The Relaxation of Thin Elliptical Cylindrical Shells Under Pure Bending Moment

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Nonlinear finite deflection thin shell theory of Reissner including initial strain is employed and this reduces the problem to a set of nonlinear ordinary differential equations by the use of restricted variational technique. The resulting two-point nonlinear boundary value problem is then solved by using a shooting technique.Through to use of this numerical technique, the effect of curvature parameter and the ellipticity ratio are examined in some detail in creep then it is found that when the deflection is large, the relaxation curves are considerable different from small deflection one and the effect of ellipticity ratio on the relaxation curves, in practical terms, are not negligible.

I. Introduction

The modern practice of engineering places a premium on design that can be termed optimum in one sense or another. From the structural point of view, optimum design often means the lightest structure consistent with integrity and safety. Thus, the determination of the stresses and displacements of structures under the nonlinear conditions can be of great importance to the engineers. The fundamental theory for large displacement problems have been available for a long time and can be found elsewhere {I, 2}. On evaluating the safety and reliability of high temperature systems, inelastic behaviour must be also considered. The intelligent use of elasticity theory can give powerful methods to pressure vessel designers who have to make decisions regarding the effects of creep, plasticity, fatique and so on in sizing parts and assessing life time {3}. Utilising numerical methods and high speed computers, inelastic analysis may be carried out for structures of any geometry and under any loading conditions.

Buckling of a thin walled cylindrical shell (tube) with infinite length is of great importance in the pipelines and marine risers, hence has been approached in several ways. Considering a thin walled cylindrical shell is loaded at its end sections by two equal and opposite bending moments M, it has been observed that its cross section alters its shape into an oval one, owing to the fact that apart from the usual bending stresses in the cross section of the shell, there arise tangential bending stresses in its meridional sections. This fact was first considered by Brazier {4}. This ovalization causes a nonlinear load deflection relation and leads to a limit load behaviour. Then further a more precise formulation of the problem was given by Reissner {5, 6}. Since then, following the Reissner type of formulation, a number of papers appeared not only for the analysis of the collapse behaviour of thin cylindrical shells including material orthotropy {7} and also the effect of combined loading conditions and bifurcation from the nonlinear state {8}. Another type of approach taken by Seide and Weingarten {9} who used the short axial wave type in a linear membrane prebuckling state in their analysis of elastic bifurcation buckling. Akselrad {IO} included the geometric nonlinearities and edge constraints in his analysis and then assumed that the cylinder would bifurcate into a short wave length pattern similar to that of cylindrical shells under axial compression. From this approximate approach, for short shells, the obtainable maximum moment is coincide with the results of Seide and Weingarten, for very long shells it approaches to the result of Brazier.

On making use of the Brazier results, some works concerning with the plastic collapse of cylindrical tubes have been published {II, I2}. In their analysis, they observed the ripples rather than the small amount of ovalisation as the primary cause of collapse and the sine wave nature of these ripples has led to the hypothesis that the tube behaves as imperfect cylinders and hence imperfections are responsible for a steady growth of these ripples which eventually lead to collapse.

The first published result using the small displacement theory of cylindrical tubes with circular cross section in creep relaxation is due to Spence and Hult {I3}. The present work may be regarded as a first study of the large displacement effects, i.e, the effect of flattening displacement in the creep relaxation and the extension of collapse behaviour of cylindrical shell with elliptic cross section to the creep regime.

2. The Problem Formulation and the Geometric Relations

It is assumed that all the assumptions made by Reissner {5} in the analysis of elastic cylindrical shell under pure bending moment are also valid in the creep regime. Therefore, the following assumptions should also be applied at high temperature.

(i) The upper fibres are stretched and lower ones compressed and so somewhere in between the tensile and compressive fibres, there is a neutral plane which is unstretched.

(ii) Normal sections of the cross section remain plane after bending and deforms into a doubly symmetrical noncircular shape which is the same throughout its entire length.

(iii) Application of the moment causes a longitudinal fibre stress and a longitudinal fibre strain as well as a circumferential bending stress and a bending curvature.

It should be also mentioned that the absence of circumferential strain and axial midsurface bending curvature change in the subsequent development in due to the assumption (iii) that is to say the assumption of inextensibility.

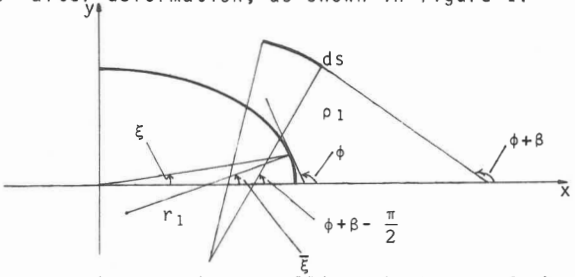
As a first approximation, an axially symmetric displacement field applies to pure bending, and the axial strain displacement may be taken as follows

$$\varepsilon=\frac{y+v}{R} \qquad (I)$$

where y is the distance of a point of the middle surface of the shell wall from the neutral surface of the shell before deformation, v is the flattening displacement, Ris a radius of the deformed center line. Including the cross sectional deformation in the external moment equilibrium we have

$$M=\oint N(y+v)\cdot ds \qquad (2)$$

where Mis an applied moment, ds is the element of arc of the deformed or undeformed circumferential curve and N is the stress resultant.It is assumed that a typical point of the undeformed cross section, (x,y) is deformed into a point (x+u, y+v) where u is the bulging displacement component. A circumferential radius of curvature r_1 changes to ρ_1 and the tangent angle ϕ changes to $\phi+\beta$ after deformation, as shown in figure I.



On writing the parametric equation of ellipse in terms of the parameter Θ as

$$x=a\cdot\cos\Theta \qquad y=b\cdot\sin\Theta \qquad (3)$$

where a and b are major and minor axes respectively, and then the straight-forward application of geometric relations, we have

$$r_o\cdot\cos\xi=a\cdot\cos\Theta \qquad r_o\cdot\sin\xi=b\cdot\sin\Theta \qquad (4)$$

hence

$$\Theta=\operatorname{arctg}\left(\frac{I}{\eta}\operatorname{tg}\xi\right) \quad \text{where} \quad \eta=\frac{b}{a} \qquad (5)$$

In terms of the variables ξ the arc element of the ellipse may be written as follows

$$ds=\frac{b(\sin^2\xi+\eta^4\cos^2\xi)^{I/2}}{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}d\xi \qquad (6)$$

The circumferential bending curvature may be defined as

$$k=\frac{d\beta}{d\xi}\cdot\frac{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}{b(\sin^2\xi+\eta^4\cos^2\xi)^{I/2}} \qquad (7)$$

For future use, the following relevant equations concerning with the geometry of undeformed and deformed cross section may be recorded as

$$-\frac{dy}{dx}=\operatorname{tg}(\pi-\phi) \qquad (8)$$

or in terms of the variables ξ and ϕ , the slope may be written as follows

$$-\operatorname{tg}\phi=\eta^2\cdot\cos\xi\ /\ \sin\xi \qquad (9)$$

and also we have

$$\cos(\bar{\xi}+\beta)=\frac{\eta^2\cos\xi\cos\beta-\sin\xi\sin\beta}{(\sin^2\xi+\eta^4\cos^2\xi)^{I/2}} \qquad (I0)$$

$$\sin(\bar{\xi}+\beta)=\frac{\sin\xi\ \cos\beta+\eta^2\cos\xi\ \cos\beta}{(\sin^2\xi+\eta^4\ \cos^2\xi\ )^{I/2}} \tag{11}$$

From the geometry of the deformed arc length, the components of displacements u and v can be expressed in the following form

$$\frac{du}{d\xi}=\frac{b}{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}\cdot\left[\sin\xi-(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}.\sin(\bar{\xi}+\beta)\right] \tag{12}$$

$$\frac{dv}{d\xi}=\frac{b}{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}\cdot\left[(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}.\cos(\bar{\xi}+\beta)-\eta^2\cos\xi\right] \tag{13}$$

3. Governing Equations

Variational formulation of governing equations in creep through the use of a variational technique, the relation between the axial strain and the cross section deformation may be written in a simple fashion. With this in mind, the strain energy expression per unit length including initial axial strain, ε_o and the initial circumferential curvature, k_o excluding the circumferential creep strain and the axial creep bending curvature, is

$$\pi_s=\oint\left(\frac{1}{2}\,C\varepsilon^2+\frac{1}{2}\,Dk^2\right)ds-\oint\left(C\varepsilon\varepsilon_o+Dkk_o\right)ds \tag{14}$$

where ε is the total axial midsurface strain, k is the total circumferential midsurface curvature.
The Euler-Lagrange equations of eq.(14) may be given as follows

$$C\left(\varepsilon-\varepsilon_o\right)=\frac{dF}{ds} \tag{15}$$

$$D\left[\frac{d}{ds}\left(\frac{d\beta}{ds}+k_o\right)\right]=F.\ \frac{\sin(\bar{\xi}+\beta)}{R} \tag{16}$$

where $C=E.h$ is the extensional stiffness, $D=\frac{Eh^3}{12(1-\nu^2)}$ is the bending stiffness, F is a Lagrange multiplier, h is the thickness of shell wall. We may also set up the following nondimensional quantity as

$$\frac{dF}{ds}=\frac{a^2c}{R}\cdot\frac{df}{ds} \tag{17}$$

Hence, a direct substitution of eq. (6) , (11), and (17) into eq. (15) and (16) and taking the derivative of them with respect to time, we have the following rate equations

$$\frac{d^2\dot{\beta}}{d\xi^2}+\frac{d\dot{\beta}}{d\xi}\cdot\left[\frac{3\sin2\xi\ .(1-\eta^2)}{2(\eta^2\cos^2\xi+\sin^2\xi)}-\frac{\sin2\xi\cdot(1-\eta^4)}{2(\sin^2\xi+\eta^4\cos^2\xi)}\right]-\frac{d\dot{k}_o}{d\xi}.b.\frac{(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}}{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}$$

$$=\frac{\dot{\beta}(-\sin\xi.\sin\beta+\eta^2\cos\xi.\cos\beta)}{(\eta^2\cos^2\xi+\sin^2\xi)^3}\cdot(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}.\alpha^2.f$$

$$+\frac{(\sin\xi.\cos\beta+\eta^2\cos\xi.\sin\beta)}{(\eta^2\cos^2\xi+\sin^2\xi)^3}\cdot(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}.\alpha^2.\dot{f} \tag{18}$$

$$\frac{d^2\dot{f}}{d\xi^2} + \frac{d\dot{f}}{d\xi}\left[\frac{3\sin 2\xi.(1-\eta^2)}{2(\eta^2\cos^2\xi+\sin^2\xi)} - \frac{\sin 2\xi\ (1-\eta^4)}{2(\sin^2\xi+\eta^4\cos^2\xi)}\right] + \frac{R}{a}\,\eta.\frac{(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}}{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}\frac{d\dot{\varepsilon}_o}{d\xi}$$

$$= -\eta^2.\frac{(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}}{(\eta^2\cos^2\xi+\sin^2\xi)^3}.\dot{\beta}(\eta^2\cos\xi.\sin\beta + \cos\beta.\sin\xi) \qquad (19)$$

4. The external moment equilibrium equation including initil strain
The external moment equilibrium equation may be written as

$$M = \oint \frac{dF}{ds}.\left(\frac{1}{C}.\frac{dF}{ds} + \varepsilon_o\right).Rds \qquad (20)$$

On defining the nondimensional parameter m as follows

$$m = \frac{M.\eta}{\pi.E.h^2.a}\sqrt{12(1-\nu^2)} \qquad (21)$$

eq. (20) may be written in terms of variable ξ in the following nondimensional form

$$m = \frac{\alpha}{\pi}.\oint\left[\left(\frac{df}{d\xi}\right)^2.\frac{1}{\eta}.\frac{(\eta^2.\cos^2\xi+\sin^2\xi)^{3/2}}{(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}} + \frac{R}{a}.\varepsilon_o.\frac{df}{d\xi}\right]d\xi \qquad (22)$$

where the curvature parameter (α) is defined as

$$\alpha = \frac{b.a}{R.h}.\sqrt{12(1-\nu^2)} \qquad (23)$$

Therefore, the rate equations of external moment equilibrium takes the following form in the first quadrant

$$\dot{m} = \frac{4\alpha}{\pi}.\int_o^{\pi/2}\left[2.\left(\frac{df}{d\xi}\right)\left(\frac{d\dot{f}}{d\xi}\right).\frac{1}{\eta}.\frac{(\eta^2\cos^2\xi+\sin^2\xi)^{3/2}}{(\sin^2\xi+\eta^4\cos^2\xi)^{1/2}} + \frac{R}{a}\left(\varepsilon_o.\frac{d\dot{f}}{d\xi} + \dot{\varepsilon}_o\frac{df}{d\xi}\right)\right]d\xi \qquad (24)$$

Eqs. (18), (19) and (24) and with the proper choice of rate boundary conditions which are defined in the following form

$$\frac{d\dot{f}(0)}{d\xi} = \dot{f}(\pi/2) = \dot{\beta}(0) = \dot{\beta}(\pi/2) = 0 \qquad (25)$$

constitute a typical two-point boundary value problem in the mathematical physics.

5. The relaxation of cylindrical shell
In the sprit of {14} we consider the following problem. 'Let an in-plane bending moment M(0) be applied uniformly to a cylindrical shell which deforms elastically resulting in a curvature. The curvature is then fixed, followed by inelastic expansion in the form of creep. The problem is to determine the relaxation of the resultant bending moment in real and / or normalised time. Having defined the relaxation creep, it is well known that several investigators ({3} and references therein) have proposed approximate constitutive equations for the stress resultants and stress couples in conjunction with the midsurface strains. Most of them are based on approximations for the creep energy dissipation rate of a shell {15} . It is convenient for this particular problem to use the approximate relations that are suggested in {15}.
In order to deal with the time dependent problem, a generalised Norton power law {3} may be used. This generalised Norton power law enters homogeneously into the rate equations. For convenience, the governing rate equations are simplified so as to remove dependence on the time function B(t) during the analysis. This is done by a normalised time-scale which is defined as

$$\tau = E\sigma_o^{n-1}\int B(t)dt \qquad (26)$$

where σ_o is an arbitrary reference stress which may be defined as

$$\sigma_o = E\frac{a}{R} \qquad (27)$$

which is in fact that elastic stress would exist in the shell bent to a radius R.
We may also define the arbitrary strain rate as

$$\dot{e}_n = \frac{a}{R}$$ and hence we have the following creep rates

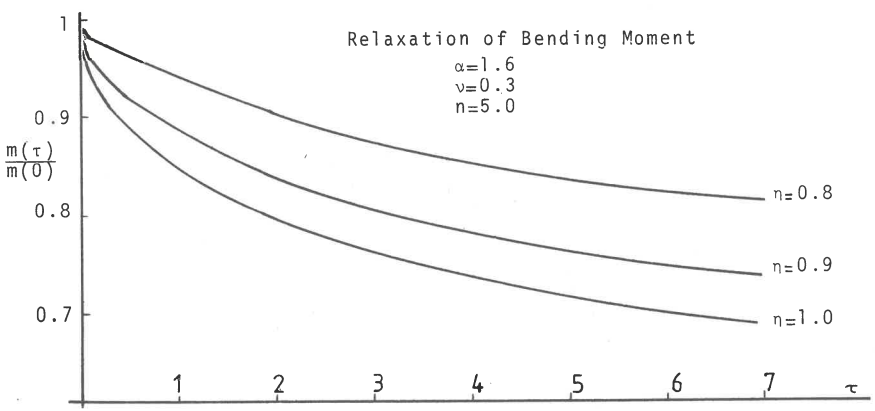
$$\dot{e}_1 = \frac{\dot{\epsilon}_o}{\frac{a}{R}} \qquad (28)$$

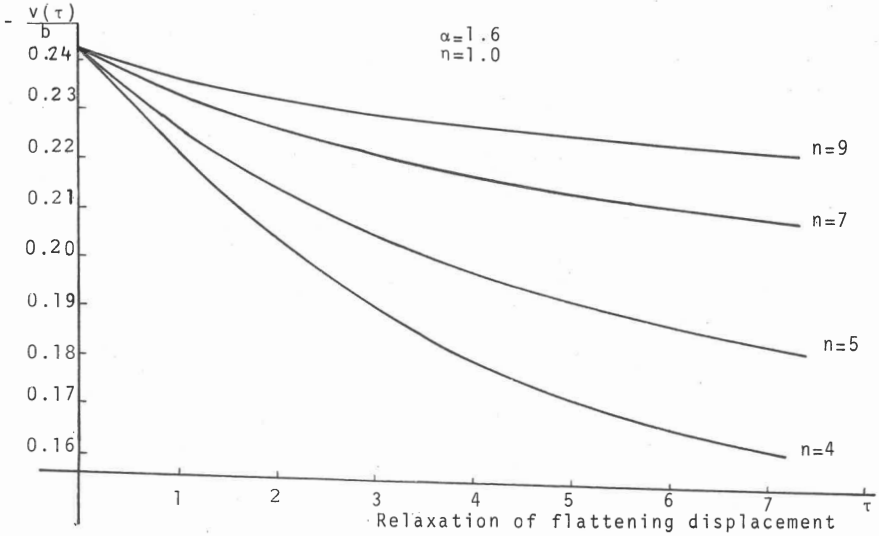
$$\dot{k}_2 = \frac{h\dot{k}_o}{4\dot{e}_n}$$

It should be noted that the dots denote differentiation with respect to the normalised time-scale, and the creep strain can be evaluated in each normalised time step.

6. Results and Discussion

Having defined the initial strain as a creep strain and then applying a shooting method to the governing equations in connection with the boundary conditions, a number of results were obtained. In the process of obtaining results, initially a small nondimensional time step of the order of 0.001 was used. After that, each time step was chosen automatically by a method suggested in {3}. In the actual numerical procedure, it was found that, the choice of time step is $\Delta\tau=0.1$ and mesh point may be taken as 40. In the computer programme, as data input, a curvature parameter α, a material parameter n, a Poisson's Ratio ν, a time step size $\Delta\tau$, a geometric parameter $\eta=\frac{b}{a}$ and *a priori* chosen normalised time at which the process was to be terminated, were chosen.
Numerical results were plotted against the normalised time scale. The main purpose here is to show the trends of results of the present model rather than to produce design data.
From these results (see figures 2, 3) we may conclude the followings. Firstly, the constant strain assumption used in the definition of relaxation in the elementary beam theory {13} does not predict the effect of curvature parameter which is in fact the dominant parameter in the present analysis. The relaxation curves are directly dependent on the curvature parameter, as a consequence of this, the analysis is to be bounded to the limit point. Secondly, a large displacement effects the relaxation curves with the increasing order of curvature parameter. This becomes more noticably true when the material parameter is increased. Due to the creep effect, the cross section of shell becomes more flattened with decreasing ratio η. Therefore, in practical terms, manufacturing inaccuracy of the cross section should be considered carefully in the creep design. The effect of η in the range shown is noticable. The rate of slope of relaxation curves decreases with a consistent manner with the curvature parameter and material index.





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