Abstract

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While the computational packages **LiE**, **Gap4**, **Chevie**, and **Magma** are sufficient for work with Lie Groups and their corresponding Lie Algebras, no such packages exist for computing the k-structure of a group or the structure of symmetric spaces. My goal is to examine the k-structure of groups and the structure of symmetric spaces and arrive at various algorithms for computing in these spaces.

ALGORITHMS FOR COMPUTING RESTRICTED ROOT SYSTEMS AND WEYL GROUPS

BY

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For Phil
and the rest of my family

Biography

Tracey Cicco was born on October 1, 1976 in Raleigh, North Carolina, where she received her elementary and secondary education. She received both her Bachelor of Science and her Master of Science degrees in Mathematics at North Carolina State University.

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Chapter 1

Introduction

In the last two decades computer algebra has had a major impact on many areas of mathematics. Best known are its accomplishments in number theory, algebraic geometry and group theory. Several people have also started to devise and implement algorithms related to Lie theory. The most noteworthy examples of this are the package LiE written by CAN (see [MLL92]) and the packages *Coxeter* and *Weyl* by J. Stembridge, which are written in Maple (see [Ste92]). In the LiE package most of the basic combinatorial aspects of Lie theory have been implemented, following the excellent description and tables in [Bou81]. These mainly describe the fine structure of a root system with its Weyl group for a maximal torus of a reductive group defined over an algebraically closed field. (Or similarly of the k-split form of a group defined over a field k). There remain many, more complex aspects of Lie theory for which it would be useful to have a computer implementation of the structure. In this thesis we lay the foundation for a computer algebra package for computations related to all k-forms of reductive groups defined over non algebraically closed fields. In the following we will call k-forms of these reductive groups: *reductive k-groups*.

For reductive k-groups there is an additional root system and Weyl group which characterizes the k-structure of the group. This additional fine structure comes from the root system of a maximal k-split torus A of G together with its Weyl group and the multiplicities of the roots. The maximal k-split torus A is contained in a maximal k-torus T and the root system and Weyl group of the maximal k-split torus A can be identified with the projection of the root system of T to A and similarly the Weyl groups can be identified with the quotient of two subgroups of the Weyl group of T. This fine structure of the two root systems with their Weyl groups and multiplicities of the roots plays a fundamental role in all studies of reductive k-groups and their applications.

In the case of reductive groups over algebraically closed fields the integrate fine structure related to the root systems with their Weyl groups has been implemented in several symbolic computation packages, like LiE, Maple, GAP4, Chevie, and Magma. These packages have become an indispensable tool for scientists in many areas of mathematics and physics. For reductive k-groups none of this fine structure has been implemented yet in a computer algebra package, although such a package would be extremely useful for many scientists as well. There are several reasons for this. The main reason is that the fine structure of these reductive k-groups a lot more complicated than that of the Lie groups, because instead of just 1 root system, there are 2 root systems which are closely entangled.

In this thesis we make the first step towards building a computer algebra package with which one can compute the fine structure of reductive k-groups.

All this fine structure of these reductive k-groups can also be computed in the Lie algebra setting, which simplifies some of the computations. To compute the fine structure of these reductive k-groups it does not suffice to compute the two root systems involved together with their Weyl groups. In many problems about these reductive k-groups one needs to know which roots project down to a root in a restricted root system and also one often needs representatives for elements of the Weyl group of the restricted root system in terms of representatives of the Weyl group of the related maximal torus. For example to compute nice bases for the root space decomposition of a reductive Lie algebra with respect to a maximal k-split torus one needs to decompose all root subspaces for roots of a maximal k-split toral subalgebra $\mathfrak{a}$ as a sum of root subspaces of a maximal toral subalgebra containing $\mathfrak{a}$. From the Γ-diagram corresponding to a reductive k-group (see section 2.9) one can easily determine this for the roots in a basis, but for all other roots we need the Weyl group and its action on the root subspaces of the maximal toral subalgebra to compute the decomposition of the root subspace of an arbitrary root of $\mathfrak{a}$. So computing the fine structure of these reductive k-groups will include computing representatives for the restricted Weyl groups in terms of Weyl group elements of the maximal toral subalgebra and also computing all the roots that project down to a root in a restricted root system.

A classification of the of the Γ-indices corresponding to reductive k-groups was given by Tits in [Tit66]. For $\mathfrak{g}$ simple defined over a field k which is algebraically closed, the real numbers, the p-adic numbers, a finite field or a number field there are 45 different types of Γ-indices which are absolutely irreducible. For each of these 45 types of Γ-indices we give an algorithm to compute their fine structure, which is roughly as follows. Let Φ denote the root system of the maximal toral subalgebra and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ a basis of Φ compatible with the Γ-index.

(1) Using the Γ-index, determine the elements of Γ.

(2) Find a basis $\bar{\Delta} = \{\lambda_1, \dots, \lambda_r\}$ of the restricted root system in terms of Δ by finding the projection of each α_j in Δ, and determine each λ_i in terms of α_j.

(3) Note the type of restricted root system, and determine a representative w_i in the Weyl group of the maximal toral subalgebra for each s_{λ_i}, with $\lambda_i \in \bar{\Delta}$. This gives representatives of the Weyl group of $\Phi(\mathfrak{a})$ in the Weyl group of the maximal toral subalgebra.

(4) Determine $\Phi(\lambda_i) := \{\alpha \in \Phi | \pi(\alpha) = \lambda_i\}$ for each λ_i in Step (1).

(5) Find the roots in $\Phi(\mathfrak{a})^+$ using the Weyl group as determined in step (3).

(6) Compute $\Phi(\lambda)$ for each $\lambda \in \Phi(\mathfrak{a})^+$ by using the fact that $\lambda = w(\lambda_i)$ for some $w \in W(\mathfrak{a})$, $\lambda_i \in \bar{\Delta}$ and using the fact that $\Phi(\lambda) = \Phi(w(\lambda_i)) = \tilde{w}\Phi(\lambda_i)$, where $\tilde{w}$ is a representative of $w \in W(\mathfrak{a})$ in the Weyl group of the maximal toral subalgebra and $\tilde{w}$ is a product of the w_i as above.

Note that $\Phi(-\lambda) = -\Phi(\lambda)$, so to determine the structure of $\Phi(\mathfrak{a})$, it suffices to determine $\Phi(\lambda)$ for $\lambda \in \Phi(\mathfrak{a})^+$.

The computation of this fine structure can be used to compute nice bases for the root space decomposition with respect to a maximal k-split toral subalgebra.

This algorithm can be implemented in any of the computer algebra packages in which the structure of Lie algebras and Lie groups has already been incorporated. We included tables for the data needed in the various steps of the algorithm and we worked out an example illustrating the algorithm (see chapter 7).

A brief summary of the contents follows. In chapter 2 we will lay the theoretical foundation for developing algorithms for a computer algebra package for the fine structure of reductive k-groups by proving a number of results about group actions on root data. We also show how these results can be used to compute the root space decomposition of a reductive Lie algebra with respect to a maximal k-split torus.

Chapter 3 outlines the six steps of the algorithm to be followed for each type of Γ-index. In chapter 4, we determine the action of Γ and compute the projection of each α_i for each type of Γ-index. This completes steps 1 and 2 of the algorithm. The results of these first two steps are included in tables at the end of chapter 4.

In chapter 5, we determine the Weyl group representative w_i for each s_{λ_i}. The findings are summarized in a table at the end of the chapter. This completes step 3 of the algorithm.

In chapter 6, we give $\Phi(\lambda_i)$ for each $\lambda_i \in \bar{\Delta}$ to complete step 4 of the algorithm. Then we determine what the admissible root strings are to complete step 5. The final component of chapter 6 is the structure of $\Phi(\mathfrak{a})^+$, which is the last step of the algorithm.

An example demonstrating each step of the algorithm and the root space decomposition is given in chapter 7.

Chapter 2

Background

2.1 Root Data

To deal with the notion of root system in reductive groups it is quite useful to work with the notion of root datum. First we review a few facts about root data. These results can be found in [Spr79, §1].

A *root datum* is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X \times X^\vee \to \mathbb{Z}$, denoted by $\langle \cdot\, , \, \cdot \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ with a bijection $\alpha \to \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$ we define endomorphisms s_α and $s_{\alpha^\vee}$ of X and $X^\vee$, respectively, by

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha, \;\; s_{\alpha^\vee}(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee. \tag{2.1}$$

The following two axioms are imposed:

(1) If $\alpha \in \Phi$, then $\langle \alpha, \alpha^\vee \rangle = 2$;

(2) if $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (2.1), that $s_\alpha^2 = 1$, $s_\alpha(\alpha) = -\alpha$ and similarly for $s_\alpha^\vee$. Put $E = X \otimes_{\mathbb{Z}} \mathbb{R}$. For a subset Ω of X we denote the subgroup of X generated by Ω by $\Omega_{\mathbb{Z}}$ and write $\Omega_{\mathbb{Q}} := \Omega_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Omega_{\mathbb{R}} := \Omega_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. We consider $\Omega_{\mathbb{Q}}$ and $\Omega_{\mathbb{R}}$ as linear subspaces of E. Let $Q := \Phi_{\mathbb{Z}}$ be the subgroup of X generated by Φ and put $V = \Phi_{\mathbb{R}} = Q \otimes_{\mathbb{Z}} \mathbb{R}$. We consider V as a linear subspace of E. Define similarly the subgroup $Q^\vee$ of $X^\vee$ and the vector space $V^\vee$. If $\Phi \neq \emptyset$, then Φ is a not necessarily reduced root system in V in the sense of Bourbaki [Bou81, Ch.VI, no. 1].

The rank of Φ is by definition the dimension of V. The root datum Ψ is called semisimple if $X \subset V$. We observe that $s_{\alpha^\vee} = {}^t s_\alpha$ and $s_\alpha(\beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$ as follows by an easy computation (c.f. Springer [Spr79, 1.4]). Let $(\cdot\ ,\ \cdot)$ be a positive definite symmetric bilinear form on E, which is $\mathrm{Aut}(\Phi)$ invariant. Now the $s_\alpha(\alpha \in \Phi)$ are Euclidean reflections, so we have

$$\langle \chi, \alpha^\vee \rangle = 2(\alpha, \alpha)^{-1}.(\chi, \alpha) \qquad (\chi \in E, \alpha \in \Phi).$$

Consequently, we can identify $\Phi^\vee$ with the set $\{2(\alpha,\alpha)^{-1}\alpha \mid \alpha \in \Phi\}$ and $\alpha^\vee$ with $2(\alpha,\alpha)^{-1}\alpha$. If $\phi \in \mathrm{Aut}(X,\Phi)$, then its transpose ${}^t\phi$ induces an automorphism of $\Phi^\vee$, so Φ induces a unique automorphism in $\mathrm{Aut}(\Psi)$, the set of automorphisms of the root datum Ψ. We shall frequently identify $\mathrm{Aut}(X,\Phi)$ and $\mathrm{Aut}(\Psi)$.

For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha \in \Phi_1$.

Example 2.1. If T is a torus in a reductive group G such that $\Phi(T)$ is a root system with Weyl group $W(T)$, then the root datum associated with the pair (G,T) is $(X^*(T)$, $\Phi(T)$, $X_*(T)$, $\Phi^\vee(T))$, where $X^*(T)$ is the set of characters of T and $X_*(T)$ is the set of 1-parameter subgroups of T. So in each of the cases that T is either a maximal torus of G, a maximal k-split torus of G, a maximal θ-split torus of G or a maximal (θ,k)-split torus of G, the above root datum exists.

Remark 1. If T_1 and T_2 are tori and ϕ is a homomorphism of T_1 into T_2, then the mapping ${}^t\phi$ of $X^*(T_2)$ into $X^*(T_1)$, defined by

$${}^t\phi(\chi_2) = \chi_2 \circ \phi, \quad \chi_2 \in X^*(T_2) \tag{2.2}$$

is a module homomorphism. If ϕ is an isomorphism, then ${}^t\phi^{-1}$ is a module isomorphism from $(X^*(T_1), \Phi(T_1))$ onto $(X^*(T_2), \Phi(T_2))$.

2.2 Actions on root data

In the study of algebraic k-groups, symmetric spaces, and symmetric k-varieties, we encounter several root systems and Weyl groups. The root datum representing the k-structure (or the symmetric space) can be obtained from a group action on the root datum of a maximal torus. In the case of the k-structure, this is the Galois group of the splitting extension.

In the case of a symmetric space, this is a group of order 2 coming from an involution and for the case of symmetric k-varieties, it is a combination of these. In this section we give some general results of a group acting on a root datum, and this can be applied to each of these cases. We will mainly focus on the action of the Galois group on this root datum. In this case the group action is obtained as follows:

Let G be a reductive k-group, T a maximal k-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, K a finite Galois extension of k which splits T and $\Gamma = \mathrm{Gal}(K/k)$ the Galois group of K/k. If $\phi \in \mathrm{Aut}(G,T)$ is defined over k, then $\phi^\star := {}^t(\phi|T)^{-1}$ satisfies ${\phi^\star}^\sigma = \phi^\star$, i.e.

$$\sigma\phi^\star = \phi^\star\sigma \text{ for all } \sigma \in \Gamma. \tag{2.3}$$

Γ acts on (X,Φ), leading to a natural restricted root system. It turns out these are precisely the restricted root systems related to a maximal k-split torus. In the next sections we will analyze this fine structure of the restricted root systems and Weyl groups.

2.3 Restricted Roots

Let Ψ be a root datum with $\Phi \neq \emptyset$, as in 2.1 and let $\mathcal{G}$ be a finite group acting on Ψ. For $\sigma \in \mathcal{G}$ and $\chi \in X$ we will also write $\sigma(\chi)$ for the element $\sigma.\chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ. Now define the following:

$$X_0 = X_0(\mathcal{G}) = \left\{ \chi \in X \mid \sum_{\sigma \in \mathcal{G}} \sigma(\chi) = 0 \right\} \tag{2.4}$$

Then X_0 is a co-torsion free submodule of X, invariant under the action of $\mathcal{G}$. Let $\Phi_0 = \Phi_0(\mathcal{G}) = \Phi \cap X_0$. This is a closed subsystem of Φ invariant under the action of $\mathcal{G}$. Denote the Weyl group of Φ_0 by W_0 and identify it with the subgroup of $W(\Phi)$ generated by the reflections $s_\alpha, \alpha \in \Phi_0$. Put $W^{\mathcal{G}} = \{w \in W \mid w(X_0) = X_0\}$, $\bar{X}_{\mathcal{G}} = X/X_0(\mathcal{G})$ and let π be the natural projection from X to $\bar{X}_{\mathcal{G}}$. If we take $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ to be the annihilator of X_0 and $Y = X^*(A)$, then Y may be identified with $\bar{X}_{\mathcal{G}} = X/X_0$. Let $\bar{\Phi}_{\mathcal{G}} = \pi(\Phi - \Phi_0(\mathcal{G}))$ denote the set of *restricted roots of* Φ *relative to* $\mathcal{G}$.

Remark 2. X_0 is the annihilator of a maximal k-split torus A of T. $\bar{\Phi}_{\mathcal{G}}$ is the root system of $\Phi(A)$ with Weyl group $\bar{W}_{\mathcal{G}}$.

We define now an order on (X, Φ) related to the action of $\mathcal{G}$ as follows.

Definition 1. A linear order on X which satisfies

$$\text{if } \chi \succ 0 \text{ and } \chi \notin X_0, \text{ then } \chi^\sigma \succ 0 \text{ for all } \sigma \in \mathcal{G} \tag{2.5}$$

is called a *$\mathcal{G}$-linear order*. A fundamental system of Φ with respect to a $\mathcal{G}$-linear order is called a *$\mathcal{G}$-fundamental system of* Φ or a *$\mathcal{G}$-basis of* Φ.

A $\mathcal{G}$-linear order on X induces linear orders on $Y = X/X_0$ and X_0, and conversely, given linear orders on X_0 and on Y, these uniquely determine a $\mathcal{G}$-linear order on X, which induces the given linear orders (i.e., if $\chi \notin X_0$, then define $\chi \succ 0$ if and only if $\pi(\chi) \succ 0$). Instead of the above $\mathcal{G}$-linear order one can give a more general definition of a linear order on X, using only the fact that X_0 is a co-torsion free submodule of X (see [Sat71, §2.1]).

In the following we give a number of properties of an $\mathcal{G}$-linear order on X.

2.4 Restricted fundamental system

Fix a $\mathcal{G}$-linear order $\succ$ on X, let Δ be a $\mathcal{G}$-fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0. Let $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ be the annihilator of X_0 and define $\bar{\Delta}_{\mathcal{G}} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to A or also a *restricted fundamental system* of $\bar{\Phi}_{\mathcal{G}}$. The following proposition lists some properties of these fundamental systems.

Proposition 1. *Let X, X_0, Φ, Φ_0, $\bar{\Phi}_{\mathcal{G}}$, etc. be defined as above and let Δ, Δ' be $\mathcal{G}$-fundamental systems of Φ. Then we have the following*

(1) $\Delta_0 = \Delta \cap \Phi_0$.

(2) $\Delta = \Delta'$ *if and only if* $\Delta_0 = \Delta'_0$ *and* $\bar{\Delta}_{\mathcal{G}} = \bar{\Delta}'_{\mathcal{G}}$.

(3) If $\bar{\Delta}_{\mathcal{G}} = \bar{\Delta}'_{\mathcal{G}}$, *then there exists a unique* $w_0 \in W_0$ *such that* $\Delta' = w_0\Delta$.

Proof. (1). Assume $\operatorname{rank}\Phi = n$, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and $\Delta_0 = \{\alpha_1, \dots, \alpha_m\}$, $m \le n$. It suffices to show that each $\alpha \in \Phi_0$ is a linear combination of the α_i's in Δ_0. Write $\alpha = \sum_{i=1}^n r_i \alpha_i$, $r_i \in \mathbb{Z}$. We may assume $\alpha \succ 0$, i.e. $r_i \ge 0$. Since $\alpha \in \Phi_0$ we have $\sum_{\sigma \in \mathcal{G}} \alpha^\sigma = 0$. Since

$\alpha_1, \ldots, \alpha_m \in \Delta_0$ we get: $\sum_{\sigma \in \mathcal{G}} \sigma(\alpha) = \sum_{\sigma \in \mathcal{G}} \sigma(r_{m+1}\alpha_{m+1} + \ldots r_n\alpha_n)$. By the definition of $\mathcal{G}$-linear order $\sigma(\alpha_j) \succ 0$ for $m+1 \leq j \leq n$ and $\sigma \in \mathcal{G}$. So if any of the $r_j \neq 0$, $m+1 \leq j \leq n$, then $\sum_{\sigma \in \mathcal{G}} \sigma(\alpha) \succ 0$, what contradicts the fact that $\alpha \in \Phi_0$.

(2). It suffices to show $\Leftarrow$. Let $\succ$ be the $\mathcal{G}$-linear order defining Δ and $\succ'$ the $\mathcal{G}$-linear order defining Δ'. Let $\Phi^+ = \{\alpha \in \Phi \mid \alpha \succ 0\}$ and $\Phi^+_{\succ'} = \{\alpha \in \Phi \mid \alpha \succ' 0\}$. We will show that $\Phi^+ = \Phi^+_{\succ'}$, what implies the result. Let $\alpha \in \Delta$. If $\alpha \in \Delta_0 = \Delta'_0$, then $\alpha \succ' 0$. If $\alpha \notin \Delta_0$, then $\pi(\alpha) \in \bar{\Delta} = \bar{\Delta}'$, hence also $\alpha \succ' 0$. Since Δ determines Φ^+, it follows that $\Phi^+ \subset \Phi^+_{\succ'}$. The same argument shows $\Phi^+_{\succ'} \subset \Phi^+$, hence $\Phi^+ = \Phi^+_{\succ'}$.

(3). Since Δ_0 and Δ'_0 are fundamental systems of Φ_0, there exists a unique $w_0 \in W_0$ such that $w_0\Delta_0 = \Delta'_0(\mathcal{G})$. But then $w_0\Delta \cap \Phi_0 = \Delta'_0(\mathcal{G})$ and $\pi(w_0\Delta) = \bar{\Delta}_{\mathcal{G}} = \bar{\Delta}'_{\mathcal{G}}$. So by (2) $\Delta' = w_0\Delta$. □

2.5 Restricted Weyl group

There is a natural (Weyl) group associated with the set of restricted roots, which is related to $W^{\mathcal{G}}/W_0$. Since W_0 is a normal subgroup of $W^{\mathcal{G}}$, every $w \in W^{\mathcal{G}}$ induces an automorphism of $\bar{X}_{\mathcal{G}} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ ($\chi \in X$). Define $\bar{W}_{\mathcal{G}} = \{\pi(w) \mid w \in W^{\mathcal{G}}\}$. We call this the *restricted Weyl group*, with respect to the action of $\mathcal{G}$ on X. It is not necessarily a Weyl group in the sense of Bourbaki [Bou81, Ch.VI,no.1]. However we can show the following.

Proposition 2. *Let X, X_0, Φ, Φ_0, $\bar{\Phi}_{\mathcal{G}}$, Δ, Δ_0, $\bar{\Delta}_{\mathcal{G}}$, W_0, $W^{\mathcal{G}}$, $\bar{W}_{\mathcal{G}}$ be defined as above and let A be the annihilator of X_0. Then we have the following:*

(1) If $w \in W^{\mathcal{G}}$, then $w(\Delta)$ is an $\mathcal{G}$-fundamental system.

(2) Let $w \in W^{\mathcal{G}}$. Then $w \in W_0$ iff $\pi(w) = 1$ iff $\pi(w)\bar{\Delta}_{\mathcal{G}} = \bar{\Delta}_{\mathcal{G}}$.

(3) $\bar{W}_{\mathcal{G}} \cong W^{\mathcal{G}}/W_0$.

(4) $W^{\mathcal{G}}/W_0 \cong N_G(A)/Z_G(A)$, where $N_G(A)$ and $Z_G(A)$ are, respectively, the normalizer and centralizer of A in G.

Proof. (1). For $w \in W^{\mathcal{G}}$ define an order $\succ_w$ on X as follows:

$$\text{if } \chi \in X \text{ and } \chi \notin X_0, \text{ then } \chi \succ_w 0 \text{ if and only if } w(\chi) \succ 0.$$

Since $w(X_0) = X_0$ the order $\succ_w$ is a $\mathcal{G}$-linear order on X and $w(\Delta)$ is a $\mathcal{G}$-fundamental system of Φ with respect to this order.

(2). If $w \in W_0$, then from the definition of $\pi(w)$ it follows that $\pi(w) = 1$, which implies that $\pi(w)\bar{\Delta}_{\mathcal{G}} = \bar{\Delta}_{\mathcal{G}}$. So it suffices to show that the latter condition implies that $w \in W_0$. Since $w(\Delta)$ and Δ are both $\mathcal{G}$-fundamental systems it follows from Proposition 1(3) that there exists $w_0 \in W_0$ such that $w_0 w(\Delta) = \Delta$, what implies that $w = w_0^{-1} \in W_0$.

(3) is immediate from (1) and (2).

(4). Let $n \in N_G(T)$ and $w \in W(T)$ the corresponding Weyl group element. Then $w(X_0) = X_0$ if and only if $n \in N_G(A)$. It follows that $w \in W^{\mathcal{G}}$ if and only if $n \in N_G(A)$. By (2) $w \in W_0$ if and only if $\pi(w) = 1$. This is true if and only if $n \in Z_G(A)$. Since $N_G(A) = (N_G(A) \cap N_G(T)) \cdot Z_G(A)$ the result follows. □

Remarks 1. (1) In the case that A is a maximal k-split torus, then $\bar{\Phi}_{\mathcal{G}}$ is actually a root system with Weyl group $\bar{W}_{\mathcal{G}}$. The general question when $\bar{\Phi}_{\mathcal{G}}$ is a root system in $Y = X/X_0$ was studied in [Sch69].

(2) In the remainder of this section we will also write $\bar{\Phi}$, $\bar{\Delta}$, $\bar{W}$ instead of $\bar{\Phi}_{\mathcal{G}}$, $\bar{\Delta}_{\mathcal{G}}$, $\bar{W}_{\mathcal{G}}$ whenever it causes no confusion.

2.6 Action of $\mathcal{G}$ on Δ

From Proposition 2 it follows that $W^{\mathcal{G}}$ acts on the set of $\mathcal{G}$-fundamental systems of Φ. There is also a natural action of $\mathcal{G}$ on this set. If Δ is a $\mathcal{G}$-fundamental system of Φ, and $\sigma \in \mathcal{G}$, then the $\mathcal{G}$-fundamental system $\sigma(\Delta) = \{\sigma(\alpha) \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ, i.e. $\sigma(\bar{\Delta}) = \bar{\Delta}$. This follows from the fact that $\alpha_i \equiv \sigma(\alpha)_i \mod X_0)$ for all $\alpha_i \in \Delta$, $\sigma \in \mathcal{G}$. From Proposition 1 it follows that there is a unique element $w_\sigma \in W_0$ such that $\sigma(\Delta) = w_\sigma \Delta$. This means we can define a new operation of $\mathcal{G}$ on X as follows:

$$[\sigma](\chi) = w_\sigma^{-1}\sigma(\chi), \quad \chi \in X, \quad \sigma \in \mathcal{G}. \tag{2.6}$$

It is easily verified that $\chi \to [\sigma](\chi)$ is an automorphism of the triple (X, Φ, Δ) and that $[\sigma][\mathcal{G}](\chi) = [\sigma\mathcal{G}](\chi)$ for all σ, $\mathcal{G} \in \mathcal{G}$, $\chi \in X$.

In the following we prove some properties of the action of $\mathcal{G}$ on Δ which will be needed later. We will assume X_0 is as defined in (2.4) and $\succ$ is a $\mathcal{G}$ order on X.

Lemma 1. *Let $\lambda_j \in \bar{\Delta}$ and $\alpha_i \in \Delta$ such that $\pi(\alpha_i) = \lambda_j$. If $\sigma \in \mathcal{G}$, then we have the following:*

(1) $\sigma(\alpha_i) = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$ for some $\alpha_p \in \pi^{-1}(\lambda_j)$, $c_{i,r}(\sigma) \in \mathbb{Z}$.

(2) $[\sigma](\alpha_i) = \alpha_p + \sum_{\alpha_r \in \Delta_0} b_{i,r}(\sigma)\alpha_r$ for some $\alpha_p \in \pi^{-1}(\lambda_j)$, $b_{i,r}(\sigma) \in \mathbb{Z}$.

Proof. Let $\operatorname{rank}(\Phi) = n$. Write $\sigma(\alpha_i) = \sum_{r=1}^{n} c_{i,r}(\sigma)\alpha_r$, where $c_{i,r}(\sigma) \in \mathbb{Z}$. Since $\alpha_i \in \Delta$ and Δ is a $\mathcal{G}$-fundamental system of Φ we may assume that $c_{i,r}(\sigma) \geq 0$ if $\alpha_i \notin \Delta_0$, and $c_{i,r}(\sigma) = 0$ if $\alpha_i \in \Delta_0$ and $\alpha_r \notin \Delta_0$. Reorder the fundamental roots, if necessary, so that $\Delta - \Delta_0 = \{\alpha_1, \ldots, \alpha_m\}$ and $\Delta_0 = \{\alpha_{m+1}, \ldots, \alpha_n\}$. Then the matrices $(c_{ij}(\sigma))_{1 \leq i,j \leq n}$ are integral, and of the form $\begin{pmatrix} A_\sigma & B_\sigma \\ 0 & D_\sigma \end{pmatrix}$, where all entries of A_σ and B_σ are ≥ 0. Since the product of the matrices $(c_{ij}(\sigma))$ and $(c_{ij}(\sigma^{-1}))$ is the identity matrix, it follows that A_σ is necessarily a permutation matrix, hence if $\alpha_i \notin \Delta_0$, $\sigma(\alpha_i) = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$. Since $\pi(\alpha_i) = \pi(\sigma(\alpha_i)) = \lambda_j$ it follows that $\alpha_p \in \pi^{-1}(\lambda_j)$.

(2). For $\sigma \in \mathcal{G}$ let $w_\sigma \in W_0$ such that $[\sigma](\alpha_i) = w_\sigma^{-1}\sigma(\alpha_i)$. Let $c_{i,r}(\sigma) \in \mathbb{Z}$ and $\alpha_p \in \pi^{-1}(\lambda_j)$ such that $\sigma(\alpha_i) = \alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r$. Then

$$[\sigma](\alpha_i) = w_\sigma^{-1}(\alpha_p + \sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r) = w_\sigma^{-1}(\alpha_p) + w_\sigma^{-1}(\sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r).$$

Since $w_\sigma^{-1} \in W_0$ it follows that $w_\sigma^{-1}(\sum_{\alpha_r \in \Delta_0} c_{i,r}(\sigma)\alpha_r) = \sum_{\alpha_r \in \Delta_0} d_{i,r}(\sigma)\alpha_r$ for some $d_{i,r}(\sigma) \in \mathbb{Z}$. Similarly $w_\sigma^{-1}(\alpha_p) = \alpha_p + \sum_{\alpha_r \in \Delta_0} e_{i,r}(\sigma)\alpha_r$ for some $e_{i,r}(\sigma) \in \mathbb{Z}$. Let $b_{i,r}(\sigma) = d_{i,r}(\sigma) + e_{i,r}(\sigma)$. Then $[\sigma](\alpha_i) = \alpha_p + \sum_{\alpha_r \in \Delta_0} b_{i,r}(\sigma)\alpha_r$. □

Lemma 2. *Let $\Omega = \Delta_0(\mathcal{G}) \cup \{[\sigma](\alpha) - \alpha \mid \alpha \in \Delta - \Delta_0(\mathcal{G})$ and $[\sigma](\alpha) \neq \alpha\}$. Then $X_0(\mathcal{G})_{\mathbb{Q}} = \Omega_{\mathbb{Q}}$ and the cardinality of $\Omega = \operatorname{rank} X_0(\mathcal{G})$.*

Proof. Ω is a linearly independent set and $\operatorname{rank} X_0(\mathcal{G}) \geq \operatorname{card} \Omega$. So it suffices to show that Ω generates $X_0(\mathcal{G})$. From the definition of $X_0(\mathcal{G})$ and $X^{\mathcal{G}}(\mathcal{G})$ it is clear that $X_0(\mathcal{G})_{\mathbb{Q}}$ is generated over $\mathbb{Q}$ by the set $\{\sigma(\alpha) - \alpha \mid \sigma \in \mathcal{G}, \alpha \in \Delta\}$. If $\alpha \in \Delta_0(\mathcal{G})$, then $\sigma(\alpha) \in \Phi \cap X_0(\mathcal{G}) = \Phi_0(\mathcal{G})$. Since $\Delta_0(\mathcal{G})$ is a fundamental system of $\Phi_0(\mathcal{G})$ it follows that $\sigma(\alpha) - \alpha \in \Delta_0(\mathcal{G})_{\mathbb{Z}} \subset \Omega_{\mathbb{Z}}$. If $\alpha \in \Delta - \Delta_0(\mathcal{G})$, then for all $\sigma \in \mathcal{G}$ we have $\pi(\alpha) = \pi(\sigma(\alpha)) = \lambda$ for some $\lambda \in \bar{\Delta}_{\mathcal{G}}$. By Lemma 1 we get $[\sigma](\alpha) \in \mathcal{G}^{-1}(\lambda)$ and $\sigma(\alpha) = [\sigma](\alpha) + \mathcal{G}$ for some $\mathcal{G} \in \Delta_0(\mathcal{G})_{\mathbb{Z}}$. But then $\sigma(\alpha) - \alpha = [\sigma](\alpha) - \alpha + \mathcal{G} \in \Omega_{\mathbb{Z}}$. □

Corollary 1. *Let* $X,\ X_0(\mathcal{G}),\ \Phi,\ \Phi_0(\mathcal{G}),\ \bar{\Phi}_{\mathcal{G}},\ \Delta,\ \Delta_0$ *be defined as above and let* $\bar{\Delta}_{\mathcal{G}} = \{\lambda_1, \dots, \lambda_r\}$ *be a restricted fundamental system of* $\bar{\Phi}_{\mathcal{G}}$, *with the* λ_i *mutually distinct. Then* $\lambda_1, \dots, \lambda_r$ *are linearly independent.*

Proof. Since Δ spans X it follows that $\bar{\Delta}_{\mathcal{G}}$ spans $\bar{X}_{\mathcal{G}}$, so $\operatorname{rank} \bar{X}_{\mathcal{G}} \leq r$. But since $\operatorname{rank} X = \operatorname{rank} X_0(\mathcal{G}) + \operatorname{rank} \bar{X}_{\mathcal{G}}$ it follows from Lemma 2 that $\operatorname{rank} \bar{X}_{\mathcal{G}} = r$, hence $\lambda_1, \dots, \lambda_r$ are linearly independent. □

The diagram automorphism $[\sigma]$ relates the simple roots in Δ, which are lying above a restricted root in $\bar{\Delta}_{\mathcal{G}}$:

Lemma 3. *Let* Δ *be a* $\mathcal{G}$*-basis of* Φ *and* $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ *such that* $\pi(\alpha) = \pi(\beta) \neq 0$. *Then there is a* $\sigma \in \mathcal{G}$ *such that* $\beta = [\sigma](\alpha)$.

Proof. For each $\sigma \in \mathcal{G}$ let $w_\sigma \in W_0$ such that $[\sigma] = w_\sigma^{-1}\sigma$. Since $\pi(\alpha) = \pi(\beta) \neq 0$ we have $\alpha \equiv \beta \mod X_0(\mathcal{G})$. But then $\sum_{\sigma \in \mathcal{G}} \sigma(\alpha) = \sum_{\sigma \in \mathcal{G}} \sigma(\beta)$. On the other hand $\sum_{\sigma \in \mathcal{G}} \sigma(\alpha) = \sum_{\sigma \in \mathcal{G}} w_\sigma[\sigma](\alpha) = \sum_{\sigma \in \mathcal{G}}[\sigma](\alpha) + \delta_1$ with $\delta_1 \in Span(\Delta_0(\mathcal{G}))$. Similarly $\sum_{\sigma \in \mathcal{G}} \sigma(\beta) = \sum_{\sigma \in \mathcal{G}}[\sigma](\beta) + \delta_2$ with $\delta_2 \in Span(\Delta_0(\mathcal{G}))$. But then we have $\sum_{\sigma \in \mathcal{G}}([\sigma](\alpha) - [\sigma](\beta)) = \delta_1 - \delta_2$. It follows that $\delta_1 = \delta_2$ and $\beta = [\sigma](\alpha)$ for some $\sigma \in \mathcal{G}$. □

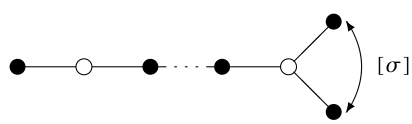
2.7 $\mathcal{G}$-indices

The actions of a finite group $\mathcal{G}$ on the root datum can be described by an index. These indices not only determine the fine structure of restricted root systems with multiplicities etc. of the corresponding k-group and symmetric variety, but also play an important role in the classifications of k-groups and symmetric varieties (or equivalently involutions of reductive groups). In this section we extend these indices to get an index which describes the action of a k-involution. Similar as for k-groups and symmetric varieties this index describes the fine structure of restricted root systems with multiplicities etc. of the corresponding symmetric k-variety, but also plays again an important role in the classification of k-involutions.

2.8 The index of $\mathcal{G}$

Throughout this section let Ψ be a semisimple root datum with $\Phi \neq \emptyset$, as in (2.1), $\mathcal{G}$ a (finite) group acting on Ψ, Δ a $\mathcal{G}$-basis of Φ and $\Delta_0 = \Delta_0(\mathcal{G}) = \Delta \cap X_0(\mathcal{G})$. Define an action of $\mathcal{G}$ on Δ, which we denote by $[\sigma]$. The action of $\mathcal{G}$ on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [Tit66] we will call the quadruple $(X, \Delta, \Delta_0, [\sigma])$ an *index of $\mathcal{G}$* or a $\mathcal{G}$-index. We will also use the name $\mathcal{G}$-diagram, following the notation in Satake [Sat71, 2.4].

As in [Tit66] we make a diagrammatic representation of the index of $\mathcal{G}$ by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\mathcal{G})$ and indicating the action of $[\sigma]$ on Δ by arrows. An example in type D_l is:



To use these $\mathcal{G}$-indices in the characterization of isomorphy classes of reductive k-groups or involutions, we need a notion of isomorphism between these indices.

Definition 2. Let Ψ and Ψ' be semisimple root data and $\mathcal{G}$ a group acting on them. A *congruence* φ of the $\mathcal{G}$-index $(X, \Delta, \Delta_0, [\sigma])$ of Ψ onto the $\mathcal{G}$-index $(X', \Delta', \Delta'_0, [\sigma]')$ of Ψ' is an isomorphism which maps $(X, \Delta, \Delta_0) \to (X', \Delta', \Delta'_0)$, and satisfies $[\sigma]' = \varphi[\sigma]\varphi^{-1}$.

For k-involutions it suffices to consider two actions of $\mathcal{G}$ on the same root datum. In that case we will also use the term *isomorphic* $\mathcal{G}$-indices instead of congruent $\mathcal{G}$-indices. In this case one can differentiate between inner and outer automorphisms.

Definition 3. Let Ψ be a root datum and $\mathcal{G}_1, \ \mathcal{G}_2 \subset \mathrm{Aut}(\Psi)$ the subgroups of $\mathrm{Aut}(\Psi)$ corresponding to actions of $\mathcal{G}$ on Ψ. Two indices $(X, \Delta, \Delta_0(\mathcal{G}_1), [\sigma]_1)$ and $(X, \Delta', \Delta'_0(\mathcal{G}_2), [\sigma]_2)$ are said to be $W(\Phi)$- (resp. $\mathrm{Aut}(\Phi)$)-*isomorphic* if there is a $w \in W(\Phi)$ (resp. $w \in \mathrm{Aut}(\Phi)$), which maps $(\Delta, \Delta_0(\mathcal{G}_1))$ onto $(\Delta', \Delta'_0(\mathcal{G}_2))$ and satisfies $w[\sigma]_1 w^{-1} = [\sigma]_2$. Instead of $W(\Phi)$-isomorphic we will also use the term *isomorphic*.

Remark 3. An index of $\mathcal{G}$ may depend on the choice of the $\mathcal{G}$-basis of Φ, i.e. for two $\mathcal{G}$-bases Δ, Δ', the corresponding indices $(X, \Delta, \Delta_0(\mathcal{G}), [\sigma])$ and $(X, \Delta', \Delta'_0(\mathcal{G}), [\sigma]')$ need not be isomorphic. However this cannot happen if $\bar{\Phi}_{\mathcal{G}}$ is a root system with Weyl group $\bar{W}_{\mathcal{G}}$:

Proposition 3. *Let Ψ be a semisimple root datum and $\mathcal{G} \subset \operatorname{Aut}(\Psi)$ a group acting on Ψ such that $\bar{\Phi}_{\mathcal{G}}$ is a root system with Weyl group $\bar{W}_{\mathcal{G}}$. If Δ, Δ' are $\mathcal{G}$-bases of Φ, then $(X, \Delta, \Delta_0(\mathcal{G}), [\sigma])$ and $(X, \Delta', \Delta_0'(\mathcal{G}), [\sigma]')$ are isomorphic.*

Proof. Let $\bar{\Delta}_{\mathcal{G}}$ and $\bar{\Delta}'_{\mathcal{G}}$ be restricted fundamental systems of $\bar{\Phi}_{\mathcal{G}}$ induced by Δ and Δ' and let $\bar{w} \in \bar{W}_{\mathcal{G}}$ such that $\bar{w}(\bar{\Delta}'_{\mathcal{G}}) = \bar{\Delta}_{\mathcal{G}}$. Since by Proposition 2(3) $\bar{W}_{\mathcal{G}} = W^{\mathcal{G}}/W_0$ there exists $w_1 \in W^{\mathcal{G}}$ such that $\pi(w_1) = \bar{w}$. By Proposition 2(1) $w_1(\Delta') \cap \Phi_0$ is a basis of Φ_0, hence there exists $w_0 \in W_0$ such that $w_0 w_1(\Delta') \cap \Phi_0 = \Delta_0(\mathcal{G})$. Let $w = w_0 w_1$. Then from Proposition 2(2) it follows that $w(\Delta') = \Delta$ and $w(\Delta_0'(\mathcal{G})) = \Delta_0(\mathcal{G})$.

It remains to show that w satisfies $[\sigma] = w[\sigma]' w^{-1}$. Let $\sigma \in \mathcal{G}$ and w_σ, $w'_\sigma \in W_0$ such that $\sigma(\Delta) = w_\sigma(\Delta)$ and $\sigma(\Delta') = w'_\sigma(\Delta')$. Then $[\sigma] = w_\sigma^{-1}\sigma$ and $[\sigma]' = (w'_\sigma)^{-1}\sigma$. Now

$$\begin{aligned} w_\sigma(\Delta) &= w_\sigma w(\Delta') = \sigma(\Delta) = \sigma w(\Delta') \\ &= \sigma w \sigma^{-1} \sigma(\Delta') = \sigma w \sigma^{-1} w'_\sigma(\Delta'). \end{aligned} \tag{2.7}$$

It follows that $w_\sigma w(\Delta') = \sigma w \sigma^{-1} w'_\sigma(\Delta')$, hence $\sigma w^{-1}\sigma^{-1} w_\sigma w(\Delta') = w'_\sigma(\Delta')$. Since both $\sigma w^{-1}\sigma^{-1} w_\sigma w$ and $w'_\sigma \in W$ it follows from (2.7) that

$$\sigma w^{-1} \sigma^{-1} w_\sigma w = w'_\sigma. \tag{2.8}$$

Now if $\chi \in X$, then

$$\begin{aligned} w[\sigma]' w^{-1}(\chi) &= w(w'_\sigma)^{-1}\sigma w^{-1}(\chi) = w w^{-1} w_\sigma^{-1} \sigma w \sigma^{-1} \sigma w^{-1}(\chi) \\ &= w_\sigma^{-1}\sigma(\chi) = [\sigma](\chi), \end{aligned} \tag{2.9}$$

what proves the result. □

Remark 4. In the case that $\bar{\Phi}_{\mathcal{G}}$ is a root system with Weyl group $\bar{W}_{\mathcal{G}}$, then the restricted root system together with the multiplicities of the roots can be easily determined from the $\mathcal{G}$-index. See for example [Hel88].

For the general congruence of the $\mathcal{G}$-indices we will use the following result:

Theorem 2.1. *Let G_1, G_2 be connected semisimple groups defined over k. For $i = 1, 2$ let T_i be a maximal k-torus of G_i, $\Psi_i = (X^*(T_i), \Phi(T_i), X_*(T_i), \Phi^\vee(T_i))$ the root datum corresponding to (G_i, T_i), $\mathcal{G}$ a (finite) group acting on Ψ_i, $X_0(\mathcal{G}, T_i) = \{\chi \in X^*(T_i) \mid \sum_{\sigma\in\mathcal{G}} \sigma(\chi) = 0\}$, $A_i =$*

$\{t \in T_i \mid \chi(t) = e$ for all $\chi \in X_0(\mathcal{G}, T_i)\}$ the annihilator of $X_0(\mathcal{G}, T_i)$, $\Delta(T_i)$ a $\mathcal{G}$-basis of $\Phi(T_i)$, $\Delta_0(T_i) = \Delta(T_i) \cap X_0(\mathcal{G})$ and $[\sigma]_i$ the action of $\mathcal{G}$ on $\Delta(T_i)$. If $\varphi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism and $\varphi^\star = {}^t(\varphi|T_1)^{-1}$ is as in (2.2), then there exists a unique $w \in W^{\mathcal{G}}(T_2)$ such that $w(\varphi^\star(\Delta(T_1))) = \Delta(T_2)$ and $\varphi^{[\star]} := w\varphi^\star$ is a congruence from $(X^*(T_1)$, $\Delta(T_1)$, $\Delta_0(T_1)$, $[\sigma]_1)$ to $(X^*(T_2)$, $\Delta(T_2)$, $\Delta_0(T_2)$, $[\sigma]_2)$.

Proof. Since $\phi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism it follows that the induced map $\varphi^\star : (X^*(T_1),\ \Phi(T_1),\ X_0(T_1)) \to (X^*(T_2),\ \Phi(T_2),\ X_0(T_2))$ is an isomorphism as well. Since $\varphi^\star(\Phi^+(T_1))$ is a set of positive roots with respect to a $\mathcal{G}$-linear order on $\Phi(T_2)$ it follows that $\varphi^\star(\Delta(T_1))$ is a $\mathcal{G}$-basis of $\Phi(T_2)$. Since $\Phi(A_2)$ is a root system with Weyl group $W(A_2)$ it follows from Proposition 2 that there exists a unique $w \in W^{\mathcal{G}}(T_2)$ such that $w(\varphi^\star(\Delta(T_1))) = \Delta(T_2)$. From Proposition 3 it follows now that the $\mathcal{G}$-indices $(X^*(T_2)$, $\Delta(T_2), \Delta_0(T_2), \phi^\star[\sigma]_1(\phi^\star)^{-1})$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$ are congruent. Let $\varphi^{[\star]} := w\varphi^\star$. With a similar argument as in (2.7) and (2.9) it follows now that $\varphi^{[\star]}$ is a congruence of the $\mathcal{G}$-indices $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$. □

Definition 4. If $\phi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a k-isomorphism as in Theorem 2.1, then we will call the congruence $\varphi^{[\star]} := w\varphi^\star$ of the $\mathcal{G}$-indices $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$ the *congruence associated with* φ.

In the cases of $\mathcal{G} = \mathcal{G}_\theta$ and $\mathcal{G} = \mathcal{G}$ we get the well known θ-index and $\mathcal{G}$-index, which are essential in the respective classifications. Since the classification of k-involutions depends on a classification of these, we will briefly review these in the next sections. First we need still a notion of irreducibility for $\mathcal{G}$-indices.

Definition 5. Let $\mathcal{G} \subset \mathrm{Aut}(X, \Phi)$ be a subgroup and Δ a $\mathcal{G}$-basis of Φ. An index $\mathcal{D} = (X$, Δ, Δ_0, $[\sigma])$ is *$\mathcal{G}$-irreducible* if Δ is not the union of two mutually orthogonal $[\sigma]$-invariant (non-empty) subsystems Δ', Δ''. The system $\mathcal{D}$ is *absolutely irreducible* if Δ is connected. In the case $\mathcal{G} = \mathcal{G}_G$ (resp. $\mathcal{G}_\theta$) we will also call an $\mathcal{G}$-irreducible index an k-irreducible index (resp. θ-irreducible index).

2.9 Γ-index

In this section we apply the above results to the case that $\mathcal{G} = \Gamma$, the Galois group of a finite splitting extension K of k for a maximal k-torus T as in 2.2. This will give us the index related to the isomorphy classes of semisimple k-groups. For the remainder of this section let G be a reductive k-group, A a k-split torus of G, $T \supset A$ a maximal k-torus, K the smallest Galois extension of k which splits T, $\Gamma = \mathrm{Gal}(K/k)$ the Galois group of K/k, $X = X^*(T)$, $\Phi = \Phi(T)$, $X_0 = X_0(\Gamma)$, $\Phi_0 = \Phi_0(\Gamma)$, etc. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$. If A is a maximal k-split torus, then G_0 is anisotropic over k and is uniquely determined (up to k-isomorphy) by the k-isomorphism class of G. In that case G_0 is also called the k-anisotropic kernel of G.

Let Δ be a Γ-basis of Φ, and let $\Delta_0 = \Delta \cap X_0$. As in (2.6) we have an action of Γ on Δ, which we denote by $[\sigma]$. The 4-tuple $(X, \Delta, \Delta_0, [\sigma])$ is called the *Γ-index of* (G, T, A). If A is a maximal k-split torus of G, then we will also call this the Γ-index of G. It was shown by Tits [Tit66] that the k-isomorphism class of G uniquely determines, up to congruence, the Γ-index of G. Using Proposition 3 this can also be seen easily as follows.

Let G_1, G_2 be connected semisimple groups defined over k and $\phi : G_1 \to G_2$ a k-isomorphism. For $i = 1, 2$ let $A_i \subset G_i$ be a maximal k-split torus, $T_i \supset A_i$ a maximal k-torus of G_i and $\Delta(T_i)$ a Γ-basis of $\Phi(T_i)$. Now $\phi(A_1)$ is a maximal k-split torus of G_2, hence there exists a $g \in G_k$ such that $\mathrm{Int}(g)\phi(A_1) = A_2$. Then $\mathrm{Int}(g)\phi(T_1) \supset A_2$ is a maximal k-torus. Let K be the smallest Galois extension of k which splits T_1 and T_2. Then there exists $x \in G_K$ such that $\mathrm{Int}(x)\,\mathrm{Int}(g)\phi(T_1) = T_2$. Let $\phi_1 = \mathrm{Int}(x)\,\mathrm{Int}(g)\phi$. Then $\phi_1 : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a K-isomorphism and by Theorem 2.1 $\varphi_1^\star = {}^t(\varphi_1|T_1)^{-1}$ as in (2.2) (modulo a Weyl group element of $W(T_2)$) is a congruence from the Γ-index of (G_1, T_1, A_1) onto the Γ-index of (G_2, T_2, A_2). Summarized we have now the following result:

Proposition 4 ([Tit66])**.** *The k-isomorphism class of G uniquely determines (up to congruence) the* Γ*-index* $(X, \Delta, \Delta_0(\Gamma), [\sigma])$ *of* G*.*

Remark 5. In the special case that G is k-anisotropic ($G = G_0$), one has $\Delta = \Delta_0(\Gamma)$, so the Γ-index of G may be abbreviated by $(X, \Delta_0(\Gamma), [\sigma])$. Applying this to the k-anisotropic kernels G_0, G_0' of G, G' it is easily seen that a congruence $\phi : (X, \Delta, \Delta_0(\Gamma), [\sigma]) \to (X', \Delta',$

$\Delta_0'(\Gamma)$, $[\sigma]')$ induces a congruence $\phi_0 : (X_0, \Delta_0(\Gamma), [\sigma]|X_0) \to (X_0', \Delta', \Delta_0'(\Gamma), [\sigma]'|X_0')$ of the Γ-index of G_0 onto the Γ-index of G_0'. The map ϕ_0 is called the *restriction* of ϕ to $(X_0, \Delta_0(\Gamma), [\sigma]|X_0)$.

2.10 Notation

The Γ-indices for k algebraically closed, the real numbers, the p-adic numbers, finite fields, and numbers fields have been classified by Tits [Tit66]. In this thesis we will derive algorithms to compute the fine structure associated with these Γ-indices. In Table 1 below we list the absolutely irreducible Γ-indices together with the associated restricted root system. In the table we use the following notation:

Let $\mathcal{D} = (X, \Delta, \Delta_0(\Gamma), [\sigma])$ be a Γ-index. For the Γ-indices we use the notation ${}^gX^t_{n,r}$. Here X denotes the type of $\Phi(T)$, i.e. one of A, $B, \dots,$ G, n the rank of Φ, r the rank of $\bar{\Delta}_\Gamma$ and g the order of the action of Γ on the Dynkin diagram. In the case that $g = 1$ (i.e. the Dynkin diagram has no nontrivial automorphism) we will omit it in the notation. Finally t denotes either the degree of the division algebra, which occurs in the definition of the considered form or the dimension of the anisotropic kernel. To differentiate between these two cases we put t between parentheses when it stands for the degree of the division algebra. In fact the degree of the division algebra is only used if X is of classical type.

Table 2.1 lists the absolutely irreducible Γ-indices and the type of restricted root system for k algebraically closed, the real numbers, the p-adic numbers, finite fields, and numbers fields. For each of these 45 cases we will examine their fine structure and give algorithms which enable one to compute it using a computer algebra package. We note that not all of these Γ-indices occurs for each of the fields we consider. For example the case ${}^2A^{(d)}_{n,p}$ only occurs if k is the p-adic numbers.

Table 2.1: Γ-indices

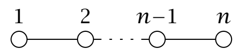
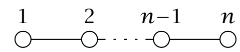
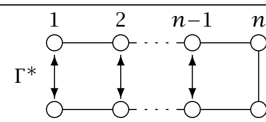
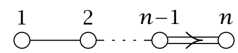
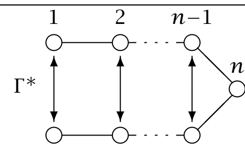
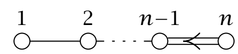
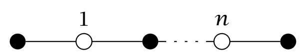
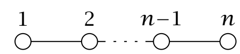
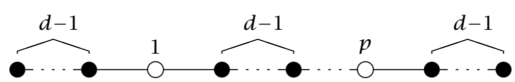
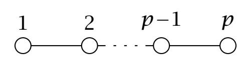
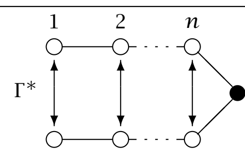
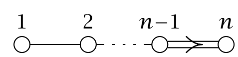
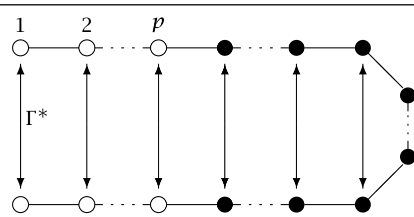
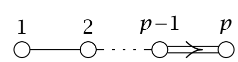
Type	Γ-index	$\Delta(\mathfrak{a})$
$A^{(1)}_{n,n}$		
${}^2A^{(1)}_{2n,n}$		
${}^2A^{(1)}_{2n-1,n}$		
$A^{(2)}_{2n+1,n}$		
$A^{(d)}_{n,p}$		
${}^2A^{(1)}_{2n+1,n}$		
${}^2A^{(1)}_{n,p}$		

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Table 2.1: *continued*

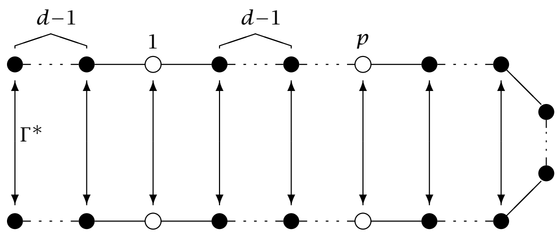
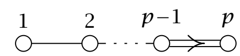
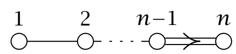
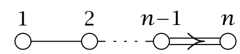
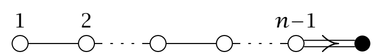
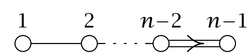
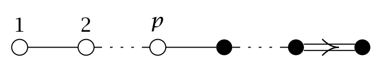
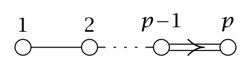
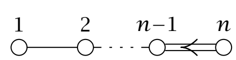
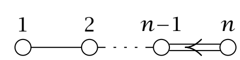
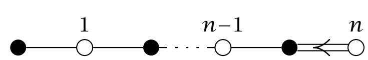
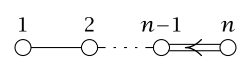
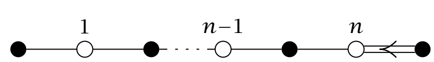
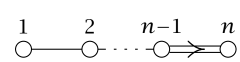
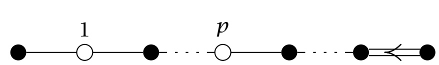
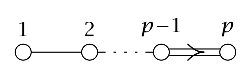
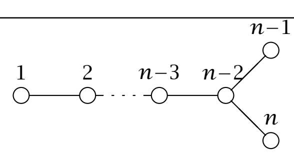
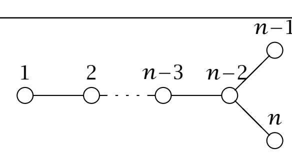
Type	Γ-index	$\Delta(\mathfrak{a})$
${}^2A_{n,p}^{(d)}$ $d\mid(n+1)$		
$B_{n,n}$		
$B_{n,n-1}$		
$B_{n,p}$		
$C_{n,n}^{(1)}$		
$C_{2n,n}^{(2)}$		
$C_{2n+1,n}^{(2)}$		
$C_{n,p}^{(2)}$		
$D_{n,n}^{(1)}$		

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Table 2.1: *continued*

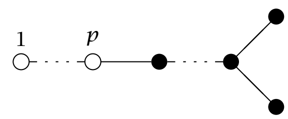
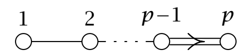
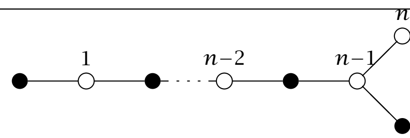
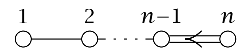
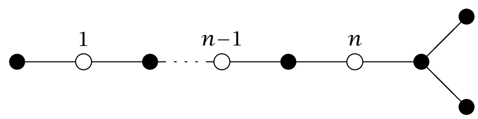
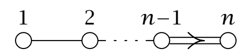
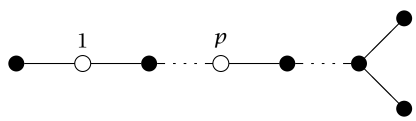
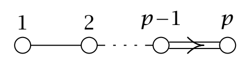
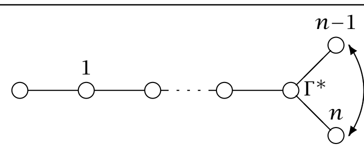
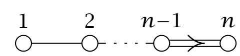
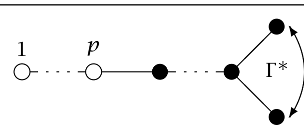
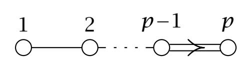
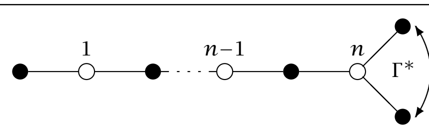
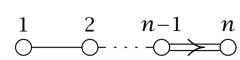
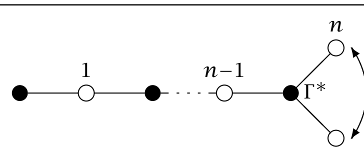
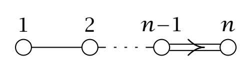
Type	Γ-index	$\Delta(\mathfrak{a})$
$D_{n,p}^{(1)}$		
$D_{2n,n}^{(2)}$		
$D_{2n+3,n}^{(2)}$		
$D_{n,p}^{(2)}$		
${}^2D_{n+1,n}^{(1)}$		
${}^2D_{n,p}^{(1)}$		
${}^2D_{2n+2,n}^{(2)}$		
${}^2D_{2n+1,n}^{(2)}$		

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Table 2.1: *continued*

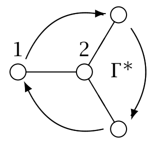
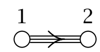
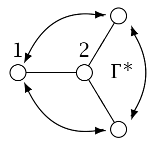
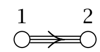
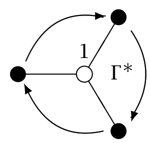
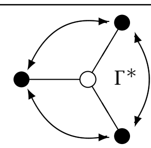
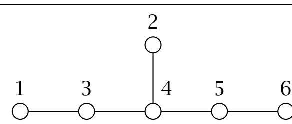
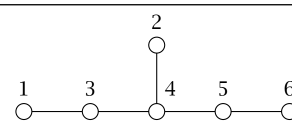
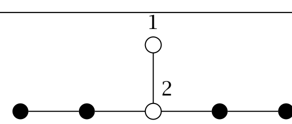
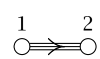
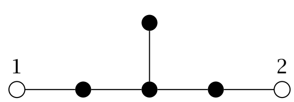
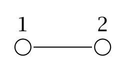
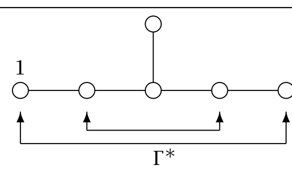
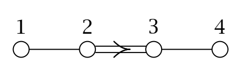
Type	Γ-index	$\Delta(\mathfrak{a})$
${}^3D_{4,2}^{(2)}$		
${}^6D_{4,2}^{(2)}$		
${}^3D_{4,1}^{(9)}$		
${}^6D_{4,1}^{(9)}$		
${}^1E_{6,6}^{0}$		
${}^1E_{6,2}^{16}$		
${}^1E_{6,2}^{28}$		
${}^2E_{6,4}^{16}$		

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Table 2.1: *continued*

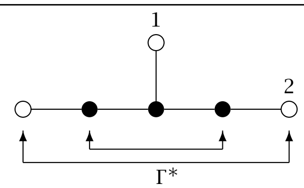
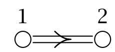
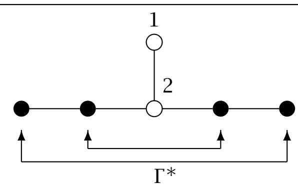
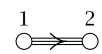
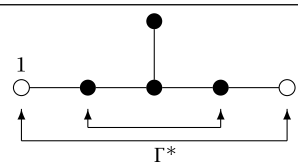
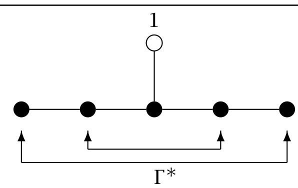
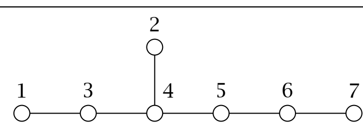
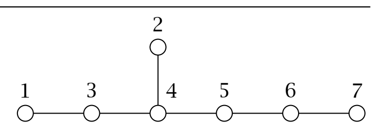
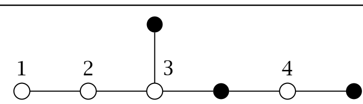
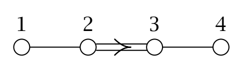
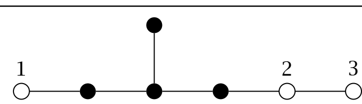
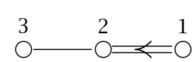
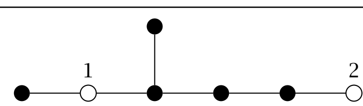
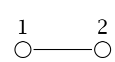
Type	Γ-index	$\Delta(\mathfrak{a})$
${}^2E_{6,2}^{16'}$		
${}^2E_{6,2}^{16''}$		
${}^2E_{6,1}^{29}$		
${}^2E_{6,1}^{35}$		
$E_{7,7}^{0}$		
$E_{7,4}^{9}$		
$E_{7,3}^{28}$		
$E_{7,2}^{31}$		

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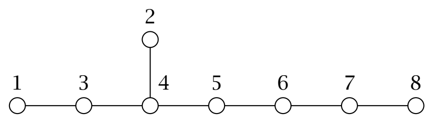
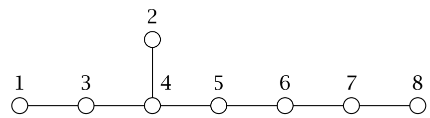
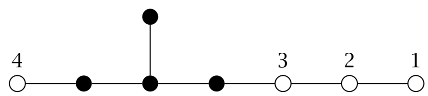
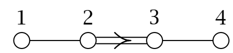
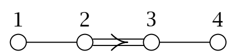
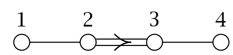
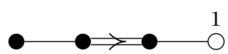
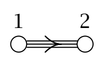
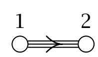
Table 2.1: *continued*

Type	Γ-index	Δ(𝔞)
$E_{8,8}^{0}$		
$E_{8,4}^{28}$		
$F_{4,4}^{0}$		
$F_{4,1}^{21}$		
$G_{2,2}^{0}$		

2.11 θ-index

In this section we apply the results to the case of symmetric spaces. In particular, we discuss the index associated with an involutorial automorphism of a reductive algebraic group. Let G be a reductive algebraic group, $\theta \in \mathrm{Aut}(G)$ an involution and T a θ-stable maximal torus of G. Write $X = X^*(T)$, $\Phi = \Phi(T)$ and let $\mathcal{I}_\theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$. In this case we will also write $X_0(\theta)$, $\bar{X}_\theta$, $\Phi_0(\theta)$, $\bar{\Phi}_\theta$, $W_1(\theta)$, $\bar{W}_\theta$, $\Delta_0(\theta)$, $\bar{\Delta}_\theta$ instead of, respectively, $X_0(\mathcal{I}_\theta)$, $\bar{X}_{\mathcal{I}_\theta}$, $\Phi_0(\mathcal{I}_\theta)$, $\bar{\Phi}_{\mathcal{I}_\theta}$, $W_0(\mathcal{I}_\theta)$, $W_1(\mathcal{I}_\theta)$, $\bar{W}_{\mathcal{I}_\theta}$, $\Delta_0(\mathcal{I}_\theta)$, $\bar{\Delta}_{\mathcal{I}_\theta}$. A $\mathcal{I}_\theta$-order on X will also be called a *θ-order* on X, a $\mathcal{I}_\theta$-basis of Φ a *θ-basis* of Φ and a $\mathcal{I}_\theta$-index a θ-index.

Let Δ be a θ-basis of Φ. To find the θ-index we need to find the action of $[-\theta]$ on (X, Φ, Δ). Since $\theta(-\Delta)$ is also a θ-basis of Φ with the same restricted basis, it follows from Proposition 1 that there is $w_0(\theta) \in W_0(\theta)$ such that $w_0(\theta)\theta(\Delta) = -\Delta$. Put $\theta^* = \theta^*(\Delta) =$

$-w_0(\theta)\theta$. Then $\theta^* = [-\theta]$. Note that $\theta^*(\Delta) \in \mathrm{Aut}(X, \Phi, \Delta) = \{\phi \in \mathrm{Aut}(X, \Phi) \mid \phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \mathrm{id}$ and $\theta^*(\Delta_0(\theta)) = \Delta_0(\theta)$.

The indices of involutions of (X, Φ) can be easily determined using the following result from [Hel88]:

Lemma 4 ([Hel88, Lemma 2.14])**.** *Let Δ be a basis of Φ, $\Delta_0 \subset \Delta$ a subset and $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta)$ such that $\theta^*(\Delta_0) = \Delta_0$, $(\theta^*)^2 = \mathrm{id}$. Let X_0 be the $\mathbb{Z}$-span of Δ_0 in X and $\Phi(\Delta_0) = \Phi \cap X_0$. Then there is an involution $\theta \in \mathrm{Aut}(X, \Phi)$ with index $(X, \Delta, \Delta_0, \theta^*)$ if and only if $\theta^*|\Delta_0 = \mathrm{id}^*(\Delta_0)$ (the opposition involution of Δ_0 with respect to $\Phi(\Delta_0)$).*

Remark 6. The above θ-index may depend on the choice of the θ-basis. However if T_θ^- is a maximal θ-split torus, then by [Ric82, 4.7] $\bar{\Phi}_\theta = \Phi(T_\theta^-)$ is a root system and by Proposition 3 the θ-index does not depend on the θ-basis. Combined with the conjugacy of the maximal θ-split tori under G_θ^0 it follows now that the θ-index is uniquely determined by the G-isomorphism class of θ:

Proposition 5 ([Hel88])**.** *Let A be a maximal θ-split torus of G, $T \supset A$ a maximal torus and Δ a θ-basis of $\Phi(T)$. The θ-index $(X,\ \Delta,\ \Delta_0,\ \theta^*)$ is uniquely determined (up to congruence) by the isomorphy class of θ.*

Remark 7. The θ-indices were classified in [Hel88]. Algorithms for the corresponding fine structures were given in [Fowler03]. Some of the cases discussed there overlap with cases discussed in this thesis. Those would be cases such that $|\Gamma| = 2$. However, not all cases such that $|\Gamma| = 2$ occur as θ-indices.

Remark 8. For symmetric k-varieties, there exists a similar index, which is a combination of the above Γ-index and θ-index. This is called a Γ_θ-index and is again determined up to congruence by the isomorphy class of the symmetric k-variety.

2.12 Root Space Decomposition

All the fine structure of a reductive k-group, a symmetric space, or a symmetric k-variety can also be computed in the Lie algebra setting, which sometimes simplifies some of the computations. This also enables us to compute some additional structure such as the root space decomposition corresponding to a maximal k-split torus, and for a θ-split or a(θ, k)-split torus as well.

Let A be a maximal k-split torus of G, $\mathfrak{a}$ the Lie algebra of A, and $\mathfrak{g}$ the Lie algebra of g. Then:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi(A)} \mathfrak{g}_\lambda.$$

Here $\Phi(\mathfrak{a})$ is the root system of $\mathfrak{a}$ in $\mathfrak{g}$. Let $T \supset A$ be a maximal k-torus with $\mathfrak{t}$ its Lie algebra and $\Phi(\mathfrak{t})$ its root system. Let $\Phi(\lambda) = \{\alpha \in \Phi(\mathfrak{t}) | \alpha|_{\mathfrak{a}} = \lambda\}$. Then we have the following result:

Theorem 2.2. *Let $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{t}$, $\Phi(\mathfrak{t})$, and $\Phi(\mathfrak{a})$ be as above. Then:*

(1) $\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_0(\Gamma)} \mathfrak{g}_\alpha.$

(2) $\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \sum_{\lambda \in \Phi(\mathfrak{a})} \mathfrak{g}_\lambda$, *with* $\mathfrak{g}_\lambda = \sum_{\alpha \in \Phi(\lambda)} \mathfrak{g}_\alpha.$

Our algorithm will also compute this root space decomposition.

Chapter 3

The Algorithm

The computation depends on the original choice of basis for the Lie algebra $\mathfrak{g}$ and the choice of the Γ-basis for $\Phi := \Phi(\mathfrak{t})$. Γ is the Galois group of a finite splitting extension K of k for a maximal k-torus T, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a Γ-basis of Φ, Δ_0 is the set of $\alpha \in \Delta$ that project to 0, and $\bar{\Delta} := \pi(\Delta - \Delta_0) = \{\lambda_1, \ldots, \lambda_r\}$ is the restricted basis.

3.1 Step One:

Using the Γ-index, determine the elements of Γ.

3.2 Step Two:

Find a basis of the restricted root system in terms of the basis of the original root system by finding the projection of each $\alpha_j \in \Delta$, and determine each λ_i in terms of α_j.

3.3 Step Three:

Note the type of restricted root system, and determine a representative $w_i \in W^\Gamma$ for each s_{λ_i}, with $\lambda_i \in \bar{\Delta}$. This gives representatives of the Weyl group of $\Phi(\mathfrak{a})$ in the Weyl group of the maximal toral subalgebra.

3.4 Step Four:

Determine $\Phi(\lambda_i) := \{\alpha \in \Phi | \pi(\alpha) = \lambda_i\}$ for each λ_i in Step One.

3.5 Step Five:

Find the roots in $\Phi(\mathfrak{a})^+$ using the Weyl group as determined in step 3.

3.6 Step Six:

We are interested in the structure of $\Phi(\mathfrak{a})$. Note that $\Phi(-\lambda) = -\Phi(\lambda)$, so it suffices to determine $\Phi(\lambda)$ for $\lambda \in \Phi(\mathfrak{a})^+$. Do this using the fact that $\lambda = w(\lambda_i)$ for some $w \in W(\mathfrak{a})$, $\lambda_i \in \bar{\Delta}$ and the fact that $\Phi(w(\lambda_i)) = \tilde{w}\Phi(\lambda_i)$, where $\tilde{w}$ is a representative of $w \in W(\mathfrak{a})$ in the Weyl group of the maximal toral subalgebra and $\tilde{w}$ is a product of the w_i as above.

Chapter 4

Techniques used for Computing the Bases and Weyl Groups

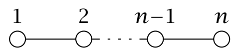
The first steps in the algorithm involve determining the Weyl group element w_σ as in 1.6 in order to recover the action of Γ. We also refer to w_σ as $w_\sigma(\Gamma)$. In most cases, this element is determined by considering that for α not in Δ_0, $w_\sigma^j(\alpha) = \alpha$ for some j, and that for α in Δ, $w_\sigma^j(\alpha) = 0$ for that same j. Recall that $\sigma = w_\sigma[\sigma]$. We then find the projection of each α using $\pi(\alpha) = \frac{1}{|\Gamma|}\Sigma_{\sigma\in\Gamma}\sigma(\alpha)$, which will be our λ_i. By obtaining λ_i in terms of α_j, we find a basis of the restricted root system in terms of the basis of the original root system. For $\alpha_j \in \Delta_0$, the projection is always 0.

In the following I list the type of Γ-index, the elements of the Galois group Γ, and give the nontrivial projections $\pi(\alpha_i)$, which give us the basis $\bar{\Delta}$ of $\Phi(\mathfrak{a})$. In doing this, the first two steps of the algorithm are completed.

4.1 A cases

There are 8 congruence classes of Γ-indices corresponding to a simple group of type A.

4.1.1 Type $A^{(1)}_{n,n}$

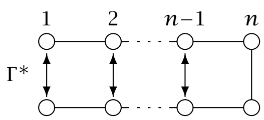


$\Gamma = \{\text{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

Here, nothing is fixed by Γ. Therefore, every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

4.1.2 Type ${}^2A^{(1)}_{2n,n}$

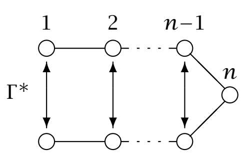


$\Gamma = \{\text{id}, \sigma\}$

$\sigma = \sigma^*$

$\lambda_i = \pi(\alpha_i) = \pi(\alpha_{2n-i+1}) = \frac{1}{2}(\alpha_i + \alpha_{2n-i+1})$

4.1.3 Type ${}^2A^{(1)}_{2n-1,n}$



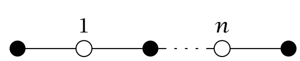
$\Gamma = \{\text{id}, \sigma\}$

$\sigma = \sigma^*$

$\lambda_i = \pi(\alpha_i) = \pi(\alpha_{2n-i}) = \frac{1}{2}(\alpha_i + \alpha_{2n-i})$

For λ_i in $\Delta(\mathfrak{a})$ with $i = 1, \cdots, n-1$, notice that there are two base roots that project down to λ_i.

4.1.4 Type $A^{(2)}_{2n+1,n}$



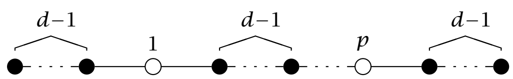
$\Gamma = \{\text{id}, \sigma\}$

$\sigma = s_1 s_3 ... s_{2n+1}$

$\lambda_i = \pi(\alpha_i) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$

α_{2i-1} is fixed for $i = 1, \cdots, n+1$. The roots α_{2i-1} project down to zero and the roots α_{2i} project down to $\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$.

4.1.5 Type $A_{n,p}^{(d)}$



$\Gamma = \{id, \sigma ..., \sigma^{d-1}\}$

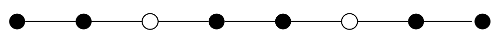
$\sigma = s_{\alpha_{d-2}} s_{\alpha_{d-3}} ... s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} ... s_{\alpha_{d-2}} s_{\alpha_{d-1}} s_{\alpha_{2d-2}} s_{\alpha_{2d-3}} ... s_{\alpha_{d+2}} s_{\alpha_{d+1}} s_{\alpha_{d+2}} s_{\alpha_{d+3}} ... s_{\alpha_{2d-2}} s_{\alpha_{2d-1}}$

$... s_{\alpha_{pd-2}} s_{\alpha_{pd-3}} ... s_{\alpha_{(p-1)d+2}} s_{\alpha_{(p-1)d+1}} s_{\alpha_{(p-1)d+2}} s_{\alpha_{(p-1)d+3}} ... s_{\alpha_{pd-2}} s_{\alpha_{pd-1}} s_{\alpha_{pd+1}} s_{\alpha_{pd+2}} s_{\alpha_{pd+1}}$

$s_{\alpha_{pd+3}} s_{\alpha_{pd+2}} s_{\alpha_{pd+1}} ... s_{\alpha_n} s_{\alpha_{n-1}} ... s_{\alpha_{pd+2}} s_{\alpha_{pd+1}}$

$\lambda_i = \pi(\alpha_{di}) = \frac{1}{2}(\alpha_{d(i-1)+1} + \alpha_{d(i-1)+2} + ... + 2\alpha_{di} + \alpha_{di+1} ... + \alpha_{d(i+1)-2} + \alpha_{d(i+1)-1})$

Example 4.1. Let us consider the Γ-diagram:



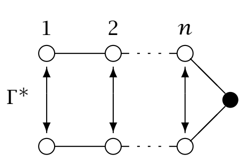
In this case, $\Gamma = \{\text{id}, \sigma\}$, and $\sigma^2 = \text{id}$.

$\sigma = s_1 s_2 s_1 s_4 s_5 s_4 s_6 s_7 s_6$.

$\lambda_1 = \pi(\alpha_3) = \frac{1}{2}(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)$

$\lambda_2 = \pi(\alpha_6) = \frac{1}{2}(\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8)$.

4.1.6 Type ${}^2A_{2n+1,n}^{(1)}$



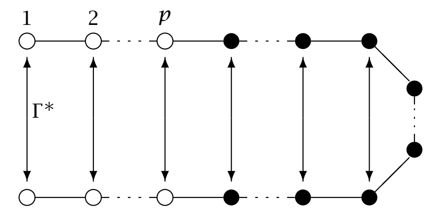
$\Gamma = \{\text{id}, \sigma\}$

$\sigma = s_{n+1} \sigma^*$

$\lambda_i = \pi(\alpha_i) = \pi_(\alpha_{2n-i+2}) = \frac{1}{2}(\alpha_i + \alpha_{2n-i+2})$

$\lambda_n = \pi(\alpha_n) = \pi(\alpha_{2n-i+2}) = \frac{1}{2}(\alpha_n + \alpha_{n+1} + \alpha_{n+2})$

4.1.7 Type ${}^2A^{(1)}_{n,p}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}} \sigma^*$

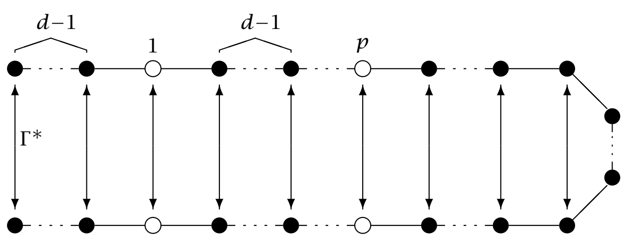
$(\lambda_p) = \pi(\alpha_p) = \frac{1}{2}(\alpha_p + \alpha_{p+1} + ... + \alpha_{p+m+1})$

$(\lambda_i) = \pi(\alpha_i) = \pi(\alpha(n - i + 1) = \frac{1}{2}(\alpha_i + \alpha_{n-i+1})$

The root α_{p+i} is fixed for $i = 1, \cdots, m$. For λ_i in $\Delta(\mathfrak{a})$ with $i = 1, \cdots, p-1$, notice that there are two base roots that project down to λ_i.

Remark 9. $n = p + m$

4.1.8 Type ${}^2A^{(d)}_{n,p}$



$\Gamma = \{\mathrm{id}, \sigma, \sigma^2, ...\sigma^{d-1}, \gamma, \gamma^2, ...\gamma^{d-1}, \gamma\sigma, \gamma\sigma^2, ..., \gamma\sigma^{d-1}, ...\gamma^{d-1}\sigma, \gamma^{d-1}\sigma^2, ..., \gamma^{d-1}\sigma^{d-1}\}$

$\sigma = s_{\alpha_{d-2}} s_{\alpha_{d-3}} ... s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} ... s_{\alpha_{d-2}} s_{\alpha_{d-1}} s_{\alpha_{2d-2}} s_{\alpha_{2d-3}} ... s_{\alpha_{d+2}} s_{\alpha_{d+1}} s_{\alpha_{d+2}} s_{\alpha_{d+3}} ...$

$s_{\alpha_{2d-2}} s_{\alpha_{2d-1}} ... s_{\alpha_{pd-2}} s_{\alpha_{pd-3}} ... s_{\alpha_{(p-1)d+2}} s_{\alpha_{(p-1)d+1}} s_{\alpha_{(p-1)d+2}} s_{\alpha_{(p-1)d+3}} ...$

$s_{\alpha_{pd-2}} s_{\alpha_{pd-1}} s_{\alpha_{pd+1}} s_{\alpha_{pd+2}} s_{\alpha_{pd+1}} s_{\alpha_{pd+3}} s_{\alpha_{pd+2}} s_{\alpha_{pd+1}} ... s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} ... s_{\alpha_{pd+2}} s_{\alpha_{pd+1}} \sigma^*$

$\gamma = s_{\alpha_{n-d+3}} s_{\alpha_{n-d+4}} ... s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-1}} s_{\alpha_{n-2}} ... s_{\alpha_{n-d+3}} s_{\alpha_{n-d+2}} s_{\alpha_{n-2d+3}} s_{\alpha_{n-2d+4}} ...$

$s_{\alpha_{n-d-1}} s_{\alpha_{n-d}} s_{\alpha_{n-d-1}} s_{\alpha_{n-d-2}} ... s_{\alpha_{n-2d+3}} s_{\alpha_{n-2d+2}} ... s_{\alpha_{n-pd+3}} s_{\alpha_{n-pd+4}} ...$

$s_{\alpha_{n-pd-1}} s_{\alpha_{n-pd+d}} s_{\alpha_{n-pd+d-1}} s_{\alpha_{n-pd+d-2}} ... s_{\alpha_{n-pd+3}} s_{\alpha_{n-pd+2}} \sigma^*$

$\lambda_i = \pi(\alpha_{di}) = \pi(\alpha_{n-di+1}) = \frac{1}{2d}(\alpha_{di-(d-1)} + 2\alpha_{di-(d-2)} + 3\alpha_{di-(d-3)} + ... + (d-1)\alpha_{di-1} + d\alpha_{di} + (d-1)\alpha_{di+1} + (d+2)\alpha_{di+2} + ... + 2\alpha_{di+(d-2)} + \alpha_{di+(d-1)} + \alpha_{n-di-d+2} + 2\alpha_{n-di-d+3} + 3\alpha_{n-di-d+4} + ... + (d-1)\alpha_{n-di} + d\alpha_{n-di+1} + (d-1)\alpha_{n-di+2} + (d+2)\alpha_{n-di+3} + ... + 2\alpha_{n-di+(d-1)} + \alpha_{n-di+d})$

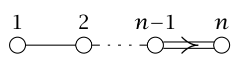
$\lambda_p = \pi(\alpha_{pd}) = \pi(\alpha_{pd+m+1}) = \frac{1}{2d}(\alpha_{dp-(d-1)} + 2\alpha_{dp-(d-2)} + 3\alpha_{dp-(d-3)} + ... + (d-1)\alpha_{dp-1} + d\alpha_{dp} + d\alpha_{dp+1} + d\alpha_{dp+2} + ... + d\alpha_{dp+m} + d\alpha_{dp+m+1} + (d-1)\alpha_{dp+m+2} + (d-2)\alpha_{dp+m+3} ... + 2\alpha_{dp+m+(d-2)} + \alpha_{dp+m+d-1}$

Remark 10. $n = 2p + m$

4.2 B case

There are 3 congruence classes of Γ-indices corresponding to a simple group of type B.

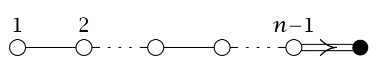
4.2.1 Type $B_{n,n}$



$\Gamma = \{\text{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

4.2.2 Type $B_{n,n-1}$



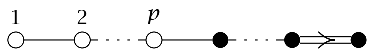
$\Gamma = \{\text{id}, \sigma\}$

$\sigma = s_n$

$\lambda_i = \pi(\alpha_i) = \alpha_i, i = 1, 2, \cdots, n-2$

$\lambda_{n-1} = \pi(\alpha_{n-1}) = \alpha_{n-1} + \alpha_n$

4.2.3 Type $B_{n,p}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m}}$
$\cdots s_{\alpha_{p+3}} s_{\alpha_{p+2}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$

$\lambda_i = \pi(\alpha_i) = \alpha_i, i = 1, 2, \cdots, p$

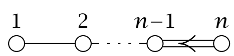
$\lambda_p = \pi(\alpha_p) = \alpha_p + ... + \alpha_n$

Remark 11. $n = p + m$

4.3 C cases

There are 4 congruence classes of Γ-indices corresponding to a simple group of type C.

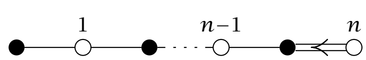
4.3.1 Type $C^{(1)}_{n,n}$



$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

4.3.2 Type $C^{(2)}_{2n,n}$



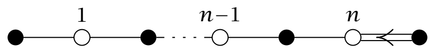
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2n-1}}$

$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, \cdots, n-1$

$\lambda_n = \pi(\alpha_{2n}) = \alpha_{2n-1} + \alpha_{2n}$

4.3.3 Type $C^{(2)}_{2n+1,n}$

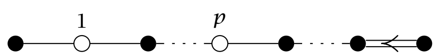


$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_1 s_3 s_5 \cdots s_{2n-1} s_{2n+1}$

$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, \cdots, n$

4.3.4 Type $C^{(2)}_{n,p}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{2p-1}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+3}} s_{\alpha_{2p+2}}$
$s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m}} \cdots s_{\alpha_{2p+3}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}}$
$s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}}$

$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, ..., p$

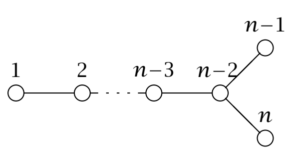
$\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + ... + 2\alpha_{n-1} + \alpha_n)$

Remark 12. $n = 2p + m$

4.4 D cases

There are 13 congruence classes of Γ-indices corresponding to a simple group of type D.

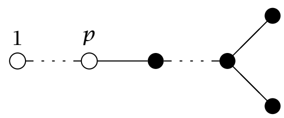
4.4.1 Type$D^{(1)}_{n,n}$



$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

4.4.2 Type $D^{(1)}_{n,p}$



$\Gamma = \{\mathrm{id}, \sigma\}$

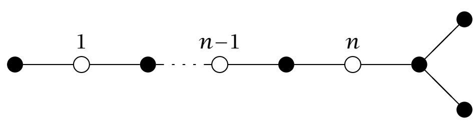
$\sigma = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+4}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}}$
$s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+4}} s_{\alpha_{p+m}}$
$s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots$

$\lambda_i = \pi(\alpha_i) = \alpha_i, i = 1, 2, \cdots, p-1$

$\lambda_p = \pi(\alpha_{p-1}) = \frac{1}{2}(2\alpha_{p+1} + 2\alpha_{p+2} + ... + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha n - 1 + \alpha_n)$

Remark 13. $n = p + m$

4.4.3 Type $D^{(2)}_{2n+3,n}$



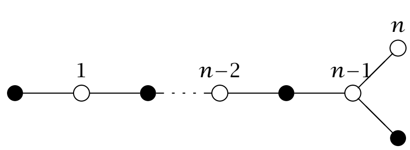
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_{2n-1}} s_{\alpha_{2n+2}} s_{\alpha_{2n+3}} s_{\alpha_{2n+1}} s_{\alpha_{2n+2}} s_{\alpha_{2n+3}} s_{\alpha_{2n+1}}$

$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i+1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots, n-1$

$\lambda_n = \pi(\alpha_{2n}) = \frac{1}{2}(\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3})$

4.4.4 Type $D^{(2)}_{2n,n}$



$\Gamma = \{\mathrm{id}, \sigma\}$

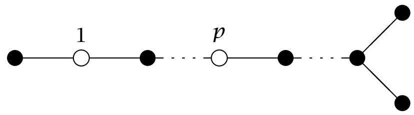
$\sigma = s_1 s_3 ... s_{2n-3} s_{2n}$

$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots, n-2$

$\lambda_{n-1} = \pi(\alpha_{2n-2}) = \frac{1}{2}(\alpha_{2n-3} + 2\alpha_{2n-2} + \alpha_{2n})$

$\lambda_n = \pi(\alpha_{2n-1}) = \alpha_{2n-1}$

4.4.5 Type $D_{n,p}^{(2)}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_1 s_3 s_5 \cdots s_{2p-1} s_{2p+1} s_{2p+2} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+4}}\ s_{\alpha_{p+3}}$

$s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}}$

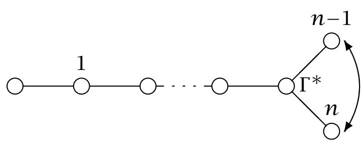
$s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots\ s_{\alpha_{p+3}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+4}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots$

$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots, p-1$

$\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$

Remark 14. $n = p + m$

4.4.6 Type ${}^2D_{n,n-1}^{(1)}$

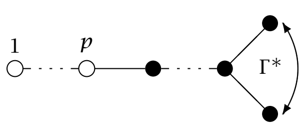


$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i, i = 1, 2, \cdots, n-1$

$\lambda_n = \pi(\alpha_{n-1}) = \pi(\alpha_n) = \frac{1}{2}(\alpha_{n-1} + \alpha_n)$

4.4.7 Type ${}^2D_{n,p}^{(1)}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+4}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}}$

$s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+m-1}} \cdots\ s_{\alpha_{p+4}}$

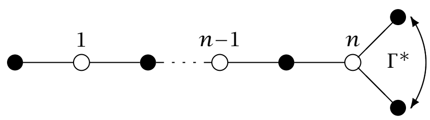
$s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots \sigma^*$

$\lambda_i = \pi(\alpha_i) = \alpha_i, i = 1, 2, \cdots, p-1$

$\lambda_p = \pi(\alpha_p) = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$

Remark 15. $n = p + m$

4.4.8 Type ${}^2D^{(2)}_{2n+2,n}$



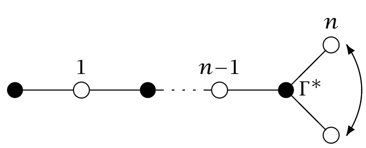
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_1 s_3 s_5 \cdots s_{2n-1} s_{2n+1} \sigma^*$

$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots n - 1$

$\lambda_n = \pi(\alpha_{2n}) = \frac{1}{4}(2\alpha_{2n-1} + 4\alpha_{2n} + 2\alpha_{2n+1} + 2\alpha_{2n+2})$

4.4.9 Type ${}^2D^{(2)}_{2n+1,n}$



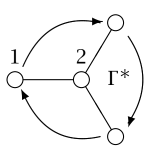
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_1 s_3 s ... s_{2n-1} \sigma^*$

$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots, n - 1$

$\lambda_n = \pi(\alpha_{2n}) = \pi(\alpha_{2n+1}) = \frac{1}{2}(\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1})$

4.4.10 Type ${}^3D^{(2)}_{4,2}$



$\Gamma = \{\mathrm{id}, \sigma, \sigma^2\}$

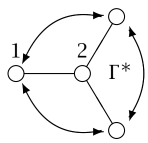
$\sigma = \sigma^*$

σ^* maps 1 to 3, 3 to 4, 4 to 1.

$\lambda_2 = \pi(\alpha_2) = \alpha_2$

$\lambda_1 = \pi(\alpha_1) = \pi(\alpha_3) = \pi(\alpha_4) = \frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4)$

4.4.11 Type ${}^6D_{4,2}^{(2)}$



$\Gamma = \{\mathrm{id}, \sigma, \gamma, \sigma^2, \sigma\gamma, \sigma^2\gamma\}$

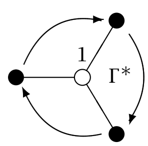
σ maps 1 to 3, 3 to 4, 4 to 1.

γ maps 3 to 1, 1 to 3.

$\lambda_2 = \pi(\alpha_2) = \alpha_2$

$\lambda_1 = \pi(\alpha_1) = \pi(\alpha_3) = \pi(\alpha_4) = \frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4)$

4.4.12 Type ${}^3D_{4,1}^{(9)}$



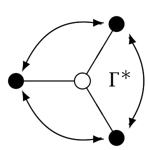
$\Gamma = \{\mathrm{id}, \sigma, \sigma^2\}$

$\sigma = s_1 s_3 s_4 \sigma^*$

σ^* maps 1 to 3, 3 to 4, 4 to 1.

$\lambda_1 = \pi(\alpha_2) = \frac{1}{3}(3\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4)$

4.4.13 Type ${}^6D_{4,1}^{(9)}$



$\Gamma = \{\mathrm{id}, \sigma, \gamma, \sigma^2, \sigma\gamma, \sigma^2\gamma\}$

$\sigma = s_1 s_3 s_4 \sigma^*$

$\gamma = s_1 s_3 s_4 \gamma^*$

σ^* maps 1 to 3, 3 to 4, 4 to 1.

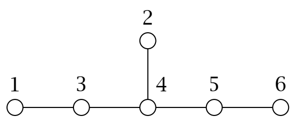
γ^* maps 3 to 1.

$\lambda_1 = \pi(\alpha_2) = \frac{1}{2}(2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4)$

4.5 E_6 cases

There are 8 congruence classes of Γ-indices corresponding to a simple group of type E_6.

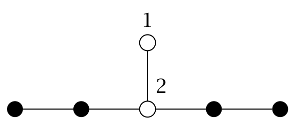
4.5.1 Type ${}^1E^0_{6,6}$



$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

4.5.2 Type ${}^1E^{16}_{6,2}$



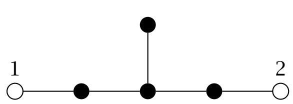
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_1 s_3 s_1 s_5 s_6 s_5$

$\lambda_1 = \pi(\alpha_2) = \alpha_2$

$\lambda_2 = \pi(\alpha_4) = \frac{1}{2}(\alpha_1 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$

4.5.3 Type ${}^1E^{28}_{6,2}$



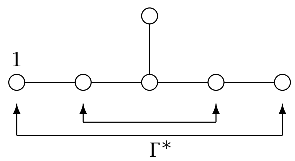
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5}$

$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$

$\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$

4.5.4 Type ${}^2E_{6,4}^{16}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = \sigma^*$

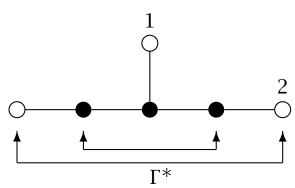
$\lambda_1 = \pi(\alpha_1) = \pi(\alpha_6) = \frac{1}{2}(\alpha_1 + \alpha_6)$

$\lambda_2 = \pi(\alpha_2) = \alpha_2$

$\lambda_3 = \pi(\alpha_3) = \pi(\alpha_5) = \frac{1}{2}(\alpha_3 + \alpha_5)$

$\lambda_4 = \pi(\alpha_4) = \alpha_4$

4.5.5 Type ${}^2E_{6,2}^{16'}$



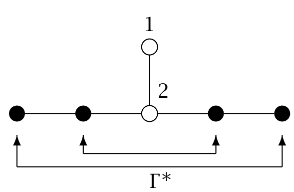
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}\sigma^*$

$\lambda_1 = \pi(\alpha_1) = \frac{1}{2}(2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$

$\lambda_2 = \pi(\alpha_6) = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$

4.5.6 Type ${}^2E_{6,2}^{16''}$



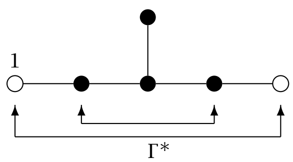
$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_5}s_{\alpha_6}s_{\alpha_5}s_{\alpha_1}s_{\alpha_3}s_{\alpha_1}\sigma^*$

$\lambda_1 = \pi(\alpha_2) = \alpha_2$

$\lambda_2 = \pi(\alpha_4) = \frac{1}{2}(\alpha_1 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$

4.5.7 Type ${}^2E_{6,1}^{29}$

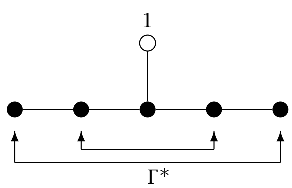


$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} \sigma^*$

$\lambda_1 = \pi(\alpha_1) = \pi(\alpha_6) = \frac{1}{2}(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6)$

4.5.8 Type ${}^2E_{6,1}^{35}$



$\Gamma = \{\mathrm{id}, \sigma\}$

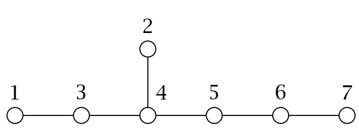
$\sigma =_{\alpha_6} s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4}$

$\lambda_1 = \pi(\alpha) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$

4.6 E_7 cases

There are 4 congruence classes of Γ-indices corresponding to a simple group of type E_7.

4.6.1 Type $E_{7,7}^{0}$

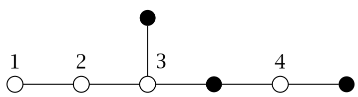


$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

Here nothing is fixed by Γ; every root projects down to itself.

4.6.2 Type $E^9_{7,4}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_2} s_{\alpha_5} s_{\alpha_7}$

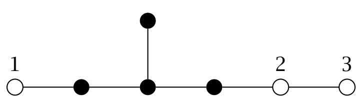
$\lambda_1 = \pi(\alpha_1) = \alpha_1$

$\lambda_3 = \pi(\alpha_3) = \alpha_3$

$\lambda_4 = \pi(\alpha_4) = \frac{1}{2}(\alpha_2 + 2\alpha_4 + \alpha_5)$

$\lambda_6 = \pi(\alpha_6) = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7)$

4.6.3 Type $E^{28}_{7,3}$



$\Gamma = \{\mathrm{id}, \sigma\}$

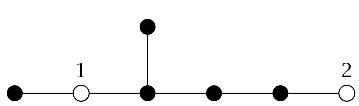
$\sigma = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5}$

$\lambda_1 = \pi(\alpha_1) = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$

$\lambda_2 = \pi(\alpha_6) = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$

$\lambda_3 = \pi(\alpha_7) = \alpha_7$

4.6.4 Type $E^{31}_{7,2}$



$\Gamma = \{\mathrm{id}, \sigma, \sigma^2, \sigma^3\}$

$\sigma = s_{\alpha_1} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$

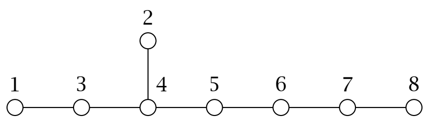
$\lambda_1 = \frac{1}{10}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)$

$\lambda_2 = \frac{1}{10}(\alpha_2 + 2\alpha_4 + 3\alpha_5 + 4\alpha_6 + 5\alpha_7)$

4.7 E_8 cases

There are 2 congruence classes of Γ-indices corresponding to a simple group of type E_8.

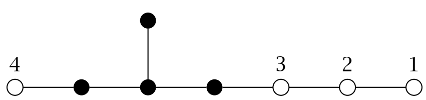
4.7.1 Type $E^0_{8,8}$



$\Gamma = \{\mathrm{id}\}$

$\lambda_i = \pi(\alpha_1) = \alpha_i$

4.7.2 Type $E^{28}_{8,4}$



$\Gamma = \{\mathrm{id}, \sigma\}$

$\sigma = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$

$\lambda_1 = \pi(\alpha_8) = \alpha_8$

$\lambda_2 = \pi(\alpha_7) = \alpha_7$

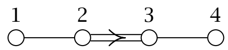
$\lambda_3 = \pi(\alpha_6) = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$

$\lambda_4 = \pi(\alpha_1) = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$

4.8 F_4 cases

There are 2 congruence classes of Γ-indices corresponding to a simple group of type F_4.

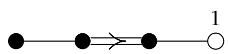
4.8.1 Type $F^0_{4,4}$



$\Gamma = \{\text{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

4.8.2 Type $F^{21}_{4,1}$



$\Gamma = \{\text{id}, \sigma\}$

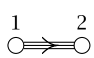
$\sigma = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_3}$

$\lambda_1 = \pi(\alpha_4) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$

4.9 G_2 case

There is 1 congruence class of Γ-indices corresponding to a simple group of type G_2.

4.9.1 Type $G^0_{2,2}$



$\Gamma = \{\text{id}\}$

$\lambda_i = \pi(\alpha_i) = \alpha_i$

Table 4.1: $w_\sigma(\Gamma)$

Type	$\Delta_0(\Gamma)$	$w_\sigma(\Gamma)$
$A^{(1)}_{n,n}$	$\emptyset$	id
${}^2A^{(1)}_{2n,n}$	$\emptyset$	id
${}^2A^{(1)}_{2n-1,n}$	$\emptyset$	id
$A^{(2)}_{2n+1,n}$	$\{\alpha_1, \alpha_3, \cdots, \alpha_{2n-2}, \alpha_{2n}\}$	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2n+1}}$
$A^{(d)}_{n,p}$	$\{\alpha_1, \alpha_2, ..., \alpha_{d-1}$ $\alpha_{d+2}, ..., \alpha_n\}$	$s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}...s_{\alpha_{d-1}}s_{\alpha_{d_2}}$ $...s_{\alpha_1}s_{\alpha_{d+1}}s_{\alpha_{d+2}}s_{\alpha_{d+1}}s_{\alpha_{d+3}}s_{\alpha_{d+2}}$ $s_{\alpha_{d+1}}\cdots s_{\alpha_n}s_{\alpha_{n-1}}\cdots s_{\alpha_{n-d+2}}$ $s_{\alpha_{n-d+1}}$
${}^2A^{(1)}_{2n+1,n}$	$\{\alpha_{n+1}\}$	$s_{\alpha_{n+1}}$
${}^2A^{(1)}_{n,p}$	$\{\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_{p+m}\}$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}$ $\cdots s_{\alpha_{p+m}}s_{\alpha_{p+m-1}}\cdots s_{\alpha_{p+1}}$
${}^2A^{(d)}_{n,p}$	$\{\alpha_1, \alpha_2, ..., \alpha_{d-1}\}$	$s_{\alpha_{d-2}}s_{\alpha_{d-3}}...s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}...s_{\alpha_{d-2}}$ $s_{\alpha_{d-1}}s_{\alpha_{2d-2}}s_{\alpha_{2d-3}}...s_{\alpha_{d+2}}s_{\alpha_{d+1}}$ $s_{\alpha_{d+2}}s_{\alpha_{d+3}}...s_{\alpha_{2d-2}}s_{\alpha_{2d-1}}...$ $s_{\alpha_{pd-2}}s_{\alpha_{pd-3}}...s_{\alpha_{(p-1)d+2}}s_{\alpha_{(p-1)d+1}}$ $s_{\alpha_{(p-1)d+2}}\ s_{\alpha_{(p-1)d+3}}...s_{\alpha_{pd-2}}s_{\alpha_{pd-1}}$ $s_{\alpha_{pd+1}}\ s_{\alpha_{pd+2}}s_{\alpha_{pd+1}}s_{\alpha_{pd+3}}s_{\alpha_{pd+2}}$ $s_{\alpha_{pd+1}}\ ...s_{\alpha_{pd+m}}\ s_{\alpha_{pd+m-1}}...\ s_{\alpha_{pd+2}}$ $s_{\alpha_{pd+1}}$, $s_{\alpha_{n-d+3}}s_{\alpha_{n-d+4}}...\ s_{\alpha_{n-1}}s_{\alpha_n}s_{\alpha_{n-1}}$ $s_{\alpha_{n-2}}\ ...\ s_{\alpha_{n-d+3}}s_{\alpha_{n-d+2}}s_{\alpha_{n-2d+3}}$ $s_{\alpha_{n-2d+4}}...\ s_{\alpha_{n-d-1}}s_{\alpha_{n-d}}s_{\alpha_{n-d-1}}$ $s_{\alpha_{n-d-2}}...s_{\alpha_{n-2d+3}}\ s_{\alpha_{n-2d+2}}...\ s_{\alpha_{n-pd+3}}$ $s_{\alpha_{n-pd+4}}...s_{\alpha_{n-pd-1}}\ s_{\alpha_{n-pd+d}}\ s_{\alpha_{n-pd+d-1}}$ $s_{\alpha_{n-pd+d-2}}...s_{\alpha_{n-pd+3}}\ s_{\alpha_{n-pd+2}}$

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Table 4.1: *continued*

Type	$\Delta_0(\Gamma)$	$w_\sigma(\Gamma)$
$B_{n,n}$	$\emptyset$	id
$B_{n,n-1}$	$\{\alpha_n\}$	s_n
$B_{n,p}$	$\{\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_{p+m}\}$	$s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$ $s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}}$ $s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m}} \cdots$ $s_{\alpha_{p+3}} s_{\alpha_{p+2}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}}$ $s_{\alpha_{p+m}}\ s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}}$ $s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$
$C^{(1)}_{n,n}$	$\emptyset$	id
$C^{(2)}_{2n,n}$	$\{\alpha_1, \alpha_3, \cdots, \alpha_{2l-3}, \alpha_{2n-1}\}$	$s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2n-3}} s_{\alpha_{2n-1}}$
$C^{(2)}_{2n+1,n}$	$\{\alpha_1, \alpha_3, ..., \alpha_{2n-1}, \alpha_{2n_1}\}$	$s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} \cdots s_{\alpha_{2n-1}} s_{\alpha_{2n+1}}$
$C^{(2)}_{n,p}$	$\{\alpha_1, \alpha_3, \cdots, \alpha_{2p-1},$ $\alpha_{2p+1}, \cdots, \alpha_{2p+m}\}$	$s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{2p-1}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}}$ $\cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+3}}$ $s_{\alpha_{2p+2}}\ s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m}} \cdots$ $s_{\alpha_{2p+3}}\ s_{\alpha_{2p+2}} \cdots s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m-1}}$ $s_{\alpha_{2p+m}}\ s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m}}$ $s_{\alpha_{2p+m-1}}\ s_{\alpha_{2p+m}}$
$D^{(1)}_{n,n}$	$\emptyset$	id
$D^{(1)}_{n,p}$	$\{\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_{p+m}\}$	$s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+4}}$ $s_{\alpha_{p+3}}\ s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}}$ $\cdots\ s_{\alpha_{p+1}}\ s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}}$ $s_{\alpha_{p+m-1}}\ \cdots\ s_{\alpha_{p+2}}\ s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots$ $s_{\alpha_{p+3}} s_{\alpha_{p+m-1}}\ \cdots\ s_{\alpha_{p+4}}\ s_{\alpha_{p+m}} s_{\alpha_{p+m-2}}$ $\cdots s_{\alpha_{p+5}} \cdots$
$D^{(2)}_{2n,n}$	$\{\alpha_1, \alpha_3, ..., \alpha_{2n-3}, \alpha_{2n}\}$	$s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2n-3}} s_{\alpha_{2n}}$

continued on next page

Table 4.1: *continued*

Type	$\Delta_0(\Gamma)$	$w_\sigma(\Gamma)$
$D^{(2)}_{2n+3,n}$	$\{\alpha_1,\alpha_3,...,\alpha_{2n+1},\alpha_{2n+2},\alpha_{2n+3}\}$	$s_{\alpha_{2n-1}}s_{\alpha_{2n+2}}s_{\alpha_{2n+3}}s_{\alpha_{2n+1}}s_{\alpha_{2n+2}}$ $s_{\alpha_{2n+3}}s_{\alpha_{2n+1}}$
$D^{(2)}_{n,p}$	$\{\alpha_1,\alpha_3,...,\alpha_{p+1},\alpha_{p+2},...,\alpha_n\}$	$s_{\alpha_1}s_{\alpha_3}s_{\alpha_5}\cdots s_{\alpha_{2p-1}}s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}$ $s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+4}}s_{\alpha_{p+3}}$ $s_{\alpha_{p+2}}s_{\alpha_{p+1}}\cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}}\cdots$ $s_{\alpha_{p+1}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+1}}s_{\alpha_{p+m-1}}$ $\cdots s_{\alpha_{p+2}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+3}}s_{\alpha_{p+m-1}}$ $\cdots\ s_{\alpha_{p+4}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+5}}\cdots$
${}^2D^{(1)}_{n+1,n}$	$\emptyset$	id
${}^2D^{(1)}_{n,p}$	$\{\alpha_{p+1},\alpha_{p+2},\cdots,\alpha_{p+m}\}$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+4}}$ $s_{\alpha_{p+3}}\ s_{\alpha_{p+2}}s_{\alpha_{p+1}}\cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}}\cdots$ $s_{\alpha_{p+1}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+1}}s_{\alpha_{p+m-1}}\cdots$ $s_{\alpha_{p+2}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+3}}s_{\alpha_{p+m-1}}\cdots$ $s_{\alpha_{p+4}}\ s_{\alpha_{p+m}}s_{\alpha_{p+m-2}}\cdots s_{\alpha_{p+5}}\cdots$
${}^2D^{(2)}_{2n+2,n}$	$\{\alpha_1,\alpha_3,...,\alpha_{2n-1},\alpha_{2n+1},\alpha_{2n+2}\}$	$s_{\alpha_1}s_{\alpha_3}s_{\alpha_5}\cdots s_{\alpha_{2n-1}}s_{\alpha_{2n+1}}s_{\alpha_{2n+2}}$
${}^2D^{(2)}_{2n+1,n}$	$\{\alpha_1,\alpha_3,\cdots,\alpha_{2n-1}\}$	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2n-1}}$
${}^3D^{(2)}_{4,2}$	$\emptyset$	id
${}^6D^{(2)}_{4,2}$	$\emptyset$	id
${}^3D^{(9)}_{4,1}$	$\{\alpha_2,\alpha_3,\alpha_4\}$	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$
${}^6D^{(9)}_{4,1}$	$\{\alpha_2,\alpha_3,\alpha_4\}$	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$
${}^1E^{0}_{6,6}$	$\emptyset$	id
${}^1E^{16}_{6,2}$	$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}$	$s_{\alpha_1}s_{\alpha_3}s_{\alpha_1}s_{\alpha_5}s_{\alpha_6}s_{\alpha_5}$
${}^1E^{28}_{6,2}$	$\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$

continued on next page

Table 4.1: *continued*

Type	$\Delta_0(\Gamma)$	$w_\sigma(\Gamma)$
${}^2E_{6,4}^{16}$	$\emptyset$	id
${}^2E_{6,2}^{16'}$	$\{\alpha_3, \alpha_4, \alpha_5\}$	$s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_4}$
${}^2E_{6,2}^{16''}$	$\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$	$s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_1}$
${}^2E_{6,1}^{29}$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	$s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5}$
${}^2E_{6,1}^{35}$	$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$\alpha_6 s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4}$
$E_{7,7}^{0}$	$\emptyset$	id
$E_{7,4}^{9}$	$\{\alpha_2, \alpha_5, \alpha_7\}$	$s_{\alpha_2} s_{\alpha_5} s_{\alpha_7}$
$E_{7,3}^{28}$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	$s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5}$
$E_{7,2}^{31}$	$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$s_{\alpha_1} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$
$E_{8,8}^{0}$	$\emptyset$	id
$E_{8,4}^{28}$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$	$s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5}$
$F_{4,4}^{0}$	$\emptyset$	id
$F_{4,1}^{21}$	$\{\alpha_1, \alpha_2, \alpha_3\}$	$s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_3}$
$G_{2,2}^{0}$	$\emptyset$	id

Table 4.2: Basis of $\Phi(\mathfrak{a})$ in terms of basis of $\Phi(\mathfrak{t})$

Type	λ
$A_{n,n}^{(1)}$	$\lambda_i = \alpha_i \ \forall i$
${}^2A_{2n,n}^{(1)}$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{n+1-i})$ for $i = 1, \cdots, n-1$ $\lambda_n = \alpha_n$
${}^2A_{2n-1,n}^{(1)}$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i+1})$

continued on next page

Table 4.2: *continued*

Type	λ
$A^{(2)}_{2n+1,n}$	$\lambda_i = \pi(\alpha_i) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$
$A^{(d)}_{n,p}$	$\lambda_i = \frac{1}{2}(\alpha_{d(i-1)+1} + \alpha_{d(i-1)+2}$ $+ \cdots + 2\alpha_{di} + \alpha_{di+1} \cdots + \alpha_{d(i+1)-2} + \alpha_{d(i+1)-1})$
${}^2A^{(1)}_{n,p}$	$\lambda_p = \frac{1}{2}(\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1})$ $\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{n-i+1})$
${}^2A^{(1)}_{2n+1,n}$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i+2})$ $\lambda_n = \pi(\alpha_n) = \frac{1}{2}(\alpha_n + \alpha_{n+1} + \alpha_{n+2})$
${}^2A^{(d)}_{n,p}$	$\lambda_i = \pi(\alpha_{di}) = \pi(\alpha_{n-di+1}) = \frac{1}{2d}(\alpha_{di-(d-1)} + 2\alpha_{di-(d-2)} + 3\alpha_{di-(d-3)} +$ $\ldots + (d-1)\alpha_{di-1} + d\alpha_{di} + (d-1)\alpha_{di+1} + (d+2)\alpha_{di+2} + \ldots + 2\alpha_{di+(d-2)}$ $+\alpha_{di+(d-1)} + \alpha_{n-di-d+2} + 2\alpha_{n-di-d+3} + 3\alpha_{n-di-d+4} + \ldots + (d-1)\alpha_{n-di}$ $+d\alpha_{n-di+1} + (d-1)\alpha_{n-di+2} + (d+2)\alpha_{n-di+3} + \ldots + 2\alpha_{n-di+(d-1)} +$ $\alpha_{n-di+d})$ $\lambda_p = \pi(\alpha_{pd}) = \pi(\alpha_{pd+m+1}) = \frac{1}{2d}(\alpha_{dp-(d-1)} + 2\alpha_{dp-(d-2)} + 3\alpha_{dp-(d-3)}$ $+ \ldots + (d-1)\alpha_{dp-1} + d\alpha_{dp} + d\alpha_{dp+1} + d\alpha_{dp+2} + \ldots + d\alpha_{dp+m} + d\alpha_{dp+m+1}$ $+(d-1)\alpha_{dp+m+2} + (d-2)\alpha_{dp+m+3} \ldots + 2\alpha_{dp+m+(d-2)} + \alpha_{dp+m+d-1}$
$B_{n,n}$	$\lambda_i = \alpha_i \ \forall i$
$B_{n,n-1}$	$\lambda_i = \alpha_i, i = 1, 2, \cdots, n-2$ $\lambda_{n-1} = \alpha_{n-1} + \alpha_n$
$B_{n,p}$	$\lambda_i = \alpha_i$ for $i = 1, \cdots, p-1$ $\lambda_p = \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1} + \alpha_{p+m}$
$C^{(1)}_{n,n}$	$\lambda_i = \alpha_i \ \forall i$
$C^{(2)}_{2n,n}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, n-1$ $\lambda_n = \alpha_{2n-1} + \alpha_{2n}$
$C^{(2)}_{2n+1,n}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, \cdots, n$

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Table 4.2: *continued*

Type	λ
$C_{n,p}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, p-1$ $\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m})$
$D_{n,n}^{(1)}$	$\lambda_i = \alpha_i \ \forall i$
$D_{n,p}^{(1)}$	$\lambda_i = \alpha_i$ for $i = 1 \cdots p-1$ $\lambda_p = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m})$
$D_{2n,n}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, n-2$ $\lambda_{n-1} = \alpha_{2n-2}$ $\lambda_n = \frac{1}{2}(\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1})$
$D_{2n+3,n}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i+1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots, n-1$ $\lambda_n = \frac{1}{2}(\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}$
$D_{n,p}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}, i = 1, 2, \cdots p-1$ $\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n$
${}^2D_{n+1,n}^{(1)}$	$\lambda_i = \alpha_i, i = 1, 2, \cdots, n-1$ $\lambda_n = \frac{1}{2}(\alpha_n + \alpha_{n+1})$
${}^2D_{n,p}^{(1)}$	$\lambda_i = \alpha_i, i = 1, 2, \cdots, p-1$ $\lambda_p = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$
${}^2D_{2n+2,n}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}), i = 1, 2, \cdots n-1$ $\lambda_n = \frac{1}{2}(\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2})$
${}^2D_{2n+1,n}^{(2)}$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, n-1$ $\lambda_l = \frac{1}{2}(\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1})$
${}^3D_{4,2}^{(2)}$	$\lambda_1 = \alpha_1$ $\lambda_2 = \frac{1}{3}(\alpha_2 + \alpha_3 + \alpha_4)$

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Table 4.2: *continued*

Type	λ
${}^{6}D_{4,2}^{(2)}$	$\lambda_1 = \alpha_1$
	$\lambda_2 = \frac{1}{3}(\alpha_2 + \alpha_3 + \alpha_4)$
${}^{3}D_{4,1}^{(9)}$	$\lambda_1 = \frac{1}{3}(3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$
${}^{6}D_{4,1}^{(9)}$	$\lambda_1 = \frac{1}{6}(6\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4)$
${}^{1}E_{6,6}^{0}$	$\lambda_i = \alpha_i \ \forall i$
${}^{1}E_{6,2}^{16}$	$\lambda_1 = \alpha_2$
	$\lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$
${}^{1}E_{6,2}^{28}$	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
${}^{2}E_{6,4}^{16}$	$\lambda_1 = \alpha_2$
	$\lambda_2 = \alpha_4$
	$\lambda_3 = \frac{1}{2}(\alpha_3 + \alpha_5)$
	$\lambda_4 = \frac{1}{2}(\alpha_1 + \alpha_6)$
${}^{2}E_{6,2}^{16'}$	$\lambda_1 = \frac{1}{2}(2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
${}^{2}E_{6,2}^{16''}$	$\lambda_1 = \alpha_2$
	$\lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$
${}^{2}E_{6,1}^{29}$	$\lambda_1 = \frac{1}{4}(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6)$
${}^{2}E_{6,1}^{35}$	$\lambda_1 = \pi(\alpha) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$
$E_{7,7}^{0}$	$\lambda_i = \alpha_i \ \forall i$
$E_{7,4}^{9}$	$\lambda_1 = \alpha_1$
	$\lambda_2 = \alpha_3$

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Table 4.2: *continued*

Type	λ
	$\lambda_3 = \frac{1}{2}(\alpha_2 + 2\alpha_4 + \alpha_5)$
	$\lambda_4 = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7)$
$E_{7,3}^{28}$	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
	$\lambda_3 = \alpha_7$
$E_{7,2}^{31}$	$\lambda_1 = \frac{1}{10}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)$
	$\lambda_2 = \frac{1}{10}(\alpha_2 + 2\alpha_4 + 3\alpha_5 + 4\alpha_6 + 5\alpha_7)$
$E_{8,8}^{0}$	$\lambda_i = \alpha_i \ \forall i$
$E_{8,4}^{28}$	$\lambda_1 = \alpha_8$
	$\lambda_2 = \alpha_7$
	$\lambda_3 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
	$\lambda_4 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
$F_{4,4}^{0}$	$\lambda_i = \alpha_i \ \forall i$
$F_{4,1}^{21}$	$\lambda_1 = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$
$G_{2,2}^{0}$	$\lambda_i = \alpha_i \ \forall i$

Chapter 5

Computing Weyl Group Elements

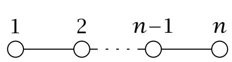
In this chapter we compute the Weyl group representative w_i in W^Γ for each reflection s_{λ_i} in the restricted root system $\Delta(\mathfrak{a})$. This is found by considering the λ_i in terms of α_j and determining $w_i \in W^\Gamma$ such that $w_i(\lambda_i) = -\lambda_i$ and $w_i(\Delta_0) = \Delta_0$.

In the following I list the Γ-index, the type of restricted root system, and the Weyl group representative w_i for each s_{λ_i}.

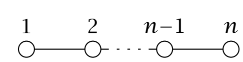
5.1 A cases

There are 8 congruence classes of Γ-indices corresponding to a simple group of type A.

5.1.1 Type $A_{n,n}^{(1)}$

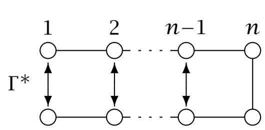


Type $\Phi(\mathfrak{a})$:

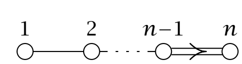


$w_i = s_{\alpha_i}$

5.1.2 Type ${}^2A_{2n,n}^{(1)}$

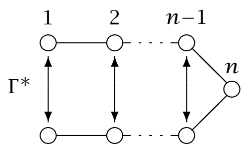


Type $\Phi(\mathfrak{a})$:

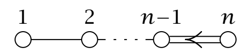


$w_i = s_{\alpha_i} s_{\alpha_{2n-i+1}}$

5.1.3 Type ${}^2A^{(1)}_{2n-1,n}$



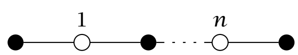
Type $\Phi(\mathfrak{a})$:



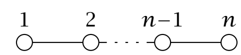
$w_i = s_{\alpha_i} s_{\alpha_{2l=n-i}}, i < n$

$w_n = s_{\alpha_n}$

5.1.4 Type $A^{(2)}_{2n+1,n}$

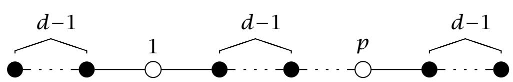


Type $\Phi(\mathfrak{a})$:

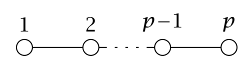


$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}$

5.1.5 Type $A^{(d)}_{n,p}$



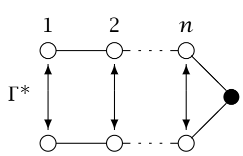
Type $\Phi(\mathfrak{a})$:



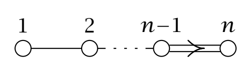
$w_i = s_{di} s_{di+1} s_{di-1} s_{di} s_{di+2} s_{di-2} s_{di+1} s_{di-1} s_{di} s_{di+3} s_{di-s} s_{di+2} s_{di-2} s_{di+1} s_{di-1}$

$s_{di} \cdots s_{di+(d-1)} s_{di-(d-1)} s_{di+(d-2)} s_{di-(d-2)} s_{di+(d-3)} s_{di-(d-3)} \cdots s_{di+1} s_{di-1} s_{di}$

5.1.6 Type ${}^2A^{(1)}_{2n+1,n}$



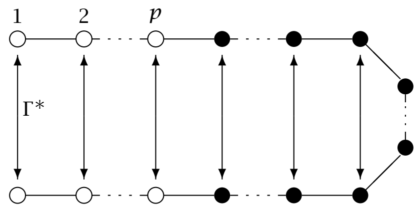
Type $\Phi(\mathfrak{a})$:



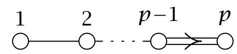
$w_i = s_i, i = 1, 2, \cdots, n-1$

$w_n = s_{\alpha_{n+2}} s_{\alpha_n} s_{\alpha_{n+1}} s_{\alpha_n} s_{\alpha_{n+2}}$

5.1.7 Type ${}^2A_{n,p}^{(1)}$



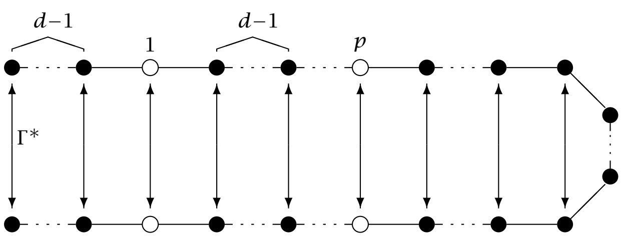
Type $\Phi(\alpha)$:



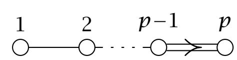
$w_i = s_{\alpha_i} s_{\alpha_{l+1-i}}$ for $i = 1, \cdots, p-1$

$w_p = s_{\alpha_{p+m+1}} s_{\alpha_{p+m}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m+1}}$

5.1.8 Type ${}^2A_{n,p}^{(d)}$



Type $\Phi(\alpha)$:



$w_i = s_{\alpha_{di}} s_{\alpha_{di+1}} s_{\alpha_{di-1}} s_{\alpha_{di+2}} s_{\alpha_{di}} s_{\alpha_{di-2}} \ldots s_{\alpha_{di+d-3}} s_{\alpha_{di+d-5}} \ldots s_{\alpha_{di-(d-5)}} s_{\alpha_{di-(d-3)}} s_{\alpha_{di+d-2}} s_{\alpha_{di+d-4}}$

$\ldots s_{\alpha_{di-(d-4)}} s_{\alpha_{di-(d-2)}} s_{\alpha_{di+d-1}} s_{\alpha_{di+d-3}} \ldots s_{\alpha_{di-(d-3)}} s_{\alpha_{di-(d-1)}} s_{\alpha_{di+d-2}} s_{\alpha_{di+d-4}} \ldots s_{\alpha_{di-(d-4)}}$

$s_{\alpha_{di-(d-2)}} s_{\alpha_{di+d-3}} s_{\alpha_{di+d-5}} \ldots \alpha_{di-(d-5)} s_{\alpha_{di-(d-3)}} \ldots s_{\alpha_{di+2}} s_{\alpha_{di}} s_{\alpha_{di-2}} s_{\alpha_{di+1}} s_{\alpha_{di-1}} s_{\alpha_{di}}$

$w_p = s_{\alpha_{pd+1}} s_{\alpha_{pd+2}} \ldots s_{\alpha_{pd-d+1}} s_{\alpha_{pd+2}} s_{\alpha_{pd+3}} \ldots s_{\alpha_{pd-d+2}} \ldots s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}} \ldots s_{\alpha_{pd-d+m}}$

$s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \ldots s_{\alpha_{pd-d+m+1}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \ldots s_{\alpha_{pd-d+m+1}} s_{\alpha_{pd+m+1}} s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m+2}} s_{\alpha_{pd+m}}$

$s_{\alpha_{pd+m-2}} \ldots s_{\alpha_{pd+m+d-2}} s_{\alpha_{pd+m+d-4}} \ldots s_{\alpha_{pd-d+m+4}} s_{\alpha_{pd+m+d-1}} s_{\alpha_{pd+m+d-3}} \ldots s_{\alpha_{pd-d+m+3}} s_{\alpha_{pd+m+d}}$

$s_{\alpha_{pd+m+d-2}} \ldots s_{\alpha_{pd-d+m+2}} s_{\alpha_{pd+m-d+1}} s_{\alpha_{pd+m-d+2}} \ldots s_{\alpha_{pd-d+1}} s_{\alpha_{pd+m+d-1}} s_{\alpha_{pd+m+d-3}} \ldots s_{\alpha_{pd+m-d+3}}$

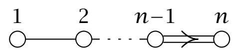
$s_{\alpha_{pd+m-d+2}} s_{\alpha_{pd+m-d+1}} \ldots s_{\alpha_{pd-d+2}} \ldots s_{\alpha_{pd+m+3}} s_{\alpha_{pd+m+1}} s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}} s_{\alpha_{pd+m-1}} \ldots s_{\alpha_{pd-2}}$

$s_{\alpha_{pd+m+2}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}} \ldots s_{\alpha_{pd-1}} s_{\alpha_{pd+m+1}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \ldots s_{\alpha_{pd}}$

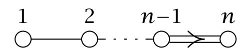
5.2 B case

There are 3 congruence classes of Γ-indices corresponding to a simple group of type B.

5.2.1 Type $B_{n,n}$

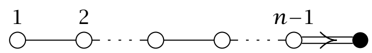


Type $\Phi(\alpha)$:

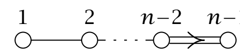


$w_i = s_i$

5.2.2 Type $B_{n,n-1}$



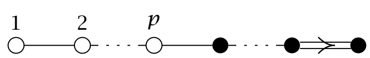
Type $\Phi(\alpha)$:



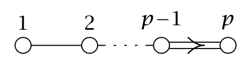
$w_i = s_i, i = 1, 2, \cdots, n-2$

$w_{n-1} = s_{n-1}s_n s_{n-1}$

5.2.3 Type $B_{n,p}$



Type $\Phi(\alpha)$:



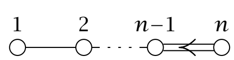
$w_i = s_{\alpha_i}, i < p$

$w_p = s_{\alpha_n}s_{\alpha_{n-1}} \cdots s_{\alpha_{p+1}}s_{\alpha_p}s_{\alpha_{p+1}} \cdots s_{\alpha_{n-1}}s_{\alpha_n}$

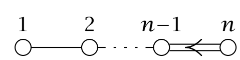
5.3 C cases

There are 4 congruence classes of Γ-indices corresponding to a simple group of type C.

5.3.1 Type $C^{(1)}_{n,n}$

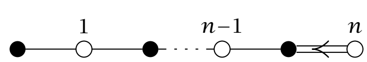


Type $\Phi(\alpha)$:

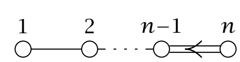


$w_i = s_{\alpha_i}$.

5.3.2 Type $C^{(2)}_{2n,n}$



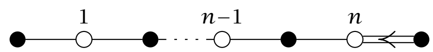
Type $\Phi(\alpha)$:



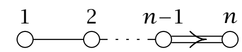
$w_i = s_{\alpha_{2i}}s_{\alpha_{2i-1}}s_{\alpha_{2i+1}}s_{\alpha_{2i}}, i < n$

$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n}}$.

5.3.3 Type $C^{(2)}_{2n+1,n}$



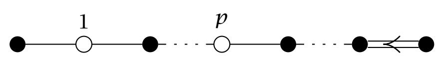
Type $\Phi(\alpha)$:



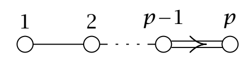
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, n-1$

$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+1}} s_{\alpha_{2n-1}} s_{\alpha_{2n}}$

5.3.4 Type $C^{(2)}_{n,p}$



Type $\Phi(\alpha)$:



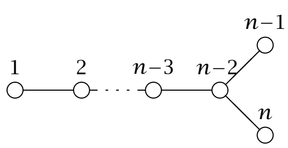
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}, i < p$

$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$

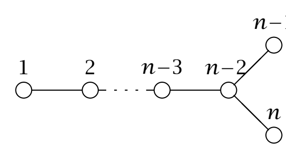
5.4 D cases

There are 13 congruence classes of Γ-indices corresponding to a simple group of type D.

5.4.1 Type $D^{(1)}_{n,n}$

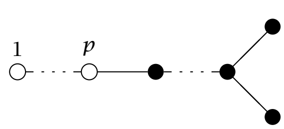


Type $\Phi(\alpha)$:

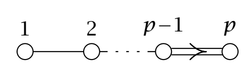


$w_i = s_{\alpha_i}$

5.4.2 Type $D^{(1)}_{n,p}$



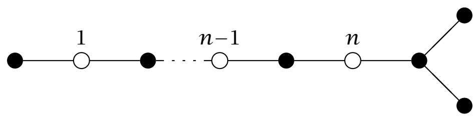
Type $\Phi(\alpha)$:



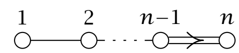
$w_i = s_{\alpha_i}, i < p$

$w_p = s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{n-2}} s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$

5.4.3 Type $D^{(2)}_{2n+3,n}$



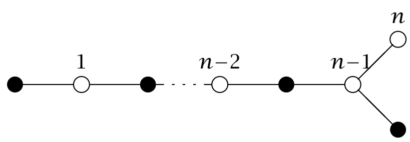
Type $\Phi(\mathfrak{a})$:



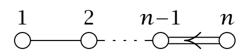
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, n-1$

$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+2}} s_{\alpha_{2n+3}} s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+2}} s_{\alpha_{2n-1}} s_{\alpha_{2n+3}} s_{\alpha_{2n+1}} s_{\alpha_{2n}}$

5.4.4 Type $D^{(2)}_{2n,n}$



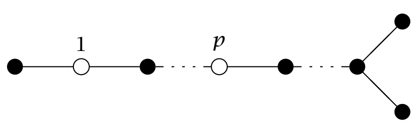
Type $\Phi(\mathfrak{a})$:



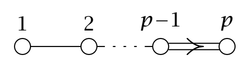
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, p-1$

$w_n = s_{\alpha_{2n}}$

5.4.5 Type $D^{(2)}_{n,p}$



Type $\Phi(\mathfrak{a})$:

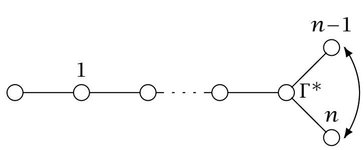


$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, p-1$

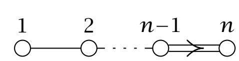
$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{n-3}} s_{\alpha_{n-2}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{n-4}} s_{\alpha_{n-3}} s_{\alpha_n} s_{\alpha_{n-1}} s_{\alpha_{n-2}} s_{\alpha_n}$

$s_{\alpha_{n-1}} s_{\alpha_{n-2}} s_{\alpha_{n-3}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$

5.4.6 Type ${}^2D^{(1)}_{n+1,n}$



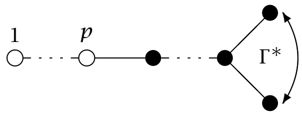
Type $\Phi(\mathfrak{a})$:



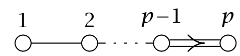
$w_i = s_{\alpha_i}, i = 1, 2, \cdots, n-1$

$w_n = s_{\alpha_{n+1}} s_{\alpha_n}$

5.4.7 Type ${}^2D^{(1)}_{n,p}$



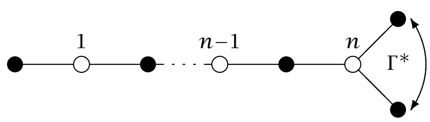
Type $\Phi(\alpha)$:



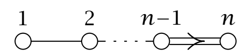
$w_i = s_{\alpha_i}, i = 1, 2, \cdots, p-1$

$w_p = s_{\alpha_p} s_{\alpha_{p+1}} s_{\alpha_{p+2}} \cdots s_{\alpha_{n-3}} s_{\alpha_{n-2}} s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-2}} s_{\alpha_{n-3}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$

5.4.8 Type ${}^2D^{(2)}_{2n+2,n}$



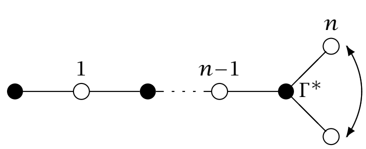
Type $\Phi(\alpha)$:



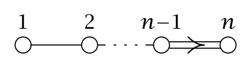
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, n-1$

$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n+2}} s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n+2}} s_{\alpha_{2n}}$

5.4.9 Type ${}^2D^{(2)}_{2n+1,n}$



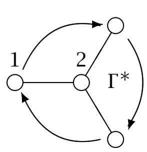
Type $\Phi(\alpha)$:



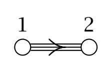
$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i < n$

$w_n = s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n}} s_{\alpha_{2n+1}}$

5.4.10 Type ${}^3D^{(2)}_{4,2}$



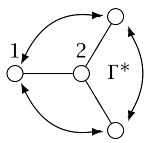
Type $\Phi(\alpha)$:



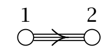
$w_1 = s_{\alpha_1}$

$w_2 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$

5.4.11 Type ${}^6D_{4,2}^{(2)}$



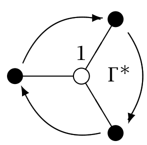
Type $\Phi(\mathfrak{a})$:



$w_1 = s_{\alpha_1}$

$w_2 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$

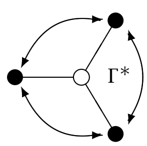
5.4.12 Type ${}^3D_{4,1}^{(9)}$



Type $\Phi(\mathfrak{a})$:

$w_1 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$

5.4.13 Type ${}^6D_{4,1}^{(9)}$



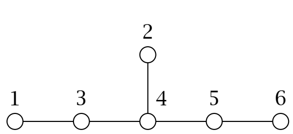
Type $\Phi(\mathfrak{a})$:

$w_1 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$

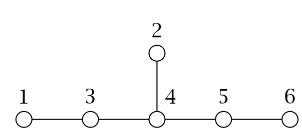
5.5 E_6 cases

There are 8 congruence classes of Γ-indices corresponding to a simple group of type E_6.

5.5.1 Type ${}^1E_{6,6}^0$

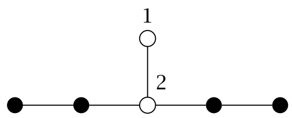


Type $\Phi(\mathfrak{a})$:

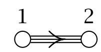


$w_i = s_{\alpha_i}$

5.5.2 Type ${}^1E_{6,2}^{16}$



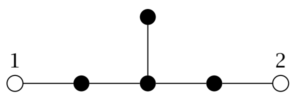
Type $\Phi(\mathfrak{a})$:



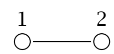
$w_1 = s_{\alpha_2}$

$w_2 = s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}$

5.5.3 Type ${}^1E_{6,2}^{28}$



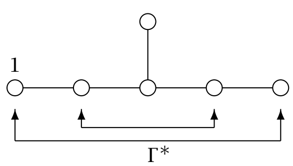
Type $\Phi(\mathfrak{a})$:



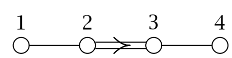
$w_1 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$

$w_2 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$

5.5.4 Type ${}^2E_{6,4}^{16}$



Type $\Phi(\mathfrak{a})$:



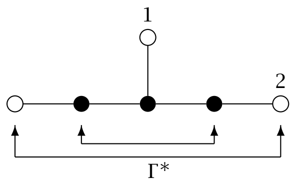
$w_1 = s_{\alpha_2}$

$w_2 = s_{\alpha_4}$

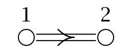
$w_3 = s_{\alpha_3}s_{\alpha_5}$

$w_4 = s_{\alpha_1}s_{\alpha_6}$

5.5.5 Type ${}^2E_{6,2}^{16'}$



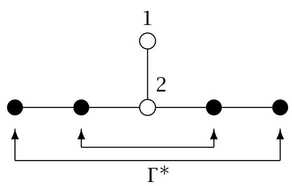
Type $\Phi(\mathfrak{a})$:



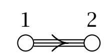
$w_1 = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}$

$w_2 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$

5.5.6 Type ${}^2E_{6,2}^{16''}$



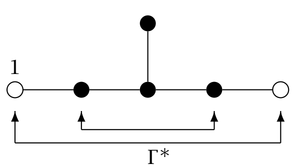
Type $\Phi(\mathfrak{a})$:



$w_1 = s_{\alpha_1}$

$w_2 = s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}$

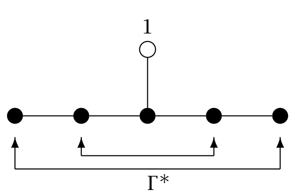
5.5.7 Type ${}^2E_{6,1}^{29}$



Type $\Phi(\mathfrak{a})$:

$w_1 = s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}$

5.5.8 Type ${}^2E_{6,1}^{35}$



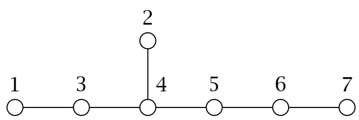
Type $\Phi(\mathfrak{a})$:

$w_1 = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}$

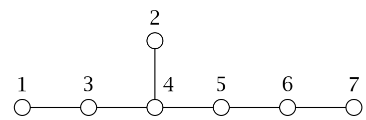
5.6 E_7 cases

There are 4 congruence classes of Γ-indices corresponding to a simple group of type E_7.

5.6.1 Type $E^0_{7,7}$

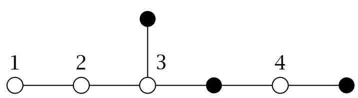


Type $\Phi(\mathfrak{a})$:

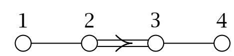


$w_i = s_{\alpha_i}$

5.6.2 Type $E^9_{7,4}$



Type $\Phi(\mathfrak{a})$:



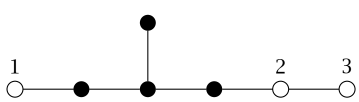
$w_1 = s_{\alpha_1}$

$w_2 = s_{\alpha_3}$

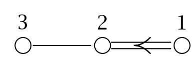
$w_3 = s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4}$

$w_4 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_6}$

5.6.3 Type $E^{28}_{7,3}$



Type $\Phi(\mathfrak{a})$:

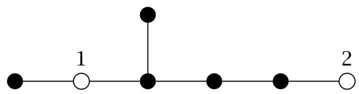


$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$

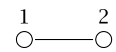
$w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$

$w_3 = s_{\alpha_7}$

5.6.4 Type $E_{7,2}^{31}$



Type $\Phi(\mathfrak{a})$:



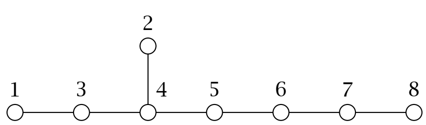
$w_1 = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}$
$s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}$

$w_2 = s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_3}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}$

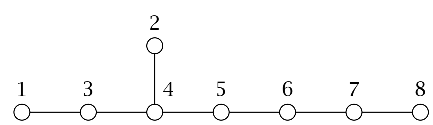
5.7 E$_8$ cases

There are 2 congruence classes of Γ-indices corresponding to a simple group of type E$_8$.

5.7.1 Type $E_{8,8}^{0}$

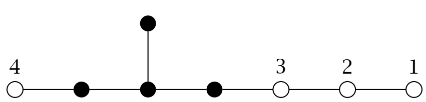


Type $\Phi(\mathfrak{a})$:

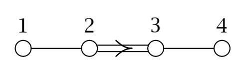


$w_i = s_{\alpha_i}$

5.7.2 Type $E_{8,4}^{28}$



Type $\Phi(\mathfrak{a})$:



$w_1 = s_{\alpha_8}$

$w_2 = s_{\alpha_7}$

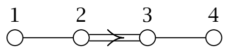
$w_3 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$

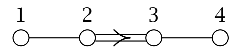
$w_4 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$

5.8 F$_4$ cases

There are 2 congruence classes of Γ-indices corresponding to a simple group of type F$_4$.

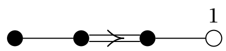
5.8.1 Type $F_{4,4}^{0}$



Type $\Phi(\mathfrak{a})$: 

$w_i = s_{\alpha_i}$

5.8.2 Type $F_{4,1}^{21}$



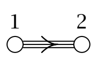
Type $\Phi(\mathfrak{a})$: 

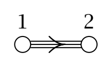
$w_1 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$

5.9 G_2 case

There is 1 congruence class of Γ-indices corresponding to a simple group of type G_2.

5.9.1 Type $G_{2,2}^{0}$



Type $\Phi(\mathfrak{a})$: 

$w_i = s_{\alpha_i}$

Table 5.1: $W(\mathfrak{a})$

Type Γ-index	s_{λ_i}-representative
$A_{n,n}^{(1)}$	$w_i = s_{\alpha_i}\ \forall i$
${}^2A_{2n,n}^{(1)}$	$w_i = s_{\alpha_i}s_{\alpha_{n+1-i}}, i < n$ $w_n = s_{\alpha_n}$
${}^2A_{2n-1,n}^{(1)}$	$w_i = s_{\alpha_i}s_{\alpha_{2n-i}}, i < n$ $w_n = s_{\alpha_n}$
$A_{2n+1,n}^{(2)}$	$w_i = s_i s_{i-1} s_{i+1} s_i$
$A_{n,p}^{(d)}$	$w_i = s_{\alpha_{di}} s_{di+1} s_{di-1} s_{di} s_{di+2} s_{di-2} s_{di+1} s_{di-1} s_{di} s_{di+3} s_{di-s} s_{di+2} s_{di-2}$ $s_{di+1}\ s_{di-1} s_{di} \cdots s_{di+(d-1)} s_{di-(d-1)} s_{di+(d-2)} s_{di-(d-2)} s_{di+(d-3)}$ $s_{di-(d-3)} \cdots\ s_{di+1} s_{di-1} s_{di}$
${}^2A_{n,p}^{(d)}$	$w_i = s_{\alpha_{di}} s_{\alpha_{di+1}} s_{\alpha_{di-1}} s_{\alpha_{di+2}} s_{\alpha_{di}} s_{\alpha_{di-2}} \dots s_{\alpha_{di+d-3}} s_{\alpha_{di+d-5}} \dots s_{\alpha_{di-(d-5)}}$ $s_{\alpha_{di-(d-3)}} s_{\alpha_{di+d-2}}\ s_{\alpha_{di+d-4}} \dots s_{\alpha_{di-(d-4)}} s_{\alpha_{di-(d-2)}} s_{\alpha_{di+d-1}} s_{\alpha_{di+d-3}} \dots$ $s_{\alpha_{di-(d-3)}}\ s_{\alpha_{di-(d-1)}}\ s_{\alpha_{di+d-2}} s_{\alpha_{di+d-4}} \dots\ s_{\alpha_{di-(d-4)}} s_{\alpha_{di-(d-2)}} s_{\alpha_{di+d-3}}$ $s_{\alpha_{di+d-5}} \dots s_{\alpha_{di-(d-5)}}\ s_{\alpha_{di-(d-3)}} \dots s_{\alpha_{di+2}} s_{\alpha_{di}} s_{\alpha_{di-2}}\ s_{\alpha_{di+1}} s_{\alpha_{di-1}} s_{\alpha_{di}}$ $w_p = s_{\alpha_{pd+1}} s_{\alpha_{pd+2}} \dots s_{\alpha_{pd-d+1}} s_{\alpha_{pd+2}} s_{\alpha_{pd+3}} \dots s_{\alpha_{pd-d+2}} \dots s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}}$ $\dots s_{\alpha_{pd-d+m}}\ s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \dots s_{\alpha_{pd-d+m+1}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \dots s_{\alpha_{pd-d+m+1}}$ $s_{\alpha_{pd+m+1}} s_{\alpha_{pd+m-1}}\ s_{\alpha_{pd+m+2}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-2}} \dots s_{\alpha_{pd+m+d-2}} s_{\alpha_{pd+m+d-4}}$ $\dots s_{\alpha_{pd-d+m+4}} s_{\alpha_{pd+m+d-1}} s_{\alpha_{pd+m+d-3}} \dots s_{\alpha_{pd-d+m+3}} s_{\alpha_{pd+m+d}} s_{\alpha_{pd+m+d-2}}$ $\dots s_{\alpha_{pd-d+m+2}} s_{\alpha_{pd+m-d+1}} s_{\alpha_{pd+m-d+2}} \dots s_{\alpha_{pd-d+1}} s_{\alpha_{pd+m+d-1}} s_{\alpha_{pd+m+d-3}}$ $\dots s_{\alpha_{pd+m-d+3}} s_{\alpha_{pd+m-d+2}} s_{\alpha_{pd+m-d+1}} \dots s_{\alpha_{pd-d+2}} \dots s_{\alpha_{pd+m+3}} s_{\alpha_{pd+m+1}}$ $s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}} s_{\alpha_{pd+m-1}} \dots s_{\alpha_{pd-2}} s_{\alpha_{pd+m+2}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} s_{\alpha_{pd+m-2}} \dots$ $s_{\alpha_{pd-1}} s_{\alpha_{pd+m+1}} s_{\alpha_{pd+m}} s_{\alpha_{pd+m-1}} \dots s_{\alpha_{pd}}$
${}^2A_{n,p}^{(1)}$	$w_i = s_{\alpha_i}s_{\alpha_{n+1-i}}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{p+m+1}} s_{\alpha_{p+m}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m+1}}$
${}^2A_{2n+1,n}^{(1)}$	$w_i = s_i, i = 1, 2, \cdots, n-1$ $w_n = s_{\alpha_{n+2}} s_{\alpha_n} s_{\alpha_{n+1}} s_{\alpha_n} s_{\alpha_{n+2}}$

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Table 5.1: *continued*

Type Γ-index	s_{λ_i}-representative
$B_{n,n}$	$w_i = s_i$
$B_{n,n-1}$	$w_i = s_i, i = 1, 2, \cdots, n-2$
	$w_{n-1} = s_{n-1}s_n s_{n-1}$
$B_{n,p}$	$w_i = s_{\alpha_i}$ for $i = 1, \cdots, p-1$
	$w_p = s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$
$C^{(1)}_{n,n}$	$w_i = s_{\alpha_i}\ \forall i$
$C^{(2)}_{2n,n}$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$
	$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n}}$
$C^{(2)}_{2n+1,n}$	$w_i = s_{2i} s_{2i-1} s_{2i+1} s_{2i}$
	$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+1}} s_{\alpha_{2n-1}} s_{\alpha_{2n}}$
$C^{(2)}_{n,p}$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$
	$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots$
	$s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$
$D^{(1)}_{n,n}$	$w_i = s_{\alpha_i}\ \forall i$
$D^{(1)}_{n,p}$	$w_i = \alpha_i$ for $i = 1 \cdots p-1$
	$w_p = s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$
$D^{(2)}_{2n,n}$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, n-1$
	$w_n = s_{\alpha_{2n}}$
$D^{(2)}_{2n+3,n}$	$w_i = s_{2i} s_{2i-1} s_{2i+1} s_{2i}$
	$w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+2}} s_{\alpha_{2n+3}}$
	$s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n+2}} s_{\alpha_{2n-1}} s_{\alpha_{2n+3}} s_{\alpha_{2n+1}} s_{\alpha_{2n}}$
$D^{(2)}_{n,p}$	$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{n-3}} s_{\alpha_{n-2}} s_{\alpha_{2p-1}} s_{\alpha_{2p}}$
	$s_{\alpha_{2p+1}} \cdots s_{\alpha_{n-4}} s_{\alpha_{n-3}} s_{\alpha_n} s_{\alpha_{n-1}} s_{\alpha_{n-2}} s_{\alpha_n} s_{\alpha_{n-1}}$
	$s_{\alpha_{n-2}} s_{\alpha_{n-3}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$
${}^2D^{(1)}_{2n+1,2n}$	$w_i = s_{\alpha_i}, i = 1, 2, \cdots, n-1$

continued on next page

Table 5.1: *continued*

Type Γ-index	s_{λ_i}-representative
	$w_n = s_{\alpha_{n+1}} s_{\alpha_n}$
${}^2D_{n,p}^{(1)}$	$w_i = s_{\alpha_i}, i = 1, 2, \cdots, p-1$ $w_p = s_{\alpha_p} s_{\alpha_{p+1}} s_{\alpha_{p+2}} \cdots s_{\alpha_{n-3}} s_{\alpha_{n-2}} s_{\alpha_{n-1}} s_{\alpha_n} s_{\alpha_{n-2}} s_{\alpha_{n-3}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$
${}^2D_{2n+2,n}^{(2)}$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}, i = 1, 2, \cdots, n-1$ $w_n = s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n+2}} s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n+1}} s_{\alpha_{2n+2}} s_{\alpha_{2n}}$
${}^2D_{2n+1,n}^{(2)}$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, n-1$ $w_n = s_{\alpha_{2n+1}} s_{\alpha_{2n}} s_{\alpha_{2n-1}} s_{\alpha_{2n}} s_{\alpha_{2n+1}}$
${}^3D_{4,2}^{(2)}$	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$
${}^6D_{4,2}^{(2)}$	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$
${}^3D_{4,1}^{(9)}$	$s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_1}$
${}^6D_{4,1}^{(9)}$	$s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_1}$
${}^1E_{6,2}^{16}$	$w_1 = s_{\alpha_2}$ $w_2 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_1} s_{\alpha_6} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4}$
${}^1E_{6,2}^{28}$	$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$ $w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$
${}^2E_{6,4}^{16}$	$w_1 = s_{\alpha_2}$ $w_2 = s_{\alpha_4}$ $w_3 = s_{\alpha_3} s_{\alpha_5}$ $w_4 = s_{\alpha_1} s_{\alpha_6}$
${}^2E_{6,2}^{16'}$	$w_1 = s_{\alpha_2} s_{\alpha_4} s_{\alpha_5} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2}$ $w_2 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$
${}^2E_{6,2}^{16''}$	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_1} s_{\alpha_6} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4}$

continued on next page

Table 5.1: *continued*

Type Γ-index	s_{λ_i}-representative
${}^2E_{6,1}^{29}$	$w_1 = s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}$ $s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}$
${}^2E_{6,1}^{35}$	$w_1 = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}$
$E_{7,7}^{0}$	$w_i = s_{\alpha_i}\ \forall i$
$E_{7,4}^{9}$	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_3}$ $w_3 = s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}$ $w_4 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_7}s_{\alpha_6}$
$E_{7,3}^{28}$	$w_1 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$ $w_2 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_3 = s_{\alpha_7}$
$E_{7,2}^{31}$	$w_1 = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}$ $s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}$ $w_2 = s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_3}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}$
$E_{8,8}^{0}$	$w_i = s_{\alpha_i}\ \forall i$
$E_{8,4}^{28}$	$w_1 = s_{\alpha_8}$ $w_2 = s_{\alpha_7}$ $w_3 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_4 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$
$F_{4,4}^{0}$	$w_i = s_{\alpha_i}\ \forall i$
$F_{4,1}^{21}$	$w_1 = s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$
$G_{2,2}^{0}$	$w_i = s_{\alpha_i}\ \forall i$

Chapter 6

The Structure of $\Phi(\mathfrak{a})^+$

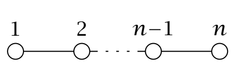
Now that we have explicitly computed each λ_i in $\Delta(\mathfrak{a})$ and each Weyl group representative in W^Γ, we can examine the structure of $\Phi(\mathfrak{a})$. Note that $\Phi(-\lambda) = -\Phi(\lambda)$, so it suffices to determine $\Phi(\lambda)$ for $\lambda \in \Phi(\mathfrak{a})^+$. As stated in step five of the algorithm, we compute the projection space $\Phi(\lambda)$ for each λ in $\Phi(\mathfrak{a})^+$ by applying each w_i as needed.

In the following I give the Γ-index for each case. Then I list $\Phi(\lambda_i)$ for each λ_i, as in Bourbaki [Bou81]. I also give $m_{\lambda_i} := |\Phi(\lambda_i)|$, the multiplicity of each λ_i. Then I state the type of restricted root system. Using the fact that $\Phi(\lambda) = \Phi(w(\lambda_i)) = \tilde{w}\Phi(\lambda_i)$, where $\tilde{w}$ is a representative of $w \in W(\mathfrak{a})$ in the Weyl group of the maximal toral subalgebra and w is a product of the w_i found in chapter 5. I then apply the Weyl group representatives as suggested by the type of restricted root system. This results in a general algorithm for finding roots of any admissible length for each case.

6.1 A cases

There are 8 congruence classes of Γ-indices corresponding to a simple group of type A.

6.1.1 Type $A^{(1)}_{n,n}$

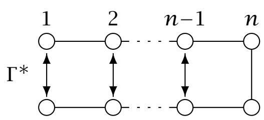


We have for $i = 1, \cdots, n$ that $\Phi(\lambda_i) = \{\alpha_i\}$.

The multiplicity of each root is $m_{\lambda_i} = 1$.

Since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.1.2 Type ${}^2A^{(1)}_{2n,n}$



Here the multiplicity of each λ_i is $m_{\lambda_i} = 2$ for $i = 1, \cdots, n$

We have that $\Phi(\lambda_i) = \{\alpha_i, \alpha_{2n-i+1}\}$ for $i = 1, \cdots, n$

$\Phi(\mathfrak{a})^+$ is of type BC_n and is compued as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1}, \alpha_{2n-i+1} + \alpha_{2n-i}\} \text{ for } i = 1, \cdots, n-1$$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_{n-1}\Phi(\lambda_n) = \{\alpha_{n-1} + \alpha_n, \alpha_{n+1} + \alpha_{n+2}\}$$

$$\Phi(\lambda_{n-1} + 2\lambda_n) = w_n\Phi(\lambda_{n-1}) = \{\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$$\{\alpha_i + \alpha_{i+1} + \alpha_{i+2}, \alpha_{2n-i-2} + \alpha_{2n-i-1} + \alpha_{2n-i}\} \text{ for } i = 1, \cdots, n-2$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + 2\lambda_n) = w_n w_{n-2}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}\}$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}\}$$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n) = w_1 w_2 ... w_{n-1} \Phi(\lambda_n) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_n + \alpha_{n+1} + \cdots + \alpha_{2n-1}\}$$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-1} \Phi(\lambda_{n-1}) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n + \alpha_{n+1}, \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{2n-1}\}$$

$$\vdots$$

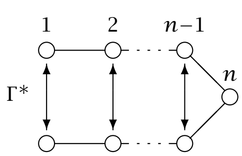
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1}) =$$

$$\{\alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n-1}\}$$

$$\Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_1 w_2 ... w_{n-2} w_{n-1} \Phi(2\lambda_n) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n}\}$$

6.1.3 Type ${}^2A^{(1)}_{2n-1,n}$



$m_{\lambda_i} = 2$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 1$.
We have that $\Phi(\lambda_i) = \{\alpha_i, \alpha_{2n-i}\}$ for $i = 1, \cdots, n$.
$\Phi(\mathfrak{a})^+$ is of type C_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}$$

$$\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1}, \alpha_{2n+i-1} + \alpha_{2n-i}\} \text{ for } i = 1, \cdots, n-1$$

$$\Phi(2\lambda_{n-1} + \lambda_n) = w_{n-1}$$

$$\Phi(\lambda_n) = \{\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$$\{\alpha_i + \alpha_{i+1} + \alpha_{i+2}, \alpha_{2n+i-2} + \alpha_{2n+i-1} + \alpha_{2n+i}\} \text{ for } i = 1, \cdots, n-2$$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_n w_{n-1}\Phi(\lambda_{n-2}) =$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2}\}$$

$$\Phi(2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) = \{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2}\}$$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_n + \alpha_{n+1} + \cdots + \alpha_{2n-1}\}$$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n + \alpha_{n+1}, \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{2n-1}\}$$

$$\vdots$$

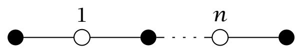
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_2 w_3 \cdots w_{n-1} w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$$\{\alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n-2}\}$$

$$\Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_1 w_2 w_3 \cdots w_{n-1} w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n-1}\}$$

6.1.4 Type $A^{(2)}_{2n+1,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n$

We have that $\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i}\}$. $\Phi(\alpha)^+$ is of type A_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}\}$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}\}$

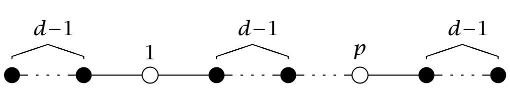
$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n) = w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$$\{\alpha_2 + \cdots + \alpha_{2n}, \alpha_2 + \cdots + \alpha_{2n+1}, \alpha_n + \cdots + \alpha_{2n+1}, \alpha_1 + \cdots + \alpha_{2n}\}$$

6.1.5 Type $A^{(d)}_{n,p}$



$m_{\lambda_i} = d^2$ for $i = 1, \cdots, n$

Remark 16. $pd + d - 1 = n$

We have that $\Phi(\lambda_i) = \{\alpha_{di}, \alpha_{di-1} + \alpha_{di}, \alpha_{di} + \alpha_{di+1} + \alpha_{di-2} + \alpha_{di-1} + \alpha_{di}, \alpha_{di-1} + \alpha_{di} + \alpha_{di+1}, \cdots, \alpha_{di-d+1} + ... + \alpha_{di}, \alpha_{di-d+2} + ... + \alpha_{di+1}, ..., \alpha_{di} + ... + \alpha_{di+d-1}, \cdots, \alpha_{di-d+1} + ... + \alpha_{di+d-1}\}$. $\Phi(\mathfrak{a})^+$ is of type A_p and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{di-d+1} + ... + \alpha_{d(i+1)}, \alpha_{di-d+1} + ... + \alpha_{d(i+1)+1}, \cdots, \alpha_{di-d+1} + ... + \alpha_{d(i+1)+(d-1)}, \alpha_{di-d+2} + ... + \alpha_{d(i+1)}, \alpha_{di-d+2} + ... + \alpha_{d(i+1)+1}, \cdots, \alpha_{di-d+2} + ... + \alpha_{d(i+1)+(d-1)}, \cdots, \alpha_{di} + ... + \alpha_{d(i+1)}, \alpha_{di} + ... + \alpha_{d(i+1)+1}, \cdots, \alpha_{di} + ... + \alpha_{d(i+1)+(d-1)}\}$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_i w_{i+1}\Phi(\lambda_{i+2}) =$$

$\{\alpha_{di-d+1} + ... + \alpha_{d(i+2)}, \alpha_{di-d+1} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di-d+1} + ... + \alpha_{d(i+2)+(d-1)}, \alpha_{di-d+2} + ... + \alpha_{d(i+2)}, \alpha_{di-d+2} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di-d+2} + ... + \alpha_{d(i+2)+(d-1)}, \cdots, \alpha_{di} + ... + \alpha_{d(i+2)}, \alpha_{di} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di} + ... + \alpha_{d(i+2)+(d-1)}\}$

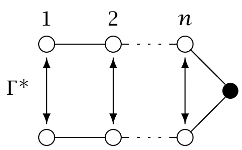
$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1 + \cdots + \lambda_{p-1} + \lambda_p) = w_1 w_2 ... w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_1 + \alpha_2 + ... + \alpha_{pd}, \alpha_1 + \alpha_2 + ... + \alpha_{pd+1}, \cdots, \alpha_1 + \alpha_2 + ... + \alpha_{pd+(d-1)}, \alpha_2 + \alpha_3 + ... + \alpha_{pd}, \alpha_2 + \alpha_3 + ... + \alpha_{pd+1}, \cdots, \alpha_2 + \alpha_3 + ... + \alpha_{pd+(d-1)}, \cdots, \alpha_d + \alpha_{d+1} + ... + \alpha_{pd}, \alpha_d + \alpha_{d+1} + ... + \alpha_{pd+1}, \cdots, \alpha_d + \alpha_{d+1} + ... + \alpha_{pd+(d-1)}\}$

6.1.6 Type $^2A^{(1)}_{2n+1,n}$



$m_{\lambda_i} = 2$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 5$.

$\Phi(\lambda_i) = \{\alpha_i, \alpha_{2n+1-i}\}$ for $i = 1, \cdots, n-1$.

$\Phi(\lambda_n) = \{\alpha_n, \alpha_{n+2}, \alpha_n + \alpha_{n+1}, \alpha_n + \alpha_{n-1}\}$

$\Phi(2\lambda_n) = \{\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$

$\Phi(\mathfrak{a})^+$ is of type BC_n and is computed as follows.

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_i\Phi(\lambda_{i+1}) = \{\alpha_{i+1} + \alpha_i, \alpha_{2n+1-i} + \alpha_{2n+2-i}\} \text{ for } i = 2, \cdots, n-1$$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_n\Phi(\lambda_{n+1}) =$$

$$\{\alpha_{n-1} + \alpha_n, \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}\}$$

$$\Phi(\lambda_{n-1} + 2\lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$$\{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2}, \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}\}$$

$$\Phi(2\lambda_{n-1} + 2\lambda_n) = w_{n-1}\Phi(2\lambda_n) = \{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}\}$$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_i + \alpha_{i+1} + \alpha_{i+2}, \alpha_{n+1-i} + \alpha_{n-i} + \alpha_{n-i-1}\} \text{ for } i = 1, \cdots, n-3$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$$\{\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{n-2}+\alpha_{n-1}+\alpha_n+\alpha_{n+1}, \alpha_{n+1}+\alpha_{n+2}+\alpha_{n+3}+\alpha_{n+4}, \alpha_{n+2}+\alpha_{n+3}+\alpha_{n+4}\}$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + 2\lambda_n) = w_{n-2}w_n\Phi(\lambda_{n-1}) =$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2}, \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \alpha_{n+4}\}$$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-1}w_n w_{n-2}\Phi(\lambda_{n-1})$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}, \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \alpha_{n+4}\}$$

$$\Phi(2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-2}w_{n-1}\Phi(2\lambda_n) =$$

$$\{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \alpha_{n+4}\}$$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1 w_2 ... w_{n-1}\Phi(\lambda_n) =$$

$$\{\alpha_1 + \alpha_2 + ... + \alpha_n, \alpha_1 + \alpha_2 + ... + \alpha_n + \alpha_{n+1}, \alpha_{n+1} + \alpha_{n+2} + ... + \alpha_{2n+1}, \alpha_{n+2} + \alpha_{n+3} + ... + \alpha_{2n+1}\}$$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_1 + \alpha_2 + ... + \alpha_{n+2}, \alpha_n + \alpha_{n+1} + ... + \alpha_{2n+1}\}$$

$$\vdots$$

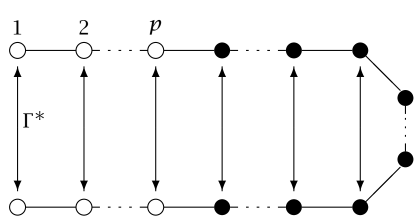
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots + \alpha_{2n}, \alpha_2 + \cdots + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{2n+1}\}$$

$$\Phi(2\lambda_1 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_1 \cdots w_{n-2} w_{n-1}\Phi(2\lambda_n) =$$

$$\{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n + \cdots + \alpha_{2n+1}\}$$

6.1.7 Type ${}^2A^{(1)}_{n,p}$



$m_{\lambda_i} = 2$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 2m$

Remark 17. $n = 2p + m$

$\Phi(\lambda_i) = \{\alpha_i, \alpha_{n+1-i}\}$ for $i = 1, \cdots, p-1$.
$\Phi(\lambda_p) = \{\alpha_p, \alpha_{p+m+1}, \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$
$\Phi(2\lambda_p) = \{\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$
$\Phi(\mathfrak{a})^+$ is of type BC_p and is computed as follows.

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-1} + \alpha_i, \alpha_{l+1-i} + \alpha_{l+2-i}\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p+m+1} + \alpha_{p+m+2}, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1} + \alpha_{p+m+2}, \cdots, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$

$$\Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m} + \alpha_{p+m+1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$$

$$\Phi(2\lambda_{p-1} + 2\lambda_p) = w_{p-1}\Phi(2\lambda_p) = \{\alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_{i-2} + \alpha_{i-1} + \alpha_i, \alpha_{l+1-i} + \alpha_{l+2-i} + \alpha_{l+3-i}\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p, \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}\}$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + 2\lambda_p) = w_{p-2}w_p\Phi(\lambda_{p-1})$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m} + \alpha_{p+m+1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}\}$

$$\Phi(\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-1} w_p w_{p-2} \Phi(\lambda_{p-1}) =$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m} + \alpha_{p+m+1} + \alpha_{p+m+2}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}\}$

$$\Phi(2\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-2} w_{p-1} \Phi(2\lambda_p) =$$

$$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3}\}$$

$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_p) = w_1 w_2 \cdots w_{p-1} \Phi(\lambda_p) =$$

$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \alpha_{p+1}, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m}, \alpha_{p+m+1} + \alpha_{p+m+2} + \cdots + \alpha_{2p+m}, \alpha_{p+m} + \alpha_{p+m+1} + \alpha_{p+m+2} + \cdots + \alpha_{2p+m}, \cdots, \alpha_{p+1} + \cdots + \alpha_{2p+m}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 \cdots w_{p-1} \Phi(\lambda_p) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m+1}, \alpha_p + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \cdots + \alpha_{2p+m}\}$$

$$\vdots$$

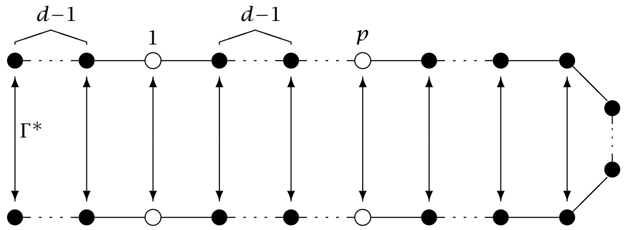
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2 w_3 \cdots w_{p-1} w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{2p+m-1}, \alpha_2 + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \cdots + \alpha_{2p+m}\}$$

$$\Phi(2\lambda_1 + \cdots + 2\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_1 \cdots w_{p-2} w_{p-1} \Phi(2\lambda_p) =$$

$$\{\alpha_1 + \cdots + \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2} + \alpha_{p+m+3} + \cdots + \alpha_{2p+m}\}$$

6.1.8 Type $^2A^{(d)}_{n,p}$



$m_{\lambda_i} = 2d^2$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 4md^2$ $\Phi(\lambda_i) = \{\alpha_{di}, \alpha_{di-1}+\alpha_{di}, \alpha_{di}+\alpha_{di+1}, \alpha_{di-2}+ \alpha_{di-1} + \alpha_{di}, \alpha_{di-1} + \alpha_{di} + \alpha_{di+1}, \cdots,$
$\alpha_{di-d+1}+\alpha_{di-d+2}+\cdots+\alpha_{di}, \alpha_{di-d+2}+\alpha_{di-d+3}+\cdots+\alpha_{di+1}, \cdots, \alpha_{di}+\cdots+\alpha_{di+d-1}, \cdots,$
$\alpha_{di-d+1} + \alpha_{di-d+2} + \cdots + \alpha_{di+d-1}, \alpha_{n-di+1}, \alpha_{n-di} + \alpha_{n-di+1}, \alpha_{n-di+1} + \alpha_{n-di}, \alpha_{n-di-1} + \alpha_{n-di}+\alpha_{n-di+1}, \alpha_{n-di}+\alpha_{n-di+1}+\alpha_{n-di+2}, \cdots, \alpha_{n-(di-d)}+\alpha_{n-(di-d)+1}+\cdots+\alpha_{n-di+1}, \alpha_{n-(di-d)+1}+ \alpha_{n-(di-d)+2} + \cdots + \alpha_{n-di}, \cdots, \alpha_{n-di+1} + \cdots + \alpha_{n-(di+d)}, \cdots, \alpha_{n-(di-d)} + \alpha_{n-(di-d)-1} + \cdots + \alpha_{n-(di+d)-1}\}, i = 1, 2, \cdots, p-1$
$\Phi(\lambda_p) = \{\alpha_{pd-d+1} + \alpha_{pd-d+2} + \cdots + \alpha_{pd}, \alpha_{pd-d+1} + \alpha_{pd-d+2} + \cdots + \alpha_{pd+1}, \cdots, \alpha_{pd-d+1} + \alpha_{pd-d+2}+\cdots+\alpha_{pd+m}, \alpha_{pd-d+2}+\alpha_{pd-d+3}+\cdots+\alpha_{pd}, \alpha_{pd-d+2}+\alpha_{pd-d+3}+\cdots+\alpha_{pd+1}, \cdots,$
$\alpha_{pd-d+2}+\alpha_{pd-d+3}+\cdots+\alpha_{pd+m}, \cdots, \alpha_{pd}, \alpha_{pd}+\alpha_{pd+1}, \cdots, \alpha_{pd}+\alpha_{pd+1}+\cdots+\alpha_{n-pd}\alpha_{pd+1}+ \alpha_{pd+2} + \cdots + \alpha_{pd+m+d}, \alpha_{pd+2} + \alpha_{pd+3} + \cdots + \alpha_{pd+m+d}, \cdots, \alpha_{pd+m+1} + \alpha_{pd+m+2} + \cdots + \alpha_{pd+m+d}, \alpha_{pd+1}+\alpha_{pd+2}+\cdots+\alpha_{pd+m+d-1}, \alpha_{pd+2}+\alpha_{pd+3}+\cdots+\alpha_{pd+m+d-1}, \cdots, \alpha_{pd+m+1}+ \alpha_{pd+m+2} + \cdots + \alpha_{pd+m+d-1}, \cdots, \alpha_{pd+1} + \alpha_{pd+2} + \cdots + \alpha_{pd+m+1}, \alpha_{pd+2} + \alpha_{pd+3} + \cdots + \alpha_{pd+m+1}, \cdots, \alpha_{pd+m+1}\}$
$\Phi(2\lambda_p) = \{\alpha_{pd-d+1}+\alpha_{pd-d+2}+\cdots+\alpha_{n-pd+1}, \alpha_{pd-d+1}+\alpha_{pd-d+2}+\cdots+\alpha_{n-pd+2}, \cdots, \alpha_{pd-d+1}+ \alpha_{pd-d+2} + \cdots + \alpha_{n-pd+d}, \alpha_{pd-d+2} + \alpha_{pd-d+3} + \cdots + \alpha_{n-pd+1}, \alpha_{pd-d+2} + \alpha_{pd-d+3} + \cdots + \alpha_{n-pd+2}, \cdots, \alpha_{pd-d+2} + \alpha_{pd-d+3} + \cdots + \alpha_{n-pd+d}, \cdots, \alpha_{pd} + \alpha_{pd+1} + \cdots + \alpha_{n-pd+1}, \alpha_{pd} + \alpha_{pd+1} + \cdots + \alpha_{n-pd+2}, \cdots, \alpha_{pd} + \alpha_{pd+1} + \cdots + \alpha_{n-pd+d}\}$
$\Phi(\mathfrak{a})^+$ is of type BC_p and is computed as follows.

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{di-d+1}+\ldots+\alpha_{d(i+1)}, \alpha_{di-d+1}+\ldots+\alpha_{d(i+1)+1}, \cdots, \alpha_{di-d+1}+\ldots+\alpha_{d(i+1)+(d-1)}, \alpha_{di-d+2}+\ldots+$

$\alpha_{d(i+1)}, \alpha_{di-d+2}+...+\alpha_{d(i+1)+1}, \cdots, \alpha_{di-d+2}+...+\alpha_{d(i+1)+(d-1)}, \cdots, \alpha_{di}+...+\alpha_{d(i+1)}, \alpha_{di}+...+\alpha_{d(i+1)+1}, \cdots, \alpha_{di}+...+\alpha_{d(i+1)+(d-1)}, \alpha_{n+1-di-d+1}+...+\alpha_{n+1-d(i+1)}, \alpha_{n+1-di-d+1}+...+\alpha_{n+1-d(i+1)+1}, \cdots, \alpha_{n+1-di-d+1}+...+\alpha_{n+1-d(i+1)+(d-1)}, \alpha_{n+1-di-d+2}+...+\alpha_{n+1-d(i+1)}, \alpha_{n+1-di-d+2}+...+\alpha_{n+1-d(i+1)+1}, \cdots, \alpha_{n+1-di-d+2}+...+\alpha_{n+1-d(i+1)+(d-1)}, \cdots, \alpha_{n+1-di}+...+\alpha_{n+1-d(i+1)}, \alpha_{n+1-di}+...+\alpha_{n+1-d(i+1)+1}, \cdots, \alpha_{n+1-di}+...+\alpha_{n+1-d(i+1)+(d-1)}\}$ for $i = 2, \cdots, p-1$

$$\Phi(\lambda_{p-1}+\lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd}, \alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+1}, \cdots, \alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m}, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd}, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+1}, \cdots, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m}, \cdots, \alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd}, \alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd+1}, \cdots, \alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd+m}, \cdots, \alpha_{pd+1}+\alpha_{pd+2}+...+\alpha_{pd+m+2d}, \alpha_{pd+1}+\alpha_{pd+2}+...+\alpha_{pd+m+2d-1}, ...\alpha_{pd+1}+\alpha_{pd+2}+...+\alpha_{pd+m+d+1}, \cdots, \alpha_{pd+2}+\alpha_{pd+2}+...+\alpha_{pd+2k+l+2d}, \alpha_{pd+2}+\alpha_{pd+2}+...+\alpha_{pd+m+2d-1}, \cdots, \alpha_{pd+2}+\alpha_{pd+2}+...+\alpha_{pd+m+d+1}, \cdots, \alpha_{pd+m+1}+\alpha_{pd+m+3}+...+\alpha_{pd+m+2d},$
$\alpha_{pd+m+1}+\alpha_{pd+m+3}+...+\alpha_{pd+m+2d-1}, \cdots, \alpha_{pd+m+1}+\alpha_{pd+m+3}+...+\alpha_{pd+m+d+1}\}$

$$\Phi(\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_{p-1}) =$$

$\{\alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+1}, \alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+2}, \cdots,$
$\alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+d}, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+1},$
$\alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+2}, \cdots, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+d}, \cdots,$
$\alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd+m+1}, \alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd+m+2}, \cdots, \alpha_{(p-1)d}+\alpha_{(p-1)d+1}+...+\alpha_{pd+m+d}, \alpha_{pd}+\alpha_{pd+1}+...+\alpha_{pd+m+2d}, \alpha_{pd}+\alpha_{pd+1}+...+\alpha_{pd+m+2d-1}, ...\alpha_{pd}+\alpha_{pd+1}+...+\alpha_{pd+m+d+1}, \alpha_{pd-1}+\alpha_{pd}+...+\alpha_{pd+m+2d}, \alpha_{pd-1}+\alpha_{pd}+...+\alpha_{pd+m+2d-1}, ...\alpha_{pd-1}+\alpha_{pd}+...+\alpha_{pd+m+d+1}, ...\alpha_{pd-d+1}+\alpha_{pd-d+2}+...+\alpha_{pd+m+2d}, \alpha_{pd-d+1}+\alpha_{pd-d+2}+...+\alpha_{pd+m+2d-1}, ...\alpha_{pd-d+1}+\alpha_{pd-d+2}+...+\alpha_{pd+m+d+1},\}$

$$\Phi(2\lambda_{p-1}+2\lambda_p) = w_{p-1}\Phi(2\lambda_p) =$$

$\{\alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+d+1}, \alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+d+2}, \cdots,$
$\alpha_{(p-1)d-d+1}+\alpha_{(p-1)d-d+2}+...+\alpha_{pd+m+d+2d}, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+d+1},$
$\alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+d+2}, \cdots, \alpha_{(p-1)d-d+2}+\alpha_{(p-1)d-d+3}+...+\alpha_{pd+m+d+2d},$

$...\alpha_{(p-1)d} + \alpha_{(p-1)d+1} + ... + \alpha_{pd+m+d+1}, \alpha_{(p-1)d} + \alpha_{(p-1)d+1} + ... + \alpha_{pd+m+d+2}, \cdots, \alpha_{(p-1)d} + \alpha_{(p-1)d+1} + ... + \alpha_{pd+m+d+2d}\}$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{di-d+1} + ... + \alpha_{d(i+2)}, \alpha_{di-d+1} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di-d+1} + ... + \alpha_{d(i+2)+(d-1)}, \alpha_{di-d+2} + ... + \alpha_{d(i+2)}, \alpha_{di-d+2} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di-d+2} + ... + \alpha_{d(i+2)+(d-1)}, \cdots, \alpha_{di} + ... + \alpha_{d(i+2)}, \alpha_{di} + ... + \alpha_{d(i+2)+1}, ..., \alpha_{di} + ... + \alpha_{d(i+2)+(d-1)}, \alpha_{n+1-(di-d+1)} + ... + \alpha_{n+1-(d(i+2))}, \alpha_{n+1-(di-d+1)} + ... + \alpha_{n+1-(d(i+2)+1)}, ..., \alpha_{n+1-(di-d+1)} + ... + \alpha_{n+1-(d(i+2)+(d-1))}, \alpha_{n+1-(di-d+2)} + ... + \alpha_{n+1-(d(i+2))}, \alpha_{n+1-(di-d+2)} + ... + \alpha_{n+1-(d(i+2)+1)}, ..., \alpha_{n+1-(di-d+2)} + ... + \alpha_{n+1-(d(i+2)+(d-1))}, \cdots, \alpha_{n+1-di} + ... + \alpha_{n+1-d(i+2)}, \alpha_{n+1-di} + ... + \alpha_{n+1-(d(i+2)+1)}, ..., \alpha_{n+1-di} + ... + \alpha_{n+1-(d(i+2)+(d-1))}\}$

for $i = 3, \cdots, p-1$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd}, \alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd+1}, \cdots, \alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd+m}, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd}, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd+1}, \cdots, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd+m}, \cdots, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd}, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd+1}, \cdots, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd+m}, \cdots, \alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{pd+m+3d}, \alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{pd+m+3d-1}, ...\alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{pd+m+2d+1},$

$\cdots, \alpha_{pd+2} + \alpha_{pd+2} + ... + \alpha_{pd+m+3d}, \alpha_{pd+2} + \alpha_{pd+2} + ... + \alpha_{pd+m+3d-1}, \cdots, \alpha_{pd+2} + \alpha_{pd+2} + ... + \alpha_{pd+m+2d+1}, \cdots, \alpha_{pd+m+1} + \alpha_{pd+m+3} + ... + \alpha_{pd+m+3d}, \alpha_{pd+m+1} + \alpha_{pd+m+3} + ... + \alpha_{pd+m+3d-1}, \cdots, \alpha_{pd+m+1} + \alpha_{pd+m+3} + ... + \alpha_{pd+m+2d+1}\}$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + 2\lambda_p) = w_{p-2}w_p\Phi(\lambda_{p-1}) =$$

$\{\alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd+m+1}, \alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd+m+2}, \cdots, \alpha_{(p-1)d-2d+1} + \alpha_{(p-1)d-2d+2} + ... + \alpha_{pd+m+d}, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd+m+1}, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd+m+2}, \cdots, \alpha_{(p-1)d-2d+2} + \alpha_{(p-1)d-2d+3} + ... + \alpha_{pd+m+d}, \cdots, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd+m+1}, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd+m+2}, \cdots, \alpha_{(p-1)d-d} + \alpha_{(p-1)d+1-d} + ... + \alpha_{pd+m+d}, \alpha_{pd} + \alpha_{pd+1} + ... + \alpha_{pd+m+3d}, \alpha_{pd} + \alpha_{pd+1} + ... +$

$\alpha_{pd+m+3d-1}, \ldots \alpha_{pd} + \alpha_{pd+1} + \ldots + \alpha_{pd+m+2d+1}, \alpha_{pd-1} + \alpha_{pd} + \ldots + \alpha_{pd+m+3d}, \alpha_{pd-1} + \alpha_{pd} + \ldots + \alpha_{pd+m+3d-1}, \ldots \alpha_{pd-1} + \alpha_{pd} + \ldots + \alpha_{pd+m+2d+1}, \ldots \alpha_{pd-d+1} + \alpha_{pd-d+2} + \ldots + \alpha_{pd+m+3d}, \alpha_{pd-d+1} + \alpha_{pd-d+2} + \ldots + \alpha_{pd+m+3d-1}, \ldots \alpha_{pd-d+1} + \alpha_{pd-d+2} + \ldots + \alpha_{pd+m+2d+1}\}$

$$\Phi(\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-1} w_p w_{p-2} \Phi(\lambda_{p-1}) =$$

$\{\alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+d+1}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+d+1}, \cdots, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+d+1}, \alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+d+2}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+d+2}, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+d+2}, \cdots,$

$\alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+2d}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+2d}, \cdots, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+2d}, \alpha_{(p-1)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+2d+1}, \alpha_{(p-1)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+2d+1},$

$\cdots, \alpha_{(p-1)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+2d+1}, \alpha_{(p-1)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+2d+2}, \alpha_{(p-1)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+2d+2}, \alpha_{(p-1)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \alpha_{pd+m+2d+2}, \cdots, \alpha_{(p-1)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+3d}, \alpha_{(p-1)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+3d}, \cdots, \alpha_{(p-)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+2d-1} + \alpha_{pd+m+3d}\}$

$$\Phi(2\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-2} w_{p-1} \Phi(2\lambda_p) =$$

$\{\alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \ldots + \alpha_{pd+m+2d+1}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \ldots + \alpha_{pd+m+2d+1}, \cdots, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d} + \alpha_{pd+m+2d+1}, \alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \ldots + \alpha_{pd+m+2d+2}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \ldots + \alpha_{pd+m+2d+2}, \cdots, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+d+1} + \ldots + \alpha_{pd+m+2d+2}, \cdots,$

$\alpha_{(p-2)d-d+1} + \alpha_{(p-2)d-d+2} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+3d-1} + \alpha_{pd+m+3d}, \alpha_{(p-2)d-d+2} + \alpha_{(p-2)d-d+3} + \ldots + \alpha_{pd+m+1} + \ldots + \alpha_{pd+m+3d-1} + \alpha_{pd+m+3d}, \cdots, \alpha_{(p-2)d} + \alpha_{(p-2)d+1} + \ldots +$

$\alpha_{pd+m+1} + ... + \alpha_{pd+m+3d-1} + \alpha_{pd+m+3d}\}$

$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_p) = w_1 w_2 ... w_{p-1} \Phi(\lambda_p) =$$

$\{\alpha_1+\alpha_2+\alpha_3+....+\alpha_{pd}, \alpha_1+\alpha_2+\alpha_3+....+\alpha_{pd+1}, \cdots, \alpha_1+\alpha_2+\alpha_3+....+\alpha_{pd+m}, \alpha_2+\alpha_3+\alpha_4+ .... + \alpha_{pd}, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+1}, \cdots, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd}, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+1}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m}, \alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd+1} + \alpha_{pd+2} + ... + \alpha_{n-1} + \alpha_n, \alpha_{pd+2} + \alpha_{pd+3} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd+2} + \alpha_{pd+3} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd+2} + \alpha_{pd+3} + ... + \alpha_{n-1} + \alpha_n, \cdots, \alpha_{pd+m+1} + \alpha_{pd+m+2} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd+m+1} + \alpha_{pd+m+2} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd+m+1} + \alpha_{pd+m+2} + ... + \alpha_{n-1} + \alpha_n\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+1}, \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+2}, \cdots, \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+d}, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+1}, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+2}, \cdots, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+d}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+1}, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+2}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+d}, \alpha_{pd-d+1} + \alpha_{pd-d+2} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd-d+1} + \alpha_{pd-d+2} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd-d+1} + \alpha_{pd-d+2} + ... + \alpha_{n-1} + \alpha_{n-2}, \alpha_{pd-d+2} + \alpha_{pd-d+3} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd-d+2} + \alpha_{pd-d+3} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd-d+2} + \alpha_{pd-d+3} + ... + \alpha_{n-1} + \alpha_{n-2}, \cdots, \alpha_{pd} + \alpha_{pd+1} + ... + \alpha_{n-d} + \alpha_{n-d+1}, \alpha_{pd} + \alpha_{pd+1} + ... + \alpha_{n-d+1} + \alpha_{n-d+2}, \cdots, \alpha_{pd} + \alpha_{pd+1} + ... + \alpha_{n-1} + \alpha_{n-2}\}$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2 w_3 ... w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+d+1}, \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+d+2}, \cdots, \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{pd+m+2d}, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+d+1}, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+d+2}, \cdots, \alpha_2 + \alpha_3 + \alpha_4 + ... + \alpha_{pd+m+2d}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+d+1}, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+d+2}, \cdots, \alpha_d + \alpha_{d+1} + \alpha_{d+2} + ... + \alpha_{pd+m+2d}, \alpha_{pd-2d+1} + \alpha_{pd-2d+2} + ... + \alpha_{n-d} +$

$\alpha_{n-d+1}, \alpha_{pd-2d+1}+\alpha_{pd-2d+2}+...+\alpha_{n-d+1}+\alpha_{n-d+2}, \cdots, \alpha_{pd-2d+1}+\alpha_{pd-2d+2}+...+\alpha_{n-1}+\alpha_{n-2}, \alpha_{pd-2d+2}+\alpha_{pd-2d+3}+...+\alpha_{n-d}+\alpha_{n-d+1}, \alpha_{pd-2d+2}+\alpha_{pd-2d+3}+...+\alpha_{n-d+1}+\alpha_{n-d+2}, \cdots, \alpha_{pd-2d+2}+\alpha_{pd-2d+3}+...+\alpha_{n-1}+\alpha_{n-2}, \cdots, \alpha_{pd-d}+\alpha_{pd-d+1}+...+\alpha_{n-d}+\alpha_{n-d+1}, \alpha_{pd-d}+\alpha_{pd-d+1}+...+\alpha_{n-d+1}+\alpha_{n-d+2}, \cdots, \alpha_{pd-d}+\alpha_{pd-d+1}+...+\alpha_{n-1}+\alpha_{n-2}\}$

$$\Phi(2\lambda_1+\cdots+2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_1\cdots w_{p-2}w_{p-1}\Phi(2\lambda_p) =$$

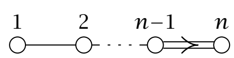
$\{\alpha_1+\alpha_2+...+\alpha_{n-d+1}, \alpha_1+\alpha_2+...+\alpha_{n-d+2}, \cdots, \alpha_1+\alpha_2+...+\alpha_n, \alpha_2+\alpha_3+...+\alpha_{n-d+1}, \alpha_2+\alpha_3+...+\alpha_{n-d+2}, \cdots, \alpha_2+\alpha_3+...+\alpha_n, \cdots, \alpha_d+\alpha_{d+1}+...+\alpha_{n-d+1}, \alpha_d+\alpha_{d+1}+...+\alpha_{n-d+2}, \cdots, \alpha_d+\alpha_{d+1}+...+\alpha_n\}$

Remark 18. $pd+m=n$

6.2 B cases

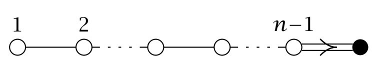
There are 3 congruence classes of Γ-indices corresponding to a simple group of type B.

6.2.1 Type $B_{n,n}$



$m_{\lambda_i}=2$ and $\Phi(\lambda_i)=\{\alpha_i\}$ for $i=1,\cdots,n$.
Since $w_i=s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a})=\Phi(\mathfrak{t})$.

6.2.2 Type $B_{n,n-1}$



$m_{\lambda_i}=1$ for $i=1,\cdots,n-2$ and $m_{\lambda_{n-1}}=2$
We have that $\Phi(\lambda_i)=\{\alpha_i\}$ for $i=1,\cdots,n-1$
$\Phi(\lambda_n)=\{\alpha_{n-1},\alpha_{n-1}+\alpha_n\}$
$\Phi(\mathfrak{a})^+$ is of type B_p and is computed as follows.

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1}\} \text{ for } i = 1, \cdots, n-2$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1}) = w_{n-2}\Phi(\lambda_{n-1}) = \{\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_{n-2} + \alpha_{n-1}\}$$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1}) = w_{n-1}\Phi(\lambda_{n-2}) = \{\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n\}$$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_i) = w_i w_{i+1}\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1} + \alpha_{i+2}\} \text{ for } i = 1, \cdots, n-3$$

$$\Phi(\lambda_{n-3} + \lambda_{n-2} + \lambda_{n-1}) = w_{n-3}w_{n-2}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}, \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1} + \alpha_{2n}\}$$

$$\Phi(\lambda_{n-3} + \lambda_{n-2} + 2\lambda_{n-1}) = w_{n-1}w_{n-3}\Phi(\lambda_{n-2}) = \{\alpha_{n-3} + \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n\}$$

$$\Phi(\lambda_{n-3} + 2\lambda_{n-2} + 2\lambda_{n-1}) = w_{n-2}w_{n-1}w_{n-3}\Phi(\lambda_{n-2}) = \{\alpha_{n-3} + 2\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n\}$$

$$\vdots$$

Roots of length $n-1$

$$\Phi(\lambda_1 + \lambda_2 + ... + \lambda_{n-2} + \lambda_{n-1}) = w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_1 + \alpha_2 + ... + \alpha_{n-2} + \alpha_{n-1}, \alpha_1 + \alpha_2 + ... + \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 + ... + \alpha_{n-2} + \alpha_{n-1} + 2\alpha_n\}$$

$$\Phi(\lambda_1 + \lambda_2 + ... + \lambda_{n-2} + 2\lambda_{n-1}) = w_{n-1}w_1 w_2 ... w_{n-3}\Phi(\lambda_{n-2}) =$$

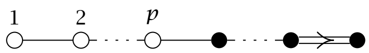
$$\{\alpha_1 + \alpha_2 + ... + \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n\}$$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1}) = w_2 w_3 ... w_{n-2}w_{n-1}w_1 w_2 ... w_{n-3}\Phi(\lambda_{n-2}) =$$

$$= \{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n\}$$

6.2.3 Type $B_{n,p}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, p-1$ and $m_{\lambda_p} = m+1$

Remark 19. $n = p + m$

$\Phi(\lambda_i) = \{\alpha_i\}$ for $i = 1, \cdots, p-1$ $\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \cdots + \alpha_{p+m}, \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$
$\Phi(\alpha)^+$ is of type B_p and is computed as follows.

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-1} + \alpha_i\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$$\Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1}) = \{\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m} + 2\alpha_{p+m+1}\}$$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-2} + \alpha_{i-1} + \alpha_i\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$\Phi(\lambda_{p-2} + \lambda_{p-1} + 2\lambda_p) = w_p w_{p-2}\Phi(\lambda_{p-1}) = \{\alpha_{p-2} + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$\Phi(\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-1}w_p w_{p-2}\Phi(\lambda_{p-1}) = \{\alpha_{p-2} + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$$\vdots$$

Roots of length p

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_p) = w_1 w_2 ... w_{p-1} \Phi(\lambda_p) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \alpha_{p+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m}, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$$\{\alpha_1 + \alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m}\}$$

$$\vdots$$

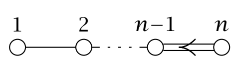
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2 w_3 ... w_{p-1} w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$$\{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m}\}$$

6.3 C cases

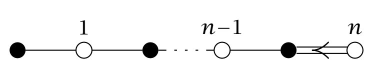
There are 4 congruence classes of Γ-indices corresponding to a simple group of type C.

6.3.1 Type $C_{n,n}^{(1)}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, n$ and $w_i = s_{\alpha_i}$ for all i, so we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.3.2 Type $C_{2n,n}^{(2)}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 2$

We have that $\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, n-1$

$\Phi(\lambda_n) = \{\alpha_{2n}, \alpha_{2n-1} + \alpha_{2n}, 2\alpha_{2n-1} + \alpha_{2n}\}$

$\Phi(\alpha)^+$ is of type C_n and is computed as follows:
Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}\}$ for $i = 1, \cdots, n-1$

$$\Phi(2\lambda_{n-1} + \lambda_n) = w_{n-1}\Phi(\lambda_n) =$$

$\{2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}\}$
Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}\}$ for $i = 1, \cdots, n-2$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_nw_{n-1}\Phi(\lambda_{n-2}) =$$

$\{\alpha_{2n+4} + \alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + \alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$\{2\alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + 2\alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, 2\alpha_{2n+5} + 2\alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}\}$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_nw_{n-1}...w_2\Phi(\lambda_1) =$$

$\{\alpha_2 + \alpha_3 + \cdots + \alpha_{2n}, \alpha_1 + \alpha_2 + \cdots + \alpha_{2n}, \alpha_2 + \alpha_3 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_nw_{n-1}w_{n-2}...w_2\Phi(\lambda_1) =$$

$\{\alpha_2 + \alpha_3 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \alpha_3 + \cdots + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2n+3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}\}$

$$\vdots$$

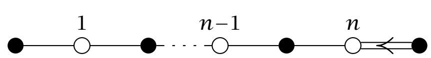
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_2w_3...w_{n-1}w_nw_{n-1}w_{n-2}...w_2\Phi(\lambda_1) =$$

$\{\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_1w_2w_3...w_{n-1}w_nw_{n-1}w_{n-2}...w_2\Phi(\lambda_1) =$$

$\{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n}, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n}\}$

6.3.3 Type $C^{(2)}_{2n+1,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n$

$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}, i = 1, \cdots, n$

$\Phi(2\lambda_n) = \{2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}\}$

$\Phi(\mathfrak{a})^+$ is of type BC_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_i\Phi(\lambda_{i+1}) =$$

$\{\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}\}$ for $i = 1, \cdots, n-2$

$$\Phi(\lambda_{n-1}+\lambda_n) = w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n}\}$

$$\Phi(\lambda_{n-1}+2\lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}\}$

$$\Phi(2\lambda_{n-1}+2\lambda_n) = w_{n-1}\Phi(2\lambda_n) =$$

$\{2\alpha_{2n-3}+2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, 2\alpha_{2n-3}+2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, 2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, 2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}\}$

Roots of length 3:

$$\Phi(\lambda_i+\lambda_{i+1}+\lambda_{i+2}) = w_i w_{i+1}\Phi(\lambda_{i+2}) =$$

$\{\alpha_{2i-1}+\alpha_{2i}+...+\alpha_{2i+5}, \alpha_{2i-1}+\alpha_{2i}+...+\alpha_{2i+4}, \alpha_{2i}+\alpha_{2i+1}+...+\alpha_{2i+5}, \alpha_{2i}+\alpha_{2i+1}+...+\alpha_{2i+4}\}$ for $i = 1,\cdots,n-3$

$$\Phi(\lambda_{n-2}+\lambda_{n-1}+\lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+...+\alpha_{2n}, \alpha_{2n-4}+\alpha_{2n-3}+...+\alpha_{2n}, \alpha_{2n-5}+\alpha_{2n-4}+...+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+...+\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2}+\lambda_{n-1}+2\lambda_n) = w_{n-2}w_n\Phi(\lambda_{n-1})$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2}+2\lambda_{n-1}+2\lambda_n) = w_{n-1}w_n w_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+2\alpha_{2n-3}+2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+2\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+2\alpha_{2n-3}+$

$2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}\}$

$$\Phi(2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-2}w_{n-1}\Phi(2\lambda_n) =$$

$\{\alpha_{2n-5} + 2\alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}\}$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1 w_2 ... w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_1 + \alpha_2 + ... + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n}, \alpha_2 + \alpha_2 + ... + \alpha_{2n}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_1 + \alpha_2 + ... + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}\}$

$$\vdots$$

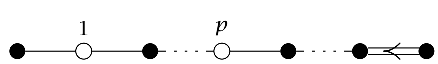
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}\}$

$$\Phi(2\lambda_1 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_1 \cdots w_{n-2} w_{n-1}\Phi(2\lambda_n) =$$

$\{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}, 2\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n} + \alpha_{2n+1}\}$

6.3.4 Type $C_{n,p}^{(2)}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, p-1$ and $m_{\lambda_p} = 2(m+1)$

Remark 20. $2p + m = n$

$$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\} \text{ for } i = 1, \cdots, p-1$$

$$\begin{aligned}\Phi(\lambda_p) =\{&\alpha_{2p}, \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots,\\
&\alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m},\\
&\alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots,\\
&\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m},\\
&\alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots,\\
&\alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}\end{aligned}$$

$\Phi(2\lambda_p) = \{2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$\Phi(\mathfrak{a})^+$ is of type $BC_p$and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 2, \cdots, p-1$

$$\Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$$\Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1}) =$$

$\{\alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-3} + \alpha_{2p-2} +$

$2\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$$\Phi(2\lambda_{p-1} + 2\lambda_p) = w_{p-1}\Phi(2\lambda_p) =$$

$\{2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, 2\alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-5} + \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-5} + \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 3, \cdots, p-1$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \cdots + 2\alpha_{2p+m-2} + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + 2\lambda_p) = w_{p-2}w_p\Phi(\lambda_{p-1}) =$$

$\{\alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-5} + \alpha_{2p-4} + \alpha_{2p-3} + \alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$$\Phi(\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-1}w_pw_{p-2}\Phi(\lambda_{p-1}) =$$

$\{\alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-5}+\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-2}w_{p-1}\Phi(2\lambda_p) =$$

$\{2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-5}+2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, 2\alpha_{2p-5}+2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1+\lambda_2+\cdots+\lambda_p) = w_1w_2...w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_2+\alpha_3+\cdots+\alpha_{2p}, \cdots, \alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \cdots, \alpha_2+\alpha_3+\cdots+\alpha_{2p}+2\alpha_{2p+1}\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{2p}, \cdots, \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \cdots, \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}+2\lambda_p) = w_pw_1w_2...w_{p-2}\Phi(\lambda_{p-1}) =$$

$\{\alpha_2+\cdots+\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_1+\alpha_2+\cdots+\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_2+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_1+\alpha_2+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\vdots$$

$$\Phi(\lambda_1+2\lambda_2+\cdots+2\lambda_{p-1}+2\lambda_p) = w_2w_3...w_{p-1}w_pw_1w_2...w_{p-2}\Phi(\lambda_{p-1}) =$$

$\{\alpha_2+\alpha_3+2\alpha_4+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_2+2\alpha_3+2\alpha_4+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\cdots+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

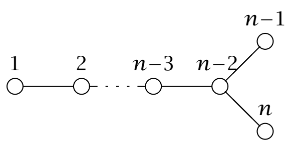
$$\Phi(2\lambda_1+\cdots+2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_1\cdots w_{p-2}w_{p-1}\Phi(2\lambda_p) =$$

$\{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

6.4 D cases

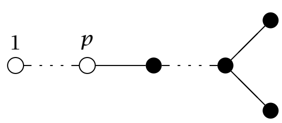
There are 11 congruence classes of Γ-indices corresponding to a simple group of type D.

6.4.1 Type $D^{(1)}_{n,n}$



$m_{\lambda_i} = 1$ and $\Phi(\lambda_i) = \{\alpha_i\}$ for all i. Since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.4.2 Type $D^{(1)}_{n,p}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, p-1$ and $m_{\lambda_p} = n - p + 1$.

Remark 21. $n = p + m$

$\Phi(\lambda_i) = \{\alpha_i\}$ for $i = 1, \cdots, p-1$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$\Phi(\mathfrak{a})^+$ is of type B_p and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-1} + \alpha_i\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(\lambda_{p-1}+\lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1}+\alpha_p, \alpha_{p-1}+\alpha_p+\alpha_{p+1}, \cdots, \alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-1}, \alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m}, \alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-3}+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \cdots, \alpha_{p-1}+\alpha_p+2\alpha_{p+1}+\cdots+2\alpha_{p+m-3}$ $+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$

$$\Phi(\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

Roots of length 3:

$$\Phi(\lambda_{i-2}+\lambda_{i-1}+\lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-2}+\alpha_{i-1}+\alpha_i\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(\lambda_{p-2}+\lambda_{p-1}+\lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}, \cdots, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-1}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-3}+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \cdots, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+2\alpha_{p+1}+\cdots+2\alpha_{p+m-3}+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$

$$\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p) = w_pw_{p-2}\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-2}+\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

$$\Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-1}w_pw_{p-2}\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-2}+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1+\lambda_2+\cdots+\lambda_p) = w_1w_2...w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_1+\alpha_2+\cdots+\alpha_p, \alpha_1+\alpha_2+\cdots+\alpha_p+\alpha_{p+1}, \cdots, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}+\alpha_{p+m},$

$\alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

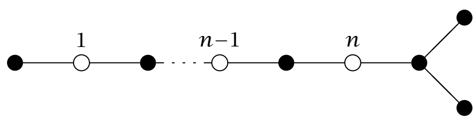
$$\{\alpha_1 + \alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2 w_3 ... w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$$\{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

6.4.3 Type $D^{(2)}_{2n+3,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 10$.

$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}, i = 1, 2, \cdots, n-1$

$\Phi(\lambda_n) = \{\alpha_{2n}, \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}\}$.

$\Phi(\mathfrak{a})^+$ is of type B_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_i \Phi(\lambda_i + 1) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}\}$ for $i = 1, \cdots, n-2$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_{n-1} \Phi(\lambda_n) =$$

$\{\alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} +$

$\alpha_{2n+2}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + 2\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-1} + 2\lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}\}$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}\}$

for $i = 1, \cdots, n-3$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+{}_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+$
$\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}+$
$\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+$
$\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+$
$\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}+\alpha_{2n+2}+$
$\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+$
$\alpha_{2n}+\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+$
$\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$\alpha_{2n-1}+\alpha_{2n}+2\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2}+\lambda_{n-1}+2\lambda_n) = w_{n-2}w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+$
$\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+$
$2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+$
$\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+$
$\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+$
$2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-5}+$
$\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+$
$2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+$
$\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-5}+\alpha_{2n-4}+$
$\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+$
$\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$
$\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-4}+$
$\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+$
$2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+$
$\alpha_{2n-2}+\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}+$
$\alpha_{2n+3}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+2\alpha_{2n-1}+2\alpha_{2n}+2\alpha_{2n+1}+\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+$

$2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-1}w_nw_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}\}$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1 w_2 ... w_{n-1} \Phi(\lambda_n) =$$

$\{\alpha_1 + \alpha_2 + ... + \alpha_{2n}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} + 2\alpha_{2n+1}, \alpha_1 + \alpha_2 + ... +$
$\alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} +$
$\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 +$
$\alpha_2 + ... + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + \alpha_3 + ... + \alpha_{2n}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... +$
$\alpha_{2n} + 2\alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + 2\alpha_{2n+1} +$
$\alpha_{2n+2} + \alpha_{2n+3}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 +$
$\alpha_3 + ... + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + \alpha_3 + ... + \alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1}) =$$

$\{\alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 +$
$... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} +$
$\alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} +$
$2\alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 +$
$\alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} +$
$\alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} +$
$2\alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 +$
$\alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} +$
$\alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} +$
$2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} +$
$\alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 +$
$\alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-4} +$
$\alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} +$
$2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 +$
$... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} +$
$\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} +$
$\alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 + ... + \alpha_{2n-4} +$
$\alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} +$
$2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_2 +$
$... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} +$

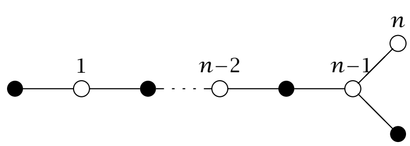
$\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_2 + ... + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + 2\alpha_{2n+1}\}$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+2} + \alpha_{2n+3}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n+1} + \alpha_{2n+3}\}$

6.4.4 Type $D^{(2)}_{2n,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 1$
$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, n-1$
$\Phi(\lambda_n) = \{\alpha_{2n}\}$

$\Phi(\mathfrak{a})^+$ is of type C_n and is computed as follows:
Roots of length 2:

$$\Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}\}$ for $i = 1, \cdots, n-2$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-2} + \alpha_{2n}, \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n}, \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(2\lambda_{n-1} + \lambda_n) = w_{n-1}\Phi(\lambda_n) = \{\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$$

Roots of length 3:

$$\Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}\}$ for $i = 1, \cdots, n-3$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_n w_{n-1}\Phi(\lambda_{n-2}) =$$

$\{\alpha_{2n+4} + \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n}, \alpha_{2n+4} + \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + \alpha_{2n+3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_n w_{n-1}\Phi(\lambda_{n-2}) =$$

$\{\alpha_{2n+4} + \alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + \alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n+5} + \alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$$\{\alpha_{2n+5} + 2\alpha_{2n+4} + 2\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n) = w_n w_{n-1} \cdots w_2 \Phi(\lambda_1) =$$

$\{\alpha_1 + \cdots + \alpha_{2n-2} + \alpha_{2n}, \alpha_2 + \cdots + \alpha_{2n-2} + \alpha_{2n}, \alpha_1 + \cdots + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \cdots + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(\lambda_1 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n) = w_{n-1}w_n w_{n-1}w_{n-2}...w_2\Phi(\lambda_1) =$$

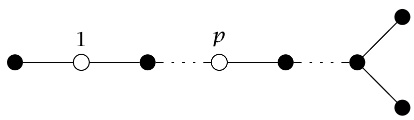
$\{\alpha_1 + \cdots + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \cdots + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \cdots + 2\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \cdots + 2\alpha_{2n+3} + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\vdots$$

$$\Phi(2\lambda_1 + \cdots + 2\lambda_{n-1} + \lambda_n) = w_1 \cdots w_{n-2} w_{n-1} \Phi(\lambda_n) =$$

$$\{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

6.4.5 Type $D_{n,p}^{(2)}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, p-1$ and $m_{\lambda_p} = 2(m+1)$

Remark 22. $n = p + m$

$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}, i = 1, \cdots, p-1$

$\Phi(\lambda_p) = \{\alpha_{2p}, \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-1}, \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-2} + \alpha_{2p+m}, \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-3} + 2\alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-3} + 2\alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-1}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-2} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{2p+m-3} + 2\alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-3} + 2\alpha_{2p+m-2} + \alpha_{2p+m-1} + \alpha_{2p+m}\}$

$\Phi(\alpha)^+$ is of type B_p and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}\}$ for $i = 1, \cdots, p-2$

$$\Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}, \cdots, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-1}, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-2} + \alpha_n, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \cdots, \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p}, \alpha_{2p-3} +$

$\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1},\cdots,\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1},\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n,\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\cdots,\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\}$

$$\Phi(\lambda_{p-1}+2\lambda_p)=w_p\Phi(\lambda_{p-1})=$$

$\{\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\}$

Roots of length 3:

$$\Phi(\lambda_{i-2}+\lambda_{i-1}+\lambda_i)=w_{i-2}w_{i-1}\Phi(\lambda_i)=$$

$\{\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4},\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}+\alpha_{2i+5},\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}+\alpha_{2i+5},\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}\}$ for $i=1,\cdots,p-3$

$$\Phi(\lambda_{p-2}+\lambda_{p-1}+\lambda_p)=w_{p-2}w_{p-1}\Phi(\lambda_p)=$$

$\{\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1},\cdots,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\cdots,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1},\cdots,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n,\cdots,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\}$

$$\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)=w_pw_{p-2}\Phi(\lambda_{p-1})=$$

$\{\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\}$

$$\Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-1}w_pw_{p-2}\Phi(\lambda_{p-1}) =$$

$\{\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-5}+\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\}$

$$\vdots$$

Roots of length p

$$\Phi(\lambda_1+\lambda_2+\cdots+\lambda_p) = w_1w_2...w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_1+\alpha_2+\alpha_3+...+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1}, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \cdots, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1}, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \cdots, \alpha_1+\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\alpha_2+\alpha_3+...+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-1}, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_n, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\cdots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n, \cdots, \alpha_2+\alpha_3+...\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n\alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_2+\alpha_3+...\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_2+$

$\alpha_3 + ...\alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-1}, \alpha_2 + \alpha_3 + ...\alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-2} + \alpha_n, \alpha_2 + \alpha_3 + ...\alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_2 + \alpha_3 + ...\alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \cdots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \cdots, \alpha_2 + \alpha_3 + ...\alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

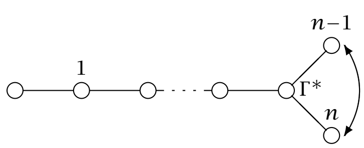
$\{\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{2p-3} + \alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_2 + \alpha_3 + ... + \alpha_{2p-3} + \alpha_{2p-2} + \alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_2 + \alpha_3 + ... + \alpha_{2p-3} + \alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\}$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2 w_3 ... w_{p-1} w_p w_1 w_2 ... w_{p-2} \Phi(\lambda_{p-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + ... + 2\alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + ... + 2\alpha_{2p-3} + 2\alpha_{2p-2} + 2\alpha_{2p-1} + 2\alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,\}$

6.4.6 Type ${}^2D^{(1)}_{n+1,n}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 2$

$\Phi(\lambda_i) = \{\alpha_i\}, i = 1, 2, \cdots, n-1$

$\Phi(\lambda_n) = \{\alpha_n, \alpha_{n+1}\}$

$\Phi(\mathfrak{a})^+$ is of type B_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1}\} \text{ for } i = 1, \cdots, n-1$$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_{n-1}\Phi(\lambda_n) = \{\alpha_{n-1} + \alpha_n, \alpha_{n-1} + \alpha_{n+1}\}$$

$$\Phi(\lambda_{n-1} + 2\lambda_n) = w_n\Phi(\lambda_{n-1}) = \{\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) = \{\alpha_i + \alpha_{i+1} + \alpha_{i+2}\} \text{ for } i = 1, \cdots, n-3$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) = \{\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_{n-2} + \alpha_{n-1} + \alpha_{n+1}\}$$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + 2\lambda_n) = w_{n-2}w_l\Phi(\lambda_{n-1}) = \{\alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-1}w_lw_{n-2}\Phi(\lambda_{n-1}) = \{\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1w_2...w_{n-1}\Phi(\lambda_n) =$$

$$\{\alpha_1 + \alpha_2 + ... + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 + ... + \alpha_{n-1} + \alpha_{n+1}\}$$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + 2\lambda_n) = w_nw_1w_2...w_{n-2}\Phi(\lambda_n - 1) =$$

$$\{\alpha_1 + \alpha_2 + ... + \alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

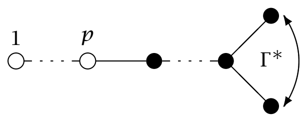
$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_n) = w_2w_3...w_{n-1}w_nw_1w_2...w_{n-1}\Phi(\lambda_{n-1}) =$$

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

$$\Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_n) = w_1w_2w_3...w_{n-1}w_nw_1w_2...w_{n-1}\Phi(\lambda_{n-1}) =$$

$$\{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$$

6.4.7 Type ${}^2D_{n,p}^{(1)}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, p-1$ and $m_{\lambda_p} = m + 1$

Remark 23. $n = p + m$

$\Phi(\lambda_i) = \{\alpha_i\}, i = 1, 2, \cdots, p-1$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{n-2} + \alpha_{n-1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{n-2} + \alpha_n \alpha_p + \alpha_{p+1} + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n\}$

$\Phi(\mathfrak{a})^+$ is of type B_p and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-1} + \alpha_i\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$$\Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1}) = \{\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) = \{\alpha_{i-2} + \alpha_{i-1} + \alpha_i\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots,$

$\alpha_{p-2} + \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$$\Phi(\lambda_{p-2} + \lambda_{p-1} + 2\lambda_p) = w_p w_{p-2}\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-2} + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

$$\Phi(\lambda_{p-2} + 2\lambda_{p-1} + 2\lambda_p) = w_{p-1} w_p w_{p-2}\Phi(\lambda_{p-1}) =$$

$$\{\alpha_{p-2} + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

$$\vdots$$

Roots of length p:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_p) = w_1 w_2 \ldots w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \alpha_{p+1}, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-2},$
$\alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-2} + \alpha_{p+m},$
$\alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_1 + \alpha_2 + \cdots + \alpha_p + \cdots + 2\alpha_{p+m-2} +$
$\alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1} + 2\lambda_p) = w_p w_1 w_2 \ldots w_{p-2}\Phi(\lambda_p - 1) =$$

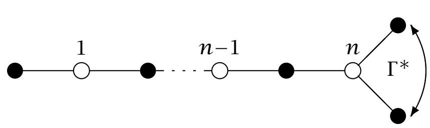
$$\{\alpha_1 + \alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_1 w_2 \ldots w_{p-1} w_p w_1 w_2 \ldots w_{p-2}\Phi(\lambda_p - 1) =$$

$$\{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

6.4.8 Type ${}^2D^{(2)}_{2n+2,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n-1$ and $m_{\lambda_n} = 8$
$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}, i = 1, 2, \cdots, n-1$

$\Phi(\lambda_n) = \{\alpha_{2n}, \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+2}, \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}, \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1} + \alpha_{2n+2}\}$

$\Phi(\mathfrak{a})^+$ is of type B_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}, \alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}, \alpha_{2i}+\alpha_{2i+1} + \alpha_{2i+2}\}$ for $i = 2, \cdots, n-1$

$$\Phi(\lambda_{n-1} + \lambda_n) = w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+2}, \alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+2}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}\}$

$$\Phi(\lambda_{n-1} + 2\lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}\}$

Roots of length 3:

$$\Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}\}$ for $i = 3, \cdots, n-1$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+2}, \alpha_{2n-5}+\alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+$

$\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+2}, \alpha_{2n-4}+\alpha_{2n-3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}\}$

$$\Phi(\lambda_{n-2} + \lambda_{n-1} + 2\lambda_n) = w_n w_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}\}$

$$\Phi(\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n) = w_{n-1} w_n w_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-4} + \alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-5} + \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}\}$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1 w_2 ... w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+2}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-2}\Phi(\lambda_{n-1}) =$$

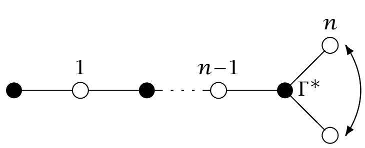
$\{\alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + \alpha_{2n-1} + 2\alpha_{2n}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_2 + \alpha_3 + ... + \alpha_{2n-5} + \alpha_{2n-4} + \alpha_{2n-3} + \alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}\}$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1} =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n} + \alpha_{2n+2}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + ... + 2\alpha_{2n-5} + 2\alpha_{2n-4} + 2\alpha_{2n-3} + 2\alpha_{2n-2} + 2\alpha_{2n-1} + 2\alpha_{2n}\}$

6.4.9 Type ${}^2D^{(2)}_{2n+1,n}$



$m_{\lambda_i} = 4$ for $i = 1, \cdots, n-1$ and $m_{\lambda_i} = 5$

$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, n-1$

$\Phi(\lambda_n) = \{\alpha_{2n}, \alpha_{2n+1}, \alpha_{2n-1} + \alpha_{2n}, \alpha_{2n-1} + \alpha_{2n+1}\}$ $\Phi(2\lambda_n) = \{\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}\}$

$\Phi(\alpha)^+$ is of type B_n and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}, \alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}\}$ for $i=2,\cdots,n-1$

$$\Phi(\lambda_{n-1}+\lambda_n) = w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n+1}, \alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-1}+2\lambda_n) = w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+3}+\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}\}$

Roots of length 3:

$$\Phi(\lambda_{i-2}+\lambda_{i-1}+\lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-5}+\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}, \alpha_{2i-5}+\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}\}$ for $i=3,\cdots,n-1$

$$\Phi(\lambda_{n-2}+\lambda_{n-1}+\lambda_n) = w_{n-2}w_{n-1}\Phi(\lambda_n) =$$

$\{\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n+1}, \alpha_{2n+5}+\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}, \alpha_{2n+5}+\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2}+\lambda_{n-1}+2\lambda_n) = w_{n-2}w_n\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+5}+\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+5}+\alpha_{2n+4}+\alpha_{2n+3}+\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}\}$

$$\Phi(\lambda_{n-2}+2\lambda_{n-1}+2\lambda_n) = w_{n-1}w_nw_{n-2}\Phi(\lambda_{n-1}) =$$

$\{\alpha_{2n+4}+\alpha_{2n+3}+2\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+4}+2\alpha_{2n+3}+2\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+5}+\alpha_{2n+4}+\alpha_{2n+3}+2\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}, \alpha_{2n+5}+\alpha_{2n+4}+2\alpha_{2n+3}+2\alpha_{2n-2}+2\alpha_{2n-1}+\alpha_{2n}+\alpha_{2n+1}\}$

$$\vdots$$

Roots of length n:

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = w_1 w_2 ... w_{n-1} \Phi(\lambda_n) =$$

$\{\alpha_1 + \cdots + \alpha_{2n-1} + \alpha_{2n}, \alpha_1 + \cdots + \alpha_{2n-1} + \alpha_{2n+1}, \alpha_2 + \cdots + \alpha_{2n-1} + \alpha_{2n}, \alpha_2 + \cdots + \alpha_{2n-1} + \alpha_{2n+1}\}$

$$\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 2\lambda_n) = w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1}) =$$

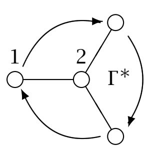
$\{\alpha_1 + \cdots + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \cdots + \alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \cdots + \alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \cdots + \alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}\}$

$$\vdots$$

$$\Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n) = w_2 w_3 ... w_{n-1} w_n w_1 w_2 ... w_{n-2} \Phi(\lambda_{n-1}) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2n-2} + 2\alpha_{2n-1} + \alpha_{2n} + \alpha_{2n+1}\}$

6.4.10 Type ${}^3D_{4,2}^{(2)}$



$m_{\lambda_1} = 3$ and $m_{\lambda_2} = 1$.
$\Phi(\lambda_1) = \{\alpha_2\}$
$\Phi(\lambda_2) = \{\alpha_1, \alpha_3, \alpha_4\}$
$\Phi(\alpha)^+$ is of type G_2 and is computed as follows:

$$\Phi(\lambda_1 + \lambda_2) = w_1 \Phi(\lambda_2) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}$$

$$\Phi(\lambda_1 + 2\lambda_2) = w_2 w_1 \Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4\}$$

$$\Phi(\lambda_1 + 3\lambda_2) = w_2 \Phi(\lambda_1) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$$

$$\Phi(2\lambda_1 + 3\lambda_2) = w_1 w_2 \Phi(\lambda_1)) = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$$

6.4.11 Type ${}^6D_{4,2}^{(2)}$

$m_{\lambda_1} = 3$ and $m_{\lambda_2} = 1$.

$\Phi(\lambda_1) = \{\alpha_2\}$

$\Phi(\lambda_2) = \{\alpha_1, \alpha_3, \alpha_4\}$

$\Phi(\mathfrak{a})^+$ is of type G_2 and is computed as follows:

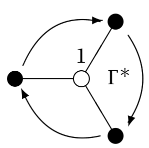
$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}$$

$$\Phi(\lambda_1 + 2\lambda_2) = w_2w_1\Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4\}$$

$$\Phi(\lambda_1 + 3\lambda_2) = w_2\Phi(\lambda_1) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$$

$$\Phi(2\lambda_1 + 3\lambda_2) = w_1w_2\Phi(\lambda_1)) = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$$

6.4.12 Type ${}^3D_{4,1}^{(9)}$



$m_{\lambda_1} = 8$.

$\Phi(\mathfrak{a})^+$ is of type BC_1.

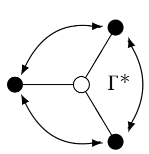
The roots are of length 1:

$\Phi(\lambda_1) =$

$\{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$

$\Phi(2\lambda_1) = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$

6.4.13 Type ${}^6D_{4,1}^{(9)}$



$m_{\lambda_1} = 8$.

$\Phi(\mathfrak{a})^+$ is of type BC_1.

The roots are of length 1:

$\Phi(\lambda_1) =$

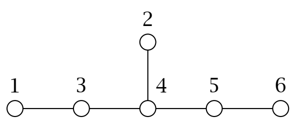
$\{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$

$\Phi(2\lambda_1) = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$

6.5 E_6 cases

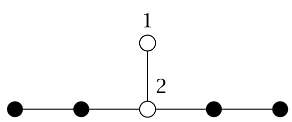
There are 8 congruence classes of Γ-indices corresponding to a simple group of type E_6.

6.5.1 Type ${}^1E^0_{6,6}$



For $i = 1, \cdots, 6$, $m_{\lambda_i} = 1$ and $\Phi(\lambda_i) = \{\alpha_i\}$, and $w_i = s_{\alpha_i}$. We have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.5.2 Type ${}^1E^{16}_{6,2}$



$m_{\lambda_1} = 1$ and $m_{\lambda_2} = 9$

$\Phi(\lambda_1) = \{\alpha_2\}$

$\Phi(\lambda_2) = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_5, \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$

$\Phi(\mathfrak{a})^+$ is of type G_2 and is computed as follows:

$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) =$$

$\{\alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$

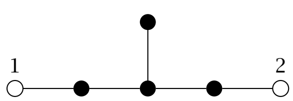
$$\Phi(\lambda_1 + 2\lambda_2) = w_2 w_1 \Phi(\lambda_2) =$$

$\{\alpha_2 + 2\alpha_4, \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$

$$\Phi(\lambda_1 + 3\lambda_2) = w_2 \Phi(\lambda_1) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

$$\Phi(2\lambda_1 + 3\lambda_2) = w_1 w_2 \Phi(\lambda_1) = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

6.5.3 Type ${}^1E^{28}_{6,2}$



$m_{\lambda_i} = 8$ for $i = 1, 2$

$\Phi(\lambda_1) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \}$

$\Phi(\lambda_2) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \}$

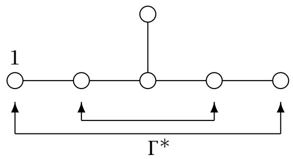
$\Phi(\mathfrak{a})^+$ is of type A_2 and is computed as follows:

Root of length 2:

$$\Phi(\lambda_1 + \lambda_2) = w_1 \Phi(\lambda_2) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$

6.5.4 Type ${}^2E_{6,4}^{16}$



$m_{\lambda_i} = 1$ for $i = 1, 2$ and $m_{\lambda_i} = 2$ for $i = 3, 4$.

$\Phi(\lambda_1) = \{\alpha_2\}$

$\Phi(\lambda_2) = \{\alpha_4\}$

$\Phi(\lambda_3) = \{\alpha_3, \alpha_5\}$

$\Phi(\lambda_4) = \{\alpha_1, \alpha_6\}$

$\Phi(\mathfrak{a})^+$ is of type F_4 and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) = \{\alpha_2 + \alpha_4\}$$

$$\Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) = \{\alpha_3 + \alpha_4, \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) = \{\alpha_1 + \alpha_3, \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) = \{\alpha_3 + \alpha_4 + \alpha_5\}$$

Roots of length 3:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) = \{\alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + \lambda_3 + \lambda_4) = w_2w_3\Phi(\lambda_4) = \{\alpha_1 + \alpha_3 + \alpha_4, \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3) = w_1w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_3\Phi(\lambda_2) = \{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

Roots of length 4:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = w_1 w_2 w_3 \Phi(\lambda_4) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) = w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) = w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_1 w_3 \Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4) = w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4) = w_4 w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$$

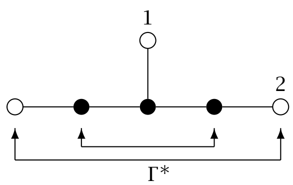
$$\Phi(\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

$$\Phi(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_1 w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

6.5.5 Type ${}^2E_{6,2}^{16'}$



The multiplicity of each λ_i is $m_{\lambda_i} = 2$ for $i = 1, \cdots, n$

$\Phi(\lambda_1) = \{\alpha_2, \alpha_2+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_3+2\alpha_4+\alpha_5\}$

$\Phi(\lambda_2) = \{\alpha_1, \alpha_6, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6\}$

$\Phi(2\lambda_2) = \{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$

$\Phi(\mathfrak{a})^+$ is of type BC_2 and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_1+\lambda_2) = w_1\Phi(\lambda_2) =$$

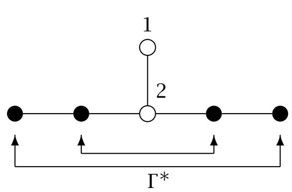
$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(\lambda_1+2\lambda_2) = w_2\Phi(\lambda_1) =$$

$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(2\lambda_1+2\lambda_2) = w_1\Phi(2\lambda_2) = \{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$$

6.5.6 Type ${}^2E_{6,2}^{16''}$



$m_{\lambda_1} = 1$ and $m_{\lambda_2} = 9$

$\Phi(\lambda_1) = \{\alpha_2\}$

$\Phi(\lambda_2) = \{\alpha_4, \alpha_3+\alpha_4, \alpha_4+\alpha_5, \alpha_1+\alpha_3+\alpha_4, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_3+$

$\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$

$\Phi(\mathfrak{a})^+$ is of type G_2 and is computed as follows:

$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) =$$

$\{\alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$

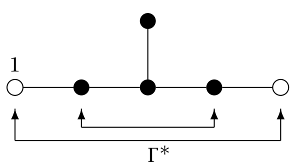
$$\Phi(\lambda_1 + 2\lambda_2) = w_2w_1\Phi(\lambda_2) =$$

$\{\alpha_2 + 2\alpha_4, \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, \alpha_2 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$

$$\Phi(\lambda_1 + 3\lambda_2) = w_2\Phi(\lambda_1) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

$$\Phi(2\lambda_1 + 3\lambda_2) = w_1w_2\Phi(\lambda_1) = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

6.5.7 Type ${}^2E_{6,1}^{29}$



$m_{\lambda_1} = 18$.

$\Phi(\mathfrak{a})^+$ is of type BC_1.

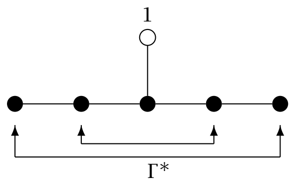
$$\Phi(\lambda_1) =$$

$\{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$

$$\Phi(2\lambda_1) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$

6.5.8 Type $^2E_{6,1}^{35}$



$m_{\lambda_1} = 9$.

$\Phi(\mathfrak{a})^+$ is of type BC_1 and is computed as follows:

$$\Phi(\lambda_1) =$$

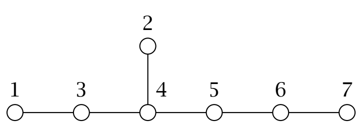
$\{\alpha_2, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(2\lambda_1) = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

6.6 E_7 cases

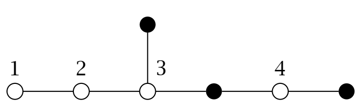
There are 4 congruence classes of Γ-indices corresponding to a simple group of type E_7.

6.6.1 Type $E_{7,7}^0$



$m_{\lambda_i} = 1$, $\Phi(\lambda_i) = \{\alpha_i\}$, and $w_i = s_{\alpha_i}$ for for $i = 1, \cdots, 7$. We have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.6.2 Type $E_{7,4}^9$



$m_{\lambda_1} = m_{\lambda_2} = 1$ and $m_{\lambda_3} = m_{\lambda_4} = 4$.

$\Phi(\lambda_1) = \{\alpha_1\}$

$\Phi(\lambda_2) = \{\alpha_3\}$

$\Phi(\lambda_3) = \{\alpha_4, \alpha_2 + \alpha_4, \alpha_4 + \alpha_5, \alpha_2 + \alpha_4 + \alpha_5\}$

$\Phi(\lambda_4) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7\}$

$\Phi(\mathfrak{a})^+$ is of type F_4 and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) = \{\alpha_1 + \alpha_3\}$$

$$\Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) = \{\alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) =$$

$$\{\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$$

$$\Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

Roots of length 3:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + \lambda_3 + \lambda_4) = w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3) = w_1w_3\Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

Roots of length 4:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) = w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) = w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_1 w_3 \Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4) = w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_2 w_1 w_3 \Phi(\lambda_2) = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4) = w_4 w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

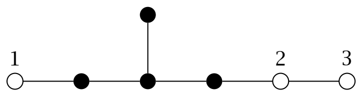
$$\Phi(\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_1 w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

6.6.3 Type $E_{7,3}^{28}$



$m_{\lambda_1} = m_{\lambda_2} = 8$ and $m_{\lambda_3} = 1$.

$\Phi(\lambda_1) = \{\alpha_1, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5\}$

$\Phi(\lambda_2) = \{\alpha_6, \alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$

$\Phi(\lambda_3) = \{\alpha_7\}$

$\Phi(\mathfrak{a})^+$ is of type C_3 and is computed as follows:

Roots of length 2:

$$\Phi(\lambda_1+\lambda_2) = w_2\Phi(\lambda_1) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(\lambda_2+\lambda_3) = w_3\Phi(\lambda_2) =$$

$\{\alpha_6+\alpha_7, \alpha_5+\alpha_6+\alpha_7, \alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7\}$

$$\Phi(2\lambda_2+\lambda_3) = w_2\Phi(\lambda_3) = \{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\}$$

Roots of length 3:

$$\Phi(\lambda_1+\lambda_2+\lambda_3) = w_3w_2\Phi(\lambda_1) =$$

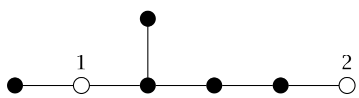
$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7\}$

$$\Phi(\lambda_1+2\lambda_2+\lambda_3) = w_2\Phi(\lambda_1+\lambda_2+\lambda_3) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$

$$\Phi(2\lambda_1 + 2\lambda_2 + \lambda_3) = w_1 w_2 \Phi(\lambda_3) = \{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

6.6.4 Type $E_{7,2}^{31}$



$m_{\lambda_1} = 5$ and $m_{\lambda_2} = 14$

$\Phi(\lambda_1) = \{\alpha_7, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$.

$\Phi(\lambda_2) = \{\alpha_3, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$.

$\Phi(\mathfrak{a})^+$ is of type A_2 and is computed as follows:

Roots of length 2:

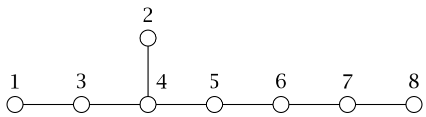
$$\Phi(\lambda_1 + \lambda_2) = w_1 \Phi(\lambda_2) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$

6.7 E_8 cases

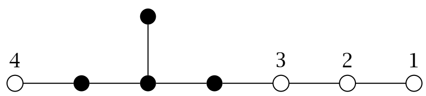
There are 2 congruence classes of Γ-indices corresponding to a simple group of type E_8.

6.7.1 Type $E_{8,8}^{0}$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, 8$, and $\Phi(\lambda_i) = \{\alpha_i\}$. Since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

6.7.2 Type $E_{8,4}^{28}$



$m_{\lambda_1} = m_{\lambda_2} = 8$ and $m_{\lambda_3} = m_{\lambda_4} = 1$.

$\Phi(\lambda_1) = \{\alpha_8\}$

$\Phi(\lambda_2) = \{\alpha_7\}$

$\Phi(\lambda_3) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$

$\Phi(\lambda_4) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$

$\Phi(\mathfrak{a})^+$ is of type F_4 and is computed as follows:
Roots of length 2:

$$\Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) = \{\alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) =$$

$\{\alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 +$

$\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$

$$\Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

Roots of length 3:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$\{\alpha_6 + \alpha_7 + \alpha_8, \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\}$

$$\Phi(\lambda_2 + \lambda_3 + \lambda_4) = w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3) = w_1w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_3\Phi(\lambda_2) = \{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

Roots of length 4:

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\}$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) = w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) = w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8\}$

$$\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4) = w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4) = w_4 w_3 w_2 w_3 w_1 w_2 w_3 \Phi(\lambda_4) =$$

$\{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8\}$

$$\Phi(\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8\}$$

$$\Phi(\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8\}$$

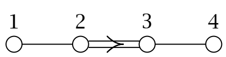
$$\Phi(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_1 w_2 w_3 w_4 w_2 w_1 w_3 \Phi(\lambda_2) =$$

$$\{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8\}$$

6.8 F_4 cases

There are 2 congruence classes of Γ-indices corresponding to a simple group of type F_4.

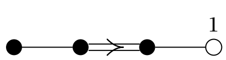
6.8.1 Type $F_{4,4}^0$



$m_{\lambda_i} = 1$, $\Phi(\lambda_i) = \{\alpha_i\}$, and $w_i = s_{\alpha_i}$ for $i = 1, 2, 3, 4$. Since $w_i = s_{\alpha_i}$ for all i, we have that

$$\Phi(\mathfrak{a}) = \Phi(\mathfrak{t}).$$

6.8.2 Type $F_{4,1}^{21}$



$m_{\lambda_i} = 2$ for $i = 1, \cdots, n$

$\Phi(\mathfrak{a})^+$ is of type BC_1, and the roots are all of length 1:

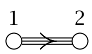
$\Phi(\lambda_1) = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\}$

$\Phi(2\lambda_1) = \{\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$

6.9 G_2 case

There is 1 congruence class of Γ-indices corresponding to a simple group of type G_2.

6.9.1 Type $G_{2,2}^0$



$m_{\lambda_i} = 1$ for $i = 1, \cdots, 2$, and $\Phi(\lambda_i) = \{\alpha_i\}$ for $i = 1, 2$. Since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

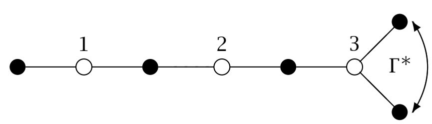
Chapter 7

An Example

To demonstrate the algorithm, I include an complete example. I end with the root space decomposition, which is obtained with ease after finding the projection spaces for each root of any admissible length.

7.1 Example ${}^2D_{8,3}^{(2)}$

Here we start off with a Lie algebra of type ${}^2D_{8,3}^{(2)}$.



7.1.1 Step 1:

To begin implementing the algorithm, we need to recover the action of Γ on $\Phi(\mathfrak{t})$. We notice that

$$\Delta_0(\Gamma) = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_8\}.$$

We calculate $w_\sigma(\Gamma) = s_{\alpha_1}s_{\alpha_3}s_{\alpha_5}s_{\alpha_7}s_{\alpha_8}$. In this case, σ=$s_{\alpha_1}s_{\alpha_3}s_{\alpha_5}s_{\alpha_7}s_{\alpha_8}\sigma^*$, and $\Gamma = \{\mathrm{id}, \sigma\}$.

7.1.2 Step 2:

The projections are found using using $\pi(\alpha) = \frac{1}{|\Gamma|}\Sigma_{\sigma\in\Gamma}\sigma(\alpha)$:

$$\pi(\alpha_2) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \lambda_1,$$

$$\pi(\alpha_4) = \frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_2, \text{ and}$$
$$\pi(\alpha_6) = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8) = \lambda_3.$$

$m_{\lambda_1} = 4$, $m_{\lambda_2} = 4$, and $m_{\lambda_3} = 8$.

7.1.3 Step 3:

The restricted root system is of type B_3.
The representatives in W^Γ for s_{λ_1}, s_{λ_2}, and s_{λ_3} are

$$w_1 = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2},$$

$$w_2 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4},$$

and

$$w_3 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_8} s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_8} s_{\alpha_6}.$$

7.1.4 Step 4:

For $\Phi(\lambda_i)$ we have:

$$\Phi(\lambda_1) = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\},$$

$$\Phi(\lambda_2) = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5\},$$

and

$$\Phi(\lambda_3) = \{\alpha_6, \alpha_5+\alpha_6, \alpha_6+\alpha_7, \alpha_6+\alpha_8, \alpha_5+\alpha_6+\alpha_7, \alpha_5+\alpha_6+\alpha_8, \alpha_6+\alpha_7+\alpha_8, \alpha_5+\alpha_6+\alpha_7+\alpha_8\}.$$

7.1.5 Step 5:

Next we find the list of positive roots, $\Phi(\mathfrak{a})^+$, by applying s_{λ_1}, s_{λ_2}, and s_{λ_3} to each λ in the restricted root system. Here we have

$$s_{\lambda_1}(\lambda_2) = \lambda_1 + \lambda_2$$

$$s_{\lambda_2}(\lambda_3) = \lambda_2 + \lambda_3 \text{ and}$$

$$s_{\lambda_3}(\lambda_2) = \lambda_2 + 2\lambda_3$$

Therefore, $\Phi(\mathfrak{a})^+ =$

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_2 + 2\lambda_3, +\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + 2\lambda_3, \lambda_1 + 2\lambda_2 + 2\lambda_3\}.$$

7.1.6 Step 6:

Now we need to compute $\Phi(\lambda_i)$ for all λ_i in $\Phi(\mathfrak{a})^+$. This is easily accomplished by looking at the previous step. We must compute

$$\Phi(\lambda_1+\lambda_2)=\Phi(s_{\lambda_1}(\lambda_2))=w_1(\Phi(\lambda_2)),$$

$$\Phi(\lambda_2+\lambda_3)=\Phi(s_{\lambda_2}(\lambda_3))=w_2(\Phi(\lambda_3)),$$

$$\Phi(\lambda_2+2\lambda_3)=\Phi(s_{\lambda_3}(\lambda_2))=w_3(\Phi(\lambda_2))$$

$$\Phi(\lambda_1+\lambda_2+\lambda_3)=\Phi(s_{\lambda_1}(\lambda_2+\lambda_3))=w_1w_2(\Phi(\lambda_3)),$$

$$\Phi(\lambda_1+\lambda_2+2\lambda_3)=\Phi(s_{\lambda_3}(\lambda_1+\lambda_2))=w_3w_1(\Phi(\lambda_2)),\text{ and}$$

$$\Phi(\lambda_1+2\lambda_2+2\lambda_3)=\Phi(s_{\lambda_2}(\lambda_1+\lambda_2+2\lambda_3))=w_2w_3w_1(\Phi(\lambda_2)).$$

We arise with

$$\Phi(\lambda_1+\lambda_2)=$$

$$\{\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\alpha_2+\alpha_3+\alpha_4+\alpha_5\},$$

$$\Phi(\lambda_2+\lambda_3)=$$

$$\{\alpha_4+\alpha_5+\alpha_6,\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha_4+\alpha_5+\alpha_6+\alpha_7,\alpha_4+\alpha_5+\alpha_6+\alpha_8,\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,$$

$$\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8,\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8,\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8\},$$

$$\Phi(\lambda_2+2\lambda_3)=$$

$$\{\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8,\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$$

$$\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8,\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8\},$$

$$\Phi(\lambda_1+\lambda_2+\lambda_3)=$$

$$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\alpha_2\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8,$$

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8,$$

$$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8,\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8\},$$

$$\Phi(\lambda_1+\lambda_2+2\lambda_3)=$$

$$\{\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$$

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8\}, \text{ and}$$

$$\Phi(\lambda_1+2\lambda_2+2\lambda_3)=$$

$$\{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$$

$$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8\}.$$

7.1.7 Root Space Decomposition:

As in Theorem 2.2, we use the above $\Phi(\lambda_i)$ to obtain the following root space decomposition.

$\mathfrak{g}_{\lambda_1} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3},$

$\mathfrak{g}_{\lambda_2} = \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5},$

$\mathfrak{g}_{\lambda_3} = \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_6+\alpha_8} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_8} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_8} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7+\alpha_8},$

$\mathfrak{g}_{\lambda_1+\lambda_2} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5},$

$\mathfrak{g}_{\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_8} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8} \oplus$
$\mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8},$

$\mathfrak{g}_{\lambda_2+2\lambda_3} = \mathfrak{g}_{\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8} \oplus$
$\mathfrak{g}_{\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8},$

$\mathfrak{g}_{\lambda_1+\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus$
$\mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_8} \oplus$
$\mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8},$

$\mathfrak{g}_{\lambda_1+\lambda_2+2\lambda_3} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus$

$\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8}$, and

$\mathfrak{g}_{\lambda_1+2\lambda_2+2\lambda_3} = \mathfrak{g}_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus$
$\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8}$.

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