

ON DISTRIBUTION FUNCTION - MOMENT RELATIONSHIPS  
IN A STATIONARY POINT PROCESS.

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SUMMARY.

Let  $G_n(t)$  denote the distribution function for the time to the  $n$ -th event in a stationary regular point process, and  $F_n(t)$  the corresponding distribution function, conditional on an event having occurred "at"  $t = 0$ . (I.e.,  $F_n(t)$  represents the distribution of the time from an "arbitrary event" to the  $n$ -th subsequent event). Let  $N(o, t)$  denote the number of events in  $(o, t)$ . Relationships between the distribution functions  $G_n$ ,  $F_n$ , and the moments (unconditional and conditional) of  $N(o, t)$  are obtained. Specifically, series are given for  $G_n(t)$  in terms of factorial moments of  $N(o, t)$ , and for  $F_n(t)$  in terms of such moments, conditional on the occurrence of an event "at"  $t = 0$ . In particular, a series given in [2] for  $F_n(t)$  is obtained as a corollary.

1. INTRODUCTION.

We shall, throughout, consider a stationary point process, and denote by  $N(s, t)$  the number of events in the (semiclosed) interval  $(s, t]$ . It will further be assumed that the mean number of events per unit time  $EN(o, 1) = \lambda$  is finite, and that there is zero probability of the occurrence of multiple events (hence the process is regular (orderly) in the sense that  $\Pr\{N(o, t) > 1\} = o(t)$  as  $t \downarrow 0$ ). Write

$$(1) \quad G_n(t) = \Pr\{N(o, t) \geq n\}, \quad n = 0, 1, 2, \dots$$

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This is the probability that the n-th event after time zero occurs no later than time t, i.e.,  $G_n(t)$  is the distribution function for the time to the n-th event after time zero (which is a well defined random variable).

We now modify  $G_n(t)$  to define a corresponding probability, but now conditioned by the occurrence of an event "at" time zero in the following precise sense:

$$(2) \quad F_n(t) = \lim_{\delta \downarrow 0} \Pr\{N(o, t) \geq n | N(-\delta, 0) \geq 1\}$$

and we shall write

$$(3) \quad F_n(t) = \Pr\{N(o, t) \geq n | \text{Event at } 0\}$$

it being understood that the right hand side of (3) is defined as the limit occurring in Equation 2; i.e., the zero probability condition that an event occurs precisely at time zero, is replaced by the positive probability condition  $N(-\delta, 0) \geq 1$ , and the limit is taken as  $\delta \downarrow 0$ .

The limit in (2) does in fact exist and moreover  $F_n(t)$  is a distribution function for each n. Further, we may interpret  $F_n(t)$  as the distribution function for the interval from time zero to the n-th event after time zero, given an event occurred "at" time zero. Alternatively  $F_n(t)$  may be regarded as the distribution function for the time between an "arbitrary event" to the n-th subsequent event. For a discussion of these points, we refer to [1].

Corresponding to (1) and (2), we define

$$(4) \quad v_n(t) = G_n(t) - G_{n+1}(t) = \Pr\{N(o, t) = n\}$$

$$(5) \quad u_n(t) = F_n(t) - F_{n+1}(t) = \Pr\{N(o, t) = n | \text{Event at } 0\}.$$

Then for fixed  $t$ ,  $v_n(t)$  is the probability distribution of the non negative integer valued random variable  $N(o, t)$ , whereas  $u_n(t)$  is the distribution of  $N(o, t)$  conditional on the occurrence of an event "at" time zero.

Correspondingly, we may define moments of  $N(o, t)$  (where they exist) and conditional moments given an event "at" time zero. It is the purpose of this paper to explore relationships between the distribution functions  $G_n(t)$ ,  $F_n(t)$  and these moments. Specifically, we will use factorial moments  $\alpha_k(t)$ ,  $\beta_k(t)$  defined as follows (writing  $N$  for  $N(o, t)$ )

$$(6) \quad \beta_k(t) = E\{N(N-1) \dots (N-k+1)\}$$

$$(7) \quad \alpha_k(t) = E\{N(N-1) \dots (N-k+1) | \text{Event at } 0\}$$

(i.e., strictly  $\alpha_k(t) = \lim_{\delta \downarrow 0} E\{N(N-1) \dots (N-k+1) | N(-\delta, 0) \geq 1\}$ , in accord-

ance with the remarks after Equation (3)). Clearly we have

$$(8) \quad \beta_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k} v_k(t) .$$

In Section 2, we shall prove some results which will include the existence of the limit in (7), and a corresponding formula for  $\alpha_k(t)$ , viz.,

$$(9) \quad \alpha_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k} u_n(t),$$

as one would intuitively expect.

In Section 3, we relate

$$(a) \quad \alpha_k(t) \text{ with } \beta_{k+1}(t)$$

$$(b) \quad G_n(t) \text{ with } \beta_1(t), \beta_2(t) \dots$$

$$(c) \quad F_n(t) \text{ with } \alpha_1(t), \alpha_2(t) \dots$$

and hence from (a) and (c),

$$(d) \quad F_n(t) \text{ with } \beta_1(t), \beta_2(t) \dots$$

The relationship (Eqn (21)) given under (d) is that given previously in [2], generalizing results of Longuet-Higgins [3] concerning zero crossing problems. However, the point of view in this present paper is that it is most natural to relate the unconditional distribution functions  $G_n(t)$  to the unconditional moments  $\beta_k(t)$ , and the conditional distribution functions  $F_n(t)$  to the conditional moments  $\alpha_k(t)$ . Then the previously known results of [2] follow as a corollary by use of the relationship given under (a).

In essence, the results of Section 3 depend on Section 2 only through the derivation of Equation (9) (Theorem 3). Thus if (9) is taken for the definition of  $\alpha_k(t)$  (rather than (7)), the Section 3 results may be obtained independently of Section 2. However, it does appear to us to be both natural and important to make the (Section 2) identification between the expressions (7) and (9).

Finally, we note that in [2], assumptions concerning the radius of convergence of the generating function  $\sum v_k(t) z^k$  were made in order to justify the calculations involved. In the present work, we use somewhat different (though closely related) assumptions. A brief comparison of these approaches is made in Section 4, where further results are also indicated.

## 2. THE CONDITIONAL FACTORIAL MOMENTS $\alpha_k(t)$ .

In this section, we prove two theorems which have independent interest and from which Equation (9) follows. A special case concerns the "renewal function"  $H(x) = \sum_1^{\infty} F_n(x)$ .

THEOREM 1. Let  $t_i$  denote the position of the  $i$ -th event after  $t = 0$  ( $i = 1, 2 \dots$ ). Then with the above notation, for  $x > 0$ ,  $h > 0$ ,

$$(10) (\lambda h)^{-1} E \left\{ \sum_{t_i \in (0, h)} \binom{N(t_i, t_i + x)}{k} \right\} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} F_n(x) \leq \infty.$$

Proof: By additivity of the mean and stationarity, there is no loss in generality in taking  $h = 1$ , and we do so. Write

$$\lambda_{i,n} = 1 \text{ if } N(t_i, t_i + x) \geq n, \lambda_{i,n} = 0 \text{ otherwise, and } N_{n,x} = \sum_{t_i \in (0,1)} \lambda_{i,n}.$$

That is  $N_{n,x}$  is the number of  $t_i \in (0, 1)$  such that  $N(t_i, t_i + x) \geq n$ . It follows from [1, Section 5, Lemma] that  $F_n(x) = \lambda^{-1} E N_{n,x}$ , and hence, since  $N_{n,x} \geq 0$ ,

$$\begin{aligned} \lambda \sum_{n=k}^{\infty} \binom{n-1}{k-1} F_n(x) &= E \left\{ \sum_{n=k}^{\infty} \binom{n-1}{k-1} N_{n,x} \right\} \\ &= E \left\{ \sum_{n=k}^{\infty} \binom{n-1}{k-1} \sum_{t_i \in (0,1)} \lambda_{i,n} \right\} \\ &= E \left\{ \sum_{t_i \in (0,1)} \sum_{n=k}^{\infty} \binom{n-1}{k-1} \lambda_{i,n} \right\} \\ &= E \left\{ \sum_{t_i \in (0,1)} \sum_{n=k}^{N(t_i, t_i + x)} \binom{n-1}{k-1} \right\} \\ &= E \left\{ \sum_{t_i \in (0,1)} \binom{N(t_i, t_i + x)}{k} \right\} \end{aligned}$$

since  $\sum_{n=k}^m \binom{n-1}{k-1} = \binom{m}{k}$ . Thus the required result follows.

COROLLARY. The "renewal function"  $H(x) = \sum_1^{\infty} F_n(x)$  satisfies

$$H(x) = (\lambda h)^{-1} E \left\{ \sum_{t_i \in (0, h)} N(t_i, t_i + x) \right\}.$$

The next result relates the  $k$ -th conditional factorial moment of  $N(0, x)$  to the left hand side of (10), under the condition that the  $(k+1)$ -st moment of  $N(0, t)$  is finite.

THEOREM 2. If  $EN^{k+1}(0, t) < \infty$  (for all  $t$ ) then for  $h > 0, x > 0$

$$(11) \quad \lim_{\delta \downarrow 0} E \left\{ \binom{N(0, x)}{k} \middle| N(-\delta, 0) \geq 1 \right\} = (\lambda h)^{-1} E \left\{ \sum_{t_i \in (0, h)} \binom{N(t_i, t_i + x)}{k} \right\}.$$

Proof: Again, we may take  $h = 1$ . Let  $\chi_{j, m} = 1$  if  $N(\frac{j-1}{m}, \frac{j}{m}) \geq 1, j = 0,$

$1 \dots m, \chi_{j, m} = 0$  otherwise. Now (with probability one) for fixed  $i$  and for  $m$  sufficiently large (depending on the particular "sample point  $\omega$ "), we have  $N(t_i, t_i + x) = N(\frac{j}{m}, \frac{j}{m} + x)$  if  $j = j(i, m)$  is that integer such that  $t_i \in (\frac{j-1}{m}, \frac{j}{m}]$ . For if  $m$  is sufficiently large, we have

$N(t_i, \frac{j}{m}) = N(t_i + x, \frac{j}{m} + x) = 0$ , remembering that  $N(s, t)$  refers to the number of events in the semiclosed interval  $(s, t]$ . Considering all of the (finite number of)  $t_i \in (0, 1)$  in this way, we see that

$$(12) \quad \sum_{j=1}^m \chi_{j, m} \binom{N(j/m, j/m + x)}{k} \rightarrow \sum_{t_i \in (0, 1)} \binom{N(t_i, t_i + x)}{k}$$

with probability one as  $m \rightarrow \infty$ . (The two sides are in fact equal for  $m \geq$  some  $M(\omega)$ ).

Now

$$\begin{aligned}
 \sum_{j=1}^m \chi_{j,m} \binom{N(j/m, j/m + x)}{k} &\leq \sum_{j=1}^m \chi_{j,m} \binom{N(o, 1 + x)}{k} \\
 &\leq N(o, 1) \binom{N(o, 1 + x)}{k} \\
 &\leq N^{k+1}(1, 1 + x)
 \end{aligned}$$

which has, by assumption, a finite mean, and thus from (12), by dominated convergence,

$$\begin{aligned}
 E \left\{ \sum_{t_i \in (o,1)} \binom{N(t_i, t_i + x)}{k} \right\} &= \lim_{m \rightarrow \infty} \sum_{j=1}^m E \left\{ \chi_{j,m} \binom{N(j/m, j/m + x)}{k} \right\} \\
 &= \lim_{m \rightarrow \infty} m E \{ \chi_{o,m} \binom{N(o, x)}{k} \} \\
 &= \lim_{m \rightarrow \infty} m E \{ \binom{N(o, x)}{k} | \chi_{o,m} = 1 \} \Pr(\chi_{o,m} = 1) \\
 &= \lambda \lim_{m \rightarrow \infty} E \{ \binom{N(o, x)}{k} | N(-\frac{1}{m}, o) \geq 1 \}
 \end{aligned}$$

since  $\Pr(\chi_{o,m} = 1) = \Pr\{N(-\frac{1}{m}, o) \geq 1\} \sim \frac{\lambda}{m}$  as  $m \rightarrow \infty$ .

Thus the conclusion of the theorem follows with  $\delta \rightarrow 0$  as  $1/m$ . For  $\delta \rightarrow 0$  in an arbitrary manner, the result follows by approximating  $\delta^{-1}$  by its integer part  $[\delta^{-1}]$ . Specifically if  $\chi_\delta = 1$  when  $N(-\delta, o) \geq 1$  and  $\chi_\delta = 0$  otherwise,

$$\lambda E \{ \binom{N(o, x)}{k} | N(-\delta, o) \geq 1 \} = \lambda E \{ \binom{N(o, x)}{k} \chi_\delta \} / \Pr(\chi_\delta = 1)$$



$$(13) \quad \sim E\left\{\binom{N(o,x)}{k} X_\delta\right\}/\delta \quad \text{as } \delta \rightarrow 0.$$

Now  $([\delta^{-1}] + 1)^{-1} \leq \delta \leq [\delta^{-1}]^{-1}$  and hence, for  $m = [\delta^{-1}]$ ,

$$X_{1/(m+1)} \leq X_\delta \leq X_{1/m}.$$

Multiplying by  $\binom{N(o,x)}{k}/\delta$  and taking expectations, we see that

$$(14) \quad [\delta(m+1)]^{-1} E\left\{\binom{N(o,x)}{k} X_{1/(m+1)}\right\}/(m+1)^{-1} \\ \leq E\left\{\binom{N(o,x)}{k} X_\delta\right\}/\delta \leq (\delta m)^{-1} E\left\{\binom{N(o,x)}{k} X_{1/m}\right\}/m^{-1}.$$

Using the conclusion of the theorem for  $\delta$  of the form  $1/m$  and (13), together with the fact that  $\delta m \rightarrow 1$ , it follows that the outer members of (14) each converge to

$$E\left\{\sum_{t_i \in (o,1)} \binom{N(t_i, t_i + x)}{k}\right\},$$

yielding the conclusion of the theorem.

We note that the assumption of the existence of the  $(k+1)$ -st moment made for all  $t$  in the theorem, need only be made for some  $t > 0$ , since it is well known that this implies existence for all  $t$ . (This is an easy consequence of Minkowski's inequality and stationarity).

Using the two results above, we now give precise conditions for the validity of (9).

**THEOREM 3.** Suppose that for some fixed  $k$ ,  $EN^{k+1}(o, t) < \infty$  for some  $t > 0$  (and hence for all  $t$ ). Then Equation (9) holds for the  $k$ -th conditional factorial moment  $\alpha_k(t)$  of  $N(o, t)$  ( $\alpha_k(t)$  being defined by (7)).

That is

$$\alpha_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k} u_k(t)$$

where

$$\begin{aligned} u_k(t) &= F_k(t) - F_{k+1}(t) \\ &= \Pr\{N(o, t) = k \mid \text{Event at } 0\} \end{aligned}$$

in the usual limiting sense.

Proof: By (7) and Theorems 1 and 2, we have

$$\begin{aligned} \alpha_k(t)/k! &= \lim_{\delta \downarrow 0} E\left\{\binom{N(o,t)}{k} \mid N(-\delta, o) \geq 1\right\} \\ (15) \quad &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} F_n(t) \\ &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} \sum_{m=n}^{\infty} u_m(t) \\ &= \sum_{m=k}^{\infty} u_m(t) \sum_{n=k}^m \binom{n-1}{k-1} \\ &= \sum_{m=k}^{\infty} \binom{m}{k} u_m(t) \end{aligned}$$

as required.

As a corollary following from Equation (15) (with  $k = 1$ ), we have for the "renewal function"  $H(x)$ ,

$$(16) \quad H(x) = \sum_1^{\infty} F_n(x) = \lim_{\delta \downarrow 0} E\{N(o, x) \mid N(-\delta, o) \geq 1\}$$

provided  $E N^2(o, t) < \infty$  for some  $t > 0$ .

### 3. SERIES EXPRESSIONS FOR $F_n(t)$ AND $G_n(t)$ .

In this section, we first relate the conditional and unconditional factorial moments of  $N(o, t)$  and then give series expressions for  $F_n(t)$ ,  $G_n(t)$  in terms of these moments. We shall use Equation (9) (shown in Theorem 3) to relate  $\alpha_k(t)$  with the  $u_n$ 's (and hence with the  $F_n$ 's). As noted previously, the results of this section may be made independent of Section 2 if one defines  $\alpha_k(t)$  by means of (9) instead of (7).

First we state three simple lemmas (the first of which has in fact already been proved in the course of Theorem 3). In the first two lemmas, we let  $\{w_k: k = 0, 1, 2 \dots\}$  be a probability distribution,

$$H_k = \sum_{j=k}^{\infty} w_j \quad \text{and} \quad \gamma_k = k! \sum_{n=k}^{\infty} \binom{n}{k} w_n,$$

the corresponding  $k$ -th factorial moment.

Lemma 1.  $\gamma_k/k! = \sum_{n=k}^{\infty} \binom{n-1}{k-1} H_n \quad (\leq \infty).$

For the right hand side may be written as

$$\begin{aligned} \sum_{n=k}^{\infty} \binom{n-1}{k-1} \sum_{m=n}^{\infty} w_m &= \sum_{m=k}^{\infty} w_m \sum_{n=k}^{\infty} \binom{n-1}{k-1} && \text{(positive terms)} \\ &= \sum_{m=k}^{\infty} \binom{m}{k} w_m \end{aligned}$$

as required.

Lemma 2.  $H_n = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \gamma_k/k!$

if this series converges absolutely.

Proof: By Lemma 1, the right hand side may be written as

$$\begin{aligned} & \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \sum_{j=k}^{\infty} \binom{j-1}{k-1} H_j \\ (16) \quad & = \sum_{j=n}^{\infty} H_j \sum_{k=n}^j (-)^{k-n} \binom{k-1}{n-1} \binom{j-1}{k-1}, \end{aligned}$$

the change in summation order being justified since the replacement of each term of the double series by its modulus yields

$$\sum_{k=n}^{\infty} \binom{k-1}{n-1} \gamma_k / k! < \infty$$

by assumption. But (10) may be written as

$$\begin{aligned} & \sum_{j=n}^{\infty} \binom{j-1}{n-1} H_j \sum_{k=n}^j (-)^{k-n} \binom{j-n}{k-n} \\ & = \sum_{j=n}^{\infty} \binom{j-1}{n-1} H_j \sum_{k=0}^{j-n} (-)^k \binom{j-n}{k} \\ & = H_n \end{aligned}$$

since

$$\sum_{k=0}^{j-n} (-)^k \binom{j-n}{k}$$

is one or zero according as  $j = n$  or  $j > n$ .

Lemma 3. For all  $t$

$$u_n(t) = F_n(t) - F_{n+1}(t) = \lambda^{-1} D^+ G_{n+1}(t), \quad n = 1, 2, \dots$$

in which  $D^+$  denotes the right hand derivative. Hence

$$G_n(t) = \lambda \int_0^t u_{n-1}(s) ds.$$

This is shown by a simple transformation of the "Palm Formulae" and restates the results of [1, Section 3.1].

We now relate the conditional and unconditional factorial moments of  $N(o, t)$ .

THEOREM 4.

$$(17) \quad \beta_{k+1}(t)/(k+1)! = \lambda \int_0^t \frac{\alpha_k(u)}{k!} du \leq \infty \quad k = 1, 2, \dots$$

Hence, if for some  $k$   $\bar{E}N^{k+1}(o, t) < \infty$  for some  $t > 0$  (and hence for all  $t > 0$ ), then  $\beta_{k+1}(t)$  is absolutely continuous with density  $\lambda(k+1)\alpha_k(t)$ .

Further, under this finiteness restriction,  $\beta_{k+1}(t)$  has, for all  $t \geq 0$ , the right hand derivative

$$(18) \quad D^+ \beta_{k+1}(t) = \lambda(k+1) \alpha_k(t).$$

Proof: By Lemmas 1 and 3

$$\begin{aligned} \beta_{k+1}(t)/(k+1)! &= \sum_{n=k+1}^{\infty} \binom{n-1}{k} G_n(t) \\ &= \lambda \sum_{n=k+1}^{\infty} \binom{n-1}{k} \int_0^t u_{n-1}(s) ds \\ &= \lambda \int_0^t \left\{ \sum_{n=k}^{\infty} \binom{n}{k} u_n(s) \right\} ds \end{aligned}$$

by virtue of the positivity of the functions involved. Thus the first statement of the theorem follows from Theorem 3, (Equation 9). (18) follows from (17) by differentiating the integral on the right, using the easily proved right-continuity of  $\alpha_k(t)$  (e.g., from (9)).

The main results of this section are given in the following theorem, and provide series expressions for  $F_n(t)$  and  $G_n(t)$  in terms of the  $\alpha_k(t)$ ,  $\beta_k(t)$ . Specifically the theorem gives conditions for the validity of the following equations (where  $n = 1, 2, 3 \dots$ ).

$$(19) \quad F_n(t) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \alpha_k(t)/k!$$

$$(20) \quad G_n(t) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-1}{n-1} \beta_k(t)/k! .$$

Equation (19) relates the conditional distribution function  $F_n(t)$  to the conditional factorial moments  $\alpha_k(t)$ , whereas (20) correspondingly relates (the unconditioned)  $G_n$  to the (unconditioned)  $\beta_k$ .

THEOREM 5. Whenever either of the series in (19) or (20) converges absolutely the corresponding equality holds.

Further, if the series in (19) converges absolutely so does that in (20) (and hence both equalities hold). Finally, if the series in (20) converges absolutely for  $0 \leq t < s$  then (19) also converges and hence both equalities hold for  $0 \leq t < s$ .

Proof: The first statement follows at once from Lemma 2 (using Equation (9) - cf. Theorem 3).

To prove the remaining statements, we first assume that the series in (19) converges absolutely. We prove that this is also true for (20). In fact, by

Theorem 4, the positivity of the functions involved, and monotonicity of  $\alpha_k(t)$

$$\begin{aligned} \sum_{k=n}^{\infty} \binom{k-1}{n-1} \beta_k(t)/k! &= \lambda \int_0^t \sum_{k=n}^{\infty} \binom{k-1}{n-1} \frac{\alpha_{k-1}(u)}{(k-1)!} du \\ &\leq \lambda t \sum_{k=n-1}^{\infty} \binom{k}{n-1} \frac{\alpha_k(t)}{k!} \\ &< \infty \end{aligned}$$

since  $\binom{k}{n-1} \sim \binom{k-1}{n-1}$  as  $k \rightarrow \infty$  and

$$\sum_{k=n}^{\infty} \binom{k-1}{n-1} \frac{\alpha_k(t)}{k!}$$

is assumed to converge. Thus the desired conclusion follows.

Finally, we assume the series in (20) converges absolutely for  $0 \leq t < s$  and show that this also holds for (19). For, again by Theorem 4

$$\int_0^t \sum_{k=n}^{\infty} \binom{k-1}{n-1} \frac{\alpha_{k-1}(u)}{(k-1)!} du < \infty \quad \text{for any } 0 \leq t < s.$$

The integrand is finite almost everywhere and hence everywhere in the interval  $(0, s)$  by monotonicity of  $\alpha_k(u)$ , from which we have when  $0 \leq t < s$

$$\sum_{k=n-1}^{\infty} \binom{k}{n-1} \frac{\alpha_k(u)}{k!} < \infty \quad \text{for any } 0 \leq u < t.$$

Using the fact that  $\binom{k}{n-1} \sim \binom{k-1}{n-1}$  as  $k \rightarrow \infty$  it follows that (19) converges absolutely, as required, when  $0 \leq t < s$ .

It is clear that we can use Theorem 4 again to express  $F_n(t)$  in terms of the  $\beta_k$  and  $G_n(t)$  in terms of the  $\alpha_k$ . The expression of  $F_n(t)$  in terms of the  $\beta_k$  is of particular interest for application to certain zero crossing problems. This expression (which was obtained by somewhat different methods in [2]) is

$$(21) \quad F_n(t) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k-2}{n-1} \frac{\beta_k'(t)}{k!},$$

in which  $\beta_k'(t)$  is written for the right hand derivative  $D^+ \beta_k(t)$ . Equation (21) is valid whenever (19) holds, and thus in particular whenever the series in (21) converges absolutely. Again the series (21) converges absolutely for  $t < s$  and equality holds whenever the series in (20) converges absolutely for  $t < s$ .

#### 4. COMPARISON WITH PREVIOUS RESULTS.

In [2], use was made of the probability generating function  $\sum v_k z^k$  of  $N(o, t)$  to obtain conditions for the validity of (21). Specifically, let  $\rho_t$  be the radius of convergence of this generating function. Then it was shown in [2] that if  $\rho_t > 2$ , the series (21) converges absolutely, and hence also so do (19) and (20).

Conversely the absolute convergence of (19), (20) or (21) (for some  $n$ ) implies  $\rho_t \geq 2$ , and in fact is a slightly stronger assumption than this, but apparently does not necessarily imply  $\rho_t > 2$ . However, if (20) is absolutely convergent at  $t = s$  and if  $\rho_s = 2$ , it follows from Theorem 5 that (19) and (21) are valid for  $t < s$ , but may conceivably not hold at  $t = s$ .

If  $\rho_t < 2$  then, from the above remarks, neither (19) nor (21) can converge absolutely. However, if  $1 < \rho_t \leq 2$  other expressions can be found for



e.g.  $F_n(t)$  in terms of the  $\alpha_k(t)$ . An investigation (involving also P. Imrey) has shown that if as long as  $\rho_t > 1$ , then, for example,

$$(22) \quad F_n(t) = \lambda^{-1} \sum_{k=n+1}^{\infty} \frac{1}{(1+q)^{k+1}} \sum_{s=n+1}^k (-)^{s-k-1} \binom{k}{s} \binom{s-2}{n-1} q^{k-s} \mu'_s(t)/s!$$

for any  $q$  chosen such that  $q > (\rho_t - 1)^{-2} - 1$ . The proof of (22) may be accomplished by means of Euler's "q-transform". It is planned to develop this and other expressions in a later paper.

## REFERENCES

- [1] Leadbetter, M.R., "On streams of events and mixtures of streams",  
J. Roy. Stat. Soc. (B) 28, (1966) p. 218-227.
- [2] Leadbetter, M.R., "On the distribution of the times between events in  
a stationary stream of events", Institute of Statistics Mimeo  
Series No. 524, University of North Carolina Statistics Department.  
Also to appear in J. Roy. Stat. Soc. (b), (1969).
- [3] Longuet Higgins, M.S., "The distribution of intervals between zeros  
of a stationary random function", Phil. Trans. Roy. Soc. 254  
(1962) p. 557-599.

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13. ABSTRACT Let $G_n(t)$ denote the distribution function for the time to the $n$ -th event in a stationary regular point process, and $F_n(t)$ the corresponding distribution function, conditional on an event having occurred "at" $t = 0$ . (I.e., $F_n(t)$ represents the distribution of the time from an "arbitrary event" to the $n$ -th subsequent event). Let $N(o, t)$ denote the number of events in $(o, t)$ . Relationships between the distribution functions $G_n$ , $F_n$ , and the moments (unconditional and conditional) of $N(o, t)$ are obtained. Specifically, series are given for $G_n(t)$ in terms of factorial moments of $N(o, t)$ , and for $F_n(t)$ in terms of such moments, conditional on the occurrence of an event "at" $t = 0$ . In particular, a series given in [2] for $F_n(t)$ is obtained as a corollary.			