Abstract

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Leibniz algebras were popularized by Jean-Louis Loday in 1993 as a non-commutative generalization of Lie Algebras. For more than a century, mathematicians have extended results from Group theory to Lie algebras and more recently to Leibniz algebras. The purpose of this work is to continue this trend of development by making contributions to the theory of formations of Leibniz algebras with special consideration given to the formation of solvable, complemented Leibniz algebras.

Formations of Leibniz Algebras

by
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Dedication

To my beautiful wife Katelin who never stopped believing in me.

To my son River who brings me joy every time I look at him.

I love you both dearly.

Biography

Justin Michael Leonesio was born on October 14, 1985 in Chicago, Illinois. One year later, he and his family moved to North Carolina. During his high school career, Justin developed a passion for mathematics and decided to pursue a career in mathematics education. Justin attended Cedarville University in Cedarville, Ohio during his freshman year of college and then transferred to Liberty University in Lynchburg, Virginia to conclude his undergraduate studies. In December of 2007, Justin graduated from Liberty, earning his Bachelor of Science degree in Mathematics along with a teacher licensure in secondary mathematics education. During the spring of 2008, Justin began his graduate studies at North Carolina State University while teaching secondary mathematics full time at Wake Christian Academy in Raleigh, North Carolina. In July of 2009, Justin met and fell in love with his future wife Katelin, and nearly three years later they were married on May 26, 2012. Justin earned his Master of Science degree in Mathematics from N.C. State in May of 2013 and then continued into the doctoral program at NCSU. On May 12, 2017, Katelin and Justin welcomed their son River into the world. Justin and his family currently live in Raleigh where he continues to teach at Wake Christian Academy while serving as chair of the school's mathematics department.

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To my son River: At the time I am writing this, you are only a few months old. You are truly a gift from God and a blessing to your mother and me. I am so grateful that you are here with us for the last leg of this journey. It has made it incredibly worthwhile to have you literally beside me as I have been writing this paper. I can not wait to see the amazing things that God has in store for your life. May the completion of this dissertation encourage you to pursue your dreams as well. I will always be there to love and support you.

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1

INTRODUCTION

1.1 A Brief History

Although Leibniz algebras are a relatively new field of study within the realm of mathematics, their origin can be traced back to the late-nineteenth century. In his home country of Norway, Marius Sophus Lie began developing the theory of continuous transformation groups with the intention of discovering symmetries within the theory of differential equations just as Galois had done with polynomial equations nearly fifty years earlier [1]. With help from Friedrich Engel, Lie published his most famous work *Theorie der Transformationsgruppen* in three volumes from 1888 to 1893. In honor of Lie's achievements, continuous transformation groups were later named Lie groups. Before his death, Lie's work on the infinitesimal transformations of Lie groups eventually led him to introduce the theory of Lie algebras to the world, which monumentally impacted the study of contemporary mathematics. For more than a century, numerous mathematicians including Killing and Cartan have devoted their lives toward the development of Lie algebra theory [12].

During the last half of the twentieth century, a trend developed in the study of Lie algebra generalizations. In 1955, Maltsev first introduced the idea of generalizing Lie algebras, and his work led to the theory of Malcev algebras [15]. One decade later, Bloh began paving the way for the eventual realization of what is known today as Leibniz algebras when he published his work on *D*-algebras [8]. Although the groundwork had been laid by Bloh during the 1960's, Leibniz algebras were not popularized until 1993 when Jean-Louis Loday published his work discussing a non-commutative generalization of Lie algebras [14]. In this treatise, Loday named his generalization after Gottfried Wilhelm Leibniz; and the name continues to be used today. Since the introduction of Leibniz algebras, many mathematicians, including several from North Carolina State University's Department of Mathematics, have focused on extending results from Lie algebras to Leibniz algebras. This work is intended to continue that trend of development by making contributions to the theory of formations of Leibniz algebras.

1.2 Leibniz Algebras

In this section, we will introduce relevant definitions, properties, and results of Leibniz algebras in the same fashion as Demir, Misra, and Stitzinger in [10]. These concepts will provide us with the necessary foundation to investigate the results addressed later in this work.

Let L be an algebra over a field $\mathbb{F}$. We will denote the left multiplication operator as L_a for $a \in L$ where $L_a(x) = [a, x]$ for all $x \in L$. Similarly, the right multiplication operator will be denoted as R_a for $a \in L$ where $R_a(x) = [x, a]$ for all $x \in L$.

Definition 1.2.1. A (left) **Leibniz algebra** L is an $\mathbb{F}$-vector space equipped with a bilinear map $[\ ,\ ] : L \times L \to L$ which satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all $a, b, c \in L$.

Definition 1.2.2. A linear operator $D : L \to L$ is a **derivation** of a Leibniz algebra L if

$$D(xy) = (D(x))y + x(D(y))$$

for all $x, y \in L$.

We will denote the collection of all derivations of a Leibniz algebra L as $\mathrm{Der}(L)$. Note from the definitions above that L_a is a derivation of L, but R_a is not a derivation of L. On the other hand, a right Leibniz algebra is defined as an $\mathbb{F}$-vector space equipped with a bilinear multiplication such that R_a is a derivation. The following example shows a left Leibniz algebra that is not necessarily a right Leibniz algebra.

Example 1.2.3. Let L be a 2-dimensional algebra with the following multiplications.

$$[x, x] = 0,\ [x, y] = 0,\ [y, x] = x,\ [y, y] = x$$

L is a left Leibniz algebra; however, L is not a right Leibniz algebra since $[[y, y], y] \neq$

$[y,[y,y]]+[[y,y],y]$, because $0=[x,y]\neq[y,x]+[x,y]=x+0=x$.

Although there are slight modifications in convention between left and right Leibniz algebras, analogous results can be proven for both; therefore, we will use the definition for left Leibniz algebras throughout the remainder of this work and will refer to them simply as Leibniz algebras.

It is clear that any Lie algebra is also a Leibniz algebra; however, the converse is not true. If a Leibniz algebra were to also satisfy the condition that $[a,a]=a^2=0$ for all $a\in L$, then it would indeed be a Lie algebra. In this case, the Leibniz identity becomes the Jacobi identity. It is important to note that since Leibniz algebras may contain at least one element a such that $[a,a]=a^2\neq 0$, then the antisymmetric property associated with Lie algebras is not a necessary condition for Leibniz algebras. In fact, the absence of antisymmetry is one of the defining characteristics of Leibniz algebras. An example of a Leibniz algebra that is not a Lie algebra is given below.

Example 1.2.4. Consider the n-dimensional cyclic Leibniz algebra L generated by a, $L=\langle a\rangle=\text{span}\{a,a^2,...,a^n\}$, with non-zero products $[a,a]=a^2$, $[a,a^2]=a^3,...,[a,a^{n-1}]=a^n$, and $[a,a^n]=a^n$. L is a Leibniz algebra but not a Lie algebra, because $a^2\neq 0$.

As seen in the cyclic example above, for any element $a\in L$ we define a^n inductively as follows: $a^1=a$ and $a^{k+1}=[a,a^k]$. Similarly, we define L^n by $L^1=L$ and $L^{k+1}=[L,L^k]$.

Definition 1.2.5. A Leibniz algebra L is **abelian** if $L^2=0$.

Proposition 1.2.6. *If L is a Leibniz algebra and $a\in L$, then $L_{a^n}=0$ for $n\in\mathbb{Z}_{>1}$.*

Proof. Let L be a Leibniz algebra, and let $a,b\in L$. First, consider the case when $n=2$. By the Leibniz identity,

$$[a^2, b] = [[a,a],b] = [a,[a,b]] - [a,[a,b]] = 0.$$

Next, assume the case when $n = k$ also holds so that $[a^k, b] = 0$ and consider the case when $n = k+1$.

$$[a^{k+1}, b] = [[a,a^k],b] = [a,[a^k,b]] - [a^k,[a,b]] = [a,0] - 0 = 0 - 0 = 0$$

Thus, by mathematical induction, $L_{a^n} = 0$ for $n \in \mathbb{Z}_{>1}$. □

Since the antisymmetric property does not hold for Leibniz algebras, the following definitions and results are needed.

Definition 1.2.7. For a Leibniz algebra L, **Leib(L)** $= \text{span}\{[a,a] \mid a \in L\}$.

Example 1.2.8. For the n-dimensional cyclic Leibniz algebra introduced in Example 1.2.4, $\text{Leib}(L) = \text{span}\{a^2, ..., a^n\} = L^2$. Here, $\text{Leib}(L)$ can be found by simply considering the general element $\beta = \beta_1 a + \beta_2 a^2 + ... + \beta_n a^n$ from L and applying Proposition 1.2.6 to the product $[\beta, \beta]$.

Example 1.2.9. Let $L = \text{span}\{a, b, c\}$ be a Leibniz algebra with non-zero products $[a,b] = c$ and $[b,a] = c$. In this case, $\text{Leib}(L) = \text{span}\{c\} = L^2$.

Definition 1.2.10. Let S be a subspace of a Leibniz algebra L.

1. S is a **subalgebra** of L if $[S,S] \subseteq S$.
2. S is a **left ideal** of L if $[L,S] \subseteq S$ and a **right ideal** of L if $[S,L] \subseteq S$.
3. S is an **ideal** of L, $S \lhd L$, if it is both a left and a right ideal of L.

Definition 1.2.11. Let L be a Leibniz algebra and $I \lhd L$, then $L/I = \{x + I \mid x \in L\}$ is called a **quotient Leibniz algebra** or **factor Leibniz algebra** of L.

It is clear that Leib(L) is a right ideal of a Leibniz algebra L from Proposition 1.2.6. Upon considering the identity

$$[a,[b,b]] = [a+[b,b], a+[b,b]] - [a,a]$$

found in [10], one can also determine that Leib(L) is a left ideal of L; therefore, Leib$(L) \lhd L$. Furthermore, it is important to note that Leib(L) is an abelian ideal of L and that Leib(L) is the minimal ideal of L such that the quotient algebra $L/$Leib(L) is a Lie algebra.

As in the case of Lie algebras, the sum and intersection of two ideals of a Leibniz algebra are also ideals; however, the product of two ideals is not necessarily an ideal as seen in the following example [10].

Example 1.2.12. Let $L = \text{span}\{x, a, b, c, d\}$ be a Leibniz algebra with non-zero multiplications $[a,b] = c$, $[b,a] = d$, $[x,a] = a = -[a,x]$, $[x,c] = c$, $[x,d] = d$, $[c,x] = d$, $[d,x] = -d$. In addition, let $I = \text{span}\{a, c, d\}$ and $J = \text{span}\{b, c, d\}$. Now, I and J are ideals of L; but $[I, J] = \text{span}\{c\}$ is not an ideal of L.

Definition 1.2.13. A Leibniz algebra **homomorphism** $\varphi : L \to L'$ is a linear mapping such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L$.

The standard homomorphisms for Lie algebras also hold for Leibniz algebras. Thus, for any Leibniz algebra homomorphism $\varphi : L \to L'$, the kernel of φ, *ker* φ, is an ideal of L; and the image of φ, *im* φ, is a subalgebra of L' [2]. Furthermore, for any ideal I of a Leibniz algebra L, the ideals of the quotient Leibniz algebra L/I are in a one-to-one correspondence with the ideals of L containing I [10].

Definition 1.2.14. Let L be a Leibniz algebra.

1. The **left center** of L is $Z^l(L) = \{x \in L \mid [x, a] = 0 \text{ for all } a \in L\}$.
2. The **right center** of L is $Z^r(L) = \{x \in L \mid [a, x] = 0 \text{ for all } a \in L\}$.
3. The **center** of L is $Z(L) = Z^l(L) \cap Z^r(L)$.

Clearly, for both Lie and Leibniz algebras, the center is an abelian ideal.

Definition 1.2.15. Let S be a subset of a Leibniz algebra L.

1. The **left centralizer** of S in L is $C_L^l(S) = \{x \in L \mid [x, s] = 0 \text{ for all } s \in S\}$.
2. The **right centralizer** of S in L is $C_L^r(S) = \{x \in L \mid [s, x] = 0 \text{ for all } s \in S\}$.
3. The **centralizer** of S in L is $C_L(S) = C_L^l(S) \cap C_L^r(S)$.

Using the Leibniz identity, one can easily determine that the centralizer of a subset S in a Leibniz algebra L is a subalgebra of L. Moreover, if $S \lhd L$, then $C_L(S) \lhd L$ [9].

Definition 1.2.16. Let H be a subalgebra of a Leibniz algebra L.

1. The **left normalizer** of H in L is $N_L^l(H) = \{x \in L \mid [x, h] \in H \text{ for all } h \in H\}$.
2. The **right normalizer** of H is $N_L^r(H) = \{x \in L \mid [h, x] \in H \text{ for all } h \in H\}$.
3. The **normalizer** of H is $N_L(H) = N_L^l(H) \cap N_L^r(H)$.

Obviously, the normalizer of a subalgebra H in a Leibniz algebra L is also a subalgebra of L; and, as seen earlier with centralizers, if $H \lhd L$, then $N_L(H) \lhd L$.

Definition 1.2.17. For a Leibniz algebra L, the series of ideals

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots \text{ where } L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}], \text{ and } L^{(i+1)} = [L^{(i)}, L^{(i)}]$$

is called the **derived series** of L.

Definition 1.2.18. A Leibniz algebra L is **solvable** if $L^{(m)} = 0$ for some $m \in \mathbb{Z}_{\geq 0}$.

As in the case of Lie algebras, the sum and intersection of two solvable ideals of a Leibniz algebra are also solvable; thus, any Leibniz algebra L contains a unique maximal solvable ideal, *rad* (L), called the **radical** of L which contains all the solvable ideals of L [10].

The following theorem and corollary from [21] further describe the relationship between a solvable Leibniz algebra and its ideals.

Proposition 1.2.19. *Let L be a Leibniz algebra and $I \lhd L$ such that I is solvable. If L/I is solvable, then L is solvable.*

Proof. Since I and L/I are solvable, then $I^{(n)} = (L/I)^{(m)} = 0$ for some n and m. Consider the canonical homomorphism $\pi : L \to L/I$. Then $\pi(L^{(m)}) = \pi(L)^{(m)} = (L/I)^{(m)} = 0$. Thus, $L^{(m)} \subseteq$ *ker* $\pi = I$. Hence, $L^{(m+n)} = (L^{(m)})^{(n)} \subseteq I^{(n)} = 0$. Therefore, L is solvable. □

Corollary 1.2.20. *Let L be a Leibniz algebra. If the Lie algebra $L/Leib(L)$ is solvable, then L is solvable.*

Proof. Since Leib(L) is an abelian ideal of L, then the result follows from Proposition 1.2.19. □

Definition 1.2.21. For a Leibniz algebra L, the series of ideals

$$L = L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots \text{ where } L^{i+1} = [L, L^i]$$

is called the **lower central series** of L.

Definition 1.2.22. A Leibniz algebra L is **nilpotent** of class c if $L^{c+1} = 0$ but $L^c \neq 0$ for some $c \in \mathbb{Z}_{\geq 1}$.

It is clear that if L is a nilpotent Leibniz algebra L of class c, then $L^c \subseteq Z^r(L)$. Using induction and repeated application of the Leibniz identity, one can also determine that

$L^c \subseteq Z^l(L)$. Therefore, $L^c \subseteq Z(L)$ and $Z(L) \neq 0$. As in the case of Lie algebras, the sum and intersection of two nilpotent ideals of a Leibniz algebra are also nilpotent; hence, any Leibniz algebra L contains a unique maximal nilpotent ideal, $nil(L)$, called the **nilradical** of L which contains all the nilpotent ideals of L [10].

In [11], Gorbatsevich proved the following proposition which shows an interesting connection between L and L^2 for Leibniz algebras.

Proposition 1.2.23. *A Leibniz algebra L is solvable if and only if L^2 is nilpotent.*

Definition 1.2.24. A Leibniz algebra L is **supersolvable** if there exists a chain

$$0 = L_0 \subset L_1 \subset \ldots \subset L_{n-1} \subset L_n = L$$

where L_i is an i-dimensional ideal of L.

Example 1.2.25. Once again, we consider the n-dimensional cyclic Leibniz algebra $L = \text{span}\{a, a^2, \ldots, a^n\}$ introduced in Example 1.2.4. Here, $L^{(1)} = [L, L] = \text{span}\{a^2, \ldots, a^n\}$ and $L^{(2)} = [L^{(1)}, L^{(1)}] = 0$; thus, L is solvable. Furthermore, $L^2 = \text{span}\{a^2, \ldots, a^n\}$, $L^3 = [L, L^2] = \text{span}\{a^3, \ldots, a^n\}$, and $L^k = [L, L^{k-1}] = \text{span}\{a^n\}$ for all $k \in \mathbb{Z}_{\geq n}$; therefore, L is supersolvable but not nilpotent.

Example 1.2.26. If we were to slightly alter the n-dimensional cyclic Leibniz algebra above to have the product $[a, a^n] = 0$, then L would be nilpotent (of class n), because $L^k = 0$ for all $k \in \mathbb{Z}_{>n}$. Moreover, L would be supersolvable; because $L = L^1 = L_n, L^2 = L_{n-1}, L^3 = L_{n-2}, \ldots, L^{n-1} = L_2, L^n = L_1$, and $L^{n+1} = L_0 = 0$.

Example 1.2.27. Recall the Leibniz algebra from Example 1.2.9 where $L = \text{span}\{a, b, c\}$ with non-zero products $[a, b] = c = [b, a]$. Here, we have $L^{(1)} = [L, L] = \text{span}\{c\}$ and $L^{(2)} = 0$. Also, $L^2 = \text{span}\{c\}$ and $L^3 = [L, L^2] = 0$. Finally, let I be the ideal $I = \text{span}\{a, c\}$ so that

$0 = L_0 \subset L_1 = \text{span}\{c\} \subset L_2 = I \subset L_3 = L$. Therefore, L is solvable, nilpotent (of class 2), and supersolvable.

Example 1.2.28. Let $L = \text{span}\{x, y\}$ with its only non-zero product as $[x, y] = y$. Here, $L^{(1)} = \text{span}\{y\}$ and $L^{(2)} = 0$; thus, L is solvable. However, $L^n = \text{span}\{y\}$ for all $n \in \mathbb{Z}_{>1}$; thus, L is not nilpotent. Since $I = \text{span}\{y\}$ is an ideal of L, then L is supersolvable.

1.3 Overview

We begin our discussion of formations in Chapter 2 by examining several classes of Leibniz algebras that were introduced in Chapter 1. Through our investigation, we determine that the class of Lie algebras are a formation within the Leibniz algebras. In addition, the Frattini subalgebra and the Frattini ideal are also defined in order to classify formations as either saturated or non-saturated.

In Chapter 3, we narrow our focus to the class of solvable, complemented Leibniz algebras. Here, we continue the work done by Yaemsiri in [23] by extending results from Lie algebras to Leibniz algebras. In [18], Towers proves that over a field of characteristic 0, a solvable Lie algebra is complemented if and only if its Frattini ideal is 0. We expand this result to Leibniz algebras. We also introduce the Cartan subalgebras of a Leibniz algebra and prove that they are the complements of the last term of the lower central series for any solvable, complemented Leibniz algebra. Furthermore, we prove that each member of the derived series of a solvable, complemented Leibniz algebra L is complemented by a Cartan subalgebra of L.

In Chapter 4, we show that the class of solvable, complemented Leibniz algebras is a formation that exhibits certain properties unlike those examined in Chapter 2. We define this type of formation to be "totally non-saturated." Moreover, we introduce formation residuals as well as $C(\mathscr{F})$, the collection of all Leibniz algebras whose formation residuals contain only complemented chief factors. We then prove that if $\mathscr{F}$ is any formation, then $C(\mathscr{F})$ is a totally non-saturated formation. Ultimately, our discussion leads us to consider several types of totally non-saturated formations from which we determine that the formation of solvable, complemented Leibniz algebras is the unique smallest totally non-saturated formation of Leibniz algebras.

CHAPTER

2

FORMATIONS

2.1 Introduction

In this chapter, we lay the foundation for the remainder of the work. First, we define what it means for a class of Leibniz algebras to be a formation. Then, we present several classical examples of formations, many of which were introduced in Chapter 1. Next, we define the Frattini subalgebra and the Frattini ideal for a given Leibniz algebra, which will enable us to introduce the classification categories of saturated and non-saturated formations. Finally, we will revisit the classical formations to determine whether they are saturated or non-saturated. Any Leibniz algebras discussed in this chapter will be finite dimensional over a field $\mathbb{F}$.

Definition 2.1.1. A formation $\mathscr{F}$ is a class of Leibniz algebras satisfying the following conditions:

1. If a Leibniz algebra $L \in \mathscr{F}$, then $f(L) \in \mathscr{F}$ for each homomorphism f of L.
2. If $H, K \lhd L$ such that $L/H, L/K \in \mathscr{F}$, then $L/(H \cap K) \in \mathscr{F}$.

Example 2.1.2. The following familiar classes of Leibniz algebras, which were introduced in Chapter 1, are classical examples of formations. For each of the examples below, the verification of the formation conditions should be clear.

$\mathscr{A}$ = the formation of abelian Leibniz algebras

$\mathscr{N}$ = the formation of nilpotent Leibniz algebras

$\mathscr{U}$ = the formation of supersolvable Leibniz algebras

$\mathscr{S}$ = the formation of solvable Leibniz algebras

Example 2.1.3. The trivial Leibniz algebra {0} is a formation.

Recall $nil(L)$, the nilradical of a Leibniz algebra L, from Chapter 1. Using the notation below, we can define a series of consecutive nilradicals for a Leibniz algebra L [20]:

$$nil_0(L) = 0, \quad nil_i(L) = nil(L/nil_{i-1}(L)) \quad \text{for } i = 1, 2, \ldots$$

Definition 2.1.4. The **nilpotent length** of a solvable Leibniz algebra L is the smallest $n \in \mathbb{N}$ such that $nil_n(L) = L$.

Example 2.1.5. The class of Leibniz algebras with nilpotent length $\leq k$ is a formation for each $k \geq 1$. We will denote these classes of Leibniz algebras as $\mathscr{N}(\leq k)$ for each $k \geq 1$, respectively.

Proposition 2.1.6. *The class of Lie algebras is a formation within the Leibniz algebras.*

Proof. Let L be a Lie algebra within the class of Leibniz algebras and let $A \lhd L$. Since L is a Lie algebra, then L/A is also a Lie algebra; because any subalgebra of a Lie algebra is also a Lie algebra. Now, let $B \lhd L$ and suppose that both L/A and L/B are Lie algebras. Consider the mapping $\gamma : L \rightarrow L/A \times L/B$ where $\gamma(x) = (x + A, x + B)$. Since both L/A and L/B are Lie algebras, then $L/A \times L/B$ must be a Lie algebra. Furthermore, since $A \cap B = ker\,\gamma$, then we have $im\,\gamma \cong L/(A \cap B)$ by the 1st Isomorphism Theorem. Since $L/A \times L/B$ is a Lie algebra, then any subalgebra of $L/A \times L/B$ must also be a Lie algebra. Therefore, $L/(A \cap B)$ is a Lie algebra. □

The format of the proof of Proposition 2.1.6 may be viewed as a general template for proving that a specific class of Leibniz algebras is, indeed, a formation.

2.2 Saturated Formations

Definition 2.2.1. The **Frattini subalgebra**, $F(L)$, of a Leibniz algebra L is the intersection of the maximal subalgebras of L.

Definition 2.2.2. The **Frattini ideal**, $\Phi(L)$, of a Leibniz algebra L is the largest ideal of L contained in $F(L)$.

In Chapter 3, we examine the relationship between the Frattini subalgebra and the Frattini ideal of a Leibniz algebra. At this time, however, our primary purpose in introducing the Frattini ideal is to define and discuss saturated formations. Before we begin that discussion, we briefly introduce two interesting properties associated with the Frattini ideal of a Leibniz algebra which will become useful later.

Proposition 2.2.3. *If L is a nilpotent Leibniz algebra over a field $\mathbb{F}$ with characteristic 0, then $L^2 = \Phi(L)$.*

Proposition 2.2.4. *If L is a Leibniz algebra, then $\Phi(L)$ is nilpotent.*

Definition 2.2.5. Let $\mathscr{F}$ be a formation of Leibniz algebras and consider any ideal M of a Leibniz algebra L such that $M \subseteq \Phi(L)$. The formation $\mathscr{F}$ is **saturated** if $L/M \in \mathscr{F}$ implies $L \in \mathscr{F}$.

Rather than considering any general ideal $M \subseteq \Phi(L)$, one may simply let M be a minimal ideal of L or let $M = \Phi(L)$ to determine whether or not a formation is saturated. The proposition below, which was proved in [5], offers an equivalent statement to the definition above.

Proposition 2.2.6. *A formation $\mathscr{F}$ is saturated iff for a Leibniz algebra L, $L/\Phi(L) \in \mathscr{F}$ implies $L \in \mathscr{F}$.*

The following proposition allows us to characterize several of the classical formations from Example 2.1.2 and Example 2.1.3 as saturated. Although it was proved for Lie algebras in [5], we provide a proof for Leibniz algebras here.

Proposition 2.2.7. *The following classes of Leibniz algebras are saturated formations:*

1. *The trivial Leibniz algebra* $\{0\}$.
2. $\mathscr{N}$ *= the formation of nilpotent Leibniz algebras.*
3. $\mathscr{U}$ *= the formation of supersolvable Leibniz algebras.*
4. $\mathscr{S}$ *= the formation of solvable Leibniz algebras.*

Proof. The formation $\{0\}$ is trivially saturated. For $\mathscr{N}$ and $\mathscr{U}$, one may use Theorem 5.5 of [4] and Theorem 6 of [9], respectively. Finally, by Proposition 2.2.4, $\Phi(L)$ is nilpotent for any Leibniz algebra L; thus, $\Phi(L)$ is also solvable. If $L/\Phi(L)$ is solvable, then L is solvable by Proposition 1.2.19. Therefore, $\mathscr{S}$ is saturated. □

Proposition 2.2.8. *The class $\mathcal{N}(\leq k)$ is a saturated formation for each $k \geq 1$.*

Proof. It has already been shown in Proposition 2.2.7 that $\mathcal{N}(1)$ is a saturated formation. Now, suppose that it holds for $k = r$ so that $\mathcal{N}(\leq r)$ is a saturated formation. Clearly, if a Leibniz algebra $L \in \mathcal{N}(\leq r+1)$, then $f(L) \in \mathcal{N}(\leq r+1)$ for each homomorphism f of L. Now, let $A \lhd L$ and $B \lhd L$ and suppose that $L/A, L/B \in \mathcal{N}(\leq r+1)$. Furthermore, let $S/A = nil(L/A)$ and $T/B = nil(L/B)$. Hence, $L/S, L/T \in \mathcal{N}(\leq r)$; and, thus, $L/(S \cap T) \in \mathcal{N}(\leq r)$. Because S/A and T/B are nilpotent ideals of L/A and L/B, respectively, there exist $m, n \in \mathbb{N}$ such that $S^m \subseteq A$ and $T^n \subseteq B$. Thus, $(S \cap T)^{m+n} \subseteq A \cap B$. Hence, $L/(A \cap B) \in \mathcal{N}(\leq r+1)$, and $\mathcal{N}(\leq r+1)$ is a formation.

Now, suppose that $L/\Phi(L) \in \mathcal{N}(\leq r+1)$. From Theorem 5.5 of [4], we can determine that $nil(L/\Phi(L)) = nil(L)/\Phi(L)$. Therefore, $L/nil(L) \in \mathcal{N}(\leq r)$ and $L \in \mathcal{N}(\leq r+1)$. Hence, $\mathcal{N}(\leq r+1)$ is saturated; and the result holds by induction. □

As we shall see in the next two examples, not all of the formations considered in this chapter are saturated.

Example 2.2.9. Recall the n-dimensional nilpotent cyclic Leibniz algebra seen in Example 1.2.26 where $L = \text{span}\{a, a^2, ..., a^n\}$. Earlier, we determined that $L^2 = \text{span}\{a^2, ..., a^n\}$. Since L is nilpotent over a field of characteristic 0, then $L^2 = \Phi(L)$ by Proposition 2.2.3. Also, $L/L^2 = L/\Phi(L)$ is abelian. Since L is not also abelian, then $\mathcal{A}$, the formation of abelian Leibniz algebras, is non-saturated.

Example 2.2.10. Let L be the n-dimensional nilpotent cyclic Leibniz algebra from Example 2.2.9. Now, $L/L^2 = L/\Phi(L) = L/\text{Leib}(L)$, which is a Lie algebra. Since L is not also a Lie algebra, then the class of Lie algebras is a non-saturated formation within the Leibniz algebras.

CHAPTER

3

COMPLEMENTED LEIBNIZ ALGEBRAS

3.1 Introduction

In this chapter, we will assume that all Leibniz algebras discussed are solvable and finite dimensional over a field $\mathbb{F}$. Recall the Frattini subalgebra $F(L)$ and the Frattini ideal $\Phi(L)$ for a Leibniz algebra L which were introduced in Chapter 2. For general Leibniz algebras, the Frattini subalgebra is not always an ideal as seen in the following Lie algebra example [16].

Example 3.1.1. Consider the 3-dimensional Lie algebra $L = \langle x, y, z \rangle = \text{span}\{x, y, z\}$ over the field $\mathbb{F}_2 = \{0, 1\}$ with multiplications $[x, y] = z$, $[y, z] = x$, and $[z, x] = y$. Here, we have

$F(L) = \langle x + y + z \rangle$. Consider $[x, x + y + z] = [x, x] + [x, y] + [x, z] = z - y \notin F(L)$. Hence, $\Phi(L) = 0$.

In the next example, we consider a basic 2-dimensional cyclic Leibniz algebra with different results [7].

Example 3.1.2. Let $L = \text{span}\{a, a^2\}$ be the 2-dimensional cyclic Leibniz algebra over a field of characteristic 0 with non-zero multiplications $[a, a] = a^2$ and $[a, a^2] = a^2$. In this case, we find that there are two maximal subalgebras: $\langle a - a^2 \rangle$ and $\langle a^2 \rangle$. This implies that $F(L) = 0$; thus, $\Phi(L) = 0$.

Since most of our results in this chapter will involve a field of characteristic 0, we will assume for the remainder of the chapter that a field $\mathbb{F}$ has characteristic 0 unless it has been specifically stated otherwise. With this constraint on $\mathbb{F}$, the following lemma, which was proved in [6], will be important for our results in the next section and beyond.

Lemma 3.1.3. *If L is a Leibniz algebra over a field of characteristics 0, then $F(L)$ is an ideal of L. Therefore, $F(L) = \Phi(L)$.*

3.2 Complemented Leibniz Algebras

Definition 3.2.1. A Leibniz algebra L is **complemented** if its subalgebra lattice is complemented; that is, given any subalgebra H in L, there exists a subalgebra K in L such that $L = \langle H, K \rangle$ and $H \cap K = 0$.

Definition 3.2.2. A Leibniz algebra L is **Φ-free** if $\Phi(L) = 0$.

Definition 3.2.3. A Leibniz algebra L is **elementary** if every subalgebra of L (including L itself) is Φ-free.

Since we are working in the context of solvable Leibniz algebras, we can weave the three definitions above together with the following theorem [7].

Theorem 3.2.4. *Let L be a solvable Leibniz algebra over a field $\mathbb{F}$ of characteristic 0. The following statements are equivalent:*

1. *L is complemented.*
2. *L is Φ-free.*
3. *L is elementary.*

Proof.

$1 \Rightarrow 2$. Let L be a complemented Leibniz algebra and assume that $\Phi(L) \neq 0$. Since $\Phi(L)$ is an ideal of L and thus a subalgebra of L, then there exists a complement subalgebra H of L such that $\langle \Phi(L), H \rangle = L$ and $\Phi(L) \cap H = 0$. Since H is a subalgebra, then $H \subseteq M$ for some maximal subalgebra M of L. Therefore, $L = \langle \Phi(L), M \rangle$; but $\Phi(L) \subseteq M$. Thus, $M = L$, which is a contradiction. Hence, $\Phi(L) = 0$.

$2 \Rightarrow 3$. Assume that $\Phi(L) = 0$. Since L is solvable, then L^2 is nilpotent by Proposition 1.2.23. Therefore, by Theorem 3.4 of [6], we know that L is elementary.

$3 \Rightarrow 1$. Assume that L is elementary; thus, $\Phi(L) = 0$. Let H be a subalgebra of L. Since $\Phi(L) = 0$, then there exists a proper subalgebra K of L such that $L = \langle H, K \rangle$. If $H \cap K \neq 0$, then there exists a proper subalgebra $K_1 \subset K$ such that $K = \langle H \cap K, K_1 \rangle$. This implies that $L = \langle H, H \cap K, K_1 \rangle = \langle H, K_1 \rangle$ and $dim(K_1) < dim(K)$. If $H \cap K_1 \neq 0$, then there exists a proper subalgebra $K_2 \subset K_1$ such that $K_1 = \langle H \cap K_1, K_2 \rangle$. This implies that $L = \langle H, H \cap K_1, K_2 \rangle = \langle H, K_2 \rangle$ and $dim(K_2) < dim(K_1)$. We may continue this process until we arrive at a proper subalgebra K_n such that $L = \langle H, K_n \rangle$ and $H \cap K_n = 0$. Therefore, L is complemented. □

Proposition 3.2.5. *A nilpotent Leibniz algebra is complemented if and only if it is abelian.*

Proof. Let L be a nilpotent complemented Leibniz algebra; thus, $\Phi(L) = 0$ by Theorem 3.2.4. Since L is nilpotent, then $L^2 = \Phi(L) = 0$ by Proposition 2.2.3. Therefore, L is abelian.

Conversely, let L be abelian and let H be a subalgebra of L. We now select K to be any complementary subspace of H so that $H \cap K = 0$. K must be a subalgebra of L, because $[K, K] = 0 \subseteq K$; therefore, K complements H in L and L is complemented. □

A result similar to the following lemma was proved by Towers in [19] for Lie algebras, but the proof is also valid for Leibniz algebras. We take a slightly different approach to the proof here by appealing to Theorem 3.2.4.

Lemma 3.2.6. *L is a complemented Leibniz algebra if and only if every subalgebra of L is also a complemented Leibniz algebra.*

Proof. L is a complemented Leibniz algebra $\Longleftrightarrow$ L is elementary $\Longleftrightarrow$ every subalgebra of L is Φ-free $\Longleftrightarrow$ every subalgebra of L is a complemented Leibniz algebra. □

Theorem 3.2.7. *Every homomorphic image of a complemented Leibniz algebra is also a complemented Leibniz algebra.*

Proof. Since every subalgebra of a Leibniz algebra can be viewed as the image of a Leibniz algebra homomorphism, then the result is clear by Lemma 3.2.6. □

Lemma 3.2.8. *If L is a complemented Leibniz algebra, then every ideal of L is complemented by a subalgebra of L.*

Proof. Since L is a complemented Leibniz algebra, the result is clear; because an ideal of L is also a subalgebra of L. □

Theorem 3.2.9. *Any ideal of a complemented Leibniz algebra is also a complemented Leibniz algebra.*

Proof. Let I be an ideal of a Leibniz algebra L. Certainly, I is also a subalgebra of L. Thus, by Lemma 3.2.6, I is a complemented Leibniz algebra. □

3.3 Cartan Subalgebras

In this section, we introduce Cartan subalgebras in order to show how they relate to complemented Leibniz algebras. As a reminder, all Leibniz algebras discussed in this section are solvable and finite dimensional over a field $\mathbb{F}$ with characteristic 0.

Definition 3.3.1. A **Cartan subalgebra** of a Leibniz algebra L is a nilpotent subalgebra of L which is self-normalizing (i.e. $N_L(C) = C$ for a nilpotent subalgebra C in L).

Unlike Lie algebras, the left and right normalizers, $N_L^l(C)$ and $N_L^r(C)$, of a subalgebra C do not necessarily coincide in Leibniz algebras due to the lack of antisymmetry. The example below provides such a case [7].

Example 3.3.2. Let $L = \text{span}\{x, y, z\}$ be a Leibniz algebra with the following non-zero products: $[z, x] = x$, $[z, y] = -y$, $[y, z] = y$, and $[z, z] = x$. Here, $C = \text{span}\{x - z\}$ is a Cartan subalgebra of L; however, $N_L^r(C) = \text{span}\{x, z\}$.

In [3], Barnes proves the existence of a Cartan subalgebra within a solvable Leibniz algebra. We will do the same, but we must first address a few important preliminary items.

Lemma 3.3.3. *If C is a Cartan subalgebra of a Leibniz algebra L and $A \lhd L$, then $(C + A)/A$ is a Cartan subalgebra of L/A.*

Proof. Let C be a Cartan subalgebra of a Leibniz algebra L and let $A \lhd L$. Now, $(C + A)/A$ is nilpotent; because $(C + A)/A \cong C/(C \cap A)$. Suppose that $x + A \in N_{L/A}((C + A)/A)$; thus,

$(x+A)(C+A) \subseteq (C+A)$ and $(C+A)(x+A) \subseteq (C+A)$. Therefore, $x(C+A) \subseteq (C+A)$ and $(C+A)x \subseteq (C+A)$; thus, $x \in N_L(C+A) = C+A$. Hence, $(C+A)/A$ is a Cartan subalgebra of L/A. □

Lemma 3.3.4. *For a Leibniz algebra L, let H be a subalgebra of L and $A \lhd L$ such that $A \subseteq H \subseteq L$. If H/A is a Cartan subalgebra of L/A and C is a Cartan subalgebra of H, then C is a Cartan subalgebra of L.*

Proof. Let L be a Leibniz algebra with subalgebra H and $A \lhd L$ such that $A \subseteq H \subseteq L$. Also, let H/A be a Cartan subalgebra of L/A and C be a Cartan subalgebra of H. Since C is a Cartan subalgebra of H, then C is certainly nilpotent. Now, suppose that $x \in N_L(C)$; thus, $x+A \in N_{L/A}((C+A)/A)$. By Lemma 3.3.3, $(C+A)/A$ is a Cartan subalgebra of H/A. Also, H/A is nilpotent; because H/A is a Cartan subalgebra of L/A. Then, H/A is a Cartan subalgebra of itself; therefore, $C+A = H$ and $x+A \in N_{L/A}(H/A) = H/A$. Hence, $x \in H$. Furthermore, $x \in N_H(C) = C$; because $x \in N_L(C)$, and C is a Cartan subalgebra of H. Thus, $N_L(C) = C$; and C is a Cartan subalgebra of L. □

Theorem 3.3.5. *If L is a solvable Leibniz algebra, then there exists a Cartan subalgebra of L.*

Proof. Let L be a solvable Leibniz algebra and let A be a minimal ideal of L. Suppose that L is a minimal counterexample. Thus, there exists a Cartan subalgebra H/A of L/A. Now, if $H \subset L$, then there exists a Cartan subalgebra C of H. By Lemma 3.3.4, C is also a Cartan subalgebra of L, which is a contradiction. Therefore, $H = L$; and L/A is nilpotent.

Let M be a maximal subalgebra of L. If $A \subseteq M$, then M/A is a maximal subalgebra of the nilpotent Leibniz algebra L/A. By Theorem 4.16 of [10], M/A is an ideal of L/A; thus, M is an ideal of L. Since M is a general maximal subalgebra of L, then by Theorem 4.16 of [10] once again, L is nilpotent, which is a contradiction. Hence, L contains a maximal subalgebra K which is not an ideal of L. For this K, we have $N_L(K) = K$ and $A \nsubseteq K$. Since A is abelian and $L = K + A$, then $K \cap A$ is an ideal of L. Because A is minimal, $K \cap A = 0$. Therefore, $K \cong L/A$, which is nilpotent; and K is a Cartan subalgebra of L. □

For our purposes in this section and beyond, we will denote the last term of the lower central series of a Leibniz algebra L as L^ω. Furthermore, for the remainder of this section, we will assume that L is a solvable, complemented Leibniz algebra and that N is a non-zero ideal of L such that N is nilpotent and L/N is nilpotent. Since L is solvable, then L^2 is nilpotent by Proposition 1.2.23. Thus L^ω is nilpotent and, by definition, L/L^ω is nilpotent. Therefore, we may hereafter replace N with L^ω.

Lemma 3.3.6. *If L is a complemented Leibniz algebra, then L^ω is abelian and L/L^ω is abelian.*

Proof. Let L be a complemented Leibniz algebra. As a member of the lower central series of L, L^ω is clearly an ideal of L. By Theorem 3.2.9, L^ω is also a complemented Leibniz algebra. Since L^ω is nilpotent, then L^ω must also be abelian by Proposition 3.2.5.

L/L^ω is a homomorphic image of L; thus, it is a complemented Leibniz algebra by Theorem 3.2.7. Since L/L^ω is nilpotent, then it is also abelian by Proposition 3.2.5. □

Proposition 3.3.7. *If L is a complemented Leibniz algebra, then $L^2 = L^\omega$.*

Proof. Let L be a complemented Leibniz algebra. By definition, $L^\omega \subseteq L^2$; and since L/L^ω is abelian, then $L^2 \subseteq L^\omega$. Therefore, $L^2 = L^\omega$. □

In order to fully lay the groundwork for Theorem 3.3.9, we must first discuss the Fitting one-component and the Fitting null-component of a vector space with respect to a linear operator. Consider the finite dimensional vector space V and let $T : V \to V$ be a linear operator on V. Then V can be represented as $V = (V_0)_T \oplus (V_1)_T$ where $T((V_0)_T) \subseteq (V_0)_T$, $T((V_1)_T) \subseteq (V_1)_T$, $(V_0)_T = \{v \in V | T^i(V) = 0 \text{ for some } i \in \mathbb{N}\}$, and $(V_1)_T = \bigcap_{i=1}^{\infty} T^i(V)$ as seen in [13].

Definition 3.3.8. The spaces $(V_0)_T$ and $(V_1)_T$ are called the **Fitting null-component** and the **Fitting one-component**, respectively, of the vector space V with respect to the linear operator T.

Theorem 3.3.9. *If L is a complemented Leibniz algebra, then L^ω is complemented in L by a Cartan subalgebra of L. Furthermore, the complements of L^ω are precisely the Cartan subalgebras of L; and all complements of L^ω are conjugate under automorphisms of L of the form $I + L_a$ for some $a \in L$.*

Proof. Let L be a complemented Leibniz algebra; thus, L^ω is abelian by Lemma 3.3.6. From Theorem 3.3.5, we know that there exists a Cartan subalgebra C of L. Since L^ω is an ideal of L, then $(C + L^\omega)/L^\omega$ is a Cartan subalgebra of L/L^ω by Lemma 3.3.3. Now, L/L^ω is nilpotent by definition; thus, L/L^ω is a Cartan subalgebra of itself. Hence, $L/L^\omega = (C + L^\omega)/L^\omega$ and $L = C + L^\omega$. Let δ be a mapping such that $\delta : C \to \text{Der}(L)$ by $\delta(c) = L_c$, the left multiplication operator. Now, δ is a homomorphism whose image $N = \delta(C)$ is nilpotent; because C is nilpotent. Therefore, $L = (C_0)_\delta \dot{+} (C_1)_\delta$, where $(C_0)_\delta$ is the Fitting null-component and $(C_1)_\delta$ is the Fitting one-component of δ acting on N. Thus, $(C_1)_\delta = [C, [C, [...[C, L]]]] \subseteq L^\omega$; and $(C_1)_\delta$ is invariant under C. Furthermore, since L^ω is abelian, then $(C_1)_\delta \lhd L$; and $L/(C_1)_\delta \cong (C_0)_\delta = C$ is nilpotent. Suppose that $(C_1)_\delta \neq L^\omega$; thus, $[C, L^\omega] \subset L^\omega$. Since L^ω is abelian, then $[L, L^\omega] = [C + L^\omega, L^\omega] = [C, L^\omega] + [L^\omega, L^\omega] \subset L^\omega$, which is a contradiction. Therefore, $(C_1)_\delta = L^\omega$ and $(C_0)_\delta = C$. Hence, all Cartan subalgebras of L are complements of L^ω.

Conversely, let K be a complement of L^ω in L such that K is not a Cartan subalgebra of L. Thus, by definition, K is nilpotent and $K \neq N_L(K)$. Hence, $N_L(K) \cap L^\omega \neq 0$. Now, let

$x \in N_L(K) \cap L^\omega$. Therefore, $[x, K] \subseteq K \cap L^\omega = 0$ and $[K, x] \subseteq K \cap L^\omega = 0$. Since $x \in L^\omega$ and L^ω is abelian, then $[x, L] = [x, L^\omega + K] = [x, L^\omega] + [x, K] = 0$ and $[L, x] = [L^\omega + K, x] = [L^\omega, x] + [K, x] = 0$. Thus, $x \in Z(L)$, the center of L, and $x \in N_L(C) = C$ for the Cartan subalgebra C earlier. Since $x \in L^\omega$, then $x \in C \cap L^\omega = 0$. Therefore, $N_L(K) \cap L^\omega = 0$, which is a contradiction. Hence, $K = N_L(K)$; and K is a Cartan subalgebra of L.

By Theorem 3.2.10 of [22], all complements of L^ω are conjugates under automorphisms of L of the form $I + L_a$ for some $a \in L$. □

Although Theorem 3.3.9 begins with the assumption that L is a complemented Leibniz algebra, this is not a necessary condition as long as L is solvable and L^ω is abelian. A similar version of the following corollary was proved for Lie algebras by Stitzinger in [17].

Corollary 3.3.10. *If L is a solvable Leibniz algebra such that L^ω is abelian, then L^ω is complemented in L by a Cartan subalgebra of L. Furthermore, the complements of L^ω are precisely the Cartan subalgebras of L; and all complements of L^ω are conjugate under automorphisms of L of the form $I + L_a$ for some $a \in L$.*

Proof. These results follow directly from the proof of Theorem 3.3.9. □

Proposition 3.3.11. *If L is a complemented Leibniz algebra and C is a Cartan subalgebra of L, then $L^\omega = [C, L^\omega]$.*

Proof. Let L be a complemented Leibniz algebra and C be a Cartan subalgebra of L. By definition, $L^\omega = [L, L^\omega]$. Since L^ω is complemented by a Cartan algebra, for example C, in L by Theorem 3.3.9, then $[L, L^\omega] \subseteq [C, L^\omega] + [L^\omega, L^\omega]$. Now, L^ω is abelian by Lemma 3.3.6; thus, $[L^\omega, L^\omega] = 0$. Furthermore, since L^ω is an ideal of L, then $[C, L^\omega] \subseteq L^\omega$. Thus, the following progression holds:

$$L^\omega = [L, L^\omega] \subseteq [C, L^\omega] + [L^\omega, L^\omega] = [C, L^\omega] \subseteq L^\omega.$$

Therefore, $L^{\omega} = [C, L^{\omega}]$. □

For a solvable, complemented Leibniz algebra L, we now define $(L^{\omega})_j = ((L^{\omega})_{j-1})^{\omega}$ for $j \in \mathbb{Z}_{>1}$ with $(L^{\omega})_1 = L^{\omega}$.

Lemma 3.3.12. *If L is a complemented Leibniz algebra, then $(L^{\omega})_j = L^{(j)}$ for each $j \in \mathbb{N}$.*

Proof. Let L be a complemented Leibniz algebra. By Proposition 3.3.7, we have $(L^{\omega})_1 = L^{\omega} = L^2 = L^{(1)}$. Since $(L^{\omega})_1$ is an ideal of L, then $(L^{\omega})_1$ is a complemented Leibniz algebra by Theorem 3.2.9. Using the information from the previous step along with Proposition 3.3.7 once again, we achieve the following result:

$$(L^{\omega})_2 = ((L^{\omega})_1)^{\omega} = ((L^{\omega})_1)^2 = [(L^{\omega})_1, (L^{\omega})_1] = [L^{(1)}, L^{(1)}] = L^{(2)}.$$

Since $(L^{\omega})_2 = L^{(2)}$ is an ideal of $(L^{\omega})_1 = L^{(1)}$, then $(L^{\omega})_2$ is also a complemented Leibniz algebra by Theorem 3.2.9. Repeating this procedure allows us to determine that $(L^{\omega})_j = L^{(j)}$ for each $j \in \mathbb{N}$. □

If a Leibniz algebra L is solvable, then there exists a smallest $n \in \mathbb{N}$ such that $(L^{\omega})_n = 0$. The lemma below shows that for complemented Leibniz algebras, we have $n = 2$.

Lemma 3.3.13. *If L is a complemented Leibniz algebra, then $(L^{\omega})_2 = 0$.*

Proof. Let L be a complemented Leibniz algebra. Then, L^{ω} is abelian by Lemma 3.3.6. Furthermore, L^{ω} is a complemented Leibniz algebra by Theorem 3.2.9. Using Proposition 3.3.7, the following equation holds:

$$(L^{\omega})_2 = ((L^{\omega})_1)^{\omega} = ((L^{\omega})_1)^2 = [(L^{\omega})_1, (L^{\omega})_1] = [L^{\omega}, L^{\omega}] = 0.$$

□

Definition 3.3.14. The **derived length** of a solvable Leibniz algebra L with dimension $n \geq 1$ is the smallest $k \in \mathbb{N}$ such that $L^{(k)} = 0$ but $L^{(k-1)} \neq 0$.

Theorem 3.3.15. *If L is a complemented Leibniz algebra, then its derived length is 2.*

Proof. This result follows directly from Lemma 3.3.12 and Lemma 3.3.13. □

Theorem 3.3.16. *If L is a complemented Leibniz algebra, then each non-zero member of the derived series is complemented by a Cartan subalgebra of L.*

Proof. Let L be complemented Leibniz algebra. By Theorem 3.3.15, the only proper non-zero member of the derived series of L is $L^{(1)}$. Since $L^{(1)} = L^{\omega}$ by Lemma 3.3.12, then the result follows by Theorem 3.3.9. □

The converse of Theorem 3.3.16 is not necessarily true as evidenced in Corollary 3.3.10 and shown in the Lie algebra example below.

Example 3.3.17. Let H be the 3-dimensional Heisenberg Lie algebra with basis elements $\{x, y, z\}$ over a field $\mathbb{F}$ where $[x, y] = z$, and $[y, z] = [x, z] = 0$. Also, let D be defined as the derivation with $D(x) = 0, D(y) = y$, and $D(z) = z$. Now, construct the Lie algebra L as the semi-direct sum of H and $\langle D\rangle$ so that $L = H \dotplus \langle D\rangle$. In this case, $L^{\omega} = L^{(1)} = \langle y, z\rangle$ and $L^{(1)}$ is complemented by the Cartan subalgebra $C = \langle x, D\rangle$. Also, $(L^{\omega})_2 = L^{(2)} = 0$. Now, the subalgebra $\langle z\rangle$ of H does not have a complement in H; thus, H is not a complemented Lie algebra. Therefore, L is not a complemented Lie algebra, despite the fact that each member of the derived series of L has a complement in L.

Corollary 3.3.18. *If L is a complemented Leibniz algebra, then each member of the derived series of L is a complemented Leibniz algebra.*

Proof. This result follows directly from Theorem 3.2.9. □

CHAPTER

4

TOTALLY NON-SATURATED FORMATIONS

4.1 The Formation of Complemented Leibniz Algebras

Throughout this chapter, we will once again assume that all Leibniz algebras discussed are solvable and finite dimensional over a field $\mathbb{F}$. In this section, we will show that the class of all solvable, complemented Leibniz algebras is a formation. This particular formation will prove to hold some interesting properties which will be examined later in the chapter. We will denote a general formation as $\mathscr{F}$ and the class of solvable, complemented Leibniz algebras as $\mathscr{C}$.

Definition 4.1.1. Let L be a Leibniz algebra with subalgebras A, B, C in L. L is **modular** if $A \subset C$, then $\langle A, B \rangle \cap C = \langle A, B \cap C \rangle$.

Since all of the Leibniz algebras discussed in this section are modular, the definition above will be useful in our investigation of $\mathscr{C}$ and particularly in the lemma below. In [18], Towers proved the following lemma for Lie algebras, but the results also hold for Leibniz algebras.

Lemma 4.1.2. *Let L be a solvable Leibniz algebra. $L \in \mathscr{C}$ if and only if L contains a minimal ideal M such that M has a complement N in $\mathscr{C}$.*

Proof. First, suppose that $L \in \mathscr{C}$, and let M be a minimal ideal of L. By Lemma 3.2.8, M is complemented by a subalgebra N in L. Since L is solvable, N must also be solvable; and N is a complemented Leibniz algebra by Lemma 3.2.6. Therefore, $N \in \mathscr{C}$.

Conversely, suppose that L contains a minimal ideal M where M is complemented by a subalgebra N of L and $N \in \mathscr{C}$. Let H be a subalgebra of L. We must find a subalgebra K in L that complements H. For $H + M$ in L, there exists a subalgebra G with $M \subset G \subset L$ such that $L = \langle (H + M), G \rangle$ and $(H + M) \cap G = M$, because $L/M \cong N$ is complemented. Thus, $M = (H + M) \cap G = (H \cap G) + M$, because $M \subset G$. Therefore, $H \cap G \subseteq M$. Hence, $H \cap G \subseteq H \cap M \subseteq H \cap G$; and $H \cap M = H \cap G$. Consequently, since $L = M + N$ and $M \subset G$, then $G = L \cap G = \langle M, N \rangle \cap G = \langle M, (N \cap G) \rangle$, because L is modular. If $H \cap M = 0$, then define $K = M + (N \cap G)$. Since $H \cap G = H \cap M$, we have

$$H \cap K = H \cap [M + (N \cap G)] = H \cap [(M + N) \cap G] = H \cap G \cap (M + N) = H \cap M \cap (M + N) = 0.$$

Furthermore, because $G = \langle M, (N \cap G) \rangle$, we have

$$\langle H,K\rangle = \langle H,M,(N\cap G)\rangle = \langle H,M,G\rangle = \langle (H+M),G\rangle = L.$$

If $H\cap M \neq 0$, then we can define $K = N\cap G$. Since $H\cap G = H\cap M$, we have

$$H\cap K = H\cap (N\cap G) = (H\cap G)\cap N = (H\cap M)\cap N = H\cap (M\cap N) = 0.$$

We claim that $L = \langle H,K\rangle$. Suppose that $M \nsubseteq \langle H,K\rangle$. Take a maximal subalgebra S of L such that $\langle H,K\rangle \subseteq S$. Now, because $G = \langle M,(N\cap G)\rangle$, we have

$$\langle H,K,M\rangle = \langle H,(N\cap G),M\rangle = \langle H,M,G\rangle = \langle (H+M),G\rangle = L.$$

Thus, we know that $M \nsubseteq S$; otherwise, $\langle H,K,M\rangle \subseteq S$. Hence, $S+M = L$ and $S\cap M = 0$. Since, $H\subseteq S$ and $H\cap M \neq 0$, we have encountered a contradiction. Therefore, $M \subseteq \langle H,K\rangle$ and

$$\langle H,K\rangle = \langle H,K,M\rangle = \langle H,(N\cap G),M\rangle = \langle H,M,G\rangle = \langle (H+M),G\rangle = L.$$

Hence, K is the complement of H and $L \in \mathscr{C}$. □

Proposition 4.1.3. *Let L be a solvable Leibniz algebra with a minimal ideal M. Either $M \subseteq \Phi(L)$, or M has a complement in L.*

Proof. If M has a complement in L, then its complement must be a maximal subalgebra of L. If M does not have a complement in L, then it must be contained in all maximal subalgebras of L. Thus, $M \subseteq \Phi(L)$. □

Definition 4.1.4. A factor algebra A/B is a **chief factor** of a Leibniz algebra L if B is an ideal of L and A/B is a minimal ideal of L/B.

Definition 4.1.5. Let L be a Leibniz algebra. A finite increasing chain of ideals of L, from 0 to L, is called a **chief series** for L if each ideal is maximal in the next.

Theorem 4.1.6. *A Leibniz algebra $L \in \mathscr{C}$ if and only if L has a chief series all of whose factors have a complement in L.*

Proof. Suppose that $L \in \mathscr{C}$. By Theorem 3.2.7, every homomorphic image of L is also in $\mathscr{C}$. Since each chief factor of L is a minimal ideal of a homomorphic image of L, then by Lemma 3.2.8, each chief factor of L has complement in L.

Conversely, suppose that L has a chief series all of whose factors have a complement in L. We will consider L to be a minimal counterexample which is not in $\mathscr{C}$. Let M be a minimal ideal of L; thus, M is a chief factor of L, and M has a complement N in L. Therefore, $N \cong L/M$. If each chief factor of L has complement in L, then each will also have a complement in L/M. Also, since N has a smaller dimension than L, then $N \in \mathscr{C}$ by induction. By Lemma 4.1.2, $L \in \mathscr{C}$, which is a contradiction. Hence, the result holds. □

Theorem 4.1.7. *$\mathscr{C}$ is a formation.*

Proof. By Theorem 3.2.7, $\mathscr{C}$ is closed under homomorphic images. Suppose that $H, K \lhd L$ such that $L/H, L/K \in \mathscr{C}$, but $L/(H \cap K) \notin \mathscr{C}$. Without loss of generality, we may assume that $H \cap K = 0$ so that the supposition translates into the following: $L/(H \cap K) = L/0 = L \notin \mathscr{C}$, where L is being considered as a minimal counterexample. Now, choose a minimal ideal M of L such that $0 \subset M \subset H$. Thus, $M = H \cap M = (H \cap M) + (H \cap K) = H \cap (M + K)$. Also, since $M \cap K = 0$, then $L/(M+K)$ is a homomorphic image of L/K; and $L/(M+K) \in \mathscr{C}$ by Theorem 3.2.7. Therefore, $L/M \in \mathscr{C}$ by induction; because L/M has a smaller dimension than L, and L was our minimal counterexample. If M does not have a complement in L, then $M \subseteq \Phi(L)$ by Proposition 4.1.3. Also, $(M + K)/K \subseteq \Phi(L/K) = 0$, because $L/K \in \mathscr{C}$. Hence, $M \subseteq K$ and $M \subseteq (H \cap K) = 0$, which is a contradiction. Therefore, M has a complement in L; and since $L/M \in \mathscr{C}$, then $L \in \mathscr{C}$ by Lemma 4.1.2. □

4.2 $\mathscr{F}$-residuals and $C(\mathscr{F})$

Although we have already discussed several examples of formations, this section will broaden our understanding of formations by introducing a method that allows us to create new formations from familiar ones.

Definition 4.2.1. Let $\mathscr{F}$ be a formation. The $\mathscr{F}$**-residual**, $L_{\mathscr{F}}$, of a Leibniz algebra L is the minimal ideal of L such that $L/L_{\mathscr{F}} \in \mathscr{F}$.

In the following four examples, we will let $\mathscr{F}$ be $\mathscr{N}$, the formation of nilpotent Leibniz algebras.

Example 4.2.2. Consider the cyclic Leibniz algebra $L = \text{span}\{x, x^2, x^3\}$ generated by x with non-zero products $[x, x] = x^2, [x, x^2] = x^3$, and $[x, x^3] = x^3$. Then, $L_{\mathscr{N}} = \text{span}\{x^3\}$.

Example 4.2.3. Similarly, let L be the 2-dimensional Leibniz algebra $L = \text{span}\{x, y\}$ with the non-zero product $[x, y] = y$. Then, $L_{\mathscr{N}} = \text{span}\{y\}$.

Example 4.2.4. Let L be a solvable, non-nilpotent Leibniz algebra. Thus, $L_{\mathscr{N}} = L^{\omega}$, the last term of the lower central series of L.

Example 4.2.5. Let L be a nilpotent Leibniz algebra. In this trivial case, $L_{\mathscr{N}} = 0$.

Definition 4.2.6. Let $\mathscr{F}$ be a formation. $C(\mathscr{F})$ is the collection of all Leibniz algebras for which the $\mathscr{F}$-residual contains only chief factors that have complements in the Leibniz algebra.

Example 4.2.7. Let $\mathscr{F} = \{0\}$. Then $L_{\{0\}} = L$ for every general Leibniz algebra L. In this case, $C(\{0\}) = \mathscr{C}$ by Theorem 4.1.6.

Proposition 4.2.8. *Let $\mathscr{F}$ and $\mathscr{F}'$ be any two formations. If $\mathscr{F} \subseteq \mathscr{F}'$, then $C(\mathscr{F}) \subseteq C(\mathscr{F}')$.*

Proof. Let L be a Leibniz algebra, and let $L \in C(\mathscr{F})$. Then, all chief factors of L below $L_{\mathscr{F}}$ have complements in L. Since $\mathscr{F} \subseteq \mathscr{F}'$, then $L_{\mathscr{F}'} \subseteq L_{\mathscr{F}}$. Thus, all chief factors of L below $L_{\mathscr{F}'}$ have complements in L and $L \in C(\mathscr{F}')$. Therefore, $C(\mathscr{F}) \subseteq C(\mathscr{F}')$. □

Proposition 4.2.9. *If $\mathscr{F}$ is a formation, then $\mathscr{F} \subseteq C(\mathscr{F})$.*

Proof. Let L be a Leibniz algebra. If $L \in \mathscr{F}$, then $L_{\mathscr{F}} = 0$. Hence, $L \in C(\mathscr{F})$; and $\mathscr{F} \subseteq C(\mathscr{F})$. □

Lemma 4.2.10. *Let L be a Leibniz algebra and $\mathscr{F}$ be a formation. If $L \in C(\mathscr{F})$, then all homomorphic images of L are in $C(\mathscr{F})$.*

Proof. Let $L \in C(\mathscr{F})$. Then, all chief factors of L below $L_{\mathscr{F}}$ have complements in L. Let $N \lhd L$, and consider the homomorphic image L/N of L. Since the ideals of L/N are in a one-to-one correspondence with the ideals of L containing N, then homomorphisms will preserve the $\mathscr{F}$-residual property and the property of a chief factor having a complement in L. Thus, $L/N \in C(\mathscr{F})$; and the result holds. □

Lemma 4.2.11. *Let L be a Leibniz algebra and $\mathscr{F}$ be a formation. If $H, K \lhd L$; $L/H \in C(\mathscr{F})$; and $L/K \in C(\mathscr{F})$, then $L/(H \cap K) \in C(\mathscr{F})$.*

Proof. Suppose $L/(H \cap K) \notin C(\mathscr{F})$. Without loss of generality, we may assume that $H \cap K = 0$ so that our supposition translates into $L/(H \cap K) = L/0 = L \notin C(\mathscr{F})$ where L is being considered as a minimal counterexample. We must show that $L \in C(\mathscr{F})$, meaning that all chief factors of L below $L_{\mathscr{F}}$ have complements in L. Obviously, if $L_{\mathscr{F}} = 0$, then $L \in C(\mathscr{F})$; therefore, let us assume that $L_{\mathscr{F}} \neq 0$.

First, we consider the case when $L_{\mathscr{F}} \cap H = 0$. If this is true, then we may replace K with $L_{\mathscr{F}}$ in the suppositions above; because $L/L_{\mathscr{F}} \in \mathscr{F} \subseteq C(\mathscr{F})$ by Proposition 4.2.9. Now, there must exist a minimal non-zero ideal M of L such that $M \subseteq L_{\mathscr{F}}$. If M does not have a complement in L, then $M \subseteq \Phi(L)$ by Proposition 4.1.3; thus, $M \subseteq (\Phi(L) \cap L_{\mathscr{F}})$. Hence,

$$M/H = (M+H)/H \subseteq (\Phi(L/H) \cap (L/H)_{\mathscr{F}}) = 0,$$

because $L/H \in C(\mathscr{F})$ and by Proposition 4.1.3. Since $M/H = 0$, then $M \subseteq H$; thus, $M \subseteq (H \cap L_{\mathscr{F}}) = 0$. We have arrived at a contradiction, because M was defined as a minimal non-zero ideal of L in $L_{\mathscr{F}}$. Therefore, M has a complement in L. Now,

$$M = M + 0 = M + (L_{\mathscr{F}} \cap H) = L_{\mathscr{F}} \cap (M+H).$$

Since $L/(M+H)$ is a homomorphic image of L/H and $L/H \in C(\mathscr{F})$, then $L/(M+H) \in C(\mathscr{F})$ by Lemma 4.2.10. Thus, $L/(L_{\mathscr{F}} \cap (M+H)) = L/M \in C(\mathscr{F})$ by induction, and all chief factors of L/M below $(L/M)_{\mathscr{F}}$ have complements in L/M. Hence, all chief factors in L between M and $L_{\mathscr{F}}$ have a complement in L; and, thus, $L \in C(\mathscr{F})$, which is a contradiction. Therefore, the result holds for this case.

Now, we consider the case when $L_{\mathscr{F}} \cap H \neq 0$. Let M be a minimal non-zero ideal of L such that $M \subseteq (L_{\mathscr{F}} \cap H)$. If M does not have a complement in L, then $M \subseteq \Phi(L)$ by Proposition 4.1.3; thus, $M \subseteq (\Phi(L) \cap L_{\mathscr{F}})$. Since $M \cap K = 0$, then

$$M/K = (M+K)/K \subseteq (\Phi(L/K) \cap (L/K)_{\mathscr{F}}) = 0$$

because $L/K \in C(\mathscr{F})$ and by Proposition 4.1.3. Since $M/K = 0$, then $M \subseteq K$; thus, $M \subseteq (H \cap K) = 0$. Once again, we have arrived at a contradiction; because M was defined as a minimal non-zero ideal of L contained in $L_{\mathscr{F}} \cap H$. Therefore, M has a complement in L. Furthermore,

$$M = M + 0 = M + (H \cap K) = H \cap (M+K).$$

Since $L/(M+K)$ is a homomorphic image of L/K and $L/K \in C(\mathscr{F})$, then $L/(M+K) \in C(\mathscr{F})$ by Lemma 4.2.10. Thus, $L/(H \cap (M+K)) = L/M \in C(\mathscr{F})$ by induction, and all chief factors of L/M below $(L/M)_{\mathscr{F}}$ have complements in L/M. Hence, all chief factors in L between M and $L_{\mathscr{F}}$ have a complement in L; and, thus, $L \in C(\mathscr{F})$. Once again, we've encountered a contradiction; and the result also holds for this case. □

Theorem 4.2.12. *If $\mathscr{F}$ is a formation, then $C(\mathscr{F})$ is also a formation.*

Proof. This result follows directly from Lemma 4.2.10 and Lemma 4.2.11. □

Theorem 4.2.13. *Let $\mathscr{F}$ be a formation. Then, $C(\mathscr{F}) = C(C(\mathscr{F}))$.*

Proof. Since $C(\mathscr{F})$ is a formation by Theorem 4.2.12, then $C(\mathscr{F}) \subseteq C(C(\mathscr{F}))$ by Proposition 4.2.9. Now, let $L \in C(C(\mathscr{F}))$. Thus, all chief factors below $L_{C(\mathscr{F})}$ have complements in L and $L/L_{C(\mathscr{F})} \in C(\mathscr{F})$. Since $\mathscr{F} \subseteq C(\mathscr{F})$ by Proposition 4.2.9, then $L_{C(\mathscr{F})} \subseteq L_{\mathscr{F}}$. Because $L/L_{C(\mathscr{F})} \in C(\mathscr{F})$, then, by definition, all chief factors below $(L/L_{C(\mathscr{F})})_{\mathscr{F}}$ have complements in $L/L_{C(\mathscr{F})}$ and, thus, in L. Hence, all chief factors in L between $L_{C(\mathscr{F})}$ and $L_{\mathscr{F}}$ have complements in L. Thus, $L \in C(\mathscr{F})$. Therefore, $C(C(\mathscr{F})) \subseteq C(\mathscr{F})$ and $C(\mathscr{F}) = C(C(\mathscr{F}))$. □

4.3 Totally Non-saturated Formations

The following definition, which was introduced in [23], will drive the remainder of our discussion on formations.

Definition 4.3.1. A formation $\mathscr{F}$ is **totally non-saturated** if either of the following equivalent statements is true:

1. If a solvable Leibniz algebra $L \notin \mathscr{F}$ and L has a minimal ideal M such that $L/M \in \mathscr{F}$, then $M \subseteq \Phi(L)$.
2. Suppose L is a solvable Leibniz algebra and M is any minimal ideal of L that has a complement in L. If $L/M \in \mathscr{F}$, then $L \in \mathscr{F}$.

The definition above introduces a third classification category for formations of Leibniz algebras. When operating within the general context of all Leibniz algebras, we may now classify a formation as either saturated, non-saturated, or totally non-saturated. Ultimately, as its name suggests, totally non-saturated formations may be considered to be as far from saturated as possible within the setting of all Leibniz algebras.

It is important to notice the distinguishing features of each formation category as we contrast the definition above with the saturated formation definition from Chapter 2. First, note that totally non-saturated formations are comprised of only solvable Leibniz algebras, whereas saturated and non-saturated formations may include Leibniz algebras which are not solvable as seen in Example 2.2.10. This restricted environment for totally non-saturated formations produces some rather interesting results. For example, while discussing the formation of solvable Leibniz algebras $\mathscr{S}$ in Chapter 2, we determined that $\mathscr{S}$ was a saturated formation; however, we will later discover that this particular formation of algebras can be given a different classification when the context is limited to only the solvable Leibniz algebras.

Another important difference can be found in the behavior of the minimal ideal of the Leibniz algebra being considered. For totally non-saturated formations, the minimal ideal has a complement in the solvable Leibniz algebra; however, for saturated formations, the minimal ideal is contained in the Frattini ideal of the Leibniz algebra. This dichotomy was also encountered in Proposition 4.1.3, which states that for a solvable Leibniz algebra L, a minimal ideal of L must either be contained in $\Phi(L)$ or have a complement in L. This proposition effectively divides formations of Leibniz algebras into either saturated or totally non-saturated cases when operating exclusively within the context of solvable Leibniz algebras. Therefore, in this restricted setting, there is no distinction between non-saturated and totally non-saturated formations.

Theorem 4.3.2. *$\mathscr{F}$ is a totally non-saturated formation if and only if $\mathscr{F} = C(\mathscr{F})$.*

Proof. Assume that $\mathscr{F}$ is a totally non-saturated formation and let $L \in C(\mathscr{F})$. We need to show that $L \in \mathscr{F}$, meaning that $L_{\mathscr{F}} = 0$. Suppose $L_{\mathscr{F}} \neq 0$. Select an ideal B of L such that $L_{\mathscr{F}}/B$ is minimal in L/B; thus, $L_{\mathscr{F}}/B$ is a chief factor of L. Since $L \in C(\mathscr{F})$, then $L_{\mathscr{F}}/B$ has a complement in L. By definition, $L/L_{\mathscr{F}} \in \mathscr{F}$; but $L/B \notin \mathscr{F}$. Since $\mathscr{F}$ is totally non-saturated, then $L_{\mathscr{F}}/B$ should be contained in $\Phi(L)$; however, it has a complement in L. By Proposition 4.1.3, we have arrived at a contradiction. Therefore, $L_{\mathscr{F}} = 0$, $L \in \mathscr{F}$, and $C(\mathscr{F}) \subseteq \mathscr{F}$. By Proposition 4.2.9, $\mathscr{F} \subseteq C(\mathscr{F})$; thus, $\mathscr{F} = C(\mathscr{F})$.

Now, assume that $\mathscr{F} = C(\mathscr{F})$. Suppose that $\mathscr{F}$ is not a totally non-saturated formation and let L be a minimal counterexample where $L \notin \mathscr{F}$, but L contains a minimal ideal M such that $L/M \in \mathscr{F}$ and M has a complement in L. Since $L_{\mathscr{F}}$ is the smallest ideal of L such that $L/L_{\mathscr{F}} \in \mathscr{F}$, then $L_{\mathscr{F}} = M$. Because M was defined to be minimal in L, all chief factors below $M = L_{\mathscr{F}}$ have a complement in L. Thus, $L \in C(\mathscr{F}) = \mathscr{F}$, which is a contradiction. Therefore, $\mathscr{F}$ is a totally non-saturated formation. □

Theorem 4.3.3. *If $\mathscr{F}$ is a formation, then $C(\mathscr{F})$ is a totally non-saturated formation.*

Proof. Let $\mathscr{F}$ be a formation and suppose that $C(\mathscr{F})$ is not a totally non-saturated formation. Now let L be a minimal counterexample where $L \notin C(\mathscr{F})$, but L contains a minimal ideal M such that $L/M \in C(\mathscr{F})$ and M has a complement in L. Since $L_{C(\mathscr{F})}$ is the smallest ideal of L such that $L/L_{C(\mathscr{F})} \in C(\mathscr{F})$, then $L_{C(\mathscr{F})} = M$. Because M was defined to be minimal in L, all chief factors below $M = L_{C(\mathscr{F})}$ have a complement in L. Hence, $L \in C(C(\mathscr{F}))$. By Theorem 4.2.13, $C(\mathscr{F}) = C(C(\mathscr{F}))$; thus, $L \in C(\mathscr{F})$, which is a contradiction. Therefore, $C(\mathscr{F})$ is a totally non-saturation formation. □

Definition 4.3.4. The **socle** of a Leibniz algebra L, $Soc(L)$, is the union of all minimal ideals of L and the direct sum of some of them.

Definition 4.3.5. The **abelian socle** of a Leibniz algebra L, $Asoc(L)$, is the union of all abelian minimal ideals of L and the direct sum of some of them.

The following lemma was proved in [6] using the definitions above.

Lemma 4.3.6. *If L is a Leibniz algebra such that $\Phi(L) = 0$, then $Asoc(L) = nil(L)$.*

Theorem 4.3.7. *Let L be a Leibniz algebra and $\mathscr{F}$ be a formation. $\mathscr{F}$ is a totally non-saturated formation if and only if $L/nil(L) \in \mathscr{F}$ implies $L/\Phi(L) \in \mathscr{F}$.*

Proof. Assume that $\mathscr{F}$ is a totally non-saturated formation and suppose that L is a minimal counterexample where $L/nil(L) \in \mathscr{F}$ but $L/\Phi(L) \notin \mathscr{F}$. Without loss of generality, we may let $\Phi(L) = 0$ so that we now have $L \notin \mathscr{F}$. Since $L_{\mathscr{F}}$ is the minimal ideal of L such that $L/L_{\mathscr{F}} \in \mathscr{F}$, then $L_{\mathscr{F}} \subseteq nil(L)$. By Lemma 4.3.6, $nil(L)$ is the direct sum of minimal abelian ideals of L; and each of those ideals is a complemented Leibniz algebra by Proposition 3.2.5. Hence, $L \in C(\mathscr{F})$. By Theorem 4.3.2, $\mathscr{F} = C(\mathscr{F})$; thus, $L \in \mathscr{F}$, which is a contradiction. Therefore, the result holds.

Conversely, suppose that $\mathscr{F}$ is not a totally non-saturated formation and let L be a minimal counterexample where $L \notin \mathscr{F}$, but L contains a minimal ideal M such that $L/M \in \mathscr{F}$ and M has a complement in L. As in the proof of Theorem 4.3.2, we have $L_{\mathscr{F}} = M$. Since M is a minimal ideal of a solvable Leibniz algebra, then M is abelian and, thus, nilpotent. Also, by Proposition 2.2.4, $\Phi(L)$ is nilpotent; hence, $M + \Phi(L) \subseteq nil(L)$. Furthermore, since $L/(M + \Phi(L))$ is a homomorphic image of L/M, then $L/(M + \Phi(L)) \in \mathscr{F}$. We now also have $L/nil(L) \in \mathscr{F}$, which implies that $L/\Phi(L) \in \mathscr{F}$ by assumption. Now, $L/(M \cap \Phi(L)) \in \mathscr{F}$. Since M has a complement in L, then $M \cap \Phi(L) = 0$ by Proposition 4.1.3. Therefore, $L \in \mathscr{F}$, which is a contradiction; and the result holds. □

Theorem 4.3.8. *Let L be a Leibniz algebra and $\mathscr{F}$ be a totally non-saturated formation. $L_{\mathscr{F}}$ is nilpotent if and only if $L_{\mathscr{F}} \subseteq \Phi(L)$.*

Proof. Assume that $\mathscr{F}$ is a totally non-saturated formation and suppose that $L_{\mathscr{F}} \subseteq \Phi(L)$ for some Leibniz algebra L. Since $\Phi(L)$ is nilpotent by Proposition 2.2.4, then $L_{\mathscr{F}}$ is nilpotent; and the result holds.

Conversely, assume that $L_{\mathscr{F}}$ is nilpotent for some Leibniz algebra L; thus, we have $L_{\mathscr{F}} \subseteq nil(L)$. Now, suppose that $L_{\mathscr{F}} \nsubseteq \Phi(L)$. By definition, $L/L_{\mathscr{F}} \in \mathscr{F}$. Since $\Phi(L)$ is also nilpotent by Proposition 2.2.4, then $\Phi(L) \subseteq nil(L)$. Therefore, $L_{\mathscr{F}} + \Phi(L) \subseteq nil(L)$. Since $L/(L_{\mathscr{F}} + \Phi(L))$ is a homomorphic image of $L/L_{\mathscr{F}}$, then $L/(L_{\mathscr{F}} + \Phi(L)) \in \mathscr{F}$. Furthermore, $L/nil(L) \in \mathscr{F}$; thus, $L/\Phi(L) \in \mathscr{F}$ by Theorem 4.3.7. Since $L_{\mathscr{F}}$ is the smallest ideal of L such that $L/L_{\mathscr{F}} \in \mathscr{F}$, then $L_{\mathscr{F}} \subseteq \Phi(L)$, which is a contradiction. Therefore, $L_{\mathscr{F}} \subseteq \Phi(L)$; and the result holds. □

At this point, we have only considered totally non-saturated formations in a general sense. We will now narrow our focus to examine and compare specific cases of totally non-saturated formations. Our investigation will begin and end with $\mathscr{C}$, the formation of solvable, complemented Leibniz algebras.

Theorem 4.3.9. *$\mathscr{C}$ is a totally non-saturated formation.*

Proof. Suppose $\mathscr{C}$ is not a totally non-saturated formation. Let L be a minimal counterexample where $L \notin \mathscr{C}$, but L contains a minimal ideal M such that $L/M \in \mathscr{C}$ and M has a complement in L. Thus, there exists a subalgebra $N \in L$ such that $L/M \cong N$ and $N \in \mathscr{C}$. By Lemma 4.1.2, we have arrived at a contradiction. Therefore, $\mathscr{C}$ is a totally non-saturated formation. □

Lemma 4.3.10. *The intersection of totally non-saturated formations is also a totally non-saturated formation.*

Proof. Let $\{\mathscr{F}_i\}$ be a collection of totally non-saturated formations with $\mathscr{F} = \cap \mathscr{F}_i$ and let L be a Leibniz algebra. If $L \in \mathscr{F}$ and M is a homomorphic image of L, then $L \in \mathscr{F}_i$ forces $M \in \mathscr{F}_i$ for all i. Hence, $M \in \mathscr{F}$. Next, consider $H, K \triangleleft L$ such that L/H, $L/K \in \mathscr{F}$. This forces L/H, $L/K \in \mathscr{F}_i$ for all i. Therefore, $L/(H \cap K) \in \mathscr{F}_i$ for all i, and $L/(H \cap K) \in \mathscr{F}$. We have shown that $\mathscr{F}$ is a formation, but we must still show that $\mathscr{F}$ is totally non-saturated. Since each $\mathscr{F}_i$ is totally non-saturated, then $\mathscr{F}_i = C(\mathscr{F}_i)$ for each i by Theorem 4.3.2. Furthermore, $\cap \mathscr{F}_i = \cap C(\mathscr{F}_i)$. Also, since $\mathscr{F} \subseteq \mathscr{F}_i$ for each i, then $C(\mathscr{F}) \subseteq C(\mathscr{F}_i)$ for each i by Proposition 4.2.8. Now, $C(\mathscr{F}) \subseteq \cap C(\mathscr{F}_i) = \cap \mathscr{F}_i = \mathscr{F} \subseteq C(\mathscr{F})$ by Proposition 4.2.9. Therefore, $\mathscr{F} = C(\mathscr{F})$; and $\mathscr{F}$ is a totally non-saturated formation by Theorem 4.3.2. □

Lemma 4.3.11. *Each formation $\mathscr{F}$ is contained in a unique minimal totally non-saturated formation, $C(\mathscr{F})$.*

Proof. We already know that $\mathscr{F} \subseteq C(\mathscr{F})$ by Proposition 4.2.9 and that $C(\mathscr{F})$ is a totally non-saturated formation by Theorem 4.3.3. Now, suppose that $\mathscr{F}$ is contained in a minimal totally non-saturated formation $\mathscr{M}$. Since $\mathscr{F} \subseteq \mathscr{M}$, then $C(\mathscr{F}) \subseteq C(\mathscr{M})$ by Proposition 4.2.8. Also, $\mathscr{M} \subseteq C(\mathscr{F})$, because $\mathscr{M}$ is minimal. Moreover, since $\mathscr{M}$ is totally non-saturated, then $C(\mathscr{M}) = \mathscr{M}$ by Theorem 4.3.2. Therefore, $\mathscr{F} \subseteq \mathscr{M} \subseteq C(\mathscr{F}) \subseteq C(\mathscr{M}) = \mathscr{M}$ and $\mathscr{M} = C(\mathscr{F})$.

In order to prove uniqueness, allow $\mathscr{F}$ to be contained in another minimal totally non-saturated formation $\mathscr{M}'$. Following the same line of reasoning given above, we have $\mathscr{F} \subseteq \mathscr{M}' \subseteq C(\mathscr{F}) \subseteq C(\mathscr{M}') = \mathscr{M}'$. Therefore, $\mathscr{M}' = C(\mathscr{F}) = \mathscr{M}$; and the result holds. □

Example 4.3.12. Recall the following symbols used for the classical formations of Leibniz algebras discussed in Chapter 2.

$\mathscr{A}$ = the formation of abelian Leibniz algebras

$\mathscr{N}$ = the formation of nilpotent Leibniz algebras

$$\mathscr{U} = \text{the formation of supersolvable Leibniz algebras}$$

$$\mathscr{S} = \text{the formation of solvable Leibniz algebras}$$

Also, recall Example 4.2.7 where $\mathscr{F} = \{0\}$. Because $L_{\{0\}} = L$ for every Leibniz algebra L, we determined that $C(\{0\}) = \mathscr{C}$. Using Proposition 4.2.8 and Lemma 4.3.11, we can order the totally non-saturated formations associated with each of the classical formations listed above. Since $\{0\} \subseteq \mathscr{A} \subseteq \mathscr{N} \subseteq \mathscr{U} \subseteq \mathscr{S}$, then $\mathscr{C} = C(\{0\}) \subseteq C(\mathscr{A}) \subseteq C(\mathscr{N}) \subseteq C(\mathscr{U}) \subseteq C(\mathscr{S})$. Furthermore, we know that $\{0\} \subset \mathscr{A} \subset \mathscr{N} \subset \mathscr{U} \subset \mathscr{S}$. The following examples investigate whether or not proper subsets may be used when ordering the totally non-saturated formations above.

Example 4.3.13. Let $\mathscr{F} = \mathscr{S}$, the formation of solvable Leibniz algebras. By Proposition 4.2.9, $\mathscr{S} \subseteq C(\mathscr{S})$; and by Theorem 4.3.3, $C(\mathscr{S})$ is a totally non-saturated formation. Since, by definition, totally non-saturated formations are comprised of only solvable Leibniz algebras, then $C(\mathscr{S}) \subseteq \mathscr{S}$ and, thus, $C(\mathscr{S}) = \mathscr{S}$. Hence, $\mathscr{S}$ is trivially a totally non-saturated formation. Under these circumstances, we have $L_{\mathscr{S}} = 0$ for each L considered. Furthermore, $\mathscr{C} \subset \mathscr{S} = C(\mathscr{S})$.

It is fascinating to note that in Chapter 2, $\mathscr{S}$ was classified as a saturated formation while operating within the general context of all Leibniz algebras; however, within the restricted setting of only the solvable Leibniz algebras, $\mathscr{S}$ may be classified as a totally non-saturated formation.

Example 4.3.14. Let $\mathscr{F} = \mathscr{A}$, the formation of abelian Leibniz algebras. By Proposition 3.2.5, $\mathscr{A} \subseteq \mathscr{C}$. Also, by Proposition 4.2.8, we have $C(\{0\} = \mathscr{C} \subseteq C(\mathscr{A})$. Since $C(\mathscr{A})$ is the unique minimal totally non-saturated formation containing $\mathscr{A}$ by Lemma 4.3.11, then $C(\mathscr{A}) \subseteq \mathscr{C}$. Therefore, $\mathscr{A} \subseteq \mathscr{C} \subseteq C(\mathscr{A}) \subseteq \mathscr{C}$. Hence, we have shown that $\mathscr{C} = C(\mathscr{A})$.

Example 4.3.15. Let $\mathscr{F} = \mathscr{N}$, the formation of all nilpotent Leibniz algebras, and consider the following set of circumstances. Let V be a vector space with p basis elements $e_1, ..., e_p$ over a field $\mathbb{F}$ of characteristic p where p is prime. Also, let H be a 3-dimensional Lie algebra of linear transformations on V with basis elements $\{x, y, z\}$ and the following left multiplications:

$$x(e_j) = e_{j+1} \text{ with subscripts mod } p,$$
$$y(e_j) = (j+1)e_{j-1} \text{ with subscripts mod } p,$$
$$z(e_j) = e_j.$$

Within this context, we also have the following:

$$[y, x](e_j) = (yx - xy)(e_j) = yx(e_j) - xy(e_j) = e_j,$$
$$[y, z](e_j) = (yz - zy)(e_j) = yz(e_j) - zy(e_j) = 0,$$
$$[z, x](e_j) = (zx - xz)(e_j) = zx(e_j) - xz(e_j) = 0.$$

Thus, $[y, x] = z$. Also, let all right multiplications $VH = 0$. Now, we define $L = V \dotplus H$ with $[h, v] = h(v)$. Here, we have $L_{\mathscr{N}} = V$. Since V is a minimal ideal of L, and V has a complement H in L, then $L \in C(\mathscr{N})$. Now, consider the ideal $\langle z \rangle \dotplus V$ of L. Since $\langle z \rangle \dotplus V$ doesn't have a complement in L, then $L \notin \mathscr{C}$ by Lemma 3.2.8. Therefore, we have shown that $\mathscr{C} \subset C(\mathscr{N})$.

Example 4.3.16. Once again, let $\mathscr{F} = \mathscr{N}$. Now, consider the solvable Leibniz algebra $L = \langle x, y, z \rangle$ with non-zero products $[x, y] = y + z$ and $[x, z] = z$. Here, we have $L_{\mathscr{N}} = \langle y, z \rangle$. Now, the subalgebra $\langle y + z \rangle$ does not have a complement in L; thus, $L \notin C(\mathscr{N})$. Therefore, we have shown that $C(\mathscr{N}) \subset C(\mathscr{S}) = \mathscr{S}$.

Based on the results from the examples above, we can conclude the following:

$$\mathscr{C} = C(\mathscr{A}) \subset C(\mathscr{N}) \subseteq C(\mathscr{U}) \subseteq C(\mathscr{S}) = \mathscr{S}.$$

Although we have shown that the formation of solvable, complemented Leibniz algebras $\mathscr{C}$ is the smallest of the totally non-saturated formations considered above, its position among all totally non-saturated formations is much more remarkable.

Theorem 4.3.17. *$\mathscr{C}$ is the unique smallest totally non-saturated formation.*

Proof. Lemma 4.3.10 and Lemma 4.3.11 together confirm the existence of a unique smallest totally non-saturated formation. In order to show that this formation is necessarily $\mathscr{C}$, we must recall the trivial formation $\mathscr{F} = \{0\}$ from Example 4.2.7. Here, we found that $C(\mathscr{F}) = C(\{0\}) = \mathscr{C}$. Now, let $\mathscr{F}'$ be a general totally non-saturated formation. Of course, $\{0\} = \mathscr{F} \subseteq \mathscr{F}'$; thus, $C(\mathscr{F}) \subseteq C(\mathscr{F}')$ by Proposition 4.2.8. Also, since $\mathscr{F}'$ is totally non-saturated, then $C(\mathscr{F}') = \mathscr{F}'$. Hence, $\mathscr{C} = C(\mathscr{F}) \subseteq C(\mathscr{F}') = \mathscr{F}'$; thus, $\mathscr{C}$ is contained in every totally non-saturated formation. Therefore, $\mathscr{C}$ is contained in the intersection of every totally non-saturated formation. Since $\mathscr{C}$ itself is also a totally non-saturated formation by Theorem 4.3.9, then $\mathscr{C}$ is the unique smallest totally non-saturated formation. □

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