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A PROBABILISTIC APPROACH TO THE TAILS OF INFINITELY DIVISIBLE LAWS

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1. Introduction. Consider an arbitrary non–degenerate infinitely divisible real random variable X. Usually this is described by the Lévy formula for its characteristic function. For any t in the real line $\mathbb{R}$, we have

$$\begin{aligned} C(t) &= E\exp(iXt) \\ &= \exp\Big\{ i\theta t - \frac{\sigma^2}{2}t^2 \\ &+ \int_{-\infty}^{0}\left(e^{ixt}-1-\frac{ixt}{1+x^2}\right)dL(x) + \int_0^\infty \left(e^{ixt}-1-\frac{ixt}{1+x^2}\right)dR(x)\Big\}, \end{aligned}$$

where $\theta \in \mathbb{R}$ and $\sigma \geq 0$ are uniquely determined constants and L and R are uniquely determined left–continuous and right–continuous functions (the so–called Lévy measures) on $(-\infty,0)$ and $(0,\infty)$, respectively, that is $L(\cdot)$ and $R(\cdot)$ are non–decreasing functions such that $L(-\infty)=0=R(\infty)$ and

$$\int_{-\epsilon}^{0} x^2\,dL(x) + \int_0^\epsilon x^2\,dR(x) < \infty \quad \text{for any} \quad \epsilon > 0. \tag{1.1}$$

Many authors have investigated the tail behavior of the distribution of X (see [1], [3], [7–11], [13–21], for instance). Of necessity, the methods normally used have a Fourier–analytic character and the results are usually formulated in terms of

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conditions on the Lévy measures, conditions that do not always allow an immediate probabilistic insight.

As an integral part of a probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables and of the corresponding lightly trimmed sums, a representation of X is given in Theorem 3 of [5]; cf. also the end of Section 2 of [6] in the present volume. Introduce the non-decreasing, non-positive, right-continuous inverse functions

$$\phi_1(s) = \inf\{x < 0 : L(x) > s\},\ \phi_2(s) = \inf\{x < 0 : -R(-x) > s\},\ 0 < s < \infty,$$

and consider two standard (intensity one) left-continuous Poisson processes $N_1(s)$ and $N_2(s)$, $s \geq 0$, and a standard normal random variable Z such that $N_1(\cdot)$, Z, and $N_2(\cdot)$ are independent. Since (1.1) implies that

$$\int_\epsilon^\infty \phi_1^2(s)\,ds + \int_\epsilon^\infty \phi_2^2(s)\,ds < \infty \quad \text{for any} \quad \epsilon > 0, \tag{1.2}$$

the random variables

$$\begin{aligned} V_j &= \int_{Y_j}^\infty (N_j(s) - s)\,d\phi_j(s) - \int_1^{Y_j} s d\phi_j(s) \\ &= \int_1^\infty (N_j(s) - s)\,d\phi_j(s) + \int_0^1 N_j(s)\,d\phi_j(s), \end{aligned} \tag{1.3}$$

where Y_j is the first jump-point of $N_j(\cdot)$, are well defined for $j = 1, 2$ (the first integrals exist as improper Riemann integrals with probability one), and the constants

$$\theta_j = -\phi_j(1) + \int_0^1 \frac{\phi_j(s)}{1 + \phi_j^2(s)}\,ds - \int_1^\infty \frac{\phi_j^3(s)}{1 + \phi_j^2(s)}\,ds,\ j = 1, 2,$$

are finite, and we have the distributional equality

$$X \stackrel{\mathcal{D}}{=} V_{1,2}(\sigma, \theta) := -V_1 + \sigma Z + V_2 + \theta - \theta_1 + \theta_2, \tag{1.4}$$

that is $E \exp(iV_{1,2}(\sigma,\theta)t) = C(t)$, $t \in I\!R$.

The aim of the present paper is to introduce a direct probabilistic approach to the investigation of the tails of X based on the representation in (1.4) and on the use of elementary calculations and inequalities for the Poisson distribution. We derive several earlier results by this approach, sometimes in a polished form, under *natural* new versions of the conditions, including Sato's [15] bounds. The fact that Sato's theorem can be obtained by this approach is particularly interesting to us. This is because we used it in the proof of Corollary 1 in [5] when proving the necessity half of the normal convergence criterion in the framework of our general probabilistic theory of convergence, also joint with Erich Haeusler. With the present proof of Sato's theorem our entire theory becomes independent of the existing literature and uses Fourier analysis only to ensure the uniqueness of ϕ_1, ϕ_2, σ, and θ on the right side of (1.4). The results are stated in Section 2, all the proofs are in Section 3.

We close the present introduction by pointing out that it is very easy to see directly that the right side of (1.4) is infinitely divisible. Indeed, for any integer $n \geq 1$, let $Z(k,n)$, $1 \leq k \leq n$, be Normal $(0,1/n)$ random variables and $N_j^{(k)}$, $1 \leq k \leq n$, $j = 1,2$, be standard Poisson processes such that $Z(1,n),\ldots,Z(n,n)$, $N_1^{(1)},\ldots,N_1^{(n)}$, $N_2^{(1)},\ldots N_2^{(n)}$ are independent, and form

$$V_j(k,n) = \int_{1/n}^{\infty} (N_j^{(k)}(u) - u)\, d\phi_j(nu) + \int_0^{1/n} N_j^{(k)}(u)\, d\phi_j(nu).$$

Then the random variables

$$V_{1,2}^{(k,n)}(\sigma) = -V_1(k,n) + \sigma Z(k,n) + V_2(k,n),\ k = 1,\ldots,n,$$

are independent and identically distributed and, interchanging integration and summation and substituting $nu = s$, we see that

$$\sum_{k=1}^{n} V_{1,2}^{(k,n)}(\sigma) \stackrel{\mathcal{D}}{=} -V_1 + \sigma Z + V_2.$$

It would be interesting to prove the converse statement (that if X is infinitely divisible then with uniquely determined ϕ_1, ϕ_2, σ, and θ we necessarily have (1.4)) *without* a recourse to the Lévy or Lévy–Hinchin formula for the characteristic function. This problem seems to be difficult.

2. Results. The first set of results is formulated for the extreme pieces V_j in (1.4), defined in (1.3). It is worthwhile to separate these because the corresponding results for the general X will follow from these special ones in an extremely elementary fashion.

Introduce the non-negative quantities

$$A_j = -\phi_j(0+) \leq \infty, \; j = 1, 2,$$

and let δ denote any finite constant. Unless otherwise stated, we do not necessarily assume that $A_j < \infty$. If $A_j = \infty$, then $1/A_j$ is interpreted as zero. In Theorem 1 below, j can be 1 or 2, and we only consider the non-degenerate case when $\phi_j \not\equiv 0$. This is equivalent to assuming that $A_j > 0$.

THEOREM 1. (i) *For every $\alpha > 1/A_j$ there exists an $x_0 = x_0(\alpha, \phi_j, \delta) > 0$ such that for all $x \geq x_0$,*

$$P\{V_j + \delta > x\} \geq \exp(-\alpha x \log x).$$

(ii) *If $A_j < \infty$, then for every $\alpha < 1/A_j$ there exists an $x_0 = x_0(\alpha, \phi_j, \delta) > 0$ such that for all $x \geq x_0$,*

$$P\{V_j + \delta \geq x\} \leq \exp(-\alpha x \log x).$$

(iii) *For every* $\tau > 0$ *there exists an* $x_0 = x_0(\tau, \phi_j, \delta) > 0$ *such that for all* $x \geq x_0$,

$$P\{V_j + \delta \leq -x\} \leq \exp(-\tau x^2).$$

(iv) *The essential infimum of* $V_j + \delta$ *is* $-\int_1^\infty s d\phi_j(s) + \delta$.

(v) *The random variable* $V_j + \delta$ *is bounded from below (with probability one) if and only if* $\int_1^\infty s d\phi_j(s) < \infty$, *which in turn is equivalent to* $-\int_1^\infty \phi_j(s) ds < \infty$.

Now we consider the general infinitely divisible random variable $X = X(L, R, \sigma, \theta)$ with characteristic function $C(\cdot)$ or, what is the same, $X = X(\phi_1, \phi_2, \sigma, \theta)$ on the left side of (1.4). We exclude the trivial case when X degenerates at θ, which happens if and only if $A_1 = A_2 = \sigma = 0$ (cf. Theorem 4 in [5]). So at least one of A_1, A_2, and σ is assumed to be positive. As usual, Φ will denote the standard normal distribution function.

THEOREM 2. (i) *If* $0 < A_2 \leq \infty$, *then for every* $\alpha > 1/A_2$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X > x\} \geq \exp(-\alpha x \log x). \tag{2.1}$$

If $0 < A_2 < \infty$, *then for every* $0 < \alpha < 1/A_2$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X \geq x\} \leq \exp(-\alpha x \log x). \tag{2.2}$$

If $A_2 = 0, A_1 > 0$, *and* $\sigma > 0$, *then for every* $0 < \epsilon < 1$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X \geq x\} \leq 1 - \Phi\left((1 - \epsilon)\frac{x}{\sigma}\right). \tag{2.3}$$

(ii) *If* $0 < A_1 \leq \infty$, *then for every* $\beta > 1/A_1$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X < -x\} \geq \exp(-\beta x \log x). \tag{2.4}$$

If $0 < A_1 < \infty$, *then for every* $0 < \beta < 1/A_1$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X \leq -x\} \leq \exp(-\beta x \log x). \tag{2.5}$$

If $A_1 = 0, A_2 > 0$, *and* $\sigma > 0$, *then for every* $0 < \epsilon < 1$ *there exists an* $x_0 > 0$ *such that for all* $x \geq x_0$,

$$P\{X \leq -x\} \leq \Phi\left(-(1-\epsilon)\frac{x}{\sigma}\right). \tag{2.6}$$

(iii) *The random variable* X *is almost surely bounded from below if and only if* $A_1 = 0, \sigma = 0$, *and* $\int_1^\infty s d\phi_2(s) < \infty$, *or, equivalently,* $-\int_1^\infty \phi_2(s) ds < \infty$, *in which case*

$$-\infty < \text{ess inf } X = \theta + \int_0^\infty \frac{\phi_2(s)}{1+\phi_2^2(s)}\, ds = \theta - \int_0^\infty \frac{x}{1+x^2}\, dR(x).$$

(iv) *The random variable* X *is almost surely bounded from above if and only if* $A_2 = 0, \sigma = 0$, *and* $\int_1^\infty s d\phi_1(s) < \infty$, *or, equivalently,* $-\int_1^\infty \phi_1(s) ds < \infty$, *in which case*

$$\text{ess sup } X = \theta - \int_0^\infty \frac{\phi_1(s)}{1+\phi_1^2(s)} ds = \theta - \int_{-\infty}^0 \frac{x}{1+x^2} dL(x) < \infty.$$

(v) *The random variable* X *is almost surely unbounded.*

Results of the type of the first two statements of parts (i) and (ii) of Theorem 2 have been proved by Kruglov [11], Ruegg [13,14], Horn [10], Steutel [16] and Elliott and Erdős [7]. Related is the work of Wolfe [18]. The principally final result has been achieved by Sato [15], who formulates it in terms of the support of the combined Lévy measure $L(-x)-R(x), x>0$. Sato's result in fact covers the multidimensional case, and hence the univariate special case of his statement is by nature two–sided. However, the univariate special case of his proof allowed Bingham, Goldie, and Teugels [2; page 342] to formulate the more precise one–sided results. Our results in (2.1), (2.2) and (2.4), (2.5) of parts (i) and (ii) of Theorem 2 above are equivalent forms of these one–sided variants of Sato's theorem. An equivalent form of Sato's two–sided univariate result has been proved later but independently by Esseen [9].

We are not aware of a precise formulation of the complementary results in part (iii) of Theorem 1 and in (2.3) and (2.6) of Theorem 2 in the literature.

Parts (iii) and (iv) of Theorem 2 are equivalent forms of results achieved jointly by the two papers of Baxter and Shapiro [1] and Tucker [17]. Independently of Tucker they have been also obtained by Esseen [8]. These two parts trivially imply part (v) of Theorem 2, which has been obtained first by Chatterjee and Pakshirajan [3] by an independent Fourier–analytic proof.

There are, of course, other works that deal with the tail behavior of $X(L,R,\sigma,\theta)$. For example, Yakymiv [19] has recently proved rather precise results for a special class of infinitely divisible variables, extending greatly the main theorem of Zolotarev [20]. It would be interesting to see if these problems can also be approached by our more direct probabilistic methods.

3. Proofs. For the sake of simplicity in the notation, we drop all the subscripts j in the proof of Theorem 1. Thus V_j, ϕ_j, N_j, Y_j, and A_j become V, ϕ, N, Y, and A in this proof.

The main trick in the proof of (i) and (iii) is to introduce the functions

$$\phi_x(s) = \begin{cases} \phi(s), & 0 < s < x^2, \\ \phi(x^2), & x^2 \leq s < \infty, \end{cases} \quad \bar{\phi}_x(s) = \begin{cases} \phi(x^2), & 0 < s < x^2, \\ \phi(s), & x^2 \leq s < \infty, \end{cases}$$

where $x > 0$, and to write (cf. (1.3))

$$V = \int_Y^\infty (N(s) - s)d\phi(s) - \int_1^Y sd\phi(s) = V(x) + \bar{V}(x), \tag{3.1}$$

where

$$\begin{aligned} V(x) &= \int_Y^\infty (N(s) - s)d\phi_x(s) - \int_1^Y sd\phi_x(x) \\ &= \int_Y^\infty N(x)d\phi_x(s) - \int_1^\infty sd\phi_x(s) \end{aligned} \tag{3.2}$$

and

$$\bar{V}(x) = \int_Y^\infty (N(s) - s)d\bar{\phi}_x(s) - \int_1^Y sd\bar{\phi}_x(s). \tag{3.3}$$

Note that for $x > 1$,

$$\int_1^\infty sd\phi_x(s) = -\phi(1) + x^2\phi(x^2) - \int_1^{x^2} \phi(s)\, ds.$$

Clearly, (1.2) implies that $x^2\phi(x^2) = o(x)$ as $x \to \infty$, and by an elementary argument based on the Cauchy–Schwarz inequality we also have

$$\int_1^{x^2} \phi(s)ds = o(x) \quad \text{as} \quad x \to \infty.$$

Hence

$$\int_1^\infty sd\phi_x(s) = o(x) \quad \text{as} \quad x \to \infty. \tag{3.4}$$

The following inequality for the Poisson distribution, stated as Inequality 2 in [12], is an easy consequence of Stirling's formula.

LEMMA 1. *For every $\lambda > 0$ there exists a constant $0 < K(\lambda) < \infty$ such that for all $y \geq 1$,*

$$P\{N(\lambda) \geq y\} \geq K(\lambda) y^{-1/2} \exp\{-y(\log y - 1 - \log \lambda)\}.$$

LEMMA 2. *For all $x \geq 1, \gamma > 0$, and $\gamma/\phi(x^2) \leq t < \infty$,*

$$E \exp(t\bar{V}(x)) \leq \exp\left(\frac{t^2}{2} f(x) e^{\gamma}\right),$$

where

$$f(x) = x^2 \phi^2(x^2) + \int_{x^2}^{\infty} \phi^2(s) ds \to 0 \quad as \quad x \to \infty.$$

Proof. Setting

$$\begin{aligned}\theta(x) &= -\bar{\phi}_x(1) + \int_0^1 \frac{\bar{\phi}_x(s)}{1 + \bar{\phi}_x^2(s)} ds - \int_1^{\infty} \frac{\bar{\phi}_x^3(s)}{1 + \bar{\phi}_x^2(s)} ds \\ &= -\frac{\phi^3(x^2)}{1 + \phi^2(x^2)} - \int_1^{\infty} \frac{\bar{\phi}_x^3(s)}{1 + \bar{\phi}_x^2(s)} ds,\end{aligned}$$

the argument used for the computation of the corresponding characteristic function in the proof of Theorem 3 in [5] gives

$$\begin{aligned}\log E \exp(t\bar{V}(x)) &= \int_0^{\infty} \left\{ \exp(t\bar{\phi}_x(s)) - 1 - t \frac{\bar{\phi}_x(s)}{1 + \bar{\phi}_x^2(s)} \right\} ds + t\theta(x) \\ &\leq \int_0^{\infty} \left\{ t\bar{\phi}_x(s) + \frac{t^2}{2} \bar{\phi}_x^2(s) e^{\gamma} - t \frac{\bar{\phi}_x(s)}{1 + \bar{\phi}_x^2(s)} \right\} ds + t\theta(x) \\ &= \frac{t^2}{2} f(x) e^{\gamma},\end{aligned}$$

where we used the inequality

$$e^y - 1 \leq y + \frac{y^2}{2} e^\gamma, \quad -\infty < y \leq \gamma. \quad \blacksquare$$

The next two lemmas together show that the right tial of V is determined by the term $V(x)$ in (1.3).

LEMMA 3. *For all $\alpha > 1/A$ and all sufficiently small $\rho > 1$,*

$$\lim_{x\to\infty} \exp(\alpha x \log x)\, P\{V(x) > \rho x\} = \infty.$$

Proof. Using (3.4), we see that for any $0 < \epsilon < 1$ and all x large enough,

$$\begin{aligned} P\{V(x) > \rho x\} &\geq P\left\{\int_Y^\infty N(s)\phi_x(s) > (1+\epsilon)\rho x\right\} \\ &\geq P\{N(\epsilon)(\phi(x^2) - \phi(\epsilon)) > (1+\epsilon)\rho x\} \\ &= P\{N(\epsilon) > \beta x\} \\ &\geq K(\epsilon)(\beta x)^{-1/2} \exp\{-\beta x(\log x + \log(\beta/\epsilon) - 1)\} \end{aligned}$$

by Lemma 1 in the last step, where $\epsilon > 0$ and $\rho > 1$ are chosen so small and x so large that

$$\frac{1}{A} < \beta = \beta(x) = \frac{(1+\epsilon)\rho}{\phi(x^2) - \phi(\epsilon)} < \alpha.$$

This inequality clearly implies the lemma. $\blacksquare$

LEMMA 4. *For all $\alpha > 0$ and $\rho > 0$,*

$$\lim_{x\to\infty} \exp(\alpha x^2)\, P\{|\bar{V}(x)| \geq \rho x\} = 0.$$

Proof. For any $0 < t < \infty$, the Markov inequality and Lemma 2 give that

$$P\{\bar{V}(x) \geq \rho x\} \leq \exp\left(-\rho t x + \frac{t^2}{2} f(x) e^\rho\right).$$

Upon choosing $t = \rho e^{-\rho} x / f(x) > 0$, since $f(x) \to 0$ as $x \to \infty$, we obtain

$$\lim_{x \to \infty} \exp(\alpha x^2) P\{\bar{V}(x) \geq \rho x\} \leq \lim_{x \to \infty} \exp\left\{x^2 \left(\alpha - \frac{\rho^2}{2e^\rho} \frac{1}{f(x)}\right)\right\} = 0.$$

On the other hand, for any $t > 0$ such that $\rho/\phi(x^2) \leq -t < 0$,

$$P\{-\bar{V}(x) \geq \rho x\} \leq \exp\left(-\rho t x + \frac{t^2}{2} f(x) e^\rho\right).$$

It is easy to see that presently the choice $t = \rho(x^2/(\epsilon^\rho f(x)))^{1/2}$ is permissible, whence we get

$$\lim_{x \to \infty} \exp(\alpha x^2) P\{-\bar{V}(x) \geq \rho x\} \leq \lim_{x \to \infty} \exp\left\{x^2 \left(\alpha + \frac{\rho^2}{2} - \frac{\rho^2}{\sqrt{\epsilon^\rho}} \frac{1}{\sqrt{f(x)}}\right)\right\} = 0$$

again since $f(x) \to 0$ as $x \to \infty$. The two limit relations prove the lemma. ∎

Proof of Theorem 1. (i) Choose $\alpha > 1/A$ and let $\epsilon > 0$ be small enough so that Lemma 3 holds for $\rho = 1 + \epsilon$. Then by (3.1),

$$\begin{aligned} P\{V > x\} &\geq P\{V(x) > (1+\epsilon)x,\ |\bar{V}(x)| < \epsilon x\} \\ &\geq P\{V(x) > (1+\epsilon)x\} - P\{|\bar{V}(x)| \geq \epsilon x\}. \end{aligned}$$

Now Lemmas 3 and 4 imply that

$$\lim_{x \to \infty} \exp(\alpha x \log x) P\{V > x\} = \infty,$$

from which the statement follows. ∎

(iii) For any $0 < \epsilon < 1$ and $x > 0$, by (3.1) we have

$$P\{V \leq -x\} \leq P\{V(x) \leq -(1-\epsilon)x\} + P\{-\bar{V}(x) \geq \epsilon x\}.$$

Since by (3.2) and (3.4),

$$V(x) = \int_Y^{x^2} N(s) d\phi(s) - o(x) \quad \text{as} \quad x \to \infty,$$

we notice that the first probability in this bound becomes zero for all x large enough. Hence the statement follows from Lemma 4. ∎

(ii) The proof of this statement is more analytical. It goes as the proof of Lemma 1 of Sato [15] and hence uses arguments from the proof of Theorem 1 of Zolotarev [21]. These calculations performed with the moment generating function actually go back to Cramér [4]. Our presentation is self-contained.

First of all note that the condition $A < \infty$ implies that

$$\int_0^\epsilon s d\phi(s) < \infty \quad \text{for all} \quad \epsilon > 0. \tag{3.5}$$

Introduce the random variable

$$W = \int_0^\infty (N(s) - s) d\phi(s)$$

and note that by (1.3),

$$W = V - \int_0^1 s d\phi(s). \tag{3.6}$$

Again, arguing as in the proof of Theorem 3 in [5], we find that

$$E \exp(tW) = \exp(\xi(t)), \quad t \in I\!R, \tag{3.7}$$

where

$$\xi(t) = \int_0^\infty \{\exp(-t\phi(s)) - 1 + t\phi(s)\} \, ds.$$

Since $|\exp(-v) - 1 + v| \leq v^2 \exp(|v|)/2$ for all $v \in I\!R$, we see that

$$|\xi(t)| \leq \left(\frac{1}{2}\int_0^\infty \phi^2(s)ds\right) t^2 e^{A|t|}, \quad t \in I\!R.$$

By (1.2) this says that, due to the condition $A < \infty$, $E\exp(tW)$ is finite for all $t \in \mathbb{R}$.

Differentiating ξ we obtain

$$\xi'(t) = -\int_0^\infty \phi(s)\{\exp(-t\phi(s)) - 1\}\,ds, \quad t \in \mathbb{R},$$

and differentiating once more,

$$\xi''(t) = \int_0^\infty \phi^2(s)\,\exp(-t\phi(s))ds > 0, \quad t \in \mathbb{R},$$

from which we see that

$$\xi'(t) \downarrow \mu := \int_0^\infty \phi(s)ds \quad \text{as} \quad t \downarrow -\infty,$$

where $-\infty \leq \mu < 0$, $\xi'(0) = 0$, and $\xi'(t) \uparrow \infty$ as $t \uparrow \infty$. For any $\mu < x < \infty$, introduce the inverse to ξ':

$$\xi'(\eta(x)) = x. \tag{3.8}$$

The function η is well defined on (μ, ∞) since ξ' is strictly increasing and continuous. Furthermore, by the inverse function theorem we have

$$\xi''(\eta(x))\eta'(x) = 1, \quad \mu < x < \infty, \tag{3.9}$$

and we know from the above that $\eta(x) > 0$ if and only if $x > 0$.

Now by (3.7) and (3.8), for any $x > 0$,

$$P\{W \geq x\} = P\{\eta(x)W \geq \eta(x)x\} \leq \exp\{\xi(\eta(x)) - \eta(x)\xi'(\eta(x))\}.$$

Observe that

$$\begin{aligned}
\xi(\eta(x)) - \eta(x)\xi'(\eta(x)) &= \int_0^{\eta(x)} \xi'(s)ds - \eta(x)\xi'(\eta(x)) \\
&= -\int_0^{\eta(x)} s\xi''(s)ds \\
&= -\int_0^{x} \eta(t)\xi''(\eta(t))\eta'(t)dt \\
&= -\int_0^{x} \eta(t)dt
\end{aligned}$$

by (3.9). Thus for all $x > 0$,

$$P\{W \geq x\} \leq \exp\left(-\int_0^x \eta(t)dt\right). \tag{3.10}$$

Since $\exp(v) - 1 \leq v \exp(v)$ for all $v > 0$,

$$\xi'(t) \leq \left(\int_0^\infty \phi^2(s)ds\right) t \exp(At) =: Bt \exp(At),\ t > 0,$$

from which it follows by (3.8) that $t \leq B\eta(t) \exp(A\eta(t))$, $t > 0$. But $\eta(t) \uparrow \infty$ as $t \uparrow \infty$, and thus the last inequality implies that for any $0 < \alpha < 1/A$ there exists an $x_0 \geq e$ such that $t \leq \exp(\eta(t)/\alpha)$ for all $t \geq x_0$. This says that $\alpha \log t \leq \eta(t)$ for all $t \geq x_0$. Substituting this bound into (3.10), we obtain for all $x \geq x_0$ that

$$\begin{aligned}
P\{W \geq x\} &\leq \exp\left\{-\int_0^{x_0} \eta(t)dt - \alpha \int_{x_0}^x \log t\, dt\right\} \\
&\leq \exp\left\{-\alpha \int_{x_0}^x \log t\, dt\right\} \\
&\leq \exp(\alpha x \log x).
\end{aligned}$$

By (3.6) this leads immdiately to the statement. ■

(iv) Starting out from (1.3) again, we now use a different decomposition of V : for any $x > 1$ we have $V = U_1(x) + U_2(x)$, where

$$U_1(x) = \int_0^x N(s)d\phi(s) - \phi(x)N(x) + \phi(x)x - \int_1^x sd\phi(s)$$

and

$$U_2(x) = \int_x^\infty (N(s) - s)d\phi(s) + \phi(x)(N(x) - x)$$
$$= \int_x^\infty \{(N(s) - N(x)) - (s - x)\}d\phi(s).$$

Since (1.2) implies that

$$E\left(\int_x^\infty (N(s) - s)d\phi(s)\right)^2 = \int_x^\infty \int_x^\infty \min(s,t)d\phi(s)d\phi(t) \to 0$$

and

$$E(\phi(x)(N(x) - x))^2 = \phi^2(x)x \to 0$$

as $x \to \infty$, we have

$$U_2(x) \to_P 0 \quad \text{as} \quad x \to \infty. \tag{3.11}$$

On the other hand, since N has independent increments, $U_1(x)$ and $U_2(x)$ are independent for each $x > 1$. Therefore, for any $\epsilon > 0$,

$$P\{V < -\int_1^x sd\phi(s) + \phi(x)x + \epsilon\}$$
$$\geq P\{U_1(x) = -\int_1^x sd\phi(s) + \phi(x)x,\ |U_2(x)| < \epsilon\}$$
$$= P\{N(x) = 0\}\ P\{|U_2(x)| < \epsilon\}$$
$$= \exp(-x)\ P\{|U_2(x)| < \epsilon\} > 0$$

for all x large enough.

If $\int_1^\infty sd\phi(s) = \infty$, then this inequality obviously implies that V is unbounded from below.

If $\int_1^\infty sd\phi(s) < \infty$, then

$$\phi(x)x \to 0 \quad \text{as} \quad x \to \infty, \tag{3.12}$$

and hence the above inequality implies that for any $\beta > 0$,

$$P\left\{V < -\int_1^\infty s d\phi(s) + \beta\right\} > 0.$$

At the same time, it follows form (3.11), (3.12), and the law of large numbers that

$$P\left\{V \geq -\int_1^\infty s d\phi(s)\right\} = P\left\{\int_0^\infty N(s) d\phi(s) \geq 0\right\} = 1,$$

and hence the statement. ■

(v) This was in fact proved above, and formally follows from (iv). ■

Proof of Theorem 2. (i) Let $\alpha > 1/A_2$ be given, fix a $\gamma \in (0,1)$, and choose $\epsilon > 0$ so small that $\alpha > \alpha/(1+\epsilon) > \alpha' > 1/A_2$ for some $\alpha > \alpha' > 1/A_2$. Then, applying (i) of Theorem 1 to V_2, for all $x > 0$ large enough we have by (1.4) that

$$\begin{aligned} P\{X > x\} &\geq P\{-V_1 + \sigma Z + \theta - \theta_1 + \theta_2 > -\epsilon x\}\, P\{V_2 > (1+\epsilon)x\} \\ &\geq (1-\gamma) \exp\{-\alpha'(1+\epsilon)x \log((1+\epsilon)x)\}. \end{aligned}$$

Multiplying this by $\exp(\alpha x \log x)$, we see that the resulting lower bound goes to infinity as $x \to \infty$, and hence (2.1) follows.

Next, suppose $0 < A_2 < \infty$, let $0 < \alpha < 1/A_2$ be given, and choose $0 < \epsilon < 1$ so small that $0 < \alpha < \alpha/(1-\epsilon) < \alpha'' < 1/A_2$ for some $\alpha < \alpha'' < 1/A_2$. Then, applying (ii) and (iii) of Theorem 1 to V_2 and V_1, respectively, for any $\tau > 0$ and for all $x > 0$ large enough,

$$\begin{aligned} P\{X \geq x\} &\leq P\{V_2 + \theta - \theta_1 + \theta_2 \geq (1-\epsilon)x\} + P\left\{-V_1 \geq \frac{\epsilon}{2}x\right\} + P\left\{\sigma Z \geq \frac{\epsilon}{2}x\right\} \\ &\leq \exp\{-\alpha''(1-\epsilon)x \log((1-\epsilon)x)\} + \exp(-\tau x^2) + \left(1 - \Phi\left(\frac{\epsilon}{2\sigma}x\right)\right). \end{aligned}$$

Multiplying this by $\exp(\alpha x \log x)$, the resulting upper bound goes to zero as $x \to \infty$, and hence (2.2) follows.

Finally, assume that $A_2 = 0$ and $A_1, \sigma > 0$ and fix $0 < \epsilon < 1$. Let $0 < \epsilon' < \epsilon$ and choose τ so that $\tau > (2\sigma^2)^{-1}$. Then, applying (iii) of Theorem 1 to V_1,

$$\begin{aligned}
P\{X \geq x\} &= P\{-V_1 + \sigma Z + \theta - \theta_1 \geq x\} \\
&\leq P\{-V_1 + \theta - \theta_1 \geq \epsilon' x\} + P\{\sigma Z \geq (1 - \epsilon')x\} \\
&\leq \exp(-\tau x^2) + 1 - \Phi((1 - \epsilon')x/\sigma).
\end{aligned}$$

Multiplying this inequality by $(1 - \Phi((1 - \epsilon)x/\sigma))^{-1}$ and using standard upper and lower estimates for $1 - \Phi(\cdot)$, we see that the resulting upper bound converges to zero as $x \to \infty$. This proves (2.3). ■

(ii) This is completely analogous to (i). ■

(iii) Sufficiency follows from part (v) of Theorem 1. Conversely, suppose that X is bounded from below. Then (2.4) implies that $A_1 = 0$. Obviously there exists a $y > 0$ such that $P\{V_2 < y\} > 0$. thus, for any $x > 0$,

$$\begin{aligned}
P\{X < -x\} &= P\{\sigma Z + V_2 + \theta + \theta_2 < -x\} \\
&\geq P\{\sigma Z + \theta + \theta_2 < -x - y\}\, P\{V_2 < y\} > 0,
\end{aligned}$$

unless $\sigma = 0$. Hence X is distributed as $V_2 + \theta + \theta_2$, and by (v) and (iv) of Theorem 1, we have $\int_1^\infty s\,d\phi_2(s) < \infty$ and

$$ess \inf X = -\int_1^\infty s\,d\phi_2(s) + \theta + \theta_2 = \theta + \int_0^\infty \frac{\phi_2(s)}{1 + \phi_2^2(s)}\,ds. \quad ■$$

(iv) This is completely analogous to (iii). ■

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