WEAK CONVERGENCE OF GENERALIZED U-STATISTICS

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Wichura (1969) has studied an invariance principle for partial sums of a multi-dimensional array of independent random variables. It is shown that a similar invariance principle holds for a broad class of generalized U-Statistics for which the different terms in the partial sums are not independent. Weak convergence of generalized U-statistics for random sample sizes is also studied. The case of (generalized) von Mises' functionals is treated briefly.

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$D^c[0,1]$ space, Generalized U-statistics, invariance principle, Gaussian processes, relative compactness, random indices and weak convergence.

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1. Introduction. Let $\{X_{ji}, i\geq 1\}, j=1,\ldots,c$, be $c(\geq 2)$ independent sequences of independent random vectors (irv), where X_{ji} has a distribution function (df) $F_j(x)$, $x\varepsilon R^p$, the $p(\geq 1)$-dimensional Euclidean space, for $j=1,\ldots,c$. Let $g(X_{ji}, i=1,\ldots, m_j, j=1,\ldots,c)$ be a Borel-measurable kernel of degree $\underset{\sim}{m} = (m_1,\ldots,m_c)$, where we assume (without any loss of generality) that $g(\ )$ is symmetric in the $m_j(\geq 1)$ arguments (vectors) of the jth set, for $j=1,\ldots,c$. Let $m_o=m_1+\ldots+m_c$, $\underset{\sim}{F}=(F_1,\ldots,F_c)$, and consider the regular functional of $\underset{\sim}{F}$.

$$\theta(\underset{\sim}{F}) = \int\cdots_{R^{pm_o}}\int g(x_{11},\ldots,x_{cm_c})\Pi_{j=1}^{c}\Pi_{i=1}^{m_j} dF_j(x_{ji}) \tag{1.1}$$

defined on $\mathcal{F} = \{\underset{\sim}{F}: |\theta(F)| < \infty\}$.

For a set of samples of sizes $\underset{\sim}{n}=(n_1,\ldots,n_c)$ with $n_j\geq m_j$, $1\leq j\leq c$, the generalized U-statistic for $\theta(\underset{\sim}{F})$ is defined by

$$U(\underset{\sim}{n}) = \Pi_{j=1}^{c}\binom{n_j}{m_j}^{-1}\sum\nolimits^*_{(\underset{\sim}{n})} g(X_{j\alpha},\ \alpha=i_{j1},\ldots,i_{jm_j},\ 1\leq j\leq c), \tag{1.2}$$

where the summation $\sum^*_{(\underset{\sim}{n})}$ extends over all $1\leq i_{j1}<\ldots<i_{jm_j}\leq n_j$, $1\leq j\leq c$. For various properties of $U(\underset{\sim}{n})$, including its asymptotic normality, we may refer to Fraser (1957) and Puri and Sen (1971), among others. For the asymptotic normality, it is assumed that $\theta(\underset{\sim}{F})$ is stationary of order zero and essentially

$$\lim_{n\to\infty} n_j/n=\lambda_j:\ 0<\lambda_j<1,\ j=1,\ldots,c, \tag{1.3}$$

where $n=n_1+\ldots+n_c$. Weak convergence of a stochastic process derived from the tail sequence of one-sample U-statistics to a Wiener process has been studied by Loynes (1970), while Miller and Sen (1972) consider a Donsker-type invariance principle for one-sample U-statistics. They show that a process derived from $\{U([k\lambda_1],\ldots,[k\lambda_c]),\ k\leq n\}$ converges weakly to a one-dimensional Wiener process. The more general and natural case of a c-dimensional time-parameter where we use the entire set $\{U(\underset{\sim}{k}):\ \underset{\sim}{m}\leq\underset{\sim}{k}\leq\underset{\sim}{n}\}$ (or the entire tail set $\{U(\underset{\sim}{k}):\ \underset{\sim}{k}\geq\underset{\sim}{n}\}$) is considered here, and it is shown that weak convergence to appropriate multi-dimensional Gaussian processes hold under no extra regularity conditions; here $\underset{\sim}{a} \leq \underset{\sim}{b}$ means that $a_i \leq b_i$ for all $1 \leq i \leq c$. It may be noted that for $c \geq 2$, the ordering of $\underset{\sim}{n}$ is not defined, and as a result, the treatment of Loynes (1970) or of Miller and Sen (1972), resting on the reverse martingale property of U-statistics, does not work out. Also, as is usually the case with generalized U-statistics, $U(\underset{\sim}{n})$ in (1.2) involves a set of summands which are not all stochastically independent. Thus, Theorem 2 (or its Corollary 1) or Wichura (1969) does not lead us to the desired result. The task is accomplished here by first extending Theorem 1 of Whichura (1969) to more general summands, and then using a decomposition of $U(\underset{\sim}{n})$ which fits into this extension.

The preliminary notions and the basic theorems are considered in section 2. The proofs of the theorems are presented in section 3. The case of generalized von Mises' (1947) functionals is treated in section 4. In the last section, the case of random sample sizes is also considered. The results are useful in the developing area of sequential inference based on generalized U-statistics.

2. Statement of the main results.

For every d_j: $0 \leq d_j \leq m_j$, $1 \leq j \leq c$, let

$$g_{d_1 \ldots d_c}(x_{ji},\ i=1,\ldots,d_j,\ 1 \leq j \leq c) = \tag{2.1}$$

$$E\{g(x_{j1},\ldots,x_{jd_j},\ X_{jd_j+1},\ldots,X_{jm_j},\ 1 \leq j \leq c)\},$$

so that $g_{00\ldots0} = \theta(F)$ and $g_{m_1 \ldots m_c}(\ \ ) = g(\ \ )$. Let then

$$\zeta_{d_1 \ldots d_c}(\underset{\sim}{F}) = Eg^2_{d_1 \ldots d_c}(X_{ji},\ 1 \leq i \leq d_j,\ 1 \leq j \leq c) - \theta^2(\underset{\sim}{F}), \tag{2.2}$$

so that $\zeta_{0\ldots0}(\underset{\sim}{F})=0$. We assume that (i)

$$\sigma_j^2 = m_j^2 \zeta_{\delta_{j1} \ldots \delta_{jc}}(\underset{\sim}{F}) > 0 \text{ for every } 1 \leq j \leq c, \tag{2.3}$$

(where $\delta_{ab}=1$ or 0 according as $a=b$ or not), and (ii)

(2.4) $$\zeta_{m_1 \ldots m_c}(\underset{\sim}{F}) < \infty \text{ i.e., } g\varepsilon L^2.$$

Later on, we shall see that (2.3) may be replaced by $\max_{1 \le j \le c} \sigma_j^2 > 0$, and the neccessary modifications are trivial. We know that for $n_j \ge m_j$, $1 \le j \le c$, under (1.3), (2.3) and (2.4),

$$r^2(\underset{\sim}{n}) = V(U(\underset{\sim}{n})) = \prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \sum_{d_1=0}^{m_1} \cdots \sum_{d_c=0}^{m_c} \prod_{j=1}^{c} \{\binom{m_j}{d_j}\binom{n_j-m_j}{m_j-d_j}\} \zeta_{d_1 \ldots d_c}(\underset{\sim}{F})$$

(2.5) $$= \sum_{j=1}^{c} n_j^{-1} \sigma_j^2 + O(n^{-2}), \quad n = n_1 + \ldots + n_c.$$

Let now $E_c = [0,1]^c$ be the c-dimensional unit cube in R^c, $\underset{\sim}{t} = (t_1, \ldots, t_c)$ εE_c, and let $[\underset{\sim}{n}\underset{\sim}{t}] = ([n_1 t_1], \ldots, [n_c t_c])$ where $[s]$ denotes the largest integer $\le s$. Then, for every $\underset{\sim}{n}(\ge \underset{\sim}{m})$, we define a process $W(\underset{\sim}{n}) = \{W(\underset{\sim}{t}; \underset{\sim}{n}) : \underset{\sim}{t} \varepsilon E_c\}$ by letting

(2.6) $$W(\underset{\sim}{t}, \underset{\sim}{n}) = \psi([\underset{\sim}{n}\underset{\sim}{t}]; n)[U([\underset{\sim}{n}\underset{\sim}{t}]) - \theta(\underset{\sim}{F})], \quad [\underset{\sim}{n}\underset{\sim}{t}] \ge \underset{\sim}{m},$$

$$= 0, \text{ otherwise,}$$

where for $\underset{\sim}{k} = (k_1, \ldots, k_c)$, $k_j > 0$, $j = 1, \ldots, c$,

(2.7) $$\psi(\underset{\sim}{k}, n) = n^{-\frac{1}{2}} \left(\sum_{j=1}^{c} \sigma_j \lambda_j^{\frac{1}{2}}\right) \left(\sum_{j=1}^{c} \sigma_j \lambda_j^{\frac{1}{2}} k_j^{-1}\right)^{-1},$$

so that $\psi(\underset{\sim}{k}, n)$ is $n^{-\frac{1}{2}}$ times a harmonic mean of $k_1, \ldots, k_c$. Consider now c independent copies of a standard Brownian motion on [0,1] and denote these by $W_j = \{W_j(t) : 0 \le t \le 1\}$, $j = 1, \ldots, c$. Finally, the space $D^c[0,1]$ of all real functions on E_c with no discontinuities of the second kind and the associated (extended) Skorokhod J_1-topology are defined as in Neuhaus (1971). Then, our first theorem of the paper is the following.

<u>THEOREM 2.1</u>. <u>Under</u> (1.3), (2.3) <u>and</u> (2.4), $W(\underset{\sim}{n})$ <u>converges in law in the extended Skorokhod</u> J_1<u>-topology on</u> $D^c[0,1]$ <u>to a Gaussian function</u> $W = \{W(\underset{\sim}{t}) : \underset{\sim}{t} \varepsilon E_c\}$, <u>where</u>

$$(2.8)\qquad W(\underset{\sim}{t}) = \begin{cases} (\sum_{j=1}^{c}\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{c}\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1})^{-1}[\sum_{j=1}^{c}\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1}W_j(t_j)],\ \underset{\sim}{t}>\underset{\sim}{0}, \\ 0,\text{ with probability 1, if } t_j=0 \text{ for some } j\text{: } 1\le j\le c. \end{cases}$$

We define a process $W^*(\underset{\sim}{n}) = \{W^*(\underset{\sim}{t};\underset{\sim}{n});\ \underset{\sim}{t}\varepsilon E_c\}$ as follows. Considering the tail set $\{U(\underset{\sim}{k}):\ \underset{\sim}{k}\ge\underset{\sim}{n}\}$, let

$$(2.9)\qquad W^*(\underset{\sim}{t};\underset{\sim}{n}) = r^{-1}(\underset{\sim}{n})[U([\underset{\sim}{n}/\underset{\sim}{t}]) - \theta(\underset{\sim}{F})],\ \underset{\sim}{t}\varepsilon E_c,$$

where $[\underset{\sim}{n}/\underset{\sim}{t}] = ([n_1/t_1],\ldots,[n_c/t_c])$. Let then

$$(2.10)\qquad \underset{\sim}{w} = (w_1,\ldots,w_c)';\ w_j = \sigma_j\lambda_j^{-\frac{1}{2}}(\textstyle\sum_{j=1}^{c}\sigma_j^2/\lambda_j)^{-\frac{1}{2}},\ 1\le j\le c;$$

$$(2.11)\qquad \underset{\sim}{W}(\underset{\sim}{t}) = (W_1(t_1),\ldots,W_c(t_c))',\ \underset{\sim}{t}\varepsilon E_c,$$

where the $W_j(t)$ are defined earlier, and let

$$(2.12)\qquad W^* = \{W^*(\underset{\sim}{t}):\ \underset{\sim}{t}\varepsilon E_c\}\ ;\ W^*(\underset{\sim}{t}) = \underset{\sim}{w}'\underset{\sim}{W}(\underset{\sim}{t}),\ \underset{\sim}{t}\varepsilon E_c.$$

Then, our second theorem of the paper is the following.

<u>THEOREM 2.2</u>. <u>Under</u> (1.3), (2.3) <u>and</u> (2.4), <u>as</u> $n\to\infty$,

$$(2.13)\qquad W^*(\underset{\sim}{n}) \overset{\mathcal{D}}{\to} W^*,\ \text{in the Skorokhod } J_1\text{-topology on } D^c[0,1].$$

Theorems 2.1 and 2.2 provide multi-sample extensions of the weak convergence results of Miller and Sen (1972) and Loynes (1970), respectively. Related results on von Mises' (1947) functionals are considered in section 4.

3. Proofs of the theorems. For simplicity of the proofs, we explicitly consider the case of c=2; an essentially same but more laborious proof holds for general $c(\ge 2)$. First, we consider certain basic lemmas needed in the subsequent steps of the proof.

Let $\mathcal{B}_n^{(j)}$ be the σ-field generated by $\{X_{j1},\ldots,X_{jn}\}$ for $n\ge 1$, $1\le j\le c$; $\mathcal{B}_{n_1n_2}^{(12)}$

denote the product σ-field $\mathcal{B}_{n_1}^{(1)} \times \mathcal{B}_{n_2}^{(2)}$ for $n_1 \geq 1$, $n_2 \geq 1$. Further, let $S_{ii'} = S(X_{11},\ldots,X_{1i}, X_{21},\ldots,X_{2i'})$, for $i,i' \geq 1$, be a sequence of random variables such that

$$E(S_{ij} | \mathcal{B}_{i'j}^{(12)}) = S_{i'j} \quad \text{a.e. (almost everywhere)} \tag{3.1}$$

for every $i \geq i' \geq 1$ and $j \geq 1$, and for every $j \geq j' \geq 1$, $n_1 \geq 1$,

$$E(\underset{\sim}{S}_{n_1}^{(j)} | \mathcal{B}_{n_1 j'}^{(12)}) = \underset{\sim}{S}_{n_1}^{(j')} \quad \text{a.e.,} \tag{3.2}$$

where $\underset{\sim}{S}_k^{(j)} = (S_{1j},\ldots,S_{kj})'$ for $k \geq 1$, $j \geq 1$. Finally, assume that for every $i \geq 1$, $j \geq 1$,

$$E(S_{ij}) = 0, \ \sigma_{ij}^2 = E(S_{ij}^2) < \infty. \tag{3.3}$$

<u>LEMMA 3.1</u>. <u>Under</u> (3.1), (3.2) <u>and</u> (3.3), <u>for every</u> $n_1 \geq 1$, $n_2 \geq 1$,

$$E[(\max_{1 \leq i \leq n_1} \max_{1 \leq j \leq n_2} |S_{ij}|)^2] \leq 16\sigma_{n_1 n_2}^2. \tag{3.4}$$

Since Doob's (1953, p.317) inequality holds for non-negative sub-martingales, the proof of the lemma follows precisely on the same line as in the proof of theorem 1 of Wichura (1969), and hence, the details are omitted. The extension of (3.4) for a general $c(\geq 2)$ is immediate; we need to replace 16 by 4^c and $\sigma_{n_1 n_2}^2$ by $E[S^2(X_{11},\ldots,X_{1n_1},\ldots,X_{c1},\ldots,X_{cn_c})]$.

Consider now a two-sample U-statistic $U_{n_1 n_2}$ and denote by $r^2(n_1,n_2) = \mathrm{Var}(U_{n_1 n_2})$; note that by (2.5), $r^2(n_1,n_2)$ is non-increasing in each of n_1 and n_2.

<u>LEMMA 3.2</u>. <u>For every</u> $N_j \geq n_j \geq m_j (\geq 1)$, $j=1,2$,

$$\begin{aligned} &E[(\max_{n_1 \leq k \leq N_1} \max_{n_2 \leq q \leq N_2} r^{-1}(n_1,n_2)|U_{kq} - U_{kN_2} - U_{N_1 q} + U_{N_1 N_2}|)^2] \\ &\leq 16 r^{-2}(n_1,n_2)[r^2(n_1,n_2) - r^2(n_1,N_2) - r^2(N_1,n_2) + r^2(N_1,N_2)] \\ &\leq 16, \ \underline{\text{uniformly in}}\ N_1(\geq n_1)\ \underline{\text{and}}\ N_2(\geq n_2). \end{aligned} \tag{3.5}$$

Proof. For every $r\geq 1$, $s\geq 1$, we let

$$(3.6)\qquad h_{rs}^{(N_1N_2)} = U_{N_1-r+1N_2-s+1}-U_{N_1-rN_2-s+1}-U_{N_1-r+1N_2-s}+U_{N_1-rN_2-s},$$

so that for every $1\leq i\leq N_1-n_1$, $1\leq j\leq N_2-n_2$,

$$(3.7)\qquad S_{ij} = \sum_{r=1}^{i}\sum_{s=1}^{j} h_{rs}^{(N_1N_2)}$$

$$= U_{N_1-i+1\ N_2-j+1}-U_{N_1-i+1N_2}-U_{N_1N_2-j+1}+U_{N_1N_2}.$$

Let, now $C_n^{(j)}$ be the σ-field generated by the unordered collection $\{X_{j1},\ldots,X_{jn}\}$ and by $X_{jn+1},X_{jn+2},\ldots$, $j=1,2$, and $C_{n_1n_2}^{(12)}$ be the product σ-field $C_{n_1}^{(1)}\times C_{n_2}^{(2)}$ for $n_1\geq m_1, n_2\geq m_2$. It follows by some standard arguments that $E(U_{kq}|C_{k'q'}) = U_{k'q'}$ a.e. for every $k\leq k'$ and $q\leq q'$. Consequently, it follows by some routine steps that for every $1\leq i\leq N_1-n_1$, $1\leq j\leq N_2-n_2$,

$$(3.8)\qquad E(S_{ij}|C^*_{i'j}) = S_{i'j}\quad \text{a.e., } i'\leq i,$$

$$(3.9)\qquad E(\underset{\sim}{S}^{(j)}_{N_1-n_1}|C^*_{N_1-n_1,j'}) = \underset{\sim}{S}^{(j')}_{N_1-n_1}\quad \text{a.e., } j\geq j',$$

where $C^*_{kq} = C^{(12)}_{N_1-k\ N_2-q}$, $1\leq k\leq N_1-n_1$, $1\leq q\leq N_2-n_2$, and $\underset{\sim}{S}_k^{(j)} = (S_{1j},\ldots,S_{kj})'$, for $k\geq 1$, $j\geq 1$. Further, by (3.7), $E(S_{ij}) = 0$ for all $i\geq 1$, $j\geq 1$. Finally, C^*_{kq} is $\uparrow$ in k and q. Consequently, the same proof as in Lemma 3.1 holds, and (3.5) follows by noting that by (3.7),

$$E(S^2_{N_1-n_1\ N_2-n_2}) = r^2(n_1,n_2)-r^2(n_1,N_2)-r^2(N_1,n_2+r^2(N_1,N_2)$$

$$(3.10)\qquad \leq r^2(n_1,n_2) \text{ for every } N_1\geq n_1,\ N_2\geq n_2,$$

as $r^2(i,j)$ is $\downarrow$ in i and j. Q.E.D.

We further note that $\{U_{kN_2}, C_{kN_2};\ n_1\leq k\leq N_1\}$ has the reverse martingale property, so that by reversing the index set, we obtain that

$$E[(\max_{n_1\le k\le N_1} r^{-1}(n_1,N_2)|U_{kN_2}-U_{N_1N_2}|)^2]$$
(3.11)
$$\le 4r^{-2}(n_1,N_2)[r^2(n_1,N_2)-r^2(N_1,N_2)] \le 4,$$

uniformly in $N_1\ge n_1$, $N_2\ge n_2$. Similarly,

$$E[(\max_{n_2\le q\le N_2} r^{-1}(N_1,n_2)|U_{N_1q}-U_{N_1N_2}|)^2]$$
(3.12)
$$\le 4r^{-2}(N_1,n_2)[r^2(N_1,n_2)-r^2(N_1,N_2)] \le 4,$$

uniformly in $N_1\ge n_1$, $N_2\ge n_2$. Finally,

(3.13) $$E[|U_{N_1N_2}-\theta(\underset{\sim}{F})|^2] = r^2(N_1,N_2) \le r^2(n_1,n_2),$$

uniformly in $N_1\ge n_1$, $N_2\ge n_2$. Hence, by Lemma 3.2, (3.11), (3.12), (3.13), the c_r- inequality and the Chebycher inequality, we obtain that for every $\varepsilon>0$, there exists a positive $K_\varepsilon(<\infty)$, such that for every $N_1\ge n_1$, $N_2\ge n_2$,

(3.14) $$P\{\max_{n_1\le k\le N_1}\ \max_{n_2\le q\le N_2}|U_{kq}-\theta(\underset{\sim}{F})|>r(n_1,n_2)K_\varepsilon\} < \varepsilon,$$

and hence,

(3.15) $$P\{\sup_{\underset{\sim}{k}\ge\underset{\sim}{n}}|U_{\underset{\sim}{k}}-\theta(\underset{\sim}{F})|>r(\underset{\sim}{n})K_\varepsilon\} < \varepsilon.$$

We now consider a typical decomposition for $U(\underset{\sim}{n})$, defined in (1.2), when c=2; the case of $c\ge 2$ follows trivially on the same line. Let us write $U^*_{00}(\underset{\sim}{n}) = \theta(\underset{\sim}{F})$ and for $\underset{\sim}{k}\ge\underset{\sim}{0}$,

(3.16) $$U^*_{\underset{\sim}{k}}(\underset{\sim}{n}) = \sum_{d_1=0}^{k_1}\sum_{d_2=0}^{k_2} (-1)^{d_1+d_2}\binom{k_1}{d_1}\binom{k_2}{d_2}U_{d_1d_2}(\underset{\sim}{n});\ \underset{\sim}{0}\le\underset{\sim}{k}\le\underset{\sim}{m},$$

where for $0\le d_1\le m_1$ and $0\le d_2\le m_2$, we have

(3.17) $$U_{d_1d_2}(\underset{\sim}{n}) = \prod_{j=1}^{2}\binom{n_j}{d_j}^{-1}\sum\nolimits^*_{(\underset{\sim}{n})} g_{d_1d_2}(X_{j\alpha_{ji}},\ 1\le i\le d_j,\ 1\le j\le 2),$$

where the summation $\sum^*_{(\underset{\sim}{n})}$ extends over all $1\le\alpha_{j1}<\ldots<\alpha_{jd_j}\le m_j$, $j=1,2$. Then, by (1.2), (3.16), (3.17) and a few routine steps, we obtain that

$$U(\underset{\sim}{n}) = \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2} \binom{m_1}{k_1}\binom{m_2}{k_2}U^*_{\underset{\sim}{k}}(\underset{\sim}{n}), \tag{3.18}$$

where each $U^*_{\underset{\sim}{k}}(\underset{\sim}{n})$ $(\underset{\sim}{0}\le\underset{\sim}{k}\le\underset{\sim}{m})$ is a generalized U-statistic. A little readjustments lead us to

$$\begin{aligned} U(\underset{\sim}{n}) = {} & \theta(\underset{\sim}{F})+U_1(n_1)+U_2(n_2) + \\ & \sum_{d_1=1}^{m_1}\sum_{d_2=1}^{m_2}\binom{m_1}{d_1}\binom{m_2}{d_2}U^*_{\underset{\sim}{d}}(\underset{\sim}{n}), \end{aligned} \tag{3.19}$$

where

$$U_1(n_1) = \binom{n_1}{m_1}^{-1}\sum^*_{(n_1)}[g_{m_1 0}(X_{1i_1},\ldots,X_{1i_{m_1}})-\theta(\underset{\sim}{F})], \tag{3.20}$$

$$U_2(n_2) = \binom{n_2}{m_2}^{-1}\sum^*_{(n_2)}[g_{0m_2}(X_{2i_1},\ldots,X_{2i_{m_2}})-\theta(\underset{\sim}{F})] \tag{3.21}$$

and the summation $\sum^*_{(n_j)}$ extends over all $1\le i_1<\ldots<i_{m_j}\le n_j$, $j=1,2$.

We first consider the proof of Theorem 2.2 which is relatively simpler. We write [by(3.19)]

$$\begin{aligned} U^*_3(\underset{\sim}{n}) &= [U(\underset{\sim}{n})-\theta(\underset{\sim}{F})] - U_1(n_1) - U_2(n_2) \\ &= \sum_{d_1=1}^{m_1}\sum_{d_2=1}^{m_2}\binom{m_1}{d_1}\binom{m_2}{d_2}U^*_{\underset{\sim}{d}}(\underset{\sim}{n}), \end{aligned} \tag{3.22}$$

and note that $U^*_3(\underset{\sim}{n})$ is a generalized U-statistic for which

$$\begin{aligned} [r^*(\underset{\sim}{n})]^2 &= E[U^*_3(\underset{\sim}{n})]^2 \\ &= r^2(\underset{\sim}{n}) - E[U_1(n_1)]^2 - E[U_2(n_2)]^2 \\ &= \binom{n_1}{m_1}^{-1}\binom{n_2}{m_2}^{-1} \sum_{d_1=1}^{m_1} \sum_{d_2=1}^{m_2}\binom{m_1}{d_1}\binom{m_2}{d_2}\binom{n_1-m_1}{m_1-d_1}\binom{n_2-m_2}{m_2-d_2}\zeta_{d_1d_2}(\underset{\sim}{F}) \\ &= n_1^{-1}n_2^{-1}m_1^2m_2^2\zeta_{11}(\underset{\sim}{F}) + O(n^{-3}). \end{aligned} \tag{3.23}$$

Thus, from (2.5) and (3.23), we note that under (1.3),

$$r^*(\underset{\sim}{n})/r(\underset{\sim}{n}) = O(n^{-\frac{1}{2}}), \tag{3.24}$$

and hence, using (3.15) for $\{U_3^*(\underset{\sim}{k}),\ \underset{\sim}{k}\geq\underset{\sim}{n}\}$ and (3.24), we conclude that for every $\varepsilon>0$ and $\eta>0$, there exists an n_0 $[=n_0(\varepsilon,\eta)]$, such that for $n\geq n_0$,

$$P\{\sup_{\underset{\sim}{k}\geq\underset{\sim}{n}}|U_3^*(\underset{\sim}{k})| > \eta.r(\underset{\sim}{n})\} < \varepsilon. \tag{3.25}$$

Therefore, under (1.3), (2.3) and (2.4), as $n\to\infty$,

$$\sup_{\underset{\sim}{k}\geq\underset{\sim}{n}} r^{-1}(\underset{\sim}{n})|U(\underset{\sim}{k})-\theta(\underset{\sim}{F})-U_1(k_1)-U_2(k_2)| \xrightarrow{P} 0. \tag{3.26}$$

Let us now define

$$W_j^*(t,n_j) = \begin{cases} \sigma_j^{-1}n_j^{\frac{1}{2}}U_j([n_j/t]), & 0<t\leq 1, \\ 0, & t=0, \text{ for } j=1,2; \end{cases} \tag{3.27}$$

$$w_j(\underset{\sim}{n}) = r^{-1}(\underset{\sim}{n})\sigma_j n_j^{-\frac{1}{2}},\ j=1,2. \tag{3.28}$$

Then, from (2.9), (3.26), (3.27) and (3.28), we obtain that

$$\sup_{\underset{\sim}{t}\in E_2}|W^*(\underset{\sim}{t};\underset{\sim}{n})-\sum_{j=1}^{2} w_j(\underset{\sim}{n})W_j^*(t_j,n_j)| \xrightarrow{P} 0 \text{ as } n\to\infty. \tag{3.29}$$

Now, by (1.3), (2.5), (2.10) and (3.28), we obtain that

$$\lim_{n\to\infty} w_j(\underset{\sim}{n}) = w_j \text{ for } j=1,2. \tag{3.30}$$

Also, $W_j^*(n_j) = \{W_j^*(t,n_j),\ 0\leq t\leq 1\}$, $j=1,2$, are stochastically independent, and by the results of Loynes (1970), as $n\to\infty$,

$$W_j^*(n_j) \xrightarrow{\mathcal{D}} W_j = \{W_j(t):\ 0\leq t\leq 1\},\ j=1,2 \tag{3.31}$$

in the Skorokhod J_1-topology on $D[0,1]$, where W_1 and W_2 are independent copies of a standard Brownian motion on [0,1]. Finally, (3.31) implies that

$\sup_{0\le t\le 1} |W_j^*(t;n_j)|$, j=1,2, are measurable and

(3.32) $$\sup_{0\le t\le 1} |W_j^*(t;n_j)| = O_p(1),\ j=1,2.$$

Consequently, (2.13) follows from (3.29), (3.30), (3.31) and (3.32). Q.E.D.

To prove (2.8) [i.e., Theorem 2.1], we note that by using (3.23) and some standard inequalities among the $\{\zeta_{d_1 d_2}(\underset{\sim}{F}): \underset{\sim}{0}\le \underset{\sim}{d}\le \underset{\sim}{m}\}$,

(3.33) $$r^*(\underset{\sim}{n}) \le C(\underset{\sim}{F})(n_1 n_2)^{-1},\ \forall\, n_1\ge m_1,\ n_2\ge m_2,$$

where $C(\underset{\sim}{F}) < \infty$ (whenever (2.4) holds) and it does not depend on (n_1,n_2). We shall show first that under (1.3), (2.3) and (2.4), as $n\to\infty$,

(3.34) $$\max_{\underset{\sim}{m}<\underset{\sim}{k}<\underset{\sim}{n}} \psi(\underset{\sim}{k},n)|U_3^*(\underset{\sim}{k})| \xrightarrow{P} 0.$$

For this, we partition the set $A(\underset{\sim}{n}) = \{\underset{\sim}{k}: \underset{\sim}{m}\le\underset{\sim}{k}\le\underset{\sim}{n}\}$ into three disjoint subsets:

(3.35) $$A_1(\underset{\sim}{n}) = \{\underset{\sim}{k}\varepsilon A(\underset{\sim}{n}): \min_{1\le j\le 2} k_j \le \varepsilon_1 n^{\frac{1}{2}}\},$$

(3.36) $$A_2(\underset{\sim}{n}) = \{\underset{\sim}{k}\varepsilon A(\underset{\sim}{n}): \varepsilon_1 n^{\frac{1}{2}} < \min_{1\le j\le 2} k_j \le [n^{4/5}]\},$$

(3.37) $$A_3(\underset{\sim}{n}) = \{\underset{\sim}{k}\varepsilon A(\underset{\sim}{n}): \min_{1\le j\le 2} k_j > [n^{4/5}]\},$$

where $\varepsilon_1(>0)$ is arbitrarily small and will be chosen later on. Let then

(3.38) $$\delta = \min_{1\le j\le 2} [\sigma_j\lambda_j^{\frac{1}{2}}/(\sigma_1\lambda_1^{\frac{1}{2}} + \sigma_2\lambda_2^{\frac{1}{2}})],$$

so that by (1.3), $0<\delta<1$. Then, by (2.7) and (3.38),

(3.39) $$\psi(\underset{\sim}{k},n) \le n^{-\frac{1}{2}}\delta^{-1}[\min_{1\le j\le 2} k_j],\ \forall\, \underset{\sim}{m}\le\underset{\sim}{k}\le\underset{\sim}{n}.$$

Consequently, by (3.35) and (3.39),

$$(3.40)\qquad \max_{\underset{\sim}{k}\varepsilon A_1(\underset{\sim}{n})} \psi(\underset{\sim}{k},n)\,|U_3^*(\underset{\sim}{k})|$$

$$\le \delta^{-1}\varepsilon_1\{\max_{\underset{\sim}{k}\varepsilon A_1(\underset{\sim}{n})}|\,U_3^*(\underset{\sim}{k})|\} \le \delta^{-1}\varepsilon_1\{\sup_{\underset{\sim}{k}\ge\underset{\sim}{m}}|U_3^*(\underset{\sim}{k})|\},$$

where by (3.15) and (3.33), we obtain for $\{U_3^*(\underset{\sim}{k}),\ \underset{\sim}{k}\ge\underset{\sim}{m}\}$ that $\sup_{\underset{\sim}{k}\ge\underset{\sim}{m}}|U_3^*(\underset{\sim}{k})| = O_p(1)$, so that (3.40) can be made arbitrarily small (in probability) by letting $\varepsilon_1(>0)$ arbitrarily small.

Again, for $\underset{\sim}{k}\varepsilon A_2(\underset{\sim}{n})$, $r^*([\varepsilon_1 n^{\frac12}],\ [\varepsilon_1 n^{\frac12}]) = O([\varepsilon_1 n^{\frac12}]^{-1})$, by (3.23) and (3.33). On the other hand, by (3.39), $\max_{\underset{\sim}{k}\varepsilon A_2(\underset{\sim}{n})}\psi(\underset{\sim}{k},n) \le n^{-\frac12}\delta^{-1}[n^{4/5}] \le \delta^{-1}n^{3/10}$. Thus, for every $\varepsilon_1>0$,

$$(3.41)\qquad \max_{\underset{\sim}{k}\varepsilon A_2(\underset{\sim}{n})}\psi(\underset{\sim}{k},n)\,.\,r^*([\varepsilon_1 n^{\frac12}],\ [\varepsilon_1 n^{\frac12}])$$

$$\le \delta^{-1}n^{3/10}\{O([\varepsilon_1 n^{\frac12}]^{-1})\}$$

$$\le \delta^{-1}\{O(1/(\varepsilon_1 n^{1/5}))\} \to 0 \text{ as } n\to\infty.$$

On the other hand, by (3.15), (3.23) and (3.33),

$$(3.42)\qquad \max_{\underset{\sim}{k}\varepsilon A_2(\underset{\sim}{n})}[r^*([\varepsilon_1 n^{\frac12}],[\varepsilon_1 n^{\frac12}])]^{-}|U_3^*(\underset{\sim}{k})| = O_p(1).$$

Consequently, by (3.41) and (3.42), we have for $n\to\infty$,

$$(3.43)\qquad \max_{\underset{\sim}{k}\varepsilon A_2(\underset{\sim}{n})}\psi(\underset{\sim}{k},\underset{\sim}{n})\,|U_3^*(\underset{\sim}{k})| = o_p(1).$$

Finally, for $\underset{\sim}{k}\varepsilon A_3(\underset{\sim}{n})$, $\psi(\underset{\sim}{k},n) \le n^{-\frac12}\max\,(k_1,k_2) \le n^{\frac12}$, while $r^*([n^{4/5}],[n^{4/5}]) = O(n^{4/5})$, so that

$$(3.44)\qquad \max_{\underset{\sim}{k}\varepsilon A_3(\underset{\sim}{n})}\psi(\underset{\sim}{k},n)\cdot r^*([n^{4/5}],[n^{4/5}]) = O(n^{-3/10}).$$

On the other hand, by (3.15), (3.23) and (3.33), as $n\to\infty$,

$$(3.45)\qquad \max_{\underset{\sim}{k}\varepsilon A_3(n)}[r^*([n^{4/5}],[n^{4/5}])]^{-1}|U_3^*(\underset{\sim}{k})|$$

$$\leq \sup_{k_j \geq n^{4/5},\ j=1,2} [r^*([n^{4/5}],[n^{4/5}])]^{-1} |U_3^*(\underset{\sim}{k})| = O_p(1).$$

Thus, from (3.44) and (3.45), we have for $n\to\infty$,

$$\max_{\underset{\sim}{k}\in A_3(\underset{\sim}{n})} \psi(\underset{\sim}{k},n)|U_3^*(\underset{\sim}{k})| = O_p(n^{-3/10}) = o_p(1), \tag{3.46}$$

and, as a result, (3.34) holds.

On D[0,1], we now consider independent processes $W_j(n_j)=\{W_j(t,n_j):\ 0\leq t\leq 1\}$, $j=1,2$, where for $0\leq t\leq 1$,

$$W_j(t,n_j) = \begin{cases} ([n_jt]/\sigma_j n_j^{\frac{1}{2}})U_j([n_jt]), & m_j\leq[n_jt]\leq n_j, \\ 0, & t \leq (m_j-1)/n_j;\ j=1,2. \end{cases} \tag{3.47}$$

Also, let for every $\underset{\sim}{t}>\underset{\sim}{0}$,

$$\begin{aligned} w_j(\underset{\sim}{t},\underset{\sim}{n}) &= \psi([\underset{\sim}{n}\underset{\sim}{t}];n)\{\sigma_j n_j^{\frac{1}{2}}/[n_jt_j]\} \\ &= n^{-\frac{1}{2}}(\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}}/[n_jt_j])^{-1}(\sigma_j n_j^{\frac{1}{2}}/[n_jt_j]) \\ &= (\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}}/[n_jt_j])^{-1}(\lambda_j^{\frac{1}{2}}\sigma_j/[n_jt_j])\ [1+o(1)]. \end{aligned} \tag{3.48}$$

Then, for every $n_jt_j\geq 1$, $1\leq j\leq 2$, $w_j(\underset{\sim}{t},\underset{\sim}{n})$ is positive and is bounded above from $(\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}})(n_j/n\lambda_j)^{\frac{1}{2}} = (\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}})[1+o(1)]$, $j=1,2$. Also, for every $\underset{\sim}{t}>\underset{\sim}{0}$,

$$\lim_{n\to\infty} w_j(\underset{\sim}{t},\underset{\sim}{n}) = (\sum_{j=1}^{2}\sigma_j\lambda_j^{\frac{1}{2}})(\sum_{j=1}^{2}\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1})^{-1}(\sigma_j\lambda_j^{-\frac{1}{2}}t_j^{-1}),\ j=1,2. \tag{3.49}$$

Finally, it follows from Miller and Sen (1972) that $W_1(n_1)$ and $W_2(n_2)$ are stochastically independent, each converging in law to a standard Brownian motion. Consequently, for every $\varepsilon>0$ and $\eta>0$, there exist a $\delta>0$ and an n_0, such that for $n\geq n_0$,

$$P\{\sup_{0\leq t\leq\delta}|W_j(t;n_j)| > \varepsilon\} < \tfrac{1}{2}\eta,\quad j=1,2, \tag{3.50}$$

$$\sup_{0\leq t\leq 1}|W_j(t;n_j)| = O_p(1),\ j=1,2. \tag{3.51}$$

Thus, (2.8) follows from (2.6), (2.7), (3.22), (3.34), (3.47), (3.48), (3.49), (3.50) and (3.51). Q.E.D.

4. *Weak convergence of von Mises' functionals.* We define the empirical distributions by

$$F_j(x;n_j) = n_j^{-1} \sum_{i=1}^{n_j} c(x-X_{ji}),\ x \in R^p,\ n_j \geq 1,\ j=1,\ldots,c, \tag{4.1}$$

where $c(u)=1$ or 0 according as all the p components of u are non-negative or at least one negative; we let $\underset{\sim}{F}(.,\underset{\sim}{n}) = (F_1(.,n_1),\ldots F_c(.,n_c))$. Then, the von Mises (1947) functional corresponding to (1.1) is

$$\theta(\underset{\sim}{F}(.,\underset{\sim}{n})) = \int\cdots\int_{R^{pm_o}} g(x_{11},\ldots,x_{cm_c}) \prod_{j=1}^{c} \prod_{i=1}^{m_j} dF_j(x_{ji};n_j). \tag{4.2}$$

Here, we define a process $\tilde{W}(\underset{\sim}{n}) = \{\tilde{W}(\underset{\sim}{t};\underset{\sim}{n});\ \underset{\sim}{t} \in E_c\}$ as in (2.6), where we replace $\{U(\underset{\sim}{k});\ \underset{\sim}{k} \in A(\underset{\sim}{n})\}$ by $\{\theta(\underset{\sim}{F}(.,\underset{\sim}{k}));\ \underset{\sim}{k} \in A(\underset{\sim}{n})\}$. Similarly, on replacing $\{U(\underset{\sim}{k});\ \underset{\sim}{k} \geq \underset{\sim}{n}\}$ by $\{\theta(\underset{\sim}{F}(.,\underset{\sim}{k});\ \underset{\sim}{k} \geq \underset{\sim}{n}\}$ in (2.9), we define $\tilde{W}^*(\underset{\sim}{n}) = \{\tilde{W}^*(\underset{\sim}{t};\underset{\sim}{n});\ \underset{\sim}{t} \in E_c\}$. Finally, we strengthen (2.4) to

$$\zeta^*(\underset{\sim}{F}) = \max_{1 \leq j \leq c}\ \max_{1 \leq \alpha_{j1} \leq \cdots \leq \alpha_{jm_j} \leq m_j} E\{g^2(X_{1\alpha_{11}},\ldots,X_{c\alpha_{cm_c}})\} < \infty. \tag{4.3}$$

THEOREM 4.1. Under (1.3), (2.3) and (4.3), as $n \to \infty$, $\tilde{W}(\underset{\sim}{n})$ and $\tilde{W}^*(\underset{\sim}{n})$ converge in law in the extended Skorokhod J_1-topology on $D^c[0,1]$ to W and W^*, respectively, which are defined in (2.8) and (2.12).

Proof. Again, we consider the case of $c=2$, and for $\underset{\sim}{0} \leq \underset{\sim}{d} \leq \underset{\sim}{m}$, define

$$V_{\underset{\sim}{d}}^*(\underset{\sim}{n}) = \int\cdots\int_{R^{pd_o}} g_{\underset{\sim}{d}}(x_{11},\ldots,x_{1d_1},x_{21},\ldots,x_{2d_2}) \prod_{j=1}^{2} \prod_{i=1}^{d_j} d[F_j(x_{ji},n_j)-F_j(x_{ji})], \tag{4.4}$$

where $d_o = d_1+d_2$. Then, we may write

$$\theta(\underset{\sim}{F}(.,\underset{\sim}{n})) = \sum_{\underset{\sim}{d}=\underset{\sim}{0}}^{\underset{\sim}{m}} \binom{m_1}{d_1}\binom{m_2}{d_2} V_{\underset{\sim}{d}}(\underset{\sim}{n}) \tag{4.5}$$

$$= \theta(\underset{\sim}{F}) + V_1(n_1) + V_2(n_2) + \sum_{\underset{\sim}{d}=\underset{\sim}{1}}^{\underset{\sim}{m}} \binom{m_1}{d_1}\binom{m_2}{d_2} V^*_{\underset{\sim}{d}}(\underset{\sim}{n}),$$

where

$$(4.6) \qquad V_1(n_1) = \int_{R^{pm_1}}\!\!\cdots\int g_{m_1 0}(x_1,\ldots,x_{m_1}) \prod_{i=1}^{m_1} dF_1(x_i,n_1) - \theta(\underset{\sim}{F});$$

$$(4.7) \qquad V_2(n_2) = \int_{R^{pm_2}}\!\!\cdots\int g_{0 m_2}(x_1,\ldots,x_{m_2}) \prod_{i=1}^{m_2} dF_2(x_i,n_2) - \theta(\underset{\sim}{F}).$$

Now, proceeding as in the proof of Lemma 2.6 of Miller and Sen (1972), we have for every $k \geq m_j$,

$$(4.8) \qquad E[V_j(k) - U_j(k)]^2 \leq C(\underset{\sim}{F}) k^{-3}, \text{ for } j=1,2,$$

where, under (4.3), $C(\underset{\sim}{F}) < \infty$, and $U_j(n_j)$, $j=1,2$, are defined in (3.20) and (3.21). Since [by (3.39)], $\psi(\underset{\sim}{k},n) \leq n^{-\frac{1}{2}}\delta^{-1}k_j$, $j=1,2$, by (4.8),

$$(4.9) \qquad \begin{aligned} &P\{\max_{\underset{\sim}{m} \leq \underset{\sim}{k} \leq \underset{\sim}{n}} \psi(\underset{\sim}{k},n) |U_j(k_j) - V_j(k_j)| > \varepsilon\} \\ &\leq \sum_{k=m_j}^{n_j} n^{-1}\delta^{-2}k^2 E[U_j(k) - V_j(\underset{\sim}{k})]^2 \varepsilon^{-2} \\ &\leq C(\underset{\sim}{F}) n^{-1}\delta^{-2}\varepsilon^{-2} \sum_{k=m_j}^{n_j} k^{-1} \\ &= C(\underset{\sim}{F})(\delta\varepsilon)^{-2} \cdot O(n^{-1}\log n), \; j=1,2 \end{aligned}$$

so that for every $\varepsilon > 0$, the right hand side of (4.9) converges to 0 as $n \to \infty$. Similarly, on noting that $r^2(\underset{\sim}{n})$, defined by (2.5) is bounded below by $\min_{1 \leq j \leq c} (\sigma_j^2/n_j)$, we obtain that as $n \to \infty$, under (1.3), for every $\varepsilon > 0$, $j=1,2$,

$$(4.10) \qquad \begin{aligned} &P\{\max_{\underset{\sim}{k} \geq \underset{\sim}{n}} r^{-1}(\underset{\sim}{n}) |U_j(k_j) - V_j(k_j)| > \varepsilon\} \\ &\leq \varepsilon^{-2} r^{-2}(\underset{\sim}{n}) \sum_{k=n_j}^{\infty} E\{[U_j(k) - V_j(k)]^2\} \\ &\leq \varepsilon^{-2} r^{-2}(\underset{\sim}{n}) C(\underset{\sim}{F}) \cdot \sum_{k=n_j}^{\infty} k^{-3} \\ &= C(\underset{\sim}{F}) \varepsilon^{-2} [O(n^{-1})] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Consequently, to prove the theorem, all we need to show is that under (1.3), (2.3) and (4.3), for every $\underset{\sim}{d} \geq \underset{\sim}{1}$,

$$P\{\max_{\underset{\sim}{m} \leq \underset{\sim}{k} \leq \underset{\sim}{n}} \psi(\underset{\sim}{k},n)|U^*_{\underset{\sim}{d}}(\underset{\sim}{k}) - V^*_{\underset{\sim}{d}}(\underset{\sim}{k})| > \varepsilon\} < \eta, \tag{4.11}$$

$$P\{\max_{\underset{\sim}{k} \geq \underset{\sim}{n}} r^{-1}(\underset{\sim}{n})|U^*_{\underset{\sim}{d}}(\underset{\sim}{k}) - V^*_{\underset{\sim}{d}}(\underset{\sim}{k})| > \varepsilon\} < \eta, \tag{4.12}$$

where both $\varepsilon(>0)$ and $\eta(>0)$ are arbitrarily small, and n is chosen adequately large. Note that $V^*_{11}(\underset{\sim}{k}) = U^*_{11}(\underset{\sim}{k})$ for all $\underset{\sim}{k} \geq \underset{\sim}{m}$. Also, by extending the proof of Lemma 2.6 of Miller and Sen (1972), we have for every $\underset{\sim}{d} \geq \underset{\sim}{1}$[under (4.3)],

$$E\{[U^*_{\underset{\sim}{d}}(\underset{\sim}{k}) - V^*_{\underset{\sim}{d}}(\underset{\sim}{k})]^2\} \leq C(\underset{\sim}{F}) k_1^{-d_1} k_2^{-d_2}\{k_1^{-2} + k_2^{-2}\}, \tag{4.13}$$

for $\underset{\sim}{k} \geq \underset{\sim}{m}$. Consequently, if $d_1 \geq 2$, $d_2 \geq 2$, the proof for (4.11) and (4.12) follow trivially by using (4.13), the Chebychev and the Bonferroni inequalities. Thus, we need to consider the case where $\min_{1 \leq j \leq 2} d_j = 1$, but $d_1 + d_2 \geq 3$. We consider explicitly the case of $\underset{\sim}{d} = (1,2)$; the case of (2,1) follows similarly, while for any other $\underset{\sim}{d}$: $1 = \min(d_1, d_2) < \max(d_1, d_2) (\geq 2)$, a similar but more laborious proof holds.

By direct evaluation from (3.16) and (4.4), we have

$$V^*_{12}(\underset{\sim}{k}) - U^*_{12}(\underset{\sim}{k}) = \frac{1}{k_2} U^*_{12}(\underset{\sim}{k}) + \frac{1}{k_2 - 1} U^{**}_{12}(\underset{\sim}{k}), \tag{4.14}$$

for every $\underset{\sim}{k} \geq \underset{\sim}{m}$, where for $m_2 \geq 2$,

$$U^{**}_{12}(\underset{\sim}{k}) = \frac{1}{k_1 k_2} \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \int \cdots \int_{R^{3p}} g_{12}(x_1, y_1, y_2) d[c(x_1 - X_{1i_1}) - F_1(x_1)] \cdot$$
$$d[c(y_1 - X_{2i_2}) - F_2(y_1)] d[c(y_2 - X_{2i_2}) - F_2(y_2)]. \tag{4.15}$$

Consequently, to prove (4.11) and (4.12) for $\underset{\sim}{d} = (1,2)$, it suffices to show that as $n \to \infty$,

$$\max_{\underset{\sim}{m} \leq \underset{\sim}{k} \leq \underset{\sim}{n}} |U^*_{12}(\underset{\sim}{k})| = o_p(1), \quad \max_{\underset{\sim}{m} \leq \underset{\sim}{k} \leq \underset{\sim}{n}} |U^{**}_{12}(\underset{\sim}{k})| = o_p(1); \tag{4.16}$$

$$\text{(4.17)} \qquad \sup_{\underset{\sim}{k} \geq \underset{\sim}{n}} |U_{12}^{*}(\underset{\sim}{k})| = o_p(1), \quad \sup_{\underset{\sim}{k} \geq \underset{\sim}{n}} |U_{12}^{**}(\underset{\sim}{k})| = o_p(1);$$

note that $\psi(\underset{\sim}{k},n)/k_2 = O(n^{-\frac{1}{2}}), \forall \underset{\sim}{k} \leq \underset{\sim}{n}$ and $r^{-1}(\underset{\sim}{n})k_2^{-1} \to 0$ as $n \to \infty$, for every $\underset{\sim}{k} \geq \underset{\sim}{n}$. Since $U_{12}^{*}(\underset{\sim}{k})$ is a generalized U-statistic, the proof of (4.16) and (4.17) for $\{U_{12}^{*}(\underset{\sim}{k})\}$ follows directly from (3.15) and the fact that as in (3.33), $E[U_{12}^{*}(\underset{\sim}{k})]^2 \leq C(\underset{\sim}{F})k_1^{-1}k_2^{-2}$ for every $\underset{\sim}{k} \geq \underset{\sim}{m}$.

Now, we define the sequence of σ-fields $\{C_{kq},\ k \geq m_1,\ q \geq m_2\}$ as in section 3 [following (3.7)]. Then it follows from (4.15) that for every $k' \geq k$ and $q' \geq q$,

$$\text{(4.18)} \qquad E[U_{12}^{**}(k,q)\,|\,C_{k'q'}] = U_{12}^{**}(k',q') \text{ a.e.}$$

Consequently, following the same method of approach as in the proof of Lemma 3.2, we obtain from Lemma 3.1 that for every $\underset{\sim}{N} \geq \underset{\sim}{n} \geq \underset{\sim}{m}$,

$$\text{(4.19)} \qquad E[(\max_{1 \leq j \leq 2}\ \max_{n_j \leq k_j \leq N_j} |U_{12}^{**}(k_1,k_2)|)^2]$$

$$\leq 16E\{E[U_{12}^{**}(\underset{\sim}{n})]^2 - E[U_{12}^{**}(n_1,N_2)]^2 - E[U_{12}^{**}(N_1 n_2)]^2 + E[U_{12}^{**}(\underset{\sim}{N})]^2\}$$

$$\leq 16E[U_{12}^{**}(\underset{\sim}{n})]^2 \leq 16C(\underset{\sim}{F})n_1^{-1}n_2^{-2}.$$

Thus, the proof of (4.16) and (4.17) for $\{U_{12}^{**}(\underset{\sim}{k})\}$ follows immediately from (4.19) and the Chebychev inequality. Q.E.D.

5. Weak convergence for random indices. We now consider the case where in (1.3) we allow $(\lambda_1,\ldots,\lambda_c)' = \underset{\sim}{\lambda}$ to be a stochastic vector with positive elements. More precisely, let $\{\underset{\sim}{N}_n = (N_n^{(1)},\ldots,N_n^{(c)})',\ n \geq 1\}$ be a sequence of vectors with positive integer valued random variables, such that

$$\text{(5.1)} \qquad n^{-1}\underset{\sim}{N}_n \overset{P}{\to} \lambda = (\lambda_1,\ldots,\lambda_c)', \text{ as } n \to \infty,$$

where λ_j, $j=1,\ldots,c$ are positive random variables defined on the same probability space as of the original $\{X_{ji},\ i \geq 1\}$, $j=1,\ldots,c$.

We define $W(\mathbf{N}_n)$ as in (2.6) when $\mathbf{N}_n \geq \mathbf{1}$; otherwise, we let $W(\mathbf{N}_n) = 0$. Similarly, we define $W^*(\mathbf{N}_n)$ as in (2.9) when $\mathbf{N}_n \geq \mathbf{1}$; otherwise, we let $W^*(\mathbf{N}_n) = 0$. Finally, we define $\tilde{W}(\mathbf{N}_n)$ and $\tilde{W}^*(\mathbf{N}_n)$ as in section 4, with $\mathbf{n}$ being replaced by $\mathbf{N}_n$. Our basic problem is to study the weak convergence of $W(\mathbf{N}_n)$, $W^*(\mathbf{N}_n)$, $\tilde{W}(\mathbf{N}_n)$ and $\tilde{W}^*(\mathbf{N}_n)$, when $n\to\infty$ and (5.1) holds.

THEOREM 5.1. *Under* (2.3), (2.4) *and* (5.1), $W(\mathbf{N}_n)$ *and* $W^*(\mathbf{N}_n)$ *converge in law in the extended Skorokhod* J_1*-topology on* $D^c[0,1]$ *to* W *and* W^*, *respectively, defined in* (2.8) *and* (2.12), *while under* (2.3), (4.3) *and* (5.1), $\tilde{W}(\mathbf{N}_n)$ *and* $\tilde{W}^*(\mathbf{N}_n)$ *weakly converge to* W *and* W^* *respectively*.

Proof. We shall only consider the case of $W(\mathbf{N}_n)$ as the other cases follow similarly. For $\mathbf{k}(\geq\mathbf{1})$, let $\{\mathbf{t}_{\mathbf{q}}=(t_{q_1}^{(1)},\ldots,t_{q_c}^{(c)})',\ \mathbf{0} \leq \mathbf{q} \leq \mathbf{k}\}$ be the set of points where $0 = t_0^{(j)}<t_1^{(j)}<\ldots<t_{k_j}^{(j)}\leq 1$, $1\leq j\leq c$. Then, by Theorem 2.1, we have for every $\mathbf{k}\geq\mathbf{1}$ and $\mathbf{t}_{\mathbf{q}}$, $\mathbf{0}\leq\mathbf{q}\leq\mathbf{k}$, as $n\to\infty$, under (1.3), (2.3) and (2.4),

$$[W(\mathbf{t}_{\mathbf{q}};\mathbf{n}):\ \mathbf{0}\leq\mathbf{q}\leq\mathbf{k}] \xrightarrow{\mathcal{D}} [W(\mathbf{t}_{\mathbf{q}}):\ \mathbf{0}\leq\mathbf{q}\leq\mathbf{k}]. \tag{5.2}$$

Also, by Lemma 3.2, (3.11) and (3.12), it follows that for every positive ε and η, there exists a $\delta>0$, such that under (1.3), as $n\to\infty$,

$$P\{\max_{\mathbf{N}:\ |\mathbf{N}-\mathbf{n}|<\delta n} r^{-1}(\mathbf{n})\,|U(\mathbf{N})-U(\mathbf{n})|>\varepsilon\} < \eta, \tag{5.3}$$

where $|\mathbf{x}|$ denotes $\max_{1\leq j\leq c}|x_j|$. In fact, (5.3) readily extends to the set of $\{r^{-1}(\mathbf{n})\,|U([\mathbf{N}\mathbf{t}_{\mathbf{q}}])-U([\mathbf{n}\mathbf{t}_{\mathbf{q}}])|:\ |\mathbf{N}-\mathbf{n}|<\delta n,\ \mathbf{0}\leq\mathbf{q}\leq\mathbf{k}\}$. Thus, by (5.2), (5.3) and theorem 2 of Mogyorodi (1967), we conclude that under (5.1), (2.3) and (2.4), as $n\to\infty$,

$$[W(\mathbf{t}_{\mathbf{q}};\mathbf{N}_n):\ \mathbf{0}\leq\mathbf{q}\leq\mathbf{k}] \xrightarrow{\mathcal{D}} [W(\mathbf{t}_{\mathbf{q}}):\ \mathbf{0}\leq\mathbf{q}\leq\mathbf{k}], \tag{5.4}$$

for every $\underset{\sim}{k} \geq \underset{\sim}{1}$ and arbitrary $\{\underset{\sim}{t}_q : \underset{\sim}{0} \leq \underset{\sim}{q} \leq \underset{\sim}{k}\} \subset [0,1]^c$. Hence, the convergence of finite-dimensional laws is established.

To establish the relative compactness (or tightness) of $W(\underset{\sim}{N}_n)$, we define the uniform topology

$$(5.5) \qquad \rho(x,y) = \sup_{\underset{\sim}{t} \in E_c} |x(\underset{\sim}{t}) - y(\underset{\sim}{t})|,$$

and the modulus of continuity (where $\delta > 0$)

$$(5.6) \qquad \omega_\delta(x) = \sup\{|x(\underset{\sim}{t}) - x(\underset{\sim}{s})| : \underset{\sim}{t}, \underset{\sim}{s} \in E_c,\ |\underset{\sim}{t} - \underset{\sim}{s}| < \delta\}.$$

Now, using our Lemma 3.2, (3.11) and (3.12) in place of Theorem 1 of Wichura (1969) and thereby extending his (2a) to our statistics, we obtain by the same technique as in his proof of Theorem 3 (on pp. 686-687) that as $n \to \infty$, under (1.3), (2.3) and (2.4), for every $\varepsilon > 0$,

$$(5.7) \qquad \lim_{\delta \downarrow 0} \limsup_n P\{\omega_\delta(W(\underset{\sim}{n})) > \varepsilon\} = 0.$$

Consider now a sequence of generalized U-statistics

$$(5.8) \quad U'([\underset{\sim}{n}\underset{\sim}{t}]) = \prod_{j=1}^{c} \binom{n_j}{m_j}^{-1} \sum\nolimits^{**}_{(\underset{\sim}{n}\underset{\sim}{t})} q^*(X_{j\alpha_{ji}},\ 1 \leq i \leq m_j,\ 1 \leq j \leq c),\ t_j \geq n_j^{-1} k_n^{(j)},\ 1 \leq j \leq c;$$

$$= 0, \text{ if } t_j < k_n^{(j)}/n_j \text{ for some } 1 \leq j \leq c,$$

where $g^*(x_{11}, \ldots, x_{cm_c}) = g(x_{11}, \ldots, x_{cm_c}) - \theta(\underset{\sim}{F})$, the summation $\sum^{**}_{(\underset{\sim}{n}\underset{\sim}{t})}$ extends over all $k_n^{(j)} + 1 \leq \alpha_{j1} < \ldots < \alpha_{jm_j} \leq [n_j t_j]$, $1 \leq j \leq c$, and

$$(5.9)\qquad \lim_{n\to\infty} k_n^{(j)} = \infty \text{ but } \lim_{n\to\infty} n^{-\frac{1}{2}} k_n^{(j)} = 0,\ 1\le j\le c.$$

Then, on replacing $U([\underset{\sim}{n}\underset{\sim}{t}])-\theta(\underset{\sim}{F})$ in (2.6) by $U'([\underset{\sim}{n}\underset{\sim}{t}])$, $\underset{\sim}{t}\varepsilon\underset{\sim}{E}_c$, we define a parallel process $W'(\underset{\sim}{n}) = \{W'(\underset{\sim}{t};\underset{\sim}{n}): \underset{\sim}{t}\varepsilon E_c\}$.

LEMMA 5.2. *Under* (1.3), (2.3) *and* (2.4), *as* $n\to\infty$,

$$(5.10)\qquad \rho(W(\underset{\sim}{n}),W'(\underset{\sim}{n})) \to 0, \text{ almost surely}.$$

Proof. As before, we consider only the case of c=2. For $\underset{\sim}{0}\le\underset{\sim}{\ell}\le\underset{\sim}{m}\le\underset{\sim}{k}_n$, $\underset{\sim}{k}-\underset{\sim}{k}_n\ge\underset{\sim}{m}-\underset{\sim}{\ell}$, we define

$$(5.11)\quad U_{\underset{\sim}{\ell}}(\underset{\sim}{k},\underset{\sim}{k}_n) = \Big\{\prod_{j=1}^{2}\binom{k_n^{(j)}}{\ell_j}\binom{k_j-k_n^{(j)}}{m_j-\ell_j}\Big\}^{-1}\hat{\sum}^{*}_{(\underset{\sim}{k})} g^*(X_{j\alpha_{ji}},\ 1\le i\le m_j,\ 1\le j\le 2),$$

where the summation $\hat{\sum}^{*}_{(\underset{\sim}{k})}$ extends over all $1\le\alpha_{j1}<\ldots<\alpha_{j\ell_j}\le k_n^{(j)}<\alpha_{ji_{\ell_j+1}}<\ldots\le\alpha_{ji_{m_j}}\le k_j$, $1\le j\le 2$. Thus $U_{\underset{\sim}{\ell}}(\underset{\sim}{k},\underset{\sim}{k}_n)$ may be interpreted as a generalized U-statistic based on four samples of sizes $(k_n^{(1)},k_1-k_n^{(1)},k_n^{(2)},k_2-k_n^{(2)})$. Consequently, as in our Lemma 3.2, (3.11), (3.12), (3.13), (3.14) and (3.15), it can be shown that for every $\varepsilon>0$, there exist a positive $K_\varepsilon(<\infty)$ and an $n_o(\varepsilon)$, such that for $n\ge n_o(\varepsilon)$, under (1.3) and (2.4),

$$(5.12)\qquad P\Big\{\max_{\underset{\sim}{0}\le\underset{\sim}{\ell}\le\underset{\sim}{m}}\ \sup_{\underset{\sim}{k}>\underset{\sim}{k}_n}|U_{\underset{\sim}{\ell}}(\underset{\sim}{k},\underset{\sim}{k}_n)| > K_\varepsilon\Big\} < \varepsilon.$$

Also, by (1.2), (5.8) and (5.11), we obtain that for $[\underset{\sim}{n}\underset{\sim}{t}] \ge \underset{\sim}{k}_n$,

$$(5.13)\ U([\underset{\sim}{n}\underset{\sim}{t}])-\theta(\underset{\sim}{F}) = \sum_{\underset{\sim}{\ell}=\underset{\sim}{0}}^{\underset{\sim}{m}}\ \prod_{j=1}^{2}\Big\{\binom{m_j}{\ell_j}\binom{k_j-k_n^{(j)}}{m_j-\ell_j}\binom{k_n^{(j)}}{\ell_j}\binom{k_j}{m_j}^{-1}\Big\}U_{\underset{\sim}{\ell}}([\underset{\sim}{n}\underset{\sim}{t}],\underset{\sim}{k}_n),$$

where we let $k_j=[n_jt_j]$, j=1,2. Consequently, by (5.12) and (5.13), as $n\to\infty$,

$$(5.14)\qquad \sup_n \max_{\underset{\sim}{k}_n<\underset{\sim}{k}<\underset{\sim}{n}} \{n^{-\frac{1}{2}}[\min(k_1,k_2)]\,|U(\underset{\sim}{k})-U'(\underset{\sim}{k})|\}$$

$$\leq \{\sup_n \max_{\underset{\sim}{k}_n<\underset{\sim}{k}<\underset{\sim}{n}} \max_{\underset{\sim}{0}<\underset{\sim}{\ell}<\underset{\sim}{m}} |U_{\underset{\sim}{\ell}}(\underset{\sim}{k},\underset{\sim}{k}_n)|\}\{O([n^{-1}(k_n^{(1)})^2],[n^{-1}(k_n^{(2)})^2])\}$$

$$= O_p((n^{-\frac{1}{2}}k_n^{(1)})^2,\ (n^{-\frac{1}{2}}k_n^{(2)})^2) = o_p(1), \text{ by (5.9)}.$$

Thus, by (2.6), (2.7), (3.39), (5.5) and (5.14), as $n\to\infty$,

$$(5.15)\qquad \sup_{n^{-1}\underset{\sim}{k}_n<\underset{\sim}{t}<\underset{\sim}{1}} |W(\underset{\sim}{t};\underset{\sim}{n}) - W'(\underset{\sim}{t};\underset{\sim}{n})| \to 0, \text{ almost surely.}$$

On the other hand, for $\underset{\sim}{t}\varepsilon E_c^{(n)} = E_c-\{\underset{\sim}{t}:\ n^{-1}\underset{\sim}{k}_n<\underset{\sim}{t}<\underset{\sim}{1}\}$, $W'(\underset{\sim}{t};\underset{\sim}{n})=0$ [by (5.8)], so that by (2.6), (2.7) and (3.39), as $n\to\infty$,

$$(5.16)\qquad \sup_{\underset{\sim}{t}\varepsilon E_c^{(n)}} |W(\underset{\sim}{t};\underset{\sim}{n})-W'(\underset{\sim}{t};\underset{\sim}{n})|$$

$$= \sup_{\underset{\sim}{t}\varepsilon E_c^{(n)}} |W(\underset{\sim}{t};\underset{\sim}{n})|$$

$$\leq \{\max_{\underset{\sim}{m}<\underset{\sim}{k}<\underset{\sim}{n}} |U(\underset{\sim}{k})-\theta(\underset{\sim}{F})|\}\{\sup_{\underset{\sim}{t}\varepsilon E_c^{(n)}} \psi([\underset{\sim}{n}\underset{\sim}{t}],n)\}$$

$$\leq \{\sup_{\underset{\sim}{k}\geq\underset{\sim}{m}} |U(\underset{\sim}{k})-\theta(\underset{\sim}{F})|\}\{O([n^{-\frac{1}{2}}k_n^{(1)}]^2,[n^{-\frac{1}{2}}k_n^{(2)}]^2)\}$$

$$= O(1)\cdot o(1) = o(1), \text{ almost surely, by (5.9) and (5.12).}$$

Hence, (5.10) follows from (5.15) and (5.16). Q.E.D.

We recall that the random vector $\underset{\sim}{\lambda}$ in (5.1) and the X_{ji}, $j\geq 1$, $i\geq 1$, are all defined on a common probability space, say, $(\Omega,\mathcal{A},P)$.

LEMMA 5.3. *If $A\varepsilon\mathcal{A}$, then for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta(>0)$, sufficiently small, such that*

(5.17) $$\limsup_n P\{\omega_\delta(W'(\underset{\sim}{n})) > \varepsilon | A\} < \eta.$$

Proof. By (5.7) and Lemma 5.2, for every $\varepsilon>0$,

(5.18) $$\lim_{\delta\downarrow 0} \limsup_n P\{\omega_\delta(W'(\underset{\sim}{n})) > \varepsilon\} < \eta.$$

Also, by definition in (5.8), $W'(\underset{\sim}{n})$ depends only on the set $\{X_{ji}, k_n^{(j)}<i\leq n_j, j=1,\ldots,c\}$. Hence, using (5.9), Rényi's (1958) idea of mixing sequence of sets and proceeding as in Lemma 3 of Blum, Hanson and Rosenblatt (1963), it follows that (5.18) implies (5.17). Q.E.D.

We return now to the proof of Theorem 5.1. By virtue of (5.1), for every $\delta'>0$,

(5.19) $$P\{|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}| > \delta'\} \to 0 \text{ as } n\to\infty.$$

Also, $\underset{\sim}{\lambda}$ is a vector of positive random variables, so that for every $\eta>0$, there exists an $a_o(\eta)$, such that

(5.20) $$P\{\min_{1\leq j\leq c} \lambda_j \leq a_o(\eta)\} \leq \frac{1}{3}\eta.$$

Then, for every $\varepsilon>0$,

(5.21) $$\begin{aligned} &P\{\rho(W(\underset{\sim}{N}_n),W'(\underset{\sim}{N}_n))>\varepsilon\} \\ &\leq P\{|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}|>\delta'\}+P\{\rho(W(\underset{\sim}{N}_n),W'(\underset{\sim}{N}_n))>\varepsilon,\ |n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}|\leq\delta'\} \\ &\leq P\{|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}|>\delta'\} + P\{\min_{1\leq j\leq c} \lambda_j\leq a_o(\eta)\} + \\ &\quad P\{\rho(W(\underset{\sim}{N}_n),W'(\underset{\sim}{N}_n))>\varepsilon,\ |n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}|\leq\delta',\ \min_{1\leq j\leq c} \lambda_j>a_o(\eta)\}. \end{aligned}$$

Thus, if we let $0<\delta'<\frac{1}{2}\, a_o(\eta)$, then by (5.19), (5.20) and Lemma 5.2, we conclude that (5.21) can be made arbitrarily small by choosing $\eta(>0)$ small and letting $n\to\infty$. Hence, to establish the relative compactness of $\{W(\underset{\sim}{N}_n)\}$, it suffices to show that

$$\limsup_n P\{\omega_\delta(W'(\underset{\sim}{N}_n)) > \varepsilon\} \to 0 \text{ as } \delta\downarrow 0, \tag{5.22}$$

for every $\varepsilon>0$. We note that by (1.2), (2.6), (2.7), (5.8), (5.13) and (5.18), for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o(\varepsilon,\eta)$, such that for $n \geq n_o(\varepsilon,\eta)$,

$$P\{\max_{\underset{\sim}{k}:\ |\underset{\sim}{k}-\underset{\sim}{n}|<\delta n} \rho(W'(\underset{\sim}{k}),W'(\underset{\sim}{n})) > \varepsilon\} < \eta. \tag{5.23}$$

Hence, using the inequality that

$$\omega_\delta(W'(\underset{\sim}{N}_n)) \leq 2\rho(W'(\underset{\sim}{N}_n),W'([n\underset{\sim}{\lambda}])) + \omega_\delta(W([n\underset{\sim}{\lambda}])), \tag{5.24}$$

it suffices to show that

$$\limsup_n P\{\omega_\delta(W'([n\underset{\sim}{\lambda}])) > \varepsilon\}\to 0 \text{ as } \delta\downarrow 0, \tag{5.25}$$

$$P\{\rho(W'(\underset{\sim}{N}_n),W'([n\underset{\sim}{\lambda}])) > \varepsilon\}\to 0 \text{ as } n\to\infty. \tag{5.26}$$

Consider then a multiple array of events

$$A(\underset{\sim}{h}) = \{\underset{\sim}{\lambda}:\ a_o(\eta)+h_j\delta' < \lambda_j \leq a_o(\eta) + (h_j+1)\delta',\ 1\leq j\leq c\},\ \underset{\sim}{h}\geq\underset{\sim}{0}, \tag{5.27}$$

and let

$$\underset{\sim}{a}(\underset{\sim}{h}) = (a_o(\eta) + (h_j+\tfrac{1}{2})\delta',\ 1\leq j\leq c),\ \underset{\sim}{h}\geq\underset{\sim}{0}. \tag{5.28}$$

Then, for every $\varepsilon>0$, $\delta>0$, by (5.20), (5.24) and (5.27),

$$(5.29)\qquad P\{\omega_\delta(W'([n\lambda])) > \varepsilon\}$$

$$\leq P\{\min_{1\leq j\leq c} \lambda_j \leq a_o(\eta)\} + P\{\omega_\delta(W'([n\lambda]))>\varepsilon,\ \lambda_j \geq a_o(\eta),\ 1\leq j\leq c\}$$

$$\leq \eta/3 + \sum_{h=0}^{\infty} P\{A(h)\}P\{\omega_{\delta'}(W'([n\lambda]))>\varepsilon|A(h)\}$$

$$\leq \eta/3 + \sum_{h=0}^{\infty} P\{A(h)\}P\{\rho(W'([n\lambda]),\ W'([na(h)])) > \tfrac{1}{3}\varepsilon \mid A(h)\}$$

$$+ \sum_{h=0}^{\infty} P\{A(h)\}P\{\omega_\delta(W'([na(h)])) > \tfrac{1}{3}\varepsilon|A(h)\},$$

where we let $P(B|A(h))=0$ when $P\{A(h)\}=0$. Now, in the same say as (5.17) follows from (5.18), it follows from (5.23) that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an n_o, such that for $A\varepsilon\mathcal{A}$ and $n\geq n_o$,

$$(5.30)\qquad P\{\max_{k:\ |k-n|<\delta n} \rho(W'(k),W'(n))>\varepsilon|A\} < \eta.$$

Since when $A(h)$ holds, $|\lambda-a(h)|<\delta'$, we obtain from (5.30) that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta'(0<\delta'<\tfrac{1}{2}a_o(\eta))$, and an $n_o(\varepsilon,\eta)$, such that for all $n\geq n_o(\varepsilon,\eta)$,

$$(5.31)\qquad P\{\rho(W'([n\lambda]),W'([na(h)])) > \tfrac{1}{3}\varepsilon|A(h)\} < \tfrac{1}{3}\eta,\ \forall\, h\geq 0.$$

Consequently, by Lemma 5.2 and (5.31), it follows that for $n \geq n_o(\varepsilon,\eta)$,

$$(5.32)\qquad P\{\omega_\delta(W'([n\lambda])) > \varepsilon\} < \eta,$$

which proves (5.25). Finally,

$$(5.33)\quad P\{\rho(W'(\underset{\sim}{N}_n),W'([n\underset{\sim}{\lambda}])) > \varepsilon\}$$

$$\leq P\{|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}|>\delta'\} + P\{\min_{1\leq j\leq c} \lambda_j \leq a_o(\eta)\}$$

$$+ \sum_{\underset{\sim}{h}=\underset{\sim}{0}}^{\infty} P\{A(\underset{\sim}{h})\}P\{\rho(W'(\underset{\sim}{N}_n),W'([n\underset{\sim}{\lambda}]))>\varepsilon,|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}| \leq \delta'|A(\underset{\sim}{h})\},$$

$$\leq P\{|n^{-1}\underset{\sim}{N}_n-\underset{\sim}{\lambda}| > \delta'\} + \eta/3 +$$

$$\sum_{\underset{\sim}{h}=\underset{\sim}{0}}^{\infty} P\{A(\underset{\sim}{h})\}P\{\max_{\underset{\sim}{k}:\ |n^{-1}\underset{\sim}{k}-\underset{\sim}{a}(\underset{\sim}{h})|<2\delta'} \rho(W'(\underset{\sim}{k}),W'([n\underset{\sim}{a}(\underset{\sim}{h})]))>\varepsilon|A(\underset{\sim}{h})\}$$

$\leq \eta$, by (5.19) and (5.30).

Hence the proof of the theorem is complete.

The theorem is of great value in studying the asymptotic distribution of generalized U-statistics, useful in the context of sequential procedures based on these statistics.

As an illustration of the uses of Theorems 2.1, 2.2, 4.1 and 5.1, we consider a simple case where p=1, c=2 and

$$(5.34)\qquad \theta(F_1,F_2) = \int_{-\infty}^{\infty} F_1(x)dF_2(x) = P\{X_{1i} \leq X_{2j}\},$$

and we assume that both F_1 and F_2 are continuous everywhere. Then $\theta(F(\cdot\underset{\sim}{n}) = U(\underset{\sim}{n})$ is the Wilcoxon rank statistic

$$(5.35)\qquad \frac{1}{n_1n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c(X_i,X_j);\quad c(u,v) = \begin{cases} 1, & u\leq v \\ 0, & u>v \end{cases}.$$

Here m_1=m_2=1, and the summands $\{c(X_i,X_j),\ 1\leq i\leq n,\ 1\leq j\leq n_2\}$ are not all independent, so that Wichura's (1969) results do not hold. If the two distributions F_1 and F_2 are mutually overlapping, then (2.3) holds, while (2.4) holds for all F_1, F_2, as $c(u,v)$ is bounded. If $F_1 \equiv F_2$, $\theta(F_1,F_2) = \frac{1}{2}$, $\sigma_1^2 = \sigma_2^2 = \frac{1}{12}$, so that the results further simplify. Theorem 5.1 for the Wilcoxon statistic is useful for the problem of sequential testing and estimating $\theta(F_1,F_2)$.

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