ON THE J-INTEGRAL FOR NONHOMOGENEOUS CRACKED COMPOSITES

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SUMMARY

The well-known path independent J-integral of fracture mechanics has played an important role in the analysis of cracked structures. Consider a homogeneous elastic cracked body in a state of plane stress, the simple relationship between the J-integral and the mode I stress intensity factor K_I is established as

$$J = K_I^2/E$$

where E is Young's modulus. Thus, once the J-integral is determined, the mode I stress intensity factor K_I can be easily evaluated. It is to be emphasized that the J-integral is path independent only when the material under consideration is homogeneous, at least in the direction of the sharp crack.

It is the main objective of the present work to develop a rigorous, efficient finite element procedure using the hybrid-displacement model for obtaining the J values of linear elastic, nonhomogeneous composites and the stress intensity factors inferred from it. The concept of the path independent J-integral for homogeneous body thereby need be extended to nonhomogeneous cracked structures.

The present hybrid-displacement finite element model is based on a modified variational principle of potential energy with relaxed continuity requirements for displacements at the interelement boundary. The proper dominar singular behavior for strains and stresses is embedded in each of circular-sector shaped "embedded-singularity" elements near the crack-tip. Continuity of displacements and tractions between these near-tip elements with "singular" field assumptions and the neighboring elements with "regular" field assumptions is enforced through an independently assumed interelement boundary displacement field and a Lagrange multiplier technique.

For simplicity, a center-cracked specimen with inclusions of different material properties is analyzed. It is found that the path-independent nature for the J-value of the present nonhomogeneous cracked composites still remains valid provided the J-integral paths never go through the inclusions. Consequently, a convenient, practical procedure for the calculation of stress intensity factors of nonhomogeneous cracked structures from the computed J values is studied. Excellent correlations between the computed results and available reference solutions can be noted.

1. Introduction

One of the most important problems in elastic fracture mechanics is to determine the stress intensity factors which represent the severity of the given cracked geometry under a prescribed loading condition. Basically, two different approaches can be followed in employing finite element procedures to arrive at the required stress intensity factors. One is the direct method in which the stress intensity factors follow from the stress field or from the displacement field. The direct method requires very small elements in the crack-tip region and a large storage capacity of the computer. The second is an indirect method in which the stress intensity factors are determined via their relation with other quantities such as the energy-release rate or the J-integral [1,2]. The method used here is based directly on computing the J-integral of the cracked body and the stress intensity factor inferred from it. The advantages of using the J-integral are as follows: (1) No extrapolations are required. (2) There is no need for a particularly fine mesh near the crack-tip. (3) The J-integral is path-independent and can be calculated on a contour well away from the crack-tip where the stresses and strains can be more accurately determined.

Eventhough several finite element approaches have been devoted to the calculation of the stress intensity factors by means of the J-integral, the work which is done on solving nonhomogeneous cracked structures is very limited. In the literature, Papaioannou et al. [3] used a rather detailed finite element patern to treat a center-cracked specimen with circular inclusions near the crack-tip through a specific model developed by Wilson [4]. Recently, Smelser et al. [5] extended the J-integral without change to bimaterials provided the bond line is straight in x_1 direction (see Fig. 1). In this paper, an attempt is made to study the concept of the J-integral of general nonhomogeneous cracked composites and seek for a possible way to estimate the stress intensity factors from it. This will be helpful in selecting realistic materials with the best resistance to crack propagations.

In as much as the most commonly used compatible displacement finite element model with polynomial basis functions in each element cannot represent the asymptotic singular stress and strain fields in the vicinity of the crack-tip, an alternative finite element formulation is needed. In such a formulation, one should be able to incorporate the asymptotic singular behaviors for stresses and strains in "singular elements" near the crack-tip and use only regular polynomials type basis functions in "regular elements" in the far field. Interelement continuity of displacements and tractions must also be maintained between the singular elements with singular field assumptions and the regular elements with regular field assumptions. One such formulation is the assumed hybrid-displacement finite element model. The basis of the hybrid-displacement model and its use in fracture problems have been discussed in detail in [6,7]. The present investigation is an application of the assumed hybrid-displacement model [7] to nonhomogeneous cracked problems.

To verify the path independent nature of the J-integral of nonhomogeneous cracked structures numerically, first of all, a center-cracked specimen with arbitrarily located inclusions is analyzed. Besides, for comparison purposes, the same specimen solved by Papaioannou et al. [3] is also investigated in this work. The influence of circular inclusions of different material properties on the mode I stress intensity factors is explored. A possible practical procedure for the evaluation of stress intensity factors of nonhomogeneous cracked structures from the computed J values as done for homogeneous structures is verified. Excellent results can be noted.

2. On the Path Independence of the J-Integral

The original line integral form for J [1,2] for two-dimensional linear or nonlinear elastic material behavior is defined by (for small deformation)

$$J = \int_\Gamma (w dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds)$$

Referring to Fig. 1, Γ is an arbitrary path of the integral from the bottom surface of the crack through material to the upper surface of the crack, x_1 and x_2 are the Cartesian coordinates, u_i are displacement components, and w is the strain energy density function defined by

$$w = w(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{ij} d\varepsilon_{ij}$$

where ε_{ij} and σ_{ij} are strain and stress tensors respectively. T_i are components of surface traction along the boundary Γ defined by $T_i = \sigma_{ij} n_j$, where n_j are the components of the outward normal to the integration path. ds is an element of path length.

Now, using equilibrium (in the absence of body forces), the usual strain-displacement relationships (small strains and rotations) and using the Green's theorem, Rice [2] showed for any closed path within a body (not jumping across the crack) $J_{\text{closed path}} = 0$. (see Fig. 1) Since along a flat crack surface, $dx_2 = 0$ and $T_i = 0$, then the contribution to J is zero as noted from the integral. Thus, $J_\Gamma + J_{\Gamma'} = J_{\text{closed path}} = 0$ or $J_\Gamma = J_{\Gamma'}$ (with reversed direction). This result shows that J is path independent when applied around a crack-tip from one crack surface to another. However, it is to be mentioned that the J-integral is path independent only when the material under consideration is homogeneous, at least in the x_1 direction [5,8]. Thus, the path independent nature of the J-integral for nonhomogeneous system should be further explored.

Without loss of generality, as shown in Fig. 2, consider a cracked structure embedded with n arbitrarily located inclusions. Following the same procedures as employed by Rice [2], after some manipulations, we arrive at the conclusions quoted as follows:

1. The J-integral (with the path Γ) is still path-independent.
2. $J_{\Gamma \ell_1}$ and $J_{\Gamma \ell_2}$ are also equal and the path independent nature remains valid provided the integrating paths (enclose the same inclusions) never go through the inclusions.
3. The relationship among J, $J_{\Gamma \ell}$ and $J_{\Gamma s_k}$ can be expressed as

$$\int_\Gamma (w dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds) = \int_{\Gamma \ell} (w dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds) - \sum_{k=1}^{n} \int_{\Gamma s_k} (w dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds)$$

or, for convenience,

$$J = J_{\Gamma \ell} - \sum_{k=1}^{n} J_{\Gamma s_k} \qquad (1)$$

The fact that J is path independent implies that J is still a crack-tip parameter for the present nonhomogeneous system, i.e., J value on a contour immediately adjacent to the crack tip can be evaluated on a larger contour from conditions far away. Consider a homogeneous linear elastic cracked body in a state of plane stress, the simple relationship between the J-integral and the mode I stress intensity factor K_I is established as [8]

$$J = K_I^{2}/E \qquad (2)$$

where E is Young's modulus. Once the J-integral is estimated, K_I can be easily determined. Thus, for practical applications, further extension of eq. (2) to nonhomogeneous cracked composites is desired.

3. Brief Description of Formulation

Early attempts to deal with fracture problems by the method of finite elements usually have met with partial success due to the presence of a mathematical singularity at the crack-tip. To get an improved technique, we can devide the present nonhomogeneous cracked structures into three regions: (a) a small region near the crack-tip where the singular, near field solution is predominant, (b) a region away from the crack-tip where no singularity exists and (c) a region of inclusions. As previously mentioned, the assumed hybrid-displacement model is imperative in region (a). The element designed in this region is singular circular-sector shaped elements embedded with the correct Williams' singularities [9]. Because of the simplicity and less cost for computer usage, however, the most commonly used compatible displacement model together with quadratic isoparametric regular elements are employed in regions (b) and (c). Thus, the total energy functional π of the whole cracked composite system can be expressed as the sum of π_{HD} (for region (a)), π_{CD} (for region (b)) and π_{CD} (for region (c)). For want of space, only the formulation of the hybrid-displacement finite element model is shown here. More detailed discussions can be found in [6,7].

The variational principle which governs the assumed hybrid-displacement finite element model is the stationary condition of a modified variational principle of potential energy for which the functional to be varied is

$$\pi_{HD}(U_i, U_{Li}, T_{Li}) = \sum_{m=1}^{p}\{\int_{Am}(\frac{1}{2}E_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} - \overline{F}_iU_i)dA - \int_{\partial Am}T_{Li}(U_i - U_{Li})ds - \int_{s\sigma m}\overline{T}_iU_{Li}ds\} \qquad (3)$$

where

U_i = independently assumed interior displacements for each element,

U_{Li} = independently assumed interelement boundary displacements which are inherently compatible,

T_{Li} = Lagrangian multiplier terms which are physically the independently assumed arbitrary interelement boundary tractions,

p = total number of singular circular-sector shaped elements,

Am = area of the mth element (m = 1, 2 ... p),

$E_{ijk\ell}$ = elasticity tensor,

$\varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ within each element Am,

$\overline{F}_i$ = body force

∂Am = entire boundary of mth element,

$s\sigma m$ = a portion of ∂Am where tractions are specified,

$\overline{T}_i$ = specified surface tractions on $s\sigma m$, In constructing the finite element equations, we assume

$$\{U_i\} = [\bar{U}]\{\beta\} \qquad \text{in } Am \qquad (4)$$

$$\{\varepsilon_{ij}\} = [W]\{\beta\} \qquad \text{in } Am \qquad (5)$$

$$\{U_{Li}\} = [L]\{q\} \qquad \text{on } \partial Am \qquad (6)$$

and

$$\{T_{Li}\} = [R]\{\alpha\} \qquad \text{on } \partial Am \qquad (7)$$

where the column matrices $\{\beta\}$ and $\{\alpha\}$ are undetermined independent parameters, and $\{q\}$ are the appropriate nodal displacements. The interpolation function [U] for the interior displacements in eq. (4) includes regular polynomial mode and a correct Williams' mode ($U_i \sim r^{\frac{1}{2}}$). [W] is derived from [U] by the use of strain-displacement relations. The function [L] contains the same behavior for two adjacent elements at the common boundaries. The traction interpolation function [R] on the element boundary is derived from a self-equilibrated stress field derived from an assumed Airy stress function. Thus, the proper $r^{-\frac{1}{2}}$ singular behavior for stresses is also contained.

Substituting eqs. (4)-(7) into eq. (3), neglecting the body force terms, π_{HD} can be written in terms of {q} only, that is

$$\pi_{HD} = \sum_{m=1}^{p} \left(\frac{1}{2}\{q\}^T[K]\{q\} - \{q\}^T\{F\}\right)$$

where

$$[K] = ([P]^{-1}[G])^T[H]([P]^{-1}[G])$$

and

$$[P] = \int_{\partial Am}[R]^T[U]ds$$

$$[G] = \int_{\partial Am}[R]^T[L]ds$$

$$[H] = \int_{Am}[W]^T[E][W]dA$$

$$\{F\} = \int_{s\sigma m}[L]^T\{\bar{T}_i\}ds$$

in the above, $\{\ \}^T$ and $[\ ]^T$ are the transpose of { } and [ ] respectively.

$[P]^{-1}$ is the inverse of [P] and [E] is the constant elastic matrix. By expressing the element nodal displacements {q} in terms of independent generalized global displacements {q*}, and using the condition of stationary of π with respect to {q*} one obtains the final algebraic equations

$$[K^*]\{q^*\} = \{F^*\}$$

where [K*] and {F*} are the global matrices which are assembled by the corresponding element matrices. Once {q*} are obtained, the corresponding matrices and parameters $\{\varepsilon_{ij}\}$, $\{\sigma_{ij}\}$, J, $J_{\Gamma\ell}$ and $J_{\Gamma s}$ can then be easily determined.

4. Results and Conclusions

In order to verify the validity of eq. (1) numerically, for simplicity, detailed analyses are first performed for the case of a center-cracked nonhomogeneous specimen with four arbitrarily located inclusions. Since the system is symmetrical, it is only necessary to analyze a quarter portion whose finite element mesh is idealized and shown in Fig. 3. For the crack-tip core region, the circular-sector shaped six-node singular elements embedded with proper singularities are used and surrounded by "regular" eight-node isoparametric elements. The number of elements and nodes are 49 and 170, respectively. All the J, $J_{\Gamma\ell}$, $J_{\Gamma s}$ paths and appropriate prescribed displacement boundary conditions are also shown in Fig. 3.

Excellent numerical data are obtained and listed in Table I. E_i and ν_i are Young's modulus and Poisson's ratio of the inclusions respectively. The accurate path independent nature of J and $J_{\Gamma\ell}$ is found. As displayed in Table I, as expected, $J_{\Gamma\ell}$ is identical to J and $J_{\Gamma s} \sim 0$ for the case where no inclusions exist (i.e. $E_i = E$). It is worthwhile to be noted that $J_{\Gamma\ell}$ could be negative under the prescribed direction as that of J.

For studying the applicability of eq. (2) to the nonhomogeneous system, analyses are also done for the problem as solved by Papaioannou et al. [3]. As seen in Fig. 4, two identical circular inclusions centered on the extension of the crack line and symmetrically located with respect to the crack are assumed to be present. A similar finite element mesh as displayed in Fig. 3 is employed. Fig. 4 also shows the computed variation of normalized stress intensity factor K_I/K_0 (K_0 denotes the stress intensity factor without inclusions) with various elastic modulus ratios in the cases of R/a = 0.5 and R/a = 0.3 respectively. The data given by Papaioannou et al. [3] are also shown for comparison purposes. Excellent agreement between the computed normalized stress intensity factors and reference solutions implies that, for the calculation of stress intensity factors, eq. (2) for homogeneous plane stress system is also applicable for the present nonhomogeneous system.

Acknowledgement

The author would like to thank Miss Chyang Wu for typing the manuscript of this work.

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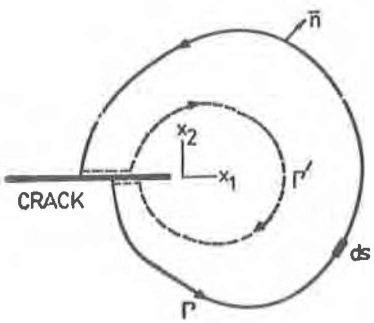
Table I. $J_{\Gamma\ell}$, $J_{\Gamma s}$ and J values ($\nu_i = \nu = 0.2$)

E_i/E	$J_{\Gamma\ell}$			$J_{\Gamma s}$	$J_{\Gamma\ell}$(ave.)-$J_{\Gamma s}$	J			J(ave.)	error (%)
0.1	-262.28	-262.07	-262.38	-279.37	16.99	17.10	16.83	16.71	16.88	0.65%
0.5	72.39	75.10	73.89	8.80	64.99	64.00	64.67	63.59	64.03	1.48%
1	117.47	115.66	114.02	0.01	115.70	118.44	117.47	115.65	117.19	——
4	106.70	102.60	98.11	-5.32	107.79	113.13	113.71	112.01	112.95	-4.57%

(unit : N/mm)



$$W = W(\epsilon_{mn}) = \int_0^{\epsilon_{mn}} \sigma_{ij}\, d\epsilon_{ij}$$

$$T_i = \sigma_{ij} n_j \; ; \; \sigma_{ij,j} = 0$$

$$J = \int_{\Gamma} \left(w\, dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds \right)$$

Fig. 1. Definition of the J-integral

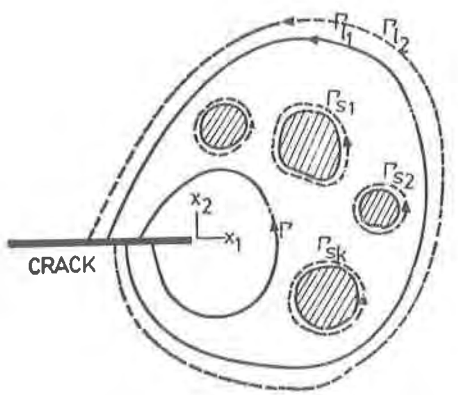


Fig. 2. Definitions of J, $J_{\Gamma\ell}$ and $J_{\Gamma s}$ for Nonhomogeneous Cracked Structures

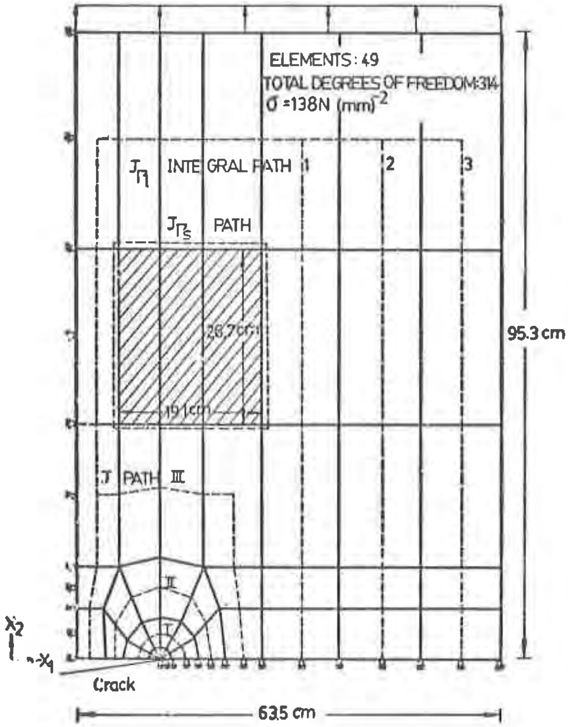


Fig. 3. Finite Element Model of Center-Cracked Specimen with Inclusions

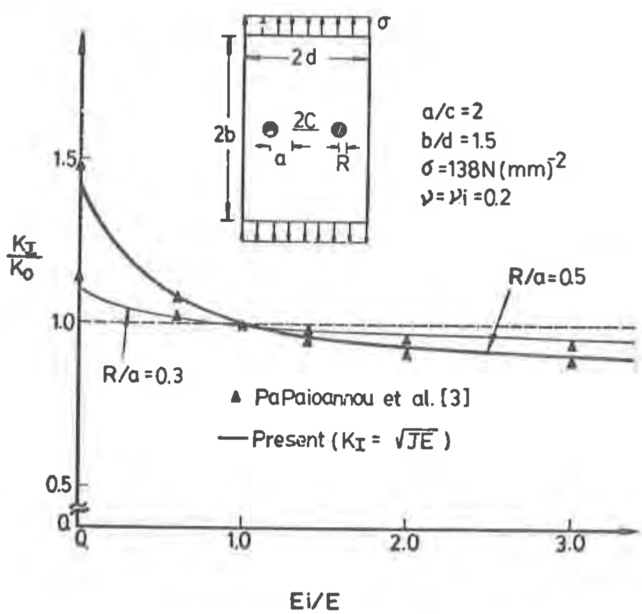


Fig. 4. Variation of Normalized Stress Intensity Factor with Elastic Modulus Ratio