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SOME LIMIT DISTRIBUTIONS CONNECTED WITH FIXED INTERVAL ANALYSIS

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The proofs of some limit theorems on the limiting distributions of statistics that enter fixed interval analysis are presented.

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SOME LIMIT DISTRIBUTIONS CONNECTED WITH FIXED INTERVAL ANALYSIS

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Summary: The proofs of some theorems (stated in Sethuraman (1963)) on the limiting distributions of some statistics that enter in the method of Fixed Interval Analysis are presented.

1. Introduction

Let (\underline{Y}, X) be a random variable taking values in $(\mathcal{Y} \times \mathcal{X})$ where \mathcal{Y} is E_k the Euclidean space of k dimensions and \mathcal{X} is a measurable space. Let E_1, E_2, \dots, E_g be g disjoint measurable sets in \mathcal{X} whose union is the whole space \mathcal{X} .

$(\underline{y}_1, x_1), (\underline{y}_2, x_2), \dots, (\underline{y}_n, x_n)$ are n independent observations on (\underline{Y}, X) . The number of x_i 's that fall in E_j is $n_j, j = 1, \dots, g$. \underline{u}_j is defined by the relation

$$\underline{u}_j = \sum^j \underline{y}_i / n_j \quad j = 1, \dots, g$$

where \sum^j is the summation over all "i" such that x_i is in E_j .

Throughout this paper it is assumed that

$$V(\underline{Y}) < \infty \quad (1)$$

and

$$\text{Prob. } (X \in E_j) = \pi_j > 0 \quad j = 1, \dots, g \quad (2)$$

where for any random variable \underline{Z} , $v(\underline{Z})$ denotes the variance covariance matrix of \underline{Z} .

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The following theorem is established in section 3.

Theorem 1. The asymptotic distribution of $(\underline{u}_1, \dots, \underline{u}_g)$ is the distribution of g independent normal distributions.

This theorem plays a fundamental role in the method of Fixed Interval Analysis, for instance see Sethuraman (1963). Interpreted in Sample Survey language this theorem, among other things, states that the post-stratified stratum means are independently distributed in the limit.

2. Notations, definitions and preliminaries.

Let $\underline{Y}(E_j)$, called the conditional random variable of \underline{Y} given that X is in E_j , denote a random variable on \underline{Y} with the distribution defined by $\text{Prob.}(\underline{Y}(E_j) \in A) = \text{Prob.}(\underline{Y} \in A, X \in E_j) / \text{Prob.}(X \in E_j)$. For any random variable \underline{Z} , $E(\underline{Z})$ denotes the vector of expectations of \underline{Z} . Define

$$E(\underline{Y}(E_j)) = \underline{\mu}_j \quad (3)$$

$$V(\underline{Y}(E_j)) = \underline{\Sigma}_j \quad (4)$$

$$p_j = n_j/n \quad (5)$$

$$\sqrt{n} (\underline{u}_j - \underline{\mu}_j) = \underline{\eta}_j(n) \quad (6)$$

$$\sqrt{n} (p_j - \pi_j) = \underline{\zeta}_j(n) \quad (7)$$

$$j = 1, \dots, g.$$

Let $\{\xi_n(\cdot, \theta)\}$, $n = 0, 1, \dots$ be a sequence of families of probability distributions on the Borelsubsets of E_m (or more generally, of any topological space) and θ vary in a compact topological space K .

Definition. $\{\xi_n(\cdot, \theta)\}$ is said to converge weakly, uniformly and continuously (in other words, in the UC^* sense) to $\xi_0(\cdot, \theta)$ with respect to θ in K

$\inf_{\theta} k_n(\theta) \rightarrow \infty$ as $n \rightarrow \infty$.

Let $MN(\alpha, L)$ stand for the multivariate normal distribution with mean vector α and variance covariance matrix L .

Lemma 1. The sequence of families of distributions of

$\left\{ \frac{Z_{n_1} + \dots + Z_{nk_n(\theta)}}{\sqrt{k_n(\theta)}} - k_n(\theta) y_n \right\} / \sqrt{k_n(\theta)}$ converges in the UC^* sense to the distribution $MN(0, V)$.

3. Main theorems.

We first prove the following lemma.

Lemma 2. The distributions of $(\eta_1(n), \dots, \eta_g(n))$ given that $\zeta(n) = \underline{z}$, $\sum_{i=1}^g z_i = 0$ converges in the UC^* sense to the distribution $MN(0, \Lambda)$ with respect to \underline{z} in any closed bounded subset of E_g , where

$$\Lambda = \begin{pmatrix} \frac{1}{n_1} \Sigma_1 & 0 & & 0 \\ 0 & \frac{1}{n_2} \Sigma_2 & & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & \frac{1}{n_g} \Sigma_g \end{pmatrix} \quad (8)$$

Proof: The event $\zeta(n) = \underline{z}$ is equivalent with probability one to the event $n_i = \lfloor n \pi_i + \sqrt{n} z_i \rfloor$ $i = 1, \dots, g$, since

$$\text{Prob.} \left\{ n \pi_i + \sqrt{n} \zeta_i(n) = \lfloor n \pi_i + \sqrt{n} \zeta_i(n) \rfloor, i = 1, \dots, g \right\} = 1.$$

The conditional distribution of y_1, \dots, y_n given that $n_i = \lfloor n \pi_i + \sqrt{n} z_i \rfloor$, $i = 1, \dots, g$ is the distribution of g independent samples of size n_1, \dots, n_g on $Y(E_1), \dots, Y(E_g)$, respectively. $\sqrt{\frac{n_1}{n}} \eta_1(n), \dots, \sqrt{\frac{n_g}{n}} \eta_g(n)$ are the

normalized means of these g independent samples. For \underline{z} in a closed bounded subset of E_g we note that $\inf_{i, \underline{z}} [n\pi_i + \sqrt{n} z_i] \rightarrow \infty$ as $n \rightarrow \infty$. Thus all the conditions of lemma 1 are satisfied. Further $[n\pi_i + \sqrt{n} z_i]/n$ tends to π_i , uniformly in \underline{z} in any closed bounded subset of E_g . Hence the conditional distributions of $(\eta_1(n), \dots, \eta_g(n))$ given that $\xi(n) = \underline{z}$ converges in the UC^* sense to the distribution $MN(0, \Lambda)$ with respect to \underline{z} in any closed bounded subset of E_g .

Theorem 3. The joint distribution of $(\eta_1(n), \dots, \eta_g(n), \xi(n))$ converges weakly to the distribution $MN(0, B)$ where

$$B = \left(\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & C \end{array} \right) \quad (9)$$

where $C = \left(\begin{array}{cccc} \pi_1(1-\pi_2) & -\pi_1\pi_2 & \dots & -\pi_1\pi_g \\ -\pi_1\pi_2 & \pi_2(1-\pi_2) & \dots & -\pi_2\pi_g \\ \cdot & \cdot & \cdot & \cdot \\ -\pi_1\pi_g & -\pi_2\pi_g & \dots & \pi_g(1-\pi_g) \end{array} \right) . \quad (10)$

Proof: This theorem is an immediate consequence of Theorem 2, lemma 2 and the observation that the distribution of $\xi(n)$ converges weakly to the distribution $MN(0, C)$.

Proof of theorem 1: Theorem 1 is contained in theorem 3.

4. References

1. Sethuraman, J. (1961). Some limit theorems for joint distributions, Sankhyā, Series A, 23, 379-386.
2. Sethuraman, J. (1963). Fixed Interval Analysis and Fractile Analysis, submitted to the Mahalanobis Birthday Volume.