Finite Elements Schemes for Inviscid Compressible Flows

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Abstract

A finite element scheme is presented for the numerical solution of the two-dimensional, unsteady Euler equations describing high-speed flows of an inviscid and compressible fluid. The proposed scheme represents an extension to the Galerkin finite element method of the well-known Lax-Wendroff finite difference method. The numerical results indicate that the proposed method possesses good shock-capturing properties without requiring the explicit use of artificial viscosity.

1. Introduction

Significant advances in the use of finite elements for the numerical simulation of inviscid, compressible flows have been made recently by paying due attention to the fundamental role of the characteristics in the solution of hyperbolic problems. Among the new proposed approaches, the characteristic Petrov-Galerkin method [1,2] and the Taylor-Galerkin method [3-6] are emerging as promising methods for the solution of unsteady inviscid flow problems.

A common feature of the above two approaches is the recognition that, in order to successfully apply finite elements to hyperbolic equations, the time discretization process needs to be analysed and treated with more care than had usually been done so far.

In the Taylor-Galerkin method, which is to be discussed herein, a Taylor series expansion in the time step is employed before the spatial discretization process in order to increase the accuracy of the classical time-stepping methods. The time derivatives of second and (possibly) third order in the Taylor series are expressed formally in terms of the original partial differential equation, so as to obtain a generalized governing equation discretized in time, but still continuous in the space variables. This generalized equation is subsequently written in a weak form and discretized in space by means of the standard Galerkin-Bubnov finite element method.

The Taylor-Galerkin method is, in fact, the proper generalization to finite elements of the concepts which are at the basis of the well-known Lax-Wendroff finite difference scheme [7]. The application of the method to non-linear hyperbolic problems is discussed in [4] for the case of a single advection equation and in [5] for the unsteady Euler equations in one space dimension. The aim of the present paper is to demonstrate its applicability in the numerical solution of the unsteady Euler equations in two dimensions.

2. Computational Scheme

The Euler equations governing the time-dependent flow in two space dimensions of an inviscid and compressible fluid are expressed in the following conservative form:

$$w_t + \underline{\nabla} \cdot \underline{f} = 0 \tag{1}$$

where

$$w = \begin{bmatrix} \rho \\ m \\ n \\ e \end{bmatrix} \quad ; \quad \underline{f} = \begin{bmatrix} m & , & n \\ m^2/\rho + p & , & mn/\rho \\ mn/\rho & , & n^2/\rho + p \\ (e+p)m/\rho & , & (e+p)n/\rho \end{bmatrix} \tag{2}$$

ρ is the density, $(m,n) = (\rho u, \rho v)$ is the linear momentum density, (u,v) the fluid velocity, $e = p/(\gamma-1) + \frac{1}{2}(m^2 + n^2)/\rho$ is the energy density, p is the pressure and γ the specific heat ratio. A perfect gas law has been assumed.

In order to derive a Taylor-Galerkin scheme for solving system (1), consider the following Taylor series expansion of the unknown w in the time step Δt:

$$w^{n+1} = w^n + \Delta t \, w_t^n + \frac{1}{2} \Delta t^2 \, w_{tt}^n + \ldots \tag{3}$$

While w_t is directly obtained from the governing system (1), w_{tt} must be evaluated by time differentiation of eq.(1) which gives:

$$w_{tt} = -\underline{\nabla} \cdot \underline{f}_t = -\underline{\nabla} \cdot [\underline{f}_w w_t] = \underline{\nabla} \cdot [\underline{f}_w \underline{\nabla} \cdot \underline{f}] \tag{4}$$

or, equivalently,

$$w_{tt} = \underline{\nabla} \cdot [\underline{A}(w) \underline{\nabla} \cdot \underline{f}] \tag{5}$$

where $\underline{A}(w) = \underline{f}_w$ is the Jacobian matrix. Now, by substituting the time derivatives in the Taylor series (3) by their values of eqs.(1) and (5), the following time-discretized equation is obtained:

$$(w^{n+1} - w^n)/\Delta t = -\underline{\nabla} \cdot \underline{f}^n + \frac{1}{2} \Delta t \, \underline{\nabla} \cdot [\underline{A}(w^n) \underline{\nabla} \cdot \underline{f}^n] \tag{6}$$

which represents a second-order accurate generalization of the familiar forward Euler time-stepping method.

The generalized equation (6) is subsequently written in a weak form by taking its scalar product with an arbitrary admissible weighting function, followed by integration by parts of the term containing second-order spatial derivatives. Finally, fully discrete equations are obtained by means of the standard Galerkin-Bubnov finite element method.

A well-known property of non-linear hyperbolic equations is that discontinuities are often generated, even starting from smooth initial data, as a consequence of non-linear interactions. Since the Galerkin approximation to the advective term, $\underline{\nabla} \cdot \underline{f}$, provides an accurate representation of the non-linear interactions, spatial oscillations are often generated near the discontinuities. We have found that such unphysical spatial oscillations (or wiggles) can be significantly reduced by replacing the standard Galerkin approximation to $\underline{\nabla}\cdot\underline{f}$, namely,

$$<\phi_K, \underline{\nabla}\cdot\underline{f}\,(\sum_J w_J\phi_J)> \tag{7}$$

where ϕ_J denotes the shape functions, by the following "relaxed" approximation

$$<\phi_K, \underline{\nabla}\cdot(\sum_J \underline{f}(w_J)\phi_J)> \tag{8}$$

Such a treatment is employed only locally in compression regions, i.e.,at nodes k for which the inequality

$$<\phi_K, \underline{\nabla}\cdot\underline{a}(w)> < 0$$

is satisfied, where $\underline{a} = (u,v)$ is the velocity field.

As shown in the next section, the locally relaxed Taylor-Galerkin method is able to give a crisp representation of moderately strong shocks without requiring the addition of a purely artificial viscosity of the Von Neumann-Richtmyer or Lapidus type.

3. Illustrative Example

As an illustration of the proposed Taylor-Galerkin method, we have considered the propagation of a discontinuity in a divergent nozzle. The computational mesh is shown in Fig. 1a and consists of 4-noded quadrilateral elements. Denoting by subscripts 1 and 2 the regions upstream and downstream of the discontinuity, the initial conditions are as follows:

$\rho_1 = 3.333$	$\rho_2 = 1.0$
$m_1 = 2.3331;\ n_1 = 0.0$	$m_2 = n_2 = 0.0$
$e_1 = 2.8523$	$e_2 = 0.2857$
$M_1 = 1.197$	$M_2 = 0.0$

M represents the Mach number and the momentum components m,n are parallel to the coordinate axes x and y, respectively. The shock speed is given by the Hugoniot-Rankine relations and is equal to one. At the inlet, the flow is supersonic and the condition $w_{inlet} = w_1$ is prescribed as inlet boundary condition. On the other hand, the flow is subsonic at the outlet, and the only prescribed condition there is that the momentum component n_2 be zero. In addition, symmetry conditions are prescribed on the upper lateral surface, and the flow is imposed to be parallel to the lower lateral surface.

The time step was taken as $\Delta t = 0.018$, corresponding to a maximum value of the Courant number c of

$$c = \lambda\ \Delta t/\Delta x = 0.464$$

$(\lambda = (|u| + a)_{max};\quad a = (\gamma p/\rho)^{1/2} = \text{sound speed})$.

Figures 1b to 1f show the computed level lines of density at t=0, 0.25, 0.50, 1.0 and 1.5, respectively. The level lines are equally spaced between the values 0.85 and 3.3, the peak value being 3.5. The flow is supersonic upstream of the shock and must remain parallel to the tube wall. So, at the point where the flow experiences an enlargement, we note the appearance of a Prandtl-Meyer expansion, and, where it takes its original direction again, we observe the presence of a detached shock.

The present example, together with other applications presented in [4,5,6], indicates that the proposed Taylor-Galerkin procedure leads to an accurate and versatile method of dealing with problems governed by non-linear hyperbolic equations, or systems of equations.

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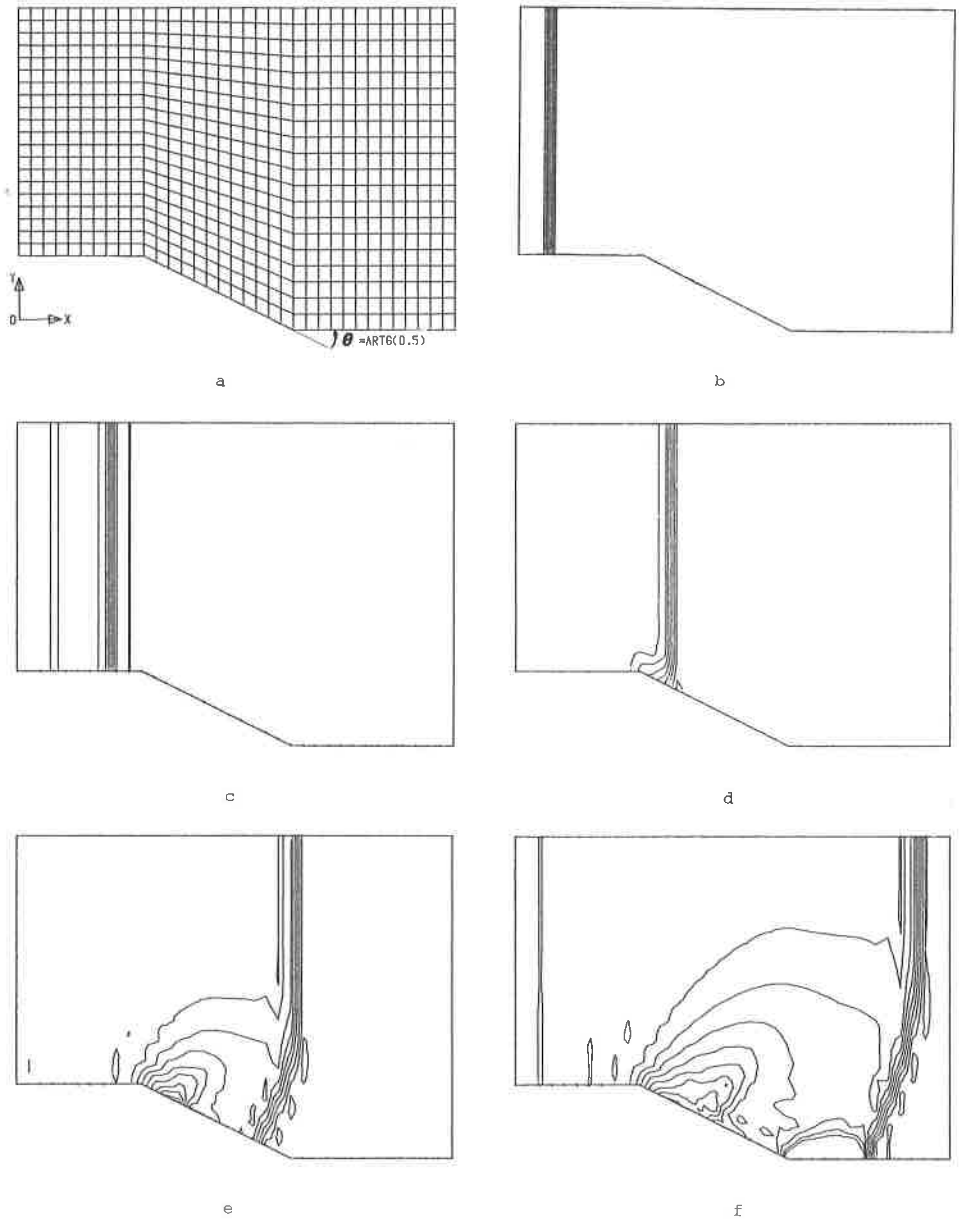

Fig. 1 : Flow through a divergent nozzle : a) finite element mesh, b) - f) density profiles at t = 0 ; 0.25 ; 0.5 ; 1.0 ; 1.5 .