

## ABSTRACT

PENG, YUANYUAN. Matching and Stabilization of Discrete Mechanical Systems. (Under the direction of Dmitry Zenkov.)

This dissertation contributes to the areas of discrete mechanics and geometric computational control theory, addressing stabilization of relative equilibria of discrete mechanical systems. Recent studies have shown that the energy shaping is an effective approach to stabilization of equilibria and relative equilibria of mechanical systems. Mechanical systems have a very rich structure, and taking this structure into account when numerical models are designed is important. Examples of quantities/structures to preserve are the momentum and, in some cases, the symplectic form and the phase volume. Generic algorithms typically do not preserve these structures. In our investigation, we develop discrete analogues of energy shaping techniques of Bloch, Leonard, and Marsden. The major contributions are the use of nonconservative forcing for energy shaping in the discrete setting and the development of discrete models that preserve the stability properties of the related continuous-time systems.

Matching and Stabilization of Discrete Mechanical Systems

by  
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## BIOGRAPHY

I was born in a small city in the North of China named Shi-Jia-Zhuang. There I spend my childhood, and there I went to the middle school and high school. Back then, I was not sure about why I should study, but I studies hard in the eyes of a child and played a lot when I did not study. Those times are obscure, confusing, vague, but very happy and unforgettable. This was the beginning of my academic career.

After high school, I studied for four years at Xi'an University for Bachelor Degree. The most important things I learned were: 1) We can use rationality to solve problems. 2) Mathematics is a good tool to use. Beauty exists in Mathematics; 3) Studying should be based on my own interests and utilitarian purpose. I need to learn how to balance between them. These four years were the beginning of my formal academic training.

After I got my Bachelor Degree, I decided to study further for my Ph.D. in the USA. I have been lucky to study at North Carolina State University. I have got fundamental and systematic training in Mathematics. This five year stretch has been a busy and beneficial period for me: I have chosen my research topics and have investigated interesting mathematics problems. Now it is almost the end of my study at NCSU, but suddenly I found that this end is just the real beginning of my academic exploration.

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# Chapter 1

## Introduction

This dissertation is devoted to the development of structure-preserving algorithms for numerical simulations of controlled mechanical system, with special attention to stabilization of equilibria and relative equilibria by the method of controlled Lagrangians. Simulation of mechanical systems is a challenging subject as there is a number of quantities one may be interested in preserving, such as the symplectic form and, for systems with symmetry, the momentum map. The former is important to preserve in order to avoid the artificial numerical dissipation, the preservation of the latter is significant in e.g. rigid body control by means of flywheels. Our strategy here is to use the discrete analogue of continuous-time mechanics, which may be interpreted as a dynamical system approach to numerical analysis and computational control theory. This chapter overviews the basics of dynamical systems, including stability analysis of discrete systems, and mechanics. The chapter is organized as follows. The first section deals with reviewing the basics of dynamical systems and mathematical background of mechanics of systems with a finite number of degrees of freedom. The second section is an overview of the method of controlled Lagrangians. The third section discusses the techniques for stability analysis of discrete system.

## 1.1 Dynamical Systems

Dynamical systems are a mathematical formalism for studying evolutionary processes. In the finite-dimensional case, when system's states are identified by means of a finite number of (real) parameters, the *phase space*, i.e., the space of all states of the system, is often assumed to be a real differentiable  $n$ -dimensional manifold  $M$ . Every state of the system is represented by a point of  $M$ . The evolution is then captured by a one-parameter family of maps  $\Phi^t : M \rightarrow M$ , where the parameter  $t$  is regarded as the time variable. There are numerous applications of dynamic system in mechanics, physics, biology, circuit theory, economics, etc.

**Continuous-Time Dynamical Systems.** In the smooth and continuous-time setting, it is assumed that  $\Phi^t : M \rightarrow M$  is a diffeomorphism for each  $t \in \mathbb{R}$ . Thus, the evolution of each state is represented by the curve  $\Phi^t x$ . It is also often assumed that the maps  $\Phi^t$  form a *one-parameter group of diffeomorphisms*, i.e.,  $\Phi^0$  is the identity map,  $(\Phi^t)^{-1} = \Phi^{-t}$ , and  $\Phi^t \circ \Phi^s = \Phi^{(t+s)}$ . Such a model clearly captures deterministic evolutionary processes: The initial state of the system determines, in principle, all past and future states.

Given a smooth dynamical system  $\Phi^t : M \rightarrow M$  as above, one defines the *velocity* at the state  $x \in M$  by the formula

$$v(x) := \left. \frac{d}{dt} \right|_{t=0} \Phi^t x \in T_x M,$$

where  $T_x M$  is the tangent space of  $M$  at  $x$ . The dynamical system can then be characterized by the differential equation  $\dot{x} = v(x)$  on the manifold  $M$ . Here and below, an overdot denotes differentiation with respect to time. This is the most typical way for dynamical systems to appear in applications in general and in mechanics in particular.

**Discrete Dynamical Systems.** Let  $\Psi : M \rightarrow M$  be a diffeomorphism. A *discrete dynamical system* is defined by taking iterations of  $\Psi$ . That is, for every  $x \in M$ , the trajectory of a discrete dynamical system is the sequence  $\Psi^n x$ , where  $n \in \mathbb{Z}$ . Discrete dynamical systems appear in a number of ways. For example, given a continuous-time system  $\Phi^t : M \rightarrow M$  and a positive

real number  $h$  (time step), one introduces a discrete dynamical system on  $M$  by the formula  $\Psi^n x = \Phi^{nh} x$ . In the important special case when  $x(t) = \Phi^t x_0$  is a  $T$ -periodic solution of a continuous-time dynamical system, one considers the *Poincaré map*  $\Psi = \Phi^T$ . The iterations of the Poincaré map of course define a discrete dynamical system. The Poincaré map and the corresponding dynamical system are important tools for studying the properties of that periodic solution.

## 1.2 An Overview of Mechanics

**Newtonian Mechanics.** By a mechanical system we mean a collection of particles (point masses) that move in the three-dimensional Euclidean space  $\mathbb{R}^3$ . Let the number of particles be  $N$ . Denote the mass and the position vector of the  $i$ -th particle by  $m_i$  and  $\mathbf{r}_i$ , respectively. The components of  $\mathbf{r}_i$  are written as  $(x_i, y_i, z_i)$ , that is  $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ . It is convenient to introduce the vector

$$\mathbf{x} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_N, y_N, z_N) \in \mathbb{R}^n, \quad n = 3N.$$

This vector captures the configuration of the system. Accordingly, the  $n$ -dimensional space  $\mathbb{R}^n$  is called the *configuration space*. By definition, the number of degrees of freedom of a mechanical system equals the dimension of the configuration space.

In the framework of Newtonian mechanics, according to Newton's laws of motions stated in [27], the motion of a system is determined by the condition that the rate of change of the linear momentum for each particle of the system is proportional to the force acting on that particle:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}}), \quad i = 1, \dots, N, \quad (1.1)$$

where the vector  $\mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}})$  is the force applied to the  $i$ -th particle and where the overdot stands for the time derive  $d/dt$ .

**Systems with Constraints.** Assume now that the configurations of the system satisfy certain restrictions called *constraints*.<sup>1</sup> This means that there is a smooth submanifold  $Q \subset \mathbb{R}^n$  such that at each moment of time  $\mathbf{x} \in Q$ . This manifold  $Q$  is called the configuration space (of a system with constraints). The constraints are assumed *ideal*, that is, they can be replaced with *reaction forces*  $\mathbf{R}_i$ ,  $i = 1, \dots, N$ , acting on the particles in a way such that  $\langle \mathbf{R}, \delta \mathbf{x} \rangle = 0$ , for all  $\mathbf{x} \in Q$ ,  $\delta \mathbf{x} \in T_{\mathbf{x}}Q$ . Here  $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_N)$ . In the presence of constraints, the equations of motion become

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{R}_i(\mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{x} \in Q, \quad i = 1, \dots, N. \quad (1.2)$$

**Lagrangian Mechanics.** Lagrange [10] obtained the form of equations of motion in a covariant, i.e., coordinate-independent, form.

By definition, the *kinetic energy* of the system of  $N$  particles is the scalar quantity

$$K = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{i=1}^N m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2).$$

The forces applied to the system are called *conservative* if there exists a function  $U(\mathbf{x})$  called the *potential energy* such that

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{x}},$$

or equivalently,

$$\mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i} = -\left( \frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i} \right), \quad i = 1, \dots, N.$$

Assume that all forces (except for the constraint reactions) applied to the system are conservative. By definition, the *Lagrangian* of the system is the difference between its kinetic and potential energies:

$$L = K - U.$$

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<sup>1</sup>More precisely, constraints discussed here are called *holonomic*, or *position* constraints. Other constraint types are not considered in this thesis.

Lagrange shows that in an arbitrary coordinate system  $q = (q^1, \dots, q^m)$ ,  $m = \dim Q$ , on the configuration space  $Q$ , the dynamics of a constrained mechanical system is given by the *Euler–Lagrange equations*

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0, \quad (1.3)$$

or, in coordinates,

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0, \quad i = 1, \dots, m. \quad (1.4)$$

**Hamilton’s Principle.** A very important interpretation of equations (1.3) is that they are equivalent to an optimization problem.

Consider two configurations of the system  $q_a$  and  $q_b$ , and the set of all continuously-differentiable paths in the configuration space  $\gamma : [a, b] \rightarrow Q$  satisfying the conditions  $\gamma(a) = q_a$  and  $\gamma(b) = q_b$ . Define the *action* functional on this set of paths by the formula

$$S(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt. \quad (1.5)$$

**Definition 1.2.1.** A *variation* of a path  $\gamma : [a, b] \rightarrow Q$  is a continuously-differentiable map  $\beta : [a, b] \times [-\varepsilon, \varepsilon] \rightarrow Q$  such that  $\beta(t, 0) = \gamma(t)$ . A *variation with fixed ends* is the one satisfying the condition  $\beta(a, s) = \gamma(a)$  and  $\beta(b, s) = \gamma(b)$  for all  $s \in [-\varepsilon, \varepsilon]$ .

**Definition 1.2.2.** A curve  $\gamma : [a, b] \rightarrow Q$  is a **critical point** of  $S(\gamma)$  if

$$\delta \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt := \frac{d}{ds} \Big|_{s=0} \int_a^b L(\beta(t, s), \dot{\beta}(t, s)) dt = 0$$

for all fixed-end variation of  $\gamma$ .

**Theorem 1.2.3 (Hamilton’s Principle).** A path  $\gamma : [a, b] \rightarrow Q$  is a solution of the Euler–Lagrange equations if and only if  $\gamma$  is a critical point of the action functional  $S(\gamma)$  within the class of paths with fixed ends.

This principle in particular clarifies the fact that the Euler–Lagrange equations have the

same structure in any coordinate system on  $Q$ . This immediately follows from the fact that definition 1.2.2 is *geometric*, i.e., independent of coordinates.

See [1, 2, 20, 22] for more detail and history of Lagrangian mechanics and Hamilton's principle.

**Forced Lagrangian Systems.** Nonconservative forces can be added to the Lagrangian formulation of mechanics as follows. Let the constrained system be subject to the nonconservative forces. These forces are represented by a one-form  $F = F_i(q, \dot{q}) dq^i \in T^*Q$ . Then the *forced Euler–Lagrange equations* are

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + F = 0, \quad (1.6)$$

or, in coordinates,

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + F_i = 0, \quad i = 1, \dots, m. \quad (1.7)$$

Hamilton's principle for forced systems reads

$$\delta \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt + \int_a^b \langle F(\gamma(t), \dot{\gamma}(t)), \delta\gamma(t) \rangle dt = 0.$$

**Systems with Symmetry.** We discuss here only systems with commutative symmetry. The general noncommutative case is technically more involved and is not imperative for this thesis.

Let  $L : TQ \rightarrow \mathbb{R}$  be a Lagrangian. Assume that the configuration manifold is a principal fiber bundle with a base space  $S$  and a group  $G$ . This means that there is a projection  $\pi : Q \rightarrow S$  and that for every point  $\phi \in S$  there exists an open neighborhood of this point  $U$  such that the set  $\pi^{-1}(U) \approx U \times G \subset Q$ . In other words,  $Q$  locally is a direct product, however, it may fail to have the structure of a direct product globally. We assume here and below that the group  $G$  is commutative. Elements of  $G$  are denoted by  $s$ . Thus, locally the configuration variables can be written as *bundle coordinates*  $(r, s)$ , where  $r = (r^1, \dots, r^\sigma) \in S$  and  $s = (s^1, \dots, s^m) \in G$ . As an example of a fiber bundle, consider the direct product  $S \times G$ . Such bundles are called *trivial*.

**Definition 1.2.4.** Let  $L : TQ \rightarrow \mathbb{R}$  be a Lagrangian. The (group) variables  $s = (s^1, \dots, s^m)$  are called **cyclic** if the Lagrangian is independent of  $s$ , i.e.,  $\partial L / \partial s^a = 0$ ,  $a = 1, \dots, m$ .

Thus, if the variables  $s$  are cyclic, the Lagrangian is a function of  $(r, \dot{r}, \dot{s})$ . The Euler–Lagrange equations in this setting become

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{s}^a} = 0. \quad (1.8)$$

The second group of equations in (1.8) states that the quantities  $\partial L / \partial \dot{s}^a$ ,  $a = 1, \dots, m$ , are conserved along the trajectories of (1.8). These conservation laws are often called *cyclic integrals* or *momentum integrals*. The quantity

$$p := \frac{\partial L}{\partial \dot{s}}(r, \dot{r}, \dot{s}) \quad (1.9)$$

is called the *conjugate momentum*. Assume that equation (1.9) can be solved for  $\dot{s}$ . We write the solutions as  $\dot{s}^a = S^a(r, \dot{r}, p)$ .

The following result is useful in stability analysis of relative equilibria of mechanical systems.

**Theorem 1.2.5.** *The dynamics on the common level set*

$$\frac{\partial L}{\partial \dot{s}^a} = p_a, \quad a = 1, \dots, m,$$

of the cyclic integrals is given by the equations

$$\frac{d}{dt} \frac{\partial R_p}{\partial \dot{r}^\alpha} - \frac{\partial R_p}{\partial r^\alpha} = 0, \quad \alpha = 1, \dots, \sigma, \quad (1.10)$$

where the **Routhian**  $R_p$  is defined by

$$R_p(r, \dot{r}) = L - \langle p, \dot{s} \rangle|_{\dot{s}=S(r, \dot{r}, p)} = L - p_a \dot{s}^a|_{\dot{s}=S(r, \dot{r}, p)}. \quad (1.11)$$

The dynamics of the  $s$  variables is given by the *reconstruction equations*

$$\dot{s}^a = -\frac{\partial R_p}{\partial p_a}, \quad a = 1, \dots, m. \quad (1.12)$$

Assume that the Lagrangian is the kinetic minus potential energy, and the kinetic energy is a quadratic form in velocity:

$$L(r, \dot{r}, \dot{s}) = \frac{1}{2}g_{\alpha\beta}(r)\dot{r}^\alpha\dot{r}^\beta + g_{a\alpha}(r)\dot{r}^\alpha\dot{s}^a + \frac{1}{2}g_{ab}(r)\dot{s}^a\dot{s}^b - U(r).$$

It is straightforward to compute that the Routhian is

$$R_p(r, \dot{r}) = \frac{1}{2}(g_{\alpha\beta} - g_{a\alpha}g^{ac}g_{c\beta})\dot{r}^\alpha\dot{r}^\beta + g_{a\alpha}g^{ac}p_c\dot{r}^\alpha - U_p(r),$$

where

$$U_p(r) = U(r) + \frac{1}{2}g^{ab}p_ap_b \quad (1.13)$$

is the called the *amended potential*.

### 1.3 Relative Equilibria and Their Stability

**Stability of Equilibria.** For a continuously-differentiable vector field  $f(x)$  on a manifold  $M$ , consider a system of ordinary differential equations

$$\frac{dx}{dt} = f(x). \quad (1.14)$$

In other words,  $M$  is the phase space for (1.14). A state  $x_e$  such that

$$f(x_e) = 0 \quad (1.15)$$

is called an *equilibrium*. A solution with initial condition  $x(0) = x_e$  is just  $x(t) = x_e$ . An equilibrium  $x_e$  is said to be *stable* if for any  $\varepsilon > 0$  it is possible to find  $\delta(\varepsilon) > 0$  such that for any initial condition  $x(0)$  with

$$d(x(0), x_e) < \delta(\varepsilon), \tag{1.16}$$

the solution of (1.14) with initial conditions  $x(0)$  is defined for all  $t > 0$  and

$$d(x(t), x_e) < \varepsilon, \quad \text{for all } t \geq 0. \tag{1.17}$$

Here,  $d(x, y)$  is an arbitrary distance function on the phase space of (1.14).<sup>2</sup>

An equilibrium is called *asymptotically stable* if it is stable and in addition

$$\lim_{t \rightarrow \infty} d(x(t), x_e) = 0$$

for all initial conditions  $x(0)$  satisfying (1.16). An equilibrium that is not stable is called *unstable*.

There are two basic methods for checking stability of equilibria. One approach is to construct a *Lyapunov function*  $V(x)$ , i.e., a continuously-differentiable function such that  $V(x_e) = 0$ ,  $V(x) > 0$  if  $x \neq x_e$ ,<sup>3</sup> and the *flow derivative*  $\dot{V}(x) := \langle dV, f(x) \rangle \leq 0$  (at least locally).

**Theorem 1.3.1. (Lyapunov [16])** *If there exists a Lyapunov function  $V(x)$  for the equilibrium  $x_e$ , this equilibrium is stable. If, in addition, the flow derivative  $\dot{V}$  is negative-definite (that is,  $-\dot{V}$  is positive-definite), the equilibrium  $x_e$  is asymptotically stable.*

The Lyapunov-function-based approach is universal: If an equilibrium is stable/asymptotically stable, there exists a Lyapunov function for this equilibrium. However, Lyapunov functions are not always easy to construct.

A useful generalization of Theorem 1.3.1 is found in Barbashin and Krasovsky [3] and LaSalle

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<sup>2</sup>This definition does not depend on the choice of a distance function because the phase space is a finite-dimensional manifold.

<sup>3</sup>Such functions are called *positive-definite*.

[11], see also LaSalle and Lefschetz [12].

**Theorem 1.3.2. (Invariance Principle)** *Suppose that  $\Omega$  is a compact subset of the phase space,  $f(x)$  is a continuously-differentiable vector field on  $\Omega$ , and that  $\Omega$  is positively invariant with respect to the dynamics of  $f(x)$ , that is, any solution starting in  $\Omega$  remains in  $\Omega$  for all future moments of time. Let  $N$  be the largest invariant set in  $\Omega$  with  $\dot{V}(x) = 0$ . Then all solutions starting in  $\Omega$  tend to  $N$  asymptotically as  $t \rightarrow \infty$ .*

In particular, if this largest invariant set is a single equilibrium, this equilibrium is asymptotically stable.

**Remark 1.3.3.** *If stability of equilibria of a mechanical system is studied, one often selects the mechanical energy as a Lyapunov function. The mechanical energy is the quantity*

$$E(q, \dot{q}) = \left\langle \frac{\partial L}{\partial \dot{q}}, \dot{q} \right\rangle - L(q, \dot{q}).$$

*If the Lagrangian is the difference of kinetic and potential energies, and the kinetic energy is a quadratic form in velocity, the mechanical energy equals the sum of kinetic and potential energies.*

The other approach is to conclude stability based on the spectrum of the linearization of (1.14) at  $x_e$ ,

$$\frac{d\xi}{dt} = D_x f(x_e) \xi, \tag{1.18}$$

where  $D_x f(x_e)$  is the Jacobian of  $f(x)$  at  $x_e$ .

**Theorem 1.3.4. (Lyapunov [16])** *If all eigenvalues of  $D_x f(x_e)$  lie in the open left half plane, then the equilibrium  $x_e$  is asymptotically stable. If an eigenvalue lie in the right half plane, the equilibrium  $x_e$  is unstable.*

This approach, in general, is not applicable if the spectrum contains pure imaginary eigenvalues. The following theorem addresses, in parts, this issue.

**Theorem 1.3.5. (Lyapunov [16])** Consider the system of differential equations

$$\dot{x} = Ax + X(x, y), \quad \dot{y} = Y(x, y), \quad (1.19)$$

where  $x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$ ,  $A$  is an  $m \times m$ -matrix, and  $X(x, y)$ ,  $Y(x, y)$  represent nonlinear terms. If all eigenvalues of the matrix  $A$  have negative real parts, and  $X(x, y)$ ,  $Y(x, y)$  vanish when  $x = 0$ , then the solution  $(x, y) = (0, 0)$  of system (1.19) is stable with respect to  $(x, y)$ , and asymptotically stable with respect to  $x$ . If a solution  $(x(t), y(t))$  of (1.19) is close enough to the solution  $(x, y) = (0, 0)$ , then

$$\lim_{t \rightarrow \infty} x(t) = 0, \quad \lim_{t \rightarrow \infty} y(t) = c,$$

where  $c$  is a constant.

**Remark 1.3.6.** Theorem 1.3.5 is called the **Lyapunov–Malkin theorem**. As stated, it was known to Lyapunov for analytic systems. Malkin [19] generalized Lyapunov’s result for time-dependent systems of differential equations.

One often needs a substitution in order to apply Theorem 1.3.5. The following lemma addresses this issue.

**Lemma 1.3.7.** Consider a system of differential equations

$$\dot{u} = Au + By + \mathcal{U}(u, y), \quad \dot{y} = \mathcal{Y}(u, y), \quad (1.20)$$

where  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$ ,  $\det A \neq 0$ , and where  $\mathcal{U}$  and  $\mathcal{Y}$  represent higher order nonlinear terms. There is a change of variables  $(u, y) = (x + \phi(y), y)$ , with suitable  $\phi(y)$ , such that

(i) In the new variables  $(x, y)$  system (1.20) becomes

$$\dot{x} = Ax + X(x, y), \quad \dot{y} = Y(x, y),$$

(ii) If  $Y(0, y) = 0$ , then  $X(0, y) = 0$  as well.

**Stability of Relative Equilibria of Mechanical Systems.** Assume now that a mechanical system has a symmetry group  $G$  as in Definition 1.2.4. The dynamics is then given by equations (1.8). A *relative equilibrium* of a mechanical system with symmetry is a trajectory whose projection onto the reduced phase space  $TQ/G$  is an equilibrium. In other words, a relative equilibrium is a trajectory whose shape position and velocity and whose group velocity are constants. By definition, a relative equilibrium is *orbitally stable* if the corresponding equilibrium in  $TQ/G$  is stable, that is, if it is stable with respect to shape variables, shape velocities, and group velocities. Stability of relative equilibria is often established by constructing a Lyapunov function for the reduced system on  $TQ/G$ . When carrying this task out, representation (1.10) is often helpful: One can construct a Lyapunov function by computing the energy associated with the Routhian. Pioneered by Routh [28], this approach is an instance, for systems with commutative symmetry, of the *energy-momentum* method for stability analysis. The latter is a very powerful technique developed in Marsden, Simo, Lewis and Posbergh [23], Simo, Posbergh and Marsden [30, 31], Simo, Lewis and Marsden [29], and Lewis and Simo [13].

## 1.4 The Method of Controlled Lagrangians

The *method of controlled Lagrangians* is a constructive approach for stabilization of equilibria and relative equilibria of mechanical system. The basic principle behind this method is to consider a class of control laws that yield closed-loop dynamics which remain in Lagrangian form. The development of the method started in papers [4] and [5].

In the controlled Lagrangian approach one considers a mechanical system with an uncontrolled (free) Lagrangian equal to kinetic energy minus potential energy. In the simplest setting we modify the kinetic energy to produce a new controlled Lagrangian which describes the dynamics of the controlled closed-loop system. The method may be extended in various ways including the modification of the potential energy.

Suppose the system has configuration space  $Q$  and a Lie group  $G$  acts freely and properly on  $Q$ , so that  $Q$  has the principal bundle structure. We assume that the group  $G$  is commutative.

It is useful to keep in mind the case in which  $Q = S \times G$  with  $G$  acting only on the second factor by the left group multiplication.

For example, for the inverted planar pendulum on a cart discussed below,  $Q = S^1 \times \mathbb{R}$  with  $G = \mathbb{R}$ , the group of real numbers under addition (corresponding to translations of the cart).

Our goal is to control the variables lying in the *shape space*  $Q/G$  using controls that act directly on the variables lying in  $G$ .<sup>4</sup> The controlled Lagrangian is constructed to be  $G$ -invariant, thus providing modified or *controlled* conservation laws.

Assume that the uncontrolled system is characterized by the kinetic energy (quadratic form)  $\frac{1}{2}g(\dot{q}, \dot{q})$  and potential energy  $U(q)$ , both  $G$ -invariant, so that the Lagrangian of the uncontrolled system is

$$L = \frac{1}{2}g(\dot{q}, \dot{q}) - U(q).$$

The key modification of the Lagrangian involves changing the kinetic energy metric  $g(\cdot, \cdot)$ . The tangent space to  $Q$  can be split into a sum of horizontal and vertical parts defined as follows: For each tangent vector  $v_q$  to  $Q$  at a point  $q \in Q$ , we can write a unique decomposition  $v_q = \text{Hor } v_q + \text{Ver } v_q$ , such that the vertical part is tangent to the orbits of the  $G$ -action and the horizontal part is kinetic-energy-metric-orthogonal to the vertical space, i.e., it is uniquely defined by the identity

$$g(v_q, w_q) = g(\text{Hor } v_q, \text{Hor } w_q) + g(\text{Ver } v_q, \text{Ver } w_q) \tag{1.21}$$

with  $v_q$  and  $w_q$  arbitrary tangent vectors to  $Q$  at the point  $q \in Q$ . This choice of horizontal space coincides with that given by the *mechanical connection*; see, for example, Marsden [1992].

For the kinetic energy of our controlled Lagrangian, we use a modified version of the right-hand side of equation (1.21). The modification consists of three ingredients:

1. A new choice of horizontal space, denoted  $\text{Hor}_\tau$ ,
2. a change  $g \rightarrow g_\sigma$  of the metric on horizontal vectors,

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<sup>4</sup>The shape space is  $S$  in the case  $Q = S \times G$ .

3. a change  $g \rightarrow g_\rho$  of the metric on vertical vectors.

Let  $\xi_Q$  denote the infinitesimal generator corresponding to  $\xi \in \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$  (see Marsden [1992] and Marsden and Ratiu [1994]). Thus, for each  $\xi \in \mathfrak{g}$ ,  $\xi_Q$  is a vector field on the configuration manifold  $Q$  and its value at a point  $q \in Q$  is denoted  $\xi_Q(q)$ .

**Definition 1.4.1.** Let  $\tau$  be a Lie-algebra-valued horizontal one-form on  $Q$ ; that is, a one-form that annihilates vertical vectors. The  **$\tau$ -horizontal space** at  $q \in Q$  consists of tangent vectors to  $Q$  at  $q$  of the form  $\text{Hor}_\tau v_q = \text{Hor} v_q - [\tau(v)]_Q(q)$ , which also defines  $v_q \mapsto \text{Hor}_\tau(v_q)$ , the  **$\tau$ -horizontal projection**. The  **$\tau$ -vertical projection operator** is defined by  $\text{Ver}_\tau(v_q) := \text{Ver}(v_q) + [\tau(v)]_Q(q)$ .

**Definition 1.4.2.** Given  $g_\sigma, g_\rho$  and  $\tau$ , the **controlled Lagrangian** equals a modified kinetic minus the given potential energy:

$$L_{\tau,\sigma,\rho}(v) = \frac{1}{2}[g_\sigma(\text{Hor}_\tau v_q, \text{Hor}_\tau v_q) + g_\rho(\text{Ver}_\tau v_q, \text{Ver}_\tau v_q)] - U(q). \quad (1.22)$$

The equations corresponding to this Lagrangian will be our closed-loop equations. The new terms appearing in those equations corresponding to the directly controlled variables are interpreted as control inputs. The modifications to the Lagrangian are chosen so that no new terms appear in the equations corresponding to the variables that are not directly controlled. We refer to this process as *matching*. Once the control law is derived using the controlled Lagrangian, the closed-loop stability of an equilibrium can be determined by energy methods, using any available freedom in the choice of  $\tau$ ,  $g_\sigma$  and  $g_\rho$ .

The advantage of this approach is that the closed-loop dynamics is itself Lagrangian, and thus stabilization can be understood in terms of energy, providing a natural Lyapunov functions and yielding a large and computable basin of attraction.

Under some reasonable assumptions on the metric  $g_\sigma$ ,  $L_{\tau,\sigma,\rho}(v)$  has the following useful structure.

**Theorem 1.4.3.** *Assume that  $g = g_\sigma$  on Hor and Hor and Ver are orthogonal for  $g_\sigma$ . Then*

$$L_{\tau,\sigma,\rho}(v) = L(v + \tau(v)_Q) + \frac{1}{2}g_\sigma(\tau(v)_Q, \tau(v)_Q) + \frac{1}{2}\varpi(v),$$

where  $v \in T_qQ$  and  $\varpi(v) = (g_\rho - g)(\text{Ver}_\tau(v), \text{Ver}_\tau(v))$ .

Locally, we will write coordinates as  $(r^\alpha, s^a)$ , where  $r^\alpha$  are coordinates for the shape space and  $s^a$  are coordinates for the symmetry group. The Lagrangian can be written

$$L(r, \dot{r}, \dot{s}) = \frac{1}{2}g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + g_{\alpha a}\dot{r}^\alpha\dot{s}^a + \frac{1}{2}g_{ab}\dot{s}^a\dot{s}^b - U(r) \quad (1.23)$$

and the controlled Lagrangian is

$$L_{\tau,\sigma}(r, \dot{r}, \dot{s}) = L(r, \dot{r}, \dot{s} + \tau_\alpha\dot{r}^\alpha) + \frac{1}{2}\sigma_{ab}\tau_\alpha^a\tau_\beta^b\dot{r}^\alpha\dot{r}^\beta,$$

where  $\sigma_{ab}$  is the group component of  $g_\sigma$ . We will let  $g^{ab}$  denote the inverse of the matrix  $g_{ab}$ , and  $\sigma^{ab}$  denote the inverse of the matrix  $\sigma_{ab}$ , etc.

First, we compare the Euler–Lagrange equations for the uncontrolled Lagrangian with the equations for the controlled Lagrangian, and observe that there are new terms in the equations for the symmetry variables. These will be interpreted as the components of the control  $u$ . The next step is to choose  $\tau$  and  $g_\sigma$  so that the Euler–Lagrange equations for both systems match. In other words, we choose them so that no new terms appear in the shape equations for the controlled Lagrangian:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} = 0, \quad (1.24)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}^a} = u_a \quad (1.25)$$

and

$$\frac{d}{dt} \frac{\partial L_{\tau,\sigma}}{\partial \dot{r}^\alpha} - \frac{\partial L_{\tau,\sigma}}{\partial r^\alpha} = 0, \quad (1.26)$$

$$\frac{d}{dt} \frac{\partial L_{\tau,\sigma}}{\partial \dot{s}^a} = u_a^{\text{diss}}. \quad (1.27)$$

Bloch, Leonard, and Marsden analyzed the conditions under which matching could be achieved, and proved the First Matching Theorem, which in local coordinates states that dynamics for the controlled system will match the dynamics for the modified Lagrangian if the following **matching assumptions** are satisfied:

**Assumption M-1.**  $\sigma_{ab}\tau_\alpha^b = -g_{\alpha a}$ ,

**Assumption M-2.**  $\sigma^{bd}(\sigma_{ad,\alpha} + g_{ad,\alpha}) = 2g^{bd}g_{ad,\alpha}$ ,

**Assumption M-3.**  $\tau_{\alpha,\delta}^b - \tau_{\delta,\alpha}^b - g^{db}g_{ad,\alpha}\tau_\delta^a = 0$ ,

where  $g_{ad,\alpha}$  is the slot derivative of the component  $g_{ad}$  with respect to  $x^\alpha$ . See [6] for details.

Alternatively, if the following *simplified matching assumptions* hold:

**Assumption S-1.**  $\sigma_{ab} = \sigma g_{ab}$ , for a scalar constant  $\sigma$ ,

**Assumption S-2.**  $g_{ab}$  is independent of  $r^\alpha$ ,

**Assumption S-3.**  $\tau_\alpha^b = -(1/\sigma)g^{ab}g_{\alpha a}$ ,

**Assumption S-4.**  $g_{\alpha a,\delta} = g_{\delta a,\alpha}$ ,

then all three of M-1–M-3 hold.

Bloch, Leonard, and Marsden [6] show the form of the control law in terms of positions and velocities of the system, along with the following theorem which shows how to stabilize the relative equilibrium. First of all, recall the definition of the amended potential  $U_p$  and the discussion of Routh reduction in Section 1.2. Let  $A_{\alpha\beta} = g_{\alpha\beta} + g_{\alpha d} [\sigma^{da} - g^{da}] g_{\beta a}$ . (Note that under the matching assumptions,  $A_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha d} g^{da} g_{\beta a} - g_{\alpha d} \tau_\beta^d$ ).

**Theorem 1.4.4.** *Suppose that the matching assumptions hold. The point  $r_e$  is a relative equilibrium if and only if it is a critical point of the amended potential  $U_p$  of  $L_{\tau,\sigma}$ . The relative equilibrium is stable if the second variation of  $E_p = \frac{1}{2}A_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + U_p$  is definite at  $r_e$ .*

**Remark 1.4.5.** *Often the Hessian of the amended potential is negative-definite at  $r_e$ . In such a case, one would require that the second variation of  $E_p$  is negative-definite.*

Thus, one can take advantage of any freedom in the choice of  $\tau$  and  $g_\sigma$  to stabilize by satisfying this theorem. Bloch, Leonard, and Marsden also go on to analyze asymptotic stability. The control law is found to be:

$$u_a = -\frac{d}{dt}(g_{ab}\tau_\alpha^b\dot{\phi}^\alpha) = -\left(g_{ab,\delta}\tau_\alpha^b\dot{r}^\alpha\dot{r}^\delta + g_{ab}\tau_{\alpha,\delta}^b\dot{r}^\alpha\dot{r}^\delta + g_{ab}\tau_\alpha^b\ddot{r}^\alpha\right). \quad (1.28)$$

We note that the matching assumptions impose restrictions on the kinetic energy metric  $g$ , thus limiting the applicability of the method. The restrictions on  $\tau$  and  $g_\sigma$  also restrict our choices when designing a controller. This motivates us to seek a relaxed matching technique, which will be the subject of the next chapter.

## 1.5 Discrete Mechanics

**Discrete Hamilton's Principle and Discrete Euler–Lagrange Equations.** A discrete analogue of Lagrangian mechanics can be obtained by considering a discretization of Hamilton's principle; this approach underlies the construction of variational integrators. See Marsden and West [25], and references therein, for a more detailed exposition of discrete mechanics. The origins of discrete mechanics are in the optimal control literature of the 1960s, see Jordan and Polak [8], Cadzow [7]. For the recent development we refer the reader to Veselov [33], Marsden and Wendlandt [34, 24], and Kane, Marsden, Ortiz and West [9].

Consider a Lagrangian mechanical system with configuration manifold  $Q$  and Lagrangian  $L : TQ \rightarrow \mathbb{R}$ . A key notion is that of a *discrete Lagrangian*, which is a map  $L^d : Q \times Q \rightarrow \mathbb{R}$  that approximates the action integral along an exact solution of the Euler–Lagrange equations

joining the configurations  $q_k, q_{k+1} \in Q$ ,

$$L^d(q_k, q_{k+1}) \approx \underset{q \in \mathcal{C}([0, h], Q)}{\text{ext}} \int_0^h L(q, \dot{q}) dt, \quad (1.29)$$

where  $\mathcal{C}([0, h], Q)$  is the space of curves  $q : [0, h] \rightarrow Q$  with  $q(0) = q_k$ ,  $q(h) = q_{k+1}$ , and  $\text{ext}$  denotes extremum.

In the discrete setting, the action integral of Lagrangian mechanics is replaced by an action sum

$$S^d(q_0, q_1, \dots, q_N) = \sum_{k=0}^{N-1} L^d(q_k, q_{k+1}),$$

where  $q_k \in Q$ ,  $k = 0, 1, \dots, N$ , is a finite sequence of points in the configuration space. The equations are obtained by the discrete Hamilton principle, which extremizes the discrete action given fixed endpoints  $q_0$  and  $q_N$ . Taking the extremum over  $q_1, \dots, q_{N-1}$  gives the *discrete Euler–Lagrange equations*

$$D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k) = 0$$

for  $k = 1, \dots, N - 1$ .

Since we are concerned with control, we need to consider the effect of external forces on Lagrangian systems. In the context of discrete mechanics, this is addressed by introducing the *discrete Lagrange–d’Alembert principle* (see Kane, Marsden, Ortiz, and West [9]), which states that

$$\delta \sum_{k=0}^{n-1} L^d(q_k, q_{k+1}) + \sum_{k=0}^{n-1} F^d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = 0$$

for all variations  $(\delta q_0, \delta q_1, \dots, \delta q_N)$  of  $(q_0, q_1, \dots, q_N)$  that vanish at the endpoints, i.e.,  $\delta q_0 = \delta q_N = 0$ . The discrete one-form  $F^d$  on  $Q \times Q$  approximates the impulse integral between the points  $q_k$  and  $q_{k+1}$ , just as the discrete Lagrangian  $L^d$  approximates the action integral. We

define the maps  $F_1^d, F_2^d : Q \times Q \rightarrow T^*Q$  by the relations

$$\begin{aligned} F_1^d(q_0, q_1) \delta q_0 &:= F^d(q_0, q_1) \cdot (\delta q_0, 0), \\ F_2^d(q_0, q_1) \delta q_1 &:= F^d(q_0, q_1) \cdot (0, \delta q_1). \end{aligned}$$

The discrete Lagrange–d’Alembert principle may then be rewritten as

$$\delta \sum_{k=0}^{n-1} L^d(q_k, q_{k+1}) + \sum_{k=0}^{n-1} \left[ F_1^d(q_k, q_{k+1}) \delta q_k + F_2^d(q_k, q_{k+1}) \delta q_{k+1} \right] = 0$$

for all variations  $\delta \mathbf{q}$  of  $\mathbf{q}$  that vanish at the endpoints. This is equivalent to the *forced discrete Euler–Lagrange equations*

$$D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k) + F_1^d(q_k, q_{k+1}) + F_2^d(q_{k-1}, q_k) = 0. \quad (1.30)$$

Equations (1.30) implicitly define the *update map*  $\Phi : Q \times Q \rightarrow Q \times Q$ , where  $\Phi(q_{k-1}, q_k) = (q_k, q_{k+1})$  and where  $Q \times Q$  replaces the velocity phase space  $TQ$  of Lagrangian mechanics.

**Discrete Mechanical Systems with Symmetry.** As in the continuous time case, we only discuss here systems with commutative symmetry.

Let  $L^d : Q \times Q \rightarrow \mathbb{R}$  be a discrete Lagrangian. Assume that the configuration manifold is a principle fiber bundle with a base space  $S$  and a commutative group  $G$ . Recall that the local configuration variables in this case are be written as  $(r, s)$ , where  $r = (r^1, \dots, r^\sigma) \in S$  and  $s = (s^1, \dots, s^m) \in G$ .

**Definition 1.5.1.** *The (group) variables  $s = (s^1, \dots, s^m)$  are called **cyclic** if the discrete Lagrangian  $L^d$  depends on  $s_k$  and  $s_{k+1}$  through the combination  $\Delta s_k = s_{k+1} - s_k$ , that is,  $L^d = L^d(r_k, r_{k+1}, \Delta s_k)$ .*

The discrete Euler–Lagrange equations in this setting become

$$D_2L^d(r_{k-1}, r_k, \Delta s_{k-1}) + D_1L^d(r_k, r_{k+1}, \Delta s_k) = 0, \quad (1.31)$$

$$D_3L^d(r_{k-1}, r_k, \Delta s_{k-1}) - D_3L^d(r_k, r_{k+1}, \Delta s_{k+1}) = 0, \quad (1.32)$$

which are the discrete analogue of equations (1.8).

Define the *discrete conjugate momentum* by the formula

$$p_k = D_3L^d(r_k, r_{k+1}, \Delta s_{k+1}). \quad (1.33)$$

Equation (1.32) then states that the discrete Euler–Lagrange equations conserve the discrete conjugate momentum, i.e.,

$$p_{k-1} = p_k.$$

along the trajectories of (1.31) and (1.32).

**Discretization of Continuous-Time Systems.** In the rest of this paper, we will adopt the notations

$$q_{k+1/2} = \frac{q_k + q_{k+1}}{2} \quad \text{and} \quad \Delta q_k = q_{k+1} - q_k.$$

Given a continuous-time mechanical system with Lagrangian  $L : TQ \rightarrow \mathbb{R}$ , we define its discrete analogue by introducing the *second-order accurate* discrete Lagrangian by the formula

$$L^d(q_k, q_{k+1}) = hL(q_{k+1/2}, \Delta q_k/h).$$

Discretization of nonconservative forces is performed in a similar manner.

## 1.6 Stability Analysis for Discrete Systems

Stability of equilibria and relative equilibria of discrete systems is checked in a manner similar to that of the continuous-time case. Let  $\Psi : M \rightarrow M$  be a diffeomorphism and

$$x_{k+1} = \Psi x_k \tag{1.34}$$

be the corresponding discrete dynamical system. Let  $x_e \in M$  be an equilibrium, that is,  $x_e = \Psi x_e$ .

A function  $V$  is called a Lyapunov function if it is positive-definite (in a neighborhood of  $x_e$ ) and  $V(x_{k+1}) \leq V(x_k)$  for all  $k \in \mathbb{Z}$ .

**Theorem 1.6.1.** *If there exists a Lyapunov function for an equilibrium  $x_e$  of (1.34), this equilibrium is stable.*

**Theorem 1.6.2.** *The equilibrium  $a \in M$  of the discrete dynamical system (1.34) is asymptotically stable if the spectrum of the linearization of (1.34) at  $a$  belongs to the open unit disk. If there is an eigenvalue outside the closed unit disk, the equilibrium is unstable.*

In a similar way, one can state the analogues of Theorems 1.3.2 and 1.3.5.

## Chapter 2

# Matching and Stabilization of Discrete Mechanical Systems

This chapter focuses on the stabilization of relative equilibria of discrete mechanical systems with  $n$ -dimensional shape space and  $m$ -dimensional group. The symmetry group is assumed commutative, and the configuration space is the direct product of the shape space and the said symmetry group. The chapter develops the basic matching conditions and stability conditions for discrete Lagrangian systems.

### 2.1 Matching with Nonconservative Forcing

#### 2.1.1 Relaxed Matching in the Continuous-Time Case

We start by reviewing the relaxed matching techniques for continuous-time Lagrangian systems. Recall that it is assumed that the configuration space  $Q$  is the direct product of an  $n$ -dimensional shape space  $S$  and an  $m$ -dimensional commutative Lie group  $G$ .

The configuration variables are written as  $q = (\phi, s) = (\phi^1, \dots, \phi^n, s^1, \dots, s^m) = (\phi^\alpha, s^a)$ ,  $\alpha = 1, \dots, n$ ,  $a = 1, \dots, m$ , where  $\phi \in S$  and  $s \in G$ . The continuous-time Lagrangian equals

the kinetic minus potential energy of the system and is given by the formula

$$L(\phi, \dot{\phi}, \dot{s}) = \frac{1}{2} [g_{\alpha\beta}(\phi) \dot{\phi}^\alpha \dot{\phi}^\beta + 2g_{\alpha a}(\phi) \dot{\phi}^\alpha \dot{s}^a + g_{ab}(\phi) \dot{s}^a \dot{s}^b] - V(\phi). \quad (2.1)$$

This Lagrangian is  $G$ -invariant. The corresponding controlled Euler–Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}^\alpha} - \frac{\partial L}{\partial \phi^\alpha} = 0, \quad (2.2)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}^a} = u_a, \quad (2.3)$$

where  $u_a$  is the control input.

Recall that the original techniques of Bloch, Leonard, and Marsden require that a set of matching conditions M-1–M-3 be satisfied. These matching conditions often are very restrictive and may be even incompatible. Addressing this incompatibility, Long *et al.* [14, 15] developed the *relaxed matching* formalism. We will now expose the basics of this formalism. It turns out that in relaxed matching, the choice of the modification to the horizontal metric,  $\rho_\sigma$ , has no effect, so we set  $\sigma = 0$  and define the controlled Lagrangian  $L_\tau$  by the following formula:

$$\begin{aligned} L_\tau(\phi, \dot{\phi}, \dot{s}) &= L(\phi, \dot{\phi}, \dot{s} + \tau \dot{\phi}) \\ &= \frac{1}{2} g_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta + g_{\alpha a} \dot{\phi}^\alpha (\dot{s}^a + \tau_\beta^a \dot{\phi}^\beta) + \frac{1}{2} g_{ab} (\dot{s}^a + \tau_\alpha^a \dot{\phi}^\alpha) (\dot{s}^b + \tau_\beta^b \dot{\phi}^\beta) - V(\phi). \end{aligned}$$

The corresponding Euler–Lagrange equations are

$$\frac{d}{dt} \frac{\partial L_\tau}{\partial \dot{\phi}^\alpha} - \frac{\partial L_\tau}{\partial \phi^\alpha} = w_\alpha, \quad (2.4)$$

$$\frac{d}{dt} \frac{\partial L_\tau}{\partial \dot{s}^a} = u_a^{\text{diss}}, \quad (2.5)$$

where  $w_\alpha$  are the components of the non-conservative force that we introduce to generalize the matching procedure of [6], and where the dissipation-emulating terms  $u_a^{\text{diss}}$  are included for asymptotic stabilization. Defining the controlled momentum  $\tilde{J}_a = \frac{\partial L_\tau}{\partial \dot{s}^a} = g_{\alpha a} \dot{\phi}^\alpha + g_{ab} \dot{s}^b +$

$g_{ab}\tau_\alpha^b\dot{\phi}^\alpha$ , equation (2.5) becomes equivalent to

$$\frac{d}{dt}\tilde{J}_a = u_a^{\text{diss}}.$$

**Theorem 2.1.1** ([15]). *The controlled Euler–Lagrange equations (2.2) and (2.3) are identical to the Euler–Lagrange equations (2.4) and (2.5) for the controlled Lagrangians  $L_\tau$  when the following conditions are satisfied:*

$$u_a = u_a^{\text{cons}} + u_a^{\text{diss}}, \quad u_a^{\text{cons}} = -\frac{d}{dt}\left(g_{ab}\tau_\alpha^b\dot{\phi}^\alpha\right), \quad (2.6)$$

$$\begin{aligned} w_\alpha &= g_{\alpha a}\tau_\beta^a\ddot{\phi}^\beta + \tau_\alpha^a u_a^{\text{diss}} + \left(\partial_\gamma(\tau_\alpha^a)\tilde{J}_a - g^{bc}\tilde{J}_c\partial_\alpha(g_{ab}\tau_\gamma^a)\right)\dot{\phi}^\gamma \\ &+ \left[\partial_\gamma(g_{\alpha a}\tau_\beta^a) - \partial_\alpha(g_{\gamma a}\tau_\beta^a + \frac{1}{2}g_{ab}\tau_\gamma^a\tau_\beta^b) + \partial_\alpha(g_{ab}\tau_\beta^a)g^{bc}g_{\gamma c} + \partial_\alpha(g_{ab}\tau_\beta^a)\tau_\gamma^b\right]\dot{\phi}^\beta\dot{\phi}^\gamma. \end{aligned} \quad (2.7)$$

Recall that the goal is to feedback-stabilize relative equilibria of system (2.2) and (2.3). We make certain assumptions which allow the stability of the shape equations to be analyzed independently of the controlled momentum level. After showing asymptotic stability with respect to the shape variables, we show that, under some additional assumptions, the controlled momentum is defined for all future values of time, and that it converges to a constant value as  $t \rightarrow \infty$ .

Define  $A_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha a}g^{ab}g_{\beta b} - g_{\alpha a}\tau_\beta^a$ .

**Assumption R-1.** *The form  $\tau$  is chosen so that  $A_{\alpha\beta}$  is symmetric in  $\alpha$  and  $\beta$ ; that is,  $g_{\alpha a}\tau_\beta^a = g_{\beta a}\tau_\alpha^a$ .*

Note that when simplified matching conditions  $\sigma_{ab} = \sigma g_{ab}$  for constant scalar  $\sigma$  and  $\tau_\alpha^b = -(1/\sigma)g^{ab}g_{\alpha a}$  of Bloch, Leonard, and Marsden hold, assumption R-1 will hold, since,

$$g_{\beta b}\tau_\alpha^b = -(1/\sigma)g_{\beta b}g^{ab}g_{\alpha a} = -(1/\sigma)g_{\alpha b}g^{ab}g_{\beta a} = g_{\alpha b}\tau_\beta^b.$$

**Assumption R-2.**  $g_{ab} = \text{const.}$

This is just one of the simplified matching conditions in Bloch, Leonard, and Marsden. There are many interesting examples where this condition is satisfied, such as the pendulum on a cart (see Section 2.3) and the spherical pendulum on a hockey puck (see [6]).

**Assumption R-3.**  $g_{\alpha a, \delta} = g_{\delta a, \alpha}$ ,

so that

$$\begin{aligned} & A_{\alpha\beta} \ddot{\phi}^\beta + g_{\alpha a} g^{ab} u_b^{\text{diss}} + U_{,\alpha} + \left[ A_{\alpha\beta,\gamma} - \frac{1}{2} g_{\gamma\beta,\alpha} + g_{\gamma b,\alpha} (g^{ab} g_{\beta a} + \tau_\beta^b) \right] \dot{\phi}^\beta \dot{\phi}^\gamma \\ &= A_{\alpha\beta} \ddot{\phi}^\beta + g_{\alpha a} g^{ab} u_b^{\text{diss}} + U_{,\alpha} + \left[ g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\gamma\beta,\alpha} - g_{\alpha a} g^{ab} g_{\beta b,\gamma} - g_{\alpha a} \tau_{\beta,\gamma}^a \right] \dot{\phi}^\beta \dot{\phi}^\gamma = 0. \end{aligned} \quad (2.8)$$

Assumption R-3 is identical to the simplified matching assumption S-4. It will always hold for systems with one shape degree of freedom.

Notice that by assuming  $g_{\alpha a, \gamma} = g_{\gamma a, \alpha}$ , the shape equations become independent of momentum. Thus, the dynamics after reduction are not dependent on the selection of momentum level. However they are still dependent on  $u^{\text{diss}}$ , which is the time derivative of the momentum. In this manner the dynamics are still coupled. Since the dynamics of the shape variable do not depend on the momentum levels, neither do the relative equilibria. So the value of the shape variable at relative equilibria is the same for all momentum levels. This can be seen, for example, in the pendulum on the cart. There is a relative equilibrium corresponding to each motion of the system with constant linear momentum and with the pendulum in the upward vertical position.

Since the dynamics of the shape variables are independent of momentum, stability with respect to the shape variables can be studied separately.

### 2.1.2 Discrete Relaxed Matching

Recall that, given a continuous-time Lagrangian system and using the mid-point approximation, a *second-order accurate* discrete Lagrangian is constructed:

$$L^d(q_k, q_{k+1}) = hL(q_{k+1/2}, \Delta q_k/h).$$

Recall that

$$q_{k+\frac{1}{2}} = \frac{q_k + q_{k+1}}{2}, \quad \Delta q_k = q_{k+1} - q_k,$$

where  $q_k = (\phi_k, s_k) = (\phi_k^1, \dots, \phi_k^n, s_k^1, \dots, s_k^m)$ . Thus, for a system with  $n$ -dimensional shape space and  $m$ -dimensional group the discrete  $G$ -invariant Lagrangian is given by the formula

$$L^d(q_k, q_{k+1}) = \frac{h}{2} \left[ g_{\alpha\beta}(\phi_{k+1/2}) \frac{\Delta\phi_k^\alpha}{h} \frac{\Delta\phi_k^\beta}{h} + 2g_{\alpha a}(\phi_{k+1/2}) \frac{\Delta\phi_k^\alpha}{h} \frac{\Delta s_k^a}{h} + g_{ab} \frac{\Delta s_k^a}{h} \frac{\Delta s_k^b}{h} \right] - hV(\phi_{k+1/2}), \quad (2.9)$$

with  $g$  and  $V$  coming from the continuous-time Lagrangian. The discrete dynamics is governed by the equations

$$\frac{\partial L^d(q_k, q_{k+1})}{\partial\phi_k^\alpha} + \frac{\partial L^d(q_{k-1}, q_k)}{\partial\phi_k^\alpha} = 0, \quad (2.10)$$

$$\frac{\partial L^d(q_k, q_{k+1})}{\partial s_k^a} + \frac{\partial L^d(q_{k-1}, q_k)}{\partial s_k^a} + u_{k,a} = 0, \quad (2.11)$$

where  $u_{k,a}$ ,  $a = 1, \dots, m$  are the discrete control inputs. The system is underactuated as the control only appears in the group directions.

### 2.1.3 Discrete Matching Conditions

As in the continuous-time case, it is assumed that relative equilibria  $\phi_k^\alpha = \phi_e, \Delta s_k = \xi_e$  of equations (2.10) and (2.11) in the absence of control inputs are unstable.

Motivated by the continuous-time matching procedure (see Section 2.1), we define the dis-

crete controlled Lagrangian  $L_\tau^d(q_k, q_{k+1})$  by the formula

$$L_\tau^d(q_k, q_{k+1}) = hL_\tau(q_{k+1/2}, \Delta q_k/h) = hL^d(\phi_{k+1/2}, \Delta\phi_k/h, (\Delta s_k + \tau(\phi_{k+1/2})\Delta\phi_k)/h). \quad (2.12)$$

The dynamics associated with the controlled Lagrangian (2.12) is:

$$\frac{\partial L_\tau^d(q_k, q_{k+1})}{\partial \phi_k^\alpha} + \frac{\partial L_\tau^d(q_{k-1}, q_k)}{\partial \phi_k^\alpha} + w_{k,\alpha} = 0 \quad (2.13)$$

$$\frac{\partial L_\tau^d(q_k, q_{k+1})}{\partial s_k^a} + \frac{\partial L_\tau^d(q_{k-1}, q_k)}{\partial s_k^a} + u_{k,a}^{\text{diss}} = 0. \quad (2.14)$$

where  $w_{k,\alpha}$  are the components of the discrete non-conservative force, and the dissipation-emulating terms  $u_{k,a}^{\text{diss}}$  are included for asymptotic stabilization.

Define the controlled momentum  $J_k$  by the formula

$$J_{k,a} = -\frac{\partial}{\partial s_k^a} L_\tau^d(q_k, q_{k+1}) = g_{\alpha a} \frac{\Delta \phi_k^\alpha}{h} + g_{ab} \frac{\Delta s_k^b + \tau_\alpha^b \Delta \phi_k^\alpha}{h}, \quad a = 1, \dots, m. \quad (2.15)$$

**Theorem 2.1.2.** *The dynamics (2.10), (2.11) are equivalent to the dynamics (2.13), (2.14) if and only if the following conditions are satisfied:*

$$u_{k,a} = u_{k,a}^{\text{cons}} + u_{k,a}^{\text{diss}}, \quad (2.16)$$

$$u_{k,a}^{\text{cons}} = -\frac{1}{h} \left[ g_{ab} \tau_\alpha^b(\phi_{k+1/2}) \Delta \phi_k^\alpha - g_{ab} \tau_\alpha^b(\phi_{k-1/2}) \Delta \phi_{k-1}^\alpha \right], \quad (2.17)$$

$$\begin{aligned} w_{k,\alpha} = & -\frac{1}{h} \partial_{\phi_k^\alpha} \left( g_{\gamma\alpha}(\phi_{k+1/2}) \tau_\beta^a(\phi_{k+1/2}) \Delta \phi_k^\beta + \frac{1}{2} g_{ab} \tau_\gamma^a(\phi_{k+1/2}) \tau_\beta^b(\phi_{k+1/2}) \Delta \phi_k^\gamma \Delta \phi_k^\beta \right) \\ & - \frac{1}{h} \left[ \partial_{\phi_k^\alpha} \left( g_{ab} \tau_\gamma^a(\phi_{k+1/2}) \Delta \phi_k^\gamma \right) \right] \left[ g^{db} \left( h J_{k,d} - g_{\beta d}(\phi_{k+1/2}) \Delta \phi_k^\beta \right) - \tau_\beta^b(\phi_{k+1/2}) \Delta \phi_k^\beta \right] \\ & - \frac{1}{h} \partial_{\phi_k^\alpha} \left( g_{\gamma\alpha}(\phi_{k-1/2}) \tau_\beta^a(\phi_{k-1/2}) \Delta \phi_{k-1}^\beta + \frac{1}{2} g_{ab} \tau_\gamma^a(\phi_{k-1/2}) \tau_\beta^b(\phi_{k-1/2}) \Delta \phi_{k-1}^\gamma \Delta \phi_{k-1}^\beta \right) \\ & - \frac{1}{h} \left[ \partial_{\phi_k^\alpha} \left( g_{ab} \tau_\gamma^a(\phi_{k-1/2}) \Delta \phi_{k-1}^\gamma \right) \right] \left[ g^{db} \left( h J_{k-1,d} - g_{\beta d}(\phi_{k-1/2}) \Delta \phi_{k-1}^\beta \right) - \tau_\beta^b(\phi_{k-1/2}) \Delta \phi_{k-1}^\beta \right]. \end{aligned} \quad (2.18)$$

*Proof.* Using decomposition (2.16) and the requirement that systems (2.10), (2.11) and (2.13), (2.14) are identical, we obtain

$$u_{k,a}^{\text{cons}} = \frac{\partial}{\partial s_k^a} \left[ L_\tau^d(q_k, q_{k+1}) - L^d(q_k, q_{k+1}) \right] - \frac{\partial}{\partial s_k^a} \left[ L_\tau^d(q_{k-1}, q_k) - L^d(q_{k-1}, q_k) \right], \quad (2.19)$$

$$w_{k,a} = -\frac{\partial}{\partial \phi_k^a} \left[ L_\tau^d(q_k, q_{k+1}) - L^d(q_k, q_{k+1}) \right] + \frac{\partial}{\partial \phi_k^a} \left[ L_\tau^d(q_{k-1}, q_k) - L^d(q_{k-1}, q_k) \right]. \quad (2.20)$$

Next, we obtain

$$\begin{aligned} L_\tau^d(q_k, q_{k+1}) - L^d(q_k, q_{k+1}) &= \frac{1}{h} \left[ g_{\alpha\alpha}(\phi_{k+1/2}) \tau_\beta^a(\phi_{k+1/2}) \Delta\phi_k^\alpha \Delta\phi_k^\beta \right. \\ &\quad \left. + g_{ab} \tau_\beta^b(\phi_{k+1/2}) \Delta\phi_k^\beta \Delta s_k^a + \frac{1}{2} g_{ab} \tau_\alpha^a(\phi_{k+1/2}) \tau_\beta^b(\phi_{k+1/2}) \Delta\phi_k^\alpha \Delta\phi_k^\beta \right]. \end{aligned} \quad (2.21)$$

Formulae (2.19) and (2.21) imply (2.17).

From the definition of the discrete controlled momentum (2.15), we obtain

$$\Delta s_k^b = g^{db} (h J_{k,d} - g_{\alpha d}(\phi_{k+1/2}) \Delta\phi_k^\alpha) - \tau_\alpha^b(\phi_{k+1/2}) \Delta\phi_k^\alpha, \quad (2.22)$$

Formulae (2.20), (2.21), and (2.22) imply (2.18).  $\square$

## 2.2 Stabilization of Relative Equilibria of Discrete Systems

Here we establish the discrete analogue of the results of Long *et al.* [14, 15].

Recall that relative equilibria of a continuous-time system with symmetry are solutions with constant shape configuration and constant group velocity. The group dynamics itself need not to be constant. A good illustration is the pendulum on a (horizontally moving) cart: The relative equilibria are uniform rigid-body motions of the system. That is, the pendulum is not moving relative to the cart, while the cart is moving at a constant speed.

In a similar manner, a relative equilibrium of a discrete mechanical system is a motion with constant shape configuration and constant group increments, i.e.,  $\phi_k^\alpha = \phi_e^\alpha$  and  $\Delta s_k^a = \Delta s_e^a$ . As

pointed out earlier, we assume, without loss of generality,  $\phi_e = 0$ .

**Theorem 2.2.1.** *Define the discrete dissipation-emulating terms by the formula*

$$u_{k,b}^{\text{diss}} = \frac{1}{2h} [c_b^d(\phi_{k+1/2})g_{\beta d}(\phi_{k+1/2})\Delta\phi_k^\beta + c_b^d(\phi_{k-1/2})g_{\beta d}(\phi_{k-1/2})\Delta\phi_{k-1}^\beta], \quad (2.23)$$

and suppose that assumptions R-1, R-2, and R-3 hold. Assume further that

1.  $A_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha a}g^{ab}g_{\beta b} - g_{\alpha a}\tau_\beta^a$  is a negative-definite matrix in a neighborhood of  $\phi = 0$ ,  
and
2.  $c_b^d$  are chosen so that  $g_{\alpha a}g^{ab}c_b^d g_{\beta d}$  is a positive-definite matrix in a neighborhood of  $\phi = 0$ .

Then for all solutions of system (2.10), (2.11) with initial shape state  $(\phi_0, \phi_1)$  sufficiently close to the origin and with  $u_{k,a}^{\text{cons}}$  defined by (2.17), the shape configuration variables converge asymptotically to  $\phi_e = 0$ .

*Proof.* Recall that the dissipation-emulating terms are defined by formula (2.23). Using the definition of the controlled momentum,  $J_{k,a} = -\partial_{s_k^a} L_\tau^d(q_k, q_{k+1})$ , equation (2.14) becomes

$$-J_{k,b} + J_{k-1,b} + \frac{1}{2h} [c_b^d(\phi_{k+1/2})g_{\beta d}(\phi_{k+1/2})\Delta\phi_k^\beta + c_b^d(\phi_{k-1/2})g_{\beta d}(\phi_{k-1/2})\Delta\phi_{k-1}^\beta] = 0. \quad (2.24)$$

The linearization of controlled momentum equation (2.24) at a relative equilibrium is:

$$-J_{k,b} + J_{k-1,b} + \frac{1}{2h} c_b^d(0)g_{\beta d}(0) [\Delta\phi_k^\beta + \Delta\phi_{k-1}^\beta] = 0. \quad (2.25)$$

Next, we linearize shape equation (2.10) (or, equivalently, equation (2.13)). Recall assumption R-2 that states  $g_{ab} = \text{const}$ . Straightforward calculation shows that the left-hand side of

(2.10) is (recall that  $f_{,\alpha}$  denotes partial differentiation with respect to  $\phi^\alpha$ ):

$$\begin{aligned}
& \frac{1}{h} \left[ \frac{1}{2} g_{\gamma\beta,\alpha}(\phi_{k+\frac{1}{2}}) \Delta\phi_k^\gamma \Delta\phi_k^\beta - g_{\alpha\beta}(\phi_{k+\frac{1}{2}}) \Delta\phi_k^\beta \right. \\
& \quad \left. + g_{\beta a,\alpha}(\phi_{k+\frac{1}{2}}) \Delta\phi_k^\beta \Delta s_k^a - g_{\alpha a}(\phi_{k+\frac{1}{2}}) \Delta s_k^a \right] - \frac{h}{2} U_{,\alpha}(\phi_{k+\frac{1}{2}}) \\
& + \frac{1}{h} \left[ \frac{1}{2} g_{\gamma\beta,\alpha}(\phi_{k-\frac{1}{2}}) \Delta\phi_{k-1}^\gamma \Delta\phi_{k-1}^\beta + g_{\alpha\beta}(\phi_{k-\frac{1}{2}}) \Delta\phi_{k-1}^\alpha \right. \\
& \quad \left. + g_{\beta a,\alpha}(\phi_{k-\frac{1}{2}}) \Delta\phi_{k-1}^\beta \Delta s_{k-1}^a + g_{\alpha a}(\phi_{k-\frac{1}{2}}) \Delta s_{k-1}^a \right] - \frac{h}{2} U_{,\alpha}(\phi_{k-\frac{1}{2}}). \tag{2.26}
\end{aligned}$$

Next, using the controlled momentum equations (2.14), the group increments  $\Delta s_k$  are eliminated from equations (2.26). It is straightforward to see that the resulting equations are independent of the controlled momenta  $J_{k-1}$  and  $J_k$ . This is an important observation that we will use later in justifying nonlinear stability of relative equilibria. But first, it is important to obtain the linearized equations and study its spectrum.

Linearizing at the origin (2.26) (recall that the equilibrium value of the shape variables  $\phi$  is zero), we obtain

$$\begin{aligned}
& \frac{1}{h^2} A_{\alpha\beta}(0) (\Delta\phi_k^\beta - \Delta\phi_{k-1}^\beta) - \frac{1}{2h} g^{ac}(0) g_{\alpha a}(0) g_{\beta b}(0) c_c^b(0) (\Delta\phi_k^\beta + \Delta\phi_{k-1}^\beta) \\
& \quad + \frac{1}{2} U_{\alpha\beta}(0) (\phi_{k+1/2} + \phi_{k-1/2}) = 0. \tag{2.27}
\end{aligned}$$

Here  $U_{\alpha\beta}(0)$  are the components of the potential energy's Hessian evaluated at the equilibrium.

When  $h \rightarrow 0$ , the linearized discrete shape equations (2.27) converge to

$$A_{\alpha\beta} \ddot{\phi}^\beta - g^{ac}(0) g_{\alpha a}(0) g_{\beta b}(0) c_c^b(0) \dot{\phi} + U_{\alpha\beta}(0) \phi^\beta = 0,$$

which are the linearized continuous-time shape equations, as should be expected.

The energy for this linearized discrete shape dynamics (2.27) is

$$E_{k,k+1} = \frac{1}{2h} A_{\alpha\beta}(0) \Delta\phi_k^\alpha \Delta\phi_k^\beta + \frac{1}{2} U_{\alpha\beta}(0) \phi_{k+\frac{1}{2}}^\alpha \phi_{k+\frac{1}{2}}^\beta. \tag{2.28}$$

By choosing the form  $\tau$  appropriately, the matrix  $A_{\alpha\beta}(0)$  becomes negative-definite, thus making the energy (2.28) negative-definite.

Using the energy  $E_{k,k+1}$  as a Lyapunov function for the linearized shape dynamics. Computing the discrete energy increment, we obtain

$$\begin{aligned}\Delta E_k &= E_{k,k+1} - E_{k-1,k} \\ &= \frac{1}{2h^2} A_{\alpha\beta}(0) \left[ \Delta\phi_k^\alpha \Delta\phi_k^\beta - \Delta\phi_{k-1}^\alpha \Delta\phi_{k-1}^\beta \right] + \frac{1}{2} U_{\alpha\beta}(0) \left[ \phi_{k+\frac{1}{2}}^\alpha \phi_{k+\frac{1}{2}}^\beta - \phi_{k-\frac{1}{2}}^\alpha \phi_{k-\frac{1}{2}}^\beta \right].\end{aligned}\quad (2.29)$$

From (2.27),

$$\begin{aligned}\frac{1}{2h^2} A_{\alpha\beta}(0) \left( \Delta\phi_k^\alpha + \Delta\phi_{k-1}^\alpha \right) \left( \Delta\phi_k^\beta - \Delta\phi_{k-1}^\beta \right) &+ \frac{1}{2} U_{\alpha\beta}(0) \left( \Delta\phi_k^\alpha + \Delta\phi_{k-1}^\alpha \right) \left( \phi_{k+\frac{1}{2}}^\beta + \phi_{k-\frac{1}{2}}^\beta \right) \\ &= \frac{1}{2h} g^{ca}(0) g_{\alpha a}(0) g_{\beta d}(0) c_c^d(0) \left( \Delta\phi_k^\alpha + \Delta\phi_{k-1}^\alpha \right) \left( \Delta\phi_k^\beta - \Delta\phi_{k-1}^\beta \right),\end{aligned}$$

which implies

$$\begin{aligned}\Delta E_k &= E_{k,k+1} - E_{k-1,k} \\ &= \frac{1}{2h^2} A_{\alpha\beta}(0) \left( \Delta\phi_k^\alpha \Delta\phi_k^\beta - \Delta\phi_{k-1}^\alpha \Delta\phi_{k-1}^\beta \right) \\ &\quad + \frac{1}{2} U_{\alpha\beta}(0) \left( \phi_{k+\frac{1}{2}}^\alpha \phi_{k+\frac{1}{2}}^\beta - \phi_{k-\frac{1}{2}}^\alpha \phi_{k-\frac{1}{2}}^\beta \right) \\ &= \frac{1}{2h} g^{ca}(0) g_{\alpha a}(0) g_{\beta d}(0) c_c^d(0) \left( \Delta\phi_k^\beta - \Delta\phi_{k-1}^\beta \right) \left( \Delta\phi_k^\alpha - \Delta\phi_{k-1}^\alpha \right).\end{aligned}\quad (2.30)$$

The discrete energy increment  $\Delta E_k$  is non-negative because of the assumption of the theorem that the matrix

$$g^{ca}(0) g_{\alpha a}(0) g_{\beta d}(0) c_c^d(0)$$

is negative-definite.

Using the discrete LaSalle principle, we conclude that the equilibrium of the linearized shape dynamics is asymptotically stable, which implies stability (nonasymptotic) of the linearized momentum dynamics. This implies that the spectrum of the linearized shape dynamics belongs

to the open unit disk. It is obvious that the spectrum of the linearized momentum dynamics consists of the unit eigenvalues.

Nonlinear stability follows from the center manifold based stability analysis. This approach originated in Lyapunov [16] in the continuous-time setting, and is applicable, almost without changes, to the discrete case. The center manifold in this setting is given by the equations  $\phi = 0$ ,  $\Delta\phi = 0$ . The dynamics on the center manifold is trivial: Each trajectory is a equilibrium  $\phi_k = 0$ ,  $\Delta\phi_k = 0$ ,  $J_k = \text{const}$ . The unit part of the spectrum of the linearization at each of these equilibria corresponds to the “momentum directions”. The stable part of the spectrum at each equilibrium corresponds to the stable manifold directions. The entire reduced phase space in a (sufficiently small) neighborhood of the equilibrium of interest is foliated by the stable manifolds. Given an initial condition near this linearly stable equilibrium, the trajectory subject to that initial condition will stay on one of the nearby stable manifolds, and therefore will converge to one of the nearby equilibria on the center manifold. This implies Lyapunov stability, which in addition is asymptotic in the directions of the stable manifolds.  $\square$

**Remark 2.2.2.** *Stability conditions for the discrete dynamics are identical to these of the corresponding continuous-time system.*

## 2.3 The Cart-Pendulum System

To illustrate the techniques developed above, we now apply them to the problem of stabilization of an inverted pendulum on a cart, see Figure 2.1. Let  $s$  denote the position of the cart on the  $s$ -axis and let  $\theta$  denote the angle of the pendulum with the upright vertical, as in Figure 2.1. The configuration space for this system is  $Q = S \times G = S^1 \times \mathbb{R}$ , with the first factor being the pendulum angle  $\theta$  and the second factor being the cart position  $s$ . The velocity phase space,  $TQ$ , has coordinates  $(\theta, s, \dot{\theta}, \dot{s})$ . The length of the pendulum is  $l$ , the mass of the pendulum is  $m$  and that of the cart is  $M$ . The symmetry group  $G$  of the pendulum-cart system is that of translation in the  $s$  variable, so  $G = \mathbb{R}$ . We do not destroy this symmetry when doing

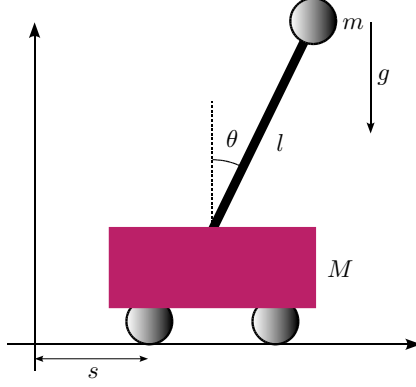


Figure 2.1: An inverted pendulum on a cart

stabilization in  $\theta$ .

The (continuous-time) Lagrangian for this system is

$$L(\theta, \dot{\theta}, \dot{s}) = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta \cos \theta \dot{s}\dot{\theta} + \gamma\dot{s}^2) - U(\theta), \quad (2.31)$$

where  $\alpha = ml^2, \beta = ml, \gamma = M + m$  are constants and  $U(\theta) = mgl(1 - \cos \theta)$  is the potential energy. Note that  $\alpha\gamma - \beta^2 > 0$ .

The upward equilibrium of the pendulum is unstable and the potential energy is positive-definite at this equilibrium.

The corresponding discrete Lagrangian reads:

$$\begin{aligned} L^d(\theta_{k+\frac{1}{2}}, \Delta\theta_k, \Delta s_k) &= hL(\theta_{k+\frac{1}{2}}, \Delta\theta_k/h, \Delta s_k/h) \\ &= \frac{1}{2h} \left[ \alpha\Delta\theta_k^2 + 2\beta \cos(\theta_{k+\frac{1}{2}})\Delta\theta_k\Delta s_k + \gamma\Delta s_k^2 \right] - U(\theta_{k+\frac{1}{2}}). \end{aligned} \quad (2.32)$$

The discrete dynamics of the pendulum on a cart is given by equations (2.10) and (2.11). Recall that the control is applied to the cart, and so the pendulum is not directly actuated.

Next we form the controlled Lagrangian as in (2.12) by modifying only the kinetic energy of the cart-pendulum system. This involves a nontrivial choice of the form  $\tau$ . The dynamics for the controlled Lagrangian is given by (2.13) and (2.14).

Motivated by the continuous-time analysis of the cart-pendulum system given in Bloch *et al.* [5], we set  $\tau(\theta_{k+\frac{1}{2}}) = \kappa \cos(\theta_{k+\frac{1}{2}})$ ,  $c_b^d = c$ , where  $c$  is a positive number. This defines the controller, and by Theorem 2.2.1, the stability condition is

$$\kappa > \frac{\alpha\gamma - \beta^2}{\beta\gamma} > 0. \quad (2.33)$$

which are identical to the continuous-time stability conditions. This is confirmed by numerical simulations of the discrete dynamics of the cart-pendulum system in Figure 2.2.

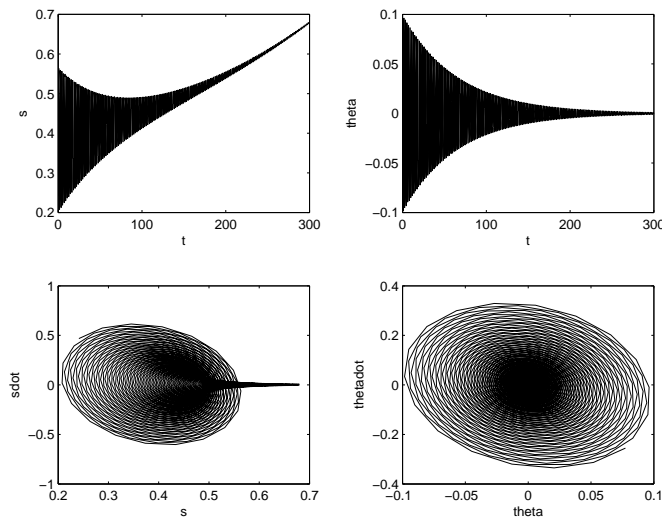


Figure 2.2: Controlled dynamics of the discrete pendulum-cart system

## Chapter 3

# Stabilization of Discrete Rigid Body

### 3.1 The Continuous-Time Satellite-Rotor System

The satellite-rotor system is a rigid body spinning around its center of mass with a rotor aligned along the third principal axis of the body as shown in Figure 3.1. The rotor is actuated, with

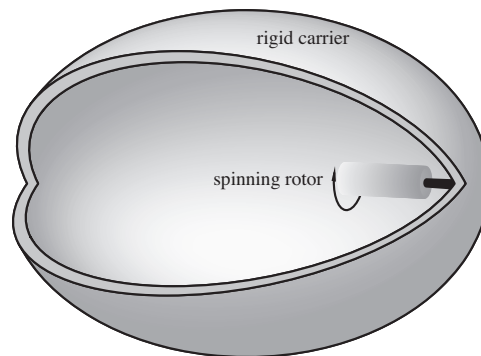


Figure 3.1: The satellite-rotor system

actuation characterized by the control torque  $u$ . The configuration space for this system is  $Q = SO(3) \times SO(2)$ , with the first factor being the satellite attitude and the second factor being the rotor angle relative to a reference direction in the satellite.<sup>1</sup>

---

<sup>1</sup>Recall that  $SO(n)$  denotes the group of  $n \times n$  rotation matrices.

Bloch et al. [4, 6] study stabilization of rotations of this system about its intermediate inertia axis. These rotations are known to be unstable. Bloch et al. [4, 6] show that stabilization of the said rotations can be achieved by applying an appropriate control torque to the rotor. This control torque is computed using the method of controlled Lagrangians.

Recall that the rotor is spinning about the third principle axis of the body. In the following, we will discuss the stability of rotations of the continuous-time satellite-rotor system by the Lagrangian methods. The emphasis is on using the formalism that facilitates a smoother transition to the discrete setting.

The Lagrangian for this system is just the kinetic energy:

$$L = \frac{1}{2} \langle (\mathbb{I} + \mathbb{J}) \Omega, \Omega \rangle + \langle \mathbb{J} \Omega, C \rangle \eta + \frac{1}{2} \langle \mathbb{J} C, C \rangle \eta^2. \quad (3.1)$$

where  $\mathbb{I}$  is the inertia operator of the satellite,  $\mathbb{J}$  is the inertia operator of the rotor,  $C$  is the rotor's axis in a matrix representation (details below),  $g \in SO(3)$  (an orthogonal matrix) is the attitude of the system,  $\Omega = g^{-1} \dot{g} \in so(3)^2$  is the body angular velocity of the system, and  $\eta \in \mathbb{R}$  is the angular velocity of the rotor relative to the spacecraft. Recall that the pairing in (3.1) and likewise formulae in this chapter is defined by the formula

$$\langle \xi, \eta \rangle = \frac{1}{2} \text{Tr } \xi \eta^T.$$

It is straightforward to show that for any square matrices  $\xi$ ,  $\eta$ , and  $\zeta$

$$\text{Tr } \xi = \text{Tr } \xi^T, \quad \text{Tr } \xi \eta \equiv \text{Tr } \eta \xi, \quad \text{and} \quad \text{Tr } \xi \eta \zeta \equiv \text{Tr } \zeta \xi \eta.$$

---

<sup>2</sup>Recall that  $so(n)$  is the Lie algebra of the group  $SO(n)$ . The elements of this Lie algebra are skew-symmetric  $n \times n$  matrices. Likewise, the elements of the dual Lie algebra  $so^*(n)$  are skew-symmetric  $n \times n$  matrices.

Since the matrix  $\Omega$  is skew-symmetric, it can be written as

$$\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix},$$

where the entries  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$  have the meaning of the components of the angular velocity vector  $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ . When the matrix representation of angular velocity is used, the inertia operators  $\mathbb{I}, \mathbb{J} : so(3) \rightarrow so^*(3)$  are given by the formulae

$$\mathbb{I}\Omega = A\Omega + \Omega A \quad \text{and} \quad \mathbb{J}\Omega = B\Omega + \Omega B, \quad (3.2)$$

where  $A = \text{diag}(A_1, A_2, A_3)$  and  $B = \text{diag}(B_1, B_2, B_3)$ , and  $B_1 = B_2$ , so we can rewrite matrix  $B$  as  $\text{diag}(B_1, B_1, B_3)$ .

The matrix  $C$  is the matrix representation of the direction of the angular velocity of the rotor relative to the body frame:

$$C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.3)$$

The dynamics of the system is

$$\dot{p} = \text{ad}_\Omega^* p, \quad \dot{\pi} = u. \quad (3.4)$$

where  $p = \partial L / \partial \Omega \in so^*(3)$  is the angular momentum of the system,  $\pi = \partial L / \partial \eta \in \mathbb{R}$  is the momentum conjugate to the rotor's angular velocity relative to the body, and  $u$  is the control torque. Here and elsewhere,  $\text{ad}_\xi \eta \equiv \xi \eta - \eta \xi$  is the adjoint action of a Lie algebra on itself, and  $\text{ad}^*$  is its dual. We use the representation in which  $p$  and  $\Omega$  are  $3 \times 3$  skew matrices, so that

$$\text{ad}_\Omega^* p = p\Omega - \Omega p. \quad (3.5)$$

Note that for non-orthogonal algebras the formula (3.5) is no longer valid.

According [4, 6] the controlled Lagrangian is:

$$L_{\tau,\sigma,\rho} = \frac{1}{2}\langle(\mathbb{I} + \mathbb{J}_{\tau,\sigma,\rho}) \Omega, \Omega\rangle + \rho(1+r)\langle\mathbb{J} \Omega, C\rangle\eta + \frac{1}{2}\rho\langle\mathbb{J} C, C\rangle\eta^2, \quad (3.6)$$

i.e., the inertia operator  $\mathbb{J}$  is replaced with

$$\mathbb{J}_{\tau,\sigma,\rho} \xi = B_{\tau,\sigma,\rho} \xi + \xi B_{\tau,\sigma,\rho},$$

where

$$B_{\tau,\sigma,\rho} = \text{diag} (B_1(\rho(1+r)^2 + \sigma r^2), B_1(\rho(1+r)^2 + \sigma r^2), B_3 + B_1(1 - \rho(1+r)^2 - \sigma r^2)).$$

The dynamics associated with the controlled reduced Lagrangian is:

$$\frac{d}{dt} \frac{\partial L_{\tau,\sigma,\rho}}{\partial \Omega} = \text{ad}_\Omega^* \frac{\partial L_{\tau,\sigma,\rho}}{\partial \Omega}, \quad \frac{d}{dt} \frac{\partial L_{\tau,\sigma,\rho}}{\partial \eta} = 0. \quad (3.7)$$

As shown in [4, 6], systems (3.4) and (3.7) are equivalent if

$$u = -\frac{d}{dt}(2B_1 r \Omega_3), \quad \sigma = -\frac{1}{r}, \quad \rho = \frac{1}{1+r},$$

and this the matching conditions are

$$\sigma = -\frac{1}{r}, \quad \rho = \frac{1}{1+r}, \quad (3.8)$$

and the control torque is

$$u = -r \text{Tr} (BC\dot{\Omega}^T) \equiv -2r B_1 \dot{\Omega}_3. \quad (3.9)$$

## 3.2 Stability of the Continuous-Time Satellite-Rotor

Observe that the last equation of system (3.7) defines the controlled conservation law

$$\frac{\partial L_{\tau,\rho,\sigma}}{\partial \eta} = c = \text{const.} \quad (3.10)$$

The coordinate form of equations (3.7) reduced to the level set of this conservation law and takes into account the matching conditions (3.8) reads

$$\begin{aligned} \dot{\Omega}_1 &= \frac{I_2 + J_2}{I_1 + J_1} \Omega_2 \Omega_3 - \frac{I_3 + J_3}{I_1 + J_1} \Omega_2 \Omega_3 + \frac{J_3(1+r)}{I_1 + J_1} \Omega_2 \Omega_3 - \frac{c(1+r)}{I_1 + J_1} \Omega_2, \\ \dot{\Omega}_2 &= \frac{I_3 + J_3}{I_2 + J_2} \Omega_1 \Omega_3 - \frac{I_1 + J_1}{I_2 + J_2} \Omega_1 \Omega_3 - \frac{J_3(1+r)}{I_2 + J_2} \Omega_1 \Omega_3 + \frac{c(1+r)}{I_2 + J_2} \Omega_1, \\ \dot{\Omega}_3 &= \frac{(I_1 - I_2) \Omega_1 \Omega_2}{I_3 - r J_3}, \end{aligned}$$

where  $I_1 > I_2 > I_3$ ,  $I_1 = A_2 + A_3$ ,  $I_2 = A_1 + A_3$ ,  $I_3 = A_1 + A_2$  and  $J_1 = J_2$ ,  $J_3$  are the eigenvalues of the inertia operators  $\mathbb{I}$  and  $\mathbb{J}$ , respectively.

It is necessary to consider the dynamics on the zero level of the conservation law (3.10). Indeed, the relative equilibria to stabilize corresponds to the solutions  $(0, M, 0)$  of these equations, which is only possible if  $c = 0$ .

We point out here that these stability conditions are identical to the spectral stability conditions. Indeed, the matrix  $\mathbf{A}$  of linearization at  $(0, M, 0)$  of the dynamics reduced to the zero level of the conservation law (3.10) is straightforward to compute. It reads

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & M \frac{(I_2 + J_2) - (I_3 + J_3) + J_3(1+r)}{I_1 + J_1} \\ 0 & 0 & 0 \\ M \frac{I_1 - I_2}{I_3 - r J_3} & 0 & 0 \end{pmatrix}.$$

The two nonzero eigenvalues of this matrix are

$$\pm \sqrt{-M^2 \frac{(I_1 - I_2)[(I_2 + J_2) - (I_3 + J_3) + J_3(1 + r)]}{(I_1 + J_1)(I_3 - rJ_3)}}. \quad (3.11)$$

Thus, for (linear) stability one has to require that

$$\frac{(I_1 - I_2)[(I_2 + J_2) - (I_3 + J_3) + J_3(1 + r)]}{(I_1 + J_1)(I_3 - rJ_3)} > 0.$$

Because of the assumption  $I_1 > I_2 > I_3$ , the latter reduces to

$$\frac{(I_2 + J_2) - (I_3 + J_3) + J_3(1 + r)}{I_3 - rJ_3} > 0,$$

which implies

$$r > \frac{I_3 - I_2 - J_2}{J_3} = -1 + \frac{A_2 - A_3 + B_1 - B_3}{2B_1}, \quad (3.12)$$

which is the stability condition of Bloch *et al.* [4, 6].

### 3.3 The Moser–Veselov Discretization of the Satellite-Rotor System

Recall that in the discrete setting, the velocity phase space  $TQ$  is replaced with  $Q \times Q$  and the states of the discrete system are written  $(q_k, q_{k+1})$ . For the satellite-rotor system  $Q = SO(3) \times SO(2)$  and thus  $q_k = (g_k, x_k)$ , where  $g \in SO(3)$  and  $x_k \in SO(2)$ . Define the system's incremental rotations by the formula  $W_{k,k+1} = g_k^{-1} g_{k+1} \in SO(3)$  and the incremental rotations of the rotor relative to the satellite by  $\Delta x_k = x_{k+1} - x_k \in SO(2)$ .

Following the approach of Veselov and Moser (see e.g. [33, 26, 21]), we construct the discrete Lagrangian and discrete controlled Lagrangian for the satellite-rotor system by replacing  $\Omega$  with  $(W_{k,k+1} - e)/h$  and  $\eta$  with  $\Delta x_k/h$  in formulae (3.1) and (3.6), where  $h > 0$  is the time-step.

Following the Moser–Veselov discretization procedure (see e.g. [21]), the discrete Lagrangian

is obtained by substituting the quantity  $(W_{k,k+1} - e)/h$  that approximates the continuous-time angular velocity  $\Omega$  in the continuous-time Lagrangian (3.1). The magnitude of the angular velocity of the rotor is replaced with  $\Delta x_k/h$ . The first term in (3.1) becomes (to keep the formulae shorter, we omit the index  $k, k+1$ )

$$\begin{aligned} & \frac{1}{4} \text{Tr}[(W - e)^T((A + B)(W - e) + (W - e)(A + B))] \\ &= \frac{1}{4} \text{Tr}[W^T(A + B)W + e^T(A + B)e - W^T(A + B)e - e^T(A + B)W \\ & \quad + W^TW(A + B) + e^Te(A + B) - W^T(A + B) - W(A + B)]. \end{aligned}$$

Neglecting the constant terms, using the properties of trace, and taking into account that  $A$  and  $B$  are symmetric, and  $W$  is orthogonal, we obtain

$$\frac{1}{4} \text{Tr}[-2W^T(A + B) - 2W(A + B)] = \frac{1}{4} \text{Tr}[-4W(A + B)] = -\text{Tr} W(A + B).$$

The coefficient next to  $\eta$  in the second term of (3.1) becomes

$$\begin{aligned} & \frac{1}{2} \text{Tr}(B(W - e) + (W - e)B)^T C = \frac{1}{2} \text{Tr}(W^T B^T + B^T W^T - 2B^T)C \\ &= \frac{1}{2} \text{Tr}[W^T BC + BW^T C - 2BC] = \frac{1}{2} \text{Tr}[BCW^T + CBW^T - 2BC] \\ &= \frac{1}{2} \text{Tr} 2BC(W^T - e) = \text{Tr} BC(W^T - e), \end{aligned}$$

where we used the properties of trace along with the fact that  $CB+BC$  for  $B = \text{diag}(B_1, B_1, B_3)$  and  $C$  defined by formula (3.3). The coefficient next to  $\eta^2$  in the third term of (3.1) becomes

$$\frac{1}{4} \text{Tr} C^T (BC + CB) = \frac{1}{4} \text{Tr}[C^T BC + C^T CB] = \frac{1}{4} \text{Tr} [(-C)B(-C)^T + CBC^T] = \frac{1}{2} \text{Tr} CBC^T.$$

Summarizing, the discrete Lagrangian for the satellite-rotor system is

$$l^d(W_{k,k+1}, \Delta x_k) = \frac{1}{h} \left[ -\text{Tr}(W_{k,k+1}(A+B)) \right. \\ \left. + \text{Tr}(BC(W_{k,k+1}^T - e))\Delta x_k + \frac{1}{2} \text{Tr}(CBC^T)\Delta x_k^2 \right], \quad (3.13)$$

where  $A$  and  $B$  are diagonal matrices that define the inertia operators  $\mathbb{I}$  and  $\mathbb{J}$  as in (3.2), and  $e$  is the identity matrix. Note also that, according to the properties of trace, the first term in (3.13) can be rewritten as

$$-\text{Tr}((A+B)W_{k,k+1}) \equiv -\text{Tr}((A+B)W_{k,k+1}^T) \equiv -\text{Tr}(W_{k,k+1}^T(A+B)).$$

The satellite-rotor system is a generalized rigid body on the group  $SE(3) \times SE(2)$ , and thus the controlled dynamics of the discrete satellite-rotor is given by the equations

$$R_{W_{k,k+1}}^* D_1 l^d(W_{k,k+1}, \Delta x_k) - \text{Ad}_{W_{k-1,k}}^* R_{W_{k-1,k}}^* D_1 l^d(W_{k-1,k}, \Delta x_{k-1}) = \bar{\lambda}_k, \quad (3.14)$$

$$D_2 l^d(W_{k-1,k}, \Delta x_{k-1}) - D_2 l^d(W_{k,k+1}, \Delta x_k) + u_k = 0. \quad (3.15)$$

where  $\bar{\lambda}_k$  is the Lagrange multiplier. Here and elsewhere,  $L_g^*$  and  $R_g^*$  are the duals of the derivatives of the left and right translations,  $\text{Ad}$  denotes the adjoint action of a Lie group on its Lie algebra, and  $\text{Ad}^*$  is its dual. Eliminating the Lagrange multiplier and simplifying, we obtain

$$M_{k+1} - W_{k-1,k}^T M_k W_{k-1,k} = 0, \quad (3.16)$$

$$D_2 l^d(W_{k-1,k}, \Delta x_{k-1}) - D_2 l^d(W_{k,k+1}, \Delta x_k) + u_k = 0, \quad (3.17)$$

where

$$M_k = R_{W_{k-1,k}}^* D_1 l^d(W_{k-1,k}, \Delta x_{k-1}) - (R_{W_{k-1,k}}^* D_1 l^d(W_{k-1,k}, \Delta x_{k-1}))^T \quad (3.18)$$

represents the discrete angular momentum of the system.

The reduced discrete controlled Lagrangian is computed just like the discrete Lagrangian and reads

$$l_{\tau,\sigma,\rho}^d(W_{k,k+1}, \Delta x_k) = \frac{1}{h} \left[ -\text{Tr}(W_{k,k+1}(A + B_{\tau,\sigma,\rho})) \right. \\ \left. + \rho(1+r) \text{Tr}(BC(W_{k,k+1}^T - e))\Delta x_k + \frac{1}{2}\rho \text{Tr}(CBC^T)\Delta x_k^2 \right]. \quad (3.19)$$

The corresponding discrete Euler–Lagrange equations are

$$M_{k+1}^c - W_{k-1,k}^T M_k^c W_{k-1,k} = 0, \quad (3.20)$$

$$D_2 l_{\tau,\sigma,\rho}^d(W_{k-1,k}, \Delta x_{k-1}) - D_2 l_{\tau,\sigma,\rho}^d(W_{k,k+1}, \Delta x_k) = 0, \quad (3.21)$$

where

$$M_k^c = R_{W_{k-1,k}}^* D_1 l_{\tau,\sigma,\rho}^d(W_{k-1,k}, \Delta x_{k-1}) - (R_{W_{k-1,k}}^* D_1 l_{\tau,\sigma,\rho}^d(W_{k-1,k}, \Delta x_{k-1}))^T.$$

As in the continuous-time case, equation (3.21) is equivalent to the controlled momentum conservation law

$$D_2 l_{\tau,\sigma,\rho}^d(W_{k,k+1}, \Delta x_k) = c = \text{const},$$

and one selects  $c = 0$  for studying the discrete analogues of rotations of the satellite about its intermediate axis of inertia. With the condition  $c = 0$  in mind, the controlled momentum conservation law becomes

$$\text{Tr} [CBC^T] \Delta x_k + (1+r) \text{Tr} [BC(W_{k,k+1}^T - e)] = 0.$$

It is straightforward to check that equations (3.17) and (3.21) are equivalent if and only if the discrete control law is

$$u_k = -\frac{1}{h} r \text{Tr} BC \left[ (W_{k,k+1}^T - e) - (W_{k-1,k}^T - e) \right] \equiv -\frac{1}{h} r \text{Tr} BC [W_{k,k+1}^T - W_{k-1,k}^T]. \quad (3.22)$$

Indeed, expanding the left-hand sides of equations (3.17) and (3.21), we have

$$\begin{aligned} & \frac{1}{h} \left[ \text{Tr}(CBC^T)\Delta x_{k-1} + \text{Tr} BC(W_{k-1,k}^T - e) \right. \\ & \quad \left. - \text{Tr}(CBC^T)\Delta x_k - \text{Tr} BC(W_{kk+1}^T - e) \right] + u_k = 0, \\ & \frac{1}{h} \left[ \text{Tr}(CBC^T)\Delta x_{k-1} + (1+r) \text{Tr} BC(W_{k-1,k}^T - e) \right. \\ & \quad \left. - \text{Tr}(CBC^T)\Delta x_k - (1+r) \text{Tr} BC(W_{kk+1}^T - e) \right] = 0. \end{aligned}$$

The last two equations are equivalent if and only if the discrete control law is given by (3.22).

Asking that (3.16) and (3.20) are equivalent requires that the matching conditions

$$\sigma = -\frac{1}{r}, \quad \rho = \frac{1}{1+r} \quad (3.23)$$

hold. Observe that these matching conditions are identical to the matching conditions of the continuous-time model, and that the discrete control input (3.22) becomes the control input of the continuous-time model (3.9) after taking the limit  $h \rightarrow 0$ .

Next, consider relative equilibria that correspond to the rotations of the satellite about the intermediate inertia axis. These relative equilibria are represented by the increment rotations

$$W_e = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{pmatrix}. \quad (3.24)$$

Since  $W_e$  is an orthogonal matrix,  $a$  and  $b$  should satisfy the condition  $a^2 + b^2 = 1$ . We would like to linearize the dynamics near the equilibrium  $M_e$ , i.e., to write the linear approximation

$$M_{k+1} - M_e = \mathbf{A}_d(M_k - M_e),$$

where  $\mathbf{A}_d$  is a linear operator that acts on matrices.

Recall that the incremental rotation  $W_{k,k+1}$  is a  $3 \times 3$  matrix satisfying the condition

$W_{k,k+1}W_{k,k+1}^T = I$ . We write it as

$$W_{k,k+1} = \begin{pmatrix} w_{11,k} & w_{12,k} & w_{13,k} \\ w_{21,k} & w_{22,k} & w_{23,k} \\ w_{31,k} & w_{32,k} & w_{33,k} \end{pmatrix}.$$

The discrete momentum  $M_k$  is a skew-symmetric  $3 \times 3$  matrix:

$$M_k = \begin{pmatrix} 0 & -m_{3,k} & m_{2,k} \\ m_{3,k} & 0 & -m_{1,k} \\ -m_{2,k} & m_{1,k} & 0 \end{pmatrix}.$$

To proceed with the linearization, we eliminate  $W_{k-1,k}$  from the equations of motion using formula (3.18). Because of skew-symmetry of  $M_k$ , this matrix relation is equivalent to three scalar equations. Adding the condition  $W_{k,k+1}W_{k,k+1}^T = I$  that is equivalent to six scalar equations, we obtain nine equations for the nine elements of the matrix  $W_{k,k+1}$ . Using the implicit function theorem we find the derivatives  $\partial w_{ab}/\partial m_c$ .

Using these derivatives and taking into account the matching conditions, we find the linearized equations of motion. The corresponding matrix  $\mathbf{A}_d$  has five non-zero components:

$$\mathbf{A}_d = \begin{pmatrix} \frac{A(r)}{D(r)} & 0 & \frac{B(r)}{D(r)} \\ 0 & 1 & 0 \\ \frac{C(r)}{D(r)} & 0 & \frac{A(r)}{D(r)} \end{pmatrix}.$$

The four nontrivial components of  $\mathbf{A}_d$  are computed to be (tedious but straightforward

calculation)

$$\begin{aligned}
A(r) &= -(1+r)B_1[b^2(A_1 - A_2) + (1+a)^2(A_2 + B_1 + A_3 + B_3)] \\
&\quad + 2[(A_2 + B_1)(a(A_2 + B_1) + A_3 + B_3) + (A_1 + B_1)((A_2 + B_1) + a(A_3 + B_3))], \\
B(r) &= 2b[(1+a)(1+r)B_1 - A_2 - B_1 + A_3 + B_3](A_2 + B_1 + A_3 + B_3), \\
C(r) &= 2b[(1+a)(1+r)B_1 - A_1 - 2B_1 - A_2](A_1 - A_2), \\
D(r) &= (1+r)B_1[b^2(A_1 + B_1) - (a(1+a)^2 + (2+a)b^2)(A_2 + B_1) - (1+a)^2(A_3 + B_3)] \\
&\quad + 2[(A_1 + B_1)(a(A_2 + B_1) + A_3 + B_3) + (A_2 + B_1)((A_2 + B_1) + a(A_3 + B_3))].
\end{aligned}$$

The characteristic polynomial of  $\mathbf{A}_d$  is

$$(1 - \lambda) \left[ \lambda^2 - 2 \frac{A(r)}{D(r)} \lambda + 1 \right]. \quad (3.25)$$

Thus, the eigenvalues of  $\mathbf{A}_d$  are 1 and  $\frac{A(r)}{D(r)} \pm \sqrt{\frac{A^2(r)}{D^2(r)} - 1}$ .

One calculates that

$$\det \mathbf{A}_d = \frac{A^2(r) - B(r)C(r)}{D^2(r)} = 1.$$

This implies that the two eigenvalues belong to the unit circle when  $B(r)C(r) < 0$ , and one of the eigenvalues is outside the closed unit disk when  $B(r)C(r) > 0$ . Therefore, the linear stability condition is  $B(r)C(r) < 0$ . Since each of the quantities  $A_1, A_3, B_1, B_3$  is positive, the linear stability condition is

$$\frac{A_2 - A_3 - aB_1 - B_3}{(1+a)B_1} < r < \frac{A_1 + A_2 + B_1 - aB_1}{(1+a)B_1}. \quad (3.26)$$

For sufficiently small time steps, the incremental rotation (3.24) is a rotation through an acute angle, and thus  $0 < a < 1$ , which implies  $1 < (1+a) < 2$ . Therefore,

$$\frac{A_2 - A_3 - B_1 - B_3}{2B_1} < \frac{A_2 - A_3 - aB_1 - B_3}{(1+a)B_1}.$$

So, substitute into the condition for  $r$ , we have:

$$\frac{A_2 - A_3 - B_1 - B_3}{2B_1} < r < \frac{A_1 + A_2 + B_1}{B_1}. \quad (3.27)$$

We thus have the following result:

**Theorem 3.3.1.** *The condition for linear stability of the relative equilibrium (3.24) of the discrete controlled satellite-rotor implies the stability condition (3.12) of the corresponding relative equilibrium of the original continuous-time controlled satellite-rotor dynamics.*

The opposite statement is clearly incorrect. Thus, if the Moser–Veselov discretization algorithm is used for simulations of the controlled dynamics of the satellite-rotor system, one has to make sure that the gain parameter  $r$  satisfies the stability condition (3.26).

### 3.4 Discretization of the Rigid Body Using Natural Charts and Stability of Relative Equilibria

The previous section revealed an interesting disagreement of the stability conditions of the discrete and continuous-time satellite-rotor dynamics. There the Moser–Veselov discretization procedure was used to obtain the discrete dynamics. In this section we study this issue in depth using the discretization technique of Marsden *et al.* [21].

Recall that the satellite rotor may be viewed as a generalized rigid body on the group  $SO(3) \times SO(2)$ . Accordingly, in this section we study stability of relative equilibria of a generalized rigid body. We use a somewhat more general discretization technique of Marsden *et al.* [21] instead of the Moser–Veselov discretization.

**Linearized Dynamics of a Generalized Rigid Body.** Let  $G$  be a Lie group and  $\mathfrak{g}$  be its Lie algebra. A generalized rigid body is a mechanical system on  $G$  whose Lagrangian is a positive-definite  $G$ -invariant metric on  $G$ . Such a metric is fully defined by a positive-definite

*inertia operator*  $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ . The definition is as follows, given a curve  $g(t)$  in  $G$ , the body angular velocity  $\Omega$  is defined by  $\Omega = g^{-1}(t)\dot{g}(t)$ , and the kinetic energy of the body is  $\frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle$ .

As discussed in the literature, the (reduced) dynamics is given by the Euler–Poincaré equations

$$\dot{p} = \text{ad}_{\mathbb{I}^{-1}p}^* p. \quad (3.28)$$

where  $p = \mathbb{I}\Omega \in \mathfrak{g}^*$  is the body momentum. For an equilibrium  $p = p_0$  of equations (3.28) (that corresponds to a steady-state rotation of the body), we introduce the variable  $\mu \in \mathfrak{g}^*$  (momentum displacement) by the formula

$$p = p_e + \mu. \quad (3.29)$$

Equations (3.28) and (3.29) imply

$$\begin{aligned} \frac{d(p_e + \mu)}{dt} &= \text{ad}_{\mathbb{I}^{-1}(p_e + \mu)}^*(p_e + \mu) \\ &= \text{ad}_{\mathbb{I}^{-1}p_e}^*(p_e) + \text{ad}_{\mathbb{I}^{-1}\mu}^* p_e + \text{ad}_{\mathbb{I}^{-1}(p_e)}^* \mu + \text{ad}_{\mathbb{I}^{-1}\mu}^* \mu. \end{aligned} \quad (3.30)$$

Since  $p_e$  is an equilibrium,  $\text{ad}_{\mathbb{I}^{-1}p_e}^* p_e = 0$ , and thus the linearization of Euler–Poincaré equations (3.28) at  $p_e$  therefore is

$$\dot{\mu} = A\mu := \text{ad}_{\mathbb{I}^{-1}\mu}^* p_e + \text{ad}_{\mathbb{I}^{-1}p_e}^* \mu. \quad (3.31)$$

**The Discrete Euler–Poincaré Equations.** We now review the theory of the discrete Euler–Poincaré equations. The exposition follows paper [21].

**Theorem 3.4.1** ([21]). *Let  $L$  be a left-invariant Lagrangian on  $G \times G$  and let  $l : (G \times G)/G \cong G \rightarrow R$  be the reduced discrete Lagrangian given by  $l(g_1^{-1}g_2) = L(g_1, g_2)$ . For any integer  $N \geq 3$ , let  $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$  be a sequence in  $G \times G$  and define  $W_{k,k+1} \equiv g_k^{-1}g_{k+1}$  to be the corresponding sequence in  $G$ . Then the following are equivalent:*

1. *The sequence  $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$  is an extremum of the action sum  $S : G^{N+1} \rightarrow R$  for*

arbitrary variations  $\delta g_k = (d/d\varepsilon)|_0 g_k^\varepsilon$  where for each  $k$ ,  $\varepsilon \mapsto g_k^\varepsilon$  is a smooth curve in  $G$  such that  $g_k^0 = g_k$ .

2. The sequence  $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$  satisfies the discrete Euler–Lagrangian equations.
3. The sequence  $\{W_{k,k+1}\}_{k=0}^{N-1}$  is an extremum of the reduced action sum  $s : G \rightarrow \mathbb{R}$  with respect to variations  $\delta W_{k,k+1}$ , induced by the variance  $\delta g_k$ , and given by:

$$\delta W_{k,k+1} = TL_{W_{k,k+1}}(\delta g_{k+1} g_{k+1}^{-1} - \text{Ad}_{W_{k,k+1}} \delta g_k g_k^{-1}). \quad (3.32)$$

4. The sequence  $\{W_{k,k+1}\}_{k=0}^{N-1}$  satisfies the discrete Euler–Poincaré equations

$$\text{Ad}_{W_{k-1,k}} TL_{W_{k-1,k}} l'(W_{k-1,k}) - TL_{W_{k,k+1}} l'(W_{k,k+1}) = 0 \quad (3.33)$$

for  $k = 1, 2, \dots, N-1$ , where the operators act on variations of the form  $\vartheta_k = g_k^{-1} \delta g_k$ .

**Discretization on Lie Groups Using Natural Charts.** The following procedure is discussed in detail in Marsden et al. [21].

For finite-dimensional Lie groups, the *exponential map*  $\exp : \mathfrak{g} \rightarrow G$  is a local diffeomorphism and thus may be used for constructing *natural charts*. For the left-invariant case, consider the map  $\phi_g : G \rightarrow \mathfrak{g}$  defined by the formula

$$\phi_g = \exp^{-1} \circ L_{g^{-1}}, \quad (3.34)$$

where  $L_g : G \rightarrow G$  denotes the left translation. Using this map, define the discrete Lagrangian,  $L^d : G \times G \rightarrow \mathbb{R}$  by

$$L^d(g_1, g_2) = L \left( \phi_g^{-1} \left[ \frac{\phi_g(g_1) + \phi_g(g_2)}{2} \right], (\phi_g^{-1}) * \left[ \frac{\phi_g(g_2) - \phi_g(g_1)}{h} \right] \right), \quad (3.35)$$

where  $h$  is the time step. Next, let

$$\eta = \frac{\phi_g(g_1) + \phi_g(g_2)}{2}. \quad (3.36)$$

We denote the Lie algebra element approximating the body velocity  $g^{-1}\dot{g}$  by

$$\zeta = \frac{\phi_g(g_2) - \phi_g(g_1)}{h}. \quad (3.37)$$

Below we use the standard formula for the derivative of the exponential map:

$$T_\eta \exp = T_e L_{g'} \text{iex}(-\text{ad}_\eta), \quad \eta \in \mathfrak{g}, \quad g' = \exp(\eta), \quad (3.38)$$

where  $\text{iex}$  is an analytic function defined by

$$\text{iex}(w) = \sum_{n=0}^{\infty} \frac{w^n}{(n+1)!}. \quad (3.39)$$

See Varadarajan [32] and Marsden et al. [21] for details.

Thus, the expression for the discrete Lagrangian becomes

$$L^d(g_1, g_2) = L(\phi_g^{-1}(\eta), T_{g'} L_g \cdot T_e L_g' \cdot \text{iex}(-\text{ad}_\eta)(\zeta)). \quad (3.40)$$

If we set  $q \equiv \phi_g^{-1}(\eta) = L_g g'$ , that formula is expressed as

$$L^d(g_1, g_2) = L(q, T_e L_q \cdot \text{iex}(-\text{ad}_\eta)(\zeta)). \quad (3.41)$$

For  $g = g_k$  or  $g = g_{k+1}$ , it can be easily verify that  $\text{ad}_\zeta \eta \equiv [\zeta, \eta] = 0$ , which implies  $\text{iex}(-\text{ad}_\eta)(\zeta) = \zeta$ . So, with  $g = g_{k+1}$ , the discrete Lagrangian is simply

$$L^d(g_k, g_{k+1}) = L(q, T_e L_q(\zeta)), \quad (3.42)$$

where

$$\eta = \frac{1}{2} \log(g_k^{-1} g_{k+1}), q \equiv \phi_{g_{k+1}}(\eta) = (g_k g_{k+1})^{1/2}, \quad \zeta = \frac{1}{h} \log(g_k^{-1} g_{k+1}) \quad (3.43)$$

and where  $\log \equiv \exp^{-1}$ . Consequently, the reduced discrete Lagrangian is given by

$$l_d(W_{k,k+1}) = l(\log(W_{k,k+1})/h), \quad (3.44)$$

where  $W_{k,k+1} = g_k^{-1} g_{k+1}$ .

Using this discrete Lagrangian and setting the Euler–Poincaré equation, we obtain the following left-invariant implicit algorithm on the Lie algebra  $\mathfrak{g}$ :

$$\chi(\text{ad}_{\xi_{k,k+1}}^*) \cdot l'(\xi_{k,k+1}/h) = \exp(\text{ad}_{\xi_{k-1,k}}^*) \cdot \chi(\text{ad}_{\xi_{k-1,k}}^*) \cdot l'(\xi_{k-1,k}/h), \quad (3.45)$$

where  $\xi_{k,k+1} \equiv \log(f_{k,k+1}) \in \mathfrak{g}$  and the function  $\chi$  is defined by the formula

$$\chi(\text{ad}_\xi) \cdot \text{iex}(-\text{ad}_\xi) = \text{id}.$$

**The Linearized Euler–Poincaré Equations on a Lie Algebra.** Now we want to linearize both sides of the implicit algorithm (3.45) at an equilibrium. Let  $\xi_e$  be an equilibrium of (3.45). Since the function  $\text{iex}$  is analytic, the inverse function  $\chi$  is also analytic, and thus it can be written as a convergent (locally) power series  $\chi(x) = \sum_{n=0}^{\infty} d_n x^n$ . Since  $\text{iex}(0) = 1$ , the first term in the expansion of  $\chi$  is 1, that is,  $d_0 = 1$ .

First we linearize left-hand side of (3.45). Setting  $l'(\xi_{k,k+1}/h) = p_{k+1}$  and  $p_{k+1} = p_e + \mu_{k+1}$ , where  $p_e$  is the equilibrium value of the momentum, and expanding the left-hand side of (3.45), we obtain:

$$\begin{aligned} \chi(\text{ad}_{\xi_{k,k+1}}) \cdot l'(\xi_{k,k+1}/h) &= \chi(\text{ad}_{\xi_{k,k+1}}^*) p_{k+1} \\ &= (d_0 + d_1 \text{ad}_{\xi_{k,k+1}}^* + d_2 \text{ad}_{\xi_{k,k+1}}^{*2} + \dots) p_{k+1}. \end{aligned}$$

Next, we linearize  $\text{ad}_{\xi_{k,k+1}}^* p_{k+1}$  at  $p_e$ :

$$\begin{aligned}
\text{ad}_{\xi_{k,k+1}}^* p_{k+1} &= \text{ad}_{\mathbb{I}^{-1}(p_0 + \mu_{k+1})}^* (p_0 + \mu_{k+1}) \\
&= \text{ad}_{\mathbb{I}^{-1}p_0}^* p_0 + \text{ad}_{\mathbb{I}^{-1}\mu_{k+1}}^* p_0 + \text{ad}_{\mathbb{I}^{-1}p_0}^* \mu_{k+1} + \text{ad}_{\mathbb{I}^{-1}\mu_{k+1}}^* \mu_{k+1} \\
&\approx \text{ad}_{\mathbb{I}^{-1}\mu_{k+1}}^* p_0 + \text{ad}_{\mathbb{I}^{-1}p_0}^* \mu_{k+1} \\
&= A\mu_{k+1},
\end{aligned}$$

where  $A$  is the operator defined by the linearization of the continuous-time system at  $p_e$ . See formula (3.31) for details.

The linearization of  $(\text{ad}_{\xi_{k,k+1}}^*)^n p_{k+1}$ ,  $n = 2, 2, 3, \dots$  therefore is given by the formula

$$(\text{ad}_{\mathbb{I}^{-1}p_e}^*)^{(n-1)} A\mu_{k+1}. \quad (3.46)$$

Expanding the analytic function  $\chi$  in a power series, we obtain the linearization of the left hand side of equation (3.45):

$$\begin{aligned}
d_0 p_e + d_0 \mu_{k+1} + d_1 A\mu_{k+1} + d_2 \text{ad}_{\mathbb{I}^{-1}p_e}^* A\mu_{k+1} + d_3 (\text{ad}_{\mathbb{I}^{-1}p_e}^*)^2 A\mu_{k+1} + \dots \\
= d_0 p_e + d_0 \mu_{k+1} + \sum_{n=1}^{\infty} d_n (\text{ad}_{\mathbb{I}^{-1}p_e}^*)^{n-1} A\mu_{k+1}.
\end{aligned}$$

Next, we treat, in a similar way, the right hand side of equation (3.45). Using the power series expansions

$$\begin{aligned}
\exp(\text{ad}_{\xi_{k-1,k}}^*) &= \sum_{n=0}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!}, \\
\chi(\text{ad}_{\xi_{k-1,k}}^*) &= \sum_{m=0}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m,
\end{aligned}$$

the right-hand side of (3.45) becomes:

$$\exp(\text{ad}_{\xi_{k-1,k}}^*) \cdot \chi(\text{ad}_{\xi_{k-1,k}}^*) = \sum_{n=0}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=0}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m. \quad (3.47)$$

Recall that  $l'(\xi_{k-1,k}/h) = p_k$  and  $p_k = p_e + \mu_k$ . Then substitute  $l'(\xi_{k-1,k}/h) = p_k$  and  $p_k = p_e + \mu_k$  into the right-hand side of (3.45). We obtain:

$$\begin{aligned} \exp(\text{ad}_{\xi_{k-1,k}}^*) \cdot \chi(\text{ad}_{\xi_{k-1,k}}^*) \cdot l'(\xi_{k-1,k}/h) &= \left[ \sum_{n=0}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=0}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m \right] p_k \\ &= d_0 p_k + \left[ \sum_{n=0}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=0}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m - d_0 \right] p_k. \end{aligned}$$

Recall that the linearization of  $(\text{ad}_{\xi_{k-1,k}}^*)^n p_k$  is  $(\text{ad}_{\mathbb{I}-1 p_e}^*)^{(n-1)} A \mu_k$ . Then the linearization on the right hand side of the (3.45) is:

$$d_0 p_e + d_0 \mu_k + \left[ \sum_{n=1}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=1}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m \right] A \mu_k. \quad (3.48)$$

Utilizing the linearization on (3.45), we obtain the linearization of the implicit Euler–Poincaré algorithm on the Lie algebra  $\mathfrak{g}$ :

$$\begin{aligned} d_0 p_e + d_0 \mu_{k+1} + \sum_{n=0}^{\infty} d_n (\text{ad}_{\mathbb{I}-1 p_e}^*)^{n-1} A \mu_{k+1} \\ = d_0 p_e + d_0 \mu_k + \left[ \sum_{n=1}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=1}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m \right] A \mu_k. \end{aligned} \quad (3.49)$$

The quantity  $d_0 p_e$  on both sides cancels out. Since  $d_0 = 1$  the linearization of the discrete dynamics is given by the operator

$$A_d = \left[ I + \sum_{n=0}^{\infty} d_n (\text{ad}_{\mathbb{I}-1 p_e}^*)^{n-1} A \right]^{-1} \left[ I + \sum_{n=1}^{\infty} \frac{(\text{ad}_{\xi_{k-1,k}}^*)^n}{n!} \cdot \sum_{m=1}^{\infty} d_m (\text{ad}_{\xi_{k-1,k}}^*)^m A \right], \quad (3.50)$$

where  $A$  is the linearization of the continuous-time dynamics. This complicated link between

the operators  $A$  and  $A_d$  makes the stability analysis of the discrete dynamics a nontrivial task. It also suggests that certain modifications to the studied discretization technique may be necessary. This last issue is investigated in the next chapter.

## Chapter 4

# Discrete Mechanics with Constraints

In this chapter we suggest a discretization technique for constrained systems that acknowledges the constrained dynamics in the following sense: First, the discrete dynamics of a system with constraints is identical to the dynamics associated with the discrete constrained Lagrangian, and second, the discrete analogue of the statement that constraints can be replaced with their reaction forces is satisfied. This development is motivated by the stability observations of Chapter 3 as well as by similar effects observed by Lynch and Zenkov [17] in the nonholonomic setting.

### 4.1 Continuous-Time Constrained Forced Systems

Consider a Lagrangian  $L : TQ \rightarrow \mathbb{R}$  and assume that the configuration variables split, in a natural way, in two groups, that is  $q = (x, y) = (x^1, \dots, x^\sigma, y^1, \dots, y^m)$ . That for example happens if the configuration manifold has the structure of the direct product,  $Q = X \times Y$ , or, more generally, if  $Q$  is a fiber bundle. Recall that  $(x, y)$  are called *bundle coordinates*.

Assume that there are ideal holonomic constraints<sup>1</sup> that in bundle coordinates  $(x, y)$  are

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<sup>1</sup>Constraints are called ideal if they can be replaced with reaction forces that produce no work on system's virtual displacements consistent with constraints.

given by

$$y = \phi(x), \tag{4.1}$$

or, in components,

$$y^1 = \phi^1(x), y^2 = \phi^2(x), \dots, y^m = \phi^m(x).$$

The constrained Lagrangian  $L_c : X \rightarrow \mathbb{R}$  is defined by the formula

$$L_c(x, \dot{x}) = L(x, \dot{x}, \phi(x), \dot{\phi}(x)) = L(x, \dot{x}, \phi(x), \partial_x \phi(x) \dot{x}). \tag{4.2}$$

For systems with ideal constraints, the dynamics of the system can be written either as the Euler–Lagrange equations on  $TX$  for the constrained Lagrangian,

$$\frac{d}{dt} \frac{\partial L_c}{\partial \dot{x}^\alpha} - \frac{\partial L_c}{\partial x^\alpha} = 0, \tag{4.3}$$

or the Euler–Lagrange equations with Lagrange multipliers

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = -\lambda_a \frac{\partial \phi^a}{\partial x^\alpha}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^a} - \frac{\partial L}{\partial y^a} = \lambda_a, \tag{4.4}$$

coupled with the constraints (4.1). See [2] for details. Here and elsewhere, the sum over the repeated index is understood, as usual. Observe that the  $y$ -equations define the Lagrange multipliers. Excluding the Lagrange multiplier, one obtains

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^a} - \frac{\partial L}{\partial y^a} \right) \frac{\partial \phi^a}{\partial x^\alpha}. \tag{4.5}$$

The two representations of dynamics (4.3) and (4.5) become equivalent after the latter are restricted to the constraint manifold specified by equations (4.1), i.e., after  $y$  and  $\dot{y}$  are eliminated from (4.5). The terms in the right-hand sides of equations (4.5) are the  $x$ -components of the reaction forces of constraints. In other words, in the continuous-time setting, ideal constraints can be replaced with constraint’s reaction forces.

On the contrary, if one uses the discretization procedure of [25] for constrained systems, the two representations of the discrete dynamics are not the same. More precisely, consider the following two discrete dynamics. The first one is obtained by computing the constrained Lagrangian of the continuous-time system and then discretizing the dynamics for this Lagrangian (using the mid-point rule). The second one is obtained by calculating the constraint reaction forces of the continuous-time system and then discretizing the resulting continuous-time system with non-conservative forces. The two discrete dynamics then generically are not the same. Below, we suggest a modified discretization procedure that makes the two discretizations equivalent.

## 4.2 Discretization Procedure I

Consider a discrete Lagrangian system specified by the discrete Lagrangian  $L_d : Q \times Q \rightarrow \mathbb{R}$ . Assume that  $Q$  is a fiber bundle as in Section 4.1, and assume the configuration variables are  $q_k = (x_k, y_k) = (x_k^1, \dots, x_k^\sigma, y_k^1, \dots, y_k^m)$ , i.e., the bundle coordinates are used. Assume also that the manifolds  $X$  and  $Y$  (the base and fiber of the bundle) are vector spaces (or commutative Lie groups). Finally, assume that the system has  $m$  (discrete) holonomic constraints

$$y_{k+\frac{1}{2}} = \phi(x_{k+\frac{1}{2}}) \tag{4.6}$$

or, in coordinates,

$$y_{k+\frac{1}{2}}^1 = \phi^1(x_{k+\frac{1}{2}}), y_{k+\frac{1}{2}}^2 = \phi^2(x_{k+\frac{1}{2}}), \dots, y_{k+\frac{1}{2}}^m = \phi^m(x_{k+\frac{1}{2}}).$$

Observe that the constraints are introduced in a somewhat different manner compared to the standard definition (see e.g. [25]): The mid-point rule is used to discretize the continuous-time constraint (4.1), whereas the standard way to introduce holonomic constraints in the discrete setting is to ask that  $y_k = \phi(x_k)$ .

Given a continuous-time Lagrangian  $L : TQ \rightarrow \mathbb{R}$ , define the discrete constrained Lagrangian  $L_c^d : X \times X \rightarrow \mathbb{R}$  by the mid-point rule

$$L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) = L^d(x_{k+\frac{1}{2}}, \Delta x_k, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+\frac{1}{2}}) \Delta x_k). \quad (4.7)$$

The motivation for this definition is that  $L_c^d$  is identical to the discretization of the constrained Lagrangian (4.2) and thus the linear stability conditions for equilibria of the continuous-time system and its discretization are the same [18].

Recall that the discrete action sum for  $L_c^d$  is defined by the formula

$$\mathcal{S}_c^d = \sum_{k=0}^{N-1} L_c^d(x_k, x_{k+1}) \quad (4.8)$$

and that a sequence  $\{x_k\}_{k=0}^N$  is a trajectory of the system defined by  $L_c^d$  if and only if  $\delta \mathcal{S}_c^d = 0$ . Evaluating  $\delta \mathcal{S}_d$  and asking that  $\delta \mathcal{S}_d = 0$  results in the discrete Euler–Lagrange equations on  $X \times X$ :

$$D_1 L_c^d(x_k, x_{k+1}) + D_2 L_c^d(x_{k-1}, x_k) = 0. \quad (4.9)$$

Using  $(x_{k+1/2}, \Delta x_k)$  as new coordinates instead of  $(x_k, x_{k+1})$ , writing the action  $\mathcal{S}_c^d$  as a function of these new coordinates, and computing  $\delta \mathcal{S}_c^d$ , we obtain

$$\begin{aligned} \delta \mathcal{S}_c^d &= \delta \sum_{k=0}^{N-1} L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) = \sum_{k=0}^{N-1} \delta L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \\ &= \sum_{k=0}^{N-1} D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \delta x_{k+\frac{1}{2}} + \sum_{k=0}^{N-1} D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \delta \Delta x_k \\ &= \sum_{k=0}^{N-1} \left[ \frac{1}{2} D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) - D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \right] \delta x_k \\ &\quad + \sum_{k=0}^{N-1} \left[ \frac{1}{2} D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) + D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \right] \delta x_{k+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{N-1} \left[ \frac{1}{2} D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) - D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) \right. \\
&\quad \left. + \frac{1}{2} D_1 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}) + D_2 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}) \right] \delta x_k.
\end{aligned}$$

Thus, when the constrained discrete Lagrangian is represented as a function of  $(x_{k+1/2}, \Delta x_k)$ , the dynamics of the corresponding constrained Lagrangian system becomes

$$\begin{aligned}
&\frac{1}{2} [D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) + D_1 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1})] \\
&\quad - D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) + D_2 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}) = 0. \quad (4.10)
\end{aligned}$$

These equations of course are equivalent to equations (4.9), as

$$D_1 L_c^d(x_k, x_{k+1}) = \frac{1}{2} D_1 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k) - D_2 L_c^d(x_{k+\frac{1}{2}}, \Delta x_k), \quad (4.11)$$

$$D_2 L_c^d(x_{k-1}, x_k) = \frac{1}{2} D_1 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}) + D_2 L_c^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}). \quad (4.12)$$

**Remark 4.2.1.** *For discretized continuous-time systems, the representation of the discrete dynamics in the form of equations (4.10), while equivalent to equations (4.9), gives a better reflection of the mechanical structure of the continuous-time dynamics (4.3).*

Now, rewrite the *discrete* dynamics obtained from the continuous-time dynamics in terms of the constrained continuous-time Lagrangian  $L_c(x, \dot{x}) = L(x, \dot{x}, \phi(x), \dot{\phi}(x))$ , where  $L(x, \dot{x}, y, \dot{y})$  is the unconstrained Lagrangian for the original continuous-time system.

Differentiating the formula  $L_c(x, \dot{x}) = L(x, \dot{x}, \phi(x), \partial_x \phi(x) \dot{x})$ , we obtain the formulae for the derivatives  $D_1 L_c^d(x_k, x_{k+1})$  and  $D_2 L_c^d(x_{k-1}, x_k)$  in terms of the continuous-time unconstrained Lagrangian  $L$  and the function  $\phi(x)$  that defines the continuous-time constraints. These formulae read:

$$\begin{aligned}
D_1 L_c^d(x_k, x_{k+1}) &= h \frac{\partial}{\partial x_k} L_c(x_{k+\frac{1}{2}}, \Delta x_k/h) \\
&= h \frac{\partial}{\partial x_k} L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h).
\end{aligned}$$

Then the discrete Euler–Lagrange equations (4.9) are equivalent to

$$\begin{aligned}
h \frac{\partial}{\partial x_k} L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \\
+ h \frac{\partial}{\partial x_k} L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) = 0. \quad (4.13)
\end{aligned}$$

Expanding, we obtain that the left-hand side of equations (4.9) is

$$\begin{aligned}
&h \left[ \frac{1}{2} D_1 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \right. \\
&\quad - \frac{1}{h} D_2 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \\
&\quad + \frac{1}{2} D_3 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \cdot \partial_x \phi(x_{k+1/2}) \\
&\quad + D_4 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \\
&\quad \quad \quad \cdot \left( -\frac{1}{h} \partial_x \phi(x_{k+1/2}) + \frac{1}{2} \partial_x^2 \phi(x_{k+1/2}) \Delta x_k/h \right) \Big] \\
&+ h \left[ \frac{1}{2} D_1 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \right. \\
&\quad + \frac{1}{h} D_2 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \\
&\quad + \frac{1}{2} D_3 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \cdot \partial_x \phi(x_{k-1/2}) \\
&\quad + D_4 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \\
&\quad \quad \quad \cdot \left( \partial_x \phi(x_{k-1/2}) (\frac{1}{h}) + \frac{1}{2} \partial_x^2 \phi(x_{k-1/2}) \Delta x_{k-1}/h \right) \Big]. \quad (4.14)
\end{aligned}$$

### 4.3 Discretization Procedure II

Recall that the right-hand side of equation (4.5),

$$F(x, \dot{x}) = - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^a} - \frac{\partial L}{\partial y^a} \right) \frac{\partial \phi^a}{\partial x^\alpha},$$

may be interpreted as an  $x$ -component of the constraint reaction. With this view of the dynamics, we follow Marsden and West and use the mid-point rule to discretize equations (4.5).

That is, we obtain the following discretization procedure:

$$\begin{aligned} D_2 L^d(x_{k-1}, x_k, y_{k-1}, y_k) + D_1 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \\ + F^d(x_{k-1}, x_k, x_{k+1}, y_{k-1}, y_k, y_{k+1}) = 0, \end{aligned} \quad (4.15)$$

where

$$L^d(x_k, x_{k+1}, y_k, y_{k+1}) = hL(x_{k+1/2}, \Delta x_k/h, y_{k+1/2}, \Delta y_k/h)$$

and where

$$\begin{aligned} F^d(x_{k-1}, x_k, x_{k+1}, y_{k-1}, y_k, y_{k+1}) &= \frac{1}{2} [F(x_{k-1/2}, \Delta x_{k-1}/h) + F(x_{k+1/2}, \Delta x_k/h)] \\ &= \frac{1}{2} [D_4 L^d(x_{k-1}, x_k, y_{k-1}, y_k) \partial_x \phi(x_{k-1/2}) + D_3 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \partial_x \phi(x_{k+1/2})]. \end{aligned} \quad (4.16)$$

We emphasize that in (4.15) it is assumed that the discrete Lagrangian  $L^d$  is represented as a function of  $(x_k, x_{k+1}, y_k, y_{k+1})$ . If one prefers a different representation in which  $L^d$  is a function of  $(x_{k+1/2}, \Delta x_k, y_{k+1/2}, \Delta y_k)$ , the right-hand side of equations (4.15) should be replaced with

$$\begin{aligned} \frac{1}{2} [D_1 L^d(x_{k+1/2}, \Delta x_k, y_{k+1/2}, \Delta y_k) + D_1 L^d(x_{k-1/2}, \Delta x_{k-1}, y_{k-1/2}, \Delta y_{k-1})] \\ - D_2 L^d(x_{k+\frac{1}{2}}, \Delta x_k, y_{k+1/2}, \Delta y_k) + D_2 L^d(x_{k-\frac{1}{2}}, \Delta x_{k-1}, y_{k-1/2}, \Delta y_{k-1}). \end{aligned} \quad (4.17)$$

This is confirmed with a calculation similar to the derivation of formula (4.10).

Equations (4.15) are incomplete. One would use the discrete constraints

$$y_{k+1/2} = \phi(x_{k+1/2}), \quad \Delta y_k = \partial_x \phi(x_{k+1/2}) \Delta x_k \quad (4.18)$$

to obtain the complete system.

Using (4.16), equations (4.15) become

$$D_2 L^d(x_{k-1}, x_k, y_{k-1}, y_k) + D_1 L^d(x_k, x_{k+1}, y_k, y_{k+1}) + \frac{1}{2} [D_4 L^d(x_{k-1}, x_k, y_{k-1}, y_k) \partial_x \phi(x_{k-\frac{1}{2}}) + D_3 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \partial_x \phi(x_{k+\frac{1}{2}})] = 0.$$

Similar to (4.15), these equations are incomplete, and one would need to substitute in the constraints (4.18) for obtaining a complete system.

In certain situations, it may be useful to rewrite the dynamics as

$$[D_2 L^d(x_{k-1}, x_k, y_{k-1}, y_k) + \frac{1}{2} D_4 L^d(x_{k-1}, x_k, y_{k-1}, y_k) \partial_x \phi(x_{k-\frac{1}{2}})] + [D_1 L^d(x_k, x_{k+1}, y_k, y_{k+1}) + \frac{1}{2} D_3 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \partial_x \phi(x_{k+\frac{1}{2}})] = 0. \quad (4.19)$$

Replacing  $y_{k+\frac{1}{2}}$  with  $\phi(x_{k+\frac{1}{2}})$  and  $\Delta y_k/h$  with  $\frac{\partial \phi}{\partial x_k}(x_{k+\frac{1}{2}}) \Delta x_k/h$ , the terms of the equations of motion become

$$\begin{aligned} & D_1 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \\ &= \frac{\partial}{\partial x_k} h L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) \\ &= h \left[ \frac{1}{2} D_1 L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) - \frac{1}{h} D_2 L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) \right] \\ &= h \left[ \frac{1}{2} D_1 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \right. \\ &\quad \left. - \frac{1}{h} D_2 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \right], \end{aligned} \quad (4.20)$$

$$\begin{aligned}
& D_3 L^d(x_k, x_{k+1}, y_k, y_{k+1}) \partial_x \phi(x_{k+1/2}) \\
&= \frac{\partial}{\partial y_k} h L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) \partial_x \phi(x_{k+1/2}) \\
&= h \left[ \frac{1}{2} D_3 L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) - \frac{1}{h} D_4 L(x_{k+\frac{1}{2}}, \Delta x_k/h, y_{k+\frac{1}{2}}, \Delta y_k/h) \right] \partial_x \phi(x_{k+1/2}) \\
&= h \left[ \frac{1}{2} D_3 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \partial_x \phi(x_{k+1/2}) \right. \\
&\quad \left. - \frac{1}{h} D_4 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \partial_x \phi(x_{k+1/2}) \right], \tag{4.21}
\end{aligned}$$

$$\begin{aligned}
& D_2 L^d(x_{k-1}, x_k, y_{k-1}, y_k) \\
&= \frac{\partial}{\partial x_k} h L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) \\
&= h \left[ \frac{1}{2} D_1 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) + \frac{1}{h} D_2 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) \right] \\
&= h \left[ \frac{1}{2} D_1 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \right. \\
&\quad \left. + \frac{1}{h} D_2 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \right], \tag{4.22}
\end{aligned}$$

$$\begin{aligned}
& D_4 L^d(x_{k-1}, x_k, y_{k-1}, y_k) \partial_x \phi(x_{k-1/2}) \\
&= \frac{\partial}{\partial y_k} h L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) \partial_x \phi(x_{k-1/2}) \\
&= h \left[ \frac{1}{2} D_3 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) \right. \\
&\quad \left. + \frac{1}{h} D_4 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, y_{k-\frac{1}{2}}, \Delta y_{k-1}/h) \right] \partial_x \phi(x_{k-1/2}) \\
&= h \left[ \frac{1}{2} D_3 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \partial_x \phi(x_{k-1/2}) \right. \\
&\quad \left. + \frac{1}{h} D_4 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \partial_x \phi(x_{k-1/2}) \right]. \tag{4.23}
\end{aligned}$$

Summarizing, the left-hand side of the discrete equations of motion (4.19) is

$$\begin{aligned}
& h \left[ \frac{1}{2} D_1 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \right. \\
& \quad + \frac{1}{h} D_2 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \\
& \quad + \frac{1}{2} D_1 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \\
& \quad - \frac{1}{h} D_2 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \\
& \quad \frac{1}{2} \left( \frac{1}{2} D_3 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \partial_x \phi(x_{k-1/2}) \right. \\
& \quad \quad + \frac{1}{h} D_4 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \partial_x \phi(x_{k-1/2}) \\
& \quad \quad + \frac{1}{2} D_3 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \partial_x \phi(x_{k+1/2}) \\
& \quad \quad \left. \left. - \frac{1}{h} D_4 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \partial_x \phi(x_{k+1/2}) \right) \right]. \quad (4.24)
\end{aligned}$$

Next, compare discrete dynamics (4.14) and discrete dynamics (4.24). Our goal is to find out when the two dynamics are identical. The results are summarized in the following statement.

**Theorem 4.3.1.** *The discretization procedures I and II become equivalent if the discrete constraint reaction force is defined by the formula*

$$\begin{aligned}
& h \left[ \frac{1}{4} D_3 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \partial_x \phi(x_{k+1/2}) \right. \\
& \quad + D_4 L(x_{k+\frac{1}{2}}, \Delta x_k/h, \phi(x_{k+\frac{1}{2}}), \partial_x \phi(x_{k+1/2}) \Delta x_k/h) \cdot \left[ \frac{1}{2h} \partial_x^2 \phi(x_{k+1/2}) \Delta x_k \right. \\
& \quad \quad \quad \left. \left. - \frac{1}{2h} \partial_x \phi(x_{k+1/2}) \right] \right. \\
& \quad + \frac{1}{4} D_3 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \partial_x \phi(x_{k-1/2}) \\
& \quad \left. + D_4 L(x_{k-\frac{1}{2}}, \Delta x_{k-1}/h, \phi(x_{k-\frac{1}{2}}), \partial_x \phi(x_{k-1/2}) \Delta x_{k-1}/h) \cdot \left[ \frac{1}{2h} \partial_x^2 \phi(x_{k-1/2}) \Delta x_{k-1} \right. \right. \\
& \quad \quad \quad \left. \left. + \frac{1}{2h} \partial_x \phi(x_{k-1/2}) \right] \right].
\end{aligned}$$

This theorem gives an alternative approach to the discretization of systems with holonomic constraints that ensures that the stability properties of equilibria of the continuous-time system and its discretization are the same. These stability properties, while formally being local, may

have a global influence on the discrete dynamics as the equilibria are a part of system's limit set.

## Conclusions and Future Work

The relaxed matching stabilization technique for relative equilibria of mechanical systems has been extended to the discrete setting. The study of the discrete-time model of the satellite-rotor led to a better understanding of discrete mechanics and stability, and triggered the development of a more consistent discretization approach. This new discretization approach contributes to our understanding of discrete constrained mechanics with nonconservative forces.

The future research plans include a deeper study of discrete matching formalism using the results of Chapter 4. Another interesting and important project is stabilization of periodic orbits of mechanical systems. We expect that the discrete matching formalism will be useful in this setting as the stability analysis of periodic orbits is an intrinsically discrete procedure.

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