A Segmental Finite Element for General Plane Elasticity Problems in Polar Coordinates

A.B. Sabir
University College, Dept. of Civil and Structural Engineering, Newport Road, Cardiff CF2 1TA, U.K.

Abstract

A segmental finite element is developed for the analysis of general plane elasticity problems. The element is bound by two concentric circular arcs and two straight radial lines. Polar coordinates are used and the shape functions are based on assumed strains. The resulting element satisfies both the exact representation of strain free rigid body modes of displacements and compatibility within the element.

This element is applied to the solution of rotationally symmetric and unsymmetric deformations. A considerable range of aspect ratio of circumferential to radial dimensions are considered. The results are compared with those obtained from other existing sector elements. It is seen that the strain based element converges faster for deflections and stresses and that only few elements are required to obtain satisfactory results.

1. Introduction

In an earlier publication Ashwell and Sabir |1| investigated the suitability of the available finite elements for curved structures. It was revealed that to obtain satisfactory converged results, the finite elements based on assumed independent polynomial displacement functions require the curved structure to be divided into a large number of elements. Shortcomings highlighted for these elements were found to be largely removed if assumed strains, rather than displacement, functions are used to develop finite elements for curved structures. An account of basic considerations was therefore carried out on circular arches, cylindrical shells and doubly curved shells and a method for the development of displacement fields was obtained, see Sabir et al |2-7|.

More recently it was shown that the above approach is not confined to curved structures and a new class of elements for general plane elasticity problems were developed using the strain approach, see Sabir |8|. A basic rectangular inplane element having only the essential nodal degrees of freedom, satisfying the requirement of strain free rigid body modes of displacements and compatibility within the element is first developed. The approach is further extended to develop elements satisfying the above basic considerations together with equilibrium. Furthermore, the opportunity was taken to show the versatility of the approach to develop a rectangular element having the inplane (drilling) rotation as a degree of freedom. The above elements were tested by applying them to the two dimensional plane elasticity problems of beams having a considerable range of aspect ratio of depth to length and Sabir and Chow |9| used these elements for the calculation of elastic buckling loads for plates with square and circular holes. Furthermore Sabir |10| applied these elements to the analysis of shear walls with openings. In all cases it was shown that satisfactory converged results can be obtained when the structure is divided into few elements.

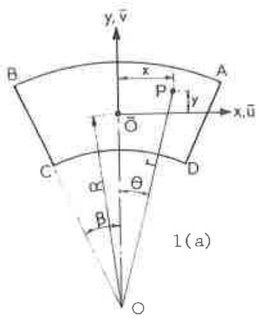
The success of the application of the strain approach to the two dimensional plane elasticity problems prompted the extension of the work to the development of finite sector elements in a polar coordinates system. This can be achieved in two ways, the first is to derive strain based elements in polar coordinates using a direct approach i.e. by giving due consideration to the constitutive equations in polar coordinates, assuming polynomial expressions for the strains and integrating the resulting equations to obtain the displacement functions. The second approach is to use the displacement fields obtained for the previously mentioned strain based rectangular inplane elements and converting the coordinate system from a Cartesian to a polar one. In this way the new functions are allowed to have variations of strains dictated by the constitutive equations in polar coordinates and also can be easily integrated over the area of a segment bound by two radial lines and two concentric circular arcs.

This second approach is used in the present paper and such an annular strain based element is developed. The resulting element satisfies compatibility within it, the exact representation of strain free rigid body modes of displacements and the constitutive equations required the three components of strains to vary lineearly in the radial direction and trigonometrically in the circumferential direction. The above element is tested by applying it to a variety of two dimensional elasticity problems of circular plates. Several types of loads are considered so that rotationally symmetric and unsymmetric cases are examined.

Convergence curves for deflections and stresses are given and compared with those obtained from other sector elements.

2. Derivation of the shape functions and stiffness matrix for the sector element

Consider the sector element shown in figure 1(a), the three components of strain at any point p in the Cartesian coordinate system x and y will be given by



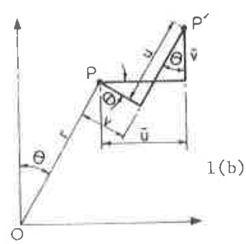


Fig. 1 Displacements of sector element

$$\varepsilon_x = \partial\bar{u}/\partial x$$
$$\varepsilon_y = \partial\bar{v}/\partial y \qquad \text{eq. (1)}$$
$$\gamma_{xy} = \partial\bar{u}/\partial y + \partial\bar{v}/\partial x$$

where ε_x and ε_y are the direct strains, γ_{xy} is the shearing strain and $\bar{u}$ and $\bar{v}$ are the translational displacements in the x and y directions respectively.

If these three strains are equal to zero, we can integrate the resulting three differential equations to obtain

$$\bar{u} = a_1 - a_3 y \qquad \text{eq. (2)}$$
$$\bar{v} = a_2 + a_3 x$$

Equations (2) represent the shape functions for the sector elements in terms of its three rigid body displacement components a_1 , a_2 and a_3. If the sector element is to have four corner nodes at A,B,C and D (figure (1a)) and each node is to have two degrees of freedom, the element shape function should contain eight independent constants. Having used three for the representation of the rigid body components, we are left with five constants to represent the straining of the element. These five are apportioned among the three strains as follows :

$$\varepsilon_x = a_4 + a_5 y$$
$$\varepsilon_y = a_6 + a_7 x \qquad \text{eq. (3)}$$
$$\gamma_{xy} = a_8$$

If we equate equations (3) and (1) and integrate we obtain

$$\bar{u} = a_4 x + a_5 xy + a_8 y/2 - a_7 y^2/2$$
$$\bar{v} = a_6 y + a_7 xy + a_8 x/2 - a_5 x^2/2 \qquad \text{eq. (4)}$$

The final shape functions will be given by adding equations (2) and (4) to obtain

$$\bar{u} = a_1 - a_3 y + a_6 y + a_7 xy + a_8 y/2 - a_7 y^2/2$$
$$\bar{v} = a_2 + a_3 x + a_6 y + a_7 xy + a_8 x/2 - a_5 x^2/2 \qquad \text{eq. (5)}$$

To convert the above in terms of a polar coordinate system we use from figure (1b).

$$x = r\sin\Theta$$
$$y = r\cos\Theta - R \qquad \text{eq. (6)}$$

where R is radius of curvature of the central circumferential line of the element and the polar coordinates r and Θ are as shown in figure (1a). Furthermore if any point p is displaced to p' with components $\bar{u}$ and $\bar{v}$ in the x and y directions then the displacement components in the r and Θ directions u and v will be given (see figure (1b)) by

$$u = \bar{u}\sin\Theta + \bar{v}\cos\Theta$$
$$v = \bar{u}\cos\Theta - \bar{v}\sin\Theta \qquad \text{eq. (7)}$$

The final shape functions will be obtained by substituting (5) into (7) and substituting in the resulting equations for x and y from equations (6), hence

$$u = a_1\sin\Theta + a_2\cos\Theta + a_3 R\sin\Theta + a_4 r\sin^2\Theta + a_5 r\sin^2\Theta(r\cos\Theta/2 - R) + a_6\cos\Theta(r\cos\Theta - R)$$
$$a_7\sin\Theta(r^2\cos^2\Theta - R^2)/2 + a_8\sin\Theta(r\cos\Theta - R/2)$$
$$v = a_1\cos\Theta - a_2\sin\Theta + a_3(R\cos\Theta - r) + a_4 r\sin\Theta\cos\Theta + a_5 r\sin\Theta(r\cos^2\Theta + r\sin^2\Theta/2) + a_6\sin\Theta(R - r\cos\Theta)$$
$$-a_7(r^2\cos^3\Theta - R^2\cos\Theta + 2r^2\sin^2\Theta\cos\Theta - 2rR)/2 + a_8(\cos 2\Theta - R\cos\Theta)/2 \qquad \text{eq. (8)}$$

The stiffness matrix K for the sector element can now be calculated from the well known expression

$$K = |C^{-1}|^T \left[\int |B^T| \, |D| \, |B| dv \right] |C^{-1}| \qquad \text{eq. (9)}$$

where D is the rigidity matrix, C is the transformation matrix and B is the strain matrix which in polar coordinates can be written as

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r} \\ \varepsilon_\Theta &= \frac{u}{r} + \frac{\partial v}{r \partial \Theta} \\ \gamma_{r\Theta} &= \frac{\partial u}{r \partial \Theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \end{aligned} \qquad \text{eq. (10)}$$

where ε_r and ε_Θ are the direct radial and circumferential strains and $\gamma_{r\Theta}$ is the shearing strain.

Hence from equations (10) and (8)

$$|B| = \begin{bmatrix} 0 & 0 & 0 & \sin^2\Theta & \sin^2\Theta(r\cos\Theta - R) & \cos^2\Theta & r\sin\Theta\cos^2\Theta & \frac{1}{2}\sin 2\Theta \\ 0 & 0 & 0 & \cos^2\Theta & \cos^2\Theta(r\cos\Theta - R) & \sin^2\Theta & r\sin^3\Theta & -\frac{1}{2}\sin 2\Theta \\ 0 & 0 & 0 & \sin 2\Theta & \sin 2\Theta(r\cos\Theta - R) & -\sin 2\Theta & -r\sin\Theta\sin 2\Theta & \cos 2\Theta \end{bmatrix} \qquad \text{eq. (11)}$$

We note that the strains vary linearly with r and in a complex trigonometrical form with Θ. The rest of the matrix manipulations to obtain K can be easily carried out and for the case of plane stress problems the results for the bracketed integral of equation (9) is given in Appendix 1. This element shall be referred to by element I.

3. Other elements used for comparison

One of the most commonly used finite elements for plane elasticity problems is the rectangular bilinear element where the shape functions are

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 xy \\ v &= a_5 + a_6 x + a_7 y + a_8 xy \end{aligned} \qquad \text{eq. (12)}$$

Raju |11| developed a sector element based on the above functions by replacing x and y with r and Θ hence he took

$$\begin{aligned} u &= a_1 + a_2 r + a_3 \Theta + a_4 r\Theta \\ v &= a_5 + a_6 r + a_7 \Theta + a_8 r\Theta \end{aligned} \qquad \text{eq. (13)}$$

We may also develop an element using the above replacement and applying it to equations (5) hence

$$\begin{aligned} u &= a_1 - a_3\Theta + a_4 r + a_5 r\Theta + a_8\Theta/2 - a_7\Theta^2/2 \\ v &= a_2 + a_3 r + a_6\Theta + a_7 r\Theta + a_8 r/2 - a_5 r^2/2 \end{aligned} \qquad \text{eq. (14)}$$

The elements obtained from the shape functions given by equations (13) and (14) shall be referred to by element II and III respectively.

4. Problems considered

The performance of the three above mentioned elements are tested by applying them to two types of problems. The first type considered is that of the annulus semi circular plate shown in figure (2). The plate is subjected to two equilibrating shearing forces applied at the free ends. The forces are assumed to be uniformly distributed along the two horizontal radial lines. This problem exhibits a rotationally unsymmetric type of deformations. The aspect ratios of the annulus were taken to have a considerable range of values. Wide, moderately wide, narrow and extremely narrow (beam like) plates are analysed. In the case of the narrow plates the deformation is mainly due to bending and the effect of radial stress and shear deformations progressively increases as the plate is made wider.

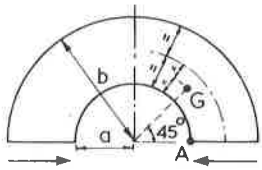


Fig. 2 An annulus subjected to shear

Convergence curves for the radial deflection at A are given in figure (3) for a moderately wide plate. The finite elements results are given in this figure as a percentage of the analytical value. This figure shows that the results obtained from the use of the strain based sector element (I) converges much faster than those obtained from elements II and III.

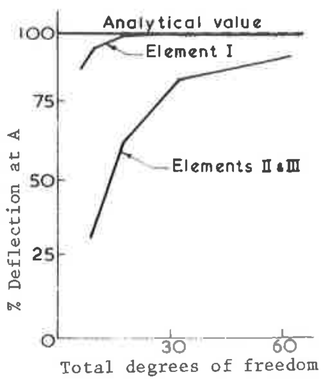


Fig. 3 Convergence curves for the radial deflection at A

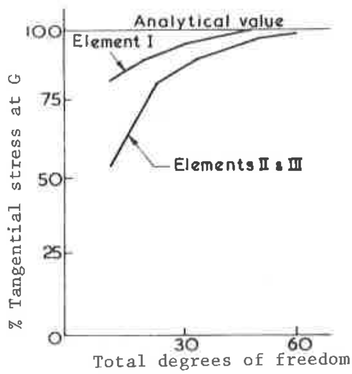


Fig. 4 Convergence curves for the shear stress at G

Similar convergence curves are obtained for wider and narrow plates and in all cases element (I) performed much better than the other two (the results for elements II and III are almost the same). Furthermore we noticed that the performance of element (I) was consistent while those for the other elements deteriorated as the plate is made narrow. Table 1 shows a sample of such results when a mesh (4 x 4) is used. Figure (4) gives convergence curve for the tangential stress at G again we see that the strain based sector element performs better than the other two.

Table 1 Deflection at A expressed as a percentage of analytical value

Element	b/a = 3	b/a = 2	b/a = 1.3
I	89.98	74.46	32.52
II	89.95	73.79	33.53
III	98.97	99.03	99.70

The other problem considered is that of a thick cylinder subjected to a uniform internal pressure. This problem exhibits a rotationally symmetric deformation and the results showed a similar good performance of the strain based sector element I.

Due to lack of space only a small sample of results are given above. During the presentation at the conference a comprehensive set of results will be shown. Convergence curves for all the cases for deflection and stresses will be used to illustrate the suitability of the strain based sector element (I) for this class of problems.

5. References

/ 1 / ASHWELL, D.G., SABIR, A.B. "Limitations of certain curved finite elements when applied to arches". Int. J.Mech.Sci., 13, 133-139 (1971).

/ 2 / SABIR, A.B., "Stiffness matrices for the general deformation (out of plane and inplane of curved beam elements based on independent strain functions". The Mathematics of finite elements and application II. Editor J.R. Whiteman, Chapter 34, 411-421, Academic Press, (1975).

/ 3 / ASHWELL, D.G., SABIR, A.B., ROBERTS, T.M. "Further studies in the application of curved finite elements to circular arches". Int.J.Mech.Sci., 13, 507-517, (1971).

/ 4 / SABIR, A.B., ASHWELL, D.G. "A comparison of curved beam finite elements when used in vibration problems". J. of Sound and Vibration, 18 (4), 555-563, (1971).

/ 5 / ASHWELL, D.G., SABIR, A.B. "A new cylindrical shell finite element based on simple independent strain functions". Int.J.Mech.Sci., 14, 171-183, (1972).

/ 6 / SABIR, A.B., CHARCHAFCHI, T.A. "Curved rectangular and general quadrilateral shell elements for cylindrical shells". The mathematics of finite elements and application IV. Editor J.R. Whiteman, 231-239, Academic Press, (1982).

/ 7 / SABIR, A.B. "A shallow shell finite element for general shell analysis". Proceedings of the Conference on Variational Methods in Engineering, University of Southampton, July, (1985).

/ 8 / SABIR, A.B. "A new class of finite elements for plane elasticity problems". The computational aspects of finite elements CAFEM, 7th Int.Conf. on Structural Mech. in Reactor Technology, Chicago, August, (1983).

/ 9 / SABIR, A.B., CHOW, F.Y. "Elastic buckling of flat panels containing circular and square holes". Int.Conf. on instability and plastic collapse of steel structures, Manchester University, (1983).

/10 / SABIR, A.B. "Strain based elements for the analysis of shear walls". Proc.Int.Conf. on Tall buildings. 447-453. Hong Kong University, December, (1984).

/11 / RAJU, I.S., KRISHNA, M., RAO, A.K. "Sector elements for matrix displacement analysis". Int.J.Numerical Methods in Engineering, 6, 553-563, (1973).

Appendix 1

The bracketed integral of equation (9) for the strain based sector element I is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & h_1 & h_2 & \nu h_1 & 0 & 0 \\ & & & & h_3 & \nu h_2 & 0 & 0 \\ \text{symmetrical} & & & & & h_1 & 0 & 0 \\ & & & & & & h_4 & 0 \\ & & & & & & & h_5 \end{bmatrix}$$

where

$$h_1 = (b^2-a^2)D\beta$$
$$h_2 = 2D(b^3-a^3)\sin\beta/3 + (b+a)h_1/2$$
$$h_3 = D(b^4-a^4)(\beta + 5\sin^3\beta\cos\beta/12 + 7\cos^3\beta\sin\beta/12 + 11\sin2\beta/8 + \sin^5\beta\cos\beta/3 - 2\cos^5\beta\sin\beta/3)$$
$$h_4 = (b^4-a^4)D(\beta + \sin2\beta/2)/4$$
$$h_5 = (1-\nu)h_1/2$$
$$D = Et^3/12(1-\nu^2)$$