An Appraisal of Numerical Solution Techniques for Elasto-Plastic and Elasto-Viscoplastic Material Problems

D.R.J. Owen, C.M.B. Gomez
Department of Civil Engineering,
University College of Swansea, Singleton Park, Swansea SA2 8PP, United Kingdom

Summary

The aim of the paper is to critically compare the relative efficiency of numerical algorithms available for the solution of nonlinear finite element problems. The methods considered include the Conjugate Newton, Quasi Newton and Secant Newton methods. The performance of these algorithms is compared against the standard Newton Raphson and Modified Newton Raphson solution processes.

Difficulties are generally encountered in the solution of structures in which limit points or unstable solution branches exist, since small load increments are required in the vicinity of such regions whose location on the load/displacement curves are generally not known prior to solution. Such problems can be alleviated by use of solution algorithms which simultaneously update both the loads and displacements during equilibrium iteration. In this paper the performance of such a scheme based on Riks' Constant Spherical Arc process is investigated.

The algorithms are appraised by application to several problems involving elasto-plastic or large deformation behaviour. The results obtained indicate the Secant Newton method to be the most efficient followed by the Quasi Newton method; both algorithms being considerably superior to the standard Newton Raphson and modified Newton methods with regard to both computational costs and stability. The Quasi Newton method operates satisfactorily without a line search whereas it is found that the performance of the Conjugate Newton method is quite sensitive to the quality of the search process. Use of the load path constraint method is shown to be particularly beneficial by application to a "snap-through" stability problem.

1. Introduction

Solution processes for nonlinear finite element problems are, at best, computationally expensive and any steps that can be taken to significantly reduce these costs are of practical importance. During the last two years or so several alternative solution procedures for nonlinear problems have been presented and the aim of this paper is to critically compare the efficiency and limits of applicability of such methods. The methods considered include the Conjugate Newton Method (CN), Quasi Newton Method (QN) and Accelerated Modified Newton Raphson or Secant Newton Method (SN). The performance of these algorithms is compared against the standard Newton Raphson (NR), Modified Newton Raphson (MN) and Initial Stiffness (IS) solution processes.

Most standard nonlinear equation solution techniques exhibit poor convergence characteristics when applied to problems possessing limit and bifurcation points or which have unstable branches to the solution path. Dramatic improvements can be made in these areas by simultaneous adjustment of both the load and displacement parameters during the iterative solution process. To this end the adaptive load techniques of Bergan (1) or the Constant Spherical Arc process originally suggested by Riks (2) can be employed. In this paper the latter approach is adopted and can be easily incorporated in most of the solution algorithms described above.

The solution algorithms are all based on the Frontal Equation Solver developed by Irons (3,5) employing the equation resolution version presented in Ref. (4). The various methods are assessed by application to both elasto-plastic and large deformation problems in continua and thin shells.

2. Solution of nonlinear equations

A displacement formulation of a nonlinear finite element problem demands the simultaneous satisfaction of the global stiffness equations $\underline{K}\ \underline{a} = \underline{f}$ and the equilibrium (or residual force) equations

$$\underline{\psi} = - \underline{r} = \int_\Omega \underline{B}^T \underline{\sigma}\, d\Omega - \underline{f} = 0 \qquad (1)$$

in which $\underline{K}$ is the assembled stiffness matrix, $\underline{a}$ is the vector of unknowns (displacements) and $\underline{f}$ is the load vector including body forces and surface tractions. The term $\underline{\sigma}$ represents the stresses satisfying the appropriate nonlinear constitutive relation and $\underline{B}$ is the strain matrix which is displacement dependent for large deformation problems.

The solution of a nonlinear problem almost invariably involves an incremental/iterative process, in which a number of iterative cycles are performed, for each increment of applied load, until the residual forces $\underline{\psi}$ vanish. Thus a nonlinear solution algorithm requires a means of determining successive "search directions" $\underline{p}_i$ so that, ultimately on the (i+1)th iteration the displacements

$$\underline{a}_{i+1} = \underline{a}_i + \underline{p}_i \qquad (2)$$

result in a stress field $\underline{\sigma}_{i+1}$ which satisfies (1).

Convergence of the solution process can be monitored in several different ways involving comparing some measure of the residual forces $\underline{\psi}$ against the current level of applied loads, $\underline{f}$. In this work convergence is deemed to have been achieved if on the mth iteration

$$||\underline{\psi}_m|| \leq \tau\ ||\underline{f}|| \qquad (3)$$

where $||\quad||$ represents the Euclidian norm and τ is a convergence factor typically specified in the range $0.01 \leq \tau \leq 0.0001$.

2.1 Newton Raphson and Modified Newton Methods

In the Newton-Raphson Method (NR) the displacements are updated between successive iterations according to (2) with a search direction $\underline{p}_i$ being sought which makes the residuals

$$\underline{\psi}_{i+1} = \underline{\psi}(\underline{a}_i + \underline{p}_i) = 0 \tag{4}$$

Expressing (4) in a truncated Taylor's series form gives the search direction

$$\underline{p}_i = - \left[\frac{\partial \underline{\psi}_i}{\partial \underline{a}_i}\right]^{-1} \underline{\psi}_i = - [\underline{K}_i]^{-1} \underline{\psi}(a_i) \tag{5}$$

where the residual Jacobian matrix coincides with the tangential stiffness matrix $\underline{K}_i$ corresponding to displacement state, $\underline{a}_i$. Thus the computation of the search direction demands a stiffness evaluation followed by a complete equation reduction and backsubstitution for each iteration of each load increment.

The Initial Stiffness Method (IS) seeks to reduce the computation costs associated with the Newton-Raphson (or tangential stiffness) method by employing the original structural stiffness matrix $\underline{K}_o$ at all times when determining the search direction in (5). In this way, after the first increment of the first iteration, only equation resolution need be performed and this approach has been shown to be unconditionally stable (6). However whereas the Newton-Raphson method possesses a quadratic rate of convergence, the Initial Stress approach has only a linear rate and convergence can be slow if the structure exhibits a high degree of nonlinearity. Therefore in general the most efficient algorithm is found by combining the two approaches and recomputing the stiffness matrix only once during a load increment. This is termed the Modified Newton method (MN).

2.1.1 Newton Methods with Line Search

Instead of using update formula (2), it is possible to follow the approach used in Conjugate Gradient (CG) methods (7) and search along the direction $\underline{p}_i$ to find the position where the potential energy of the structure is minimised. Thus now

$$\underline{a}_{i+1} = \underline{a}_i + S_i \underline{p}_i \tag{6}$$

In the Conjugate Gradient Method the search parameter S_i is determined explicitly, but here S_i is determined from the minimum energy condition

$$G(S) = \underline{p}_i^T \underline{\psi}(\underline{a}_i + S_i \underline{p}_i) = 0 \tag{7}$$

This technique of line search is also used in the Conjugate Newton and Quasi Newton methods and the actual computation of S_i is discussed in the next section.

2.2 Conjugate Newton Method

The Conjugate Newton Method (CN) was proposed by Irons (8) in an attempt to combine the best features of the Modified Newton Method and the Conjugate Gradient technique. The basic idea is that the new search direction $\underline{p}_i$ should be easy to compute through the resolution of a system of equations, as in the MN method, and that it should be K-orthogonal to the previous value $\underline{p}_{i-1}$, as in the CG method. The algorithm can be summarised as

$$\underline{a}_{i+1} = \underline{a}_i + S_i \underline{p}_i \tag{8}$$

$$\underline{p}_i = \underline{p}_i^* + \beta \underline{p}_{i-1} \tag{9}$$

where $\underline{p}_i^*$ is the standard NR iterative displacement

$$\underline{p}_i^* = [\underline{K}_T^o]^{-1} \underline{r}_i \tag{10}$$

and $\underline{r}_i = \underline{r}(\underline{a}_i) = -\underline{\psi}(\underline{a}_i)$. The term $\underline{K}_T^{\,o}$ denotes the tangential stiffness matrix at the beginning of the increment. The parameter β is equal to 0 for i=1 and for i>1 is such that

$$\underline{p}_i^{\,T}\,\underline{K}_T^{\,o}\,\underline{p}_{i-1} = 0 \tag{11}$$

2.2.1 The Search parameter S_i and parameter β

In each iteration a search is performed along the direction $\underline{p}_i$ in order to find the value of S_i at which (7) is satisfied. The scalar function, G(S), is simultaneously evaluated for a series of trial values of S (say, {S}={0.25,1.0,2.0,4.0}) and the value of S_i corresponding to G(S) = 0 determined by interpolation.

Substituting from (9) and (10) in (11) results in

$$\beta = \frac{\{[\underline{K}_T^{\,o}]^{-1}\,\underline{r}_i\}^T\,\underline{K}_T^{\,o}\,\underline{p}_{i-1}}{\underline{p}_{i-1}^{\,T}\,\underline{K}_T^{\,o}\,\underline{p}_{i-1}} \tag{12}$$

Defining $\underline{\gamma}_i$ as the change in residual forces occurring between iterations i and i+1, then use of (8) and (5) gives

$$\underline{\gamma}_i = \underline{r}_{i+1} - \underline{r}_i = S_i[\underline{K}_T^{\,i}]\,\underline{p}_i \tag{13}$$

where $\underline{K}_T^{\,i}$ is the tangential stiffness matrix corresponding to the ith iteration. The value of the residual change, $\underline{\gamma}_i$ will depend on the approximation taken for $\underline{K}_T^{\,i}$. For $\underline{K}_T^{\,i} = \underline{K}_T^{\,o}$ equation (12) reduces to

$$\beta = -\frac{\{[\underline{K}_T^{\,o}]^{-1}\,\underline{r}_i\}^T\,\underline{\gamma}_{i-1}}{\underline{p}_{i-1}^{\,T}\,\underline{\gamma}_{i-1}} \tag{14}$$

2.3 Quasi Newton Method

This is a matrix update method that avoids the full reformulation and inversion of the stiffness matrix as demanded by the NR technique and yet does not operate with a constant stiffness matrix as does the IS method. It seeks to update the stiffness matrix from $\underline{K}^{i-1}$ to $\underline{K}^{i}$ so as to satisfy equation (13). Several options remain open for the choice of update algorithm but experience to date indicates the BFGS (Broyden-Fletcher-Goldfarb-Shanno) - Rank 2 method to be the most efficient for nonlinear structural problems. Reports of its successful application can be found in Refs. (9-11).

The BFGS algorithm for matrix updating is

$$[\underline{K}^i]^{-1} = (\underline{I} + \underline{w}_i\underline{v}_i^{\,T})\,[\underline{K}^{i-1}]^{-1}\,(\underline{I} + v_i w_i^{\,T}) \tag{15}$$

where
$$\underline{w}_i = \frac{\underline{p}_{i-1}}{\underline{p}_{i-1}^{\,T}\,(\underline{r}_{i-1} - r_i)} \tag{16}$$

and
$$\underline{v}_i = \underline{r}_i - \underline{r}_{i-1}\left(1 + \left[\frac{S_{i-1}\,p_{i-1}^{\,T}\,(\underline{r}_{i-1} - \underline{r}_i)}{\underline{p}_{i-1}^{\,T}\,\underline{r}_{i-1}}\right]^{\frac{1}{2}}\right) \tag{17}$$

Commencing from the tangential stiffness at the beginning of the load increment, (15) gives the following expression for the current stiffness for the ith iteration.

$$[\underline{K}^i]^{-1} = \left[\prod_{j=i}^{1}(\underline{I} + \underline{w}_j\underline{v}_j^{\,T})\right][\underline{K}_T^{\,o}]^{-1}\left[\prod_{j=1}^{i}(\underline{I} + \underline{v}_j\underline{w}_j^{\,T})\right] \tag{18}$$

The computational procedure followed in solution is outlined below:

(a) At the end of the (i-1)th iteration, the accumulated displacements are $\underline{a}_{i-1}$ and the residual forces are $\underline{r}_{i-1}$

(b) Compute the search direction as

$$\underline{p}_{i-1} = [\underline{K}^{i-1}]^{-1} \underline{r}_{i-1} \qquad (19)$$

In computation, the stiffness matrix $\underline{K}^{i-1}$ is never evaluated explicitly: Instead $\underline{p}_{i-1}$ is computed directly as follows.

(i) Evaluate
$$\underline{b}_{i-1} = \left[\prod_{j=1}^{i-1} (\underline{I} + \underline{v}_j \underline{w}_j^T)\right] \underline{r}_{i-1} \qquad (20a)$$

which merely involves vector multiplication.

(ii) Compute
$$\underline{c}_{i-1} = [\underline{K}_T^{\,o}]^{-1} \underline{b}_{i-1} \qquad (20b)$$

This step corresponds to equation resolution with only reduction of the terms in the vector $\underline{b}_{i-1}$ being necessary.

(iii) Evaluate
$$\underline{p}_{i-1} = \left[\prod_{j=i-1}^{1} (\underline{I} + \underline{w}_j \underline{v}_j^T)\right] \underline{c}_{i-1} \qquad (20c)$$

Again this step merely involves vector multiplication. Note that for i=1 the search direction $\underline{p}_o$ is calculated according to (10).

(c) Carry out a line search according to Section 2.2.1 to find the value of S_{i-1} which satisfies (7).

(d) Update the displacements according to

$$\underline{a}_i = \underline{a}_{i-1} + S_{i-1} \underline{p}_{i-1} \qquad (21)$$

(e) Compute the new residuals, $\underline{r}_i$ according to (1) and check for convergence according to (3). If the solution has not converged to the specified tolerance, return to Step (b) and carry out a further iteration.

In Step (b) the number of products increases rapidly with the number of iterations performed. In practice it may be advisable to recompute the tangential stiffness matrix and restart the above computational sequence if convergence has not been achieved after a specified number of iterations (say 10-15).

The success of the algorithm is also linked to the condition number C_i of the matrix $\underline{A}^i = (\underline{I} + \underline{v}_i \underline{w}_i^T)$ and can be expressed as

$$C^i = \left[\frac{\underline{p}_i^T \, \underline{\gamma}_i}{\underline{p}_i^T \, \underline{K}^{i-1} \, \underline{p}_i}\right]^{\frac{1}{2}} \qquad (22)$$

If the condition number becomes too large the updated inverse matrix will be nearly singular, in which case the BFGS update should be avoided and the algorithm restarted with a recomputed tangential stiffness matrix.

2.4 Secant Newton (Accelerated Newton) Method

Another matrix update method has been proposed by Crisfield (12) which seeks to satisfy "secant" equation (13) rather than orthogonality condition (11). The update algorithm is based on a formula proposed by Fletcher (13) and can be written

$$[\underline{K}^i] = [\underline{K}^{i-1}] + \frac{1}{\underline{p}_{i-1}^T \, \underline{\psi}_{i-1}} \underline{\psi}_{i-1} \underline{\psi}_{i-1}^T + \frac{1}{\underline{p}_{i-1}^T \, \underline{\gamma}_i} \underline{\gamma}_i \underline{\gamma}_{i-1}^T \qquad (23)$$

Crisfield further proposed that $[\underline{K}^{i-1}]$ be replaced by $[\underline{K}_T{}^o]$ the tangential stiffness matrix at the beginning of the increment. Again the updated stiffness matrix is never computed explicitly and the search direction is computed directly from the following relationships

$$\underline{p}_i = e_i \underline{p}_{i-1} + h_i \underline{p}_i{}^* \tag{24}$$

$$\underline{p}_i{}^* \,\hat{=}\, - [\underline{K}_T{}^o]^{-1} \, \underline{\psi}_i \tag{25}$$

in which e_i and h_i are the scalar quantities

$$h_i = - d_i / b_i ; \quad e_i = h_i \left(1 - \frac{c_i}{b_i} \right) - 1 \tag{26}$$

and b_i, c_i and d_i are scalar products given by

$$b_i = \underline{p}_{i-1}{}^T \, \underline{\gamma}_i ; \quad c_i = \underline{p}_i{}^{*T} \, \underline{\gamma}_i ; \quad d_i = \underline{p}_{i-1}{}^T \, \underline{\psi}_{i-1} \tag{27}$$

when i=1, $e_i = 0$ and $h_i = 1$.

Evaluation of the search direction involves only the equation resolution demanded in (25) together with some vector products. The displacements are then updated according to the Newton relationship given by (2).

3. Load Path Constraining Methods

In the vicinity of limit points or bifurcation points the structural performance becomes highly nonlinear and the need for the application of small load increments in such regions is obvious. Indeed in structures where a "snap-through" behaviour occurs it is necessary to reduce the load level if it is required to maintain equilibrium throughout the load/displacement history. Riks (2) has proposed a method in which both the load and displacement levels are simultaneously varied during iteration to convergence. Equilibrium equations (1) are modified to include the load level as an additional variable so that

$$\underline{\psi}_i{}^m = \int_\Omega \underline{B}^T \, \underline{\sigma} \, d\Omega - \lambda_i{}^m \, \underline{q} = 0 \tag{28}$$

where $\lambda_i{}^m$ is a parameter that defines the total applied loads on increment m at iteration i, as a function of some fixed reference load $\underline{q}$. Solution for the additional variable $\lambda_i{}^m$ is afforded by introduction of a constraint equation which constrains the load path to lie on a spherical arc during each load increment.

$$(\Delta \underline{a}_i{}^m)^T \, \Delta \underline{a}_i{}^m + (\Delta \lambda_i{}^m)^2 \, \underline{q}^T \, \underline{q} = (\Delta L^m)^2 \tag{29}$$

in which $\Delta \underline{a}_i{}^m$ is the total incremental displacement vector at iteration i during load step m, $\Delta\lambda_i{}^m$ is the corresponding incremental load factor and ΔL^m is the arc length for load increment m. Constraint equation (29) is illustrated graphically in Fig. 1.

Crisfield (14) proposed the simpler constraint equation

$$(\Delta \underline{a}_i{}^m)^T \, (\Delta \underline{a}_i{}^m) = (\Delta L^o)^2 \tag{30}$$

where ΔL^o is the basic arc length computed on the first iteration of a load step.

Upon a variation of load level $\delta\lambda_i$, (28) gives (dropping the load step superscript m for convenience)

$$\underline{\psi}_i \, (\lambda_i + \delta\lambda_i) = \underline{\psi}_i \, (\lambda_i) - \delta\lambda_i \, \underline{q} \tag{31}$$

with the corresponding iterative displacement, assuming a NR algorithm, of

$$\underline{p}^*(\lambda_i + \delta\lambda_i) = \underline{p}_i{}^*(\lambda_i) + \delta\lambda_i \, \underline{p}_T \tag{32}$$

where $\underline{p}_T$ is the tangent displacement vector (constant through the increment if a MN method is used) and $\underline{p}_i^*(\lambda_i)$ is the standard search direction of the NR method as given by (10). The total incremental displacement is then given by

$$\Delta\underline{a}_{i+1} = \Delta\underline{a}_i + \underline{p}^*(\lambda_i + \delta\lambda_i) = \Delta\underline{a}_i + \underline{p}_i^* + \delta\lambda_i\,\underline{p}_T \tag{33}$$

The value of $\delta\lambda_i$ is determined by substituting (33) into constraint condition (30) to give the following quadratic equation

$$c_1\,\delta\lambda_i^{\,2} + c_2\,\delta\lambda_i + c_3 = 0 \tag{34}$$

in which

$$c_1 = \underline{p}_T^{\,T}\,\underline{p}_T \qquad c_2 = 2(\Delta\underline{a}_i + \underline{p}_i^*)^T\,\underline{p}_T \tag{35}$$

$$c_3 = (\Delta\underline{a}_i + \underline{p}_i^*)^T\,(\Delta\underline{a}_i + \underline{p}_i^*) - (\Delta L^o)^2$$

From the solution of (34) the value of $\delta\lambda_i$ is chosen which gives the angle between $\Delta\underline{a}_{i+1}$ and $\Delta\underline{a}_i$ to be less than $\frac{\pi}{2}$, (14).

4. Numerical Examples

The first example considered is that of an axisymmetric cylinder under applied internal pressure subject to plane strain conditions in the axial direction. The cylinder was modelled by four 8-node isoparametric elements, resulting in 46 degrees of freedom and a Von Mises yield surface was assumed for elasto-plastic analysis. In Tables I and II results are presented corresponding to different load incrementation and convergence tolerance limits. In each case the numerical results are in accordance with the theoretical values (15).

From Table I it is seen that for small increments of load all the algorithms employed give a comparable efficiency except for the SN method which is considerably superior. Table II shows that for loading applied in large steps the SN method is still the most efficient closely followed by the QN method. Both approaches are greatly superior to the standard Newton methods NR and IS.

The second problem considered is that of the elasto-plastic failure of a plane strain notched beam subjected to pure bending. Solution was obtained for three degrees of mesh refinement (118, 236 and 396 degrees of freedom) and one idealisation is shown in Fig. 2 together with the material properties assumed and geometric dimensions. Table III shows a comparison of the results obtained by various solution algorithms for the 118 d.o.f. idealisation. The results show, as expected, a slow rate of convergence of the IS method which becomes worse with increasing spread of plasticity. The number of iterations on the 12th increment of load is three times larger than for the SN method and is directly reflected in the computation time. Both the NR and MN techniques proved to be unstable giving divergent results on the 3rd and 4th load increments respectively.

Results are presented for the QN method for the case where line searches are adopted (Version I) and when the line search process is suppressed (Version II). The results show the use of line searches to be beneficial since they result in a reduction in the number of iterations which is reflected in a 30% saving in computation time. Once again the SN method proved to be the most efficient of the algorithms tested. These trends are confirmed by Table IV where results are presented for the 236 and 396 d.o.f. idealisations.

It is worth noting that, after numerical experiments, the most suitable trial values for the search parameter were found to be $|S_i| = |0,1.0,2.0,4.0,6.0|$. Of these, the solution procedure rarely used more than the first two and usually chose $S_i = 0.5$.

The next problem considered is illustrated in Fig. 3. The cylindrical shell roof was analysed elasto-plastically using 6 Semiloof thin shell elements (5) in one symmetric quadrant. Table V compares the performance of the standard IS method against that of the SN approach which was found to be the most economical algorithm in the previous problems. The SN method is seen to require far fewer iterations for convergence than the IS approach particularly as collapse conditions are approached.

The final example considered illustrates use of the load path constraining method described in Section 3. Large deformation numerical solutions are undertaken for a shallow arch which exhibits a "snap-through" behaviour. The circular arch of span 2540 mm and rise 76 mm is pinned at its ends and loaded centrally by a vertical point load. The arch is rectangular in section with depth 76.2 mm and breadth 25.4 mm. The elastic modulus is 7000 dN/mm^2 and Poisson's ratio is 0.3. The problem was first solved using the load path constraint method in conjunction with the MN algorithm described in Section 2.1. The loading was applied in specified increments and it is seen from Table VI that during iteration to convergence the load parameter λ^m is drastically adjusted and in fact becomes negative during the snap-through stage. In fact, when the determinant of the stiffness matrix changes sign the code automatically reverses the sense of the applied load increments. The problem was then reanalysed using the NR, IS and QN methods in turn but incrementing the prescribed central vertical displacement rather than by load application. Suitable displacement increments were chosen by reference to the previous constrained load path solution. The computation costs for all methods are comparable but the great advantage of the load path constraint approach is the ease with which a complex load/displacement path can be followed without the need for a careful selection of the applied load increment factors.

5. Discussion of Results and Conclusions

In this paper we have attempted to compare the performance of several algorithms which have recently been presented for the numerical solution of nonlinear problems. From the results discussed (and other experience to date) the SN method appears to be the most efficient followed by the SN method; both algorithms being considerably superior to the standard NR and MN methods with regard to both computational costs and stability. The relative efficiency is a function of the convergence limit specified and the SN and QN methods become even more attractive the smaller the tolerance value τ specified.

The performance of both the CG and CN methods are sensitive to the quality of line search. On the other hand the QN method is not so dependent on a line search and Table III shows that the method performs satisfactorily without a line search process (i.e. S_i = 1.0). However a line search accelerates convergence and it was found that the value of S_i usually lay in the range 0.0 to 1.0. An important computational feature of all the algorithms presented is that symmetry of the equations is retained throughout the solution process.

The load path constraint method summarised in Section 3 can be used in conjunction with other of the algorithms considered and has the practical advantage of being easily implemented into any existing code. It alleviates the need for an accurate estimate of load incrementation in problems with limit points or unstable branches, which is particularly important for pressure loaded structures where displacement controlled loading is impossible. Loading by incrementally prescribed displacements is also not possible in structures which exhibit a "snap-back" phenomenon as sometimes occurs in thin shells.

Further computational economies can be made for elasto-plastic analysis by combining these improved algorithms with a substructuring technique whereby the stiffness equations of the (assumed) elastic part of the structure are assembled and reduced once only during the computation process. Further results covering the application of these nonlinear solution algorithms will be available for presentation at the conference.

6. References

/1/ BERGAN, P. G., "Automated incremental-iterative solution schemes", Proc. Int. Conf. on Numerical Methods for Nonlinear Problems, Swansea, U.K., September (1980).

/2/ RIKS, E., "An incremental approach to the solution of snapping and buckling problems", Int. J. Solids and Structures, 15, 529-551 (1979).

/3/ IRONS, B. M., "A frontal solution program", Int. J. Num. Meth. in Engng., 2, 5-32 (1970).

/4/ OWEN, D. R. J., HINTON, E., "Finite Elements in Plasticity: Theory and Practice", Pineridge Press, Swansea, U.K. (1980).

/5/ IRONS, B. M., AHMAD, S., "Techniques of Finite Elements", Ellis Horwood, Chichester, U.K. (1980).

/6/ ARGYRIS, J. H., SCHARPF, D. W., "Methods of elasto-plastic analysis", ISD, ISSC Symp. on Finite Element Tech., Stuttgart (1969).

/7/ STIEFEL, E., HESTENES, H., "Methods of conjugate gradient for solving linear systems" J. Res. National Bureau of Standards, 409-436 (1952).

/8/ IRONS, B. M., ELSAWAF, A. F., "The conjugate Newton method for solving finite element equations", In: Formulation and Computational Algorithms in Finite Element Analysis, M.I.T. Press, 655-672 (1977).

/9/ GERADIN, M., HOGGE, M. A., "Quasi-Newton iteration in nonlinear structural dynamics", Fifth Int. Conf. on Structural Mechanics in Reactor Technology, Berlin, Germany, August (1979).

/10/ MATTHIES, H., STRANG, G., "The solution of nonlinear finite element equations, Int. J. Num. Meth. in Engng., 14, 1613-1626 (1979).

/11/ BATHE, K. J., CIMENTO, A. P., "Some practical procedures for the solution of nonlinear finite element equations", Comp. Meth. Applied Mechanics and Engng., 22, 59-85 (1980).

/12/ CRISFIELD, M. A., "Iterative solution procedures for linear and nonlinear structural analysis", TRRL Lab. Report 900, Transport and Road Research Laboratory, Crowthorne, Berkshire, U.K. (1979).

/13/ FLETCHER, R., "A new approach to variable metric algorithms", Computer Journal, 13, 317-322 (1970).

/14/ CRISFIELD, M. A., "Incremental/Iterative solution procedures for nonlinear structural analysis", Proc. Int. Conf. on Numerical Methods for Nonlinear Problems, Swansea, U.K., September (1980).

/15/ HODGE, P. G., WHITE, G. N., "A quantitative comparison of flow and deformation theories of plasticity", J. Appl. Mech., 17, 180-184 (1950).

TABLE I   Solution comparison for incrementally loaded thick cylinder.
Convergence tolerance factor τ = 0.001 (Execution times in ICL 1904S secs.)

Load factor	NR	IS	SN	CN		QN	
	NI	NI	NI	NI	NG	NI	NG
0.5	1	1	1	1	1	0	1
0.6	2	4	3	2	7	1	8
0.8	3	9	5	3	10	1	9
0.9	2	14	6	2	6	1	7
TIME	30	37	19	36		34	

Problem Details

Elastic modulus = 2.1×10^4 dN/mm^2

Poisson's ratio = 0.3

Uniaxial yield stress = 24.0 dN/mm^2

No strain hardening

Internal radius = 100 mm

External radius = 200 mm

Reference internal pressure = 20 dN/mm^2

NI = number of iterations per load increment

NG = number of residual evaluations per load increment

TABLE II   Solution comparison for a single load step application (Load factor = 0.9) for two different convergence tolerance factors, τ. (Execution times in ICL 1904S secs)

τ	NR	IS	SN	CN		QN	
	NI	NI	NI	NI	NG	NI	NG
0.001	29	29	10	8	35	6	18
TIME	99	30	13	35		19	
0.0001	40*	55	13	17	67	8	22
TIME	135	54	15	63		22	

*Solution not converged

TABLE III   Solution comparison for the notched beam problem of Fig. 2 using 118 d.o.f. mesh and convergence tolerance factor τ = 0.001. Reference applied load = 2.39 KN (Execution times in CDC 7600 secs.)

Load factor	w_A (mm)	IS	QN(I)		QN(II)		SN	
		NI	NI(NG)	Time	NI	Time	NI	Time
1.0	0.0164	1	- (1)	0.308	1	0.308	1	0.344
2.0	0.0329	1	- (1)	0.602	1	0.602	1	0.538
3.0	0.0498	19	3(11)	1.566	4	1.525	7	1.530
3.25	0.0541	14	1(4)	2.124	2	2.181	4	2.066
3.50	0.0584	13	1(4)	2.683	1	2.710	4	2.601
3.75	0.0627	13	1(4)	3.242	1	3.239	4	3.135
4.00	0.0670	13	1(4)	3.801	1	3.768	4	3.670
4.20	0.0706	14	1(4)	4.361	2	4.424	5	4.319
4.40	0.0747	17	3(12)	5.344	7	5.770	5	4.967
4.60	0.0815	24	5(13)	6.574	6	6.972	8	5.960
4.80	0.0887	25	4(11)	7.637	12	9.109	8	6.957
5.00	0.0958	27	3(9)	8.536	11	11.973	9	8.072
Exec. time		23.418	8.536		11.973		8.072	

TABLE IV Solution comparison for the notched beam problem of Fig. 2 for two different mesh refinements and τ = 0.0001. Reference applied load = 2.39 KN (Execution times in CDC 7600 secs.)

Load factor	236 d.o.f.					396 d.o.f.					
	IS	QN(I)		SN		IS		QN(I)		SN	
	NI	NI(NG)	Time	NI	Time	NI	Time	NI(NG)	Time	NI	Time
1.0	1	- (1)	0.747	- -	0.743	1	1.377	- (1)	1.372	1	1.367
2.0	27	4(13)	3.235	12	2.997	27	14.155	4(13)	5.776	8	4.796
3.0	43	7(18)	6.725	10	6.217	38	32.112	6(16)	11.314	12	9.853
3.25	34	7(16)	10.077	12	8.953	33	47.741	8(18)	17.880	12	14.906
3.50	43	6(16)	13.203	12	12.172	23*	64.562	9(20)	25.074	13	20.366
3.75	43	9(20)	17.296	12	15.392			8(19)	31.749	9	24.619
4.00	43	9(20)	21.392	9	17.896			9(21)	39.061	10	29.277
4.20	14*	8(19)	25.171	12	21.121		*Time limit exceeded	10(23)	46.999	12	34.747
4.40		11(24)	30.005	14	24.825			9(21)	54.308	14	41.026
4.60		10(23)	34.522	19	29.496			10(23)	62.267	14	47.310
4.80		11(25)	39.416	12	32.495			14(31)	72.877	16	54.422
5.00		11(25)	44.308	15	36.460			13(29)	82.809	17	61.940
5.20		12(27)	49.578	16	40.662			14(31)	93.425	16	69.057
5.40		12(27)	54.868	16	44.873			5(11)*	104.0	17	76.598
Time	64	54.868		44.873		64.562		104.0		76.598	

TABLE V Solution comparison for the cylindrical shell roof problem of Fig. 3 for convergence tolerance factor, τ = 0.01. Reference applied pressure = 4 KN/m^2. Vertical central displacement, w. (Execution times in CDC 7600 secs.)

Load factor λ	SN			IS		
	w (mm)	NI	Inc. Time	w (mm)	NI	Inc. Time
0.4	39.34	1	1.065	39.34	1	1.029
0.5	49.18	1	0.257	49.18	1	0.220
0.525	51.64	1	0.257	51.64	1	0.218
0.550	54.10	1	0.255	54.10	1	0.217
0.575	56.56	1	0.257	56.56	1	0.219
0.600	60.15	6	0.887	59.18	6	1.137
0.620	63.70	4	0.890	63.39	8	1.511
0.640	69.89	6	1.312	69.50	16	2.985
0.660	79.82	8	1.742	79.29	25	4.687
TIME	6.922			12.223		

TABLE VI Solution comparison for snap-through behaviour of a shallow arch
(Execution times in CDC 7600 secs.)

λ^m	$\Delta\lambda_o^m$	$\Delta\lambda_c^m$	Load path constraint method with MN iteration		NR	IS	QN(I)	
			w (mm)	Time (NI)	Time (NI)	Time (NI)	Time	NI(NG)
0.274	0.3	0.274	7.01	0.170(1)	0.204(1)	0.168(1)	0.157	- (2)
0.457	0.2	0.183	12.80	0.328(1)	0.398(1)	0.330(2)	0.302	- (2)
0.636	0.2	0.179	19.72	0.485(1)	0.592(1)	0.491(2)	0.445	- (2)
0.807	0.2	0.171	28.64	0.643(1)	0.787(1)	0.697(3)	0.590	- (2)
0.890	0.1	0.083	35.55	0.800(1)	0.981(1)	0.948(4)	0.733	- (2)
0.966	0.1	0.076	45.70	0.958(1)	1.174(1)	1.334(7)	0.878	- (2)
0.876	0.1	- 0.090	74.22	1.247(4)	1.368(1)	*	1.125	2(8)
0.761	0.1	- 0.115	86.97	1.458(2)	1.562(1)		1.268	- (2)
0.674	0.1	- 0.087	97.42	1.615(1)	1.756(1)		1.411	- (2)
0.589	0.1	- 0.085	109.97	1.773(1)	1.951(1)	No convergence	1.610	1(4)
0.601	0.1	0.012	133.19	2.282(9)	2.146(1)		1.875	2(7)
0.761	0.1	0.160	142.46	2.441(1)	2.341(1)		2.018	- (2)
0.878	0.1	0.117	147.70	2.597(1)	2.535(1)		2.160	- (2)
0.988	0.1	0.110	151.33	2.755(1)	2.729(1)		2.303	- (2)
1.087	0.1	0.099	154.41	2.867(1)	2.840(1)		2.446	- (2)

Reference load value = 30,000 KN, τ = 0.01, w = vertical central deflection

λ^m Load factor after convergence on increment m

$\Delta\lambda_o^m$ Applied load factor for increment m

$\Delta\lambda_c^m$ Incremental load factor after convergence for increment m

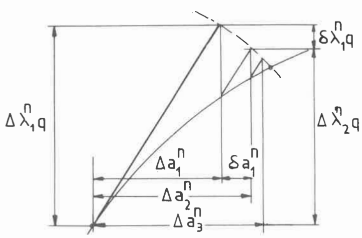

Fig. 1 One dimensional geometrical representation of a constant spherical arc load incrementation scheme

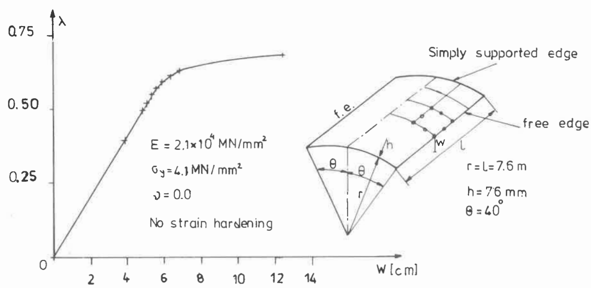

Fig. 3 Load/displacement characteristics of a cylindrical shell roof assuming elasto-plastic behaviour

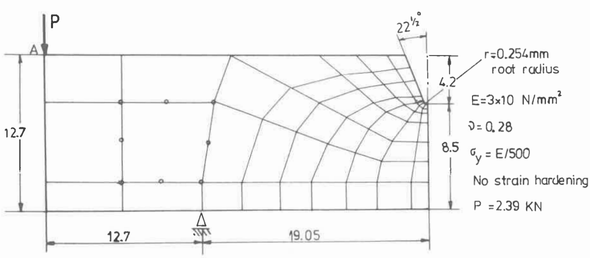

Fig. 2 Geometry and material properties for the elasto-plastic analysis of a notched beam in pure bending (396 d.o.f. mesh)