THE FINITE ELEMENT METHOD AND LOCAL BOUNDS FOR BOUNDARY VALUE PROBLEMS OF ELASTIC STRUCTURES

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SUMMARY

As the numerical results for elastic structures are in general approximate it is desirable to have bounds for the unknown stresses or displacements. Using a Hilbert space where every state of a linear elastic structure is associated with an element f of this space, global bounds for an unknown state f are easy to obtain, if two approximate or comparison states $f^{\approx}$ and $f^{\sim}$ are known

$$\|f - f^{\sim}\| \leqslant \|f^{\approx} - f^{\sim}\| \qquad \text{or} \qquad \|f - f^{\approx}\| \leqslant \|f^{\approx} - f^{\sim}\|$$

where the elastic energy norm is used and where $f^{\sim}$ has to fulfill at least the geometrical and $f^{\approx}$ at least the static conditions of the given problem. Unfortunately finite global bounds do not exclude infinite local field quantities (e.g. stress). In this sense global bounds are useless but nevertheless they provide some indication on the overall accuracy of the analysis.

Due to Weber local bounds for a field quantity $\mathscr{F}(p)$ of f can be obtained in a point p, if for a certain Green-state $\dot{f}$ comparison states $\dot{f}^{\sim}$ and $\dot{f}^{\approx}$ are available. To avoid infinite integrals, the singularity is separated from $\dot{f}^{\sim}$ and $\dot{f}^{\approx}$. This leads to the regular parts $\check{f}^{\sim}$ and $\check{f}^{\approx}$ satisfying now inhomogeneous conditions on the boundary. The desired difference between the field quantity $\mathscr{F}$ and a special approximate quantity $\mathscr{F}_N$ is then

$$|\mathscr{F}_{(p)} - \mathscr{F}_{N(p)}| \leqslant \|f^{\approx} - f^{\sim}\| \cdot \|\check{f}^{\approx} - \check{f}^{\sim}\|.$$

Usually these four comparison states are not known and the problem has been to find them, especially $\check{f}^{\approx}$ and $\check{f}^{\sim}$. In this paper a method will be presented by which the four comparison states can be computed.

Starting point is the minimal problem that results from the demand of minimal bounds

$$\|f^{\approx} - f^{\sim}\| \cdot \|\check{f}^{\approx} - \check{f}^{\sim}\| \to \text{Min.}$$

A transformation with the theorems of Colonetti, of Betti-Maxwell-Rayleigh and of Southwell-Eshelby-Rieder leads to four separate minimal problems for the calculation of the four comparison states. In particular we have for $f^{\sim}$ and $\check{f}^{\sim}$ the Dirichlet's principle and for $f^{\approx}$ and $\check{f}^{\approx}$ Castigliano's principle.

The difficulty of the four variational problems arises from the fact that in general inhomogeneous essential boundary conditions are prescribed, which have to be fulfilled *exactly* (in particular for $\check{f}^{\sim}$ and $\check{f}^{\approx}$ there are already rational or logarithmic functions on the boundaries prescribed).

To solve these variational problems a finite element method will be established using elements which are able to satisfy prescribed inhomogeneous boundary conditions exactly.

It will then be shown that the finite element analysis for the computation of $f^{\sim}$ and $\check{f}^{\sim}$ can be nearly synchronized: the final two systems of algebraic equations differ only on the right hand side. Therefore it is only necessary to set up just one coefficient matrix and to invert it once. As the same applies for $f^{\approx}$ and $\check{f}^{\approx}$ we can state that the computational work for local bounds is not much more than that for global bounds, where only $f^{\sim}$ and $f^{\approx}$ are needed. Several numerical examples will be given.

NOTATIONS

$\underline{\varepsilon}$ Tensor (underlined greek letters)

$\underline{a}$ Vector (underlined small latin letters)

$\underline{A}$ Matrix (underlined capital latin letters)

$\underline{a}^T$ Vector transponed

$\underline{A}^T$ Matrix transponed

$\underline{\varepsilon} \cdot\cdot \underline{\sigma} \equiv \varepsilon_{ij}\,\sigma_{ji}$ Double scalar product of two tensors of order two with Einstein's sum-convention; i=1,2,3; j=1,2,3

$\underline{n} \cdot \underline{\sigma} \;\hat{=}\; n_i\,\sigma_{ij}$ Scalar product of a vector and a tensor

$\underline{a} \cdot \underline{b} \equiv a_i\,b_i$ Scalar product of two vectors

$\underline{a}\,\underline{b} \equiv (a_i\,b_j)$ Dyade product of two vectors

$\underline{a} \times \underline{b} \equiv (\epsilon_{ijk}\,a_j b_k)$ Vector product

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k = 123,\ 231,\ 312 \\ -1 & \text{if } i,j,k = 213,\ 321,\ 132 \\ 0 & \text{if two indices are equal} \end{cases}$$

Symbols for differentiation:

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z} = \partial_i \equiv \frac{\partial}{\partial x_i}$$

$$\mathrm{Div}\,\underline{\sigma} = \nabla \cdot \underline{\sigma} \;\hat{=}\; \partial_i\,\sigma_{ij}$$

$$\mathrm{Ink}\,\underline{\varepsilon} = \nabla \times \underline{\varepsilon} \times \nabla \;\hat{=}\; -\,\epsilon_{ikl}\,\epsilon_{jmn}\,\partial_k\,\partial_m\,\varepsilon_{ln}$$

$$\mathrm{Def}\,\underline{w} = \tfrac{1}{2}\,(\nabla\underline{w} + \underline{w}\,\nabla) \;\hat{=}\; \tfrac{1}{2}\,(\partial_i w_j + \partial_j w_i)$$

1. INTRODUCTION

By the dual analysis for elastic structures (Castigliano's principle and Dirichlet's principle) we obtain global bounds to the solution or bounds to the influence coefficients [1]. This provides already a control on the overall accuracy of finite element results. But we will have no information wether the stress in any considered point of the structure is extraordinary high or even infinite. It is therefore desired to have local bounds for any fieldvalue (in particular for stresses) of the structure. This is possible with the use of Green's function [2,3,4,5]. But the difficulty in this concept is how the admissible approximate Green's functions can be found and how the enormous computational work can be reduced towards a practical utility of the method. This paper will bring new considerations in this field.

2. GENERATION OF FORMULAS FOR LOCAL BOUNDS

A general state of a linear elastic body is determined by geometric and static conditions, which are imposed on the body.

Geometric conditions are given by prescribed displacements $\underline{w}_s$ on the part S_w of the boundary and prescribed extra strains (non-elastic strains) $\underline{\varepsilon}^Q$ in the interior V of the body. Extra strains could be caused by a temperature field, magnetization or plastic deformation. With $\underline{\varepsilon}^Q$ we introduce now the incompatibility tensor

(2.1) $$\underline{\eta} = -\mathit{Ink}\,\underline{\varepsilon}^Q$$

From the mathematical point of view $\underline{\eta}$ and $\underline{w}_s$ mean the inhomogenities of the compatibility conditions in the interior and on the surface of the body.

(2.2) $$\mathit{Ink}\,\underline{\varepsilon} = \underline{\eta} = -\mathit{Ink}\,\underline{\varepsilon}^Q \qquad \text{in V}$$

(2.3) $$\underline{w} = \underline{w}_s \qquad \text{on } S_w$$

$\Big\}$ geometric conditions

Static conditions are given by prescribed forces $\underline{p}_s$ on the surface S_p and $\underline{p}$ in the interior V. Corresponding to the extra strains we introduce the extra stresses $\underline{\sigma}^K$.

(2.4) $$\underline{p} = \mathit{Div}\,\underline{\sigma}^K$$

From the mathematical point of view the forces $\underline{p}$ and $\underline{p}_s$ mean the inhomogenities of the equilibrium conditions in the interior and on the surface of the body.

(2.5) $$\mathit{Div}\,\underline{\sigma} = -\underline{p} = -\mathit{Div}\,\underline{\sigma}^K \qquad \text{in V}$$

(2.6) $$\underline{n}\cdot\underline{\sigma} = \underline{p}_s \qquad \text{on } S_p$$

$\Big\}$ static conditions

where $\underline{n}$ is the normal unit vector.

With every elastic state of the body we associate now an element f of a Hilbert-space $\mathcal{H}$. As scalar product of two states $\overset{1}{f}$ and $\overset{2}{f}$ we take the elastic interaction energy [6].

$$(2.7)\quad \{\overset{1}{f},\overset{2}{f}\} = \int_V \overset{1}{\underline{\sigma}}\cdot\cdot\overset{2}{\underline{\varepsilon}}\,dV = \int_V \overset{1}{\underline{\varepsilon}}\cdot\cdot\overset{2}{\underline{\sigma}}\,dV$$

with

$$(2.8)\quad \overset{1}{\underline{\sigma}} = Ink\,\overset{1}{\underline{\chi}} - \overset{1}{\underline{\sigma}}{}^K \qquad \chi = \text{stressfunction}$$

and

$$(2.9)\quad \overset{2}{\underline{\varepsilon}} = Def\,\overset{2}{\underline{w}} - \overset{2}{\underline{\varepsilon}}{}^Q$$

a partial integration yields

$$(2.10)\quad \{\overset{1}{f},\overset{2}{f}\} = [\overset{2}{f},\overset{1}{p}] + [\overset{1}{f},\overset{2}{w}] = [\overset{1}{f},\overset{2}{p}] + [\overset{2}{f},\overset{1}{w}]$$

where we introduced the abreviations

$$(2.11)\quad [\overset{2}{f},\overset{1}{p}] = \int_V \overset{2}{\underline{w}}\cdot\overset{1}{\underline{p}}\,dV + \int_{S_p} \overset{2}{\underline{w}}\cdot\underline{p}_s\,dS$$

$$(2.12)\quad [\overset{1}{f},\overset{2}{w}] = -\int_V (Ink\,\overset{1}{\underline{\chi}} - \overset{1}{\underline{\sigma}}{}^K)\cdot\cdot\overset{2}{\underline{\varepsilon}}{}^Q\,dV + \int_{S_w} (Ink\,\overset{1}{\underline{\chi}} - \overset{1}{\underline{\sigma}}{}^K)\cdot\underline{n}\cdot\overset{2}{\underline{w}}_s\,dS$$

In Fig.1 we introduce for the unknown state f some comparison or approximate states satisfying not all the prescribed geometric and static conditions. For this comparison states it is in general possible to calculate the stress tensor $\underline{\sigma}$ (or the strain tensor), usual by using stress or displacement functions (see (2.8) and (2.9)). With this comparison states we can derive how much in a point $\underline{x}$ the approximate fieldvalue $\mathcal{F}_i$ differs from the unknown fieldvalue at the worst [2,3,4,5]

$$(2.13)\quad |\mathcal{F}(\underline{x}) - \mathcal{F}_i(\underline{x})| \le \rho_i\, |f^{\approx} - f^{\sim}|\cdot|\check{f}^{\approx} - \check{f}^{\sim}|$$

with the norm notation $|f| = \{f,f\}$ and

$$(2.14)\quad \mathcal{F}_1 = [\overset{\circ}{f}{}^{\sim},p] - [\overset{\circ}{f}{}^{\approx},w] + \{f^{\sim},\check{f}^{\approx}-\check{f}^{\sim}\} \qquad \rho_1 = 1$$

$$(2.15)\quad \mathcal{F}_2 = [\overset{\circ}{f}{}^{\sim},p] - [\overset{\circ}{f}{}^{\approx},w] + \{f^{\approx},\check{f}^{\approx}-\check{f}^{\sim}\} \qquad \rho_2 = 1$$

$$(2.16)\quad \mathcal{F}_3 = [\overset{\circ}{f}{}^{\sim},p] - [\overset{\circ}{f}{}^{\approx},w] + \tfrac{1}{2}\{f^{\sim}+f^{\approx},\check{f}^{\approx}-\check{f}^{\sim}\} \qquad \rho_3 = \tfrac{1}{2}$$

i.e. we have upper and lower bounds for $\mathcal{F}_i$. For the value $\mathcal{F}_3$ we have smaller bounds. But this is connected with more computational work for the approximate value $\mathcal{F}_3$.

The type of the obtained fieldvalue $\mathcal{F}$ (displacement,stress etc.) depends on the singularity which we use for the state $\overset{\infty}{f}$.

3. COMPUTATION WITH THE FINITE ELEMENT METHOD

To calculate the bounds in (2.13) we need the comparison states $f^{\approx}, f^{\sim}, \check{f}^{\approx}, \check{f}^{\sim}$. This states are in general not known and we will determine them from the condition that the bounds should become a minimum

(3.1) $$D = |f^{\approx} - f^{\sim}| \cdot |\breve{f}^{\approx} - \breve{f}^{\sim}| \rightarrow Min$$

or squared

(3.2) $$D^2 = (|f^{\approx}|^2 + |f^{\sim}|^2 - 2\{f^{\approx}, f^{\sim}\}) \cdot (|\breve{f}^{\approx}|^2 + |\breve{f}^{\sim}|^2 - 2\{\breve{f}^{\approx}, \breve{f}^{\sim}\}) \rightarrow Min$$

For the scalar products $\{f^{\approx}, f^{\sim}\}$ and $\{\breve{f}^{\approx}, \breve{f}^{\sim}\}$ we apply (2.1o)

(3.3) $$D^2 = (|f^{\approx}|^2 - 2[f^{\approx}, w] + |f^{\sim}|^2 - 2[f^{\sim}, p]) \cdot (|\breve{f}^{\approx}|^2 - 2[\breve{f}^{\approx}, \breve{w}] + |\breve{f}^{\sim}|^2 - 2[\breve{f}^{\sim}, \breve{p}]) \rightarrow Min$$

and we can deduce four different variational problems

(3.4) $$G^{\approx} = |f^{\approx}|^2 - 2[f^{\approx}, w] \rightarrow Min$$

(3.5) $$G^{\sim} = |f^{\sim}|^2 - 2[f^{\sim}, p] \rightarrow Min$$

(3.6) $$\breve{G}^{\approx} = |\breve{f}^{\approx}|^2 - 2[\breve{f}^{\approx}, \breve{w}] \rightarrow Min$$

(3.7) $$\breve{G}^{\sim} = |\breve{f}^{\sim}|^2 - 2[\breve{f}^{\sim}, \breve{p}] \rightarrow Min$$

The minimization of $G^{\sim}$ and $\breve{G}^{\sim}$ corresponds with Dirichlet's principle and leads to the determination of $f^{\sim}$ and $\breve{f}^{\sim}$. In the variational process the demanded geometric conditions in Fig.1 for $f^{\sim}$ and $\breve{f}^{\sim}$ are essential whereas the static conditions are natural.

The minimization of $G^{\approx}$ and $\breve{G}^{\approx}$ corresponds with Castigliano's principle and leads to the determination of $f^{\approx}$ and $\breve{f}^{\approx}$. In the variational process the demanded static conditions in Fig.1 for $f^{\approx}$ and $\breve{f}^{\approx}$ are essential whereas the geometric conditions are natural.

For all the four variational problems we can write

(3.8) $$G^* = |f^*|^2 - 2[f^*, t] \rightarrow Min$$

The finite element method will play a fundamental part for the solution of this problem and will moreover lead to some essential reduction in the computational work as we will see lateron. For this purpose we will divide the state f^* into two parts

(3.9) $$f^* = f_S^* + f_V^*$$

where

(3.1o) $$f_S^* = f_B^* + f_{Extra}^*$$

should fulfill the essential conditions and

(3.11) $\quad f_V^* = \beta_\alpha^* b_\alpha \qquad \alpha = 1(1)N, \quad \beta_\alpha^* = \text{constants}$

should approach the natural conditions in the variational process but without violating the already satisfied essential conditions. The state f^*_{Extra} meets the essential conditions in the interior (if they are inhomogeneous at all). For $f^\sim$ resp. $f^\approx$ the state f^*_{Extra} will be represented by $-\underline{\sigma}^K$ resp. $-\underline{\varepsilon}^Q$ (see (2.8) resp. (2.9)). The state f_B^* takes care of the essential boundary conditions. To find a representation for the state f_B^* that is actually the main difficulty of methods establishing bounds by means of Green's function, because there are inhomogeneous boundary conditions to be satisfied exactly. For $\breve{f}^\sim$ and $\breve{f}^\approx$ (fig.1) we have even prescribed rational or logarithmic functions on the boundaries, as the singularity state $\overset{\infty}{f}$ is seperated. Because of this prescribed boundary functions, it is in general impossible to find admissible variational functions covering the whole domain. The interpolation problem will be easier, if the domain is seperated in subdomains. Then the inhomogeneous boundary conditions have only to be satisfied on the part of the boundary which coincides with the surface of the body. For various shapes of finite elements the author has developed interpolation functions which interpolate not only nodal values but also boundary functions and their derivatives. [7,8]

As the states $\beta_\alpha^* b_\alpha^*$ must satisfy the homogeneous essential conditions (natural conditions are not specified) for any constants β_α^* the states b_α are load- or self-stress-states (see Fig.1 and [9]). This states are easy to represent by usual conforming finite elements [1o]. In the following we will use (3.9) together with (3.11) for the variational process

(3.12) $\quad f^* = f_S^* + \beta_i^* b_i^* \qquad i = 1(1)N$

In particular we have

(3.13) $\quad f^\sim = f_S^\sim + \beta_i^\sim b_i' \quad i = 1(1)q^\sim \qquad f^\approx = f_S^\approx + \beta_j^\approx b_j'' \quad j = 1(1)q^\approx$

(3.14) $\quad \breve{f}^\sim = \breve{f}_S^\sim + \breve{\beta}_m^\sim \breve{b}_m' \quad m = 1(1)\breve{q}^\sim \qquad \breve{f}^\approx = \breve{f}_S^\approx + \breve{\beta}_n^\approx \breve{b}_n'' \quad n = 1(1)\breve{q}^\approx$

The comparison states $f^\sim$ and $\breve{f}^\sim$ are characterized by different inhomogeneous geometric conditions, which are satisfied by different states $f_S^\sim$ and $\breve{f}_S^\sim$. But we can however use the same unit states b multiplied with different parameters. As the same applies for $f^\approx$ and $\breve{f}^\approx$ we have

(3.15)
$$b'_i = \breve{b}'_m \qquad i = m = 1(1)\, q^{\sim}$$
$$b''_j = \breve{b}''_n \qquad j = n = 1(1)\, q^{\approx}$$

This circumstances will turn out as a great advantage in the further computation.

To calculate the value G^* of the functional (3.8) we use now (3.12) and obtain

(3.16)
$$G^* = \beta_i^* \{ b_i^*, b_j^* \} \beta_j^* + 2\beta_i^* \{ b_i^*, f_s^* \} - 2\beta_i^* [b_i^*, t] + \{ f_s^*, f_s^* \} - 2 \, [f_s^*, t] \rightarrow Min \qquad i,j = 1(1)\, q^*$$

For the sake of simpler presentation we introduce the following matrices and vectors

(3.17) $$(\underline{B}^*)_{ij} = \{ b_i, b_j \}$$

(3.18) $$(\underline{k}^*)_i = \beta_i^*$$

(3.19) $$(\underline{r}^*)_i = -\{ b_i^*, f_s^* \} + [b_i^*, t] = (-\underline{r}_1^* + \underline{r}_2^*)_i$$

(3.2o) $$\gamma^* = \{ f_s^*, f_s^* \} - 2 [f_s^*, t]$$

where $\underline{r}_1^* = \{ b_i^*, f_s^* \} \qquad \underline{r}_2^* = [b_i^*, t]$

Then we can write for the funtional (3.16)

(3.21) $$G^* = \underline{k}^* \underline{B}^* \underline{k}^* - 2\, \underline{k}^* \underline{r}^* + \gamma^* \rightarrow Min$$

The condition for the minimum
$$\frac{\partial G^*}{\partial \beta_i^*} = 0 \qquad i = 1(1)\, q^*$$
yields the system of equations

(3.22) $$\underline{B}^* \underline{k}^* = \underline{r}^*$$

By inverting $\underline{B}^*$ we get the unknown parameter

(3.23) $$\underline{k}^* = (\underline{B}^*)^{-1} \underline{r}^*$$

We multiply now (3.22) with the calculated vector $\underline{k}^*$ of (3.23)

(3.24) $$\underline{k}^* \underline{B}^* \underline{k}^* = \underline{k}^* \underline{r}^* = \underline{r}^* (\underline{B}^*)^{-1} \underline{r}^*$$

The calculation of the functional simplifies a great deal if (3.24) is substituted into (3.21)

(3.25) $\qquad G^{*} = -\underline{k}^{*}\underline{r}^{*} + \gamma^{*}$

or with (3.23)

(3.26) $\qquad G^{*} = -\underline{r}^{*}(\underline{B}^{*})^{-1}\underline{r}^{*} + \gamma^{*}$

The value G^{*} is then - with the exception of γ^{*} - a by-product when the system of equations (3.22) is solved : we just have to carry out the scalar product of the vectors of the right side before and after the inversion.

The four variational problems (3.4) till (3.7) cause four systems of equations like (3.22). But we will see that it is not necessary to have four systems. It is enough to invert two matrices B^{*}, because (3.15) yields

(3.27) $\qquad \underline{B}^{\sim} = \underline{\breve{B}}^{\sim} \qquad$ and $\qquad \underline{B}^{\approx} = \underline{\breve{B}}^{\approx}$

This means that only two coefficient matrices have to be build up and inverted, which reduces the computation work by nearly one half.

With (3.27) we have for the four funtionals (3.4)-(3.7)

$$
(3.28)\qquad
\left\{
\begin{aligned}
G^{\sim} &= -\underline{r}^{\sim}(\underline{B}^{\sim})^{-1}\underline{r}^{\sim} + \gamma^{\sim} \\
\breve{G}^{\sim} &= -\underline{\breve{r}}^{\sim}(\underline{B}^{\sim})^{-1}\underline{\breve{r}}^{\sim} + \breve{\gamma}^{\sim} \\
G^{\approx} &= -\underline{r}^{\approx}(\underline{B}^{\approx})^{-1}\underline{r}^{\approx} + \gamma^{\approx} \\
\breve{G}^{\approx} &= -\underline{\breve{r}}^{\approx}(\underline{B}^{\approx})^{-1}\underline{\breve{r}}^{\approx} + \breve{\gamma}^{\approx}
\end{aligned}
\right.
$$

where

$$
(3.29)\qquad
\left\{
\begin{aligned}
(\underline{r}^{\sim})_i &= -\{b'_i, \mathfrak{f}_s^{\sim}\} + [b'_i, p] = (-\underline{r}_1^{\sim} + \underline{r}_2^{\sim})_i \\
(\underline{r}^{\approx})_j &= -\{b''_j, \mathfrak{f}_s^{\approx}\} + [b''_j, w] = (-\underline{r}_1^{\approx} + \underline{r}_2^{\approx})_j \\
(\underline{\breve{r}}^{\sim})_i &= -\{b'_i, \breve{\mathfrak{f}}_s^{\sim}\} + [b'_i, \breve{p}] = (-\underline{\breve{r}}_1^{\sim} + \underline{\breve{r}}_2^{\sim})_i \\
(\underline{\breve{r}}^{\approx})_j &= -\{b''_j, \breve{\mathfrak{f}}_s^{\approx}\} + [b''_j, \breve{w}] = (-\underline{\breve{r}}_1^{\approx} + \underline{\breve{r}}_2^{\sim})_j
\end{aligned}
\right.
$$

$$
\begin{aligned}
\gamma^{\sim} &= \{\mathfrak{f}_s^{\sim}, \mathfrak{f}_s^{\sim}\} - 2[\mathfrak{f}_s^{\sim}, p] \\
\gamma^{\approx} &= \{\mathfrak{f}_s^{\approx}, \mathfrak{f}_s^{\approx}\} - 2[\mathfrak{f}_s^{\approx}, w] \\
\breve{\gamma}^{\sim} &= \{\breve{\mathfrak{f}}_s^{\sim}, \breve{\mathfrak{f}}_s^{\sim}\} - 2[\breve{\mathfrak{f}}_s^{\sim}, \breve{p}] \\
\breve{\gamma}^{\approx} &= \{\breve{\mathfrak{f}}_s^{\approx}, \breve{\mathfrak{f}}_s^{\approx}\} - 2[\breve{\mathfrak{f}}_s^{\approx}, \breve{w}]
\end{aligned}
$$

$i = 1(1)q^{\sim}$
$j = 1(1)q^{\approx}$

With (3.1)-(3.7) and the values of (3.28) we have for the bounds of (2.3)

$$D = \sqrt{(G^{\approx} + G^{\sim}) \cdot (\breve{G}^{\approx} + \breve{G}^{\sim})} \tag{3.30}$$

The approximate fieldvalue $\mathcal{F}_3$ from (2.16) we calculate with the four comparison states we obtained from the condition that the bounds should become a minimum.

$$\mathcal{F}_3 = [\mathring{f}^{\sim}, p] - [\mathring{f}^{\approx}, w] + \tfrac{1}{2}\{f^{\sim} + f^{\approx}, \breve{f}^{\approx} - \breve{f}^{\sim}\} \tag{3.31}$$

We have now according to Fig.1

$$\mathring{f}^{\sim} = \overset{\infty}{f} + \breve{f}^{\sim} \qquad \mathring{f}^{\approx} = \overset{\infty}{f} + \breve{f}^{\approx}$$

and according to (2.10)

$$\{\breve{f}^{\approx}, f^{\sim}\} = [f^{\sim}, \breve{p}] + [\breve{f}^{\approx}, w]$$

$$\{f^{\approx}, \breve{f}^{\sim}\} = [\breve{f}^{\sim}, p] + [f^{\approx}, \breve{w}]$$

This we insert in (3.31)

$$\mathcal{F}_3 = [\overset{\infty}{f}, p] - [\overset{\infty}{f}, w] + \tfrac{1}{2}\{f^{\approx}, \breve{f}^{\approx}\} - \tfrac{1}{2}\{f^{\sim}, \breve{f}^{\sim}\}$$
$$+ \tfrac{1}{2}[\breve{f}^{\sim}, p] - \tfrac{1}{2}[\breve{f}^{\approx}, w] + \tfrac{1}{2}[f^{\sim}, \breve{p}] - \tfrac{1}{2}[f^{\approx}, \breve{w}]$$

With (3.13)-(3.15) and (3.29) we obtain

$$\begin{aligned}
\mathcal{F}_3 = {} & [\overset{\infty}{f}, p] - [\overset{\infty}{f}, w] \\
& + \tfrac{1}{2}[\breve{f}_s^{\sim}, p] - \tfrac{1}{2}[\breve{f}_s^{\approx}, w] + \tfrac{1}{2}\underline{\breve{k}}^{\sim}\underline{r}_2^{\sim} - \tfrac{1}{2}\underline{\breve{k}}^{\approx}\underline{r}_2^{\approx} \\
& + \tfrac{1}{2}[f_s^{\sim}, \breve{p}] - \tfrac{1}{2}[f_s^{\approx}, \breve{w}] + \tfrac{1}{2}\underline{k}^{\sim}\underline{\breve{r}}_2^{\sim} - \tfrac{1}{2}\underline{k}^{\approx}\underline{\breve{r}}_2^{\approx} \\
& + \tfrac{1}{2}\{f_s^{\approx}, \breve{f}_s^{\approx}\} + \tfrac{1}{2}\underline{\breve{k}}^{\approx}\underline{r}_1^{\approx} + \tfrac{1}{2}\underline{k}^{\approx}\underline{\breve{r}}_1^{\approx} + \tfrac{1}{2}\underline{k}^{\approx}\underline{B}^{\approx}\underline{\breve{k}}^{\approx} \\
& - \tfrac{1}{2}\{f_s^{\sim}, \breve{f}_s^{\sim}\} - \tfrac{1}{2}\underline{\breve{k}}^{\sim}\underline{r}_1^{\sim} - \tfrac{1}{2}\underline{k}^{\sim}\underline{\breve{r}}_1^{\sim} - \tfrac{1}{2}\underline{k}^{\sim}\underline{B}^{\sim}\underline{\breve{k}}^{\sim}
\end{aligned}$$

Because of

$$\begin{aligned}
\underline{k}^{\sim}\underline{B}^{\sim}\underline{\breve{k}}^{\sim} = \underline{\breve{k}}^{\sim}\underline{B}^{\sim}\underline{k}^{\sim} = \underline{\breve{k}}^{\sim}\underline{r}^{\sim} = \underline{k}^{\sim}\underline{\breve{r}}^{\sim} &= \underline{r}^{\sim}(\underline{B}^{\sim})^{-1}\underline{\breve{r}}^{\sim} \\
&= \underline{\breve{r}}^{\sim}(\underline{B}^{\sim})^{-1}\underline{r}^{\sim} \\
\underline{k}^{\approx}\underline{B}^{\approx}\underline{\breve{k}}^{\approx} = \underline{\breve{k}}^{\approx}\underline{B}^{\approx}\underline{k}^{\approx} = \underline{\breve{k}}^{\approx}\underline{r}^{\approx} = \underline{k}^{\approx}\underline{\breve{r}}^{\approx} &= \underline{r}^{\approx}(\underline{B}^{\approx})^{-1}\underline{\breve{r}}^{\approx} \\
&= \underline{\breve{r}}^{\approx}(\underline{B}^{\approx})^{-1}\underline{r}^{\approx}
\end{aligned}$$

and $\underline{r}^{*} = -\underline{r}_1^{*} + \underline{r}_2^{*}$ (see (3.29)) some terms disappear

$$\mathcal{F}_3 = [\overset{\infty}{f}, p] - [\overset{\infty}{f}, w] + \tfrac{1}{2}[\breve{f}_S^{\sim}, p] + \tfrac{1}{2}[f_S^{\sim}, \breve{p}] - \tfrac{1}{2}[\breve{f}_S^{\approx}, w] - \tfrac{1}{2}[f_S^{\approx}, \breve{w}]$$

$$+ \tfrac{1}{2}\breve{\underline{r}}^{\sim}(\underline{B}^{\sim})^{-1}\underline{r}^{\sim} - \tfrac{1}{2}\breve{\underline{r}}^{\approx}(\underline{B}^{\approx})^{-1}\underline{r}^{\approx} + \tfrac{1}{2}\{f_S^{\approx}, \breve{f}_S^{\approx}\} - \tfrac{1}{2}\{f_S^{\sim}, \breve{f}_S^{\sim}\}$$

In a similar way we can derive formulas for $\mathcal{F}_1$ and $\mathcal{F}_2$.

$$\mathcal{F}_1 = [\overset{\infty}{f}, p] - [\overset{\infty}{f}, w] + [\breve{f}_S^{\sim}, p] + [f_S^{\sim}, \breve{p}] - \{f_S^{\sim}, \breve{f}_S^{\sim}\} + \breve{\underline{r}}^{\sim}(\underline{B}^{\sim})^{-1}\underline{r}^{\sim}$$

$$\mathcal{F}_2 = [\overset{\infty}{f}, p] - [\overset{\infty}{f}, w] - [\breve{f}_S^{\approx}, w] - [f_S^{\approx}, \breve{w}] + \{f_S^{\approx}, \breve{f}_S^{\approx}\} - \breve{\underline{r}}^{\approx}(\underline{B}^{\approx})^{-1}\underline{r}^{\approx}$$

Finally we present some numerical results. For a built-in uniformly loaded square plate (Kirchhoff's theory) we have calculated the bounds for the displacement w and the moment $M_x = M_y$ in the center of the plate. For this purpose the following singularities were used:

I. $\overset{\infty}{w} = \frac{1}{8\pi N}\,\frac{1}{2}\,r^2 \ln\left(\frac{r}{a}\right)^2$ (generating bounds for the displacement)

II. $\overset{\infty}{w} = \frac{1}{8\pi N}\left[(1+\nu)\ln\left(\frac{r}{a}\right)^2 + 2(1-\nu)\frac{x^2}{r^2}\right]$ (gen. bounds for the moment)

$\nu = 0.3$ a= arbitrary length N = flexural rigidity p = load

The domain was divided in four elements.(Fig.2.). The results are compiled in Fig.3.

Elastic state		Satisfied respectively approached ($\rightarrow$) geometric values in V	geometric values on S	static values in V	static values on S
General elastic state $f = f' + f''$	f	$\underline{\varepsilon}^Q$	$\underline{w}_s$	$\underline{p}$	$\underline{p}_s$
Load-stress-state of f (characterized by hom. geometric cond)	f'	0	0	$\underline{p}$	$\underline{p}_s$
Self-stress-state of f (characterized by hom. static cond.)	f''	$\underline{\varepsilon}^Q$	$\underline{w}_s$	0	0
Comparison state of f	$f^{\sim}$	$\underline{\varepsilon}^Q$	$\underline{w}_s$	$\rightarrow \underline{p}$	$\rightarrow \underline{p}_s$
Comparison state of f	$f^{\approx}$	$\rightarrow \underline{\varepsilon}^Q$	$\rightarrow \underline{w}_s$	$\underline{p}$	$\underline{p}_s$
<u>Green's state</u> $\mathring{f} = \overset{\infty}{f} + \check{f}$ with singular part $\overset{\infty}{f}$ and regular part $\check{f}$	$\mathring{f}$	0	0	0	0
<u>Fundamental state</u> satisfies everywhere the homogen. eq. (2.2) and (2.5) except in the point $\underline{x}_i$ where there is a singularity	$\overset{\infty}{f}$	0	$\overset{\infty}{\underline{w}}_s$	0	$\overset{\infty}{\underline{p}}_s$
<u>Regular state</u> cares in Green's state $\mathring{f}$ for the satisfaction of the boundary conditions	$\check{f}$	0	$-\overset{\infty}{\underline{w}}_s$	0	$-\overset{\infty}{\underline{p}}_s$
Green's comparison state of $\mathring{f}$ $\mathring{f}^{\sim} = \overset{\infty}{f} + \check{f}^{\sim}$	$\mathring{f}^{\sim}$	0	0	$\rightarrow 0$	$\rightarrow 0$
Green's comparison state of $\mathring{f}$ $\mathring{f}^{\approx} = \mathring{f} + \check{f}^{\approx}$	$\mathring{f}^{\approx}$	$\rightarrow 0$	$\rightarrow 0$	0	0
Regular comparison state of $\check{f}$	$\check{f}^{\sim}$	0	$-\overset{\infty}{\underline{w}}_s$	$\rightarrow 0$	$\rightarrow -\overset{\infty}{\underline{p}}_s$
Regular comparison state of $\check{f}$	$\check{f}^{\approx}$	$\rightarrow 0$	$\rightarrow -\overset{\infty}{\underline{w}}_s$	0	$-\overset{\infty}{\underline{p}}_s$

Fig. 1

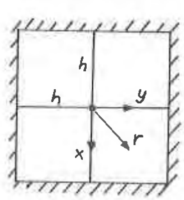

Fig. 2

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	$\mathcal{F}_1$ = Moment	$\mathcal{F}_1$ = Displacement
Functional $G^{\sim}$	$-2.1196\ h^6p^2 10^{-2}/N$	$-2.1196\ h^6p^2 10^{-2}/N$
$\breve{G}^{\sim}$	$13.8694\ h^2 10^{-2}/N$	$3.0203\ h^2 10^{-2}/N$
$G^{\approx}$	$2.5221\ h^6p^2 10^{-2}/N$	$2.5221\ h^6p^2 10^{-2}/N$
$\breve{G}^{\approx}$	$-13.2250\ h^2 10^{-2}/N$	$-2.9879\ h^2 10^{-2}/N$
Bounds D	$0.509\ h^2p 10^{-2}/N$	$0.114\ h^4p 10^{-2}/N$
Fieldvalue $\mathcal{F}_1$	$9.017\ h^2p 10^{-2}/N$	$1.936\ h^4p 10^{-2}/N$
Fieldvalue $\mathcal{F}$ ("exact")	9.24 ″	2.016 ″
Deviation $\left\lvert\frac{\mathcal{F}_1-\mathcal{F}}{\mathcal{F}}\right\rvert$ 100%	2.4%	4.0%
Deviation maximal $\left\lvert\frac{D}{\mathcal{F}}\right\rvert$ 100%	5.5%	5.7%

Fig. 3