FRACTIONAL MOMENTS OF A QUADRATIC FORM IN NONCENTRAL NORMAL RANDOM VARIABLES

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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF TABLES

APPENDIX TABLES

1. INTRODUCTION

1.1 Statement of the Problem

Let the kx1 vector random variable (r.v.) $\underline{X}$ be normally distributed with vector mean $\underline{\xi}$ and variance matrix D, where D is a nonsingular diagonal matrix with diagonal elements σ_i^2, $i = 1,\ldots,k$. Let Q be the quadratic form $\underline{X}'\underline{X}$. In compact notation, we write

$$Q = \underline{X}'\underline{X}, \quad \text{where } \underline{X} \text{ is } N[\underline{\xi}, D].$$

The problem is to determine the fractional moments of Q, i.e., $E(Q^r)$, $r > 0$.

More generally, suppose $\underline{X}_1$ is $N[\underline{\xi}, V]$, and $Q_1 = \underline{X}_1'A\underline{X}_1$, where A is a symmetric nonnegative-definite matrix. If V is nonsingular, it is well known that we may, by means of a nonsingular linear transformation, express Q_1 in the form

$$Q_1 = \Sigma\, a_i u_i \qquad (a_i > 0),$$

where the u_i are independent noncentral chi-square r.v.'s, and the a_i are the nonzero characteristic roots of AV. When so expressed, Q_1 is said to be in canonical form.

In our problem,

$$Q = \underline{X}'\underline{X} = \sum_{i=1}^{k} \sigma_i^2 \left(\frac{X_i}{\sigma_i} \right)^2 ,$$

where the $\left(\frac{X_i}{\sigma_i} \right)^2$ are independent noncentral chi-square r.v.'s, each with one degree of freedom, and the σ_i^2 are, of course, the nonzero characteristic roots of ID. Thus Q is seen to be in canonical form, and hence any result that we might obtain pertaining to Q would pertain to Q_1 as well.

Let $Z = Q^{\frac{1}{2}}$. Clearly, Z may be interpreted as the distance from an appropriate origin to a k-dimensional normal random point. We should like, in particular, to determine the integer moments of Z. The even moments of Z are, of course, the integer moments of Q, while the odd moments of Z are the "half-fractional" moments of Q, *i.e.*, $E(Q^r)$, where r is an odd multiple of $\frac{1}{2}$. Now the characteristic function for Q is known [see, *e.g.*, Patnaik (1949)] and hence the integer moments of Q are essentially known. In effect, then, we are primarily interested in the half-fractional moments of Q, although the treatment will usually be more general.

1.2 Literature Search

A search of the literature suggests an apparent lack of interest in fractional moments of Q; yet, in the past year, the author was twice asked to consider special cases of the problem. In only one paper, an internal report of Space Technology Laboratories by Scheuer (1961), was related work found. Scheuer treats the case $\underline{\xi} = \underline{0}$ and k = 2; for completeness, a brief look at his work is included in Section 3. An almost trivial case, $\underline{\xi} = \underline{0}$ and $D = I\sigma^2$, has undoubtedly been solved many times. But aside from these two special cases, the problem appears not to have been treated.

Perhaps the apparent lack of interest in fractional moments stems, in reality, from the complicated nature of the distribution of Q. Helpful to this study, however, is the fact that explicit expressions for the distribution function of Q have been recently given by Shah and Khatri (1961) and Imhof (1961).

1.3 Distance Problems, in General

The problem stated in the first paragraph of Section 1.1 is quite general in that practically all distance problems in normal r.v.'s can be reduced to that form. To see this, suppose that we are interested in the distance between two points, $\underline{Y}_1$ and $\underline{Y}_2$, in k-dimensional space, where

$$\begin{pmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{pmatrix} \text{ is } N\left[\begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \begin{pmatrix} V_1 & C \\ C' & V_2 \end{pmatrix}\right]. \tag{1.3.1}$$

Clearly, $Q = (\underline{Y}_1 - \underline{Y}_2)'(\underline{Y}_1 - \underline{Y}_2)$, and we have the most general distance problem in normal r.v.'s that can be formulated.

To reduce this problem to the desired form, we first make the linear transformation $\underline{Y} = \underline{Y}_1 - \underline{Y}_2$. Then

$$Q = \underline{Y}'\underline{Y}, \quad \text{and} \quad \underline{Y} \text{ is } N[\underline{\mu}, V], \tag{1.3.2}$$

where $\underline{\mu} = \underline{\mu}_1 - \underline{\mu}_2$ and $V = V_1 + V_2 - C - C'$. We require only that V be nonsingular, and, hence, positive definite. This requirement is certainly met if the variance matrix in (1.3.1) is nonsingular (hence, positive definite). For suppose there exists a vector $\underline{u} \neq 0$ such that $\underline{u}'V\underline{u} = 0$. Then

$$\begin{pmatrix} \underline{u} \\ -\underline{u} \end{pmatrix}' \begin{pmatrix} V_1 & C \\ C' & V_2 \end{pmatrix} \begin{pmatrix} \underline{u} \\ -\underline{u} \end{pmatrix} = \underline{u}'V\underline{u} = 0,$$

which is not possible.

Next we make an orthogonal transformation. It is well known that for every real symmetric matrix A there exists an orthogonal matrix P such that $P'AP = D$, where D is a diagonal matrix whose diagonal elements are the characteristic roots of A. Therefore, let P be such a matrix for V, and let us make the orthogonal transformation $\underline{X} = P'\underline{Y}$. Then

$$Q = \underline{X}'\underline{X}, \quad \text{and} \quad \underline{X} \text{ is } N[\underline{\xi}, D],$$

where $\underline{\xi} = P'\underline{\mu}$, $D = P'VP$, and the reduction is complete.

In theory, at least, finding such an orthogonal matrix P is simple enough. One determines the characteristic roots of V, and a corresponding set of orthonormal characteristic vectors. The characteristic vectors constitute the columns of P. In practice, the computation of P may be tedious if k is large; various computing techniques exist, but we shall not consider the matter further here.

In (1.3.2), we required that V be nonsingular. The reason for this requirement becomes clear if we suppose that V is singular (hence, positive semidefinite). In this case, there exists an orthogonal matrix P_1 such that

$$P_1'V\, P_1 = \begin{pmatrix} D_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix},$$

where D_1 is a diagonal matrix whose diagonal elements are the nonzero characteristic roots of V. If we make the transformation $\underline{X} = P_1'\underline{Y}$, we see that

$$\underline{X} = \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix} \quad \text{is} \quad N\left[\begin{pmatrix} \underline{\xi}_1 \\ \underline{\xi}_2 \end{pmatrix}, \begin{pmatrix} D_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}\right],$$

and

$$Q = \underline{X}'\underline{X} = \underline{X}_1'\underline{X}_1 + \underline{X}_2'\underline{X}_2 = \underline{X}_1'\underline{X}_1 + \underline{\xi}_2'\underline{\xi}_2 .$$

Now, because of the presence of the constant term $\underline{\xi}_2'\underline{\xi}_2$, investigation of the fractional moments of this r.v. is an altogether different problem from that proposed. Furthermore, it appears to be of less practical interest, and, accordingly, we shall not consider it further.

1.4 Organization of the Paper

Because of recently given explicit expressions for the distribution function of Q, mentioned in Section 1.2, it is not difficult to write explicit expressions for fractional moments of Q. Rather, it is in the numerical evaluation of such expressions that one encounters difficulty. After developing, in Section 2, some properties of non-central chi-square required in later parts of the work, we turn in Section 3 to consider two special cases of the more general problem for which exact solutions are not particularly difficult to evaluate numerically. In Section 4, we develop several approximate solutions to the general problem, while, in Section 5, exact solutions to the general problem are considered. In Section 6, comparisons of exact and approximate solutions for a set of numerical examples are given. Section 7 contains a concluding summary of the paper. The work also contains three appendices: Appendix 9.1 contains material relating to the confluent hypergeometric function; Appendix 9.2 gives a convenient approximation for the ratio of two gamma functions; Appendix 9.3 contains a proof of an important result obtained formally in Section 5.

2. NONCENTRAL CHI-SQUARE

2.1 Introduction

In general, we shall be dealing with a quadratic form which is a linear combination of independent noncentral chi-square random variables. Because of this fact, we shall pause briefly to establish a few properties of the noncentral chi-square distribution.

Let the kx1 vector $\underline{X}$ be $N[\underline{\xi}, I]$. It is well known that $\underline{X}'\underline{X}$ is a noncentral chi-square r.v. having k degrees of freedom and noncentrality parameter λ, where, consistent with recent practice, we define $\lambda = \frac{1}{2}\,\underline{\xi}'\underline{\xi}$.

More generally, let U be a noncentral chi-square r.v. having ν degrees of freedom (ν not necessarily an integer) and noncentrality parameter λ. In compact notation, we write

$$U \text{ is } \chi^2(\nu, \lambda).$$

If $\lambda=0$, then U is the well-known central chi-square r.v. [designated here by $\chi^2(\nu, o)$ or simply $\chi^2(\nu)$].

2.2 Probability Density Function, Moment Generating Function, and Cumulants

The probability density function (pdf) of U is variously derived and presented in the literature. Fisher (1928), p. 663, equation (B), obtained it as a particular case of the distribution of the multiple correlation coefficient. He gives (in the present notation)

$$f(u) = \frac{1}{2}\left(\frac{1}{i}\sqrt{\frac{u}{2\lambda}}\right)^{\frac{\nu-2}{2}} e^{-\frac{1}{2}(u+2\lambda)}\, J_{\frac{\nu-2}{2}}\left(i\sqrt{2\lambda u}\right) \qquad (\nu > 0; \lambda, u \geq 0), \tag{2.2.1}$$

where $J_\eta(x)$ is the Bessel function of the first kind of order η. (In Fisher's expression, there is a slight but obvious sign error in the exponent of the exponential.) Equation (2.2.1) is now more popularly given in terms of $I_\eta(x)$, the modified Bessel function of the first kind of order η,

$$f(u) = \tfrac{1}{2}\left(\frac{u}{2\lambda}\right)^{\frac{\nu-2}{4}} e^{-(\lambda+\frac{u}{2})} I_{\frac{\nu}{2}-1}(\sqrt{2\lambda u}) \qquad (\nu > 0;\lambda,u \geq 0). \tag{2.2.2}$$

The pdf of U has long been viewed as being somewhat other than attractive. One recent author, Lindgren (1960), derives

$$f(u) = \frac{u^{\frac{\nu-2}{2}} e^{-(\lambda+\frac{u}{2})}}{\sqrt{2^\nu \pi}} \sum_{i=o}^{\infty} \frac{(2\lambda u)^i \Gamma(i+\frac{1}{2})}{(2i)!\Gamma(\frac{\nu}{2}+i)} \qquad (\nu > 0;\lambda,u \geq 0), \tag{2.2.3}$$

and then states,

> The expression finally obtained for the noncentral chi-square density is still not especially pleasant to behold or to work with. However, various tables are available; . . . .

Had Lindgren made use of the duplication formula for the gamma function,

$$\Gamma(\tfrac{1}{2})\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}),$$

his expression for f(u) could have been reduced to a more appealing one, viz.

$$f(u) = \sum_{i=o}^{\infty}\left[\frac{e^{-\lambda}\lambda^i}{i!}\right]\left[\frac{u^{\frac{\nu+2i}{2}-1} e^{-\frac{u}{2}}}{2^{\frac{\nu+2i}{2}}\Gamma\left(\frac{\nu+2i}{2}\right)}\right] \qquad (\nu > 0;\lambda,u \geq 0) \tag{2.2.4}$$

$$= \sum_{i=o}^{\infty} P_\lambda(i) g_{\nu+2i}(u) \qquad (\nu > 0;\lambda,u \geq 0).$$

In this form, $f(u)$ is not particularly unpleasant to behold. The function $g_{\nu+2i}(u)$ is the pdf of a central chi-square r.v. with $\nu+2i$ degrees of freedom, while its coefficient $P_\lambda(i)$ is the i-th probability term of a Poisson r.v. with parameter λ. Thus the pdf for a noncentral chi-square r.v. is the sum of an infinite number of central chi-square pdf's, each weighted by a term from the Poisson distribution. It appears that Robbins and Pitman (1949) were the first to view the non-central chi-square in this light.

Equation (2.2.4) may be quickly obtained from (2.2.2), and vice versa, with the aid of the following relationship [see Sneddon (1961), p. 127, equation (33.4)]:

$$I_\eta(x) = \left(\frac{x}{2}\right)^\eta \sum_{i=0}^{\infty} \frac{(x^2/4)^i}{i!\,\Gamma(\eta+i+1)} .$$

Moreover, for some purposes, (2.2.4) is not at all unpleasant to work with. Consider, for example, the ease with which we derive the moment generating function (mgf) for U. Formally,

$$M_U(t) = \int_0^\infty e^{ut} \sum_{i=o}^\infty P_\lambda(i) g_{\nu+2i}(u)du$$

$$= \sum_{i=o}^\infty P_\lambda(i) \int_0^\infty e^{ut} g_{\nu+2i}(u)du$$

$$= \sum_{i=o}^\infty P_\lambda(i)\, M_{\chi^2(\nu+2i)}(t)$$

$$= \sum_{i=o}^\infty P_\lambda(i)(1-2t)^{-\frac{\nu+2i}{2}} \qquad (t < \tfrac{1}{2})$$

$$= \frac{e^{-\lambda}}{(1-2t)^{\nu/2}} \sum_{i=o}^\infty \frac{[\lambda/(1-2t)]^i}{i!}$$

$$= \frac{e^{2\lambda t/(1-2t)}}{(1-2t)^{\nu/2}} \qquad (t < \tfrac{1}{2}). \qquad (2.2.5)$$

The change in order of integration and summation is justified, of course, because the integrand is nonnegative. An elegant derivation of (2.2.4) is afforded by direct derivation of $M_U(t)$ from knowledge of the distribution of $\underline{X}$ (i.e., $M_U(t) = E[\exp(\underline{X}'\underline{X}t)]$), followed by straightforward application of the customary inversion formula to $\Phi_U(t) = M_U(it)$.

Readily derived, using the mgf, is the reproductive property of noncentral chi-square, i.e., a sum of independent noncentral chi-square r.v.'s is itself a noncentral chi-square r.v.: specifically,

$$\Sigma\, \chi^2(\nu_i,\lambda_i) = \chi^2(\Sigma\nu_i,\Sigma\lambda_i).$$

We shall have need of the cumulants of U. By definition, the cumulant generating function is the natural logarithm of $M_U(t)$. Thus

$$C_U(t) = \frac{2\lambda t}{1-2t} - \frac{\nu}{2}\log(1-2t)$$

$$= 2\lambda t \sum_{i=o}^{\infty} (2t)^i + \frac{\nu}{2} \sum_{i=1}^{\infty} \frac{(2t)^i}{i} \qquad (|t| < \tfrac{1}{2})$$

$$= \sum_{j=1}^{\infty} \frac{2^{j-1}t^j}{j} (2\lambda j) + \sum_{i=1}^{\infty} \frac{2^{i-1}t^i}{i} (\nu)$$

$$= \sum_{i=1}^{\infty} 2^{i-1} (\nu+2i\lambda) \frac{t^i}{i} \qquad (|t| < \tfrac{1}{2}).$$

Hence, the r-th cumulant of U is seen to be

$$\kappa_r = 2^{r-1}(r-1)!(\nu+2r\lambda) \qquad (r = 1,2,\ldots). \tag{2.2.6}$$

2.3 Moments and Moment Recursion

It is clear that we may obtain integer moments of U from (2.2.5) or (2.2.6). We shall, however, also require fractional moments of U. In general,

$$E(U^r) = \int_o^{\infty} u^r \sum_{i=o}^{\infty} P_\lambda(i) g_{\nu+2i}(u)du$$

$$= \sum_{i=o}^{\infty} P_\lambda(i) \int_o^{\infty} u^r g_{\nu+2i}(u)du$$

$$= \sum_{i=o}^{\infty} P_\lambda(i)\, E[\chi^2(\nu+2i)]^r$$

$$= \sum_{i=o}^{\infty} P_\lambda(i) \left\{ 2^r \frac{\Gamma\left(\frac{\nu+2i}{2} + r\right)}{\Gamma\left(\frac{\nu+2i}{2}\right)} \right\}$$

(At this point, it is interesting to note that for r = 0,1,2,..., the part in braces is a polynomial in i of degree r. Accordingly, the r-th integer moment of a noncentral chi-square r.v. is expressible as a linear combination of the moments of orders 0,1,...,r of a Poisson r.v.)

$$= 2^r e^{-\lambda} \frac{\Gamma(\frac{\nu}{2}+r)}{\Gamma(\frac{\nu}{2})} \left[1 + \frac{(\frac{\nu}{2}+r)}{(\frac{\nu}{2})} \frac{\lambda}{1!} + \frac{(\frac{\nu}{2}+r)(\frac{\nu}{2}+r+1)}{(\frac{\nu}{2})(\frac{\nu}{2}+1)} \frac{\lambda^2}{2!} + \ldots \right]$$

$$= 2^r e^{-\lambda} \frac{\Gamma(\frac{\nu}{2}+r)}{\Gamma(\frac{\nu}{2})} F(\tfrac{\nu}{2}+r;\, \tfrac{\nu}{2};\, \lambda),$$

where F(a;c;x) is the confluent hypergeometric function (Kummer's function), some of whose properties are given in Section 9.1.1 of Appendix 9.1. Using now equation (9.1.1.3) of that section, we have, finally,

$$E(U^r) = 2^r \frac{\Gamma(\frac{\nu}{2}+r)}{\Gamma(\frac{\nu}{2})} F(-r;\, \tfrac{\nu}{2};\, -\lambda) \qquad (r > -\tfrac{\nu}{2}) \,. \qquad (2.3.1)$$

We shall now derive a useful second-order recursion for the moments of U. In equation (9.1.1.9), let $a = -(r+1)$, $c = \frac{\nu}{2}$, and replace x by $-\lambda$ to obtain

$$(\tfrac{\nu}{2}+r+1)F(-r-2;\tfrac{\nu}{2};-\lambda) = (\tfrac{\nu}{2}+\lambda+2r+2)F(-r-1;\tfrac{\nu}{2};-\lambda)-(r+1)F(-r;\tfrac{\nu}{2};-\lambda).$$

In this recursion equation, substitute for $F(\cdot;\frac{\nu}{2};-\lambda)$ from (2.3.1) to obtain

$$(\tfrac{\nu}{2}+r+1)\,\frac{\Gamma(\frac{\nu}{2})\,E(U^{r+2})}{2^{r+2}\Gamma(\frac{\nu}{2}+r+2)} = (\tfrac{\nu}{2}+\lambda+2r+2)\,\frac{\Gamma(\frac{\nu}{2})\,E(U^{r+1})}{2^{r+1}\Gamma(\frac{\nu}{2}+r+1)} - (r+1)\,\frac{\Gamma(\frac{\nu}{2})E(U^{r})}{2^{r}\;\Gamma(\frac{\nu}{2}+r)}\;,$$

which upon simplification reduces to

$$E(U^{r+2}) = [\nu+2\lambda+4(r+1)]E(U^{r+1})-2(r+1)(\nu+2r)E(U^{r}) \qquad (r > -\tfrac{\nu}{2}), \tag{2.3.2}$$

the desired recursive relationship. The results (2.3.1) and (2.3.2) are believed to be new.

2.4 Remarks

Of course, for $\lambda=0$, all numbered relations in this chapter specialize to the corresponding expressions for central chi-square. However, although (2.3.2) specializes to a valid central chi-square moment recursion of order two, there does exist an obvious first-order recursion, viz.

$$E[\chi^2(\nu)]^{r+1} = (\nu+2r)\;E[\chi^2(\nu)]^{r}\;. \tag{2.4.1}$$

Possibly the most troublesome aspect of the noncentral chi-square distribution has been the numerical evaluation of $F(u) = P(U \leq u)$ (as, for instance, when one wishes to investigate the power function for some statistical test based upon chi-square theory). Various more tractable r.v.'s, approximating U in some sense, have been offered with the hope that their distribution functions would serve as acceptably good approximations, for practical purposes, to that of U. Without doubt, Patnaik's (1949) approximation has become the most popular of these (it is also one of the earliest and very likely the simplest).

More recently, an approximation by Pearson (1959) has been gaining favor. In fact, the proposal, justification, and comparison of such approximations have occupied a fair amount of space in statistical journals in the past decade. At last, however, in November 1962, extensive tables of the cumulative noncentral chi-square distribution were published by Haynam, Govindarajulu, and Leone (1962). The tables referenced by Lindgren [*cf*. the quotation following (2.2.3) of this chapter] are those by Fix (1954), a short and limited set.

Fortified by this brief consideration of the properties of noncentral chi-square, let us now consider two special cases of the problem at hand.

3. TWO SPECIAL CASES OF THE GENERAL PROBLEM

3.1 Introduction

For certain choices of the values of the parameters k, $\underline{\xi}$, and D, the general problem may be reduced to one for which the exact solution may be expressed in terms of well-tabled special functions. Therefore, before proceeding to an attack on the general problem, we shall, in this section, consider two special cases; the first is believed to be new, while the second is the work of Scheuer (1961), and its inclusion here is for the purpose of completeness.

3.2 Special Case 1 [k and $\underline{\xi}$ General, $D = I\sigma^2$]

We have that $\underline{X}$ is $N[\underline{\xi}, I\sigma^2]$, and

$$Q = \underline{X}'\underline{X} = \sigma^2 \sum_{i=1}^{k} \left(\frac{X_i}{\sigma}\right)^2$$

$$= \sigma^2 \sum_{i=1}^{k} \chi^2(1,\lambda_i) = \sigma^2\chi^2(k,\lambda),$$

where $\lambda_i = \dfrac{\xi_i^2}{2\sigma^2}$, and $\lambda = \sum_{i=1}^{k} \lambda_i = \dfrac{\underline{\xi}'\underline{\xi}}{2\sigma^2}$. By equation (2.3.1),

$$E(Q^r) = (2\sigma^2)^r \frac{\Gamma(\frac{k}{2} + r)}{\Gamma(\frac{k}{2})} F(-r;\tfrac{k}{2};-\lambda) \qquad (r > -\tfrac{k}{2}), \qquad (3.2.1)$$

and by (2.3.2), we have the following moment recursion:

$$E(Q^{r+2}) = \sigma^2[k+2\lambda+4(r+1)]E(Q^{r+1})-2\sigma^4(r+1)(k+2r)E(Q^r) \qquad (r > -\tfrac{k}{2}),$$

$$(3.2.2)$$

which together with the zero-th and first moments of Q, will generate all integer moments of Q [note that (3.2.1) and (3.2.2) are not confined to integer moments]. Although it is clear that the integer moments of Q may be obtained from (2.2.5) or (2.2.6), it is worth noting that (3.2.2) generates them more conveniently.

For moments of Z $(=Q^{\frac{1}{2}})$, we have

$$E(Z^n) = (\sigma\sqrt{2})^n \frac{\Gamma(\frac{k+n}{2})}{\Gamma(\frac{k}{2})} F(-\tfrac{n}{2};\tfrac{k}{2};-\lambda) \qquad (n > -k), \qquad (3.2.3)$$

and

$$E(Z^{n+4}) = \sigma^2[k+2\lambda+2(n+2)]E(Z^{n+2})-\sigma^4(n+2)(k+n)E(Z^n) \qquad (n > -k). \qquad (3.2.4)$$

Thus (3.2.4), together with the zero-th and first three integer moments of Z, will generate all integer moments of Z. It is worthwhile, perhaps, to remark that, although the variance of Q tends to infinity with k, the variance of Z remains finite; in fact, it is not difficult to show that

$$\mathrm{Var}(Z) \longrightarrow \tfrac{1}{2}\sigma^2 \quad \text{as} \quad k \longrightarrow \infty.$$

Tables of the confluent hypergeometric function are available [see Rushton and Lang (1954) and Slater (1960)]. Alternatively, one may, for k a positive integer and n a positive odd integer, express $F(-\frac{n}{2};\frac{k}{2};-\lambda)$ in terms of more extensively tabled special functions (recall, again, that the case of n even is not of interest because the even moments of Z are the integer moments of Q and these are essentially known). In particular, when k and n are each positive odd integers,

$F(-\frac{n}{2};\frac{k}{2};-\lambda)$ may be expressed in terms of the error function, $\mathrm{erf}(\sqrt{\lambda})$, defined by (9.1.1.6), or in terms of the incomplete gamma function, $\gamma(\frac{1}{2},\lambda)$, defined by (9.1.1.7). When k and n are positive even and odd integers, respectively, $F(-\frac{n}{2};\frac{k}{2};-\lambda)$ may be expressed in terms of modified Bessel functions of the first kind of orders zero and one, alone. Proofs of the claims of the last two sentences constitute Section 9.1.2 of Appendix 9.1.

The first three integer moments of Z are readily seen to be

$$E(Z^1) = \sigma\sqrt{2}\ \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\ F(-\tfrac{1}{2};\tfrac{k}{2};-\lambda),$$

$$E(Z^2) = \sigma^2(k+2\lambda),$$

$$E(Z^3) = (\sigma\sqrt{2})^3\ \frac{\Gamma(\frac{k+3}{2})}{\Gamma(\frac{k}{2})}\ F(-\tfrac{3}{2};\tfrac{k}{2};-\lambda)\ .$$

In Section 9.1.3 of Appendix 9.1, the two confluent hypergeometric functions appearing in the expressions for $E(Z^1)$ and $E(Z^3)$ are given Bessel or error function representations for k = 1,2,3 (i.e. for $\underline{X}$ univariate, circular, or spherical normal). These choices for k will serve for distance problems that one is most apt to encounter of the present type. For k = 1, see (9.1.3.1) and (9.1.3.2); for k = 2, (9.1.3.3) and (9.1.3.4); for k = 3, (9.1.3.5) and (9.1.3.6).

Finally, if $\underline{\xi} = \underline{0}$, then $Q = \sigma^2\chi^2(k,0)$, and special case 1 is reduced to an almost trivial problem. In particular, (3.2.3) specializes to

$$E(Z^n) = (\sigma\sqrt{2})^n \frac{\Gamma(\frac{k+n}{2})}{\Gamma(\frac{k}{2})} \qquad (n > -k),$$

and, rather than (3.2.2), we have, by (2.4.1),

$$E(Z^{n+2}) = \sigma^2(k+n)\, E(Z^n) \qquad (n > -k).$$

In fact, specializing further to the physically meaningful dimensions, we have that

k	$E(Z)$	$Var(Z)$
1	$\sqrt{\frac{2}{\pi}}\,\sigma = 0.7979\sigma$	$(1 - \frac{2}{\pi})\,\sigma^2 = 0.3634\sigma^2$
2	$\sqrt{\frac{\pi}{2}}\,\sigma = 1.2533\sigma$	$(2 - \frac{\pi}{2})\,\sigma^2 = 0.4292\sigma^2$
3	$2\sqrt{\frac{2}{\pi}}\,\sigma = 1.5958\sigma$	$(3 - \frac{8}{\pi})\,\sigma^2 = 0.4535\sigma^2$.

3.3 Special Case 2 [$k = 2$, $\underline{\xi} = \underline{0}$, $D = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$]

We have that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ is } N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right],$$

and without loss of generality, suppose $\sigma_1^2 \geq \sigma_2^2$. We have

$$Q = X_1^2 + X_2^2 = \sigma_1^2\left(\frac{X_1}{\sigma_1}\right)^2 + \sigma_2^2\left(\frac{X_2}{\sigma_2}\right)^2,$$

where $\left(\frac{X_1}{\sigma_1}\right)^2$ and $\left(\frac{X_2}{\sigma_2}\right)^2$ are independent $\chi^2(1)$ r.v.'s. Under a transformation to polar coordinates, say $\frac{X_1}{\sigma_1} = R\cos\Theta$, $\frac{X_2}{\sigma_2} = R\sin\Theta$, it is well known that R^2 is χ^2 (2) and Θ is uniform on $(0,2\pi)$, and that R and Θ are independent r.v.'s. Thus

$$E(Q^r) = E[\sigma_1^2 R^2 \cos^2\Theta + \sigma_2^2 R^2 \sin^2\Theta]^r$$

$$= \sigma_1^{2r} E(R^{2r}) E[1-h^2\sin^2\Theta]^r, \quad \text{where} \quad 0 \leqq h^2 = \frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2} < 1.$$

Clearly,

$$E(R^{2r}) = 2^r\,\Gamma(r+1) \qquad (r > -1).$$

Now

$$E[1+h^2\sin^2\Theta]^r = \frac{1}{2\pi}\int_0^{2\pi}(1-h^2\sin^2\theta)^r d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}(1-h^2\sin^2\theta)^r d\theta$$

$$(t = \sin^2\theta)$$

$$= \frac{1}{\pi}\int_0^1 t^{-\frac{1}{2}}\,(1-t)^{-\frac{1}{2}}\,(1-h^2t)^r dt$$

$$= F(-r,\tfrac{1}{2};1;h^2),$$

where $F(a,b;c;x)$ is the well-known hypergeometric function [see Erdelyi, (1953), p. 59, equation (10), for this integral representation].

We have, then

$$E(Q^r) = (2\sigma_1^2)^r\,\Gamma(r+1)\,F(-r;\tfrac{1}{2};1;h^2) \qquad (r > -1)\ ,$$

and for moments of Z,

$$E(Z^n) = (\sigma_1\sqrt{2})^n\,\Gamma(\tfrac{n}{2}+1)\,F(-\tfrac{n}{2},\tfrac{1}{2};1;h^2) \qquad (n > -2). \tag{3.3.1}$$

In the recursion [see Erdelyi (1953), p. 103, equation (31)]

$$(c-a)F(a-1,b;c;x) = [c-2a-(b-a)x]F(a,b;c;x) + a(1-x)F(a+1,b;c;x), \tag{3.3.2}$$

let $a = -(\frac{n}{2}+1)$, $b = \frac{1}{2}$, $c = 1$, replace x by h^2, and substitute for $F(\cdot,\frac{1}{2};1;h^2)$ from (3.3.1) to obtain the moment recursion

$$E(Z^{n+4}) = \sigma_1^2(n+3)(2-h^2)E(Z^{n+2}) - \sigma_1^4(n+2)^2(1-h^2)E(Z^n) \qquad (n > -2). \tag{3.3.3}$$

The recursion (3.3.3), together with the zero-th and first three integer moments of Z, will yield all integer moments of Z. Now the hypergeometric function is not extensively tabled. Happily, however, it turns out that the odd moments of Z may be expressed in terms of the well-tabled complete elliptic integrals of the first and second kinds, K(h) and E(h), respectively, since

$$K(h) = \frac{\pi}{2}\,F(\tfrac{1}{2},\tfrac{1}{2};1;h^2), \quad E(h) = \frac{\pi}{2}\,F(-\tfrac{1}{2},\tfrac{1}{2};1;h^2) \tag{3.3.4}$$

[see e.g., Sneddon (1961), p. 42, examples (viii) and (ix)]. Thus the first three integer moments of Z are

$$E(Z^1) = \sigma_1 \sqrt{\frac{\pi}{2}}\, F(-\tfrac{1}{2},\tfrac{1}{2};1;h^2)$$

$$= \sigma_1 \sqrt{\frac{2}{\pi}}\, E(h);$$

$$E(Z^2) = \sigma_1^2 + \sigma_2^2;$$

$$E(Z^3) = \sigma_1^3 \sqrt{2\pi}\, [\tfrac{3}{2}\, F(-\tfrac{3}{2},\tfrac{1}{2};1;h^2)],\ \text{and by (3.3.2)},\quad a = -\tfrac{1}{2},$$

$$= \sigma_1^3 \sqrt{2\pi}\, [(2-h^2)F(-\tfrac{1}{2},\tfrac{1}{2};1;h^2) - \tfrac{1}{2}\,(1-h^2)F(\tfrac{1}{2},\tfrac{1}{2};1;h^2)]$$

$$= \sigma_1^3 \sqrt{\frac{2}{\pi}}\, [2(2-h^2)E(h) - (1-h^2)K(h)],\quad \text{by (3.3.4)},$$

$$= \sigma_1 \sqrt{\frac{2}{\pi}}\, [2(\sigma_1^2 + \sigma_2^2)E(h) - \sigma_2^2\, K(h)].$$

This last expression, for $E(Z^3)$, is developed in detail here because it does not agree with that given by Scheuer, the error apparently being his.

3.4 Remarks

These two special cases should serve to answer moment questions for a fair share of the distance problems that one will encounter. Attempts were made to generalize Scheuer's work to higher dimensions (i.e., $k = 3,4,\ldots$), but with little real success. Scheuer's work does not generalize in the two-dimensional case for noncentral r.v.'s because then, of course, the polar transformation variates R and Θ are not independent.

Finally, in this section and elsewhere in the present work, we often encounter expressions of the type

$$\frac{\Gamma(x+r)}{\Gamma(x)} . \tag{3.4.1}$$

When r is not an integer, the numerical evaluation of (3.4.1) can be a nuisance. Presented as Appendix 9.2 is an exceptionally good approximation for (3.4.1).

We next turn to consider several approximate solutions for the more difficult general problem.

4. THE GENERAL PROBLEM, APPROXIMATE SOLUTIONS

4.1 Introduction

In general, the quadratic form Q is a linear combination of independent noncentral chi-square r.v.'s, in contrast to the simpler special cases dealt with in the previous section. In this section we shall consider several approximate solutions for the positive fractional moments of Q, while in the following section exact solutions will be investigated. In Section 6, the quality of the approximate solutions will be judged on the basis of comparisons of the approximate and exact solutions for a set of numerical examples.

As in Section 1, we have that $\underline{X}$ is $N[\underline{\xi}, D]$, where D is diag(σ_i^2), $i = 1,\ldots,k$, and

$$Q = \underline{X}'\underline{X} = \sum_{i=1}^{k} \sigma_i^2 \left(\frac{X_i}{\sigma_i} \right)^2 = \sum_{i=1}^{k} \sigma_i^2 \chi^2(1,\lambda_i),$$

where $\lambda_i = \frac{1}{2}\left(\frac{\xi_i}{\sigma_i} \right)^2$. Again, the integer moments of Q are readily obtained. In particular, by relation (2.2.6), the r-th cumulant of Q is given by

$$\kappa_r = 2^{r-1}(r-1)! \sum_{i=1}^{k} \sigma_i^{2r}(1+2r\lambda_i) \qquad (r = 1,2,3,\ldots). \qquad (4.1.1)$$

Specifically,

$$\left.\begin{aligned}
\kappa_1 &= \mu_1' = \Sigma\ \sigma_i^2(1+2\lambda_i), \\
\kappa_2 &= \mu_2 = 2\ \Sigma\ \sigma_i^4(1+4\lambda_i), \\
\kappa_3 &= \mu_3 = 8\ \Sigma\ \sigma_i^6(1+6\lambda_i), \\
\kappa_4 &= \mu_4 - 3\mu_2^2 = 48\ \Sigma\ \sigma_i^8(1+8\lambda_i)\ .
\end{aligned}\right\} \qquad (4.1.2)$$

However, there seems to be no easy way to obtain fractional moments of Q.

Following Patnaik (1949) and Pearson (1959), we might try approximating Q by more tractable central chi-square related r.v.'s, using the method of matching moments. On the other hand, because of the new "machinery" developed in Section 2 [specifically, (2.3.1) and (2.3.2)], we might try to extend the Patnaik and Pearson approximations by fitting more general, noncentral chi-square related, r.v.'s. The approximating r.v.'s would be of the following forms:

$Y_1 = a_1\chi^2(\nu_1)$---------2-moment approx. (Patnaik)

$Y_2 = a_2\chi^2(\nu_2)+b_1$------3-moment approx. (Pearson)

$Y_3 = a_3\chi^2(\nu_3,\zeta_1)$------3-moment approx. (Patnaik extension)

$Y_4 = a_4\chi^2(\nu_4,\zeta_2)+b_2$---4-moment approx. (Pearson extension)

We should like to determine the parameters of each of the above approximating r.v.'s, and an explicit expression for a general moment of each.

4.2 Patnaik's Approximation $[Y_1 = a_1\chi^2(\nu_1)]$

To determine the parameters a_1 and ν_1, Patnaik equates the first two cumulants of Q, given by (4.1.2), to the first two cumulants of Y_1, and solves to obtain

$$a_1 = \frac{\kappa_2}{2\kappa_1} \quad \text{and} \quad \nu_1 = \frac{2\kappa_1^2}{\kappa_2} .$$

Then, for the moments of Y_1, we have by (2.3.1),

$$E(Y_1^r) = E[a_1\chi^2(\nu_1)]^r = \left(\frac{\kappa_2}{\kappa_1}\right)^r \frac{\Gamma\left(\frac{\kappa_1^2}{\kappa_2} + r\right)}{\Gamma\left(\frac{\kappa_1^2}{\kappa_2}\right)} \qquad \left(r > -\frac{\kappa_1^2}{\kappa_2}\right), \tag{4.2.1}$$

and by (2.4.1),

$$E(Y_1^{r+1}) = a_1(\nu_1+2r)E(Y_1^r) \qquad \left(r > -\frac{\nu_1}{2}\right) . \tag{4.2.2}$$

Patnaik (1949) offered his two-moment central chi-square approximation primarily for the purpose of evaluating the distribution function of a single noncentral chi-square r.v. At the same time, however, he pointed out that Y_1 might also be used to approximate a linear combination of independent noncentral chi-square r.v.'s. His approximation is well known and widely used due to its simplicity; however, the recent publication of extensive tables of the cumulative noncentral chi-square distribution by Haynam, et al. (1962) makes the primary intended use of Y_1 passe. More recently, Pearson (1959) offered his three-moment central chi-square approximation Y_2, again for the same primary purpose as Patnaik's approximation, and again, for this purpose, it too is passé. Imhof (1961) used Pearson's approximation to evaluate $F(u) = P\{\Sigma a_i \chi^2(\nu_i,\lambda_i) \leqq u\}$ and found, as did Johnson (1959) for $F(u) = P\{\chi^2(\nu,\lambda) \leqq u\}$, that in general Pearson's approximation is superior to Patnaik's.

4.3 Pearson's Approximation [$Y_2 = a_2\chi^2(\nu_2)+b_1$]

Pearson considers the r.v.

$$Y_2 = \left(\frac{\chi^2(\nu_2) - \nu_2}{\sqrt{2\nu_2}}\right)\sqrt{\kappa_2} + \kappa_1 .$$

Clearly, the first two cumulants of Y_2 agree with the first two cumulants of Q. Equating the third cumulants of Y_2 and Q, he obtains

$$8\nu_2\left(\frac{\kappa_2}{2\nu_2}\right)^{\frac{3}{2}} = \kappa_3$$

yielding

$$\nu_2 = 8\,\frac{\kappa_2^3}{\kappa_3^2} . \tag{4.3.1}$$

It follows that

$$a_2 = \left(\frac{\kappa_2}{2\nu_2}\right)^{\frac{1}{2}} = \frac{\kappa_3}{4\kappa_2} , \tag{4.3.2}$$

$$b_1 = \kappa_1 - a_2\nu_2 = \kappa_1 - \frac{2\kappa_2^2}{\kappa_3} . \tag{4.3.3}$$

Now this approximation is fully applicable for the approximate evaluation of $F(q) = P(Q \leqq q)$; however, we shall soon see that b_1 must be restricted to nonnegative values for the purpose of approximating fractional moments of Q. Moreover, because of the presence of b_1, it is clear that our prior knowledge of the moments of central chi-square will be of little value in determining $E(Y_2^r)$ when r is not an integer.

We have $Y_2 = a_2\chi^2(\nu_2)+b_1$. The pdf for Y_2 is readily shown to be

$$g(y) = \frac{(y-b_1)^{\frac{\nu_2}{2}-1} e^{-\frac{1}{2a_2}(y-b_1)}}{\Gamma\left(\frac{\nu_2}{2}\right)(2a_2)^{\nu_2/2}} \qquad (y \geqq b_1) .$$

Hence,

$$E(Y_2^r) = \int_{b_1}^{\infty} y^r g(y)dy,$$

and now it is clear that b_1 must be restricted to nonnegative values in order to assure that $E(Y_2^r)$ be real when r is fractional. When $b_1 = 0$,

$$E(Y_2^r) = (2a_2)^r \frac{\Gamma\left(\frac{\nu_2}{2}+r\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} \qquad \left(r > -\frac{\nu_2}{2}\right) .$$

Less trivially, suppose $b_1 > 0$. Making the transformation $t = \frac{y-b_1}{b_1}$, we obtain

$$E(Y_2^r) = \frac{b_1^r}{\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{b_1}{2a_2}\right)^{\frac{\nu_2}{2}} \int_0^{\infty} t^{\frac{\nu_2}{2}-1} (t+1)^r e^{-\left(\frac{b_1}{2a_2}\right)t} dt$$

$$= b_1^{\frac{\nu_2}{2}+r} (2a_2)^{-\frac{\nu_2}{2}} U\left(\frac{\nu_2}{2};\ r+\frac{\nu_2}{2}+1;\ \frac{b_1}{2a_2}\right) \qquad (|r| < \infty), \tag{4.3.4}$$

where the function $U(a;c;x)$ is also known as a confluent hypergeometric function [see Slater (1960), p. 38, equation (3.1.19), for its integral representation]. This function is expressible in terms of the simpler Kummer's function [see (9.1.1.15)]. Although (4.3.4) exists for all real r, this fact does not, of course, mean that (4.3.4) may be regarded as a suitable approximation for all moments of Q.

In the recursion (9.1.1.16), let $a = \frac{\nu_2}{2}$, $c = r + \frac{\nu_2}{2} + 2$, replace x by $b_1/2a_2$, and substitute for $U\left(\frac{\nu_2}{2}\,;\,\cdot\,;\,\frac{b_1}{2a_2}\right)$ from (4.3.4) to obtain, after simplification, the moment recursion

$$E(Y_2^{r+2}) = [2a_2\left(r+\frac{\nu_2}{2}+1\right) + b_1]E(Y_2^{r+1})-2a_2b_1(r+1)E(Y_2^r) \quad (|r| < \infty). \tag{4.3.5}$$

Consider now the implication of the imposed restriction that $b_1 \geqq 0$, which by (4.3.3) is equivalent to

$$\kappa_1\kappa_3 - 2\kappa_2^2 \geqq 0 \; . \tag{4.3.6}$$

Thus, only when (4.3.6) holds will the Pearson approximation be applicable for approximating fractional moments of Q.

We next consider a logical extension of Patnaik's approximation.

4.4 Patnaik-extension Approximation [$Y_3 = a_3\chi^2(\nu_3,\zeta_1)$]

Rather than solving for the parameter a_3, we shall solve for $K = \frac{1}{a_3}$. Equating the first three cumulants of KQ and $\chi^2(\nu_3,\zeta_1)$ we obtain the system

$$\left.\begin{aligned} K\kappa_1 &= \nu_3 + 2\zeta_1, &\quad (a)\\ K^2\kappa_2 &= 2\nu_3 + 8\zeta_1, &\quad (b)\\ K^3\kappa_3 &= 8\nu_3 + 48\zeta_1. &\quad (c)\end{aligned}\right\} \qquad (4.4.1)$$

Observe that, with respect to the right-hand sides of (4.4.1), 8[(b)-(a)] = (c), yielding a quadratic equation in K with respect to the left-hand sides, viz.

$$\kappa_3 K^2 - 8\kappa_2 K + 8\kappa_1 = 0.$$

Solving this system of equations, we get

$$K = \frac{2}{\kappa_3}\left[2\kappa_2 \pm \sqrt{4\kappa_2^2 - 2\kappa_1\kappa_3}\right], \qquad (4.4.2)$$

$$\nu_3 = \frac{K}{2}\,[4\kappa_1 - K\kappa_2], \qquad (4.4.3)$$

$$\zeta_1 = \frac{K}{4}\,[K\kappa_2 - 2\kappa_1]\ . \qquad (4.4.4)$$

Clearly, we must require that $\nu_3 > 0$, $\zeta_1 \geqq 0$, and that the radicand in (4.4.2) be nonnegative. Note, with some anticipation, that this last requirement is a happy complementation to (4.3.6), i.e., here we require that

$$2\kappa_2^2 - \kappa_1\kappa_3 \geqq 0. \qquad (4.4.5)$$

With the inequalities just imposed on ν_3 and ζ_1, we obtain from (4.4.3) and (4.4.4) the following inequalities on K:

$$2\kappa_1 \leqq K\kappa_2 < 4\kappa_1\ . \qquad (4.4.6)$$

Substituting for K, from (4.4.2), in (4.4.6) we obtain

$$\kappa_1\kappa_3 \leqq 2\kappa_2^2 \pm \kappa_2\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} < 2\kappa_1\kappa_3. \tag{4.4.7}$$

We shall now show that the smaller solution for K is inadmissible, being always below the lower bound.

Suppose

$$\kappa_1\kappa_3 \leqq 2\kappa_2^2 - \kappa_2\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} .$$

Multiplying through by -2 and adding $4\kappa_2^2$ to each side we obtain

$$4\kappa_2^2 - 2\kappa_1\kappa_3 \geqq 2\kappa_2\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} ,$$

or

$$\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} \geqq 2\kappa_2 ,$$

which is not possible because, by (4.1.1), the κ_r are all strictly positive.

For the larger solution for K, it is easily shown that the left-hand inequality of (4.4.7) is always satisfied. We shall now show that the right-hand inequality of (4.4.7) is also satisfied. We have to show that

$$2\kappa_2^2 + \kappa_2\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} < 2\kappa_1\kappa_3 . \tag{4.4.8}$$

To establish (4.4.8), suppose (as will be proved shortly) that

$$2\kappa_1\kappa_3 - 3\kappa_2^2 > 0. \tag{4.4.9}$$

Then

$$4\kappa_2^2 - 2\kappa_1\kappa_3 = \kappa_2^2 + (3\kappa_2^2-2\kappa_1\kappa_3) < \kappa_2^2 .$$

Hence

$$\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} < \kappa_2 .$$

Then

$$2\kappa_1\kappa_3 > 3\kappa_2^2 = 2\kappa_2^2 + \kappa_2^2 > 2\kappa_2^2 + \kappa_2\sqrt{4\kappa_2^2-2\kappa_1\kappa_3} .$$

Thus (4.4.8) is established, given (4.4.9).

To prove (4.4.9), we substitute for κ_1, κ_2, and κ_3 from (4.1.2), obtaining

$$\sum_{i,j=o}^{k} f(i,j) > 0, \qquad (4.4.10)$$

where

$$f(i,j) = 4\sigma_i^2\sigma_j^6(1+2\lambda_i)(1+6\lambda_j) - 3\sigma_i^4\sigma_j^4(1+4\lambda_i)(1+4\lambda_j).$$

The truth of (4.4.10) is readily established. First, for i=j, we have

$$f(i,i) = \sigma_i^8(1+8\lambda_i) > 0,$$

and hence

$$\sum_{i=1}^{k} f(i,i) > 0.$$

To sweep through the remaining k(k-1) terms, consider jointly the index pairs (i,j) and (j,i), i≠j. We have

$$f(i,j) + f(j,i) = \sigma_i^2\sigma_j^2 \Big\{ 4(1+2\lambda_i+6\lambda_j+12\lambda_i\lambda_j)\sigma_j^4 +$$

$$4(1+2\lambda_j+6\lambda_i+12\lambda_i\lambda_j)\sigma_i^4 - 6(1+4\lambda_i+4\lambda_j+16\lambda_i\lambda_j)\sigma_i^2\sigma_j^2 \Big\} .$$

Now it is not difficult to show that the part in braces is a positive-definite quadratic form in σ_i^2 and σ_j^2. Hence,

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{k} f(i,j) > 0.$$

Thus the truth of inequality (4.4.10) is established, and with it the admissibility of the larger solution for K.

By way of a summary for approximation Y_3, we have

$$\left.\begin{aligned}
Y_3 &= \frac{1}{K}\chi^2(\nu_3,\zeta_1),\\
&\text{where}\\
K &= \frac{2}{\kappa_3}\left[2\kappa_2 + \sqrt{4\kappa_2^2 - 2\kappa_1\kappa_3}\right],\\
\nu_3 &= \frac{K}{2}\left[4\kappa_1 - K\kappa_2\right],\\
\zeta_1 &= \frac{K}{4}\left[K\kappa_2 - 2\kappa_1\right].
\end{aligned}\right\} \qquad (4.4.11)$$

We recall that approximation Y_3 is applicable for approximating moments of Q only when (4.4.5) is satisfied. Then, by (2.3.1),

$$E(Y_3^r) = E[\frac{1}{K}\chi^2(\nu_3,\zeta_1)]^r$$

$$= \left(\frac{2}{K}\right)^r \frac{\Gamma\left(\frac{\nu_3}{2}+r\right)}{\Gamma\left(\frac{\nu_3}{2}\right)} F(-r;\frac{\nu_3}{2};-\zeta_1) \qquad \left(r > -\frac{\nu_3}{2}\right), \qquad (4.4.12)$$

and by (2.3.2),

$$E(Y_3^{r+2}) = \frac{1}{K}[\nu_3+2\zeta_1+4(r+1)]E(Y_3^{r+1}) - \frac{2}{K^2}(r+1)(\nu_3+2r)E(Y_3^r) \qquad \left(r > -\frac{\nu_3}{2}\right). \qquad (4.4.13)$$

Looking again at the complementary nature of the inequalities (4.3.6) and (4.4.5), it is easily verified that, under equality in each, Y_2 and Y_3 specialize to the same r.v. (in fact, to Y_1). Thus, combining Y_2 and Y_3, with their respective regions of applicability, we obtain a single three-moment approximation of full applicability for approximating moments of Q. Letting Y be the resulting r.v., we have

$$Y = \begin{cases} Y_2, & \text{if } \Delta < 0, \\ Y_3, & \text{if } \Delta \geqq 0, \end{cases} \tag{4.4.14}$$

where

$$\Delta = 2\kappa_2^2 - \kappa_1\kappa_3 , \tag{4.4.15}$$

the discriminant of applicability.

At this point, the following conjecture regarding the distribution function of Q is suggested.

<u>Conjecture</u>: For the purpose of approximating $F(q) = P(Q \leqq q)$, the r.v. Y is, in general, superior to Y_2 [recall that Y_2 is fully applicable for the approximate evaluation of F(q)]. In particular,

$$F(q) \doteq P(Y \leqq q) = \begin{cases} P\left\{ \chi^2(\nu_2) \leqq \dfrac{q-b_1}{a_2} \right\}, & \Delta < 0, \\ P\left\{ \chi^2(\nu_3, \zeta_1) \leqq \dfrac{q}{a_3} \right\}, & \Delta \geqq 0. \end{cases} \tag{4.4.16}$$

This conjecture will be tested by numerical examples in Section 6.

4.5 Pearson-extension Approximation $[Y_4=a_4\chi^2(\nu_4,\zeta_2)+b_2]$

An approximation of this nature, as it turns out, is much too involved to warrant serious consideration. Moreover, it would necessarily have restricted regions of applicability for approximating either fractional moments or the distribution function of Q. Furthermore, we could hardly expect to find a counterpart of Y_3 to serve in complementary fashion to Y_4 as Y_3 does to Y_2. Accordingly, approximation Y_4 will not be considered further.

4.6 Summary

In summary, then, for purposes of approximating fractional moments of Q, we have two approximations of full applicability. The first, a two-moment approximation based upon Patnaik's r.v., Y_1, *viz*.

$$E(Q^r) \doteq E(Y_1^r), \tag{4.6.1}$$

where $E(Y_1^r)$ is given by (4.2.1). The second, a three-moment approximation based in part upon Pearson's r.v., Y_2, and in part upon the newly proposed r.v. Y_3, *i.e*.

$$E(Q^r) \doteq E(Y^r) = \begin{cases} E(Y_2^r), & \Delta < 0, \\ \\ E(Y_3^r), & \Delta \geqq 0, \end{cases} \tag{4.6.2}$$

where $E(Y_2^r)$, $E(Y_3^r)$, and Δ are given by (4.3.4), (4.4.12), and (4.4.15), respectively.

In order to judge the quality of the approximations (4.6.1) and (4.6.2), we now turn to consider exact solutions for $E(Q^r)$, even though their numerical evaluation may be difficult.

5. THE GENERAL PROBLEM, EXACT SOLUTIONS

5.1 Introduction

In the search for exact solutions to the general problem, we are aided considerably by the work presented in two recent papers. Shah and Khatri (1961) and Imhof (1961) give the distribution function for a nonnegative-definite quadratic form in noncentral normal r.v.'s. Thus, by definition,

$$E(Q^r) = \int_0^{\infty} q^r dF(q) \tag{5.1.1}$$

where $F(q) = P\left\{ Q \leqq q \right\}$. Alternatively,

$$E(Q^r) = \int_0^{\infty} [1-F(q^{1/r})]dq \qquad (r > 0) \tag{5.1.2}$$

where (5.1.2) is a straightforward generalization of a corresponding relation for $r = 1$ given by Parzen (1960), p. 211, equation (2.39). The choice between (5.1.1) and (5.1.2) would be based upon efficiency of numerical evaluation.

5.2 F(q) as Given by Shah and Khatri

In the notation used here, Shah and Khatri (1961) give F(q) as an infinite double series:

$$F(q) = \left(\prod_{i=1}^{k} \sigma_i^2 \right) e^{-\lambda} \sum_{i,j=0}^{\infty} \frac{(-1)^i 2^{j-i} q^{i+j} E[Q*^i L*^{2j}]}{i!(2j)!\Gamma(i+j+\frac{k}{2}+1)} \tag{5.2.1}$$

where $\lambda = \frac{1}{2} \sum_1^k \left(\frac{\xi_i}{\sigma_i} \right)^2$, $Q* = \frac{1}{2} \sum_1^k \frac{(x_i - \xi_i)^2}{\sigma_i^4}$, and $L* = \frac{1}{\sqrt{2}} \sum_1^k \frac{(x_i - \xi_i)\xi_i}{\sigma_i^3}$.

They show that the series in (5.2.1) is absolutely convergent, and state that, in practice, one would compute a finite number of terms and then give upper and lower bounds on $F(q)$. We require an "exact" evaluation of $F(q)$, i.e., for any preassigned small quantity $\delta > 0$, we must be able to evaluate enough terms so that the absolute value of the error committed is not greater than δ. Success in this regard would depend sharply upon our being able to supply the joint moments of Q^* and L^*.

Shah and Khatri state, "The joint moments of Q^* and L^* are easy to obtain from the joint cumulants of Q^* and L^*." Now the joint cumulants of Q^* and L^* are readily obtained, and, hence, theoretically at least, the joint moments may be obtained. However, bivariate moments expressed in terms of bivariate cumulants have been tabled only for μ'_{ij}, $i+j \leqq 6$ [see Cook (1951)]. Such a limited tabulation would hardly suffice for our purposes. Moreover, in view of the present techniques discussed in the literature [see, e.g., Kendall and Stuart (1958)], we would have to regard the prospect of generating further moment-cumulant relationships as quite unattractive. Hence, it would seem that we must either abandon further consideration of (5.2.1) in conjunction with either (5.1.1) or (5.1.2), or we must find a simpler device for generating moments in terms of cumulants. A simpler device was discovered, and it was to have been presented here as an appendix, but, as it turns out, the discovery is essentially a "rediscovery" of a device published by David (1952).

5.3 F(q) as Given by Imhof

Again, in the notation used here, Imhof (1961) gives F(q) as an integral,

$$F(q) = \tfrac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\sin[A(t) - \frac{tq}{2}]}{t\, \rho(t)}\, dt, \qquad (5.3.1)$$

where

$$A(t) = \tfrac{1}{2} \sum_{i=1}^{k} \left[\nu_i \tan^{-1}(\sigma_i^2 t) + \frac{\xi_i^2 t}{1+\sigma_i^4 t^2} \right], \qquad (5.3.2)$$

and

$$\rho(t) = \left[\prod_{i=1}^{k} (1+\sigma_i^4 t^2)^{\frac{\nu_i}{4}} \right] \exp\left(\tfrac{1}{2} \sum_{i=1}^{k} \frac{(\sigma_i \xi_i t)^2}{1+\sigma_i^4 t^2} \right) \qquad (5.3.3)$$

(note that $\rho(0) = 1$, and $\rho(t)$ is a monotonic increasing function), where the ν_i are the degrees of freedom associated with chi-square r.v.'s in the quadratic form

$$Q^* = \sum_{i=1}^{k} \sigma_i^2 \chi^2(\nu_i, \lambda_i) \qquad (\nu_i > 0,\ i = 1, \ldots, k).$$

Clearly, $Q^* = Q$ when the ν_i are set equal to unity, and for greater generality in work to follow, the ν_i are retained. Actually, Imhof's expression for the distribution function, serves for a wider class of quadratic forms than we consider here [see Imhof's paper].

Imhof further gives

$$f(q) = \frac{dF(q)}{dq} = \frac{1}{2\pi} \int_0^{\infty} \frac{\cos[A(t) - \frac{1}{2}tq]}{\rho(t)}\, dt, \qquad (5.3.4)$$

although, possibly due to a printing error, the factor $\frac{1}{2}$ before the integral sign was not given. Primarily because of the convergence factor $\frac{1}{t}$ present in the integrand of (5.3.1), but absent in (5.3.4), the moment expression (5.1.2) would be preferable to (5.1.1), should we choose to use Imhof's expression for F(q) rather than that given by Shah and Khatri.

5.4 The Choice

Thus, to evaluate $E(Q^r)$, at least at the present stage of development, we have a choice of the following nature: (i) we may write a program for the numerical evaluation of the integral of an infinite double series, with a side program for evaluating the joint moments of Q^* and L^*, or (ii) we may write a program for evaluating what is essentially a double integral. Before choosing between (i) and (ii), we note that the absolutely convergent double series (5.2.1) may, of course, be expressed as a single series. Even so, it seems clear that the advantage rests with (ii) rather than (i).

As it turns out, the choice is well made because, as we shall now see, the double integral may be reduced to a single integral.

5.5 Reduction to a Single Integral (I)

Upon substituting (5.3.1) into (5.1.2), we obtain

$$\pi E(Q^r) = \int_0^\infty \left[\frac{\pi}{2} + \int_0^\infty \frac{\sin[A(t) - \frac{t}{2} q^{1/r}]}{t\,\rho(t)}\,dt\right] dq. \tag{5.5.1}$$

Now, because q appears in the integrand of (5.5.1) in a particularly simple way, we might hope that by changing the order of integration, we would be able to perform the q integration analytically and thus reduce (5.5.1) to a single integral.

With this idea in mind, we should like to replace the $\frac{\pi}{2}$ in (5.5.1) by an integral with respect to t of some function of q and t. It turns out that

$$\int_0^\infty \frac{\sin(\frac{t}{2}\, q^{1/r})}{t}\, dt, \tag{5.5.2}$$

which is equal to $\frac{\pi}{2}$ for all $q > 0$, is a good choice. Making the substitution and employing the rule for the sine of a difference between two angles, we obtain

$$\pi E(Q^r) = \int_0^\infty \int_0^\infty [f_1(t)\cos(\tfrac{t}{2}\, q^{1/r}) + f_2(t)\sin(\tfrac{t}{2}\, q^{1/r})]\, dt\, dq, \tag{5.5.3}$$

where

$$f_1(t) = \frac{\sin A(t)}{t\rho(t)} \quad \text{and} \quad f_2(t) = \frac{1}{t}\left(1 - \frac{\cos A(t)}{\rho(t)}\right). \tag{5.5.4}$$

Proceeding now in a formal way, we have that

$$\pi E(Q^r) = \int_0^\infty \left[f_1(t) \int_0^\infty \cos(\tfrac{t}{2}\, q^{1/r})\, dq + f_2(t) \int_0^\infty \sin(\tfrac{t}{2}\, q^{1/r})\, dq \right] dt$$

$$\overset{(x=q^{1/r})}{=} \int_0^\infty r \left\{ f_1(t) \int_0^\infty x^{r-1} \cos(\tfrac{t}{2}\, x)\, dx \right.$$

$$\left. + f_2(t) \int_0^\infty x^{r-1} \sin(\tfrac{t}{2}\, x)\, dx \right\} dt. \tag{5.5.5}$$

However, the two integrals inside the braces exist only for $0 < r < 1$ [see e.g., Gröbner and Hofreiter (1961), p. 118, (10a) and (10b)]. Of particular interest in this study, though, is the case $r = \frac{1}{2}$.

For $r = \frac{1}{2}$, we obtain the result

$$\pi E(Q^{\frac{1}{2}}) = \int_0^{\infty} \frac{\Gamma(\frac{3}{2})}{t^{\frac{1}{2}}} [f_1(t)+f_2(t)]dt \tag{5.5.6}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^{\infty} \frac{\rho(t)+\sin A(t)-\cos A(t)}{t^{3/2}\,\rho(t)}\, dt. \tag{5.5.7}$$

It can be proved that (5.5.1), with $r = \frac{1}{2}$, is in fact equal to (5.5.6), however, we shall not pause now to do so. Rather, we shall proceed to a more general result and prove that result rigorously.

5.6 Reduction to a Single Integral (II)

The results presented in this section follow in elegant fashion as a direct consequence of two key suggestions offered by Mr. A. A. Fojt.

Again, we shall proceed in a formal way, without regard for the validity of steps taken, deferring a rigorous proof of the end result until later.

We have seen that the integrals inside the braces in (5.5.5) do not exist, in particular, for $r \geq 1$. Mr. Fojt suggested that the difficulty might be circumvented by temporarily introducing some convergence factor, say $e^{-\alpha x}$, with $\alpha > 0$. Then

$$\left.\begin{aligned}\int_0^\infty x^{r-1}e^{-\alpha x}\sin(\tfrac{t}{2}\,x)dx &= \frac{\Gamma(r)}{\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}}\sin(r\phi),\\ \int_0^\infty x^{r-1}e^{-\alpha x}\cos(\tfrac{t}{2}\,x)dx &= \frac{\Gamma(r)}{\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}}\cos(r\phi),\end{aligned}\right\} \quad (5.6.1)$$

where $\phi = \tan^{-1}(\frac{t}{2\alpha})$. [See, e.g., Gröbner and Hofreiter (1961), p. 139, (5a) and (5b).] Now, with these results, we might hope that

$$\pi E(Q^r) = \lim_{\alpha \to 0} \int_0^\infty \frac{\Gamma(r+1)}{\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}} [f_1(t)\cos(r\phi)+f_2(t)\sin(r\phi)]dt. \quad (5.6.2)$$

If passage to the limit under the integral sign were made at this point, we would get

$$\int_0^\infty \frac{2^r\Gamma(r+1)}{t^r} [f_1(t)\cos(r\,\tfrac{\pi}{2}) + f_2(t)\sin(r\,\tfrac{\pi}{2})]dt, \quad (5.6.3)$$

and it is readily seen that, for $r = \frac{1}{2}$, this result agrees with (5.5.6). However, for $r \geqq 1$, we still have problems near the origin ($t=0$). It is not difficult to see that, for t small enough,

$$\left.\begin{aligned}\sin A(t) &= ct + O(t^3), \text{ c a constant},\\ \cos A(t) &= 1 + O(t^2),\\ \rho(t) &= 1 + O(t^2).\end{aligned}\right\} \quad (5.6.4)$$

It follows that

$$f_1(t) = O(1), \text{ and } f_2(t) = O(t) \quad (t \text{ small}). \quad (5.6.5)$$

Clearly, the integral in (5.6.3) would be well behaved, in the neighborhood of $t = 0$, if the integrand acts like $\frac{1}{t^{1-\theta}}$, $\theta > 0$. For example, suppose $r = \frac{3}{2}$. Then the integrand behaves like $t^{-3/2}$, the trouble stemming from the first term alone; for $t = \frac{5}{2}$, we have trouble stemming from the second term as well.

Mr. Fojt now suggested that the difficulty might be avoided by subtracting from $f_1(t)$ and $f_2(t)$ the leading terms in their power series expansions for t small, these terms to be treated separately.

To establish the nature of the $f_1(t)$ and $f_2(t)$ power series expansions for t small, we shall find it helpful to obtain first the characteristic function for $Q^*/2$.

$$\Phi_{\frac{Q*}{2}}(t) = M_{Q*}\left(\frac{it}{2}\right) = \prod_{j=1}^{k} M_{\chi^2(\nu_j,\lambda_j)}\left(\tfrac{1}{2}\, i\sigma_j^2 t\right)$$

$$= \frac{\exp\left(\sum_{j=1}^{k} \frac{i\lambda_j\sigma_j^2 t}{1-i\sigma_j^2 t}\right)}{\prod_{j=1}^{k} (1-i\sigma_j^2 t)^{\frac{\nu_j}{2}}}$$

$$= \frac{\exp\left(\frac{1}{2} \sum_{j=1}^{k} \frac{i\xi_j^2 t}{1-i\sigma_j^2 t}\right)}{\prod_{j=1}^{k} (1-i\sigma_j^2 t)^{\frac{\nu_j}{2}}}, \qquad (5.6.6)$$

where $M_{\chi^2(\nu,\lambda)}(t)$ is given by (2.2.5), and $\xi_j^2 = 2\sigma_j^2\lambda_j$. It turns out that

$$\left.\begin{aligned} &\left|\Phi_{\frac{Q*}{2}}(t)\right| = \frac{1}{\rho(t)}, \\ \text{and} \qquad & \\ &\arg\left[\Phi_{\frac{Q*}{2}}(t)\right] = A(t). \end{aligned}\right\} \tag{5.6.7}$$

Although Imhof did not point out explicitly the relations (5.6.7), it is clear that he was aware of them. He does, in fact, give other relations that enable us to verify readily (5.6.7). It follows from (5.6.7) that

$$\Phi_{\frac{Q*}{2}}(t) = \frac{e^{iA(t)}}{\rho(t)} = \frac{\cos A(t)+i\sin A(t)}{\rho(t)} \quad . \tag{5.6.8}$$

By definition,

$$\Phi_{\frac{Q*}{2}}(t) = \sum_{j=o}^{\infty} \frac{\mu_j'}{j!} (i\, \tfrac{t}{2})^j, \tag{5.6.9}$$

where $\mu_j' = E(Q*^j)$.

It follows then, from (5.6.8) and (5.6.9), that

$$\left.\begin{aligned} &\frac{\sin A(t)}{\rho(t)} = \sum_{\gamma=o}^{\infty} (-1)^{\gamma} \frac{\mu_{2\gamma+1}'}{(2\gamma+1)!} (\tfrac{t}{2})^{2\gamma+1}, \\ \text{and} \qquad & \\ &\frac{\cos A(t)}{\rho(t)} = \sum_{\gamma=o}^{\infty} (-1)^{\gamma} \frac{\mu_{2\gamma}'}{(2\gamma)!} (\tfrac{t}{2})^{2\gamma}, \end{aligned}\right\} \tag{5.6.10}$$

and, for t small, these expansions exist. The corresponding expansions for $f_1(t)$ and $f_2(t)$ follow immediately from (5.6.10), *viz*.,

$$\left.\begin{aligned} f_1(t) &= \frac{1}{t} \sum_{\gamma=0}^{\infty} (-1)^{\gamma} \frac{\mu'_{2\gamma+1}}{(2\gamma+1)!} \left(\frac{t}{2}\right)^{2\gamma+1}, \\ &\text{and} \\ f_2(t) &= \frac{1}{t} \sum_{\gamma=1}^{\infty} (-1)^{\gamma-1} \frac{\mu'_{2\gamma}}{(2\gamma)!} \left(\frac{t}{2}\right)^{2\gamma}. \end{aligned}\right\} \tag{5.6.11}$$

Let us define

$$\left.\begin{aligned} k_1(t,r) &= \frac{1}{t} \sum_{\gamma=0}^{[\frac{r-1}{2}]} (-1)^{\gamma} \frac{\mu'_{2\gamma+1}}{(2\gamma+1)!} \left(\frac{t}{2}\right)^{2\gamma+1}, \\ &\text{and} \\ k_2(t,r) &= \frac{1}{t} \sum_{\gamma=1}^{[\frac{r}{2}]} (-1)^{\gamma-1} \frac{\mu'_{2\gamma}}{(2\gamma)!} \left(\frac{t}{2}\right)^{2\gamma}, \end{aligned}\right\} \tag{5.6.12}$$

where the symbol [a], as used for the upper limits of summation in (5.6.12), means "the largest integer $\leqq$ a" (the sums being zero if the lower limit exceeds the upper limit). It further follows that, for t small,

$$\left.\begin{aligned} &\frac{1}{t^r}\,[f_1(t)-k_1(t,r)] \\ &\frac{1}{t^r}\,[f_2(t)-k_2(t,r)] \end{aligned}\right\} = O\left(\frac{1}{t^{1-\theta}}\right), \ \theta > 0. \tag{5.6.13}$$

Let us now designate the integrand of (5.6.2) by $h(t,\alpha,r)$, and rewrite it as

$$h(t,\alpha,r) = H(t,\alpha,r) + K(t,\alpha,r), \tag{5.6.14}$$

where

$$H(t,\alpha,r) = \frac{\Gamma(r+1)}{\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}} \left\{ [f_1(t)-k_1(t,r)]\cos(r\phi) \right.$$

$$\left. + [f_2(t)-k_2(t,r)]\sin(r\phi)\right\}, \qquad (5.6.15)$$

$$K(t,\alpha,r) = \frac{\Gamma(r+1)}{\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}} [k_1(t,r)\cos(r\phi)+k_2(t,r)\sin(r\phi)], \quad (5.6.16)$$

and we recall that $\phi = \tan^{-1}(\frac{t}{2\alpha})$. Then (5.6.2) may be rewritten

$$\pi E(Q^r) = \lim_{\alpha\to 0}\int_0^\infty [H(t,\alpha,r) + K(t,\alpha,r)]dt$$

$$= \lim_{\alpha\to 0}\int_0^\infty H(t,\alpha,r)dt + \lim_{\alpha\to 0}\int_0^\infty K(t,\alpha,r)dt. \qquad (5.6.17)$$

We know, by (5.6.13), that if passage to the limit under the integral sign were now to be made, for the first term of (5.6.17) alone, the resulting integral would be well behaved for t small. But what is the nature of the second term? Happily, it turns out that

$$\int_0^\infty K(t,\alpha,r)dt \equiv 0 \quad (\text{for } \alpha > 0;\ r > 0,\ r \neq 1,2,\ldots). \qquad (5.6.18)$$

To see the truth of this claim, we note that

$$\int_0^\infty \frac{t^{2\gamma+1}\cos(r\phi)}{t\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}}dt = C_1(\alpha)\int_0^{\pi/2} \cos^{r-2}\phi\cos(r\phi)\tan^{2\gamma}\phi\, d\phi$$

$$= 0, \text{ for all } \gamma < \frac{r-1}{2},$$

providing $\alpha > 0$, $r > 0$, and $r \neq 1,2,\ldots$. [See, e.g., Gröbner and Hofreiter (1961), p. 110, (15a).] Similarly, we note that

$$\int_0^\infty \frac{t^{2\gamma}\sin(r\phi)}{t\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}}\,dt = C_2(\alpha)\int_0^{\pi/2} \cos^{r-2\gamma-1}\phi\sin(r\phi)\sin^{2\gamma-1}\phi\,d\phi$$

$$= 0, \text{ for all } \gamma < \frac{r}{2},$$

providing $\alpha > 0$, $r > 0$, $r \neq 1,2,\ldots$. [See, e.g., Gröbner and Hofreiter (1961), p. 109, (10b).] The vanishing of these two integrals clearly implies the truth of (5.6.18).

Hence, (5.6.17) reduces to

$$\pi E(Q^r) = \lim_{\alpha \to 0} \int_0^\infty H(t,\alpha,r)dt$$

$$= \int_0^\infty H(t,0,r)dt, \tag{5.6.19}$$

and we have now obtained a fairly simple single integral expression for the fractional moments of Q*. A formal proof that this single integral does, in fact, represent the fractional moments of Q* is given as Appendix 9.3.

We conclude this section by giving a more detailed expression for (5.6.19):

$$\pi E(Q^r) = \int_0^\infty \frac{2^r\Gamma(r+1)}{t^{r+1}} \left\{ \left[\frac{\sin A(t)}{\rho(t)} - \sum_{\gamma=0}^{[\frac{r-1}{2}]} (-1)^\gamma \frac{\mu'_{2\gamma+1}}{(2\gamma+1)!} \left(\frac{t}{2}\right)^{2\gamma+1}\right]\cos\left(r\,\frac{\pi}{2}\right)\right.$$

$$\left. - \left[\frac{\cos A(t)}{\rho(t)} - \sum_{\gamma=0}^{[\frac{r}{2}]} (-1)^\gamma \frac{\mu'_{2\gamma}}{(2\gamma)!} \left(\frac{t}{2}\right)^{2\gamma}\right]\sin\left(r\,\frac{\pi}{2}\right)\right\} dt, \tag{5.6.20}$$

where $A(t)$ and $\rho(t)$ are given by (5.3.2) and (5.3.3) respectively.

6. NUMERICAL COMPARISONS

6.1 Introduction

With this section, we come, at last, to the consideration of numerical comparisons of approximate with exact solutions for the fractional moments of Q, and, somewhat incidentally, for the distribution function, F(q), of Q. Moment comparisons are confined to the case $E(Q^{\frac{1}{2}}) = E(Z)$, and these, with three exceptions, for two-dimensional (k=2) examples. More elaborate comparisons were planned; however, funds available for numerical purposes were exhausted before such comparisons could be made. Nevertheless, those comparisons judged to be most stringent (in some sense) were accomplished first, and it is believed that the conclusions drawn from these comparisons would have been supported by the other comparisons.

6.2 Distribution Function of Q

With regard to the conjecture of Section 4.4, consider the approximation (4.4.16) for F(q). When $\Delta < 0$, the approximating r.v. Y, (4.4.14), is given by Pearson's r.v. Y_2, the quality of Y_2 for approximating F(q) having been considered by Imhof (1961). In particular, we should like to consider some examples where $\Delta > 0$ (then $Y = Y_3$) so that comparisons between Y_2 and Y_3 might be made.

Of the examples considered by Imhof, three (<u>his</u> Q_5, Q_6, $\frac{1}{2}Q_5+\frac{1}{2}Q_6$) were found to have $\Delta > 0$. Moreover, for these three examples, Pearson's approximation is seen (see Imhof) to give particularly close agreement with the exact values determined by (5.3.1). Thus these examples were judged to be a fairly stiff test for the r.v. Y_3, in that Y_2 leaves little latitude for improvement. Exact values for $F(q)$ were obtained using (5.3.1), while the various approximate values, based upon the three r.v.'s Y_1 (Patnaik's), Y_2, and Y_3, were determined by suitable specializations (of the parameters) of (5.3.1). The results obtained are presented as Table 6.1.

We note that, without exception, the approximate values based upon Y_3 are closest to the exact values. While these results do support the conjecture, we do not, of course, conclude that determinations of $F(q)$ based upon Y_3 would <u>always</u> be closer to exact values than those based upon Y_2 (or even those based upon Y_1). In fact, it may well be, for practical applications, that Y_3 is not enough better than Y_2 to warrant serious consideration of the "combined" r.v. Y over Y_2 alone, if one is merely interested in approximate values of $F(q)$. In any case, we shall not pursue the matter further here, it being quite incidental to this study.

6.3 Fractional Moments of Q

For moment approximations, as noted in Section 4, the situation is a bit different from that just considered in the previous section, in that Y_2 applies <u>only</u> when $\Delta < 0$ (and Y_3, as before, applies only when $\Delta \geqq 0$). The comparison here, then, is essentially Y versus Y_1, <u>i.e.</u>, the three-moment approximation versus the two-moment approximation.

In an attempt to avoid comparisons that might be particularly favorable toward Y, rather than Y_1, we deliberately attempted to bias matters in favor of Y_1, trusting that the superiority of Y would show through in spite of such bias (we were not disappointed).

Table 6.1. Values (approximate and exact) of $F(q) = P(Q \leq q)$

Quadratic Form	q	Approximate			Exact
		$P(Y_1 \leq q)$	$P(Y_2 \leq q)$	$P(Y_3 \leq q)$	$P(Q \leq q)$
$Q_a = (.7)\chi^2(6,3) + (.3)\chi^2(2,1)$	2	.0046	.0072	.0061	.0061
	10	.5955	.5911	.5913	.5913
	20	.9770	.97793	.97792	.97792
$Q_b = (.7)\chi^2(1,3) + (.3)\chi^2(1,1)$	1	.0281	.0485	.0448	.0452
	6	.6052	.5916	.5924	.5924
	15	.9754	.97771	.97766	.97761
$Q_c = \frac{1}{2} Q_a + \frac{1}{2} Q_b$	3.5	.0395	.0440	.0437	.0437
	8	.05899	.5847	.5848	.5848
	13	.9526	.9539	.9538	.9538

Of the odd moments of Z [i.e., $E(Q^r)$, $r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$], it seems clear from considerations of interpolation vs. extrapolation that a choice to consider the first ($r = \frac{1}{2}$), or possibly the next ($r = \frac{3}{2}$), would be a choice favorable to Y_1. We chose the first ($r = \frac{1}{2}$). Having selected the moment to be calculated, we then faced the more difficult problem of choosing the actual examples (specific quadratic forms).

To aid in this choice, we determined the Δ values for all two-dimensional quadratic forms formed by taking all possible combinations (864) of the following values for the parameters:

$$\sigma_1^2 = 1;$$

$$\sigma_2^2 = 1,\ 2,\ 4,\ 9,\ 16,\ 25;$$

$$\lambda_1, \lambda_2 = 0,\ \frac{1}{32},\ \frac{1}{16},\ \frac{1}{8},\ \frac{1}{4},\ \frac{1}{2},\ 1,\ 2,\ 4,\ 8,\ 16,\ 32$$

$$(\nu_1 = \nu_2 = 1)$$

This unwieldy set of examples was then cut in half by an arbitrary decision to consider only every other value for λ_1 (starting with $\frac{1}{32}$). Also, the value $\sigma_2^2 = 1$ was judged to be relatively uninteresting (special case 1 of Section 3 applying to such examples). Thus the number of combinations was reduced to 360. Now the advantage, if any, of Y over Y_1 tends to vanish as Δ tends to zero (recall that $Y_3 = Y_2 = Y_1$ for $\Delta = 0$). Thus, for each (σ_2^2, λ_1) combination, it was decided to select, from among the 12 possible values for λ_2, that value of λ_2 such that $|\Delta|$ is smallest, or next to smallest, depending upon which choice gave the "nicer to work with" values for the ξ_i (recall $\xi_i = \sigma_i\sqrt{2\lambda_i}$)

--this rule for selection was followed, but with the added provision that, when possible, alternating selections should correspond to Δ values alternating in sign. In this manner, then, the number of two-dimensional examples to be treated was fixed at 30, approximately half with $\Delta > 0$ (implying that $Y = Y_3$) and half with $\Delta < 0$ (implying that $Y = Y_2$). In one instance, the rule yielded an example for which $\Delta = 0$. The example was retained, serving as a check. In addition, the three quadratic forms shown in Table 6.1 were included as examples.

Exact values of $E(Q^{\frac{1}{2}})$ were determined by (5.5.7), while the approximate values, values of $E(Y_1^{\frac{1}{2}})$ and $E(Y^{\frac{1}{2}})$, were determined in accordance with Section 4.6. The results obtained are presented as Table 6.2.

On inspecting Table 6.2, one is immediately struck by the apparent excellence of the values based upon the Patnaik (Y_1) approximation, so much so that it would seem foolish to strive for a more sophisticated approximation. In this respect the results presented are misleading, the examples having been selected so as to insure reasonably close agreement of approximate and exact values. What is important, though, is that, in spite of our attempts to bias matters in favor of Y_1, in only six cases are the approximate values based upon Y_1 closer to the exact values than those based upon Y.

6.4 Conclusions

In view of the results obtained, it does seem quite safe to conclude (as one feels intuitively) that the moments of Y will, in most cases, serve as better approximations to the corresponding moments of Q (or Z) than those of Y_1. We hasten, however, to add a rather obvious

Table 6.2. Values (approximate and exact) of $E(Q^{\frac{1}{2}}) = E(Z)$

Quadratic Form	Approximate		Exact
	$E(Y_1^{\frac{1}{2}})$	$E(Y^{\frac{1}{2}})$	$E(Q^{\frac{1}{2}})$
$Q_a=(.7)\chi^2(6,3)+(.3)\chi^2(2,1)$	3.021	3.020 (+)[a]	3.020
$Q_b=(.7)\chi^2(1,3)+(.3)\chi^2(1,1)$	2.290	2.282 (+)	2.281
$Q_c=\frac{1}{2}\ Q_a + \frac{1}{2}\ Q_c$	2.728	2.727 (+)	2.727
$Q_1=\chi^2(1,\frac{1}{32})+2\chi^2(1,\frac{1}{4})$	1.765	1.778 (-)	1.772
$Q_2=\chi^2(1,\frac{1}{8})+2\chi^2(1,1)$	2.437	2.423 (+)	2.423
$Q_3=\chi^2(1,\frac{1}{2})+2\chi^2(1,\frac{1}{8})$	1.904	1.904 (o)	1.901
$Q_4=\chi^2(1,2)+2\chi^2(1,\frac{1}{16})$	2.521	2.512 (+)	2.508
$Q_5=\chi^2(1,8)+2\chi^2(1,0)$	4.249	4.244 (+)	4.243
$Q_6=\chi^2(1,32)+2\chi^2(1,4)$	9.046	9.045 (+)	9.045
$Q_7=\chi^2(1,\frac{1}{32})+4\chi^2(1,\frac{1}{8})$	2.095	2.141 (-)	2.123
$Q_8=\chi^2(1,\frac{1}{8})+4\chi^2(1,\frac{1}{2})$	2.648[b]	2.643 (+)	2.652
$Q_9=\chi^2(1,\frac{1}{2})+4\chi^2(1,\frac{1}{8})$	2.318[b]	2.360 (-)	2.337
$Q_{10}=\chi^2(1,2)+4\chi^2(1,2)$	4.706	4.694 (+)	4.693
$Q_{11}=\chi^2(1,8)+4\chi^2(1,2)$	5.8898	5.8922 (-)	5.8914
$Q_{12}=\chi^2(1,32)+4\chi^2(1,\frac{1}{8})$	8.302	8.301 (+)	8.301

Table 6.2. (continued)

Quadratic Form	Approximate		Exact
	$E(Y_1^{\frac{1}{2}})$	$E(Y^{\frac{1}{2}})$	$E(Q^{\frac{1}{2}})$
$Q_{13}=\chi^2(1,\frac{1}{32})+9\chi^2(1,\frac{1}{8})$	2.908	2.963 (-)	2.953
$Q_{14}=\chi^2(1,\frac{1}{8})+9\chi^2(1,\frac{1}{2})$	3.752[b]	3.729 (+)	3.751
$Q_{15}=\chi^2(1,\frac{1}{2})+9\chi^2(1,\frac{1}{8})$	3.081	3.157 (-)	3.133
$Q_{16}=\chi^2(1,2)+9\chi^2(1,2)$	6,574	6.540 (+)	6.542
$Q_{17}=\chi^2(1,8)+9\chi^2(1,2)$	7.496	7.499 (-)	7.504
$Q_{18}=\chi^2(1,32)+9\chi^2(1,8)$	14.5489[b]	14.5488 (+)	14.5493
$Q_{19}=\chi^2(1,\frac{1}{32})+16\chi^2(1,\frac{1}{8})$	3.758	3.780 (-)	3.808
$Q_{20}=\chi^2(1,\frac{1}{8})+16\chi^2(1,\frac{1}{2})$	4.894[b]	4.846 (+)	4.876
$Q_{21}=\chi^2(1,\frac{1}{2})+16\chi^2(1,\frac{1}{8})$	3.895	3.976 (-)	3.963
$Q_{22}=\chi^2(1,2)+16\chi^2(1,2)$	8.521	8.454 (+)	8.467
$Q_{23}=\chi^2(1,8)+16\chi^2(1,\frac{1}{2})$	6.449	6.533 (-)	6.530
$Q_{24}=\chi^2(1,32)+16\chi^2(1,8)$	18.015	18.011 (+)	18.011
$Q_{25}=\chi^2(1,\frac{1}{32})+25\chi^2(1,\frac{1}{8})$	4.626	4.639 (-)	4.674
$Q_{26}=\chi^2(1,\frac{1}{8})+25\chi^2(1,\frac{1}{2})$	6.052	5.980 (+)	6.014
$Q_{27}=\chi^2(1,\frac{1}{2})+25\chi^2(1,\frac{1}{8})$	4.739	4.808 (-)	4.812
$Q_{28}=\chi^2(1,2)+25\chi^2(1,\frac{1}{2})$	6.395[b]	6.377 (+)	6.437
$Q_{29}=\chi^2(1,8)+25\chi^2(1,\frac{1}{2})$	7.380	7.483 (-)	7.494
$Q_{30}=\chi^2(1,32)+25\chi^2(1,8)$	21.663	21.653 (+)	21.654

[a]The symbols (+),(-),(o) mean that $Y=Y_3,Y_2,Y_1$, respectively.

[b]Patnaik value closer to exact value.

comment: Even as extrapolation is usually risky, while interpolation is generally sound, so it must also be in applications of the moment approximations proposed in this study. In particular, the early fractional moments of Y [$E(Y^r)$, $0 < r < 4$, say] should serve well for approximating those same moments of Q, but we would place little reliance on the approximation for moments of higher order ($r > 4$, say).

However, interest usually centers on lower moments, the first four integer moments (beyond the zero-th) being those most often investigated.

If one must evaluate many moments, or moments of orders much beyond those upon which the approximating r.v. Y is based, then he would do well to enlist the aid of a modern electronic computer so as to obtain exact evaluations based on (5.6.20).

7. SUMMARY

Fully stated, the problem treated in this thesis is that of determining positive fractional moments of a nonnegative-definite quadratic form in nonsingular noncentral normal random variables. By way of summary, let us review the contents of the several sections and appendices.

Section 1: The stage is quickly set by an immediate statement of the problem in its simplest symbolic format; interest in fractional moments of a quadratic form Q is motivated by an interest in integer moments of random distance, a topic that has apparently received little attention in the past, perhaps because of the complicated nature of the distributions involved. All distance problems in nonsingular normal random variables are shown to be reducible, by well-known linear transformations, to the same format as that given to the statement of the problem of the thesis.

Section 2: The noncentral chi-square distribution may be regarded as being the basic statistical distribution underlying quadratic forms in normal r.v.'s. The section is a brief resume of the nature and properties of the noncentral chi-square r.v.; two properties are newly established, *viz*., a general expression for its moments, and a second-order moment recursion.

Section 3: Each of Sections 3, 4, and 5 contains a different approach to the problem. The approach of Section 3 might be labeled "approach by specialization." By suitable specializations of the parameters k, $\underline{\xi}$, and D (see first paragraph of Section 1), we hoped to obtain cases admitting fairly simple exact solutions for fractional moments. The two special cases that are treated in the section yield exact solutions that are shown to be expressible in terms of well-tabled special mathematical functions, viz., the error and modified Bessel functions, and the complete elliptic integrals. The first of these cases is believed to be new, while the second is the work of Scheuer (1961), being included for purposes of completeness.

Section 4: This section might be labeled "approach by approximation," in the sense that we try to approximate Q by various more tractable r.v.'s whose fractional moments are relatively easy to obtain. This rather popular approach to dealing with noncentral chi-square r.v.'s has its foundation in the work of Patnaik (1949), who proposed a central chi-square r.v. for the purpose of evaluating (approximately) the distribution function of a noncentral chi-square r.v. We have considered three such approximating r.v.'s, Y_1 (Patnaik's two-moment approximation), Y_2 (Pearson's three-moment approximation), and Y_3 (a newly proposed three-moment approximation). The most interesting result obtained in the section is that, although neither Y_2 nor Y_3, individually, provides a fully applicable approximation for fractional moments of Q, Y_2 and Y_3 together yield a r.v. Y of full applicability. It is this fact that gave rise to the conjecture that Y is, in general, superior to Y_2 (alone) for the purpose of evaluating (approximately) the distribution function of Q.

Section 5: At the very outset (before much of the research reported in this thesis had been accomplished), it seemed quite unlikely that a significant contribution toward a tractable exact solution of the general problem would be possible. It was clear that an exact expression for the fractional moments of Q could be written explicitly, because explicit expressions for the distribution function of Q, F(q), had been recently published. Thus

$$E(Q^r) = \int_0^{\infty} q^r dF(q),$$

but the expressions did present a formidable sight. However, with the decision to use F(q) as given by Imhof (1961), and with the observation that $E(Q^r)$ could be expressed alternatively as

$$E(Q^r) = \int_0^{\infty} [1-F(q^{1/r})]dq,$$

matters did look better. We had what was essentially a double integral, and, for a modern high-speed electronic computer, that shouldn't present too much of a problem. But it did!

After many long weeks of programming and "debugging," it became clear that the double-integral evaluations would consume too much machine time. The thought of possibly reducing the double integral to a single integral by changing the order of integration returned to mind; the idea had occurred earlier, but, at that time, it didn't look promising and was quickly dismissed. But now there was stronger motivation for taking a more critical look. Some success followed, in that the reduction was apparently possible for $0 < r \leq 1$, but not for $r > 1$. This didn't seem right. The author's understanding of the nature of Fubini's theorem led him to believe that the reduction should be possible for all $r > 0$. The author put the matter to a numerical analyst (and friend). As noted in the text of the chapter, this friend made two key suggestions; the first seemed to accomplish little by itself, but, when it was considered together with the second suggestion, the reduction ultimately followed in fine fashion. This reduction of the exact expression for the fractional moments of Q to a single integral (along with the rigorous proof that the single integral is correct--Appendix 9.3) is considered to be an important contribution of this thesis. This single integral is no worse than a great many others that are routinely evaluated by electronic computers.

Section 6: In this section, a few numerical comparisons are given which support the conjecture that the r.v. Y is, in general, superior to Y_2 for evaluating (approximately) $F(q)$. Other comparisons made in the section support the hypothesis that the r.v. Y is, in general, superior to Y_1 for evaluating (approximately) $E(Q^r)$, provided r is not too large, of course. In fact, for either purpose, it is believed that approximations based upon Y will be quite reliable.

Appendix 9.1: This appendix gives various properties of the confluent hypergeometric function. The function arises in Sections 2, 3, and 4. The two theorems stated and proved in the appendix appear to be of a fairly general and useful nature, and we would hope that they may find important applications in areas other than that treated here.

Appendix 9.2: This appendix contains a very useful approximation for the ratio of two gamma functions. It would be presumptuous to imagine that it has not been previously discovered. Nevertheless, it is included because it has obvious application here, and partly because of the refinements given that enable one to obtain any required accuracy.

Appendix 9.3: With this appendix, we conclude the thesis. It contains the rigorous proof that the single integral obtained in Section 5 does, in fact, represent fractional moments of Q.

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9. APPENDICES

9.1 The Confluent Hypergeometric Function (Kummer's Function)

9.1.1 Properties and Relationships

The few properties and relationships of the confluent hypergeometric function presented in this first section are well known and may be found, for the most part, in standard works on the subject. In particular, see Erdelyi (1953), Slater (1960), Tricomi (1954).

Definition:

$$F(a;c;x) = \sum_{n=o}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}\ , \qquad (9.1.1.1)$$

where $(a)_n$ is Pochhammer's symbol:

$$\left.\begin{aligned} &(a)_o = 1, \\ &(a)_n = a(a+1)(a+2)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\ . \end{aligned}\right\} \qquad (9.1.1.2)$$

The defining series is absolutely and uniformly convergent for all values of a, c, and x, real or complex, excluding $c = 0,-1,-2,\ldots$.

Kummer's transformation (Kummer's first theorem):

$$F(a;c;x) = e^x F(c-a;c;-x). \qquad (9.1.1.3)$$

Special cases:

$$F(a;a;x) = e^x. \qquad (9.1.1.4)$$

$$F(\tfrac{1}{2};\tfrac{3}{2};-x^2) = \frac{1}{x}\ \mathrm{erf}(x), \qquad (9.1.1.5)$$

$$\text{where } \operatorname{erf}(x) \equiv \int_0^x e^{-u^2}\,du \tag{9.1.1.6}$$

$$= \tfrac{1}{2}\,\gamma(\tfrac{1}{2},x^2) \equiv \tfrac{1}{2}\int_0^{x^2} t^{-\frac{1}{2}}e^{-t}dt \quad (x>0), \tag{9.1.1.7}$$

where $\gamma(\cdot,\cdot)$ is the incomplete gamma function.

$$F(\tfrac{1}{2}+\nu;1+2\nu;2x) = (\tfrac{2}{x})^{\nu}\; e^{x}\Gamma(\nu+1)I_{\nu}(x), \tag{9.1.1.8}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind of order ν.

Recursions:

$$(c-a)F(a-1;c;x)+(2a-c+x)F(a;c;x)-aF(a+1;c;x) = 0. \tag{9.1.1.9}$$

$$c(c-1)F(a;c-1;x)-c(c-1+x)F(a;c;x)+(c-a)xF(a;c+1;x)=0. \tag{9.1.1.10}$$

$$(a-c+1)F(a;c;x)-aF(a+1;c;x)+(c-1)F(a;c-1;x) = 0. \tag{9.1.1.11}$$

$$cF(a;c;x)-cF(a-1;c;x)-xF(a;c+1;x) = 0. \tag{9.1.1.12}$$

$$c(a+x)F(a;c;x)-(c-a)xF(a;c+1;x)-acF(a+1;c;x) = 0. \tag{9.1.1.13}$$

$$(a-1+x)F(a;c;x)+(c-a)F(a-1;c;x)-(c-1)F(a;c-1;x) = 0. \tag{9.1.1.14}$$

U-function:

The U-function is also known as a confluent hypergeometric function. It is related to $F(a;c;x)$ as follows [see Slater (1960), p. 5]:

$$U(a;c;x) = \frac{\pi}{\sin\pi c}\left\{\frac{F(a;c;x)}{\Gamma(1+a-c)\Gamma(c)} - \frac{x^{1-c}F(1+a-c;2-c;x)}{\Gamma(a)\Gamma(2-c)}\right\}. \tag{9.1.1.15}$$

The U-function satisfies the following recursion:

$$(c-a-1)U(a;c-1;x)+(1-c-x)U(a;c;x)+xU(a;c+1;x) = 0. \qquad (9.1.1.16)$$

9.1.2 Representation of $F(-\frac{n}{2};\frac{k}{2};-x)$ in Terms of Bessel or Error Functions, General

Note: To facilitate the proofs in this section, the symbol ϵ is used to mean "may be expressed in terms of."

Theorem 9.1.2.1: $F(-\frac{n}{2};\frac{k}{2};-x) \in I_0(\frac{x}{2})$ and $I_1(\frac{x}{2})$, whenever n and k are positive odd and even integers, respectively, and $x \neq 0$.

Proof: We first establish three preliminary results:

By (9.1.1.12), with $a = \frac{3}{2}$ and $c = 1$,

$$F(\tfrac{3}{2};1;x) = F(\tfrac{1}{2};1;x) + xF(\tfrac{3}{2};2;x).$$

Also, by (9.1.1.10), with $a = \frac{3}{2}$ and $c = 2$,

$$F(\tfrac{3}{2};1;x) = (x+1)F(\tfrac{3}{2};2;x) - \tfrac{x}{4}F(\tfrac{3}{2};3;x).$$

Eliminating $F(\frac{3}{2};2;x)$ from these two equations, we obtain

$$F(\tfrac{3}{2};1;x) = (x+1)F(\tfrac{1}{2};1;x) + \frac{x^2}{4} F(\tfrac{3}{2};3;x). \qquad (9.1.2.1)$$

Eliminating $F(\frac{3}{2};1;x)$ from the same equations, we obtain

$$F(\tfrac{3}{2};2;x) = F(\tfrac{1}{2};1;x) + \tfrac{x}{4}F(\tfrac{3}{2};3;x). \qquad (9.1.2.2)$$

And, by (9.1.1.13), with $a = \frac{1}{2}$ and $c = 1$, followed by substitution from (9.1.2.1), we obtain

$$F(\tfrac{1}{2};2;x) = F(\tfrac{1}{2};1;x) - \tfrac{x}{4}F(\tfrac{3}{2};3;x). \qquad (9.1.2.3)$$

We now proceed with the proof:

$$F(-\frac{n}{2};\frac{k}{2};-x) = e^{-x}F(\frac{k+n}{2};\frac{k}{2};x), \text{ by } (9.1.1.3).$$

Let n and k take the values stated in the theorem. Then

$$F(\frac{k+n}{2};\frac{k}{2};x) \in F(\frac{k+n}{2};2;x) \text{ and } F(\frac{k+n}{2};1;x), \quad x \neq 0,$$

either immediately (if k = 2 or 4) or by repeated application of (9.1.1.10). Similarly, by repeated application of (9.1.1.9),

$$F(\frac{k+n}{2};1;x) \in F(\frac{3}{2};1;x) \text{ and } F(\frac{1}{2};1;x)$$

$$\in F(\frac{1}{2};1;x) \text{ and } F(\frac{3}{2};3;x), \text{ by } (9.1.2.1).$$

Also, by repeated application of (9.1.1.9),

$$F(\frac{k+n}{2};2;x) \in F(\frac{3}{2};2;x) \text{ and } F(\frac{1}{2};2;x)$$

$$\in F(\frac{1}{2};1;x) \text{ and } F(\frac{3}{2};3;x),$$

by (9.1.2.2) and (9.1.2.3).

Hence, for n, k, and x as stated in the theorem,

$$F(-\frac{n}{2};\frac{k}{2};-x) \in F(\frac{1}{2};1;x) \text{ and } F(\frac{3}{2};3;x)$$

$$\in I_0(\frac{x}{2}) \text{ and } I_1(\frac{x}{2}), \text{ by } (9.1.1.8).$$

<u>Theorem 9.1.2.2</u>: $F(-\frac{n}{2};\frac{k}{2};-x) \in \text{erf}(\sqrt{x})$, whenever n and k are each positive odd integers and $x > 0$.

<u>Proof</u>: Let n, k, and x take the values stated in the theorem. Then

$$F(-\tfrac{n}{2};\tfrac{k}{2};-x) \in F(-\tfrac{n}{2};\tfrac{3}{2};-x) \text{ and } F(-\tfrac{n}{2};\tfrac{1}{2};-x),$$

either immediately (if k = 1 or 3) or by repeated application of (9.1.1.10). Similarly, by repeated application of (9.1.1.9),

$$F(-\tfrac{n}{2};\tfrac{1}{2};-x) \in F(\tfrac{1}{2};\tfrac{1}{2};-x) \text{ and } F(\tfrac{3}{2};\tfrac{1}{2};-x),$$

and by (9.1.1.13), with $a = c = \frac{1}{2}$,

$$F(\tfrac{3}{2};\tfrac{1}{2};-x) \in F(\tfrac{1}{2};\tfrac{1}{2};-x) \text{ and } F(\tfrac{1}{2};\tfrac{3}{2};-x).$$

Therefore

$$F(-\tfrac{n}{2};\tfrac{1}{2};-x) \in F(\tfrac{1}{2};\tfrac{1}{2};-x) \text{ and } F(\tfrac{1}{2};\tfrac{3}{2};-x).$$

Also, by repeated application of (9.1.1.9),

$$F(-\tfrac{n}{2};\tfrac{3}{2};-x) \in F(\tfrac{1}{2};\tfrac{3}{2};-x) \text{ and } F(\tfrac{3}{2};\tfrac{3}{2};-x).$$

Thus for n, k, and x as specified in the theorem,

$$\begin{aligned} F(-\tfrac{n}{2};\tfrac{k}{2};-x) &\in F(\tfrac{1}{2};\tfrac{3}{2};-x) \text{ and } F(\tfrac{1}{2};\tfrac{1}{2};-x) \text{ and } F(\tfrac{3}{2};\tfrac{3}{2};-x) \\ &\in \operatorname{erf}(\sqrt{x}) \text{ and } e^{-x}, \text{ by (9.1.1.5) and (9.1.1.4)}, \\ &\in \operatorname{erf}(\sqrt{x}). \end{aligned}$$

Also, by (9.1.1.7), it is further clear that

$$F(-\tfrac{n}{2};\tfrac{k}{2};-x) \in \gamma(\tfrac{1}{2},x).$$

9.1.3 Representation of $F(-\frac{n}{2};\frac{k}{2};-\lambda)$ in Terms of Bessel or Error Functions, Specific Cases

In this section, six specific cases ($n = 1,3$; $k = 1,2,3$) of $F(-\frac{n}{2};\frac{k}{2};-\lambda)$ are expressed in terms of Bessel or error functions, consistent with Theorems 9.1.2.1 and 9.1.2.2, for use in conjunction with special case 1 of Section 3.

(i) $n = 1$, $k = 1$:

$$F(-\tfrac{1}{2};\tfrac{1}{2};-\lambda) = e^{-\lambda} + 2\sqrt{\lambda}\,\mathrm{erf}(\sqrt{\lambda}). \qquad (9.1.3.1)$$

Proof: In (9.1.1.12), let $a = c = \frac{1}{2}$ and replace x by $-\lambda$ to obtain

$$F(-\tfrac{1}{2};\tfrac{1}{2};-\lambda) = F(\tfrac{1}{2};\tfrac{1}{2};-\lambda) + 2\lambda F(\tfrac{1}{2};\tfrac{3}{2};-\lambda).$$

The result follows from (9.1.1.4) and (9.1.1.5).

(ii) $n = 3$, $k = 1$:

$$F(-\tfrac{3}{2};\tfrac{1}{2};-\lambda) = (\lambda+1)e^{-\lambda} + \sqrt{\lambda}(2\lambda+3)\mathrm{erf}(\sqrt{\lambda}). \qquad (9.1.3.2)$$

Proof: In (9.1.1.9), let $a = -\frac{1}{2}$, $c = \frac{1}{2}$, and replace x by $-\lambda$ to obtain

$$F(-\tfrac{3}{2};\tfrac{1}{2};-\lambda) = (\lambda+\tfrac{3}{2})F(-\tfrac{1}{2};\tfrac{1}{2};-\lambda) - \tfrac{1}{2}F(\tfrac{1}{2};\tfrac{1}{2};-\lambda).$$

The result follows from (9.1.3.1) above and (9.1.1.4).

(iii) $n = 1$, $k = 2$:

$$F(-\tfrac{1}{2};1;-\lambda) = e^{-\frac{\lambda}{2}}\,[(\lambda+1)I_0(\tfrac{\lambda}{2})+\lambda I_1(\tfrac{\lambda}{2})]. \qquad (9.1.3.3)$$

Proof: The result follows from application of (9.1.1.8) to (9.1.2.1), followed by (9.1.1.3).

(iv) n = 3, k = 2:

$$F(-\tfrac{3}{2};1;-\lambda) = \tfrac{1}{3}e^{-\frac{\lambda}{2}}[(2\lambda^2+6\lambda+3)I_o(\tfrac{\lambda}{2})+\lambda(\lambda+2)I_1(\tfrac{\lambda}{2})]. \qquad (9.1.3.4)$$

Proof: In (9.1.1.9), let $a = \frac{3}{2}$, $c = 1$, and replace x by λ to obtain

$$F(\tfrac{5}{2};1;\lambda) = \tfrac{2}{3}[(\lambda+2)F(\tfrac{3}{2};1;\lambda)-\tfrac{1}{2}F(\tfrac{1}{2};1;\lambda)].$$

The result follows from (9.1.2.1) and (9.1.1.8), and, subsequently, from (9.1.1.3).

(v) n = 1, k = 3:

$$F(-\tfrac{1}{2};\tfrac{3}{2};-\lambda) = \tfrac{1}{2}e^{-\lambda} + \frac{1}{2\sqrt{\lambda}}(2\lambda+1)\mathrm{erf}(\sqrt{\lambda}). \qquad (9.1.3.5)$$

Proof: In (9.1.1.9), let $a = \frac{1}{2}$, $c = \frac{3}{2}$, and replace x by $-\lambda$ to obtain

$$F(-\tfrac{1}{2};\tfrac{3}{2};-\lambda) = \tfrac{1}{2}F(\tfrac{3}{2};\tfrac{3}{2};-\lambda) + (\lambda+\tfrac{1}{2})F(\tfrac{1}{2};\tfrac{3}{2};-\lambda).$$

The result follows from (9.1.1.4) and (9.1.1.5).

(vi) n = 3, k = 3:

$$F(-\tfrac{3}{2};\tfrac{3}{2};-\lambda) = \tfrac{1}{8}(2\lambda+5)e^{-\lambda} + \frac{1}{8\sqrt{\lambda}}(4\lambda^2+12\lambda+3)\mathrm{erf}(\sqrt{\lambda}). \qquad (9.1.3.6)$$

Proof: In (9.1.1.9), let $a = \frac{1}{2}$, $c = \frac{3}{2}$, and replace x by $-\lambda$ to obtain

$$F(-\tfrac{3}{2};\tfrac{3}{2};-\lambda) = \tfrac{1}{2}(\lambda+\tfrac{5}{2})F(-\tfrac{1}{2};\tfrac{3}{2};-\lambda) - \tfrac{1}{4}F(\tfrac{1}{2};\tfrac{3}{2};-\lambda).$$

The result follows from (9.1.3.5) above and (9.1.1.5).

9.2 A Convenient Approximation

Theorem 9.2.1: For $r \geqq 0$, and x large, $x > 0$,

$$\frac{\Gamma(x+r)}{\Gamma(x)} \doteq x^r e^{\frac{r(r-1)}{2x}}$$

Proof: Employing Stirling's asymptotic series for $\log \Gamma(x)$ [see National Bureau of Standards (1964)], we have that

$$\begin{aligned}
\log \frac{\Gamma(x+r)}{\Gamma(x)} &\sim [(x+r-\tfrac{1}{2})\log(x+r-1) - (x+r-1) + \frac{1}{12(x+r-1)} + O(\frac{1}{x^3})] \\
&\quad - [(x-\tfrac{1}{2})\log(x-1) - (x-1) + \frac{1}{12(x-1)} + O\left(\frac{1}{x^3}\right)] \\
&= (x+r-\tfrac{1}{2})\log(x+r-1) - (x-\tfrac{1}{2})\log(x-1)-r \\
&\quad - \frac{r}{12(x+r-1)(x-1)} + O\left(\frac{1}{x^3}\right) \\
&= (x+r-\tfrac{1}{2})\log x(1+\frac{r-1}{x}) - (x-\tfrac{1}{2})\log x(1-\frac{1}{x}) - r + O\left(\frac{1}{x^2}\right) \\
&= (x+r-\tfrac{1}{2})[\log x + \frac{r-1}{x} - \tfrac{1}{2}(\frac{r-1}{x})^2 + O\left(\frac{1}{x^3}\right)] \\
&\quad - (x-\tfrac{1}{2})[\log x - \frac{1}{x} - \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right)] - r + O\left(\frac{1}{x^2}\right) \\
&= r\log x + \frac{r(r-1)}{2x} + O\left(\frac{1}{x^2}\right),
\end{aligned}$$

and the result follows.

Table 9.2.1 gives some indication of the quality of the approximation, as measured by the smallest integer value of x such that the relative error of approximation is less than ϵ. We observe, in particular, surprisingly good agreement for $r = \frac{1}{2}$; this is also the case for any value of r in the closed interval (0,1). The quality of the approximation drops off rapidly as r increases beyond unity. However, there are two convenient devices that will enable one to improve the approximation when r is "too" large or x "too" small.

<u>When r is "too" large</u>: We may separate r into its integer part, [r], and its nonnegative fractional part, $f_r = r - [r]$. Then

$$\frac{\Gamma(x+r)}{\Gamma(x)} = (x)_{[r]} \frac{\Gamma[(x+[r])+f_r]}{\Gamma(x+[r])} .$$

This device has the desirable feature of yielding a larger "effective" value of x while giving a smaller "effective" value for r.

Example: $r = \frac{7}{2}$.

$$\frac{\Gamma(x+\frac{7}{2})}{\Gamma(x)} = (x)_3 \frac{\Gamma[(x+3)+\frac{1}{2}]}{\Gamma(x+3)} ,$$

and the approximation is applied to $\frac{\Gamma[(x+3)+\frac{1}{2}]}{\Gamma(x+3)}$.

<u>When x is "too" small</u>: Then

$$\frac{\Gamma(x+r)}{\Gamma(x)} = \frac{(x)_n}{(x+r)_n} \frac{\Gamma[(x+n)+r]}{\Gamma(x+n)} .$$

Table 9.2.1. Values of x, A, and B such that $\frac{|A-B|}{A} \leqq \epsilon$, where

$$A = \frac{\Gamma(x+r)}{\Gamma(x)} \quad \text{and} \quad B = x^r e^{\frac{r(r-1)}{2x}}$$

r	ϵ = .01			ϵ = .001		
	x	A	B	x	A	B
0.5	1	8.862×10^{-1}	8.825×10^{-1}	2	1.3293×10^{0}	1.3285×10^{0}
1.5	4	8.724×10^{0}	8.786×10^{0}	11	3.7710×10^{1}	3.7748×10^{1}
2.5	11	4.714×10^{2}	4.759×10^{2}	35	7.6384×10^{3}	7.6460×10^{3}
3.5	21	5.179×10^{4}	5.227×10^{4}	66	2.4932×10^{6}	2.4957×10^{6}

r	ϵ = .0001				ϵ = .00001	
	x	A	B	x	A	B
0.5	4	1.93862×10^{0}	1.93847×10^{0}	9	2.958643×10^{0}	2.958621×10^{0}
1.5	36	2.18241×10^{2}	2.18262×10^{2}	112	1.189260×10^{3}	1.189272×10^{3}
2.5	112	1.34981×10^{5}	1.34994×10^{5}	354	2.370304×10^{6}	2.370328×10^{6}
3.5	208	1.32530×10^{8}	1.32543×10^{8}	661	7.474381×10^{9}	7.474456×10^{9}

Example: $x = \frac{1}{4}$; $r = \frac{1}{2}$; $n = 2$, say (n chosen to suit one's needs).

$$\frac{\Gamma(\frac{1}{4}+\frac{1}{2})}{\Gamma(\frac{1}{4})} = \frac{(\frac{1}{4})_2}{(\frac{3}{4})_2} \frac{\Gamma(\frac{9}{4}+\frac{1}{2})}{\Gamma(\frac{9}{4})} = \frac{5}{21} \frac{\Gamma(\frac{9}{4}+\frac{1}{2})}{\Gamma(\frac{9}{4})} ,$$

and the approximation is applied to $\frac{\Gamma(\frac{9}{4}+\frac{1}{2})}{\Gamma(\frac{9}{4})}$.

And, of course, these two devices may both be used to give an approximation of desired quality. In such a case,

$$\frac{\Gamma(x+r)}{\Gamma(x)} = (x)_{[r]} \frac{(x+[r])_n}{(x+r)_n} \frac{\Gamma[(x+[r]+n)+f_r]}{\Gamma(x+[r]+n)}$$

$$= \frac{(x)_{n+[r]}}{(x+r)_n} \frac{\Gamma[(x+[r]+n)+f_r]}{\Gamma(x+[r]+n)} .$$

9.3 Proof of Equation (5.6.19)

Note: Throughout this appendix, unless otherwise indicated, the limits on all integrals are zero to infinity and may not be shown explicitly.

9.3.1 Introduction

Let

$$F(t,q,\alpha,\beta,r) = e^{-\alpha q^{1/r}} e^{-\beta t^N}[f_1(t)\cos(\tfrac{t}{2}q^{1/r})+f_2(t)\sin(\tfrac{t}{2}q^{1/r})], \tag{9.3.1.1}$$

where $r > 0$, $r \neq 1,2,\ldots$, $\alpha \geqq 0$, $\beta \geqq 0$, N is some positive <u>even</u> integer larger than r, and recall that

$$f_1(t) = \frac{\sin A(t)}{\rho(t)} \quad \text{and} \quad f_2(t) = \frac{1}{t}(1 - \frac{\cos A(t)}{\rho(t)}),$$

with $A(t)$ and $\rho(t)$ being given by (5.3.2) and (5.3.3).

We shall prove that

$$\iint F(t,q,o,o,r)dtdq = \int H(t,o,r)dt$$

[cf. (5.5.3) and (5.6.19)]. In particular, we shall prove the following sequence of steps (where $\underset{\gamma}{L}$ means $\lim\limits_{\gamma \to o}$).

$$\pi E(Q^r) \overset{0}{=} \iint F(t,q,o,o,r)dtdq \qquad \text{(true by hypothesis)}$$

$$\overset{1}{=} \iint \underset{\alpha\beta}{LL}\, F(t,q,\alpha,\beta,r)dtdq \qquad \text{(obviously true)}$$

$$\overset{2}{=} \int \underset{\alpha}{L} \int \underset{\beta}{L}\, F(t,q,\alpha,\beta,r)dtdq \qquad \text{(obviously true)}$$

$$\overset{3}{=} \underset{\alpha\beta}{LL} \iint F(t,q,\alpha,\beta,r)dtdq$$

$$\overset{4}{=} \underset{\alpha\beta}{LL} \iint F(t,q,\alpha,\beta,r)dqdt$$

$$\overset{5}{=} \underset{\alpha\beta}{LL} \int F_1(t,\alpha,\beta,r)dt$$

$$\overset{6}{=} \underset{\alpha}{L} \int \underset{\beta}{L}\, F_1(t,\alpha,\beta,r)dt$$

$$\overset{7}{=} \underset{\alpha}{L} \int h(t,\alpha,r)dt$$

$$\overset{8}{=} \underset{\alpha}{L} \int [H(t,\alpha,r)+K(t,\alpha,r)]dt$$

$$\overset{9}{=} \underset{\alpha}{L} \int H(t,\alpha,r)dt + \underset{\alpha}{L} \int K(t,\alpha,r)dt$$

$$\stackrel{10}{=} \int L_{\alpha} H(t,\alpha,r)dt$$

$$\stackrel{11}{=} \int H(t,o,r)dt.$$

The β convergence factor, $e^{-\beta t^N}$, has been introduced in order to establish the absolute convergence of the iterated integral (dtdq), required for application of Fubini's theorem [change of order of integration (step 4)].

In the proof that follows, we shall call upon many properties of various functions. Some of these properties are obvious, others are not. If, for example, a bracketed expression such as [9.3.4.3] should follow some claimed property of some function, it means that that property is established in Subsection 3 of Section 9.3.4 of this appendix. If some function is said to be $L_1(o,\infty)$, say, we mean that that function is Legesgue integrable over (in this case) the positive real line. Also, for convenience, we shall simply write L.D.C.T. when appealing to the Lebesgue dominated convergence theorem [see e.g., Kolmogorov and Fomin (1960), p. 56, Theorem 1].

Finally, before proceeding with the proof itself, the author wishes to point out that the proof was established under the direction of Dr. W. R. Haseltine, many of the essential ideas being his.

9.3.2 Proof, Part I

Let

$$g(q,\alpha,\beta,r) = \int F(t,q,\alpha,\beta,r)dt.$$

It will be shown in Section 10.3.3 that

$$\underset{\beta}{L}\, g(q,\alpha,\beta,r) = g(q,\alpha,o,r), \tag{9.3.2.1}$$

and that there exists an $A > 0$ and a $B > 0$ such that

$$|g(q,\alpha,\beta,r)| < \frac{A}{B+q^{1+\delta}} \qquad (\delta > 0,\ \alpha \geqq 0,\ 0 < \beta < \beta_1 < \infty). \tag{9.3.2.2}$$

Further, (9.3.2.2) holds for $\beta = 0$ by (9.3.2.1). (continuity)

Clearly

$$\frac{A}{B+q^{1+\delta}} \text{ is } L_1(o,\infty) \text{ in } q.$$

We have that

$$\underset{\alpha\beta}{LL} \iint F(t,q,\alpha,\beta,r)dtdq = \underset{\alpha\beta}{LL} \int g(q,\alpha,\beta,r)dq$$

$$= \underset{\alpha}{L} \int \underset{\beta}{L}\, g(q,\alpha,\beta,r)dq \qquad \text{(by L.D.C.T.)}$$

$$= \underset{\alpha}{L} \int g(q,\alpha,o,r)dq \qquad \text{(by (9.3.2.1))}$$

$$= \int \underset{\alpha}{L}\, g(q,\alpha,o,r)dq \qquad \text{(by L.D.C.T.)}$$

$$= \int \underset{\alpha}{L} \int F(t,q,\alpha,o,r)dtdq$$

$$= \iint F(t,q,o,o,r)dtdq.$$

Thus, conditional upon (9.3.2.1) and (9.3.2.2), the propriety of passing the limits outside the integral signs (steps 1 through 4) is established.

Now the bracketed expression in F (9.3.1.1) is clearly bounded since $f_1(t)$ and $f_2(t)$ are bounded, and

$$e^{-\alpha q^{1/r}} \text{ is } L_1(o,\infty) \text{ in } q, \qquad e^{-\beta t^N} \text{ is } L_1(o,\infty) \text{ in } t.$$

Thus, with $\alpha > 0$ and $\beta > 0$, it is clear that the iterated integral

$$\iint |F(t,q,\alpha,\beta,r)| \, dtdq$$

exists. Hence, by Fubini's theorem [see, e.g., Munroe (1953), p. 207],

$$\underset{\alpha\beta}{LL} \iint F(t,q,\alpha,\beta,r)dtdq = \underset{\alpha\beta}{LL} \iint F(t,q,\alpha,\beta,r)dqdt \qquad \text{(step 4)}$$

$$= \underset{\alpha\beta}{LL} \int e^{-\beta t^N}\left[\int F(t,q,\alpha,o,r)dq\right]dt$$

$$= \underset{\alpha\beta}{LL} \int e^{-\beta t^N} h(t,\alpha,r)dt, \qquad \text{(step 5)}$$

where $h(t,\alpha,r)$ is, as previously designated in Section 5, the integrand of (5.6.2). Clearly,

$$|e^{-\beta t^N} h(t,\alpha,r)| \leqq |h(t,\alpha,r)|,$$

and, for $\alpha > 0$, $h(t,\alpha,r)$ is $L_1(o,\infty)$ in t. [9.3.4.3] Hence by L.D.C.T.,

$$\underset{\alpha\beta}{LL} \int e^{-\beta t^N} h(t,\alpha,r)dt = \underset{\alpha}{L} \int \underset{\beta}{L}\, e^{-\beta t^N} h(t,\alpha,r)dt \qquad \text{(step 6)}$$

$$= \underset{\alpha}{L} \int h(t,\alpha,r)dt. \qquad \text{(step 7)}$$

We have seen (Section 5.6) that we may write

$$h(t,\alpha,r) = H(t,\alpha,r) + K(t,\alpha,r), \qquad \text{(step 8)}$$

where H and K are given by (5.6.15) and (5.6.16). It turns out that

$H(t,\alpha,r)$ is $L_1(o,\infty)$ for all $\alpha \geqq 0$. [9.3.4.3]. Because h is L_1 for $\alpha > 0$ and H is L_1 for $\alpha \geqq 0$, it follows that K is L_1 for $\alpha > 0$. Thus step 9 is valid. But we have also seen (Section 5.6) that

$$\int K(t,\alpha,r)dt \equiv 0 \qquad (\text{for } \alpha > 0;\ r > 0,\ r \neq 1,2,\ldots).$$

Hence $\underset{\alpha}{L} \int K(t,\alpha,r)dt = 0$.

Now clearly $|H(t,\alpha,r)| \leqq |H(t,o,r)|$. Hence, by L.D.C.T.,

$$\underset{\alpha}{L}\int H(t,\alpha,r)dt = \int \underset{\alpha}{L}\, H(t,\alpha,r)dt \qquad \text{(step 10)}$$

$$= \int H(t,o,r)dt. \qquad \text{(step 11)}$$

Thus, conditional upon (9.3.2.1) and (9.3.2.2), the proof is complete.

9.3.3 Proof, Part II

We now turn to establish the truth of (9.3.2.1) and (9.3.2.2). Let us define

$$I_1(v,\beta) = \int g_N f_1 \cos(vt)dt,$$

and

$$I_2(v,\beta) = \int g_N f_2 \sin(vt)dt,$$

where $g_N = g_N(t,\beta) = e^{-\beta t^N}$, $f_1 = f_1(t)$, $f_2 = f_2(t)$, and $v = \frac{1}{2}\, q^{1/r}$. Note that, apart from the (v,q) change of variable,

$$I_1(v,\beta) + I_2(v,\beta) = g(q,o,\beta,r),$$

subject to the separate existence of I_1 and I_2.

Before proceeding further, it is perhaps well to remark that, whereas $f_1(t)$ is L_1 (and thus well behaved), $f_2(t)$ is not L_1, and, because of this fact, the proofs to follow are not as neat as they otherwise could have been.

9.3.3.1 Proof of (9.3.2.1). Clearly, $|I_1(v,\beta)| \leqq \int |f_1| dt$, for all $v \geqq 0$, $\beta \geqq 0$. But $f_1(t)$ is $L_1(o,\infty)$. [9.3.4.2] Hence by L.D.C.T., $I_1(v,\beta)$ exists and

$$\underset{\beta}{L}\, I_1(v,\beta) = I_1(v,o). \tag{9.3.3.1.1}$$

For $v > 0$, $I_2(v,o)$ exists as an improper Riemann integral. [9.3.4.4] For $\beta > 0$ and fixed $v > 0$, we see that

$$\begin{aligned}
|I_2(v,\beta)-I_2(v,o)| &= \left| \int_o^{C/v} f_2 \sin(vt)(g_N-1)dt + \int_{C/v}^{\infty} g_N \frac{\sin(vt)}{t}\, dt \right. \\
&\qquad \left. - \int_{C/v}^{\infty} \frac{\sin(vt)}{t}\, dt + \int_{C/v}^{\infty} \frac{\cos A(t)\sin(vt)}{t\rho(t)} (1-g_N)dt \right| \\
&\leqq \int_o^{C/v} [1-g_N(\tfrac{C}{v},\beta)]\, |f_2 \sin(vt)|\, dt + \left| \int_{C/v}^{\infty} g_N \frac{\sin(vt)}{t}\, dt \right| \\
&\qquad + \left| \int_{C/v}^{\infty} \frac{\sin(vt)}{t}\, dt \right| + \int_{C/v}^{\infty} \frac{dt}{t\rho(t)} \quad .
\end{aligned}$$

We may choose C so large that each of the last three integrals is less than $\frac{\epsilon}{4}$, $\epsilon > 0$. Now $|f_2 \sin(vt)| \leqq 2v$. [9.3.4.2] Therefore the first of these four integrals is less than

$$2C\left[1-g_N(\tfrac{C}{v},\beta)\right] = 2C\left[1-e^{-\beta(\frac{C}{v})^N}\right]. \qquad (9.3.3.1.2)$$

It is clear that there exists a β_1 so small that (9.3.3.1.2) is also less than $\frac{\epsilon}{4}$ for all β such that $0 < \beta < \beta_1$. Hence, for $v > 0$,

$$\underset{\beta}{L}\, I_2(v,\beta) = I_2(v,o). \qquad (9.3.3.1.3)$$

It is readily seen that $I_2(o,\beta) = 0$ for all $\beta \geqq 0$. Thus (9.3.3.1.3) holds for all $v \geqq 0$.

It follows from (9.3.3.1.1) and (9.3.3.1.3) that

$$\underset{\beta}{L}\, g(q,o,\beta,r) = g(q,o,o,r),$$

and since

$$g(q,\alpha,\beta,r) = e^{-\alpha q^{1/r}}\, g(q,o,\beta,r),$$

the result (9.3.2.1) also follows.

9.3.3.2 <u>Proof of (9.3.2.2)</u>. We shall establish bounds of two types for $g(q,o,\beta,r)$, from which the result (9.3.2.2) will be seen to follow.

Clearly $|I_1(v,\beta)|$ is bounded for all $v \geqq 0$, $\beta \geqq 0$ (<u>cf</u>. first paragraph of previous subsection).

For $v > 0$,

$$|I_2(v,\beta)| = \left|\int_o^{C/v} g_N f_2 \sin(vt)dt + \int_{C/v}^{\infty} g_N \frac{\sin(vt)}{t}\,dt - \int_{C/v}^{\infty} g_N \frac{\cos A(t)\sin(vt)}{t\rho(t)}\,dt\right|$$

$$\leqq \int_o^{C/v} |f_2 \sin(vt)|dt + \left|\int_{C/v}^{\infty} g_N \frac{\sin(vt)}{t}\,dt\right| + \int_{C/v}^{\infty} \frac{dt}{t\rho(t)} \quad .$$

The first integral $\leqq \int_0^{C/v} 2v dt = 2C$. Given $C > 0$, the third integral is bounded uniformly in v for $0 < v < v_1 < \infty$ (v_1 arbitrarily large, but finite), because $\frac{1}{t\rho(t)} = O\left(\frac{1}{t^{1+\eta}}\right)$, for t large. [9.3.4.2] In the second integral, let $C = 2n\pi$, where n is some positive integer, and make the transformation $x = vt$ to obtain

$$\left| \int_{2n\pi}^{\infty} e^{-\beta\left(\frac{x}{v}\right)^N} \frac{\sin x}{x} dx \right| \leqq 2 \frac{e^{-\beta\left(\frac{2n\pi}{v}\right)^N}}{2n\pi} \leqq \frac{1}{n\pi} .$$

Hence, recalling that $I_2(o,\beta) = 0$ for all $\beta \geqq 0$, it follows that

$$|I_2(v,\beta)| < M \qquad (\text{for all } v,\ 0 \leqq v < v_1 < \infty;\ \beta \geqq 0). \qquad (9.3.3.2.1)$$

From (9.3.3.2.1), and the fact that $|I_1(v,\beta)|$ is bounded for all $v \geqq 0$ and $\beta \geqq 0$, it follows that

$$|g(q,o,\beta,r)| < B_1 \qquad (0 \leqq q < q_1 < \infty,\ \beta \geqq 0). \qquad (9.3.3.2.2)$$

Thus a simple bound on $g(q,o,\beta,r)$ has been obtained. Now we seek a bound of a different character.

In what follows, we shall be concerned with derivatives of the products $g_N f_j$, $j = 1,2$. By Leibniz' rule,

$$(g_N f_j)^{(m)} = \sum_{s=o}^{m} \binom{m}{s} g_N^{(s)} f_j^{(m-s)}. \qquad (9.3.3.2.3)$$

It may be shown (by mathematical induction) that

$$g_N^{(p)} = g_N \sum_{k=1}^{p} A(k,p,N)\, \beta^{\frac{p}{N}} (\beta t^N)^{\frac{Nk-p}{N}} . \qquad (9.3.3.2.4)$$

Clearly, for $\beta > 0$, each term in $g_N^{(p)}$ is $L_1(-\infty,\infty)$ in t if we require that $Nk-p \geqq 0$, thus if $p \leqq N$ (recall that N is an even positive integer). Also, it may be shown that, for $p \geqq 1$, the $f_j^{(p)}$, $j = 1,2$, are $L_1(o,\infty)$. [9.3.4.5]

Because the integrands in $I_1(v,\beta)$ and $I_2(v,\beta)$ are seen to be even functions of t, we may write

$$I_1(v,t) = \tfrac{1}{2}\,\mathcal{R}\int_{-\infty}^{\infty} e^{ivt} g_N f_1 dt,$$

and

$$I_2(v,t) = \tfrac{1}{2}\,\mathcal{I}\int_{-\infty}^{\infty} e^{ivt} g_N f_2 dt.$$

Let v and β be strictly positive. Then, by integration by parts ($u = g_N f_j$), we obtain

$$\int_{-\infty}^{\infty} e^{ivt} g_N f_j dt = \left[\frac{e^{ivt} g_N f_j}{iv}\right]_{-\infty}^{\infty} + \frac{i}{v}\int_{-\infty}^{\infty} e^{ivt}(g_N f_j)' dt .$$

Now the term in brackets vanishes. Similarly, integrating by parts N times, we have that

$$\int_{-\infty}^{\infty} e^{ivt} g_N f_j dt = \left(\frac{i}{v}\right)^N \int_{-\infty}^{\infty} e^{ivt}(g_N f_j)^{(N)} dt.$$

Because N is an even positive integer, it follows that

$$\left.\begin{aligned} I_1(v,\beta) &= \mathcal{R}\left(\frac{i}{v}\right)^N \int_{o}^{\infty} e^{ivt}(g_N f_1)^{(N)} dt, \\ &\text{and} \\ I_2(v,\beta) &= \mathcal{I}\left(\frac{i}{v}\right)^N \int_{o}^{\infty} e^{ivt}(g_N f_2)^{(N)} dt . \end{aligned}\right\} \qquad (9.3.3.2.5)$$

Now, were we to substitute for $(g_N f_j)^{(N)}$, from (9.3.3.2.3), in (9.3.3.2.5), and then consider termwise integration, it is clear that all integrals, with one possible exception, are bounded independently of β, for $0 < \beta < \beta_1 < \infty$ [this follows because f_1 is L_1 and the $f_j^{(p)}$, $p \geqq 1$, are L_1]. The sole possible exception is

$$\int_0^\infty e^{ivt} f_2 g_N^{(N)} dt \ .$$

We now show that even this integral is bounded independently of β, for $0 < \beta < \beta_1 < \infty$.

Let

$$D_k = \max_{x=\beta t^N} \left[x^{k-1} e^{-\frac{x}{2}} \right],$$

and let

$$D_{(N)} = \sum_{k=1}^{N} |A(k,N,N)| D_k .$$

Then

$$|g_N^{(N)}| < \beta \ D_{(N)} \ e^{-\frac{\beta}{2} t^N} .$$

We have seen that $f_2(t)$ is bounded, $|f_2(t)| < b_2$, say. Then

$$\left| \int_0^{\infty} e^{ivt} f_2 g_N^{(N)} dt \right| < b_2 \beta D_{(N)} \int_0^{\infty} e^{-\frac{\beta}{2} t^N} dt$$

$$\overset{(y=\frac{\beta}{2} t^N)}{=} \frac{b_2}{N} 2^{\frac{1}{N}} \beta^{\frac{N-1}{N}} D_{(N)} \int_0^{\infty} y^{\frac{1}{N}-1} e^{-y} dy$$

$$= \frac{b_2}{N} 2^{\frac{1}{N}} \beta^{\frac{N-1}{N}} D_{(N)} \Gamma(\tfrac{1}{N}),$$

and this function of β is certainly bounded for $0 < \beta < \beta_1 < \infty$.

It follows then that we have established that

$$|I_1(v,\beta)+I_2(v,\beta)| < \frac{M_1}{v^N} \qquad (\text{for all } v > 0,\ 0 < \beta < \beta_1 < \infty),$$

from which it further follows that

$$|g(q,o,\beta,r)| < \frac{B_2}{q^{1+\delta}} \qquad (q > 0,\ 0 < \beta < \beta_1 < \infty) \qquad (9.3.3.2.6)$$

(recall that $N > r$ and $v = \frac{1}{2} q^{1/r}$), where $\delta > 0$. Thus we have a second bound for $g(q,o,\beta,r)$.

We have left only to see that (9.3.3.2.2), together with (9.3.3.2.6), yields (9.3.2.2). Let q_o be the solution of

$\frac{B_2}{q^{1+\delta}} = B_1$, i.e., $q_o = \left(\frac{B_2}{B_1}\right)^{\frac{1}{1+\delta}}$ (implicit, here, is that $q_o \leqq q_1$ [cf. (9.3.3.2.2)]; recall that q_1 may be arbitrarily large).

For $0 \leqq q \leqq q_o$,

$$|g(q,o,\beta,r)| < B_1 = \frac{2B_2}{\frac{2B_2}{B_1}} \leqq \frac{2B_2}{\frac{B_2}{B_1} + q^{1+\delta}} \quad .$$

For $q \geqq q_o$,

$$|g(q,o,\beta,r)| < \frac{B_2}{q^{1+\delta}} = \frac{2B_2}{2q^{1+\delta}} \leqq \frac{2B_2}{\frac{B_2}{B_1} + q^{1+\delta}} \quad .$$

Thus, letting $A = 2B_2$ and $B = \frac{B_2}{B_1}$, we have

$$|g(q,o,\beta,r)| < \frac{A}{B+q^{1+\delta}} \qquad (0 < \beta < \beta_1 < \infty),$$

and the desired result (9.3.2.2) follows since

$$|g(q,\alpha,\beta,r)| \leqq |g(q,o,\beta,r)| \quad .$$

9.3.4 Properties of Functions

We shall now establish the several properties of the various functions of t, $0 \leqq t < \infty$, cited in Sections 9.3.2 and 9.3.3. Clearly, all of the functions with which we deal are continuous in the open interval (o,∞). It follows that all such functions are $L_1(o,\infty)$ provided that they are $O\left(\frac{1}{t^{1-\delta_1}}\right)$, $\delta_1 > 0$, for t small, and $O\left(\frac{1}{t^{1+\delta_2}}\right)$, $\delta_2 > 0$, for t large.

9.3.4.1 The Functions A(t) and ρ(t).

$A(0) = 0$.

$A(t) = O(t)$, t small.

$\rho(t)$ is a monotonic increasing function of t.

$\rho(0) = 1.$

$\rho(t) = O(1)$, t small.

$\frac{1}{\rho(t)} = O\left(\frac{1}{t^\eta}\right)$, $\eta > 0$, t large. This property may be seen as follows: The exponential part of $\rho(t)$ behaves like a constant (in fact, like $e^{\Sigma\lambda_i}$) for t large. The product part, $\Pi\,(1+\sigma_i^4 t^2)^{\frac{\nu_i}{4}}$, is larger than $(\sigma_1^2 t)^{\nu_1/2}$, from which the result follows.

9.3.4.2 <u>The Functions $f_1(t)$ and $f_2(t)$.</u>

$f_1(t)$ is a bounded function of t.

$f_1(t) = O(1)$, t small [see (5.6.5)].

$|f_1(t)| \leqq \frac{1}{t\rho(t)} = O\left(\frac{1}{t^{1+\eta}}\right)$, $\eta > 0$, t large.

$f_1(t)$ is $L_1(o,\infty)$.

$f_2(t)$ is a bounded function of t.

$f_2(t) = O(t)$, t small [see (5.6.5)].

$0 < f_2(t) < \frac{2}{t}$, $0 < t < \infty$.

$|f_2(t)\sin(vt)| \leqq 2v$, $0 \leqq t, v < \infty$.

9.3.4.3 <u>The Functions $h(t,\alpha,r)$ and $H(t,\alpha,r)$.</u> Because $f_1(t)$ and $f_2(t)$ are bounded, it is clear that there exists a constant $C_1 > 0$ such that

$$|h(t,\alpha,r)| \leqq \frac{C_1}{\left(\alpha^2 + \frac{t^2}{4}\right)^{\frac{r}{2}}} = O\left(\frac{1}{\alpha^r}\right), \text{ t small.}$$

Also, because $|f_1(t)| \leqq \frac{1}{t\rho(t)} \leqq \frac{1}{t}$ and $|f_2(t)| < \frac{2}{t}$, it is clear that there exists a constant $C_2 > 0$ such that

$$|h(t,\alpha,r)| < \frac{C_2}{t\left(\alpha^2+\frac{t^2}{4}\right)^{\frac{r}{2}}} = O\left(\frac{1}{t^{1+r}}\right), \text{ t large.}$$

Thus it follows that $h(t,\alpha,r)$ is $L_1(o,\infty)$ for all $\alpha > 0$ (recall that $r > 0$).

Now, with respect to $H(t,\alpha,r)$, it follows from (5.6.13) that

$$|H(t,\alpha,r)| = O\left(\frac{1}{t^{1-\theta}}\right), \quad \theta > 0, \quad \text{t small.}$$

Also, from (5.6.12), it is clear that

$$k_1(t,r) = O(t^{2[\frac{r-1}{2}]}) = O(t^{r-1-\delta_1}), \qquad \delta_1 > 0, \text{ t large,}$$

and

$$k_2(t,r) = O(t^{2[\frac{r}{2}]-1}) = O(t^{r-1-\delta_2}), \qquad \delta_2 > 0, \text{ t large}$$

(recall that $r \neq 1,2,3,\ldots$). Therefore,

$$|H(t,\alpha,r)| = O\left(\frac{1}{t^{1+\delta_3}}\right), \qquad \delta_3 > 0, \text{ t large.}$$

Hence it follows that $H(t,\alpha,r)$ is $L_1(o,\infty)$ for all $\alpha \geqq 0$.

9.3.4.4 <u>The Function $I_2(v,o)$.</u> We have that

$$I_2(v,o) = \int_o^{\infty} \left[\frac{\sin(vt)}{t} - \frac{\sin(vt)\cos A(t)}{t\rho(t)}\right] dt.$$

Clearly,

$$\frac{\sin(vt)\cos A(t)}{t\rho(t)} \leqq \begin{cases} \left|\frac{\sin(vt)}{t}\right| = O(v), & t \text{ small}, \\ \\ \frac{1}{t\rho(t)} = O\left(\frac{1}{t^{1+\eta}}\right), \ \eta > 0, & t \text{ large}. \end{cases}$$

Therefore

$$\frac{\sin(vt)\cos A(t)}{t\rho(t)} \text{ is } L_1 \ (o,\infty) \text{ in } t, \text{ for all } v \geqq 0.$$

For $v > 0$, $\int_0^\infty \frac{\sin(vt)}{t}\,dt$ exists $(= \frac{\pi}{2})$ as an improper Riemann integral, but $\sin(vt)/t$ is not L_1.

Hence, for $v > 0$, it follows that $I_2(v,o)$ exists as an improper Riemann integral.

9.3.4.5 <u>The Functions $f_j^{(p)}(t)$, $p \geqq 1$, $j = 1,2$.</u> Consider the first derivatives of $f_1(t)$ and $f_2(t)$:

$$f_1'(t) = \frac{1}{\rho(t)}\left[\frac{A'(t)\cos A(t)}{t} - \frac{\sin A(t)}{t^2} - \frac{\sin A(t)}{t}\left(\frac{\rho'(t)}{\rho(t)}\right)\right],$$

$$f_2'(t) = \frac{1}{\rho(t)}\left[\frac{\cos A(t)}{t^2} + \frac{A'(t)\sin A(t)}{t} + \frac{\cos A(t)}{t}\left(\frac{\rho'(t)}{\rho(t)}\right)\right] - \frac{1}{t^2}\,.$$

It is readily seen that $A'(t)$ and $\frac{\rho'(t)}{\rho(t)}$ $[= \frac{d\log\rho(t)}{dt}]$ are rational functions of t, being $O\left(\frac{1}{t^2}\right)$ and $O\left(\frac{1}{t}\right)$, respectively, for t large. Because they are rational functions, it follows that $A''(t)$ and $\left(\frac{\rho'(t)}{\rho(t)}\right)'$ are $O\left(\frac{1}{t^3}\right)$ and $O\left(\frac{1}{t^2}\right)$, respectively, for t large, and so on.

Thus it is clear that the terms of $f_1'(t)$ and $f_2'(t)$ are, apart from the factor $\frac{1}{\rho(t)}$, rational functions or products of sines and cosines by rational functions, all rational functions being $O\left(\frac{1}{t^2}\right)$, t large. Clearly, the factor $\frac{1}{\rho(t)}$ and its derivatives can only aid convergence because

$$\frac{1}{\rho(t)} = O\left(\frac{1}{t^\eta}\right),$$

$$\left(\frac{1}{\rho(t)}\right)' = -\frac{1}{\rho(t)}\left(\frac{\rho'(t)}{\rho(t)}\right) = O\left(\frac{1}{t^{1+\eta}}\right),$$

etc., $\eta > 0$, t large.

Hence, $f_1''(t)$ and $f_2''(t)$ are $O\left(\frac{1}{t^3}\right)$, t large. Similarly, we conclude, for $p \geqq 1$, that

$$f_j^{(p)}(t) = O\left(\frac{1}{t^{p+1}}\right), \quad t \text{ large}, \; j = 1,2.$$

Because $f_1(t)$ and $f_2(t)$ are analytic at $t = 0$ [in particular, their power series expansions (5.6.11) are valid in some neighborhood of the origin], it follows that $f_j^{(p)}(t)$ is $O(1)$, t small, $j = 1,2$.

Hence, for $p \geqq 1$,

$$f_j^{(p)}(t) \text{ is } L_1(o,\infty), \; j = 1,2.$$