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SOME REMARKS ON THE EQUIVALENCE OF GAUSSIAN PROCESSES

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A. GUALTIEROTTI AND S. CAMBANIS[1]

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1. INTRODUCTION. Let $X = \{X_t, t\in T\}$ and $Y = \{Y_t, t\in T\}$ be real, zero-mean Gaussian processes with respective covariances R_X and R_Y, defined on a probability space (Ω, A, P), where T is an arbitrary index set. Denote by P_X and P_Y the probabilities induced on $(R^T, C(R^T))$ by P, X and Y respectively, where $C(R^T)$ is the σ-algebra generated by the cylinder sets of the set R^T of real functions on T. $L_2(X)$ and $L_2(Y)$ will denote the subspaces of $L_2(\Omega, A, P)$ generated by X and Y respectively, and H(X) and H(Y) the corresponding reproducing kernel Hilbert spaces (RKHS's). The associated canonical isometries will be denoted by U_X and U_Y respectively ($U_X X_t = R_X(t,\cdot)$, $t\in T$, and similarly for U_Y). We say that the processes X and Y are equivalent, or that the probabilities P_X and P_Y are equivalent, $P_X \sim P_Y$, if P_X and P_Y are mutually absolutely continuous. The following properties are well known:

(i) $P_X \sim P_Y$ if and only if $Y_t = FX_t$, $t\in T$, where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$ (i.e., F has bounded inverse and $I-F^*F$ is Hilbert-Schmidt) [6]; or equivalently if and only if $Y_t = X_t - AX_t$, $t\in T$, where A is a Hilbert-Schmidt operator in $L_2(X)$ and I-A has bounded inverse, and the equality is in law, i.e. $P_Y = P_{(I-A)X}$ [9].

(ii) If $P_X \sim P_Y$ then $sH(X) = sH(Y)$, where s indicates that what follows is considered as a set and not as a space [7, p. 181].

The first question considered in this note is the converse of (ii), i.e., if X and Y have the same RKHS's, under what additional condition on the RKHS's are they equivalent? The answer is given in Propositions 1 and 2 and results in a characterization of the norms in a RKHS corresponding to equivalent Gaussian processes.

The fact, mentioned in (i), that all Gaussian processes equivalent to X_t

are of the form $X_t - AX_t$, with A a Hilbert-Schmidt operator in $L_2(X)$, raises the problem of expressing AX_t in a more explicit way in terms of the process X. This is done in Propositions 4, 5 and 6.

2. THE RKHS OF EQUIVALENT GAUSSIAN PROCESSES. Here, and in the next section, we adopt the notation of the introduction.

PROPOSITION 1. $P_X \sim P_Y$ if and only if $sH(X) = sH(Y)$ and (a) the identity J on $sH(X) = sH(Y)$ is an equivalence operator from $H(X)$ to $H(Y)$, or (b) for every f in $sH(X)$

$$||f||^2_{H(Y)} = ||f||^2_{H(X)} + \langle f \otimes f, \Lambda \rangle_{H(X) \otimes H(X)}$$

for some $\Lambda \in H(X) \otimes H(X)$ which is symmetric and such that $- R_X \ll \Lambda$.

PROOF. Suppose first that $sH(X) = sH(Y)$ and J is an equivalence. For every $\xi \in L_2(Y)$ we have $\langle \xi, Y_t \rangle_{L_2(Y)} = (U_Y \xi)(t) = (J^{-1} U_Y \xi)(t) = \langle U_X^* J^{-1} U_Y \xi, X_t \rangle_{L_2(X)}$. Let $F^* = U_X^* J^{-1} U_Y$. Since J is an equivalence, so is J^{-1}, and since U_X, U_Y are unitary, F^* is an equivalence and so is F. It now follows from $\langle \xi, Y_t \rangle_{L_2(Y)} = \langle F^* \xi, X_t \rangle_{L_2(X)} = \langle \xi, FX_t \rangle_{L_2(Y)}$, for all ξ in $L_2(Y)$, that $Y_t = FX_t$. Thus $P_X \sim P_Y$.

Conversely, suppose that $P_X \sim P_Y$. Then $Y_t = FX_t$ where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$. For every f in $H(Y)$ we have $f(t) = \langle U_Y^* f, Y_t \rangle_{L_2(Y)} = \langle F^* U_Y^* f, X_t \rangle_{L_2(X)} = [U_X F^* U_Y^* f](t) = [J U_X F^* U_Y^* f](t)$. Thus $J U_X F^* U_Y^* = I_{H(Y)}$ and $J = U_Y (F^*)^{-1} U_X^*$. Since F is an equivalence, so is $(F^*)^{-1}$ and since U_X, U_Y are unitary, J is an equivalence.

Finally, (b) is equivalent to (a) as it follows from Property (i) and the fact that Hilbert-Schmidt operators on RKHS's have kernels in the direct product of the considered RKHS's [1].

The characterization of Proposition 1 is particularly useful if the elements in the common RKHS can be obtained in the way described in the following Proposition.

PROPOSITION 2. Suppose there exists a pair (H,L), where H is a Hilbert space and L a unitary map from H to H(X). Then $P_X \sim P_Y$ if and only if sH(X) = sH(Y) and for all h in H,

$$\text{(1)} \qquad ||Lh||^2_{H(Y)} = ||Lh||^2_{H(X)} + \langle LKh, Lh\rangle_{H(X)} ,$$

where K is a self-adjoint, Hilbert-Schmidt operator on H such that $-1 < \sigma(K)$.

REMARK 1. Condition (1) is equivalent to

(a) $||Lh||^2_{H(Y)} = ||h||^2_H + \langle Kh,h\rangle_H$, or

(b) $\langle Lh,Lh'\rangle_{H(Y)} = \langle Lh,LH\rangle_{H(X)} + \langle LKh,Lh'\rangle_{H(X)} = \langle h,h'\rangle_H + \langle Kh,h'\rangle_H$.

PROOF OF PROPOSITION 2. Suppose first that $P_X \sim P_Y$. Then J is an equivalence and J can be decomposed as J = UW, with $W = (J^*J)^{\frac{1}{2}}$ and U unitary. (W: H(X) → H(X) and U: H(X) → H(Y)). Since W is onto, for every h in H there is a g in H such that Lh = WLg. Thus every h in H can be obtained as L^*WLg, for some g in H. Set $S = L^*WL$. Then $J = ULSL^*$ and thus for h in H, $||JLh||_{H(Y)} = ||ULSL^*Lh||_{H(Y)} = ||LSh||_{H(X)} = ||Sh||_H$. Now it follows from $S = L^*U^*JL$ that S is an equivalence, since it is obtained from an equivalence operator J by "unitary multiplication." Thus $I_H - S^*S$ is equal to a self-adjoint, Hilbert-Schmidt operator -K that does not have -1 among its eigenvalues. Hence, as

$$||JLh||^2_{H(Y)} = ||Sh||^2_H = \langle (I+K)h,h\rangle_H = ||h||^2_H + \langle Kh,h\rangle_H ,$$

and (1) follows from (a) of Remark 1.

Conversely, suppose that sH(X) = sH(Y) and (1) holds. Define a unitary operator T: H → H(Y) by $T\{(I_H+K)^{\frac{1}{2}}h\} = JLh$. This definition makes sense since T is obviously onto and by (1), $||T\{(I_H+K)^{\frac{1}{2}}h\}||^2_{H(Y)} = ||JLh||^2_{H(Y)} = ||(I_H+K)^{\frac{1}{2}}h||^2_H$. But then $J = T(I_H+K)^{\frac{1}{2}}L^*$, where T and L^* are unitary and $(I_H+K)^{\frac{1}{2}}$ is an equivalence. Hence, J is an equivalence and $P_X \sim P_Y$.

REMARK 2. The existence of the assumed pair (H,L) in Proposition 2 is not as restrictive as it appears. In fact, whenever $H(X)$ (or equivalently $L_2(X)$) is separable, this assumption is satisfied and one can take H to be an L_2 space. This follows from Theorem 2 of [3]. Indeed, if $H(X)$ is separable, then, for an arbitrary measure space $(E,\mathcal{E},\mu)$ such that $L_2(\mu)$ is separable and infinite dimensional, we have for all t in T, $X_t = \int_E \phi_t(u)dZ(u)$, where $\phi_t \in L_2(\mu)$ and Z is an orthogonal random measure on $\mathcal{E}$ such that $L_2(Z) = L_2(X)$. This implies that there is a unitary map $A: L_2(\mu) \to L_2(X)$ such that $A\phi_t = X_t$, $t\in T$, and clearly $L: L_2(\mu) \to H(X)$ defined by $L = U_X A$ is unitary. The Hilbert space H is clearly non-unique. However in some specific cases, H can be chosen in a natural way. In fact, most of the known RKHS's are obtained in this way. We list some examples below.

EXAMPLES. 1) Let X have orthogonal increments on $[0,1]$ and start almost surely at the origin, with $R_X(s,t) = F(s\wedge t)$, F continuous. Then $H(X) = \{\int_0^t \phi(u)dF(u),\ \phi\in L_2(dF)\}$, $\langle\int_0^\cdot \phi dF, \int_0^\cdot \psi dF\rangle_{H(X)} = \langle\phi,\psi\rangle_{L_2(dF)}$, and $L: L_2(dF) \to H(X)$ defined by $[L\phi](t) = \int_0^t \phi dF$ is an isometry. This example includes the Wiener process $(F(u) = u)$.

2) Let X have the covariance $R_X(s,t) = F(s\wedge t)\, G(s\vee t)$, where F and G are continuous with bounded variation on $[0,1]$, F is strictly positive, except at the origin, and G is strictly positive. Suppose further that $H(u) = F(u)/G(u)$ is strictly increasing. Then $H(X) = \{G(t)\int_0^t \phi(u)dH(u),\ \phi\in L_2(dH)\}$ and $\langle G(\cdot)\int_0^\cdot \phi dH, G(\cdot)\int_0^\cdot \psi\, dH\rangle_{H(X)} = \langle\phi,\psi\rangle_{L_2[0,1]}$. Thus $L: L_2(dH) \to H(X)$ defined by $[L\phi](t) = G(t)\int_0^t \phi dH$ is unitary.

3) Let X be a linear operation on a stationary process with spectral measure μ. Then $R_X(s,t) = \int_{-\infty}^{\infty} \phi_s(\lambda)\overline{\phi}_t(\lambda)d\mu(\lambda)$ and if H is the closure in $L_2(\mu)$

of the linear span of $\{\phi_t, t\in T\}$ then $H(X) = \{\int_{-\infty}^{\infty} \phi_t\overline{\phi}d\mu,\ \phi\in H\}$ and $<\int\phi_{\cdot}\overline{\phi}d\mu\ \ \int\phi_{\cdot}\overline{\psi}d\mu>_{H(X)} =$ $= <\phi,\psi>_{L_2(\mu)}$. Thus $L : H \to H(X)$ defined by $[L\phi](t) = \int_{-\infty}^{\infty} \phi_t\overline{\phi}d\mu$ is unitary. This includes the case where X is stationary ($\phi_t(\lambda) = e^{it\lambda}$) and then $H = L_2(\mu)$ if $T = R$.

4) Let X be a linear operation on a harmonizable process with two-dimensional spectral measure r. Then $R_X(s,t) = \iint_{-\infty}^{\infty} \phi_s(u)\overline{\phi}_t(v)dr(u,v)$ and if H is the closure in the Hilbert space $\Lambda_2(r)$ [5] of the linear span of $\{\phi_t, t\in T\}$ then

$H(X) = \{\iint_{-\infty}^{\infty} \phi_t(u)\overline{\phi}(v)\ dr(u,v),\ \ \phi\in H\}$ and

$\left\langle\iint\phi_{\cdot}\overline{\phi}dr, \iint\phi_{\cdot}\overline{\psi}dr\right\rangle_{H(X)} = \left\langle\phi,\psi\right\rangle_{\Lambda_2(r)} = \iint_{-\infty}^{\infty} \phi(u)\overline{\psi}(v)dr(u,v)$. Thus $L: H \to H(X)$ defined by $[L\phi](t) = \iint_{-\infty}^{\infty}\phi_t(u)\overline{\phi}(v)dr(u,v)$ is unitary. This includes the case where X is harmonizable ($\phi_t(u) = e^{itu}$) and then $H = \Lambda_2(r)$ if $T = R$.

REMARK 3. The existence of the pair (H,L) as described in Proposition 2 is easily seen to be equivalent to the existence of a representation of the covariance R_X of the form $R_X(s,t) = <\phi_s,\phi_t>_H$ where $\{\phi_t, t\in T\} \subseteq H$.

Proposition 2 can be also expressed in terms of covariances and it then contains as particular cases the results of [10] and [8].

PROPOSITION 3. Suppose there exists a pair (H,L), where H is a Hilbert space and L a unitary map from H to $H(X)$. Let $\phi_t = L^*R_X(t,\cdot)$ or $R_X(s,t) = <\phi_s,\phi_t>_H$ (see Remark 3). Then $P_X \sim P_Y$ if and only if

(2) $$R_Y(s,t) = R_X(s,t) - <M\phi_s,\phi_t>_H$$

where M is self-adjoint, Hilbert-Schmidt and such that $\sigma(M) < 1$.

PROOF. Assume first that $P_X \sim P_Y$. Then $Y_t = FX_t$ where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$. It follows that $Y_t = FU_X^*L\phi_t$ and $R_Y(s,t) = <L^*U_XF^*FU_X^*L\phi_s,\phi_t>_H$. Thus (2) is valid with $M = L^*U_X(I_{L_2(X)} - F^*F)U_X^*L$. Since $I_{L_2(X)} - F^*F$ is self-adjoint, Hilbert-Schmidt with $\sigma(I_{L_2(X)} - F^*F) < 1$, and U_X, L are unitary, it follows that M is self-adjoint Hilbert-Schmidt with

$\sigma(M) < 1$.

Conversely, assume that (2) is valid. Then $\phi_t = L^* U_X X_t$ yields $\langle M\phi_s,\phi_t\rangle_H = \langle U_X^* LML^* U_X X_s, X_t\rangle_{L_2(X)}$ and (2) is written $R_Y(s,t) = $ $= \langle (I_{L_2(X)} - U_X^* LML^* U_X)X_s, X_t\rangle_{L_2(X)}$. Since M is self-adjoint, Hilbert-Schmidt with $\sigma(M) < 1$, and U_X, L are unitary, it follows that $U_X^* LML^* U_X$ is self-adjoint, Hilbert-Schmidt. Also $I_{L_2(X)} - U_X^* LML^* U_X = U_X^* L(I_H - M)L^* U_X > 0$ and hence its square root F_0 is an equivalence. We then have $\langle Y_s, Y_t\rangle_{L_2(Y)} = R_Y(s,t) = $ $= \langle F_0 X_s, F_0 X_t\rangle_{L_2(X)}$ which implies that the map $F_0 X_t \to Y_t$ extends to a unitary operator U from $L_2(X)$ to $L_2(Y)$ and then $Y_t = FX_t$ where $F = UF_0$ is an equivalence. Thus $P_X \sim P_Y$.

REMARK 4. The relationship between the operators F, K and M of Property (i) and Propositions 2 and 3 respectively is as follows

$$F = U[U_X^* L(I_H + K)^{-1} L^* U_X]^{\frac{1}{2}} = U[U_X^* L(I_H - M)L^* U_X]^{\frac{1}{2}}$$

$$K = L^* U_X[(F^* F)^{-1} - I_{L_2(X)}]U_X^* L = (I_H - M)^{-1} - I_H$$

$$M = L^* U_X(I_{L_2(X)} - F^* F)U_X^* L = I_H - (I_H + K)^{-1} .$$

We also have

$$R_Y(s,t) = \langle \psi_s, \psi_t\rangle_H \quad \text{where} \quad \psi_t = (I_H - M)\phi_t .$$

PROOF. The expressions relating F and M are derived in the proof of Proposition 3. It then suffices to derive the relationship between K and M. Since $sH(X) = sH(Y)$ we have $R_Y(t,\cdot) \in H(X)$. Let $\psi_t = L^* R_Y(t,\cdot)$. Then $R_Y(s,t) = \langle \psi_s, \psi_t\rangle_H$. By Remark 1 we obtain $R_X(s,t) = \langle R_X(s,\cdot), R_Y(t,\cdot)\rangle_{H(Y)} = $ $= \langle JL\phi_s, JL\psi_t\rangle_{H(Y)} = \langle \phi_s, (I_H + K)\psi_t\rangle_H$. Since $R_X(s,t) = \langle \phi_s, \phi_t\rangle_H$ we have $\langle \phi_s, \phi_t\rangle_H = \langle \phi_s, (I_H + K)\psi_t\rangle_H$. Since $\{R_X(t,\cdot), t \in T\}$ is complete in $H(X)$ and L is unitary, $\{\phi_t, t \in T\}$ is complete in H and hence $\phi_t = (I_H + K)\psi_t$.

We now have $R_Y(s,t) = \langle R_Y(s,\cdot), R_X(t,\cdot)\rangle_{H(X)} = \langle L\psi_s, L\phi_t\rangle_{H(X)} = \langle \psi_s, \phi_t\rangle_H = $ $\langle (I_H + K)^{-1}\phi_s, \phi_t\rangle$ and since $R_X(s,t) = \langle \phi_s, \phi_t\rangle_H$ it follows from (2) by inspection that $M = I_H - (I_H + K)^{-1}$. Hence $K = (I_H - M)^{-1} - I_H$ and also $\psi_t = (I_H + K)^{-1}\phi_t =$ $= (I_H - M)\phi_t$.

3. REPRESENTATION OF EQUIVALENT GAUSSIAN PROCESSES. When a pair (H,L) as described in Propositions 2 and 3 exists, then one can obtain explicit representations of the process AX_t in Property (i). Here it is more appropriate to consider the unitary map $V: H \to L_2(X)$ related to L by $V = U_X^* L$.

PROPOSITION 4. Suppose that there exists a pair (H,V), where H is a Hilbert space and V a unitary map from H to $L_2(X)$. If A is a Hilbert-Schmidt operator in $L_2(X)$, then $AX_t = V\bar{A}h_t$, where $\bar{A}$ is a Hilbert-Schmidt operator in H and $h_t = V^* X_t$.

PROOF. If $\bar{A} = V^* AV$, then $\bar{A}$ is Hilbert-Schmidt in H and $AX_t = AVh_t =$ $= VV^* AVh_t = V\bar{A}h_t$.

EXAMPLES. 5) Let X be as in Example 1. Then $V: L_2(dF) \to L_2(X)$ is defined by $V\phi = \int_0^1 \phi(u)dX_u$. Consequently $h_t = I_t$, the indicator function of [0,t] and $AX_t = \int_0^1 [\bar{A}I_t](u)dX_u$. Since a Hilbert-Schmidt operator in $L_2(dF)$ is of integral type with kernel $\alpha(u,v)$ in $L_2(dF \times dF)$ we finally have

$$AX_t = \int_0^1 \int_0^t \alpha(u,v)dF(v)dX_u \quad .$$

This result is obtained for the Wiener process in [9].

6) Consider the case where the covariance of X has the representation $R_X(s,t) = \int_E \phi_s(u)\overline{\phi}_t(u)d\mu(u)$ with $(E,\mathcal{E},\mu)$ a finite (for simplicity) measure space and $\phi_t \in L_2(\mu)$. This includes both Examples 2 and 3. Then there is an orthogonal random measure Z on $\mathcal{E}$ such that $X_t = \int_E \phi_t(u)dZ(u)$ [5]. Let H be the closure in $L_2(\mu)$ of the linear span of $\{\phi_t, t\epsilon T\}$. Then $V: H \to L_2(X)$ is defined by $V\phi = \int_E \phi(u)dZ(u)$. Consequently $h_t = \phi_t$ and $AX_t = \int_E [\bar{A}\phi_t](u)dZ(u)$. As in Example 5, $\bar{A}$ is of integral type with kernel $\alpha(u,v)$ in $L_2(\mu\times\mu)$ and finally we have

$$AX_t = \int_E\int_E \alpha(u,v)\overline{\phi}_t(v)d\mu(v)dZ(u) \quad .$$

In the case of Example 2, $H = L_2(dH)$ and $AX_t = G(t)\int_0^1\int_0^t \alpha(u,v)dH(v)dZ(u)$.

7) Consider the case where the covariance of X has the representation $R_X(s,t) = \iint_{EE} \phi_s(u)\overline{\phi}_t(v)dr(u,v)$ with $(E,\mathcal{E})$ a measurable space, r a two-dimensional spectral measure on $\mathcal{E}\times\mathcal{E}$ and ϕ_t in $\Lambda_2(r)$. Then there is a random measure Z on $\mathcal{E}$ such that $X_t = \int_E \phi_t(u)dZ(u)$ [5]. Let H be the closure in $\Lambda_2(r)$ of the linear span of $\{\phi_t, t\in T\}$. Then $V: H \to L_2(X)$ is defined by $V\phi = \int_E \phi(u)dZ(u)$. Consequently $h_t = \phi_t$ and $AX_t = \int_T [\overline{A}\phi_t](u)dZ(u)$, where $\overline{A}$ is a Hilbert-Schmidt operator in $\Lambda_2(r)$. However, no kernel representation of $\overline{A}$ seems to be available because Λ_2 is a more complicated space than L_2. Nevertheless, as follows from (i) of the Lemma at the end of this section, if E is an interval, $\overline{A}$ is the limit in the operator norm of a sequence of Hilbert-Schmidt operators $\{A_n\}_{n=1}^{\infty}$ in $L_2(\mu)$ with kernels $\{\alpha_n\}_{n=1}^{\infty}$ and we thus have

$$AX_t = \lim_{n\to\infty} \int_E \{\iint_{EE} \phi_t(v)\overline{\alpha}_n(w,u)dr(v,w)\}dZ(u)$$

where the limit is in $L_2(P)$. Consider now the important particular case where there is a measurable process $\{Z_u, u\in E\}$ and a measure μ on $\mathcal{E}$ equivalent to the Lebesgue measure such that $\int_E R_Z(u,u)d\mu(u) < \infty$ and for every $B\in\mathcal{E}$, $Z(B) = \int_B Z_u d\mu(u)$. Then $X_t = \int_E \phi_t(u)Z_u d\mu(u)$ and

$$AX_t = \lim_{n\to\infty} \int_E g_t^{(n)}(u)Z_u d\mu(u)$$

where $g_t^{(n)}(u) = \iint_{EE} \phi_t(v)\overline{\alpha}_n(w,u)R_Z(v,w)d\mu(v)\, d\mu(w)$ is in $L_2(\mu)$, the integral is defined almost surely, i.e., on the paths of Z, and the limit is in $L_2(P)$. As particular cases of this example we obtain the following representations.

PROPOSITION 5. Let X be mean square continuous, T an interval and A a Hilbert-Schmidt operator on $L_2(X)$. Then

$$AX_t = \int_T g_t(u)X_u d\mu(u) = \lim_{n\to\infty} \int_T g_t^{(n)}(u)X_u d\mu(u)$$

where the measure μ on the Borel sets of T satisfies (ii) of the Lemma, $g_t\in\Lambda_2(R_X\cdot\mu\times\mu)$, the first integral is defined in quadratic mean, $g_t^{(n)}\in L_2(\mu)$, the second integral is defined almost surely, i.e., on the paths of X, and the limit is in $L_2(P)$.

PROOF. X has a measurable modification which is henceforth considered. There exist finite measures μ, equivalent to the Lebesgue measure, and such that $\int_T R_X(t,t)d\mu(t) < \infty$ [2]. Let $dr(u,v) = R_X(u,v)d\mu(u)d\mu(v)$. Since X is mean square continuous, it has a representation $X_t = \int_T \phi_t(u)X_u d\mu(u)$ with $\{\phi_t, t\in T\} \subset \Lambda_2(r)$ [2]. The result then follows from the last case considered in Example 7.

PROPOSITION 6. Let X be mean square continuous on $[0,1]$, $X_0 = 0$ a.s., and R_X of bounded variation on $[0,1] \times [0,1]$. Let A be a Hilbert-Schmidt operator on $L_2(X)$. Then

$$AX_t = \int_0^1 g_t(u)dX_u = \lim_{n\to\infty} \int g_t^{(n)}(u)dX_u$$

where $g_t\in\Lambda_2(d^2R_X)$, μ corresponds to d^2R_X as in (i) of Lemma, $g_t^{(n)}\in L_2(\mu)$ are of the form $g_t^{(n)}(u) = \int_0^t \int_0^1 \overline{\alpha}_n(w,u)d^2R_X(v,w)$, and $\alpha_n\in L_2(\mu\times\mu)$.

PROOF. The proof is obvious from Example 7 and the observation that $X_t = \int_0^1 I_t(u)dX_u$, I_t the characteristic function of the interval $[0,t]$, i.e., $\phi_t = I_t$.

LEMMA. Let E be an interval, $\mathcal{E}$ its Borel sets, r a finite, two-dimensional spectral measure on $\mathcal{E}\times\mathcal{E}$, and K a Hilbert-Schmidt operator on $\Lambda_2(r)$. If

(i) μ is the finite measure defined on $\mathcal{E}$ by $\mu(B) = |r|(E\times B)$ for all $B\in\mathcal{E}$, or if

(ii) $dr(u,v) = R(u,v)d\mu(u)d\mu(v)$, where R is a covariance and μ a finite measure on $\mathcal{E}$, equivalent to the Lebesgue measure and such that $\int_E R(u,u)d\mu(u) < \infty$,

then $L_2(\mu) \subset \Lambda_2(r)$ and there is a sequence of Hilbert-Schmidt operators $\{K_n\}_{n=1}^{\infty}$ on $L_2(\mu)$ with kernels $\{k_n\}_{n=1}^{\infty}$, that are defined from $\Lambda_2(r)$ to $L_2(\mu)$ by $[K_n f](u) = \langle f(\cdot),k_n(\cdot,u)\rangle_{\Lambda_2(r)}$ and are such that $K_n \to K$ in the operator norm in $\Lambda_2(r)$.

PROOF. Both (i) and (ii) imply that $L_2(\mu) \subset \Lambda_2(r)$ and that there is a sequence

$\{f_n\}_{n=1}^{\infty}$ in $L_2(\mu)$ which is orthonormal and complete in $\Lambda_2(r)$. For (ii) this is shown in [2] and for (i) it is shown as Theorem 2 of [4].

In the sequel $<\cdot,\cdot>$ and $||\cdot||$ denote inner product and norm in $\Lambda_2(r)$. Since K is Hilbert-Schmidt we have $\sum_{n,m=1}^{\infty}|<Kf_n,f_m>|^2 < \infty$. For every $f\in\Lambda_2(r)$ we have $f = \sum_{n=1}^{\infty}<f,f_n>f_n$ and thus $Kf = \sum_{n=1}^{\infty}<f,f_n>Kf_n = \sum_{m=1}^{\infty}\{\sum_{n=1}^{\infty}<Kf_n,f_m><f,f_n>\}f_m$. Define $k_N(u,v) = \sum_{n,m=1}^{N}<Kf_n,f_m>f_n(u)f_m(v)$. Since k_N is in $L_2(\mu\times\mu)$, it defines a (finite rank) Hilbert-Schmidt operator K_N on $L_2(\mu)$. K_N is also defined from $\Lambda_2(r)$ to $L_2(\mu)$ by

$$[K_Nf](u) = <f(\cdot),k_N(\cdot,u)> = \sum_{n,m=1}^{N}<Kf_n,f_m><f,f_n>f_m .$$

Then

$$||Kf-K_Nf||^2 \le ||f||^2 \sum_{n,m=N+1}^{\infty}|< Kf_n,f_m>|^2$$

which implies that $K_N \to K$ in the operator norm in $\Lambda_2(r)$.

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