0) + \lambda z^{R_1+1} Q^*(s, R_1-1, \mu_0) - \mu_1 z^{r_1+1} Q^*(s, r_1+1, \mu_1) - z^2}{\lambda z^2 - (s + \mu_0 + \lambda)z + \mu_0}$$

$$(3.5.8) \quad q^*(s, z, \mu_i) = [\mu_i z^{r_i+1} Q^*(s, r_i+1, \mu_i) - \lambda z^{R_i+1} Q^*(s, R_i-1, \mu_{i-1})$$

$$+ \lambda z^{R_{i+1}+1} Q^*(s, R_{i+1}-1, \mu_i)$$

$$- \mu_{i+1} z^{r_{i+1}+1} Q^*(s, r_{i+1}+1, \mu_{i+1})] / [\lambda z^2 - (s + \mu_i + \lambda)z + \mu_i]$$

$$(i = 1, 2, \dots, N-1)$$

$$(3.5.9) \quad q^*(s, z, \mu_N) = \frac{\mu_N z^{r_N+1} Q^*(s, r_N+1, \mu_N) - \lambda z^{R_N+1} Q^*(s, R_N-1, \mu_{N-1})}{\lambda z^2 - (s + \mu_N + \lambda)z + \mu_N}.$$

Let the zeros of the quadratic  $\lambda z^2 - (s + \mu_i + \lambda)z + \mu_i$  be denoted by  $v_i(s)$  and  $\delta_i(s)$  where

$$(3.5.10) \quad \begin{cases} v_i(s) = \frac{1}{2\lambda} [(s + \mu_i + \lambda) + \{(s + \mu_i + \lambda)^2 - 4\mu_i \lambda\}^{1/2}] \\ \delta_i(s) = \frac{1}{2\lambda} [(s + \mu_i + \lambda) - \{(s + \mu_i + \lambda)^2 - 4\mu_i \lambda\}^{1/2}] \end{cases}$$

$$(i = 0, 1, 2, \dots, N)$$

and where in (3.5.10) we choose that value of the square root for which the real part is positive.

Using the same kind of argument as in Section 1.5, that is, that the zeros of the denominator of  $q^*(s, z, \mu_i)$  must coincide with the zeros of the numerator for those zeros for which  $q^*(s, z, \mu_i)$  converges, and by considering (3.5.7), (3.5.8) and (3.5.9) we arrive at the following set of  $(2N+1)$  linear equations in  $(2N+1)$  unknowns

$$\begin{aligned}
& \mu_0 Q^*(s, 1, \mu_0) + \lambda v_0^{R_1} Q^*(s, R_1 - 1, \mu_0) - \mu_1 v_0^{r_1} Q^*(s, r_1 + 1, \mu_1) = v_0 \\
& \mu_0 Q^*(s, 1, \mu_0) + \lambda \delta_0^{R_1} Q^*(s, R_1 - 1, \mu_0) - \mu_1 \delta_0^{r_1} Q^*(s, r_1 + 1, \mu_1) = \delta_0 \\
(3.5.11) \quad & \mu_i v_i^{r_i} Q^*(s, r_i + 1, \mu_i) - \lambda v_i^{R_i} Q^*(s, R_i - 1, \mu_{i-1}) + \lambda v_i^{R_{i+1}} Q^*(s, R_{i+1} - 1, \mu_i) \\
& \qquad \qquad \qquad - \mu_{i+1} v_i^{r_{i+1}} Q^*(s, r_{i+1} + 1, \mu_{i+1}) = 0 \\
& \mu_i \delta_i^{r_i} Q^*(s, r_i + 1, \mu_i) - \lambda \delta_i^{R_i} Q^*(s, R_i - 1, \mu_{i-1}) + \lambda \delta_i^{R_{i+1}} Q^*(s, R_{i+1} - 1, \mu_i) \\
& \qquad \qquad \qquad - \mu_{i+1} \delta_i^{r_{i+1}} Q^*(s, r_{i+1} + 1, \mu_{i+1}) = 0 \\
& \qquad \qquad \qquad (i = 1, 2, \dots, N-1) \\
& \mu_N \delta_N^{r_N} Q^*(s, r_N + 1, \mu_N) - \lambda \delta_N^{R_N} Q^*(s, R_N - 1, \mu_{N-1}) = 0
\end{aligned}$$

But the p.d.f. of the length of the busy period  $f_b(t)$  is given by

$$\begin{aligned}
f_b(t) &= Q'(t, 0, \mu_0) \\
&= \mu_0 Q(t, 1, \mu_0) \text{ on using (3.5.1)}
\end{aligned}$$

or in terms of Laplace transform we have

$$(3.5.12) \quad f_b^*(s) = \mu_0 Q^*(s, 1, \mu_0)$$

where  $Q^*(s, 1, \mu_0)$  can be obtained from (3.5.11).

Thus we have shown how, in principle, to obtain the Laplace transform of the distribution of the busy period.

In the special case where  $\mu$  takes on two values  $\mu_0$  and  $\mu_1$  there is only one pair of integers  $(r_1, R_1)$ ,  $r_1 \geq 0$ ,  $r_1 < R_1$ , resulting in three equations in three unknowns. The first two of these equations are exactly the same as the first two equations of (3.5.11) and the third one can be obtained from the last equation of (3.5.11) by merely replacing  $\mu_N$ ,  $r_N$ ,  $R_N$  and  $\delta_N$  by  $\mu_1$ ,  $r_1$ ,  $R_1$  and  $\delta_1$  respectively. In this case we have

$$(3.5.13) \quad f_b^*(s) = \frac{\mu_0}{R_1 R_1} \frac{R_1^{-1} R_1^{-1}}{\lambda(v_0^{-1} - \delta_0^{-1})} [(v_0^{-1} - \delta_0^{-1})^{R_1-1} (v_0^{-1} - \delta_0^{-1})^{R_1-1}] \\ + \left(\frac{\mu_0}{\lambda}\right)^{r_1-1} \frac{\delta_1^{-1} (v_0^{-1} - \delta_0^{-1})^{R_1-r_1} (v_0^{-1} - \delta_0^{-1})^{R_1-r_1}}{(v_0^{-1} - \delta_0^{-1})^{R_1-r_1} (v_0^{-1} - \delta_0^{-1})^{R_1-r_1}}$$

and the expected value  $E_b(t)$  is given by

$$(3.5.14) \quad E_b(t) = \begin{cases} \frac{1}{\mu_0 - \lambda} - \frac{R_1 - r_1}{\mu_1 - \lambda} \frac{\mu_1 - \mu_0}{\lambda} \cdot \frac{b_0^{R_1+r_1}}{b_0^{R_1} - b_0^{r_1}}, & b_0 = \lambda/\mu_0 \neq 1 \\ \frac{1}{\mu_0} \left[ \frac{1}{2}(R_1+r_1-1) + \frac{b_1}{1-b_1} \right], & b_0 = 1 \end{cases}$$

$$(b_1 < 1).$$

3.6 The Laplace transform of the distribution of time the system spends in the subset of states  $\{n, \mu_i\} (i = 0, 1, \dots, N)$  before it goes out of the subset for the first time. It is easy to see that

these problems are exactly the same as those dealt with in Section 2.6 except for the differences in the parameters. Thus in this case the solution to the distribution problem connected with the subset of states  $\{n, \mu_i\}$  can be obtained from the solution concerning the subset of states  $\{(n, \lambda_i)\}$  in Section 2.6 by merely putting  $\lambda_i = \lambda$  and  $\mu = \mu_i$  for  $i = 0, 1, \dots, N$ .

3.7 The case when N is infinite. As in Section 2.7, the existence of steady state conditions when N is infinite depends upon the convergence of the series [see (3.2.26)]

$$(3.7.1) \quad \sum_{i=1}^{\infty} (R_i - r_i) \frac{(b_{i-1} - b_i)}{b_i(1-b_{i-1})(1-b_i)} \frac{P(r_i+1, \mu_i)}{P(0, \mu_0)}$$

where the ratio  $P(r_i+1, \mu_i)/P(0, \mu_0)$  involves only the  $b_i$ 's,  $r_i$ 's and  $R_i$ 's and is given by the relations (3.2.31).

If the above series converges all the appropriate formulae developed in this chapter are still valid when there are infinite number of loops.

3.8 Remarks. On putting  $\mu_1 = \mu_2 = \dots = \mu_N = \mu_0$ , all the corresponding results or formulas for a simple queue, subject to the usual restriction  $b_0 < 1$  for the existence of equilibrium conditions, can be obtained from this chapter.

It is also possible to superimpose a cost structure on the problem and define a suitable cost function. The parameters involved

in the cost function are clearly  $r_i$ ,  $R_i$  and  $b_i$  ( $i = 1, 2, \dots, N$ ).

Appendix (b) gives a brief discussion of the problem for the special case where  $N = 1$ , that is, the case where  $\mu$  takes on two values.

## CHAPTER IV

### A SINGLE-SERVER QUEUE WITH VARIABLE ARRIVAL RATE

4.1 Introduction. In this chapter we attempt to solve a single-server queueing process under the assumptions (a) the service-time is exponentially distributed with parameter  $\mu$  and (b) the arrival process is a Poisson process with variable parameter  $\lambda$ . The meaning of the assumption (b) can best be explained as follows: The probability of an arrival in a small interval of time of length  $\delta t$  is  $\lambda \delta t$  where  $\lambda$  may assume more than one value say  $\lambda_0, \lambda_1, \dots, \lambda_N$ . If at any time  $t$  the arrival rate is  $\lambda_m$  ( $m = 0, 1, \dots, N$ ), then in the next interval of time of length  $\delta t$  the probability that  $\lambda_m$  will change to some other value is  $\kappa_m \delta t$ . Given that  $\lambda_m$  has changed during the interval  $\delta t$ , we denote by  $p_{mr}$  the conditional probability that  $\lambda$  changes from  $\lambda_m$  to  $\lambda_r$ . Clearly  $p_{mm} = 0$  and  $\sum_{r=0}^N p_{mr} = 1$  ( $m = 0, 1, \dots, N$ ).

We shall first deal with problems arising from the above arrival process, for example, the distribution of the inter-arrival time, the distribution of the number of arrivals in an interval of length  $t$  etc. In this whole chapter we shall consider the case where  $\lambda$  takes only two values  $\lambda_0$  and  $\lambda_1$  but the methods developed are perfectly general, at least in principle. When  $\lambda$  takes  $k$  values where  $k > 2$  we need to solve a  $k$ -th degree polynomial the roots of which have to be found.

If  $\lambda$  assumes only two values,  $\lambda_0$  and  $\lambda_1$  we have corresponding to these two values  $\kappa_0$  and  $\kappa_1$  as explained in the first paragraph and

the conditional transition probabilities are given by  $p_{01} = 1$ ,  $p_{10} = 1$ . The stationary probability that the arrival rate is  $\lambda_i$  is denoted by  $\pi_i$ , where

$$(4.1.1) \quad \pi_0 = \frac{\kappa_1}{\kappa_0 + \kappa_1} \quad \text{and} \quad \pi_1 = \frac{\kappa_0}{\kappa_0 + \kappa_1} .$$

4.2 The distribution problems concerning inter-arrival times: The first problem we wish to study is

(A) The inter-arrival time distribution:- Let  $Q_t(\lambda_i)$  ( $i = 0, 1$ ) denote the probability that there is no arrival in an interval of time  $(0, t)$  and that the arrival rate at time  $t$  is  $\lambda_i$ , where it is to be understood that we are concerned with only an interval of time of length  $t$  no matter where it is placed in the time axis. By employing the '  $\delta t$ ' technique, clearly we have the following two differential equations:

$$(4.2.1) \quad \begin{aligned} Q_t'(\lambda_0) + (\lambda_0 + \kappa_0) Q_t(\lambda_0) - \kappa_1 Q_t(\lambda_1) &= 0 \\ Q_t'(\lambda_1) + (\lambda_1 + \kappa_1) Q_t(\lambda_1) - \kappa_0 Q_t(\lambda_0) &= 0 \end{aligned}$$

where (4.2.1) are to be solved subject to the initial conditions

$$(4.2.2) \quad \begin{aligned} \tau_0 \equiv Q_0(\lambda_0) &= \frac{\pi_0 \lambda_0}{\pi_0 \lambda_0 + \pi_1 \lambda_1} \\ \tau_1 \equiv Q_0(\lambda_1) &= \frac{\pi_1 \lambda_1}{\pi_0 \lambda_0 + \pi_1 \lambda_1} \end{aligned} ,$$

$\pi_0$  and  $\pi_1$  being given by (4.1.1) .

Denoting the Laplace transform of  $Q_t(\lambda_i)$  with respect to  $t$  by  $Q_s^*(\lambda_i)$ ,  $\text{Re}(s) > 0$ , we have on taking the Laplace transform of (4.2.1) and making use of (4.2.2)

$$(4.2.3) \quad \begin{aligned} (s + \lambda_0 + \kappa_0)Q_s^*(\lambda_0) - \kappa_1 Q_s^*(\lambda_1) &= \tau_0 \\ - \kappa_1 Q_s^*(\lambda_1) + (s + \lambda_1 + \kappa_1)Q_s^*(\lambda_1) &= \tau_1 \end{aligned}$$

from which we obtain

$$(4.2.4) \quad \begin{aligned} Q_s^*(\lambda_0) &= \frac{\tau_0(s + \lambda_1) + \kappa_1}{s^2 + (\lambda_0 + \lambda_1 + \kappa_0 + \kappa_1)s + (\lambda_0\lambda_1 + \lambda_0\kappa_1 + \lambda_1\kappa_0)} \\ Q_s^*(\lambda_1) &= \frac{\tau_1(s + \lambda_0) + \kappa_0}{s^2 + (\lambda_0 + \lambda_1 + \kappa_0 + \kappa_1)s + (\lambda_0\lambda_1 + \lambda_0\kappa_1 + \lambda_1\kappa_0)} \end{aligned}$$

With some algebraical manipulations we can express (4.2.4) in an alternative form

$$(4.2.5) \quad \begin{aligned} Q_s^*(\lambda_0) &= \left[ \frac{\tau_0(\lambda_1 - r_0) + \kappa_1}{r_1 - r_0} \right] \frac{1}{s + r_0} + \left[ \tau_0 - \frac{\tau_0(\lambda_1 - r_0) + \kappa_1}{r_1 - r_0} \right] \frac{1}{s + r_1} \\ Q_s^*(\lambda_1) &= \left[ \frac{\tau_1(\lambda_0 - r_0) + \kappa_0}{r_1 - r_0} \right] \frac{1}{s + r_0} + \left[ \tau_1 - \frac{\tau_1(\lambda_0 - r_0) + \kappa_0}{r_1 - r_0} \right] \frac{1}{s + r_1} \end{aligned}$$

where

$$(4.2.6) \quad r_i = \frac{1}{2} \left[ (\lambda_0 + \lambda_1 + \kappa_0 + \kappa_1) \mp \sqrt{\{(\lambda_0 - \lambda_1) + (\kappa_0 - \kappa_1)\}^2 + 4\kappa_0\kappa_1} \right] \quad (i=0, 1)$$

with  $r_0$  having the negative sign in front of the radical.

If we denote by  $Q_t(\lambda)$  for the probability of no arrival in  $(0, t)$ , then, clearly  $Q_t(\lambda) = Q_t(\lambda_0) + Q_t(\lambda_1)$  or in terms of Laplace transform

$$(4.2.7) \quad Q_s^*(\lambda) = Q_s^*(\lambda_0) + Q_s^*(\lambda_1) .$$

Making use of (4.2.5) and noting that  $\tau_0 + \tau_1 = 1$ , we obtain

$$(4.2.8) \quad Q_s^*(\lambda) = \left( \frac{c - r_0}{r_1 - r_0} \right) \frac{1}{s + r_0} + \left( 1 - \frac{c - r_0}{r_1 - r_0} \right) \frac{1}{s + r_1}$$

where  $c = \lambda_0 \tau_1 + \lambda_1 \tau_0 + \kappa_0 + \kappa_1$ . Taking the inverse transform of (4.2.8) we get

$$(4.2.9) \quad Q_t(\lambda) = \left( \frac{c - r_0}{r_1 - r_0} \right) e^{-r_0 t} + \left( 1 - \frac{c - r_0}{r_1 - r_0} \right) e^{-r_1 t}$$

Let  $F(t)$  be the distribution function (d.f.) of the inter-arrival time, then clearly  $F(t) = 1 - Q_t(\lambda)$  and therefore the p.d.f.  $f(t)$  of the inter-arrival time is given by

$$(4.2.10) \quad f(t) = \left( \frac{c - r_0}{r_1 - r_0} \right) r_0 e^{-r_0 t} + \left( 1 - \frac{c - r_0}{r_1 - r_0} \right) r_1 e^{-r_1 t}$$

which is a mixture of two exponential distributions. The expected value  $E(t)$  is given by

$$E(t) = \frac{c}{r_0 r_1} = \frac{\tau_0 \lambda_1 + \tau_1 \lambda_0 + \kappa_0 + \kappa_1}{\lambda_0 \lambda_1 + \lambda_0 \kappa_1 + \lambda_1 \kappa_0}$$

Denoting by  $\bar{\lambda}$  for  $\pi_0 \lambda_0 + \pi_1 \lambda_1$  we have

$$\begin{aligned} \tau_0 \lambda_1 + \tau_1 \lambda_0 + \kappa_0 + \kappa_1 &= \frac{\pi_0 \lambda_0 \lambda_1}{\bar{\lambda}} + \frac{\pi_1 \lambda_1 \lambda_0}{\bar{\lambda}} + (\kappa_0 + \kappa_1) \\ &= \frac{\lambda_0 \lambda_1}{\bar{\lambda}} + (\kappa_0 + \kappa_1), \text{ since } \pi_0 + \pi_1 = 1 \end{aligned}$$

$$\text{and } \lambda_0 \lambda_1 + \lambda_0 \kappa_1 + \lambda_1 \kappa_0 = \lambda_0 \lambda_1 + (\kappa_0 + \kappa_1) \bar{\lambda}.$$

that is,

$$(4.2.11) \quad E(t) = \frac{1}{\bar{\lambda}}.$$

It can easily be shown that if  $\lambda_0 = \lambda_1$  ( $= \lambda$  say), then  $f(t) = \lambda e^{-\lambda t}$ . Furthermore, if  $\kappa_0 \rightarrow \infty$ ,  $\kappa_1 \rightarrow \infty$  and  $\pi_0$  is constant, then  $f(t) \rightarrow \bar{\lambda} e^{-\bar{\lambda} t}$ .

Let us denote by  $f(t, \lambda_i)$  dt the probability that the length of the inter-arrival time lies in the interval  $(t, t + dt)$  and that at its termination the arrival rate is  $\lambda_i$ . Making the same kind of argument as in the derivation of  $f(t)$  we obtain by considering (4.2.5)

$$(4.2.12) \quad \left. \begin{aligned} f(t, \lambda_0) &= \left[ \frac{\tau_0(\lambda_1 - r_0) + \kappa_1}{r_1 - r_0} \right] r_0 e^{-r_0 t} + \left[ \tau_0 - \frac{\tau_0(\lambda_1 - r_0) + \kappa_1}{r_1 - r_0} \right] r_1 e^{-r_1 t} \\ f(t, \lambda_1) &= \left[ \frac{\tau_1(\lambda_0 - r_0) + \kappa_0}{r_1 - r_0} \right] r_0 e^{-r_0 t} + \left[ \tau_1 - \frac{\tau_1(\lambda_0 - r_0) + \kappa_0}{r_1 - r_0} \right] r_1 e^{-r_1 t} \end{aligned} \right\}$$

(B) The joint distribution of two successive inter-arrival times:-

Let  $t_1$  and  $t_2$  be the lengths of two successive inter-arrival times respectively and denote by  $h(t_1, t_2)$  their joint p.d.f. Also let  $h(t_1, \lambda_i, t_2) dt_1 dt_2$  denote the probability that the length of the first inter-arrival time lies in the interval  $(t_1, t_1 + dt_1)$  and that at its termination the arrival rate is  $\lambda_i$  (which is the same as saying that the next arrival time starts with an arrival rate  $\lambda_i$ ) and that the length of the next arrival time lies in  $(t_2, t_2 + dt_2)$ . We then have

$$(4.2.13) \quad \begin{aligned} h(t_1, t_2) &= h(t_1, \lambda_0, t_2) + h(t_1, \lambda_1, t_2) \\ &= h(t_2/t_1, \lambda_0) f(t_1, \lambda_0) + h(t_2/t_1, \lambda_1) f(t_1, \lambda_1) \end{aligned}$$

where  $f(t_1, \lambda_i)$  ( $i = 0, 1$ ) are given by (4.2.12).

By virtue of the process being Markov process it is clear that whatever be  $t_1$ , the length of the next inter-arrival time depends only on the arrival rate at its start, that is,  $h(t_2/t_1, \lambda_1)$  depends only on  $\lambda_1$ . We can therefore write

$$(4.2.14) \quad h(t_1, t_2) = h(t_2/\lambda_0) f(t_1, \lambda_0) + h(t_2/\lambda_1) f(t_1, \lambda_1).$$

On the other hand the conditional distribution of  $t_2$  given  $t_1$  does depend on both  $t_1$  and the arrival rate at its start. We can write (4.2.14) as

$$h(t_2/t_1) f(t_1) = \left[ h(t_2/\lambda_0) f(\lambda_0/t_1) + h(t_2/\lambda_1) f(\lambda_1/t_1) \right] f(t_1),$$

that is,

$$(4.2.15) \quad h(t_2/t_1) = h(t_2/\lambda_0) f(\lambda_0/t_1) + h(t_2/\lambda_1) f(\lambda_1/t_1).$$

The only difference in the derivation of  $f(t_1)$  and  $h(t_2/\lambda_1)$  lies in the initial conditions.  $h(t_2/\lambda_0)$  can be obtained from (4.2.10) by merely replacing  $t$ ,  $\tau_0$  and  $\tau_1$  by  $t_2$ , 1 and 0 respectively and recalling that in (4.2.10)  $c = \tau_0 \lambda_1 + \tau_1 \lambda_0 + \kappa_0 + \kappa_1$ , that is,

$$(4.2.16) \quad h(t_2/\lambda_0) = \left( \frac{c_0 - r_0}{r_1 - r_0} \right) r_0 e^{-r_0 t_2} + \left( 1 - \frac{c_0 - r_0}{r_1 - r_0} \right) r_1 e^{-r_1 t_2}$$

where  $c_0 = \lambda_1 + \kappa_0 + \kappa_1$ . Similarly

$$(4.2.17) \quad h(t_2/\lambda_1) = \left( \frac{c_1 - r_0}{r_1 - r_0} \right) r_0 e^{-r_0 t_2} + \left( 1 - \frac{c_1 - r_0}{r_1 - r_0} \right) r_1 e^{-r_1 t_2}$$

where  $c_1 = \lambda_0 + \kappa_0 + \kappa_1$ . Put

$$(4.2.18) \left\{ \begin{array}{l} a_0 = \frac{c_0 - r_0}{r_1 - r_0}, \quad a_1 = 1 - a_0 \\ b_0 = \frac{c_1 - r_0}{r_1 - r_0}, \quad b_1 = 1 - b_0 \\ \alpha_0 = \frac{\tau_0(\lambda_1 - r_0) + \kappa_1}{r_1 - r_0}, \quad \alpha_1 = \tau_0 - \alpha_0 \\ \beta_0 = \frac{\tau_1(\lambda_0 - r_0) + \kappa_0}{r_1 - r_0}, \quad \beta_1 = \tau_1 - \beta_0 \end{array} \right.$$

Where  $\tau_0$  and  $\tau_1$  are given by (4.2.2) and  $c_0$  and  $c_1$  are as given in (4.2.16) and (4.2.17) respectively.

Making use of (4.2.16), (4.2.17), (4.2.12) and (4.2.14) we obtain for the joint distribution of  $t_1$  and  $t_2$

$$(4.2.19) \begin{aligned} h(t_1, t_2) = & (a_0 \alpha_0 + b_0 \beta_0) r_0 e^{-r_0 t_1} r_0 e^{-r_0 t_2} \\ & + (a_1 \alpha_1 + b_1 \beta_1) r_1 e^{-r_1 t_1} r_1 e^{-r_1 t_2} \\ & + (a_0 \alpha_1 + b_0 \beta_1) r_1 e^{-r_1 t_1} r_0 e^{-r_0 t_2} \\ & + (a_1 \alpha_0 + b_1 \beta_0) r_0 e^{-r_0 t_1} r_1 e^{-r_1 t_2} . \end{aligned}$$

Integrating  $h(t_1, t_2)$  with respect to  $t_1$  and  $t_2$  separately we obtain the marginal distributions of  $t_1$  and  $t_2$  and, as expected, these are the same and given by (4.2.10). Thus we have shown that  $t_1$  and  $t_2$  are

identically distributed but no independent.

Now

$$\begin{aligned}
 (4.2.20) \quad \text{Cov}(t_1, t_2) &= E(t_1 t_2) - E(t_1) E(t_2) \\
 &= (a_1 \alpha_0 + b_0 \beta_0) / r_0^2 + (a_1 \alpha_1 + b_1 \beta_1) / r_1^2 \\
 &\quad + (a_0 \alpha_1 + b_0 \beta_1 + a_1 \alpha_0 + b_1 \beta_0) / r_0 r_1 - \frac{1}{\bar{\lambda}^2}
 \end{aligned}$$

where  $\bar{\lambda}$  is as given in (4.2.11). It can easily be shown that

$$\text{Cov}(t_1, t_2) = 0 \quad \text{when } \lambda_0 = \lambda_1, \quad \text{and}$$

$$\text{Cov}(t_1, t_2) \longrightarrow 0 \text{ as } \kappa_0 \longrightarrow \infty, \quad \kappa_1 \longrightarrow \infty \text{ and } \pi_0$$

constant.

#### 4.3 The distribution problems concerning the number of arrivals:

(A) The generating function of the distribution of the number of arrivals in a time interval of length t.

Let  $p_t(n, \lambda_i)$  ( $i = 0, 1$ ) denote the probability that there are exactly  $n$  arrivals in an interval of time  $(0, t)$  and that the arrival rate at time  $t$  is  $\lambda_i$ , where again by time  $t$  we mean an interval of time of length  $t$  no matter where it is placed on the time axis. By employing the ' $\delta t$ ' technique, we have the following two set of differential equations

$$\begin{aligned}
 (4.3.1) \quad & \frac{d}{dt} p_t(0, \lambda_0) + (\lambda_0 + \kappa_0) p_t(0, \lambda_0) - \kappa_1 p_t(0, \lambda_1) = 0 \\
 & \left\{ \begin{aligned}
 & p_t'(n, \lambda_0) + (\lambda_0 + \kappa_0) p_t(n, \lambda_0) - \kappa_1 p_t(n, \lambda_1) \\
 & \quad - \lambda_0 p_t(n-1, \lambda_0) = 0 \quad (n \geq 1)
 \end{aligned} \right.
 \end{aligned}$$

$$(4.3.2) \begin{cases} p_t'(0, \lambda_1) + (\lambda_1 + \kappa_1) p_t(0, \lambda_1) - \kappa_0 p_t(0, \lambda_0) = 0 \\ p_t'(n, \lambda_1) + (\lambda_1 + \kappa_1) p_t(n, \lambda_1) - \kappa_0 p_t(n, \lambda_0) - \lambda_1 p_t(n-1, \lambda_1) = 0 \end{cases}$$

$$(n \geq 1)$$

Where (4.3.1) and (4.3.2) are to be solved subject to the initial conditions

$$(4.3.3) \begin{cases} p_0(n, \lambda_i) = 0, \quad n > 0 \\ p_0(0, \lambda_i) = \pi_i \quad (i = 0, 1) \end{cases},$$

$\pi_i$  is given by (4.1.1). Define

$$(4.3.4) \quad G_t(z, \lambda_i) = \sum_{n=0}^{\infty} z^n p_t(n, \lambda_i); \quad \omega$$

then (4.3.1) and (4.3.2) lead to

$$(4.3.5) \begin{cases} \frac{\partial G_t(z, \lambda_0)}{\partial t} + (\lambda_0 - \lambda_0 z + \kappa_0) G_t(z, \lambda_0) - \kappa_1 G_t(z, \lambda_1) = 0 \\ \frac{\partial G_t(z, \lambda_1)}{\partial t} + (-\lambda_1 z + \kappa_1) G_t(z, \lambda_1) - \kappa_0 G_t(z, \lambda_0) = 0 \end{cases}$$

and (4.3.3) leads to

$$(4.3.6) \quad G_0(z, \lambda_i) = \pi_i.$$

Denoting the Laplace transform of  $G_t(z, \lambda_i)$  with respect to  $t$  by  $G_s^*(z, \lambda_i)$ ,  $\text{Re}(s) > 0$ , we have from (4.3.5) and (4.3.6)

$$(4.3.7) \quad \begin{cases} (s + \kappa_0 + \lambda_0 - \lambda_0 z) G_S^*(z, \lambda_0) - \kappa_1 G_S^*(z, \lambda_1) = \pi_0 \\ -\kappa_0 G_S^*(z, \lambda_0) + (s + \kappa_1 + \lambda_1 - \lambda_1 z) G_S^*(z, \lambda_1) = \pi_1 \end{cases}$$

from which we obtain

$$(4.3.8) \quad \begin{cases} G_S^*(z, \lambda_0) = \frac{\pi_0 (s + \lambda_1 - \lambda_1 z) + \kappa_1}{s^2 + (\kappa_0 + \kappa_1 + (\lambda_0 + \lambda_1)(1-z)) \{s + (1-z) \{ \kappa_0 \lambda_1 + \kappa_1 \lambda_0 + \lambda_0 \lambda_1 (1-z) \} \}} \\ G_S^*(z, \lambda_1) = \frac{\pi_1 (s + \lambda_0 - \lambda_0 z) + \kappa_0}{s^2 + (\kappa_0 + \kappa_1 + (\lambda_0 + \lambda_1)(1-z)) \{s + (1-z) \{ \kappa_0 \lambda_1 + \kappa_1 \lambda_0 + \lambda_0 \lambda_1 (1-z) \} \}} \end{cases}$$

But the probability of having exactly  $n$  arrivals in an interval of time  $(0, t)$  is clearly given by

$$p_t(n) = p_t(n, \lambda_0) + p_t(n, \lambda_1) .$$

Hence if we denote by  $G_t(z)$  the generating function of the  $p_t(n)$ 's and  $G_S^*(z)$  the Laplace transform with respect to  $t$  of  $G_t(z)$  we then have

$$G_S^*(z) = G_S^*(z, \lambda_0) + G_S^*(z, \lambda_1)$$

which on using (4.3.8) gives

$$(4.3.9) \quad G_S^*(z) = \frac{s + \kappa_0 + \kappa_1 + (\pi_0 \lambda_1 + \pi_1 \lambda_0)(1-z)}{s^2 + (\kappa_0 + \kappa_1 + (\lambda_0 + \lambda_1)(1-z)) \{s + (1-z) \{ \kappa_0 \lambda_1 + \kappa_1 \lambda_0 + \lambda_0 \lambda_1 (1-z) \} \}} \\ = \frac{s + c(z)}{\{s + r_0(z)\} \{s + r_1(z)\}}$$

where

$$(4.3.10) \quad \begin{cases} c(z) = \kappa_0 + \kappa_1 + (\pi_0 \lambda_1 + \pi_1 \lambda_0)(1-z) \\ r_i(z) = \frac{1}{2} \left[ \left\{ (\kappa_0 + \kappa_1) + (\lambda_0 + \lambda_1)(1-z) \right\} \mp \sqrt{\left\{ (\kappa_0 - \kappa_1) + (\lambda_0 - \lambda_1)(1-z) \right\}^2 + 4\kappa_0 \kappa_1} \right] \end{cases}$$

(i = 0, 1)

with  $r_0(z)$  having the negative sign in front of the radical.

On further simplification the expression for  $G_s^*(z)$  given by (4.3.9) can be written as

$$(4.3.11) \quad G_s^*(z) = A_0(z) \frac{1}{s + r_0(z)} + A_1(z) \frac{1}{s + r_1(z)}$$

where

$$(4.3.12) \quad \begin{cases} A_0(z) + A_1(z) = 1, \text{ and} \\ A_0(z) = \frac{c(z) - r_0(z)}{r_1(z) - r_0(z)}. \end{cases}$$

Taking the inverse transform of (4.3.11) we obtain the expression for the required generating function

$$(4.3.13) \quad G_t(z) = A_0(z) e^{-r_0(z)t} + A_1(z) e^{-r_1(z)t}.$$

The mean and the variance of the number of arrivals in  $(0, t)$  are respectively given by

$$(4.3.14) \quad \begin{cases} E(n) = \bar{\lambda}t, \text{ where } \bar{\lambda} = \pi_0 \lambda_0 + \pi_1 \lambda_1, \text{ and} \\ \text{Var}(n) = \bar{\lambda}t + \frac{1}{2} \frac{(\lambda_0 - \lambda_1)^2}{(\kappa_0 + \kappa_1)} \left[ 1 - (\pi_1 - \pi_0)^2 \right] t - 2\pi_0 \pi_1 \frac{(\lambda_0 - \lambda_1)^2}{(\kappa_0 + \kappa_1)^2} \left( 1 - e^{-(\kappa_0 + \kappa_1)t} \right). \end{cases}$$

Under the condition that  $\kappa_0 = \kappa_1 (= \kappa \text{ say})$ , we have

$$(4.3.15) \left\{ \begin{array}{l} E(n) = \frac{1}{2} (\lambda_0 + \lambda_1)t \quad \text{and} \\ \text{Var}(n) = \frac{1}{2}(\lambda_0 + \lambda_1)t + \frac{(\lambda_0 - \lambda_1)^2}{4\kappa} \int_0^t (1 - e^{-2\kappa x}) dx \end{array} \right.$$

It has been verified that

- (i)  $\lambda_0 = \lambda_1 (= \lambda \text{ say})$  implies  $G_t(z) = e^{-\lambda(1-z)t}$ , and  
(ii)  $\kappa_0 \rightarrow \infty, \kappa_1 \rightarrow \infty$  and  $\pi_0$  constant implies  $G_t(z) \rightarrow e^{-\bar{\lambda}(1-z)t}$ .

Similarly by separately considering  $G_s^*(z, \lambda_0)$  and  $G_s^*(z, \lambda_1)$  as given by (4.3.8), we obtain

$$(4.3.16) \left\{ \begin{array}{l} G_s^*(z, \lambda_0) = R_0(z) \frac{1}{s + r_0(z)} + R_1(z) \frac{1}{s + r_1(z)} \\ G_s^*(z, \lambda_1) = Q_0(z) \frac{1}{s + r_0(z)} + Q_1(z) \frac{1}{s + r_1(z)} \end{array} \right.$$

where

$$(4.3.17) \left\{ \begin{array}{l} R_0(z) + R_1(z) = \pi_0, \quad R_0(z) = \frac{c_0(z) - \pi_0 r_0(z)}{r_1(z) - r_0(z)} \\ Q_0(z) + Q_1(z) = \pi_1, \quad Q_0(z) = \frac{c_1(z) - \pi_1 r_0(z)}{r_1(z) - r_0(z)} \\ c_0(z) = \pi_0 \lambda_1 (1-z) + \kappa_1, \quad c_1(z) = \pi_1 \lambda_0 (1-z) + \kappa_0 \end{array} \right.$$

Taking the inverse transforms of (4.3.16) we obtain

$$(4.3.18) \left\{ \begin{array}{l} G_t(z, \lambda_0) = R_0(z) e^{-r_0(z)t} + R_1(z) e^{-r_1(z)t} \\ G_t(z, \lambda_1) = Q_0(z) e^{-r_0(z)t} + Q_1(z) e^{-r_1(z)t} \end{array} \right.$$

(B) The generating function of the joint distribution of the number of arrivals in two successive non-overlapping intervals of lengths

$t_1$  and  $t_2$  :- Let  $h(n_1, n_2)$  denote the joint probability of finding  $n_1$  arrivals in an interval of time of length  $t_1$  and  $n_2$  arrivals in the next successive interval of length  $t_2$ . Also let  $h(n_1, \lambda_1, n_2)$  denote the probability that there are  $n_1$  arrivals in  $t_1$ , that at the end of  $t_1$  the arrival rate is  $\lambda_1$ , and that there are  $n_2$  arrivals in the next successive interval of length  $t_2$ . It is to be understood that both  $h(n_1, n_2)$  and  $h(n_1, \lambda_1, n_2)$  do depend explicitly on  $t_1$  and  $t_2$  but we have suppressed them for convenience of notation.

Clearly,

$$(4.3.19) \quad h(n_1, n_2) = h(n_1, \lambda_0, n_2) + h(n_1, \lambda_1, n_2) \\ = h(n_2/n_1, \lambda_0) p(n_1, \lambda_0) + h(n_2/n_1, \lambda_1) p(n_1, \lambda_1)$$

where  $p(n_1, \lambda_1)$  is the probability of finding  $n_1$  arrivals in  $t_1$  and that at its termination the arrival rate is  $\lambda_1$ .

Making the same kind of argument as in section 4.2, the conditional distribution of  $n_2$  given both  $n_1$  and  $\lambda_1$  depends only on  $\lambda_1$ . That is, we can write

$$(4.3.20) \quad h(n_1, n_2) = h(n_2/\lambda_0) p(n_1, \lambda_0) + h(n_2/\lambda_1) p(n_1, \lambda_1) .$$

Define

$$(4.3.21) \quad \left\{ \begin{array}{l} H(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} h(n_1, n_2) \\ W(z_2/\lambda_1) = \sum_{n_2=0}^{\infty} z_2^{n_2} h(n_2/\lambda_1) \\ G(z_1, \lambda_i) = \sum_{n_1=0}^{\infty} z_1^{n_1} p(n_1, \lambda_i) \quad (i = 0, 1). \end{array} \right.$$

Making use of (4.3.21) it is easy to see that (4.3.20) leads to

$$(4.3.22) \quad H(z_1, z_2) = W(z_2/\lambda_0)G(z_1, \lambda_0) + W(z_2/\lambda_1)G(z_1, \lambda_1).$$

The expressions for  $G(z_1, \lambda_i)$  ( $i = 0, 1$ ) can be obtained from (4.3.18) by merely replacing  $t$  by  $t_1$  and  $z$  by  $z_1$ . Also the derivation of the formula for  $W(z_2/\lambda_i)$  is exactly the same as that of  $G_t(z)$ , {see (4.3.13)} except for the initial conditions. More specifically  $W(z_2/\lambda_0)$  is obtained from (4.3.13) by merely replacing  $t$ ,  $\pi_0$  and  $\pi_1$  respectively by  $t_2$ , 1 and 0. Similarly for  $W(z_2/\lambda_1)$  we replace  $t$ ,  $\pi_0$  and  $\pi_1$  of (4.3.13) by  $t_2$ , 0 and 1 respectively, that is

$$\begin{aligned}
 W(z_2/\lambda_0) &= M_0(z_2)e^{-r_0(z_2)t_2} + M_1(z_2)e^{-r_1(z_2)t_2} \\
 W(z_2/\lambda_1) &= N_0(z_2)e^{-r_0(z_2)t_2} + N_1(z_2)e^{-r_1(z_2)t_2} \\
 \text{where } M_0(z_2) &= \frac{\ell_0(z_2) - r_0(z_2)}{r_1(z_2) - r_0(z_2)}, \quad M_0(z_2) + M_1(z_2) = 1 \\
 N_0(z_2) &= \frac{\ell_0(z_2) - r_0(z_2)}{r_1(z_2) - r_0(z_2)}, \quad N_0(z_2) + N_1(z_2) = 1 \\
 \ell_0(z_2) &= \kappa_0 + \kappa_1 + \lambda_1(1 - z_2) \\
 \text{and } \ell_1(z_2) &= \kappa_0 + \kappa_1 + \lambda_0(1 - z_2) .
 \end{aligned}
 \tag{4.3.23}$$

Now we are in a position to write down the final expression for  $H(z_1, z_2)$  and this is given by

$$\begin{aligned}
 (4.3.24) \quad H(z_1, z_2) &= [R_0(z_1)M_0(z_2) + Q_0(z_1)N_0(z_2)]e^{-r_0(z_1)t_1 - r_0(z_2)t_2} \\
 &+ [R_1(z_1)M_1(z_2) + Q_1(z_1)N_0(z_2)]e^{-r_1(z_1)t_1 - r_1(z_2)t_2} \\
 &+ [R_0(z_1)M_1(z_2) + Q_0(z_1)N_1(z_2)]e^{-r_0(z_1)t_1 - r_1(z_2)t_2} \\
 &+ [R_1(z_1)M_0(z_2) + Q_1(z_1)N_0(z_2)]e^{-r_1(z_1)t_1 - r_0(z_2)t_2}
 \end{aligned}$$

where  $R_0(z)$ ,  $R_1(z)$ ,  $Q_0(z)$  and  $Q_1(z)$  are given by (4.3.17) and  $M_0(z)$ ,  $M_1(z)$ ,  $N_0(z)$  and  $N_1(z)$  by (4.3.23). Also  $r_0(z)$  and  $r_1(z)$  are given by (4.3.10).

On putting  $z_1=1$  and  $z_2=1$  separately we obtain the generating functions of  $n_1$  and  $n_2$  respectively and each is found to coincide with (4.3.13).

As a check we calculated  $E(n)$  and  $\text{Var}(n)$  and obtained the same expressions as given by (4.3.14). Also the expression for the covariance between  $n_1$  and  $n_2$  is given by

$$(4.3.25) \text{cov}(n_1, n_2) = \frac{\pi_0 \pi_1}{(\kappa_0 + \kappa_1)^2} (\lambda_0 - \lambda_1)^2 [1 - e^{-(\kappa_0 + \kappa_1)t_1}] [1 - e^{-(\kappa_0 + \kappa_1)t_2}]$$

Clearly (i)  $\lambda_0 = \lambda_1$  implies  $\text{cov}(n_1, n_2) = 0$ , and

(ii)  $\kappa_0 \rightarrow \infty, \kappa_1 \rightarrow \infty$  and  $\pi_0$  constant implies  $\text{cov}(n_1, n_2) = 0$ .

#### 4.4 The server's idle fraction and discussion of the queue equations.

Let  $P(n, \lambda_i)$  ( $i = 0, 1$ ) denote the stationary probability that there are  $n$  units in the system and that the arrival rate is  $\lambda_i$ . It is easy to see that the system satisfies the following two sets of equations

$$\mu P(1, \lambda_0) - (\lambda_0 + \kappa_0) P(0, \lambda_0) + \kappa_1 P(0, \lambda_1) = 0$$

(4.4.1)

$$\mu P(n+1, \lambda_0) - (\lambda_0 + \kappa_0) P(n, \lambda_0) + \kappa_1 P(n, \lambda_1) + \lambda_0 P(n-1, \lambda_0) = 0$$

$n \geq 1$

$$\mu P(1, \lambda_1) - (\lambda_1 + \kappa_1) P(0, \lambda_1) + \kappa_0 P(0, \lambda_0) = 0$$

(4.4.2)

$$\mu P(n+1, \lambda_1) - (\lambda_1 + \kappa_1) P(n, \lambda_1) + \kappa_0 P(n, \lambda_0) + \lambda_1 P(n-1, \lambda_1) = 0$$

$n \geq 1$  .

Define

$$(4.4.3) \quad h(z, \lambda_i) = \sum_{n=0}^{\infty} z^n P(n, \lambda_i) \quad (i = 0, 1).$$

Making use of (4.4.3) equations (4.4.1) and (4.4.2) lead to the following two equations

$$(4.4.4) \quad \begin{cases} \{\lambda_0 z^2 - (\mu + \lambda_0 + \kappa_0)z + \mu\}h(z, \lambda_0) + \kappa_1 zh(z, \lambda_1) = \mu(1-z)P(0, \lambda_0) \\ \kappa_0 zh(z, \lambda_0) + \{\lambda_1 z^2 - (\mu + \lambda_1 + \kappa_1)z + \mu\}h(z, \lambda_1) = \mu(1-z)P(0, \lambda_1) \end{cases}$$

from which we obtain

$$(4.4.5) \quad \begin{cases} h(z, \lambda_0) = \Delta_0(z)/\Delta(z) \\ h(z, \lambda_1) = \Delta_1(z)/\Delta(z) \end{cases}$$

where

$$\begin{aligned} \Delta_0(z) &= \mu(1-z)P(0, \lambda_0) & \kappa_1 z \\ \Delta_1(z) &= \mu(1-z)P(0, \lambda_1) & \lambda_1 z^2 - (\mu + \lambda_1 + \kappa_1)z + \mu \\ & \lambda_0 z^2 - (\mu + \lambda_0 + \kappa_0)z + \mu & \mu(1-z)P(0, \lambda_0) \\ & \kappa_0 z & \mu(1-z)P(0, \lambda_1) \end{aligned}$$

and

$$\Delta(z) = \begin{matrix} \lambda_0 z^2 - (\mu + \lambda_0 + \kappa_0)z + \mu & \kappa_1 z \\ \kappa_0 z & \lambda_1 z^2 - (\mu + \lambda_1 + \kappa_1)z + \mu \end{matrix}.$$

Clearly  $\lim_{z \rightarrow 1} h(z, \lambda_i) = \pi_i$ , where  $\pi_i$  is given by (4.1.1).

From (4.4.5) we have

$$\begin{aligned} \pi_0 &= \lim_{z \rightarrow 1} \frac{\Delta_0(z)}{\Delta(z)} \quad (\text{form } \frac{0}{0}) \\ &= \frac{\kappa_0 \mu [P(0, \lambda_0) + P(0, \lambda_1)]}{\kappa_1 (\mu - \lambda_0) + \kappa_0 (\mu - \lambda_1)} \\ &= \frac{\kappa_1 P(0)}{\kappa_1 (1 - \rho_0) + \kappa_0 (1 - \rho_1)} = \frac{\pi_0 P(0)}{\pi_0 (1 - \rho_0) + \pi_1 (1 - \rho_1)} \end{aligned}$$

where  $\rho_i = \lambda_i / \mu$  ( $i = 0, 1$ ) and  $P(0) = P(0, \lambda_0) + P(0, \lambda_1)$  represents the fraction of time in which the server is idle, that is,

$$\begin{aligned} (4.4.6) \quad P(0) &= \pi_0 (1 - \rho_0) + \pi_1 (1 - \rho_1) \\ &= (1 - \bar{\rho}) \quad \text{where } \bar{\rho} = \bar{\lambda} / \mu \end{aligned}$$

and also we assume  $\bar{\rho} < 1$ , for otherwise there would be no steady state conditions. The result (4.4.6) is, of course, obvious from the fact that the expected number of arrivals per unit time is  $\bar{\lambda}$  (see (4.3.14)) and the capacity of the number of services per unit time is  $\mu$ .

Unfortunately, when we perform the same operation on  $h(z, \lambda_1)$  we arrive at the same result, which of course, is obvious from (4.4.4) when we put  $z = 1$ . Thus we are unable to determine  $P(0, \lambda_0)$  and  $P(0, \lambda_1)$  separately. Furthermore, at present, the solution seems intractable.

If we are interested only in the mean number of units in the system we can consider a modified process where there is a limited waiting room, of size  $N$ , say. This means we shall have a set of  $2N+2$  homogeneous linear equations in  $2N+2$  unknowns and the coefficient matrix is of rank  $2N+1$ . For small values of  $N$ , it is easy to get explicit solution for the unknowns and for large values of  $N$  we can at least write the solution in determinantal form.

Knowing the solution of the  $P(n, \lambda_1)$  for  $n = 0, 1, \dots, N$ ,  $i = 0, 1$  we can find the mean number of units in the system. Then letting  $N \rightarrow \infty$  the corresponding result will give the mean number of units in the system for our original process. This is as far as we can go concerning this problem, at present. It may be noted that we are faced with the same difficulty in the problem where the arrival rate  $\lambda$  is constant but the service rate is a random variable.

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## APPENDIX

### A NOTE ON COST STRUCTURE AND OPTIMIZATION

The cost function which we construct and consider here is of the simplest possible type: The contributions of the various components to the total cost are assumed to be linear with respect to their average values.

If the presence of a single waiting customer at the station costs  $C_c$  per unit time, the contribution to the total cost due to the average number of units waiting for service ( $L_s$ ) amounts to  $C_c L_s$ . First, we shall consider

(a) the queueing process considered in Chapter I, that is, the case where the arrival rate is controllable and assumes two values  $\lambda_0$  and  $\lambda_1$ .

In this process if we assume that the management has to pay a penalty cost of  $C_a$  per customer turned away, then the cost per unit time associated with this source is  $C_a P(\lambda_1)(\lambda_0 - \lambda_1)$ , where  $P(\lambda_1)$  is the fraction of time in which the arrival rate is  $\lambda_1$ .

Clearly the above two costs are competing with each other and the problem is how to choose  $r, R$  and  $\lambda_1$  ( or  $\delta = \rho_0 - \rho_1$ ) for minimizing the total cost. This total cost is given by

$$C(r, R, \delta) = C_c L_s + C_a P(\lambda_1) (\lambda_0 - \lambda_1)$$

$$= C_c L_s + C_a \mu P(\lambda_1) \delta$$

where

$$L_s = \left[ \frac{\rho_0}{(1-\rho_0)^2} - \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \left\{ \frac{(R-\rho_0 r)}{(1-\rho_0)} + \frac{1}{2} (R+r)(R-r-1) \frac{(\rho_0 - \rho_1)}{(1-\rho_1)} \right. \right. \\ \left. \left. - (R-\rho_1 r) \frac{(1-\rho_0)}{(1-\rho_1)^2} \right\} \right] / \left[ \frac{1}{(1-\rho_0)} - (R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \frac{(\rho_0 - \rho_1)}{(1-\rho_1)} \right], \rho_0 \neq 1$$

$$= \left[ \rho_0 m^{-2} (m+\delta)^2 - \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \left\{ (R-\rho_0 r) m^{-1} (m+\delta)^2 + \frac{1}{2} (R+r)(R-r-1) \delta (m+\delta) \right. \right. \\ \left. \left. - (R-\rho_1 r) m \right\} \right] / \left[ m^{-1} (m+\delta)^2 - (R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \delta (m+\delta) \right], m = 1 - \rho_0$$

$$- (R-\rho_1 r) m \} / \left[ m^{-1} (m+\delta)^2 - (R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \delta (m+\delta) \right], m = 1 - \rho_0$$

and

$$P(\lambda_1) = \frac{(R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \frac{1-\rho_0}{1-\rho_1}}{\frac{1}{1-\rho_0} - (R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \frac{\rho_0 - \rho_1}{1-\rho_1}}$$

$$= \frac{(R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} m(m+\delta)}{m^{-1} (m+\delta)^2 - (R-r) \frac{\rho_0^{R+r}}{r \cdot R \rho_0 - \rho_0} \delta (m+\delta)}$$

As an approximation for the choice of the optimal values of  $r$ ,  $R$  and  $\delta$ , it is suggested that the cost  $C(r, R, \delta)$  be differentiated with respect to  $r$ ,  $R$  and  $\delta$  separately and in each case set the resulting equation equal to zero. The following are the resulting equations

$$\begin{aligned}
& C_c \left( [m^{-1}(m+\delta)^2 - (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta)] \left[ \frac{\rho_o^{2R+r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} \{ (R-\rho_o r)m^{-1}(m+\delta)^2 \right. \right. \\
& \quad \left. \left. + \frac{1}{2}(R+r)(R-r-1)\delta(m+\delta) - (R-\rho_1 r)m \right] + \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \{ \rho_o m^{-1}(m+\delta)^2 \right. \right. \\
& \quad \left. \left. + \frac{1}{2}(2r+1)\delta(m+\delta) - m \rho_1 \right] \right] - [\rho_o m^{-2}(m+\delta)^2 - \\
& \quad \frac{\rho_o^{R+r}}{r - \rho_o} \{ (R-\rho_o r)m^{-1}(m+\delta)^2 + \frac{1}{2}(R+r)(R-r-1)\delta(m+\delta) - (R-\rho_1 r)m \} ] \\
& \quad \left[ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) + (R-r)\delta(m+\delta) \frac{\rho_o^{2R+r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} \right] \\
& - C_a \mu \left( [m^{-1}(m+\delta)^2 - (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta)] [m\delta(m+\delta) \left\{ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \right. \right. \right. \\
& \quad \left. \left. + (R-r) \frac{\rho_o^{2R+r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} \right\}] + [(R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} m\delta(m+\delta)] \left[ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) \right. \right. \\
& \quad \left. \left. + (R-r)\delta(m+\delta) \frac{\rho_o^{2R+r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} \right] \right) = 0.
\end{aligned}$$

$$\begin{aligned}
& c_c \left( [\rho_o m^{-2(m+\delta)} - \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \{ (R-\rho_o r)m^{-1(m+\delta)} + \right. \\
& + \frac{1}{2}(R+r)(R-r-1)\delta(m+\delta) - (R-\rho_o r)m \} ] [ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) \\
& - (R-r)\delta(m+\delta) \frac{\rho_o^{R+2r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} ] - [m^{-1(m+\delta)} - (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) ] \\
& [ \frac{\rho_o^{R+2r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} \{ (R-\rho_o r)m^{-1(m+\delta)} + \frac{1}{2}(R+r)(R-r-1)\delta(m+\delta) \\
& - (R-\rho_o r)m \} + \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \{ m^{-1(m+\delta)} + \frac{1}{2}(2R-1)\delta(m+\delta) - m \} ] ) \\
& + c_a \mu \left( [m^{-1(m+\delta)} - (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) ] [m\delta(m+\delta) \{ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} + \right. \\
& + (R-r) \frac{\rho_o^{R+2r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} ] + [ (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} m\delta(m+\delta) ] \\
& \left. [ \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) + (R-r)\delta(m+\delta) \frac{\rho_o^{R+2r} \log \rho_o}{(\rho_o^r - \rho_o^R)^2} ] \right) = 0 \\
& c_c \left( [m^{-1(m+\delta)} - (R-r) \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \delta(m+\delta) ] [2\rho_o m^{-2(m+\delta)} \right. \\
& - \frac{\rho_o^{R+r}}{\rho_o^r - \rho_o^R} \{ 2m^{-1(m+\delta)} + \frac{1}{2}(R+r)(R-r-1)(m+2\delta) \} ]
\end{aligned}$$

$$\begin{aligned}
& -[\rho_0 m^{-2}(m+\delta)^2 - \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (R-\rho_0 r)m^{-1}(m+\delta)^2 + \\
& + \frac{1}{2}(R+r)(R-r-1)\delta(m+\delta) - (R-\rho_1 r)m ] [2m^{-1}(m+\delta) \\
& - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (m+2\delta) ] \\
& + c_a \mu \left( [m^{-1}(m+\delta)^2 - (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} \delta(m+\delta)] [(R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} m(m+2\delta)] \right. \\
& \left. - [(R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} m\delta(m+\delta)] [2m^{-1}(m+\delta) + (R-r) \frac{\rho_0^{R+r}}{\rho_0^r - \rho_0^R} (m+2\delta)] \right) = 0.
\end{aligned}$$

By means of a computer it may be possible to obtain an approximate optimal solution. Next we shall consider

(b) the special case of the queueing process considered in Chapter III. This is the case where the service rate is changeable and assumes two values  $\mu_0$  and  $\mu_1$ .

In this process we assume that the additional cost per unit time for bringing in an auxiliary server with capacity  $\epsilon = \mu_1 - \mu_0$  is  $C_b \epsilon$ . The contribution to the total cost from this source is  $P(\mu_1) C_b \epsilon$ , where  $P(\mu_1)$  is the fraction of time in which the service rate is  $\mu_1$ . The total cost is, therefore, given by

$$C(r, R, \epsilon) = C_c L_s + C_b P(\mu_1) \epsilon$$

where

$$L_s = \left[ \frac{b_o}{(1-b_o)^2} - \frac{b_1 b_o^{R+r-1}}{b_o^r - b_o^R} \left\{ \frac{b_o (R-b_o r)}{b_1 (1-b_o)} + \frac{1}{2}(R+r)(R-r-1) \frac{(b_o - b_1)}{b_1 (1-b_1)} \right. \right. \\ \left. \left. - (R-b_1 r) \frac{(1-b_o)}{(1-b_1)^2} \right\} \right] / \left[ \frac{1}{1-b_o} - (R-r) \frac{b_o^{R+r-1}}{b_o^r - b_o^R} \frac{(b_o - b_1)}{(1-b_1)} \right],$$

$$P(\mu_1) = (R-r) \frac{P(r+1, \mu_1)}{(1-b_1)}$$

$$= \frac{(R-r) \frac{b_1 b_o^{R+r-1}}{b_o^r - b_o^R} \frac{1-b_o}{1-b_1}}{\frac{1}{1-b_o} (R-r) \frac{b_o^{R+r-1}}{b_o^r - b_o^R} \frac{b_o - b_1}{1-b_1}}$$

where  $b_o \neq 1$  and  $b_1 = \lambda / (\mu_o + \epsilon)$ .

The problem of finding optimum values of  $r$ ,  $R$  and  $\epsilon$  is similar to the first one and we shall not pursue further.