MODELING STRUCTURES FOR SIMULATING NONLINEAR RESPONSE

R. J. MELOSH, F. HATT

MARC Analysis Research Corporation,
260 Sheridan Avenue, Palo Alto, California 94306, U.S.A.

SUMMARY

This paper proposes a quantitative basis for ab initio measurement of the quality of a finite element model of a structural system. The measure is suitable for characterizing the fidelity representation of stiffness, mass, and some types of damping. Thereby, it provides the basis for automatic development of alternate models of a structure.

The modeling problem is formulated as an optimization problem. Finite element model parameters are the design variables. For stiffness modeling, skin gages of surfaces and cross-sectional areas of line elements are to be determined to most closely represent a detailed description of geometry. For mass modeling, the values of lumped masses are to be calculated for any number of nodes less than, or equal to, those in the detailed description. Using both sets of design variables permits modeling Rayleigh or hysteresis damping. Like the final modeling data, the detailed description is defined to be in the form of finite element model data.

A collection of mathematical programming techniques in the code MODEL, provides the means for solving the optimization problem. When the number of linearly independent equality constraints (material disposition measures) is less than the number of design variables, a quadratic programming is recognized. The objective function is the Euclidean norm of the vector of differences of the guessed and unknown design variables. The solution is obtained by the LaGrange multiplier method, when it is unique, or by Wolfe's method. If more constraint equations than design variables are present, the optimization problem is solved in a least squares sense.

Example modelings include cases where the model has fewer finite elements than that of the detailed description, different geometry, and changed topology. Simple systems with the goal of stiffness or mass modeling are used to illustrate the nature of results developed by the approach.

The modeling procedure uniquely defines parameters of a finite element model for any structural arrangement. Because the process is independent of the stress and strain state variables and can be applied to any geometry, it is suitable for modeling systems which undergo material or geometric non-linearity. Because it can deal with stiffness, mass, or damping, it is useful for finite element modeling for either static or dynamic response. Extension of implementation to admit an extremely large number of finite elements will assure that behavior of models produced by the computer, approach behavior of the most detailed structure.

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Section 1
INTRODUCTION

Finding the most relevant mathematical representation of a physical system is the elusive objective of modeling.

The difficulty is represented by progress in finite element analysis of automobiles. Computer codes became available for analyzing an automobile as a linear system about 1959. It was seven years later, however, when industry engineers were confident that they could set up an analysis and predict its accuracy from the beginning. Analyzing a number of automobiles using finite elements, they established guidelines for predicting behavior within 10% for stresses and 5% for displacements. Though they are confident that they can perform an adequate job for modeling linear systems, they are still developing understanding of modeling for the nonlinear system after 10 years of effort toward this objective.

The modeling task can be illustrated by the modeling of a low aspect ratio wing. The primary structure consists of skins of constant thickness supported by a network of spars and ribs. It is required to model this wing with a solid plate of stepped stiffnesses. In this case, the modeling problem is to choose the thicknesses such that the plate will behave most like the prototype wing. (Note that the problem is not how many thickness steps should be used for the model; that is a problem of discretization, rather than modeling.)

If the structure only behaves in the linear range, the modeling procedure is relatively simple. It is only necessary to match the stiffness distribution. In this example, the plate thicknesses are chosen so that the moment of inertia of the wing matches the moment of inertia of the plate at each corresponding point. This implies that the material used in modeling the wing is the same as that of the prototype.

The results of this matching is excellent agreement between plate and wing deformations. This type of modeling has served as a low cost procedure for preliminary design analysis of low aspect ratio wings in the aircraft industries since the B-70 wing design of the early 1960's.

The use of the concept of matching stiffness in mathematical models of structures is not uncommon. It occurs in representing stiffened sheets as stringers and shear panels, tension field beams by trusses, geometrically orthotropic shells as materially orthotropic shells, bridge trusses as beams, and reinforced concrete beams by beams of transformed section.

When the behavior of the structure is to penetrate the nonlinear range, the stiffness match is insufficient. In the wing, for example, large life forces cause buckling of the upper skin. This loss of stiffness is not reflected in the matching plate model at the same deflections. In the wing, the lower skin can yield from lifting forces. The plate model, because it is much thinner, has a much lower peak stress and yielding will occur at higher loads. The same type of arguments apply to all the stiffness models mentioned above.

The only known way of modeling for the nonlinear systems is to match the disposition of material in detail and thereby model stiffness changes due to large deflection and inelastic material effects.[1]

Since costs of numerical analysis of nonlinear systems are much greater than for linear, engineering judgement has served as the basis for modeling. This report projects a quantitative basis for modeling for nonlinear analysis, which can be reduced to modeling of stiffness for linear systems. It addresses the modeling of structures of known initial configuration and unknown character of the nonlinear behavior. The modeling process implies that the structure is composed of a single material throughout. It neglects the effects of joint compliances and joint inefficiencies. The process is currently limited to models composed of membranes and rods. The report encompasses the modeling process. The next section discusses the basis for modeling. The third section provides results for the modeling of simple structures.

Section 2
A MODELING PROCESS

This section details a process for modeling based on matching measures of the material disposition alone. The process provides a basis for quantifying the quality of modeling independent of the original geometry of the structure or the nature of the nonlinear changes it undergoes.

The Material Disposition Basis for Modeling

Regardless of the character of the structural response, the mathematical model must represent the disposition of the resisting material to some degree. Thus, a problem invariant modeling method can be based on measures of the material arrangement.

The material disposition can be characterized by its volume moments,

$$T_n = \int_{V_o} F_n(x, y, z)\, dV_o \qquad (1)$$

T_n is the n^{th} moment of volume and,

V_o and dV_o is the initial and differential volume of material.

The function F(x, y, z) can be expressed as follows:

$$F_n = x^k y^l z^m \qquad (2)$$

$$k + l + m = n$$

where k, l and m represent integer exponents that add up to the value of n.

In structural nomenclature, T_o equals the volume of V_o and T_1 are the three statical moments. For n = 2, the expression for F_n are modified so that T_n represent the polar moments of inertia around the three coordinate axes.

If all moments of two representations are matched, T_n, n = 0, 1, 2,..∞, then the material disposition of the model perfectly matches that of its prototype.

To represent the moments in a form invariant with the location of the origin, the moments can be referenced to the centroid, i.e.,

$$T_n = \int_{V_o} F_n(x', y', z')\, dV_o$$

where the primed coordinates are defined by the centroidal coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, with

$$\begin{aligned} x' &= x - \bar{x}, & \bar{x} &= \frac{1}{V_o} \int_{V_o} x dV \\ y' &= y - \bar{y}, & \bar{y} &= \frac{1}{V_o} \int_{V_o} y dV \qquad (3) \\ z' &= z - \bar{z}, & \bar{z} &= \frac{1}{V_o} \int_{V_o} z dV \end{aligned}$$

To define the components of the moments independently in terms of its invariants, they are related to principal axes at the centroid.

A unique principal orientation is found when the three mixed second order moments are zero. Thus, the moments are given by:

$$T'_n = \int_{V_o} F_n\, (x', y', z')\, dV_o$$

where

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} \lambda_1, & \lambda_2, & \lambda_3, \\ \mu_1, & \mu_2, & \mu_3, \\ \nu_1, & \nu_2, & \nu_3, \end{bmatrix} \begin{Bmatrix} x-\bar{x} \\ y-\bar{y} \\ z-\bar{z} \end{Bmatrix} \qquad (4)$$

with λ_i, μ_i, ν_i, = 1, 2, 3 direction cosines relating the base vectors in the initial x, y, z coordinate systems to those along the principal axes of the system.

The equations (3) and (4) provide the basis for determining the volume moments, the evaluation of the integrals is cumbersome even just for line and surface elements. Accordingly, the integrals are evaluated numerically using Gauss integration.[2]

Gauss integration is performed by evaluating each moment component independently. In general, a particular component is given by:

$$U_n = \int_{V_o} x'^{q}\, y'^{r}\, z'^{s}\; dV = \sum_{e=1}^{E} V_e \left(\sum_{p=1}^{P} w_p\, x_p'^{q}\, y_p'^{r}\, z_p'^{s} \right)_e \qquad (5)$$

$$n = q + r + s$$

where

U_n is the U component of the n^{th} moment,

q, r, s, are each greater than or equal to zero,

V_e is a multiplier corresponding to the cross-sectional area for rods and the thickness for triangular membranes,

E is the total number of elements,

P is the number of Gauss points for the element and,

w_p is the weighting factor for the p^{th} point.

The Finite Element Modeling Problem

The problem is to select cross-sectional areas of line elements and thicknesses of flat membranes such that the moments of a model match those of the prototype.

Thus, the problem is to choose areas and thicknesses to minimize

$$D = \sum_{e=1}^{E} (g_e^g \quad g_e^s)^2 \qquad (6)$$

where

D is the "distance" from the analyst's guess to the solution,

E is the total number of elements (triangular membrane elements, plus line element),

g_e^g is the analyst's guess for the e^{th} gage membrane thickness or line element cross-sectional area, and

g_e^s is the unknown element gage of the e^{th} element.

The choice of gages are constrained to match the volume moment requirements of the prototype structure. Eq. (6) thus, must be minimized subject to matching conditions,

$$[C]\ (g) = (U_p) \qquad (7)$$

up to the modeling level required by the specification of n, and the existence conditions,

$$g_e \geq 0 \quad e = 1,\ 2,\ ,,,E \qquad (8)$$

where

U_p are the volume moments of the prototype, and

C matrix provides volume moment coefficients for unit density for the unknown gages of the model.

C need not be a square matrix. Even if it is square, it is usually not symmetric.

Equations (7) are a maximum of (n + 1) (n + 2) (n + 3)/6 linearly independent equations for a three-dimensional structure. If the structure is planar, Eqs. (7) provide a maximum of (n + 1) (n + 2)/2 independent equations.

Because the constraint equations are linear inequalities in the thicknesses and areas and the object function quadratic, the problem is as a quadratic programming problem.

Section 3
SOLUTION TECHNIQUES

The approach to solving the modeling problem depends on the relation between the number of constraints and the number of design variables. Three techniques provide for all possibilities.

Solution with More Constraints than Model Gages

If the number of matching constraints (Eq. (7)) is greater than, or equal to, the number of design variables, a best solution can be developed in the least squares sense. The distance function and the existence conditions are ignored and the variables are obtained by,

$$(g) \;=\; [C^t\ C]^1\ [C^t]\ (U_p) \qquad (9)$$

here, C^t is the transpose of the C matrix and C^{-1} is its inverse.

Equation (9) implies equal weighting of the prototype volume moments, U_n. If the number of linearly independent matching constraints equals the number of design variables, Eqs. (9) reduces to,

$$(g) = [\ C\ ]^{1} (U_p) \qquad (10)$$

and a single solution exists which satisfies all the matching equations. Again, the solution obtained may violate the existence conditions.

Solution with Less Constraints than Model Gages

If the number of design variables is less than the number of constraints, the programming problem can be solved by the LaGrange multiplier method. If the solution is unacceptable because it involves a negative element gage, solution can be developed by Wolfe's quadratic minimization method.

In LaGrange's method, the solution is generated by minimizing Eqs. (6) subject to the constraints of Eq. (7) using LaGrange multipliers. Thereby

$$\begin{bmatrix} 2I & C \\ & \\ C & 0 \end{bmatrix} (g) = \begin{Bmatrix} 2_g^g \\ \\ U_p \end{Bmatrix} \qquad (11)$$

where λ are the LaGrange multipliers, and I is an identity matrix of order M + L. This provides a direct solution process which gives acceptable answers whenever the solution satisfies Equation (8). The solution is guaranteed to be associated with the absolute minimum of Eqs. (6).

If some of the design variables are less than zero from the solution of Eq. (11), the solution is obtained by solving a sequence of linear programming problems. The inequalities are replaced by equalities by introducing slack variables to the list of design variables. In addition, the requirement that

$$[s_x, s_y] (g) = 0 \qquad (12)$$

with $s_{x_e} = 0$ if $s_{y_e} = 0$, e = 1, 2...E, and conversely,

where the s_x and s_y are E slack variables is added to insure that either a slack variable is, or its corresponding design variable is zero.

The solution is obtained by Wolfe's long method.[3] This is a three step method which uses the simplex method to solve a linear programming problem in each step. The results of the three steps is a set of gages which satsifies Eqs. (15) and Eqs. (12). Since these are the Kuhn-Tucker optimality conditions of the problem defined by Eqs. (6), (7), and (8), this is the desired absolute minimum solution. Thus, the gages are those which meet all the constraints and are closest to the analyst's guess.

Section 4
ILLUSTRATIVE PROBLEMS

This section provides results of modeling using the procedure of Section 3. The examples exhibit applications in which the model has fewer elements and changed geometry and topology. They include a case in which the mass of the structure is replaced by an equivalent set of lumped masses.

Truss Structure

Figure 1 provides results for modeling a three-dimensional box truss by a truss with one less element. Each element of the prototype truss has a cross-section of $3.2500m^2$. This truss is to be modeled by a truss which omits one of the internal diagonal elements of the box.

The model preserves area moments through second order. A unique and acceptable model is achieved using the LaGrange multiplier process.

The Table of Figure 1 cites the element areas for the model. These areas retain the symmetry of the prototype across the plane defined by joints 1, 4, 5, and 8 as required.

When the model includes all the elements of the prototype (a reflective model) the analysis replicates the prototype.

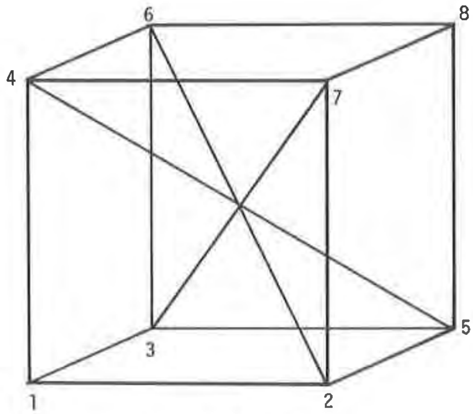
Stiffened Sheet

Figure 2 a illustrates a membrane reinforced by stringers arounds its rectangular boundary. The membrane is stepped. The objective of modeling is to define models of different elements and geometries.

Figure 2b and 2c show two modeling alternatives. In both models fourth moments were matched because the "reflective" analysis did not emulate the prototype until fourth moments were matched.

Figure 2b and 2c results were developed by the least square solution. Thus, they do not match area moments precisely. This explains why the membrane only model possesses a greater structural volume than the prototype.

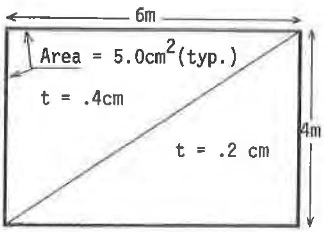
Figure 2d illustrates a design of the prototype of Figure 2a when the geometry is changed. This interesting design is produced by solving the quadratic programming problem. This design matches area moments through the second order. A solution satisfying these conditions is found which involves only two of the stringers and two of the membrane elements. The other elements have zero volume.



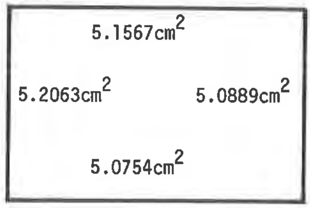
Element	Area in Prototype	Area of Model*
1-8	$5.00 cm^2$	$6.67 cm^2$
1-2	$5.00 cm^2$	$4.04 cm^2$
1-4	$5.00 cm^2$	$5.96 cm^2$
1-3	$5.00 cm^2$	$4.04 cm^2$
4-7	$5.00 cm^2$	$5.96 cm^2$
4-6	$5.00 cm^2$	$5.96 cm^2$
7-8	$5.00 cm^2$	$4.04 cm^2$
7-2	$5.00 cm^2$	$4.04 cm^2$
8-6	$5.00 cm^2$	$4.04 cm^2$
8-5	$5.00 cm^2$	$5.96 cm^2$
3-6	$5.00 cm^2$	$4.04 cm^2$
3-5	$5.00 cm^2$	$5.96 cm^2$
2-5	$5.00 cm^2$	$5.96 cm^2$
2-6	$5.00 cm^2$	$6.67 cm^2$
3-7	$5.00 cm^2$	$6.67 cm^2$
5-4	$5.00 cm^2$	$.0 cm^2$

*Based on matching through second moments.

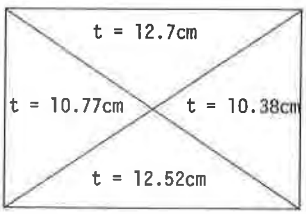
Figure 1. Modeling a Three-Dimensional Truss



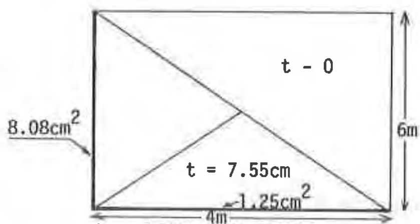
a. Prototype Reinforced Membrane



b. Prototype Modeled by Stringers Only



c. Prototype Modeled by Membranes Only



d. Stiffened Membrane Model of Changed Geometry

Figure 2. Stiffened Membrane Modeling

Mass Lumping

Figure 3 illustrates a structure whose distributed mass is to be represented by a set of point masses. Point masses are permitted at all the nodes of the prototype.

The modeling process lumps mass simply by specifying a model which has very short elements at each site where mass is to be lumped. The volume of this element times the mass density defines the mass point.

Figure 3 shows the masses produced by the least square solutions. The lumped mass results are given, as well as results from two modeling systems. In all cases, density is assumed to be one.

A comparison of the four and five mass solutions indicate that the analysis tends to sacrifice total mass to preserve higher moments. The five mass model preserves total mass, while the four does not, yet corner masses are about the same. By contrast, the lumped mass model preserves total mass. It misrepresents the higher mass moments.

Section 5
SUMMARY

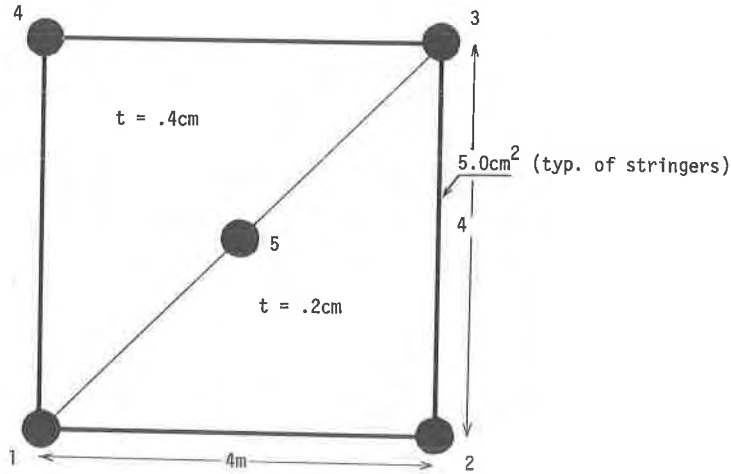
This report describes the basis for replacing a finite element representation of a given prototype structure, with a finite element representation of a model. The modeling is achieved by choosing design variables of the model so that area moments of the prototype are matched. When this would not result in a uniquely defined model, it is also required that the model parameters be as close as possible to those guessed by the analyst.

The modeling process has the following features:

1. It assigns cross-sectional areas of line elements and/or membranes such that area moments are matched.

2. It permits modeling the prototype with fewer or more elements, with elements of different connectivity or type, and with different joint locations.

3. It includes three algorithms so the complete problem span is covered: models with more, equal, or less element gages than linearly independent area moment constraint equations.



Joint	Lumped Mass	Least Square(1)	Least Square(2)
1	20.800	13.734	13.811
2	20.533	13.468	13.614
3	20.800	13.734	13.811
4	21.667	14.001	14.008
5	1.600	29.856	0.

(1) Matching 1^{st} moments: 6 equations, 5 unknowns and
Matching 2^{nd} moments: 7 equations, 5 unknowns.

(2) Matching 2^{nd} moments: 7 equations, 4 unknowns.

Figure 3. Mass Lumping of Stiffened Sheet

4. It encompasses systems of line and/or membrane elements and is invariant with loading or response characteristics of the system. It is restricted to prototypes of a single material composition, with joints with negligible compliance.

The process provides a quantitive basis for modeling a prototype which, if area moment matching capabilities were extended to infinite order, can guarantee to replicate the prototype.

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References

[1] R. J. Melosh and F. Hatt, "Modeling Structures for Simulating Nonlinear Response," NASA-CR-2693, MARC Analysis Research Corporation, May 1976.

[2] Cowper, G. R., "Gaussian Quadrature Formulas for Triangles", as reproduced in Strang, Gilbert and Fix, G. S., An Analysis of the Finite Element Method, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1973, pg. 184.

[3] Kunzi, H. P., Tzschach, H. G., and Zehnder, C. A., Numerical Methods of Mathematical Optimization, Academic Press, New York, pp. 74-80.