ABSTRACT

Glotov, Petr. **Time Reversal of Electromagnetic Waves in Randomly Layered Media.** (Under the direction of Jean-Pierre Fouque.)

Time reversal is a general technique in wave propagation in inhomogeneous media when a signal is recorded at points of a device called time reversal mirror, gets time reversed and radiated back in the medium. The resulting field has a property of refocusing. Time reversal in acoustics has been extensively studied both experimentally and theoretically. In this thesis we consider the problem of time reversal of electromagnetic waves in inhomogeneous layered media. We use Markov process model for the medium parameters which allows us to exploit diffusion approximation theorem. We show that the field generated by the time reversal mirror focuses at a point of initial source inside of the medium. The size of the focusing spot is of the kind that it is smaller than the one that would be obtained if the medium were homogeneous meaning that the super resolution phenomenon is observed.

TIME REVERSAL OF ELECTROMAGNETIC WAVES IN RANDOMLY LAYERED MEDIA

by

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BIOGRAPHY. I graduated from Moscow State University in 1998. In 1997-2001 I worked part time as software engineer at Sukhoi Design Bureau and with a Maplesoft affiliated research group at Moscow University. In 2001 I entered the Graduate School at Department of Mathematics at North Carolina State University where I worked with Prof. Jean-Pierre Fouque on wave propagation in random media.

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. . . When I start describing the magnetic field moving through space, I speak of the $\boldsymbol{E}$- and $\boldsymbol{B}$ fields and wave my arms and you may imagine that I can see them. I'll tell you what I see. I see some kind of vague shadowy, wiggling lines – here and there is an E and B written on them somehow, and perhaps some of the lines have arrows on them – an arrow here or there which disappears when I look too closely at it . . . I cannot really make a picture that is even nearly like the true waves. So if you have some difficulty in making such a picture, you should not be worried that your difficulty is unusual. . .

R. Feynman, [2], v. 2, sec. 20-3

1 Introduction

Electromagnetic field and its dynamics is the driving force of a great number of natural phenomena. It is described by a system of partial differential equations, called Maxwell equations (published in 1873):

$$\nabla \times \boldsymbol{E} = -\mu \partial_t \boldsymbol{H} \tag{1}$$

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho \tag{2}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{(s)} + \sigma \boldsymbol{E} + \epsilon \partial_t \boldsymbol{E} \tag{3}$$

$$\nabla \cdot (\mu \boldsymbol{H}) = 0 \tag{4}$$

Evolution of electromagnetic field depends on the media which enters in the governing equations in terms of the coefficients ϵ(permittivity) and μ(permeability). In our case we will set conductivity σ to be zero. The electromagnetic field is a mean of transfer of information.

Also it can be used to obtain information about the medium. Propagation of electromagnetic waves in homogeneous media is well studied and exact methods of solution have been developed ([6]). On the other hand waves in inhomogeneous media are very complex because of multi scattering. Different approximations are used in order to obtain a solution. For example in radar imaging systems it is reasonable to use Born approximation for the scattered electromagnetic field, which basically keeps track only of one scattering event and if the target is compact in shape than multiple scattering does not contribute much to the resulting field. Born approximation briefly consists of the following equations:

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s \quad \boldsymbol{E}^i,\ \boldsymbol{E}^s \text{ are incident and scattered fields} \tag{5}$$

$$\left(\nabla^2 - \frac{1}{c(\boldsymbol{x})^2}\partial_t\right)\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{J}(t,\boldsymbol{x}) \tag{6}$$

$$\left(\nabla^2 - \frac{1}{c_0^2}\partial_t\right)\boldsymbol{E}^i(t,\boldsymbol{x}) = \boldsymbol{J}(t,\boldsymbol{x}) \tag{7}$$

$$\frac{1}{c(\boldsymbol{x})^2} = \frac{1}{c_0^2} + V(\boldsymbol{x}) \tag{8}$$

then

$$\left(\nabla^2 - \frac{1}{c_0^2}\partial_t^2\right)\boldsymbol{E}^s = V(\boldsymbol{x})\partial_t^2\boldsymbol{E} \tag{9}$$

whose solution can be written as

$$\boldsymbol{E}^s = \int g(t-\tau,\boldsymbol{x}-\boldsymbol{z})V(\boldsymbol{z})\partial_t^2\boldsymbol{E}\,d\tau d\boldsymbol{z} \tag{10}$$

where g is corresponding Green's function. Born approximation consists in using $\boldsymbol{E}^i$ instead of $\boldsymbol{E}$:

$$\boldsymbol{E}^s \simeq \boldsymbol{E}^s_B = \int g(t-\tau,\boldsymbol{x}-\boldsymbol{z})V(\boldsymbol{z})\partial_t^2\boldsymbol{E}^i\,d\tau d\boldsymbol{z} \tag{11}$$

It makes the problem linear in $V(\boldsymbol{x})$. This method is not a good choice when the medium has a lot of inhomogeneities since multiple scattering events become significant. When the medium is very complex and inhomogeneous, so that it is hard to solve the problem even numerically, it is a good idea to describe such a medium as random and then to try to obtain statistical properties of the field. Some quantities have a property of having point distributions which means that their value does not depend on the particular outcome of the random media. We refer to Ch. A of [1] for more specifics and an overview of other methods and approaches. We next describe some basic notions required for statistical description of waves in random media.

1.1 Markov processes

Markov process $Y(z)$ is a set of random variables taking values in an auxiliary space S such that the sigma-algebras (information) generated by $\{Y(s), s \geq z\}$ and $\{Y(s), s \leq z\}$ are independent given the value $Y(z)$. For the case of layered random medium we can say that "medium to the right of z is independent of medium to the left once we know the medium at the point z". As an example one can think of a stack of sheets each made of different kind of material (chosen randomly and independently) and having random thickness with exponential distribution. The fact that thickness distribution is exponential is important, it provides the Markov property for the process. If say all the sheets had the same thickness then such a process by itself would not be Markovian since by looking to the left we could find out the location of the last discontinuity and thus we could predict where the next discontinuity would take place. In this case knowing all the past and the present would not be the same as knowing just the present, meaning that Markov property would not be satisfied for this media. However, we could consider the process $(Y(z), \tau(z))$ where $\tau(z)$ denotes the distance from the previous jump. This process is Markovian: the present state contains all the information about the "past" to the left that is relevant for description of the "future" to the right. We next describe the notions of semi-group and infinitesimal generator of a Markov process. Let ϕ be a real-valued function on S. Then $Y(z)$ acts on ϕ in the following way, defining the operator P_s:

$$(P_s\phi)(x) = \mathbb{E}[\phi(Y(z+s))|Y(z) = x] \tag{12}$$

We have assumed here that the process $Y(z)$ is homogeneous: the conditional distribution function $F_{Y(z+s)}(\cdot|Y(z))$ of $Y(z+s)$ conditioned on $Y(z)$ does not depend on z but only on s. The family of operators P_s constitutes a semi-group:

$$P_{s+h} = P_s P_h \tag{13}$$

Indeed, taking conditional expectation wrt $Y(z+s)$ we obtain

$$\begin{aligned} P_{s+h}\phi(x) = \mathbb{E}[\phi(Y(z+s+h))|Y(z) = x] =& \mathbb{E}[\mathbb{E}[\phi(Y(z+s+h))|Y(z+s)]|Y(z) = x] \\ =& \mathbb{E}[P_h\phi(Y(z+s))|Y(z) = x] = P_s P_h \phi(x) \end{aligned} \tag{14}$$

The infinitesimal generator of the semi-group is defined by

$$\frac{dP_s}{ds} = \mathcal{L}P_s = P_s\mathcal{L} \tag{15}$$

and then P_s in terms of $\mathcal{L}$ is

$$P_s = e^{s\mathcal{L}} \tag{16}$$

1.2 Martingales, martingale problems and diffusion approximation theorem

A random process $M(z)$ is a martingale if the expectation of the value of the process at some point in the future given the past and the present is equal to the value of the process at the present: $\mathbb{E}[M(z+h)|M(z'), z' \leq z] = M(z)$. For a homogeneous Markov process $Y(z)$ with infinitesimal generator $\mathcal{L}$ the process $M(z)$

$$M(z) = \phi(Y(z)) - \phi(Y(0)) - \int_0^z \mathcal{L}\phi(Y(s))ds \tag{17}$$

is a martingale. On the other hand, if we are given an operator $\mathcal{L}$ and the process $M(z)$ defined above is a martingale under a probability measure $\mathbb{P}$ for any ϕ from a class of functions which is large enough then the process Y is a Markov process with infinitesimal generator $\mathcal{L}$. In our case some quantities describing wave propagation satisfy equations of the kind

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon}F\left(X^\varepsilon(z), Y\left(\frac{z}{\varepsilon^2}\right), \frac{z}{\varepsilon}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d. \tag{18}$$

The joint process $(X^\varepsilon, Y^\varepsilon, \tau)$ is Markovian with the generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}F(y, x, \tau) \cdot \nabla_x + \frac{1}{\varepsilon^2}\mathcal{L}_Y + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau} \tag{19}$$

where $\mathcal{L}_Y$ is the infinitesimal generator of Y. The theorem below shows that the processes X^ε themselves converge in distribution to a diffusion process and gives the expression of the limiting generator. The proof is based on construction of a set of test functions $\phi^\varepsilon(x, y, \tau)$ of particular kind and an operator $\mathcal{L}$ such that $\phi^\varepsilon(x, y, \tau) \to \phi(x)$, $\mathcal{L}^\varepsilon\phi^\varepsilon(x, y, \tau) \to \mathcal{L}\phi(x)$ when $\varepsilon \to 0$.

Diffusion Approximation Theorem (with fast phase) [1]. *Consider the system*

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon}F\left(X^\varepsilon(z), Y\left(\frac{z}{\varepsilon^2}\right), \frac{z}{\varepsilon}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d. \tag{20}$$

Assume that Y is a Markov, stationary, ergodic process on a compact space with generator $\mathcal{L}_Y$ satisfying the Fredholm alternative. $F(x, y, \tau)$ is smooth, periodic with respect to τ with period Z_0, has bounded partial derivatives in x and satisfies the centering condition $\mathbb{E}[F(x, Y(0))] = 0$ where $\mathbb{E}$ denotes the expectation with respect to the invariant probability measure of Y. Then the random processes $(X^\varepsilon(z))_{z\geq 0}$ converge in distribution to the Markov diffusion process X with generator:

$$\mathcal{L}f(x) = \frac{1}{Z_0}\int_0^{Z_0}\int_0^\infty du\mathbb{E}[F(x, Y(0), \tau) \cdot \nabla(F(x, Y(u), \tau) \cdot \nabla f(x))]d\tau. \tag{21}$$

In this thesis we deal with medium which is layered ($(\epsilon,\mu) = (\epsilon,\mu)(z)$) and random, i.e. the electromagnetic parameters of the medium are an outcome of some random process. The medium and the field length scales are related as ε^2 and ε meaning that the wavelengths we deal with are much larger than the characteristic size of inhomogeneities and at the same time are much smaller than length of wave propagation (when we pass to the limit $\varepsilon \to 0$). This particular choice of scales allows using the above theorem and thus computing statistics of quantities we are interested in. At the same time we should note that since the governing system is hyperbolic and we are interested in the field in finite time, the quantities of our interest do not depend on the medium which is far enough from the source and the mirror, and so the medium needs to be layered only in a certain volume around the mirror and the source. This fact may become important in applications.

1.3 Time Reversal

We track down the electric field in a special experiment which will be described now. The picture is shown on Fig. 1. The space $-L \leq z \leq 0$ is filled with random medium. The medium outside this slab is homogeneous. From now on small bold symbols as well as letters with subscript t represent vectors in transverse planes $z = const$, while z-components of vectors are in regular font. A point source located at $S = (\boldsymbol{x}_s, z_s)$ generates a current $(\boldsymbol{J}_{0,t}, J_{0,z})$ at time t_s. The electric field is then recorded at points of the mirror M, time reversed, and a current proportional to the result is generated at each point of the mirror. We analyze the resulting field. We show that the field focuses at the source point and we derive some approximations which indicate that in this way we obtain super resolution effect. In the following several chapters we provide the analysis of this problem. The acoustic case is treated in [1] and we follow the general pattern developed there. Historically, time reversal in

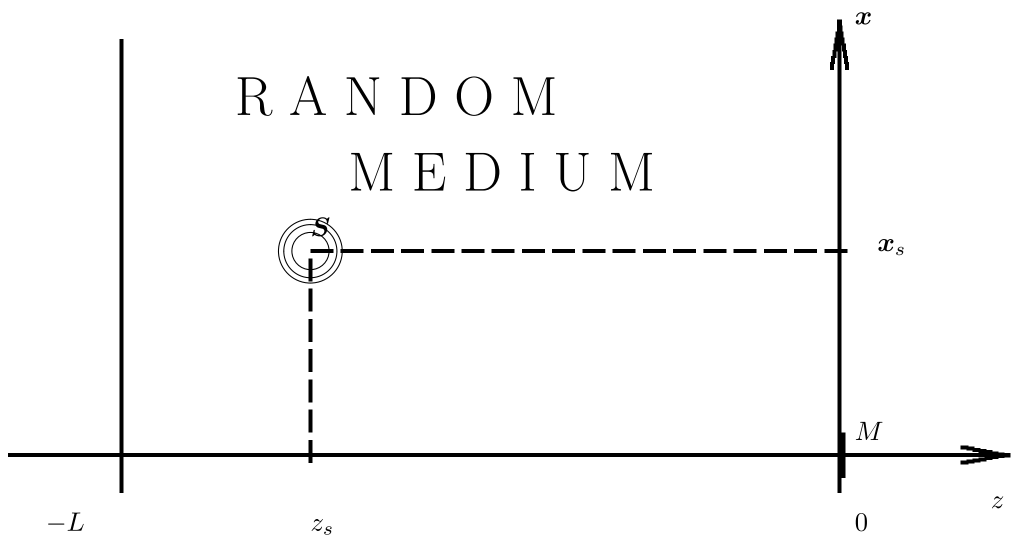

Figure 1: Time Reversal experiment setup

ultrasound acoustics has been experimentally investigated by M.Fink and his collaborators [7], and also by group of W.Kuperman [5].

2 Transformations of Maxwell equations

In this section we describe some preliminary transformations we apply to Maxwell equations.

2.1 Maxwell equations in Fourier domain

We perform a special form of Fourier transform which for the $\boldsymbol{E}$ vector is given by

$$\hat{\boldsymbol{E}}(\omega, \boldsymbol{\kappa}, z) = \int \boldsymbol{E}(\boldsymbol{x}, z, t) e^{\frac{i\omega}{\varepsilon}(t - \boldsymbol{\kappa}\cdot\boldsymbol{x})} dt d\boldsymbol{x} \tag{22}$$

Then Maxwell equations in Fourier domain are written as

$$\left(\frac{i\omega}{\varepsilon}\boldsymbol{\kappa} + \boldsymbol{z}_0 \partial_z\right) \times \hat{\boldsymbol{E}} = \frac{i\omega}{\varepsilon}\mu \hat{\boldsymbol{H}} \tag{23}$$

$$\left(\frac{i\omega}{\varepsilon}\boldsymbol{\kappa} + \boldsymbol{z}_0\partial_z\right) \cdot (\epsilon\hat{\boldsymbol{E}}) = \hat{\rho} \tag{24}$$

$$\left(\frac{i\omega}{\varepsilon}\boldsymbol{\kappa} + \boldsymbol{z}_0\partial_z\right) \times \hat{\boldsymbol{H}} = \hat{\boldsymbol{J}}^{(s)} + \sigma\hat{\boldsymbol{E}} - \frac{i\omega}{\varepsilon}\omega\epsilon\hat{\boldsymbol{E}} \tag{25}$$

$$\left(\frac{i\omega}{\varepsilon}\boldsymbol{\kappa} + \boldsymbol{z}_0\partial_z\right) \cdot (\mu\hat{\boldsymbol{H}}) = 0 \tag{26}$$

We then take dot products of these equation with $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^\perp$ where for any vector $\boldsymbol{w}$ we have defined $\boldsymbol{w}^\perp = \boldsymbol{w} \times \boldsymbol{z}_0$. We define ([3]) components as

$$\hat{E}_1 = \boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{E}}_t \tag{27}$$

$$\hat{E}_2 = \boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{E}}_t^\perp \tag{28}$$

$$\hat{H}_1 = \boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{H}}_t^\perp \tag{29}$$

$$\hat{H}_2 = -\boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{H}}_t \tag{30}$$

Then the original vector quantities can be expressed as

$$\hat{\boldsymbol{E}} = \hat{\boldsymbol{E}}_t + \boldsymbol{z}_0\hat{E}_z = \boldsymbol{\kappa}_0\hat{E}_1 - \boldsymbol{\kappa}_0^\perp\hat{E}_2 + \boldsymbol{z}_0\hat{E}_z \tag{31}$$

$$\hat{\boldsymbol{H}} = \hat{\boldsymbol{H}}_t + \boldsymbol{z}_0\hat{H}_z = -\boldsymbol{\kappa}_0\hat{H}_2 - \boldsymbol{\kappa}_0^\perp\hat{H}_1 + \boldsymbol{z}_0\hat{H}_z \tag{32}$$

From Maxwell equations it follows that

$$\hat{E}_z = -\frac{\boldsymbol{\kappa} \cdot \hat{\boldsymbol{H}}_t^\perp}{\epsilon} = -\frac{\kappa\hat{H}_1}{\epsilon} \tag{33}$$

2.2 Two systems and homogenization

From the system (23)-(26) by taking dot products with $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^\perp$ we can derive that the components defined above satisfy the following two systems of equations:

$$\frac{d\hat{E}_1}{dz} = \frac{i\omega}{\varepsilon}(\mu - \frac{\kappa^2}{\epsilon})\hat{H}_1 + \frac{\kappa}{\epsilon}\hat{J}_z e^{\frac{i\omega}{\varepsilon}(t_s - \boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}\delta(z - z_s) \tag{34}$$

$$\frac{d\hat{H}_1}{dz} = \frac{i\omega}{\varepsilon}\epsilon\hat{E}_1 - \boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{J}}_t e^{\frac{i\omega}{\varepsilon}(t_s - \boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}\delta(z - z_s) \tag{35}$$

and

$$\frac{d\hat{E}_2}{dz} = \frac{i\omega}{\varepsilon}\mu\hat{H}_2 \tag{36}$$

$$\frac{d\hat{H}_2}{dz} = \frac{i\omega}{\varepsilon}(\epsilon - \frac{\kappa^2}{\mu})\hat{E}_2 - \boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{J}}_t^{\perp} e^{\frac{i\omega}{\varepsilon}(t_s - \boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}\delta(z - z_s) \tag{37}$$

The medium is described by the parameters ϵ and μ which are random processes:

$$\epsilon = \bar{\epsilon}\left(1 + \eta\left(\frac{z}{\varepsilon^2}\right)\right) \tag{38}$$

$$\mu = \bar{\mu}\left(1 + \nu\left(\frac{z}{\varepsilon^2}\right)\right) \tag{39}$$

Here η and ν are some homogeneous Markov processes. Also, their inverses satisfy

$$\frac{1}{\epsilon} = \frac{1}{\bar{\epsilon}_1}\left(1 + \eta_1\left(\frac{z}{\varepsilon^2}\right)\right) \tag{40}$$

$$\frac{1}{\mu} = \frac{1}{\bar{\mu}_1}\left(1 + \nu_1\left(\frac{z}{\varepsilon^2}\right)\right) \tag{41}$$

We next apply homogenization techniques ([1]) to the systems above. We change variables by

$$\begin{bmatrix} \hat{E}_i \\ \hat{H}_i \end{bmatrix} = \begin{bmatrix} \xi_i^{1/2} e^{\frac{i\omega\lambda_i z}{\varepsilon}} & -\xi_i^{1/2} e^{-\frac{i\omega\lambda_i z}{\varepsilon}} \\ \xi_i^{-1/2} e^{\frac{i\omega\lambda_i z}{\varepsilon}} & \xi_i^{-1/2} e^{-\frac{i\omega\lambda_i z}{\varepsilon}} \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} \tag{42}$$

where

$$\lambda_1 = \lambda_1(\kappa) = \sqrt{\bar{\epsilon}(\bar{\mu} - \frac{\kappa^2}{\bar{\epsilon}_1})} \tag{43}$$

$$\lambda_2 = \lambda_2(\kappa) = \sqrt{\bar{\mu}(\bar{\epsilon} - \frac{\kappa^2}{\bar{\mu}_1})} \tag{44}$$

$$\xi_1 = \xi_1(\kappa) = \sqrt{\frac{\bar{\mu} - \kappa^2/\bar{\epsilon}_1}{\bar{\epsilon}}} = \frac{\lambda_1(\kappa)}{\bar{\epsilon}} \tag{45}$$

$$\xi_2 = \xi_2(\kappa) = \sqrt{\frac{\bar{\mu}}{\bar{\epsilon} - \kappa^2/\bar{\mu}_1}} = \frac{\bar{\mu}}{\lambda_2(\kappa)} \tag{46}$$

Applying the ansatz (42) into (34),(35) and (36),(37) we get the equations for A_i, B_i:

$$\frac{d}{dz}\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \frac{i\omega}{\varepsilon}\begin{bmatrix} m_i & n_i e^{-2i\omega\lambda_i z/\varepsilon} \\ -n_i e^{2i\omega\lambda_i z/\varepsilon} & -m_i \end{bmatrix}\begin{bmatrix} A_i \\ B_i \end{bmatrix} \tag{47}$$

where

$$m_1 = \frac{1}{2}\Big(\big(\bar{\mu}\nu - \frac{\kappa^2}{\bar{\epsilon}_1}\eta_1\big)\frac{1}{\xi_1} + \eta\bar{\epsilon}\xi_1\Big) \tag{48}$$

$$n_1 = \frac{1}{2}\Big(\big(\bar{\mu}\nu - \frac{\kappa^2}{\bar{\epsilon}_1}\eta_1\big)\frac{1}{\xi_1} - \eta\bar{\epsilon}\xi_1\Big) \tag{49}$$

$$m_2 = \frac{1}{2}\Big(\frac{1}{\xi_2}\bar{\mu}\nu + \xi_2\big(\bar{\epsilon}\eta - \frac{\kappa^2}{\bar{\mu}_1}\nu_1\big)\Big) \tag{50}$$

$$n_2 = \frac{1}{2}\Big(\frac{1}{\xi_2}\bar{\mu}\nu - \xi_2\big(\bar{\epsilon}\eta - \frac{\kappa^2}{\bar{\mu}_1}\nu_1\big)\Big) \tag{51}$$

The advantage of this form is that the expectation of the rhs is zero, which is a necessary condition for application of some other asymptotic methods.

We then introduce reflection coefficients $R_i = A_i/B_i$ which satisfy corresponding Riccati equations:

$$\frac{dR_i}{dz} = \frac{i\omega}{\varepsilon}\left(2m_i R_i + n_i\left(e^{-\frac{2i\omega\lambda_i z}{\varepsilon}} + R_i^2 e^{\frac{2i\omega\lambda_i z}{\varepsilon}}\right)\right) \tag{52}$$

$$\frac{dT_i}{dz} = \frac{i\omega}{\varepsilon}T_i\left(m_i + n_i e^{\frac{2i\omega\lambda_i z}{\varepsilon}}R_i\right) \tag{53}$$

The expectations of rhs of these equations are also zero.

3 Time Reversal

Briefly the time reversal (TR) experiment consists of the following two stages. At first a source inside of the inhomogeneous medium generates a current at some unknown time t_s and location $(\mathbf{x_s}, z_s)$. This in turn generates an electromagnetic field throughout the medium, and this field is recorded at points of a TR mirror. Then at each point of the mirror the signal is being time reversed and a source generates a current which is proportional to the time reversed signal. We want to analyze the field in the medium.

The jumps in the field components are

$$\left[\hat{E}_1\right]_{z_s} = \frac{\kappa}{\epsilon}\hat{J}_z(z_s)e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{54}$$

$$\left[\hat{H}_1\right]_{z_s} = -\kappa_0\cdot\hat{\boldsymbol{J}}_t(z_s)e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{55}$$

$$\left[\hat{E}_2\right]_{z_s} = 0 \tag{56}$$

$$\left[\hat{H}_2\right]_{z_s} = -\kappa_0\cdot\hat{\boldsymbol{J}}_t^{\perp}(z_s)e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{57}$$

The jumps for waves coefficients are then the following:

$$[A_i]_{z_s} = \frac{1}{2}(\xi_j^{-1/2}[\hat{E}_i]_{z_s} + \xi_j^{1/2}[\hat{H}_i]_{z_s})e^{-\frac{i\omega\lambda_i z_s}{\varepsilon}} \tag{58}$$

$$[B_i]_{z_s} = \frac{1}{2}(-\xi_j^{-1/2}[\hat{E}_i]_{z_s} + \xi_j^{1/2}[\hat{H}_i]_{z_s})e^{\frac{i\omega\lambda_i z_s}{\varepsilon}} \tag{59}$$

or in terms of the current

$$[A_1]_{z_s} = \frac{1}{2}(\xi_1^{-1/2}\frac{\kappa}{\epsilon(z_s)}\hat{J}_z(z_s) - \xi_1^{1/2}\kappa_0\cdot\hat{\boldsymbol{J}}_t(z_s))e^{-\frac{i\omega\lambda_1 z_s}{\varepsilon}}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{60}$$

$$[B_1]_{z_s} = \frac{1}{2}(-\xi_1^{-1/2}\frac{\kappa}{\epsilon(z_s)}\hat{J}_z(z_s) - \xi_1^{1/2}\kappa_0\cdot\hat{\boldsymbol{J}}_t(z_s))e^{\frac{i\omega\lambda_1 z_s}{\varepsilon}}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{61}$$

$$[A_2]_{z_s} = -\frac{1}{2}\xi_2^{1/2}\kappa_0\cdot\hat{\boldsymbol{J}}_t^{\perp}(z_s)e^{-\frac{i\omega\lambda_2 z_s}{\varepsilon}}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \tag{62}$$

$$\begin{aligned}[B_2]_{z_s} &= -\frac{1}{2}\xi_2^{1/2}\kappa_0\cdot\hat{\boldsymbol{J}}_t^{\perp}(z_s)e^{\frac{i\omega\lambda_2 z_s}{\varepsilon}}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)} \\ &= \frac{1}{2}\xi_2^{1/2}\kappa_0^{\perp}\cdot\hat{\boldsymbol{J}}_t(z_s)e^{\frac{i\omega\lambda_2 z_s}{\varepsilon}}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}\end{aligned} \tag{63}$$

A source at a point z_s creates a current which in turn creates a jump of the solution. We introduce the $P_i(a,b)$ – the propagator matrix which describes the flow given by (47).

$P_i(z_0, z)$ satisfies

$$\frac{\partial P_i}{\partial z} = \frac{i\omega}{\varepsilon} \begin{bmatrix} m_i & n_i e^{-2i\omega\lambda_i z/\varepsilon} \\ -n_i e^{2i\omega\lambda_i z/\varepsilon} & -m_i \end{bmatrix} P_i, \qquad P(z_0, z = z_0) = \mathrm{Id} \tag{64}$$

Using symmetries in the previous equation we can show that P_i has the form

$$P_i(z_0, z) = \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} (z, z_0) \tag{65}$$

where the column vector $(\alpha, \beta)^T$ solves (47) with the initial conditions

$$\alpha(z_0, z) = 1, \qquad \beta(z_0, z) = 0 \tag{66}$$

We employ propagator to get the effect of a source at the observation point by using the following equation:

$$P_i(z_s, 0) \left(P_i(-L, z_s) \begin{bmatrix} A_i(-L) \\ B_i(-L) \end{bmatrix} + J_{i,s} \right) = \begin{bmatrix} A_i(0) \\ B_i(0) \end{bmatrix} \tag{67}$$

where J_s is the jump at the source point. Different kinds of experiments provide us with some conditions at end points which are specific for those experiments, but P_i's are the same. These equations allow us to express observations in terms of the jump and P_i's. Next we compute the observations of the two stages of TR experiment.

3.1 Stage 1: Signal at the mirror

To compute the signal at the mirror we use the fact that we know part of the waves at the end points:

$$P_i(z_s, 0) \left(P_i(-L, z_s) \begin{bmatrix} 0 \\ B_i(-L) \end{bmatrix} + J_{i,s} \right) = \begin{bmatrix} A_i(0) \\ 0 \end{bmatrix} \tag{68}$$

This equation allows us to express $A_i(0), B_i(-L)$ ($A_i(0)$ is used to get the observation signal) in terms of the source and propagator matrices:

$$
\begin{aligned}
\begin{bmatrix} A_i(0) \\ B_i(-L) \end{bmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - P_i(z_s,0)P_i(-L,z_s) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} P_i(z_s,0)J_s \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - P_i(-L,0) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} P_i(z_s,0)J_s \\
&= \begin{bmatrix} \frac{\overline{\alpha_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} & -\frac{\overline{\beta_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} \\ -\frac{\beta_i(z,0)}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} & -\frac{\overline{\alpha_i(z,0)}}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} \end{bmatrix} J_s
\end{aligned}
\tag{69}
$$

This gives us the following expression for $A_i(0)$:

$$
A_i(0) = \begin{bmatrix} T_{g,i}(z_s) & -R_{g,i}(z_s) \end{bmatrix} J_s = T_{g,i}(z_s)[A_i]_{z_s} - R_{g,i}(z_s)[B_i]_{z_s} \tag{70}
$$

where

$$
T_{g,i}(z) = \frac{\overline{\alpha_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)} + \overline{\alpha_i(-L,z)}\ \overline{\alpha_i(z,0)}} \tag{71}
$$

$$
R_{g,i}(z) = \frac{\overline{\beta_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)} + \overline{\alpha_i(-L,z)}\ \overline{\alpha_i(z,0)}} \tag{72}
$$

$$
J_s = \begin{bmatrix} [A]_{z_s} \\ [B]_{z_s} \end{bmatrix} \tag{73}
$$

Next we compute the signal in time domain.

$$
\begin{aligned}
&E_z(t, z=0, \boldsymbol{x}) \\
&= -\frac{1}{(2\pi\varepsilon)^3} \int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})} \omega^2 \frac{\kappa \hat{H}_1(z=0,\omega,\boldsymbol{\kappa})}{\epsilon(z=0)} d\omega d\boldsymbol{\kappa} \\
&= -\frac{1}{(2\pi\varepsilon)^3} \int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})} \omega^2 \frac{\kappa \xi_1^{-1/2} A_1(z=0,\omega,\boldsymbol{\kappa})}{\epsilon(z=0)} d\omega d\boldsymbol{\kappa}
\end{aligned}
$$

$$= -\frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})}\omega^2\frac{\kappa\xi_1^{-1/2}}{\epsilon(z=0)}e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}$$

$$\times\left(T_{g,1}S_{a,1}(\omega,\boldsymbol{\kappa})e^{-\frac{i\omega\lambda_1 z_s}{\varepsilon}} - R_{g,1}S_{b,1}(\omega,\boldsymbol{\kappa})e^{\frac{i\omega\lambda_1 z_s}{\varepsilon}}\right)d\omega d\boldsymbol{\kappa} \tag{74}$$

$$\boldsymbol{E}_t(t,z=0,\boldsymbol{x})$$

$$= \frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})}\omega^2$$

$$\times\left(\boldsymbol{\kappa}_0\hat{E}_1(z=0,\omega,\boldsymbol{\kappa}) - \boldsymbol{\kappa}_0^{\perp}\hat{E}_2(z=0,\omega,\boldsymbol{\kappa})\right)d\omega d\boldsymbol{\kappa}$$

$$= \frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})}\omega^2$$

$$\times\left(\boldsymbol{\kappa}_0\xi_1^{1/2}A_1(z=0,\omega,\boldsymbol{\kappa}) - \boldsymbol{\kappa}_0^{\perp}\xi_2^{1/2}A_2(z=0,\omega,\boldsymbol{\kappa})\right)d\omega d\boldsymbol{\kappa}$$

$$= \frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})}\omega^2 e^{\frac{i\omega}{\varepsilon}(t_s-\boldsymbol{\kappa}\cdot\boldsymbol{x}_s)}$$

$$\times\left(\boldsymbol{\kappa}_0\xi_1^{1/2}\left(T_{g,1}S_{a,1}e^{-\frac{i\omega\lambda_1 z_s}{\varepsilon}} - R_{g,1}S_{b,1}e^{\frac{i\omega\lambda_1 z_s}{\varepsilon}}\right)\right.$$

$$\left.-\boldsymbol{\kappa}_0^{\perp}\xi_2^{1/2}\left(T_{g,2}S_{a,2}e^{-\frac{i\omega\lambda_2 z_s}{\varepsilon}} - R_{g,2}S_{b,2}e^{\frac{i\omega\lambda_2 z_s}{\varepsilon}}\right)\right)d\omega d\boldsymbol{\kappa} \tag{75}$$

where

$$S_{a,1}(\omega,\boldsymbol{\kappa}) = \frac{1}{2}(\xi_1^{-1/2}\frac{\kappa}{\epsilon(z_s)}J_z - \xi_1^{1/2}\boldsymbol{\kappa}_0\cdot\hat{\mathbf{J}}_t) \tag{76}$$

$$S_{b,1}(\omega,\boldsymbol{\kappa}) = \frac{1}{2}(-\xi_1^{-1/2}\frac{\kappa}{\epsilon(z_s)}J_z - \xi_1^{1/2}\boldsymbol{\kappa}_0\cdot\hat{\mathbf{J}}_t) \tag{77}$$

$$S_{a,2}(\omega,\boldsymbol{\kappa}) = -\frac{1}{2}\xi_2^{1/2}\kappa_0\cdot\hat{\mathbf{J}}_t^{\perp} \tag{78}$$

$$S_{b,2}(\omega,\boldsymbol{\kappa}) = -\frac{1}{2}\xi_2^{1/2}\kappa_0\cdot\hat{\mathbf{J}}_t^{\perp} \tag{79}$$

3.2 Stage 2: Time Reversal

We use the time reversed electric field from the first stage as a source for the current at the second stage:

$$\boldsymbol{J}_{TR}(\boldsymbol{x},z=0,t) = \boldsymbol{E}(\boldsymbol{x},z=0,-t)G_1(-t)G_2(\boldsymbol{x}) \tag{80}$$

where G_1 and G_2 are some appropriate window functions. Our next goal is to compute the field generated by this source. We first compute Fourier transform of the current:

$$
\begin{aligned}
&\hat{J}_{TR,z}(\omega,\boldsymbol{\kappa})\\
&= \int e^{\frac{i\omega}{\varepsilon}(t-\boldsymbol{\kappa}\cdot\boldsymbol{x})} E_z(-t,z=0,\boldsymbol{x}) G_1(t) G_2(\boldsymbol{x}) dt d\boldsymbol{x}\\
&= -\frac{1}{(2\pi\varepsilon)^3} \int \hat{G}_1\left(\frac{\omega+\omega'}{\varepsilon}\right) \hat{G}_2\left(\frac{\omega'\boldsymbol{\kappa}'-\omega\boldsymbol{\kappa}}{\varepsilon}\right) \omega'^2 \frac{\kappa'\xi_1^{-1/2}}{\epsilon(z=0)}\\
&\times e^{\frac{i\omega'}{\varepsilon}(t_s-\boldsymbol{\kappa}'\cdot\boldsymbol{x}_s)} \Big(T_{g,1}(z_s,\omega',\boldsymbol{\kappa}')\, S_{a,1}(z_s,\omega',\boldsymbol{\kappa}') e^{\frac{i\omega\lambda_1 z_s}{\varepsilon}}\\
&\qquad -R_{g,1}(z_s,\omega',\boldsymbol{\kappa}')\, S_{b,1}(z_s,\omega',\boldsymbol{\kappa}') e^{-\frac{i\omega\lambda_1 z_s}{\varepsilon}}\Big) d\omega' d\boldsymbol{\kappa}'
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
&\hat{\boldsymbol{J}}_{TR,t}(\omega,\boldsymbol{\kappa})\\
&= \frac{1}{(2\pi\varepsilon)^3} \int \hat{G}_1\left(\frac{\omega+\omega'}{\varepsilon}\right) \hat{G}_2\left(\frac{\omega'\boldsymbol{\kappa}'-\omega\boldsymbol{\kappa}}{\varepsilon}\right) \omega'^2 e^{\frac{i\omega'}{\varepsilon}(t_s-\boldsymbol{\kappa}'\cdot\boldsymbol{x}_s)}\\
&\times \Big(\boldsymbol{\kappa}_0'\xi_1^{1/2}\Big[T_{g,1}(z_s,\omega',\boldsymbol{\kappa}')\, S_{a,1}(z_s,\omega',\boldsymbol{\kappa}')\, e^{\frac{i\omega\lambda_1 z_s}{\varepsilon}}\\
&\qquad\qquad -R_{g,1}(z_s,\omega',\boldsymbol{\kappa}')\, S_{b,1}(z_s,\omega',\boldsymbol{\kappa}')\, e^{-\frac{i\omega\lambda_1 z_s}{\varepsilon}}\Big]\\
&- \boldsymbol{\kappa}'^{\perp}_0\xi_2^{1/2}\Big[T_{g,2}(z_s,\omega',\boldsymbol{\kappa}')\, S_{a,2}(z_s,\omega',\boldsymbol{\kappa}')\, e^{\frac{i\omega\lambda_2 z_s}{\varepsilon}}\\
&\qquad\qquad -R_{g,2}(z_s,\omega',\boldsymbol{\kappa}')\, S_{b,2}(z_s,\omega',\boldsymbol{\kappa}')\, e^{-\frac{i\omega\lambda_2 z_s}{\varepsilon}}\Big]\Big) d\omega' d\boldsymbol{\kappa}'
\end{aligned}
\tag{82}
$$

We next compute the field generated by this current inside of the medium at an arbitrary depth z. First we find the waves at the end points from the equation

$$
P_i(-L,0)\begin{bmatrix} 0 \\ B_{i,TR}(-L) \end{bmatrix} + J_{i,TR} = \begin{bmatrix} A_{i,TR}(0) \\ 0 \end{bmatrix} \tag{83}
$$

$$
\begin{bmatrix} A_{i,TR}(0) \\ B_{i,TR}(-L) \end{bmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - P_i(-L,0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]^{-1} J_{TR} \tag{84}
$$

This gives the field in the medium:

$$\begin{bmatrix} A_{i,TR}(z) \\ B_{i,TR}(z) \end{bmatrix} = P_i(-L,z) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} A_{i,TR}(0) \\ B_{i,TR}(-L) \end{bmatrix}$$

$$= P_i(-L,z) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - P_i(-L,0) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} J_{TR} \tag{85}$$

$$= \begin{bmatrix} 0 & -\frac{\overline{\beta_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} \\ 0 & -\frac{\overline{\alpha_i(-L,z)}}{\beta_i(z,0)\overline{\beta_i(-L,z)}+\overline{\alpha_i(z,0)\alpha_i(-L,z)}} \end{bmatrix} J_{TR} \tag{86}$$

$$J_{TR} = \begin{bmatrix} [A_{i,TR}]_0 \\ [B_{i,TR}]_0 \end{bmatrix} \tag{87}$$

It happens that here we have the same expressions as in Stage 1:

$$\begin{bmatrix} A_{i,TR}(z) \\ B_{i,TR}(z) \end{bmatrix} = -\left[B_{i,TR}\right]_0 \begin{bmatrix} R_{g,i}(z) \\ T_{g,i}(z) \end{bmatrix} \tag{88}$$

The jumps $\left[B_{i,TR}\right]_0$ are given by the time reversal current (81) and (82) using (61) and (63) (we don't plug it in yet):

$$\left[B_{1,TR}\right]_{z_s=0} = \frac{1}{2}(-\xi_1^{-1/2}\frac{\kappa}{\epsilon}\hat{J}_{TR,z} - \xi_1^{1/2}\boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{J}}_{TR,t}) \tag{89}$$

$$\left[B_{2,TR}\right]_{z_s=0} = -\frac{1}{2}\xi_2^{1/2}\boldsymbol{\kappa}_0 \cdot \hat{\boldsymbol{J}}_{TR,t}^{\perp} = \frac{1}{2}\xi_2^{1/2}{\boldsymbol{\kappa}_0}^{\perp} \cdot \hat{\boldsymbol{J}}_{TR,t} \tag{90}$$

In time domain we get

$$\begin{aligned} \boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) =& \frac{1}{(2\pi\varepsilon)^3} \int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}\omega_1^2 \left(\boldsymbol{\kappa}_{1,0}\hat{E}_1(z,\omega_1,\boldsymbol{\kappa}_1) - \boldsymbol{\kappa}_{1,0}^{\perp}\hat{E}_2(z,\omega_1,\boldsymbol{\kappa}_1)\right) d\omega_1 d\boldsymbol{\kappa}_1 \\ =& \frac{1}{(2\pi\varepsilon)^3} \int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}\omega_1^2 \left(\boldsymbol{\kappa}_{1,0} \left(\xi_1^{1/2}A_1(z,\omega_1,\boldsymbol{\kappa}_1)e^{-\frac{i\omega_1\lambda_1 z}{\varepsilon}} - \xi_1^{1/2}B_1(z,\omega_1,\boldsymbol{\kappa}_1)e^{\frac{i\omega_1\lambda_1 z}{\varepsilon}}\right)\right. \\ & \left. -\boldsymbol{\kappa}_{1,0}^{\perp}\left(\xi_2^{1/2}A_2(z,\omega_1,\boldsymbol{\kappa}_1)e^{-\frac{i\omega_1\lambda_2 z}{\varepsilon}} - \xi_2^{1/2}B_2(z,\omega_1,\boldsymbol{\kappa}_1)e^{\frac{i\omega_1\lambda_2 z}{\varepsilon}}\right)\right) d\omega_1 d\boldsymbol{\kappa}_1 \\ =& \text{continued as equation (245)} \end{aligned} \tag{91}$$

$$
\begin{aligned}
E_{TR,z}(t,z,\boldsymbol{x}) =& -\frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}\omega_1^2\frac{\kappa_1\hat{H}_{1,TR}(z,\omega_1,\kappa_1)}{\epsilon(z)}d\omega_1 d\boldsymbol{\kappa}_1\\
=& -\frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}\omega_1^2\\
&\times\frac{\kappa_1\left(\xi_1^{-1/2}A_{1,TR}(z,\omega_1,\kappa_1)e^{-\frac{i\omega_1\lambda_1 z}{\varepsilon}}+\xi_1^{1/2}B_{1,TR}(z,\omega_1,\kappa_1)e^{\frac{i\omega_1\lambda_1 z}{\varepsilon}}\right)}{\epsilon(z)}d\omega_1 d\boldsymbol{\kappa}_1\\
=& -\frac{1}{(2\pi\varepsilon)^3}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}\omega_1^2\left[B_{1,TR}\right]_0\\
&\times\frac{\kappa_1\left(\xi_1^{-1/2}T_{g,1}(z,\omega_1,\kappa_1)e^{-\frac{i\omega_1\lambda_1 z}{\varepsilon}}+\xi_1^{1/2}R_{g,1}(z,\omega_1,\kappa_1)e^{\frac{i\omega_1\lambda_1 z}{\varepsilon}}\right)}{\epsilon(z)}d\omega_1 d\boldsymbol{\kappa}_1\\
=&\text{continued as equation (246)}
\end{aligned} \tag{92}
$$

Here the window functions Fourier transforms are defined as

$$\hat{G}_1(\omega) = \int G_1(t)e^{i\omega t}dt \tag{93}$$

$$\hat{G}_2(k) = \int G_2(x)e^{-ik\cdot x}dx \tag{94}$$

Changing ω_2 to $-\omega_2$ we get equations (247) and (248).

3.3 Zoom in with ω's and $\boldsymbol{\kappa}$'s

Window terms decay when their arguments go to infinity (which happens when $\varepsilon \to 0$) so we make change of variables:

$$\omega_1 = \omega + \varepsilon h/2 \tag{95}$$

$$\omega_2 = \omega - \varepsilon h/2 \tag{96}$$

$$\boldsymbol{\kappa}_1 = \boldsymbol{\kappa} + \varepsilon \boldsymbol{l}/2 \tag{97}$$

$$\boldsymbol{\kappa}_2 = -\boldsymbol{\kappa} + \varepsilon \boldsymbol{l}/2 \tag{98}$$

This gives equations (249) and (250).

3.4 $\varepsilon \to 0$: the leading order

Since there are terms with nonzero limit as $\varepsilon \to 0$ inside of the integrals, we can cancel terms of the order ε, for example those having dot product of two almost perpendicular vectors:

$$\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0^{\perp} \cdot \left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0 = O(\varepsilon) \tag{99}$$

Next step we make is we use the following Taylor expansion of the exponents:

$$\lambda_i\left(|\boldsymbol{\kappa} + \delta\boldsymbol{\kappa}|\right) = \lambda_i\left(|\boldsymbol{\kappa}|\right) + \frac{d\lambda_i}{d\kappa}\boldsymbol{\kappa}_0 \cdot \delta\boldsymbol{\kappa} \tag{100}$$

For the derivatives of the λ's we have

$$\frac{d\lambda_1(\kappa)}{d\kappa} = -\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)} \tag{101}$$

$$\frac{d\lambda_2(\kappa)}{d\kappa} = -\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\kappa}{\lambda_2(\kappa)} \tag{102}$$

This gives for example

$$e^{-\frac{i\omega}{\varepsilon}\left(\lambda_1(|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}|)z - \lambda_1(|-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}|)z_s\right)} = e^{-\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z-z_s)} e^{\frac{i\omega}{2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)}(1+O(\varepsilon)) \tag{103}$$

where $\kappa = |\boldsymbol{\kappa}|$. As result we get (251) and (252). We have canceled $O(\varepsilon)$ terms, and the ones we had left are of lower order.

We next consider the case of homogeneous medium.

4 Homogeneous medium

In homogeneous medium ($\epsilon(z) = \bar{\epsilon} = \bar{\epsilon}_1,\ \mu(z) = \bar{\mu} = \bar{\mu}_1$) we have $T_{g,i} = 1$ and $R_{g,i} = 0$. It also implies that $\lambda_1(\kappa) = \lambda_2(\kappa)$.

4.1 Recorded field

We first compute the field at the first stage of the TR experiment.

$$E_z(t,z=0,\boldsymbol{x}) = -\frac{1}{(2\pi\varepsilon)^3}\int \omega^2 \frac{\kappa\xi_1^{-1/2}}{\epsilon(z=0)} S_{a,1}(\omega,\boldsymbol{\kappa}) e^{\frac{i\omega}{\varepsilon}(t_s - t + \boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s) - \lambda_1(\kappa)z_s)} d\omega d\boldsymbol{\kappa} \tag{104}$$

$$\begin{aligned}\boldsymbol{E}_t(t,z=0,\boldsymbol{x}) = \frac{1}{(2\pi\varepsilon)^3}\int \omega^2 &\Big(\boldsymbol{\kappa}_0 \xi_1^{1/2} S_{a,1} e^{\frac{i\omega}{\varepsilon}(t_s - t + \boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s) - \lambda_1(\kappa)z_s)} \\ &- \boldsymbol{\kappa}_0^{\perp} \xi_2^{1/2} S_{a,2} e^{\frac{i\omega}{\varepsilon}(t_s - t + \boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s) - \lambda_2(\kappa)z_s)}\Big) d\omega d\boldsymbol{\kappa}\end{aligned} \tag{105}$$

We now apply stationary phase approximation: for E_z and the first term in the expression for $\boldsymbol{E}_t$ the phase is

$$\phi = \frac{i\omega}{\varepsilon}(t_s - t + \boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s) - \lambda_1(\kappa)z_s) \tag{106}$$

We first compute the fast phase approximation wrt $\boldsymbol{\kappa}$ for a fixed ω. The stationary point is

$$\boldsymbol{\kappa}_s = (\boldsymbol{x}-\boldsymbol{x}_s)\frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{\sqrt{z_s^2 + |\boldsymbol{x}-\boldsymbol{x}_s|^2}} = (\boldsymbol{x}-\boldsymbol{x}_s)\frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{SM} \tag{107}$$

and is independent of ω.

To compute the approximation we need the determinant of Hessian of ϕ wrt $\boldsymbol{\kappa}$:

$$\begin{aligned}\det H\phi &= \det\left(\omega z_s \frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{1}{\lambda_1(\kappa)}\begin{bmatrix} 1 + \frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{1}{\lambda_1(\kappa)^2}\kappa_1^2 & \frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{1}{\lambda_1(\kappa)^2}\kappa_1\kappa_2 \\ \frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{1}{\lambda_1(\kappa)^2}\kappa_1\kappa_2 & 1 + \frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{1}{\lambda_1(\kappa)^2}\kappa_2^2 \end{bmatrix}\right) \\ &= \omega^2 z_s^2 \frac{\bar{\epsilon}^3\bar{\mu}}{\bar{\epsilon}_1^2\lambda_1(\kappa)^4} = \omega^2 z_s^2 \frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa)^4}\end{aligned} \tag{108}$$

All the eigenvalues of the Hessian matrix are positive.

The stationary point is the same for both fast phase terms in the $\boldsymbol{E}_t$ integrals and we compute

$$\lambda_1(\kappa_s) = \lambda_2(\kappa_s) = -\frac{z_s\sqrt{\bar{\epsilon}\bar{\mu}}}{SM} \tag{109}$$

$$\xi_1(\kappa_s) = -\frac{z_s}{SM}\sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}} \tag{110}$$

$$\xi_2(\kappa_s) = -\frac{SM}{z_s}\sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}} \tag{111}$$

The value of the phase at κ_s is

$$\phi(\boldsymbol{\kappa}_s) = \frac{i\omega}{\varepsilon}\left(t_s - t + |\boldsymbol{x}-\boldsymbol{x}_s|^2\frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{SM} + z_s^2\frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{SM}\right) = \frac{i\omega}{\varepsilon}\left(t_s - t + SM\sqrt{\bar{\mu}\bar{\epsilon}}\right) \tag{112}$$

So we have the following approximation:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int \kappa\xi_1(\boldsymbol{\kappa})^{-1/2}S_{a,1}(\omega,\boldsymbol{\kappa})e^{\frac{i\omega}{\varepsilon}(t_s-t+\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s)-\lambda_1(\kappa)z_s)}d\boldsymbol{\kappa}\right)e^{-\frac{i\omega}{\varepsilon}(t_s-t+SM\sqrt{\bar{\mu}\bar{\epsilon}})}\\
&=\frac{2\pi}{-\omega z_s\frac{\sqrt{\bar{\epsilon}\bar{\mu}}}{\lambda_1(\kappa_s)^2}}e^{i(2\cdot 2-2)\frac{\pi}{4}}\kappa_s\xi_1(\kappa_s)^{-1/2}\frac{1}{2}\left(\xi_1(\kappa_s)^{-1/2}\frac{\kappa_s}{\epsilon(z_s)}\hat{J}_z - \xi_1(\kappa_s)^{1/2}\frac{\boldsymbol{\kappa}_s}{\kappa_s}\cdot\hat{\boldsymbol{J}}_t\right)\\
&=\frac{i\pi\bar{\mu}\bar{\epsilon}}{\omega\ SM^3}\left(\frac{\bar{\epsilon}|\boldsymbol{x}-\boldsymbol{x}_s|^2\hat{J}_z}{\epsilon(z_s)} + z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t\right)
\end{aligned}\tag{113}$$

Similarly

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int \frac{\boldsymbol{\kappa}}{\kappa}\xi_1(\boldsymbol{\kappa})^{1/2}S_{a,1}(\omega,\boldsymbol{\kappa})e^{\frac{i\omega}{\varepsilon}(t_s-t+\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s)-\lambda_1(\kappa)z_s)}d\boldsymbol{\kappa}\right)e^{-\frac{i\omega}{\varepsilon}(t_s-t+SM\sqrt{\bar{\mu}\bar{\epsilon}})}\\
&=\frac{2\pi}{-\omega z_s\frac{\sqrt{\bar{\epsilon}\bar{\mu}}}{\lambda_1(\kappa_s)^2}}e^{i(2\cdot 2-2)\frac{\pi}{4}}\frac{\boldsymbol{\kappa}_s}{\kappa_s}\xi_1(\kappa_s)^{1/2}\frac{1}{2}\left(\xi_1(\kappa_s)^{-1/2}\frac{\kappa_s}{\epsilon(z_s)}\hat{J}_z - \xi_1(\kappa_s)^{1/2}\frac{\boldsymbol{\kappa}_s}{\kappa_s}\cdot\hat{\boldsymbol{J}}_t\right)\\
&=-\frac{i\pi\bar{\mu}z_s}{\omega\ SM^3}\left(\frac{\bar{\epsilon}\hat{J}_z}{\epsilon(z_s)} + \frac{z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t}{|\boldsymbol{x}-\boldsymbol{x}_s|^2}\right)(\boldsymbol{x}-\boldsymbol{x}_s)
\end{aligned}\tag{114}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int \frac{\boldsymbol{\kappa}^\perp}{\kappa}\xi_2(\boldsymbol{\kappa})^{1/2}S_{a,2}(\omega,\boldsymbol{\kappa})e^{\frac{i\omega}{\varepsilon}(t_s-t+\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{x}_s)-\lambda_2(\kappa)z_s)}d\boldsymbol{\kappa}\right)e^{-\frac{i\omega}{\varepsilon}(t_s-t+SM\sqrt{\bar{\mu}\bar{\epsilon}})}\\
&=\frac{2\pi}{-\omega z_s\frac{\sqrt{\bar{\epsilon}\bar{\mu}}}{\lambda_2(\kappa_s)^2}}e^{i(2\cdot 2-2)\frac{\pi}{4}}\frac{\boldsymbol{\kappa}_s^\perp}{\kappa_s}\xi_2(\kappa_s)^{1/2}\left(-\frac{1}{2}\xi_2(\kappa_s)^{1/2}\frac{\boldsymbol{\kappa}_s\cdot\hat{\boldsymbol{J}}_t^\perp}{\kappa_s}\right)\\
&=-\frac{i\pi\bar{\mu}(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t^\perp}{\omega\ SM|\boldsymbol{x}-\boldsymbol{x}_s|^2}(\boldsymbol{x}-\boldsymbol{x}_s)^\perp
\end{aligned}\tag{115}$$

Plugging in we get

$$E_z(t, z=0, \boldsymbol{x})$$

$$= -\frac{\pi\,\bar{\mu}\,\varepsilon}{(2\pi\varepsilon)^3\,SM^3}\int i\omega\left(|\boldsymbol{x}-\boldsymbol{x}_s|^2\hat{J}_z + z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t\right)e^{\frac{i\omega}{\varepsilon}(t_s-t+SM\sqrt{\bar{\mu}\bar{\epsilon}})}d\omega \tag{116}$$

$$\begin{aligned}&\boldsymbol{E}_t(t,z=0,\boldsymbol{x})\\&=\frac{\pi\,\bar{\mu}\,\varepsilon}{(2\pi\varepsilon)^3\,SM}\int i\omega\left[-\frac{z_s}{SM^2}\left(\hat{J}_z+\frac{z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t}{|\boldsymbol{x}-\boldsymbol{x}_s|^2}\right)(\boldsymbol{x}-\boldsymbol{x}_s)\right.\\&\qquad\left.+\frac{(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\hat{\boldsymbol{J}}_t^{\perp}}{|\boldsymbol{x}-\boldsymbol{x}_s|^2}(\boldsymbol{x}-\boldsymbol{x}_s)^{\perp}\right]e^{\frac{i\omega}{\varepsilon}(t_s-t+SM\sqrt{\bar{\mu}\bar{\epsilon}})}d\omega\end{aligned} \tag{117}$$

We can cancel the remaining fast phase by picking $t = t_s + SM\sqrt{\bar{\mu}\bar{\epsilon}} + \varepsilon T$. We assume that the source current is of the form

$$\boldsymbol{J} = \varepsilon^2\begin{bmatrix}\boldsymbol{f}_t\\ f_z\end{bmatrix}\left(\frac{t-t_s}{\varepsilon}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}_s\right)\delta(z-z_s) \tag{118}$$

and so

$$\hat{\boldsymbol{J}} = \varepsilon^3\begin{bmatrix}\hat{\boldsymbol{f}}_t\\ \hat{f}_z\end{bmatrix}(\omega)\delta(z-z_s)\boldsymbol{j} \tag{119}$$

We get

$$E_z(t_s+SM\sqrt{\bar{\mu}\bar{\epsilon}}+\varepsilon T, z=0,\boldsymbol{x}) = -\frac{\pi\,\bar{\mu}\,\varepsilon}{(2\pi)^3\,SM^3}\left(|\boldsymbol{x}-\boldsymbol{x}_s|^2 f_z'(T) + z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{f}_t'(T)\right) \tag{120}$$

$$\begin{aligned}\boldsymbol{E}_t(t_s+SM\sqrt{\bar{\mu}\bar{\epsilon}}+\varepsilon T, z=0,\boldsymbol{x}) =&\frac{\pi\,\bar{\mu}\,\varepsilon}{(2\pi)^3\,SM}\left[-\frac{z_s}{SM^2}\left(f_z'(T)+\frac{z_s(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{f}_t'(T)}{|\boldsymbol{x}-\boldsymbol{x}_s|^2}\right)(\boldsymbol{x}-\boldsymbol{x}_s)\right.\\&\left.+\frac{(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{f}_t'(T)^{\perp}}{|\boldsymbol{x}-\boldsymbol{x}_s|^2}(\boldsymbol{x}-\boldsymbol{x}_s)^{\perp}\right]\end{aligned} \tag{121}$$

As a check, we must be able to obtain a vector expression for $\boldsymbol{E}$, meaning it may only depend on the $\boldsymbol{SM}$ vector and its orientation wrt $\boldsymbol{f}$ (free space). Indeed, we can compute that the quantities above give for $\boldsymbol{E}$

$$\boldsymbol{E}(t_s + |\boldsymbol{x} - \boldsymbol{x}_s|^2 \frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{SM} + \varepsilon T, z = 0, \boldsymbol{x}) = \frac{\pi\, \bar{\mu}\, \varepsilon}{(2\pi)^3\, SM} \left(-\boldsymbol{f}'(T) + \boldsymbol{SM}_0 \left(\boldsymbol{f}'(T) \cdot \boldsymbol{SM}_0\right)\right) \tag{122}$$

4.2 TR field

TR field is given by

$$E_{TR,z}(t, z, \boldsymbol{x}) = -\frac{1}{2\, \bar{\epsilon}\, (2\pi)^6 \varepsilon^3} \int \overline{\hat{G}_1\,(h)} \hat{G}_2\,(h\boldsymbol{\kappa} + \omega \boldsymbol{l}) \omega^4 \left\{ \kappa \sqrt{\xi_1(\kappa)} + \frac{\kappa^3 \xi_1(\kappa)^{-3/2}}{\bar{\epsilon}^2} \right\}$$
$$e^{\frac{ih}{2}(t_s - t + (\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_1(\kappa)(z+z_s)) + \frac{i\omega}{2}\left(\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s) + \boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s)\right)}$$
$$e^{\frac{i\omega}{\varepsilon}(-(t+t_s) + (\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_1(\kappa)(z-z_s))} \overline{S_{a,1}\,(z_s, \omega, -\boldsymbol{\kappa})} d\boldsymbol{\kappa} d\boldsymbol{l} d\omega dh \tag{123}$$

$$\boldsymbol{E}_{TR,t}(t, z, \boldsymbol{x}) = \frac{1}{2} \frac{1}{(2\pi)^6 \varepsilon^3} \int \overline{\hat{G}_1\,(h)} \hat{G}_2\,(h\boldsymbol{\kappa} + \omega \boldsymbol{l}) \omega^4$$
$$\Bigg[-\left\{ \frac{\kappa^2}{\sqrt{\xi_1\,(\kappa)}\bar{\epsilon}^2} + \xi_1\,(\kappa)^{3/2} \right\}$$
$$\times\, e^{\frac{ih}{2}(t_s - t + (\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_1(\kappa)(z+z_s)) + \frac{i\omega}{2}\left(\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s) + \frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)\right)}$$
$$\times\, e^{\frac{i\omega}{\varepsilon}(-(t+t_s) + (\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_1(\kappa)(z-z_s))} \overline{S_{a,1}\,(z_s, \omega, -\boldsymbol{\kappa})} \boldsymbol{\kappa}_0$$
$$-\, \xi_2\,(\kappa)^{3/2}\, e^{\frac{ih}{2}(t_s - t + (\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_2(\kappa)(z+z_s)) + \frac{i\omega}{2}\left(\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s) + \frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}(z+z_s)\right)}$$
$$\times\, e^{\frac{i\omega}{\varepsilon}(-(t+t_s) + (\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa} - \lambda_2(\kappa)(z-z_s))} \overline{S_{a,2}\,(z_s, \omega, -\boldsymbol{\kappa})} \boldsymbol{\kappa}_0^{\perp} \Bigg]$$
$$d\boldsymbol{\kappa} d\boldsymbol{l} d\omega dh \tag{124}$$

For general and fixed t, x and z both fast phase terms have the same stationary point

$$\boldsymbol{\kappa}_s = \pm(\boldsymbol{x} - \boldsymbol{x}_s) \frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{\sqrt{(z - z_s)^2 + |\boldsymbol{x} - \boldsymbol{x}_s|^2}} = \pm(\boldsymbol{x} - \boldsymbol{x}_s) \frac{\sqrt{\bar{\mu}\bar{\epsilon}}}{SM} \tag{125}$$

and if we pick the particular $t = -t_s + \varepsilon T$, $\boldsymbol{x} = \boldsymbol{x}_s + \varepsilon \boldsymbol{X}$ and $z = z_s + \varepsilon Z$ we can cancel the fast phase at all. Now we need to show that this gives us *focusing* - meaning that the amplitudes are of order one. We compute the vector quantity $\boldsymbol{E}_{TR}(t,z,\boldsymbol{x})$ as

$$\begin{aligned}\boldsymbol{E}_{TR}(t,z,\boldsymbol{x}) =& \frac{1}{2(2\pi)^6\varepsilon^3}\int \overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)Z)}e^{ih(t_s+\boldsymbol{x}_s\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)z_s)+i\omega\left(\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s+\boldsymbol{l}\cdot\boldsymbol{x}_s\right)}\omega^4\\ &\left[-\frac{1}{\bar{\epsilon}}\left\{\kappa\sqrt{\xi_1(\kappa)}+\frac{\kappa^3\xi_1(\kappa)^{-3/2}}{\bar{\epsilon}^2}\right\}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{z}_0-\left\{\frac{\kappa^2}{\sqrt{\xi_1\left(\kappa\right)}\bar{\epsilon}^2}+\xi_1\left(\kappa\right)^{3/2}\right\}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{\kappa}_0\right.\\ &\left.-\,\xi_2\left(\kappa\right)^{3/2}\overline{S_{a,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{\kappa}_0^{\perp}\right]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh\end{aligned}$$

After change of variables $\boldsymbol{l}\mapsto\boldsymbol{k}=h\boldsymbol{\kappa}+\omega\boldsymbol{l}$ and integration wrt h and $\boldsymbol{k}$ we get

$$\begin{aligned}\boldsymbol{E}_{TR}(t,z,\boldsymbol{x}) =& \frac{1}{2(2\pi)^3\varepsilon^3}\int G_1\left(t_s-z_s\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa)}\right)G_2\left(\boldsymbol{x}_s+\boldsymbol{\kappa}\frac{z_s}{\lambda_1(\kappa)}\right)e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)Z)}\omega^2\\ &\left[-\frac{1}{\bar{\epsilon}}\left\{\kappa\sqrt{\xi_1(\kappa)}+\frac{\kappa^3\xi_1(\kappa)^{-3/2}}{\bar{\epsilon}^2}\right\}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{z}_0-\left\{\frac{\kappa^2}{\sqrt{\xi_1\left(\kappa\right)}\bar{\epsilon}^2}+\xi_1\left(\kappa\right)^{3/2}\right\}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{\kappa}_0\right.\\ &\left.-\,\xi_2\left(\kappa\right)^{3/2}\overline{S_{a,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\boldsymbol{\kappa}_0^{\perp}\right]d\boldsymbol{\kappa}d\omega \qquad (126)\end{aligned}$$

The domain for the $\boldsymbol{\kappa}$ integration is restricted to the propagating modes only. Change of variable $\boldsymbol{\kappa}\mapsto\boldsymbol{y}=\boldsymbol{x}_s+\boldsymbol{\kappa}\frac{z_s}{\lambda_1(\kappa)}$ gives

$$\begin{aligned}\boldsymbol{E}_{TR}(t,z,\boldsymbol{x}) =& -\frac{\bar{\mu}^2}{2(2\pi)^3\varepsilon^3}\int G_1\left(t_s+\frac{SM}{\bar{c}}\right)G_2\left(\boldsymbol{y}\right)e^{i\omega\left(-T-\frac{\boldsymbol{SM}\cdot(\boldsymbol{X},Z)}{\bar{c}SM}\right)}\omega^2\\ &\times\frac{1}{SM^2}\left[\overline{\hat{\boldsymbol{J}}^s(\omega)}-\frac{\left(\overline{\hat{\boldsymbol{J}}^s(\omega)}\cdot\boldsymbol{SM}\right)\boldsymbol{SM}}{SM^2}\right]d\boldsymbol{y}d\omega \qquad (127)\end{aligned}$$

where $\boldsymbol{SM}(\boldsymbol{y})=(\boldsymbol{y}-\boldsymbol{x}_s,-z_s)$ is the vector from the source $(\boldsymbol{x}_s,z_s)$ to the mirror point $(\boldsymbol{y},0)$. Now we look at the far field of a sinusoidal waveform (following [1]):

$$\boldsymbol{f}=\boldsymbol{f}_0\left(\frac{t}{T_w}\right)e^{i\omega_0 t}+c.c. \qquad (128)$$

$$G_2(\boldsymbol{y})=g_2\left(\frac{\boldsymbol{y}}{a}\right) \qquad (129)$$

$$a\ll z_s \qquad (130)$$

Changing $\boldsymbol{y}\mapsto\boldsymbol{u}=\frac{\boldsymbol{y}}{a}$ and $\omega\mapsto\gamma=T_w(\omega+\omega_0)$ we get

$$\boldsymbol{E}_{TR}(t,z,\boldsymbol{x})=-\frac{\bar{\mu}^2a^2}{2(2\pi)^3}\int G_1\left(t_s+\frac{SM}{\bar{c}}\right)g_2\left(\boldsymbol{u}\right)e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T-\frac{\boldsymbol{SM}\cdot(\boldsymbol{X},Z)}{\bar{c}SM}\right)}\left(\frac{\gamma}{T_w}-\omega_0\right)^2$$

$$\times \frac{1}{SM^2}\left[\hat{\boldsymbol{f}}_0(-\gamma) - \frac{\left(\hat{\boldsymbol{f}}_0(-\gamma)\cdot \boldsymbol{SM}\right)\boldsymbol{SM}}{SM^2}\right] d\boldsymbol{u} d\gamma \tag{131}$$

In the last integral $\boldsymbol{SM}$ denotes vector $(a\boldsymbol{u} - \boldsymbol{x}_s, -z_s)$.

After expanding the terms in the exponential in (131) into Taylor series wrt $a\boldsymbol{u}$ we get the approximation

$$\begin{aligned}\boldsymbol{E}_{TR}(t,z,\boldsymbol{x}) = &\frac{\bar{\mu}^2 a^2 \omega_0^2}{2(2\pi)^3 OS^2}\left(\boldsymbol{f}_0 - \frac{(\boldsymbol{f}_0\cdot\boldsymbol{OS})\,\boldsymbol{OS}}{OS^2}\right)\Bigg|_{\left(-\frac{T}{T_w}+\frac{(\boldsymbol{X},Z)\cdot\boldsymbol{OS}}{\bar{c}OST_w}\right)} e^{i\omega_0\left(T-\frac{\boldsymbol{OS}\cdot(\boldsymbol{X},Z)}{\bar{c}OS}\right)}\\ &\times G_1\left(t_s+\frac{OS}{\bar{c}}\right)\hat{g}_2\left(\frac{a\omega_0}{\bar{c}OS}\left(\frac{\boldsymbol{OS}\cdot(\boldsymbol{X},Z)}{OS^2}\boldsymbol{x}_s - \boldsymbol{X}\right)\right) + (\omega_0\mapsto -\omega_0)\end{aligned} \tag{132}$$

Its magnitude is given by

$$\begin{aligned}|\boldsymbol{E}_{TR}(t,z,\boldsymbol{x})| = &\frac{\bar{\mu}^2 a^2 \omega_0^2}{2(2\pi)^3 OS^2}\left|\left[\boldsymbol{f}_0 - \frac{(\boldsymbol{f}_0\cdot\boldsymbol{OS})\,\boldsymbol{OS}}{OS^2}\right]\left(-\frac{T}{T_w}+\frac{(\boldsymbol{X},Z)\cdot\boldsymbol{OS}}{\bar{c}OST_w}\right)\right|\left|\cos\left(\omega_0\left(T-\frac{\boldsymbol{OS}\cdot(\boldsymbol{X},Z)}{\bar{c}OS}\right)\right)\right|\\ &\times G_1\left(t_s+\frac{OS}{\bar{c}}\right)\left|\Re\hat{g}_2\left(\frac{a\omega_0}{\bar{c}OS}\left(\frac{\boldsymbol{OS}\cdot(\boldsymbol{X},Z)}{OS^2}\boldsymbol{x}_s-\boldsymbol{X}\right)\right)\right|\\ = &\frac{\bar{\mu}^2 a^2 \omega_0^2}{2(2\pi)^3 OS^2}\left|\left[\boldsymbol{f}_0 - (\boldsymbol{f}_0\cdot\boldsymbol{e}_3)\,\boldsymbol{e}_3\right]\left(-\frac{T}{T_w}+\frac{(\boldsymbol{X},Z)\cdot\boldsymbol{e}_3}{\bar{c}T_w}\right)\right|\left|\cos\left(\omega_0\left(T-\frac{\boldsymbol{e}_3\cdot(\boldsymbol{X},Z)}{\bar{c}}\right)\right)\right|\\ &\times G_1\left(t_s+\frac{OS}{\bar{c}}\right)\left|\Re\hat{g}_2\left(\frac{a\omega_0}{\bar{c}OS}\left(\left(\frac{|z_s|(\boldsymbol{X},Z)\cdot\boldsymbol{e}_1}{OS}\right)\frac{\boldsymbol{x}_s}{|\boldsymbol{x}_s|}-(\boldsymbol{e}_2\cdot(\boldsymbol{X},Z))\frac{\boldsymbol{x}_s^\perp}{|\boldsymbol{x}_s|}\right)\right)\right|\end{aligned} \tag{133}$$

where

$$\boldsymbol{e}_1 = (z_s\boldsymbol{x}_s, -|\boldsymbol{x}_s|^2)_0 \tag{134}$$

$$\boldsymbol{e}_2 = \frac{\boldsymbol{x}_s^\perp}{|\boldsymbol{x}_s|} \tag{135}$$

$$\boldsymbol{e}_3 = \frac{\boldsymbol{OS}}{OS}. \tag{136}$$

$\boldsymbol{e}_1$ is the unit vector orthogonal to $\boldsymbol{OS}$ and $\boldsymbol{e}_2$. Formula (133) is in agreement with Rayleigh resolution formula: the size of the focal spot is of the order of $\frac{\lambda_0 OS^2}{a|z_s|}$ in $\boldsymbol{e}_1$ direction and of $\frac{\lambda_0 OS}{a}$ in $\boldsymbol{e}_2$ direction. Rayleigh formula says that the size of the focal spot of a beam with carrier wavelength λ_0 focused with a system of size a from a distance L is of the order of $\lambda_0 L/a$. Also the angle formulas from section 15.3.2 [1] apply here.

5 Random medium

In random medium we should analyze the general expressions (251) and (252). The fast phase term is the same as in the homogeneous medium case, so we have to pick parameters in the same way as we search for a nonzero limit: $t = -t_s + \varepsilon T$, $\boldsymbol{x} = \boldsymbol{x}_s + \varepsilon \boldsymbol{X}$ and $z = z_s + \varepsilon Z$.

5.1 $E_{TR,z}$ in random medium

We first deal with $E_{TR,z}$ and we cancel the exponents with ε factor :

$$
\begin{aligned}
E_{TR,z}(t,z,\boldsymbol{x}) = &-\frac{1}{2\epsilon(z_s+\varepsilon Z)}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{ih(t_s+\boldsymbol{x}_s\cdot\boldsymbol{\kappa})+i\omega\boldsymbol{l}\cdot\boldsymbol{x}_s}e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa})}\\
&\overline{\hat{G}_1(h)}\hat{G}_2(h\boldsymbol{\kappa}+\omega\boldsymbol{l})\omega^4\left\{\kappa^3\xi_1(\kappa)^{-3/2}(\epsilon(0))^{-2}+\kappa\sqrt{\xi_1(\kappa)}\right\}\\
&\Bigg[\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{\frac{i\omega}{\varepsilon}\lambda_1(\kappa)2z_s+i\omega\lambda_1(\kappa)Z}\overline{S_{a,1}(z_s,\omega,-\boldsymbol{\kappa})}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{ih\lambda_1(\kappa)z_s+i\omega\lambda_1(\kappa)Z-i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{b,1}(z_s,\omega,-\boldsymbol{\kappa})}\\
&+\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-ih\lambda_1(\kappa)z_s-i\omega\lambda_1(\kappa)Z+i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{a,1}(z_s,\omega,-\boldsymbol{\kappa})}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{i\omega}{\varepsilon}\lambda_1(\kappa)2z_s-i\omega\lambda_1(\kappa)Z}\overline{S_{b,1}(z_s,\omega,-\boldsymbol{\kappa})}\Bigg]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh
\end{aligned}
\tag{137}
$$

Instead of dealing with the propagators coefficients directly we express $T_{g,i}$ and $R_{g,i}$ in terms of reflection and transmission coefficients:

$$R_g(z,\omega,\kappa) = \frac{\widetilde{T}_{\omega,\kappa}(z,0)R_{\omega,\kappa}(-L,z)}{1-\widetilde{R}_{\omega,\kappa}(z,0)R_{\omega,\kappa}(-L,z)} = \sum_{m=0}^{\infty}\widetilde{T}_{\omega,\kappa}(z,0)\widetilde{R}_{\omega,\kappa}(z,0)^m R_{\omega,\kappa}(-L,z)^{m+1} \quad (138)$$

$$T_g(z,\omega,\kappa) = \frac{\widetilde{T}_{\omega,\kappa}(z,0)}{1-\widetilde{R}_{\omega,\kappa}(z,0)R_{\omega,\kappa}(-L,z)} = \sum_{n=0}^{\infty}\widetilde{T}_{\omega,\kappa}(z,0)\widetilde{R}_{\omega,\kappa}(z,0)^n R_{\omega,\kappa}(-L,z)^{n} \quad (139)$$

We look for the expectation of $E_{TR,z}$, so we have to study the expectation of each term in (137). We for example pick the term with the product of two transmission coefficients:

$$\begin{aligned}
&\overline{T_{g,1}\left(z_s,\omega-\frac{\varepsilon h}{2},-\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2}\right)}T_{g,1}\left(z_s+\varepsilon Z,\omega+\frac{\varepsilon h}{2},\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2}\right) = \\
&\sum_{m,n=0}^{\infty}\overline{\widetilde{T}_1\left(\omega-\frac{\varepsilon h}{2},-\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},z_s,0\right)\widetilde{R}_1\left(\omega-\frac{\varepsilon h}{2},-\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},z_s,0\right)}^{m}\overline{R_1\left(\omega-\frac{\varepsilon h}{2},-\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},-L,z_s\right)}^{m} \\
&\times\widetilde{T}_1\left(\omega+\frac{\varepsilon h}{2},\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},z_s+\varepsilon Z,0\right)\widetilde{R}_1\left(\omega+\frac{\varepsilon h}{2},\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},z_s+\varepsilon Z,0\right)^n R_1\left(\omega+\frac{\varepsilon h}{2},\boldsymbol{\kappa}+\frac{\varepsilon \boldsymbol{l}}{2},-L,z_s+\varepsilon Z\right)^n
\end{aligned} \quad (140)$$

We can drop the εZ term in the boundary because of continuity, and since the coefficients for two separate regions are independent, the expectation goes to each of them. So we need to study the expectations of products of the kinds

$$\left(\overline{TR}^m\right)\left(\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)\times\left(TR^n\right)\left(\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)$$

and

$$\overline{R\left(\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)}^m\times R\left(\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)^n.$$

5.1.1 Expectation of $\overline{R}^p R^q$

We switch to the magnitude dependence in slowness. Let

$$U_{p,q} = R\left(\omega+\frac{1}{2}\varepsilon h,\kappa+\frac{1}{2}\varepsilon l,z_0,z\right)^p\overline{R\left(\omega-\frac{1}{2}\varepsilon h,\kappa-\frac{1}{2}\varepsilon l,z_0,z\right)}^q \quad (141)$$

Differentiating and using (52) (and expanding $\lambda_1\left(\kappa \pm \frac{1}{2}\varepsilon l\right)$) we obtain

$$\begin{aligned} \frac{dU^{\varepsilon}_{p,q}}{dz} = \frac{i\omega}{\varepsilon}\Bigg[& 2(p-q)m_{1,\kappa}U_{p,q} \\ & + e^{\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left(pU_{p+1,q}e^{ih\lambda_1(\kappa)z - i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z} - qU_{p,q-1}e^{-ih\lambda_1(\kappa)z + i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z}\right) \\ & + e^{-\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left(pU_{p-1,q}e^{-ih\lambda_1(\kappa)z + i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z} - qU_{p,q+1}e^{ih\lambda_1(\kappa)z - i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z}\right)\Bigg] \end{aligned} \tag{142}$$

with the initial condition $U^{\varepsilon}_{p,q}(z=z_0) = \mathbf{1}_0(p)\mathbf{1}_0(q)$. We next perform the following Fourier transform:

$$V^{\varepsilon}_{p,q} = \frac{1}{4\pi^2}\int\int e^{-ih(\tau-(p+q)\lambda_1(\kappa)z)+i\omega l\left(\eta-(p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{z\kappa}{\lambda_1(\kappa)}\right)} U_{p,q}\,dh\,dl \tag{143}$$

Then $V_{p,q}$'s satisfy the following differential equations:

$$\begin{aligned} \frac{\partial V^{\varepsilon}_{p,q}}{\partial z} &= -(p+q)\lambda_1(\kappa)\frac{\partial V_{p,q}}{\partial \tau} - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial V_{p,q}}{\partial \eta} \\ &+ \frac{i\omega}{\varepsilon}\left[2(p-q)m_{1,\kappa}V_{p,q} + e^{\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}}n_{1,\kappa}\left(pV_{p+1,q} - qV_{p,q-1}\right) + e^{-\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}}n_{1,\kappa}\left(pV_{p-1,q} - qV_{p,q+1}\right)\right] \end{aligned} \tag{144}$$

We now apply the infinite-dimensional version [4] diffusion approximation theorem in complex case [1]. The limit diffusion process is

$$\begin{aligned} dV_{p,q} &= -(p+q)\lambda_1(\kappa)\frac{\partial V_{p,q}}{\partial \tau}dz - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial V_{p,q}}{\partial \eta}dz \\ &\quad + \text{martingale part} \\ &\quad + \omega^2\left(-V_{p,q}\left((q^2+p^2)(2\gamma_{m_1}+\gamma_{n_1}) - 4pq\gamma_{m_1}\right) + pq\gamma_{n_1}\left(V_{p+1,q+1} + V_{p-1,q-1}\right)\right)dz \end{aligned} \tag{145}$$

Taking expectation we get

$$\frac{\partial \mathbb{E}V_{p,q}}{\partial z} = -(p+q)\lambda_1(\kappa)\frac{\partial \mathbb{E}V_{p,q}}{\partial \tau} - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial \mathbb{E}V_{p,q}}{\partial \eta}$$

$$+\,\omega^2\left[-\mathbb{E}V_{p,q}\left((q^2+p^2)(2\gamma_{m_1}+\gamma_{n_1})-4pq\gamma_{m_1}\right)+pq\gamma_{n_1}\left(\mathbb{E}V_{p+1,q+1}+\mathbb{E}V_{p-1,q-1}\right)\right] \tag{146}$$

The only nonzero diagonal subsystem satisfies

$$\frac{\partial f_p}{\partial z}=-2p\lambda_1(\kappa)\frac{\partial f_p}{\partial \tau}-2p\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial f_p}{\partial \eta}+\omega^2p^2\gamma_{n_1}\left(-2f_p+f_{p+1}+f_{p-1}\right) \tag{147}$$

$$f_p(z=0)=\mathbf{1}_0(p)\delta(\tau)\delta(\eta)/\omega \tag{148}$$

Defining a Markov process $(N_z)_{z\geq z_0}$ on $\mathbb{N}$ with the generator

$$\mathcal{L}\phi(N)=\omega^2\gamma_{n_1}N^2\left(-2\phi(N)+\phi(N-1)+\phi(N+1)\right) \tag{149}$$

we find

$$\begin{aligned}
f_p(z)&=\frac{1}{\omega}\mathbb{E}\left[\mathbf{1}_{N_z=0}\delta\left(\tau-2\lambda_1(\kappa)\int_{z_0}^{z}N_sds\right)\delta\left(\eta-2\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)}\int_{z_0}^{z}N_sds\right)\Big|N_{z_0}=p\right]\\
&=\frac{1}{\omega}\mathbb{E}\left[\mathbf{1}_{N_z=0}\delta\left(\tau-2\lambda_1(\kappa)\int_{z_0}^{z}N_sds\right)\Big|N_{z_0}=p\right]\delta\left(\eta-\tau\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right)\\
&=\frac{1}{\omega}\mathcal{W}_p^{(1)}\left(\omega,\kappa,\tau,z_0,z\right)\delta\left(\eta-\tau\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right)
\end{aligned} \tag{150}$$

Finally we have

$$\begin{aligned}
&\mathbb{E}\left(R^p\right)\left(\omega+\frac{1}{2}\varepsilon h,\kappa+\frac{1}{2}\varepsilon l,z_0,z\right)\overline{\left(R^p\right)\left(\omega-\frac{1}{2}\varepsilon h,\kappa-\frac{1}{2}\varepsilon l,z_0,z\right)}\xrightarrow{\varepsilon\to0}\\
&\int\mathcal{W}_p^{(1)}\left(\omega,\kappa,\tau,z_0,z\right)e^{i\tau\left[h-\omega l\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2ipz\left[-h\lambda_1(\kappa)+\omega l\frac{\bar\epsilon}{\bar\epsilon_1}\frac{\kappa}{\lambda_1(\kappa)}\right]}
\end{aligned} \tag{151}$$

5.1.2 Expectation of $\overline{TR}^pTR^q$

Lets denote

$$U_{p,q}=\left(TR^p\right)\left(\omega+\frac{1}{2}\varepsilon h,\kappa+\frac{1}{2}\varepsilon l,z_0,z\right)\overline{\left(TR^q\right)\left(\omega-\frac{1}{2}\varepsilon h,\kappa-\frac{1}{2}\varepsilon l,z_0,z\right)} \tag{152}$$

Using (52) and (53) as above we derive the equations

$$\frac{dU^{\varepsilon}_{p,q}}{dz} = \frac{i\omega}{\varepsilon}\Bigg[2(p-q)m_{1,\kappa}U_{p,q}$$
$$+ e^{\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left((p+1)U_{p+1,q}e^{ih\lambda_1(\kappa)z - i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z} - qU_{p,q-1}e^{-ih\lambda_1(\kappa)z + i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z}\right)$$
$$+ e^{-\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left(pU_{p-1,q}e^{-ih\lambda_1(\kappa)z + i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z} - (q+1)U_{p,q+1}e^{ih\lambda_1(\kappa)z - i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa l}{\lambda_1(\kappa)}z}\right)\Bigg] \tag{153}$$

with the initial condition $U^{\varepsilon}_{p,q}(z = z_0) = \mathbf{1}_0(p)\mathbf{1}_0(q)$. Taking the same as above Fourier transform we get:

$$V^{\varepsilon}_{p,q} = \frac{1}{4\pi^2}\int\int e^{-ih(\tau-(p+q)\lambda_1(\kappa)z) + i\omega l\left(\eta - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{z\kappa}{\lambda_1(\kappa)}\right)} U_{p,q}\,dh\,dl \tag{154}$$

$$\frac{\partial V^{\varepsilon}_{p,q}}{\partial z} = -(p+q)\lambda_1(\kappa)\frac{\partial V_{p,q}}{\partial \tau} - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial V_{p,q}}{\partial \eta}$$
$$+ \frac{i\omega}{\varepsilon}\Bigg[2(p-q)m_{1,\kappa}V_{p,q} + e^{\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left((p+1)V_{p+1,q} - qV_{p,q-1}\right)$$
$$+ e^{-\frac{2i\omega\lambda_1(\kappa)z}{\varepsilon}} n_{1,\kappa}\left(pV_{p-1,q} - (q+1)V_{p,q+1}\right)\Bigg] \tag{155}$$

We now apply the diffusion approximation theorem in complex case [1]. The limit diffusion process is

$$dV_{p,q} = -(p+q)\lambda_1(\kappa)\frac{\partial V_{p,q}}{\partial \tau}dz - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial V_{p,q}}{\partial \eta}dz$$
$$+ \text{martingale part}$$
$$+ \omega^2\left(-V_{p,q}\left(2(p-q)^2\gamma_{m_1} + \left(p^2+q^2+p+q+1\right)\gamma_{n_1}\right)\right.$$
$$\left. + pq\gamma_{n_1}V_{p-1,q-1} + (p+1)(q+1)\gamma_{n_1}V_{p+1,q+1}\right)dz \tag{156}$$

Taking expectation we get

$$\frac{\partial \mathbb{E}V_{p,q}}{\partial z} = -(p+q)\lambda_1(\kappa)\frac{\partial \mathbb{E}V_{p,q}}{\partial \tau} - (p+q)\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial \mathbb{E}V_{p,q}}{\partial \eta}$$

$$+ \omega^2 \left(-\mathbb{E}V_{p,q}\left(2(p-q)^2\gamma_{m_1} + \left(p^2+q^2+p+q+1\right)\gamma_{n_1}\right)\right.$$

$$\left. + pq\gamma_{n_1}\mathbb{E}V_{p-1,q-1} + (p+1)(q+1)\gamma_{n_1}\mathbb{E}V_{p+1,q+1}\right) dz \tag{157}$$

These uncouple into subsystems for $\mathbb{E}V_{p,p+n}$ and the only nonzero values are those of $f_p = \mathbb{E}V_{p,p}$ which satisfy

$$\frac{\partial f_p}{\partial z} = -2p\lambda_1(\kappa)\frac{\partial f_p}{\partial \tau} - 2p\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\frac{\partial f_p}{\partial \eta}$$

$$+ \omega^2\gamma_{n_1}\left(-\left(2p^2+2p+1\right)f_p + p^2 f_{p-1} + (p+1)^2 f_{p+1}\right) \tag{158}$$

$$f_p(z=0) = \mathbf{1}_0(p)\delta(\tau)\delta(\eta)/\omega \tag{159}$$

Defining a Markov process $(N_z)_{z\geq z_0}$ on $\mathbb{N}$ with the generator

$$\mathcal{L}\phi(N) = \omega^2\gamma_{n_1}\left(-\left(2N^2+2N+1\right)\phi(N) + N^2\phi(N-1) + (N+1)^2\phi(N+1)\right) \tag{160}$$

we find

$$\begin{aligned}
f_p(z) &= \frac{1}{\omega}\mathbb{E}\left[\mathbf{1}_{N_z=0}\delta\left(\tau - 2\lambda_1(\kappa)\int_{z_0}^z N_s ds\right)\delta\left(\eta - 2\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\int_{z_0}^z N_s ds\right)\Big| N_{z_0} = p\right] \\
&= \frac{1}{\omega}\mathbb{E}\left[\mathbf{1}_{N_z=0}\delta\left(\tau - 2\lambda_1(\kappa)\int_{z_0}^z N_s ds\right)\Big| N_{z_0} = p\right]\delta\left(\eta - \tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right) \\
&= \frac{1}{\omega}\mathcal{W}_p^{(T,1)}\left(\omega,\kappa,\tau,z_0,z\right)\delta\left(\eta - \tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right)
\end{aligned} \tag{161}$$

Finally we have

$$\mathbb{E}\left(TR^p\right)\left(\omega + \frac{1}{2}\varepsilon h, \kappa + \frac{1}{2}\varepsilon l, z_0, z\right)\overline{\left(TR^p\right)\left(\omega - \frac{1}{2}\varepsilon h, \kappa - \frac{1}{2}\varepsilon l, z_0, z\right)} \stackrel{\varepsilon\to 0}{\longrightarrow}$$

$$\int \mathcal{W}_p^{(T,1)}\left(\omega,\kappa,\tau,z_0,z\right)e^{i\tau\left[h - \omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau \times e^{2ipz\left[-h\lambda_1(\kappa) + \omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]} \tag{162}$$

5.1.3 Expectation of $\overline{\widetilde{T}\widetilde{R}^p}\widetilde{T}\widetilde{R}^p$

Introduce the "left going propagator" $P_i^L(z,0)$, $z \le 0$ satisfying

$$\frac{dP_i^L}{dz} = \frac{i\omega}{\varepsilon}\begin{bmatrix} m_i & n_i e^{-2i\omega\lambda_i z/\varepsilon} \\ -n_i e^{2i\omega\lambda_i z/\varepsilon} & -m_i \end{bmatrix} P_i^L \tag{163}$$

$$P_i^L(z=0,0) = I \tag{164}$$

Then $P_i^L(z,0) = P_i(z,0)^{-1}$ and P_i^L can be written as

$$P_i^L(z,0) = \begin{bmatrix} \gamma_i & \overline{\delta}_i \\ \delta_i & \overline{\gamma}_i \end{bmatrix}(z,0) = \begin{bmatrix} \overline{\alpha}_i & -\overline{\beta}_i \\ -\beta_i & \alpha_i \end{bmatrix}(z,0) \tag{165}$$

The adjoint reflection coefficients $\widetilde{R}_i(z,0) = -\frac{\beta_i(z,0)}{\alpha_i(z,0)} = \frac{\delta_i(z,0)}{\gamma_i(z,0)}$ satisfy the Riccati equation

$$\frac{d\widetilde{R}_i(z,0)}{dz} = \frac{d}{dz}\left(\frac{\delta_i(z,0)}{\gamma_i(z,0)}\right) = -\frac{i\omega}{\varepsilon}\left(2m_i\left(\frac{z}{\varepsilon^2}\right)\widetilde{R}_i + n_i\left(\frac{z}{\varepsilon^2}\right)\left(e^{\frac{2i\omega\lambda_i z}{\varepsilon}} + \widetilde{R}_i^2 e^{-\frac{2i\omega\lambda_i z}{\varepsilon}}\right)\right) \tag{166}$$

$$\widetilde{R}_i(z=0,0) = 0 \tag{167}$$

Changing variables $z \mapsto z = z_0 - y$ we obtain the equation

$$\frac{d\widetilde{R}_i}{dy} = \frac{i\omega}{\varepsilon}\left(2m_i\left(\frac{z_0-y}{\varepsilon^2}\right)\widetilde{R}_i + n_i\left(\frac{z_0-y}{\varepsilon^2}\right)\left(e^{\frac{2i\omega\lambda_i z_0}{\varepsilon}} e^{-\frac{2i\omega\lambda_i y}{\varepsilon}} + \widetilde{R}_i^2 e^{-\frac{2i\omega\lambda_i z_0}{\varepsilon}} e^{\frac{2i\omega\lambda_i y}{\varepsilon}}\right)\right) \tag{168}$$

$$z_0 \le y \le 0 \tag{169}$$

$$\widetilde{R}_i(y=z_0) = 0. \tag{170}$$

This gives

$$\frac{d\left[\widetilde{R}_i e^{-\frac{2i\omega\lambda_i z_0}{\varepsilon}}\right]}{dy} = \frac{i\omega}{\varepsilon}\left(2m_i\left(\frac{z_0-y}{\varepsilon^2}\right)\left[\widetilde{R}_i e^{-\frac{2i\omega\lambda_i z_0}{\varepsilon}}\right] + n_i\left(\frac{z_0-y}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega\lambda_i y}{\varepsilon}} + \left[\widetilde{R}_i e^{-\frac{2i\omega\lambda_i z_0}{\varepsilon}}\right]^2 e^{\frac{2i\omega\lambda_i y}{\varepsilon}}\right)\right) \tag{171}$$

which has the same form as the Riccati equation (52) for the coefficient R_i meaning that since the noise is stationary $R_i(z,0)$ and $\widetilde{R}_i(z,0)e^{-\frac{2i\omega\lambda_i z}{\varepsilon}}$ have the same distribution. Also, $T(z,0)=\widetilde{T}(z,0)$. Hence

$$\begin{aligned}
\mathbb{E}\left[\overline{\widetilde{T}\widetilde{R}^p}\widetilde{T}\widetilde{R}^p\right] &= \mathbb{E}\left[\overline{TR^p}TR^p\right]e^{\frac{2piz\left((\omega+\frac{\varepsilon h}{2})\lambda_1(\kappa+\frac{\varepsilon l}{2})-(\omega-\frac{\varepsilon h}{2})\lambda_1(\kappa-\frac{\varepsilon l}{2})\right)}{\varepsilon}} \\
&= \mathbb{E}\left[\overline{TR^p}TR^p\right]e^{2piz\left(h\lambda_1(\kappa)-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right)}+O(\varepsilon) \\
&= \int \mathcal{W}_p^{(T,1)}\left(\omega,\kappa,\tau,z,0\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2ipz\left[h\lambda_1(\kappa)-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]}+O(\varepsilon)
\end{aligned} \tag{172}$$

5.1.4 Expectations of $T_g\overline{T_g}$ and $R_g\overline{R_g}$

Combining the formulas from the previous sections we get

$$\begin{aligned}
&\mathbb{E}\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z_s+\varepsilon Z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xrightarrow{\varepsilon\to 0} \\
&\sum_{n=0}^{\infty}\int \mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2inz_s\left[h\lambda_1(\kappa)-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]} \\
&\times\int \mathcal{W}_n^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2inz_s\left[-h\lambda_1(\kappa)+\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]} \\
&=\sum_{n=0}^{\infty}\int \mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_n^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau
\end{aligned} \tag{173}$$

Similarly

$$\begin{aligned}
&\mathbb{E}\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z_s+\varepsilon Z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xrightarrow{\varepsilon\to 0} \\
&\sum_{n=0}^{\infty}\int \mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2inz_s\left[h\lambda_1(\kappa)-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]} \\
&\times\int \mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau\times e^{2i(n+1)z_s\left[-h\lambda_1(\kappa)+\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]} \\
&=e^{2iz_s\left[-h\lambda_1(\kappa)+\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)}\right]}\sum_{n=0}^{\infty}\int \mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega l\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\kappa}{\lambda_1(\kappa)^2}\right]}d\tau
\end{aligned} \tag{174}$$

5.1.5 Expectation of $E_{TR,z}$

We need to switch to magnitudes using

$$\left|\boldsymbol{\kappa}+\frac{\varepsilon\boldsymbol{l}}{2}\right| = \kappa + \frac{\varepsilon\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\kappa}}{2} + O(\varepsilon^2) \tag{175}$$

Then the limit is

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) =& \frac{1}{2\epsilon(z_s)}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{ih(t_s+\boldsymbol{x}_s\cdot\boldsymbol{\kappa})+i\omega\boldsymbol{l}\cdot\boldsymbol{x}_s}e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa})}\\
&\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)\omega^4\Big\{\kappa^3\xi_1(\kappa)^{-3/2}\left(\epsilon\left(0\right)\right)^{-2}+\kappa\sqrt{\xi_1(\kappa)}\Big\}\\
&\Bigg[-e^{ih\lambda_1(\kappa)z_s+i\omega\lambda_1(\kappa)Z-i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}e^{2iz_s\left[-h\lambda_1(\kappa)+\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]}\\
&\times\sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)^2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]}d\tau\\
&+e^{-ih\lambda_1(\kappa)z_s-i\omega\lambda_1(\kappa)Z+i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&\times\sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_n^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)^2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]}d\tau\Bigg]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh\end{aligned} \tag{176}$$

We change variables $\boldsymbol{l}\mapsto\boldsymbol{k}=\omega\boldsymbol{l}+h\boldsymbol{\kappa}$ and get

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) =& \frac{1}{2\epsilon(z_s)}\frac{1}{(2\pi)^6\varepsilon^3}\int\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(\boldsymbol{k}\right)\omega^2\Big\{\kappa^3\xi_1(\kappa)^{-3/2}\left(\epsilon\left(0\right)\right)^{-2}+\kappa\sqrt{\xi_1(\kappa)}\Big\}\\
&\Bigg[-e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}+\lambda_1(\kappa)Z)}e^{ih\left(t_s-\frac{z_s}{\bar{c}^2\lambda_1(\kappa)}+\frac{\tau}{\bar{c}^2\lambda_1(\kappa)^2}\right)+i\boldsymbol{k}\cdot\left(\boldsymbol{x}_s+\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}z_s}{\lambda_1(\kappa)}-\tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}}{\lambda_1(\kappa)^2}\right)}\\
&\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)\\
&+e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)Z)}e^{ih\left(t_s-\frac{z_s}{\bar{c}^2\lambda_1(\kappa)}+\frac{\tau}{\bar{c}^2\lambda_1(\kappa)^2}\right)+i\boldsymbol{k}\cdot\left(\boldsymbol{x}_s+\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}z_s}{\lambda_1(\kappa)}-\tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}}{\lambda_1(\kappa)^2}\right)}\\
&\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_n^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)\Bigg]d\boldsymbol{\kappa}d\boldsymbol{k}d\omega dhd\tau\end{aligned} \tag{177}$$

Integrating wrt h and $\boldsymbol{k}$ we get

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) =& \frac{1}{2\epsilon(z_s)}\frac{1}{(2\pi)^3\varepsilon^3}\int G_1\left(t_s-\frac{z_s}{\bar{c}^2\lambda_1(\kappa)}+\frac{\tau}{\bar{c}^2\lambda_1(\kappa)^2}\right)G_2\left(\boldsymbol{x}_s+\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}z_s}{\lambda_1(\kappa)}-\tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}}{\lambda_1(\kappa)^2}\right)\omega^2\\
&\left\{\kappa^3\xi_1(\kappa)^{-3/2}\left(\epsilon\left(0\right)\right)^{-2}+\kappa\sqrt{\xi_1(\kappa)}\right\}\\
&\Bigg[-e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}+\lambda_1(\kappa)Z)}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)\\
&+e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)Z)}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)\Bigg]d\boldsymbol{\kappa}d\omega d\tau
\end{aligned}
\tag{178}
$$

5.2 $\boldsymbol{E}_{TR,t}$ in random medium

The field is given by (with $t=-t_s+\varepsilon T$, $\boldsymbol{x}=\boldsymbol{x}_s+\varepsilon\boldsymbol{X}$ and $z=z_s+\varepsilon Z$)

$$
\begin{aligned}
\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) =& \frac{1}{2}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{ih(t_s+\boldsymbol{x}_s\cdot\boldsymbol{\kappa})+i\omega\boldsymbol{l}\cdot\boldsymbol{x}_s}e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa})}\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)\omega^4\\
&\Bigg[\left\{\kappa^2\frac{1}{\sqrt{\xi_1\left(\kappa\right)}}\epsilon\left(0\right)^{-2}+\xi_1\left(\kappa\right)^{3/2}\right\}\\
&\Bigg(\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{\frac{i\omega}{\varepsilon}\lambda_1(\kappa)2z_s+i\omega\lambda_1(\kappa)Z}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{ih\lambda_1(\kappa)z_s+i\omega\lambda_1(\kappa)Z-i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&-\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-ih\lambda_1(\kappa)z_s-i\omega\lambda_1(\kappa)Z+i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&+\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{i\omega}{\varepsilon}\lambda_1(\kappa)2z_s-i\omega\lambda_1(\kappa)Z}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\Bigg)\boldsymbol{\kappa}_0
\end{aligned}
$$

$$
\begin{aligned}
&- \xi_2\left(\kappa\right)^{3/2}\left(-\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)\right.\\
&\quad\times e^{\frac{i\omega}{\varepsilon}\lambda_2(\kappa)2z_s+i\omega\lambda_2(\kappa)Z}\overline{S_{a,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&+\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)\\
&\quad\times e^{ih\lambda_2(\kappa)z_s+i\omega\lambda_2(\kappa)Z-i\omega\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}z_s}\overline{S_{b,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&+\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)\\
&\quad\times e^{-ih\lambda_2(\kappa)z_s-i\omega\lambda_2(\kappa)Z+i\omega\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}z_s}\overline{S_{a,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&-\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon \boldsymbol{l}\right)\\
&\quad\left.\times e^{-\frac{i\omega}{\varepsilon}\lambda_2(\kappa)2z_s-i\omega\lambda_2(\kappa)Z}\overline{S_{b,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\right){\boldsymbol{\kappa}_0}^{\perp}\Bigg]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh \qquad (179)
\end{aligned}
$$

The coefficients have the same structure as the ones in the expression for $E_{TR,z}$, and since the equations for the reflection and transmission coefficients are also similar, we can use the results from above. For the expectation we get

$$
\begin{aligned}
\mathbb{E}\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) =& \frac{1}{2}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{ih(t_s+\boldsymbol{x}_s\cdot\boldsymbol{\kappa})+i\omega\boldsymbol{l}\cdot\boldsymbol{x}_s} e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa})}\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)\omega^4 \\
& \Bigg[\left\{\kappa^2\frac{1}{\sqrt{\xi_1\left(\kappa\right)}}\epsilon\left(0\right)^{-2}+\xi_1\left(\kappa\right)^{3/2}\right\} \\
& \times\Bigg(-e^{ih\lambda_1(\kappa)z_s+i\omega\lambda_1(\kappa)Z-i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}e^{2iz_s\left[-h\lambda_1(\kappa)+\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]} \\
& \times\sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)^2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]}d\tau \\
& -e^{-ih\lambda_1(\kappa)z_s-i\omega\lambda_1(\kappa)Z+i\omega\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}z_s}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)} \\
& \times\sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,1)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n}^{(1)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)^2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right]}d\tau\Bigg)\boldsymbol{\kappa}_0 \\
& -\xi_2\left(\kappa\right)^{3/2} \\
& \times\Bigg(e^{ih\lambda_2(\kappa)z_s+i\omega\lambda_2(\kappa)Z-i\omega\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}z_s}\overline{S_{b,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}e^{2iz_s\left[-h\lambda_2(\kappa)+\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}\frac{\bar{\mu}}{\bar{\mu}_1}\right]} \\
& \sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,2)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n+1}^{(2)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)^2}\frac{\bar{\mu}}{\bar{\mu}_1}\right]}d\tau \\
& +e^{-ih\lambda_2(\kappa)z_s-i\omega\lambda_2(\kappa)Z+i\omega\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}z_s}\overline{S_{a,2}\left(z_s,\omega,-\boldsymbol{\kappa}\right)} \\
& \times\sum_{n=0}^{\infty}\int\mathcal{W}_n^{(T,2)}\left(\omega,\kappa,\tau,z_s,0\right)*_\tau\mathcal{W}_{n}^{(2)}\left(\omega,\kappa,\tau,-L,z_s\right)e^{i\tau\left[h-\omega\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)^2}\frac{\bar{\mu}}{\bar{\mu}_1}\right]}d\tau\Bigg)\boldsymbol{\kappa}_0{}^{\perp}\Bigg]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh
\end{aligned}
\tag{180}
$$

Integrating as in the z component case we obtain

$$
\begin{aligned}
\mathbb{E}\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = &-\frac{1}{2}\frac{1}{(2\pi)^3\varepsilon^3}\int\Bigg[G_1\left(t_s-\frac{z_s}{\bar{c}^2\lambda_1(\kappa)}+\frac{\tau}{\bar{c}^2\lambda_1(\kappa)^2}\right)G_2\left(\boldsymbol{x}_s+\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}z_s}{\lambda_1(\kappa)}-\tau\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}}{\lambda_1(\kappa)^2}\right)\omega^2\\
&\left\{\kappa^2\frac{1}{\sqrt{\xi_1(\kappa)}}\epsilon(0)^{-2}+\xi_1(\kappa)^{3/2}\right\}\\
&\times\Bigg(e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}+\lambda_1(\kappa)Z)}\overline{S_{b,1}(z_s,\omega,-\boldsymbol{\kappa})}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_{n+1}^{(1)}(\omega,\kappa,\tau,-L,z_s)\\
&+e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_1(\kappa)Z)}\overline{S_{a,1}(z_s,\omega,-\boldsymbol{\kappa})}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,1)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_n^{(1)}(\omega,\kappa,\tau,-L,z_s)\Bigg)\boldsymbol{\kappa}_0\\
&+G_1\left(t_s-\frac{z_s}{\bar{c}^2\lambda_2(\kappa)}+\frac{\tau}{\bar{c}^2\lambda_2(\kappa)^2}\right)G_2\left(\boldsymbol{x}_s+\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}z_s}{\lambda_2(\kappa)}-\tau\frac{\bar{\mu}}{\bar{\mu}_1}\frac{\boldsymbol{\kappa}}{\lambda_2(\kappa)^2}\right)\xi_2(\kappa)^{3/2}\\
&\times\Bigg(e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}+\lambda_2(\kappa)Z)}\overline{S_{b,2}(z_s,\omega,-\boldsymbol{\kappa})}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,2)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_{n+1}^{(2)}(\omega,\kappa,\tau,-L,z_s)\\
&+e^{i\omega(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}-\lambda_2(\kappa)Z)}\overline{S_{a,2}(z_s,\omega,-\boldsymbol{\kappa})}\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,2)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_n^{(2)}(\omega,\kappa,\tau,-L,z_s)\Bigg)\boldsymbol{\kappa}_0{}^{\perp}\Bigg]d\boldsymbol{\kappa}d\omega d\tau
\end{aligned}
\tag{181}
$$

5.3 High frequency wave form

We further compute the field with the source satisfying (128)–(130) and $|z_s| \ll L_{loc}^{(i)}$.

5.3.1 Approximations for $\mathcal{W}_p^{(i)}$ and $\mathcal{W}_p^{T,i}$

Denote

$$L_{loc}^{(i)} = L_{loc}^{(i)}(\omega,\kappa) = \frac{1}{\omega^2\gamma_{n_i}(\kappa)} \tag{182}$$

Using the same method as in [1] we obtain

$$\mathcal{W}_p^{(i)}(\omega,\kappa,\tau,-L,z_s) \overset{L\to\infty}{\longrightarrow} P_p\left(\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)\frac{1}{2L_{loc}^{(i)}\lambda_i(\kappa)} \tag{183}$$

$$P_p(u) = \frac{d}{du}\left[\left(\frac{u}{1+u}\right)^p\mathbf{1}_{[0,\infty]}(u)\right] \tag{184}$$

Then

$$\mathcal{W}_0^{(i)}(\omega,\kappa,\tau,-L,z_s)=\delta(\tau) \tag{185}$$

$$\mathcal{W}_1^{(i)}(\omega,\kappa,\tau,-L,z_s)=\frac{1}{2L_{loc}^{(i)}\lambda_i(\kappa)}\frac{1}{\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^2}\mathbf{1}_{[0,\infty]}(\tau) \tag{186}$$

$$\mathcal{W}_2^{(i)}(\omega,\kappa,\tau,-L,z_s)=\frac{\tau}{2\left(L_{loc}^{(i)}\right)^2\lambda_i(\kappa)^2}\frac{1}{\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^3}\mathbf{1}_{[0,\infty]}(\tau) \tag{187}$$

Using the probabilistic interpretation (161) as in [1] we find the approximations

$$\mathcal{W}_0^{(T,i)}(\omega,\kappa,\tau,z_s,0)\approx\left(1-\frac{|z_s|}{L_{loc}^{(i)}}\right)\delta(\tau) \tag{188}$$

$$\mathcal{W}_1^{(T,i)}(\omega,\kappa,\tau,z_s,0)\approx\frac{1}{2L_{loc}^{(i)}\lambda_i(\kappa)}\mathbf{1}_{[0,2|z_s|\lambda_i(k)]}(\tau) \tag{189}$$

5.3.2 Convolutions

$$\begin{aligned}\mathcal{W}_g^{(R,i)}(\omega,\kappa,\tau,z_s,0)&\equiv\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,i)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_{n+1}^{(i)}(\omega,\kappa,\tau,-L,z_s)\\&\approx\frac{1}{2\lambda_i(\kappa)L_{loc}^{(i)}\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^2}-\frac{|z_s|}{L_{loc}^{(i)}}\frac{1-\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}}{L_{loc}^{(i)}\lambda_i(\kappa)\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^3}\end{aligned} \tag{190}$$

$$\begin{aligned}\mathcal{W}_g^{(T,i)}(\omega,\kappa,\tau,z_s,0)&\equiv\sum_{n=0}^{\infty}\mathcal{W}_n^{(T,i)}(\omega,\kappa,\tau,z_s,0)*_\tau\mathcal{W}_n^{(i)}(\omega,\kappa,\tau,-L,z_s)\\&\approx e^{-\frac{|z_s|}{L_{loc}^{(i)}}}\delta(\tau)+\frac{|z_s|}{L_{loc}^{(i)}}\frac{1}{2L_{loc}^{(i)}\lambda_i(\kappa)\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^2} &&(191)\\&\approx\left(1-\frac{|z_s|}{L_{loc}^{(i)}}\right)\delta(\tau)+\frac{|z_s|}{L_{loc}^{(i)}}\frac{1}{2L_{loc}^{(i)}\lambda_i(\kappa)\left(1+\frac{\tau}{2L_{loc}^{(i)}\lambda_i(\kappa)}\right)^2} &&(192)\end{aligned}$$

5.3.3 Integration

We change variables as $\tau \mapsto v_1 = -\frac{z_s}{\bar{c}^2\lambda_1(\kappa)} + \frac{\tau}{\bar{c}^2\lambda_1(\kappa)^2}$, $\boldsymbol{\kappa} \mapsto \boldsymbol{y} = \boldsymbol{x}_s - v_1\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\bar{c}^2\boldsymbol{\kappa}$, $\boldsymbol{y} \mapsto \boldsymbol{u} = \frac{\boldsymbol{y}}{a}$, $\omega \mapsto \gamma = T_w(\omega + \omega_0)$. Then

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_{c,1} - \frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}, \quad \text{where } \boldsymbol{\kappa}_{c,1} = \frac{\bar{\epsilon}_1\bar{\mu}}{v_1}\boldsymbol{x}_s \tag{193}$$

$$\lambda_1(\kappa) = \lambda_1(\kappa_{c,1}) + a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u} + O(|a\boldsymbol{u}|^2) \tag{194}$$

$$\lambda_1(\kappa_{c,1}) = \sqrt{\bar{\epsilon}\bar{\mu}\left(1 - \bar{\epsilon}_1\bar{\mu}\frac{|\boldsymbol{x}_s|^2}{v_1^2}\right)} \tag{195}$$

$$\tau = \lambda_1(\kappa_{c,1})\left(z_s + v_1\bar{c}^2\lambda_1(\kappa_{c,1})\right) + O(a|\boldsymbol{u}|) \tag{196}$$

The inverse of the Jacobian (in mixed variables) is

$$J^{-1} = \left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2\frac{a^2\lambda_1(\kappa)^2}{\bar{c}^2v_1^2} \tag{197}$$

Then the approximation for $E_{TR,z}$ is

$$\begin{aligned}
&\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = \left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2\frac{a^2}{4\bar{c}^2\epsilon(z_s)(2\pi)^3}\int\frac{\lambda_1(\kappa_{c,1})^2}{v_1^2}G_1(t_s+v_1)g_2(\boldsymbol{u})\left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2}+1\right)\left(\frac{\gamma}{T_w}-\omega_0\right)^2\\
&\times\Bigg[\left(\kappa_{c,1}^2\frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} - \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right)e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}\right)+Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)}\\
&\quad\times\left(\frac{1}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^2} - \frac{|z_s|}{L_{loc}^{(1)}}\frac{1-\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}}{L_{loc}^{(1)}\lambda_1(\kappa_{c,1})\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^3}\right)\\
&+\left(\kappa_{c,1}^2\frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} + \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right)e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}\right)-Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)}\\
&\times\left(\left(1-\frac{|z_s|}{L_{loc}^{(1)}}\right)\delta\left(\lambda_1(\kappa_{c,1})\left(z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})\right)\right) + \frac{|z_s|}{L_{loc}^{(1)}}\frac{1}{2L_{loc}^{(1)}\lambda_1(\kappa_{c,1})\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^2}\right)\Bigg]\\
&\quad d\boldsymbol{u}d\gamma dv_1 + O(a^3)
\end{aligned} \tag{198}$$

Similarly we obtain for $\boldsymbol{E}_{TR,t}$

$$\mathbb{E}\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = -\frac{1}{2}\frac{1}{(2\pi)^3}\frac{a^2}{\bar{c}^2}\Bigg[\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2 \int \frac{\lambda_1(\kappa_{c,1})^2}{v_1^2} G_1(t_s{+}v_1) g_2(\boldsymbol{u}) \frac{\xi_1(\kappa_{c,1})}{\kappa_{c,1}} \left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2}+1\right)\left(\frac{\gamma}{T_w}-\omega_0\right)^2$$

$$\times\Bigg(\Bigg(\kappa_{c,1}^2\frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} - \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\Bigg) e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}\right)+Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)}$$

$$\times\left(\frac{1}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^2} - \frac{|z_s|}{L_{loc}^{(1)}}\frac{1-\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}}{L_{loc}^{(1)}\lambda_1(\kappa_{c,1})\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^3}\right)$$

$$+\Bigg(\kappa_{c,1}^2\frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} + \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\Bigg) e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}\right)-Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)}$$

$$\times\left(\left(1-\frac{|z_s|}{L_{loc}^{(1)}}\right)\delta\left(\lambda_1(\kappa_{c,1})\left(z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})\right)\right) + \frac{|z_s|}{L_{loc}^{(1)}}\frac{1}{2L_{loc}^{(1)}\lambda_1(\kappa_{c,1})\left(1+\frac{z_s+v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^2}\right)\Bigg)$$

$$\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_1}a\boldsymbol{u}\right)_0 d\boldsymbol{u}d\gamma dv_1$$

$$+\left(\frac{\bar{\mu}_1}{\bar{\mu}}\right)^2\int\frac{\lambda_2(\kappa_{c,2})^2}{v_2^2}G_1(t_s+v_2)g_2(\boldsymbol{u})\frac{\xi_2(\kappa_{c,2})^2}{\kappa_{c,2}}\boldsymbol{\kappa}_{c,2}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}^{\perp}(\gamma)}\left(\frac{\gamma}{T_w}-\omega_0\right)^2$$

$$\times\Bigg(e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,2}-\frac{\bar{\epsilon}\bar{\mu}_1}{v_2}a\boldsymbol{u}\right)+Z\left(\lambda_2(\kappa_{c,2})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_2(\kappa_{c,2})v_2}\boldsymbol{\kappa}_{c,2}\cdot\boldsymbol{u}\right)\right)}$$

$$\times\left(\frac{1}{2\lambda_2(\kappa_{c,2})L_{loc}^{(2)}\left(1+\frac{z_s+v_2\bar{c}^2\lambda_2(\kappa_{c,2})}{2L_{loc}^{(1)}}\right)^2} - \frac{|z_s|}{L_{loc}^{(2)}}\frac{1-\frac{z_s+v_2\bar{c}^2\lambda_2(\kappa_{c,2})}{2L_{loc}^{(2)}}}{L_{loc}^{(2)}\lambda_2(\kappa_{c,2})\left(1+\frac{z_s+v_2\bar{c}^2\lambda_2(\kappa_{c,2})}{2L_{loc}^{(2)}}\right)^3}\right)$$

$$+e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,2}-\frac{\bar{\epsilon}\bar{\mu}_1}{v_2}a\boldsymbol{u}\right)-Z\left(\lambda_2(\kappa_{c,2})+a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_2(\kappa_{c,2})v_2}\boldsymbol{\kappa}_{c,2}\cdot\boldsymbol{u}\right)\right)}$$

$$\times\left(\left(1-\frac{|z_s|}{L_{loc}^{(2)}}\right)\delta\left(\lambda_2(\kappa_{c,2})\left(z_s+v_2\bar{c}^2\lambda_2(\kappa_{c,2})\right)\right) + \frac{|z_s|}{L_{loc}^{(2)}}\frac{1}{2L_{loc}^{(2)}\lambda_1(\kappa_{c,2})\left(1+\frac{z_s+v_2\bar{c}^2\lambda_2(\kappa_{c,2})}{2L_{loc}^{(2)}}\right)^2}\right)\Bigg)$$

$$\left(\boldsymbol{\kappa}_{c,2}-\frac{\bar{\epsilon}_1\bar{\mu}}{v_2}a\boldsymbol{u}\right)_0^{\perp} d\boldsymbol{u}d\gamma dv_2\Bigg] + O(a^3) \quad (199)$$

where for the second integral term we used

$$v_2 = -\frac{z_s}{\bar{c}^2\lambda_2(\kappa)} + \frac{\tau}{\bar{c}^2\lambda_2(\kappa)^2} \tag{200}$$

$$\boldsymbol{\kappa} \mapsto \boldsymbol{y} = \boldsymbol{x}_s - v_2\frac{\bar{\mu}}{\bar{\mu}_1}\bar{c}^2\boldsymbol{\kappa} \tag{201}$$

$$\boldsymbol{y} \mapsto \boldsymbol{u} = \frac{\boldsymbol{y}}{a} \tag{202}$$

$$\omega \mapsto \gamma = T_w(\omega + \omega_0) \tag{203}$$

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_{c,2} - \frac{\bar{\mu}_1\bar{\epsilon}}{v_2}a\boldsymbol{u}, \quad \text{where } \boldsymbol{\kappa}_{c,2} = \frac{\bar{\mu}_1\bar{\epsilon}}{v_2}\boldsymbol{x}_s \tag{204}$$

$$\lambda_2(\kappa) = \lambda_2(\kappa_{c,2}) + a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_2(\kappa_{c,2})v_2}\boldsymbol{\kappa}_{c,2}\cdot\boldsymbol{u} + O(|a\boldsymbol{u}|^2) \tag{205}$$

$$\lambda_2(\kappa_{c,2}) = \sqrt{\bar{\epsilon}\bar{\mu}\left(1 - \bar{\mu}_1\bar{\epsilon}\frac{|\boldsymbol{x}_s|^2}{v_2^2}\right)} \tag{206}$$

$$\tau = \lambda_2(\kappa_{c,1})\left(z_s + v_2\bar{c}^2\lambda_2(\kappa_{c,2})\right) + O(a|\boldsymbol{u}|) \tag{207}$$

We next consider different cases similar to ones in [1].

Refocusing of the front. If we assume that the support of G_1 is narrow of the the form $[T_0-\delta, T_0+\delta]$ with $\epsilon \ll \delta \ll 1$ then to have focusing we need to pick T_0 so that the argument of the delta function would run through 0:

$$z_s + v_1\bar{c}^2\lambda_1(\kappa_{c,1}) = 0 \tag{208}$$

Solving for v_1 we get

$$v_{1,0} = \frac{\sqrt{z_s^2 + \frac{\bar{\epsilon}_1}{\bar{\epsilon}}|\boldsymbol{x}_s|^2}}{\bar{c}} \tag{209}$$

and then $T_0^{(1)} = t_s + \frac{\sqrt{z_s^2+\frac{\bar{\epsilon}_1}{\bar{\epsilon}}|\boldsymbol{x}_s|^2}}{\bar{c}}$ and

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) =$$

$$\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2\frac{a^2\lambda_1(k_{c,1})^2}{4\bar{c}^2\epsilon(z_s)v_{1,0}^2(2\pi)^3}G_1\left(T_0^{(1)}\right)\int g_2(\boldsymbol{u})\left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2}+1\right)\left(\kappa_{c,1}^2\frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} + \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right)$$

$$\times\left(\frac{\gamma}{T_w}-\omega_0\right)^2 e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1}-\frac{\bar\epsilon_1\bar\mu}{v_{1,0}}a\boldsymbol{u}\right)-Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar\epsilon\bar\mu}{\lambda_1(\kappa_{c,1})v_{1,0}}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)} e^{-\frac{|z_s|}{L^{(1)}_{loc}(\kappa_{c,1})}} d\boldsymbol{u}d\gamma+O(a^3) \tag{210}$$

In the last equation $\boldsymbol{\kappa}_{c,1}$ is computed at $v_{1,0}$ and we plugged more precise exponential expression for the last factor in the integral since it comes actually from the approximation (191). Integrating we get to the leading order in a:

$$\begin{aligned}\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = &-\left(\frac{\bar\epsilon_1}{\bar\epsilon}\right)^2\frac{a^2\lambda_1(k_{c,1})^2\omega_0^2}{4\bar c^2\epsilon(z_s)v_{1,0}^2(2\pi)^3}e^{-\frac{|z_s|}{L^{(1)}_{loc}(\kappa_{c,1})}}G_1\left(T_0^{(1)}\right)\hat g_2\left(\frac{\bar\mu\omega_0 a}{v_{1,0}}\left(\bar\epsilon_1\boldsymbol{X}+\frac{\bar\epsilon Z}{\lambda_1(\kappa_{c,1})}\boldsymbol{\kappa}_{c,1}\right)\right)\\ &\times\left(\frac{\kappa_{c,1}^2}{\epsilon(z_s)}f_{0,z}+\xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{f}_{0,t}\right)\Bigg|_{\frac{1}{T_w}(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}-Z\lambda_1(\kappa_{c,1}))} e^{i\omega_0(T-\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}+Z\lambda_1(\kappa_{c,1}))}\\ &\times\left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2}+1\right)+(\omega_0\mapsto-\omega_0)+O(a^3)\end{aligned} \tag{211}$$

As for $\boldsymbol{E}_{TR,t}$ we perform the same kind of manipulations over both of the integral terms, assuming G_1 has support $[T_0^{(1)}-\delta,T_0^{(1)}+\delta]\cup[T_0^{(2)}-\delta,T_0^{(2)}+\delta]$. We get

$$\begin{aligned}\mathbb{E}\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = &\frac{a^2\omega_0^2}{4\bar c^2(2\pi)^3}\Bigg[\left(\frac{\bar\epsilon_1}{\bar\epsilon}\right)^2\frac{\lambda_1(k_{c,1})^2}{v_{1,0}^2}e^{-\frac{|z_s|}{L^{(1)}_{loc}(\kappa_{c,1})}}G_1\left(T_0^{(1)}\right)\hat g_2\left(\frac{\bar\mu\omega_0 a}{v_{1,0}}\left(\bar\epsilon_1\boldsymbol{X}+\frac{\bar\epsilon Z}{\lambda_1(\kappa_{c,1})}\boldsymbol{\kappa}_{c,1}\right)\right)\\ &\times\left(\frac{\kappa_{c,1}^2}{\epsilon(z_s)}f_{0,z}+\xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{f}_{0,t}\right)\Bigg|_{\frac{1}{T_w}(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}-Z\lambda_1(\kappa_{c,1}))} e^{i\omega_0(T-\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}+Z\lambda_1(\kappa_{c,1}))}\\ &\times\frac{\xi_1(\kappa_{c,1})}{\kappa_{c,1}}\left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2}+1\right)(\boldsymbol{\kappa}_{c,1})_0\\ &+\left(\frac{\bar\mu_1}{\bar\mu}\right)^2\frac{\lambda_2(k_{c,2})^2}{v_{2,0}^2}e^{-\frac{|z_s|}{L^{(2)}_{loc}(\kappa_{c,2})}}G_1\left(T_0^{(2)}\right)\hat g_2\left(\frac{\bar\epsilon\omega_0 a}{v_{2,0}}\left(\bar\mu_1\boldsymbol{X}+\frac{\bar\mu Z}{\lambda_2(\kappa_{c,2})}\boldsymbol{\kappa}_{c,2}\right)\right)\\ &\times\left(\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{f}_{0,t}{}^{\perp}\right)\Bigg|_{\frac{1}{T_w}(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,2}-Z\lambda_2(\kappa_{c,2}))} e^{i\omega_0(T-\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,2}+Z\lambda_2(\kappa_{c,2}))}\frac{\xi_2(\kappa_{c,2})^2}{\kappa_{c,2}}(\boldsymbol{\kappa}_{c,2})_0^{\perp}\Bigg]\end{aligned} \tag{212}$$

Long coda [1] refocusing. Here we assume that we record a part of coda: $G_1(t)=\mathbf{1}_{[T_1,T_2]}(t)$

[1] Coda is an incoherent part of a signal in time domain, it follows coherent part and its waveform looks like noise.

where $\max(T_0^{(1)}, T_0^{(2)}) < T_1 < T_2$. The integration of (198) and (199) in this case goes different way then in the front focusing. Here argument of the delta functions does not take on the value of 0 and other terms come into play. We only compute the leading in $|z_s|/L_{loc}^{(i)}$ terms. We compute the γ_{n_1} dependence on κ:

$$\gamma_{n_1}(\kappa) = \left(\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\right) \times \frac{\left(\gamma_\eta + \gamma_{\eta_1} - 2\gamma_{\eta_1\eta}\right)\kappa^4 + \bar{\epsilon}_1\bar{\mu}\left(\gamma_{\nu\eta} + \gamma_{\eta\nu} + 2\gamma_{\eta_1\eta} - 2\gamma_\eta - \gamma_{\eta_1\nu} - \gamma_{\nu\eta_1}\right)\kappa^2 + \bar{\epsilon}_1^2\bar{\mu}^2\left(\gamma_\nu + \gamma_\eta - \gamma_{\nu\eta} - \gamma_{\eta\nu}\right)}{4\left(\bar{\epsilon}_1\bar{\mu} - \kappa^2\right)} \tag{213}$$

The leading term of $E_{TR,z}$ is

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = \left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2 \frac{a^2}{4\bar{c}^2\epsilon(z_s)(2\pi)^3} \int \frac{\lambda_1(\kappa_{c,1})^2}{v_1^2} G_1(t_s + v_1) g_2(\boldsymbol{u}) \left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2} + 1\right)\left(\frac{\gamma}{T_w} - \omega_0\right)^2 \times \left(\kappa_{c,1}^2 \frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} - \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right) e^{i\left(\frac{\gamma}{T_w} - \omega_0\right)\left(-T + \boldsymbol{X}\cdot\left(\boldsymbol{\kappa}_{c,1} - \frac{\bar{\epsilon}_1\bar{\mu}}{v_1} a\boldsymbol{u}\right) + Z\left(\lambda_1(\kappa_{c,1}) + a\frac{\bar{\epsilon}\bar{\mu}}{\lambda_1(\kappa_{c,1})v_1}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)} \times \frac{1}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1 + \frac{z_s + v_1\bar{c}^2\lambda_1(\kappa_{c,1})}{2L_{loc}^{(1)}}\right)^2} d\boldsymbol{u}\, d\gamma\, dv_1 + O(a^3) \tag{214}$$

Upon the change $v_1 \mapsto w_1 = \frac{1}{v_1}$ we get

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = -\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2 \frac{a^2}{4\bar{c}^2\epsilon(z_s)(2\pi)^3} \int \lambda_1(\kappa_{c,1})^2 G_1(t_s + \frac{1}{w_1}) g_2(\boldsymbol{u}) \left(\frac{\kappa_{c,1}^2}{\xi_1(\kappa_{c,1})^2\epsilon(0)^2} + 1\right)\left(\frac{\gamma}{T_w} - \omega_0\right)^2 \times \left(\kappa_{c,1}^2 \frac{\overline{\hat{f}_{0,z}(\gamma)}}{\epsilon(z_s)} - \xi_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right) e^{i\left(\frac{\gamma}{T_w} - \omega_0\right)\left(-T + \boldsymbol{X}\cdot(\boldsymbol{\kappa}_{c,1} - \bar{\epsilon}_1\bar{\mu}w_1 a\boldsymbol{u}) + Z\left(\lambda_1(\kappa_{c,1}) + a\frac{\bar{\epsilon}\bar{\mu}w_1}{\lambda_1(\kappa_{c,1})}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)} \times \frac{1}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1 + \frac{z_s + \frac{\bar{c}^2\lambda_1(\kappa_{c,1})}{w_1}}{2L_{loc}^{(1)}}\right)^2} d\boldsymbol{u}\, d\gamma\, dw_1 + O(a^3) \tag{215}$$

We transform it to

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = -\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2 \frac{a^2}{4\bar{c}^2\bar{\epsilon}\epsilon(0)^2\epsilon(z_s)^2(2\pi)^3}$$
$$\times \int G_1(t_s + \frac{1}{w_1})g_2(\boldsymbol{u})e^{i\left(\frac{\gamma}{T_w}-\omega_0\right)\left(-T+\boldsymbol{X}\cdot(\boldsymbol{\kappa}_{c,1}-\bar{\epsilon}_1\bar{\mu}w_1 a\boldsymbol{u})+Z\left(\lambda_1(\kappa_{c,1})+a\frac{\bar{\epsilon}\bar{\mu}w_1}{\lambda_1(\kappa_{c,1})}\boldsymbol{\kappa}_{c,1}\cdot\boldsymbol{u}\right)\right)}\left(\frac{\gamma}{T_w}-\omega_0\right)^2$$
$$\frac{\left(\bar{\epsilon}^2\kappa_{c,1}^2+\epsilon(0)^2\lambda_1(\kappa_{c,1})^2\right)\left(\bar{\epsilon}\kappa_{c,1}^2\overline{\hat{f}_{0,z}(\gamma)}-\epsilon(z_s)\lambda_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\overline{\hat{\boldsymbol{f}}_{0,t}(\gamma)}\right)}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1+\frac{z_s+\frac{\bar{c}^2\lambda_1(\kappa_{c,1})}{w_1}}{2L_{loc}^{(1)}}\right)^2} d\boldsymbol{u}d\gamma dw_1 + O(a^3)$$
$$(216)$$

For $L_{loc}^{(1)}$ we have

$$L_{loc}^{(1)}(\omega,\kappa) = \left(\frac{4\bar{\epsilon}_1}{\omega^2\bar{\epsilon}}\right)\frac{\bar{\epsilon}_1\bar{\mu}-\kappa^2}{A_4\kappa^4+\bar{\epsilon}_1\bar{\mu}A_2\kappa^2+\bar{\epsilon}_1^2\bar{\mu}^2A_0} \tag{217}$$

where

$$A_4 = \gamma_\eta + \gamma_{\eta_1} - 2\gamma_{\eta_1\eta} \tag{218}$$

$$A_2 = \gamma_{\nu\eta} + \gamma_{\eta\nu} + 2\gamma_{\eta_1\eta} - 2\gamma_\eta - \gamma_{\eta_1\nu} - \gamma_{\nu\eta_1} \tag{219}$$

$$A_0 = \gamma_\nu + \gamma_\eta - \gamma_{\nu\eta} - \gamma_{\eta\nu} \tag{220}$$

Also

$$\boldsymbol{\kappa}_{c,1} = \bar{\epsilon}_1\bar{\mu}w_1\boldsymbol{x}_s \tag{221}$$

$$\lambda_1(\kappa_{c,1}) = \sqrt{\bar{\epsilon}\bar{\mu}\left(1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2\right)} = \frac{\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2}}{\bar{c}} \tag{222}$$

We integrate in $\boldsymbol{u}$ and γ and get

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = -\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2 \frac{a^2\omega_0^2}{4\bar{c}^2\bar{\epsilon}\epsilon(0)^2\epsilon(z_s)^2(2\pi)^3}\int G_1(t_s+\frac{1}{w_1})\hat{g}_2\left(\bar{\mu}\omega_0 a w_1\left(\bar{\epsilon}_1\boldsymbol{X}-\frac{\bar{\epsilon}Z}{\lambda_1(\kappa_{c,1})}\boldsymbol{\kappa}_{c,1}\right)\right)$$

$$\frac{\left(\bar{\epsilon}^2\kappa_{c,1}^2+\epsilon(0)^2\lambda_1(\kappa_{c,1})^2\right)\left(\bar{\epsilon}\kappa_{c,1}^2\hat{f}_{0,z}(.)-\epsilon(z_s)\lambda_1(\kappa_{c,1})\boldsymbol{\kappa}_{c,1}\cdot\hat{\boldsymbol{f}}_{0,t}(.)\right)\Big|_{\frac{1}{Tw}(-T+\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}+Z\lambda_1(\kappa_{c,1}))}}{2\lambda_1(\kappa_{c,1})L_{loc}^{(1)}\left(1+\frac{z_s+\frac{\bar{c}^2\lambda_1(\kappa_{c,1})}{w_1}}{2L_{loc}^{(1)}}\right)^2}$$

$$e^{-i\omega_0(\boldsymbol{X}\cdot\boldsymbol{\kappa}_{c,1}+Z\lambda_1(\kappa_{c,1}))}dw_1 e^{i\omega_0 T}+O(a^3)$$

$$= -\left(\frac{\bar{\epsilon}_1}{\bar{\epsilon}}\right)^2\frac{a^2}{4\bar{c}^2\bar{\epsilon}\epsilon(0)^2\epsilon(z_s)^2(2\pi)^3}\int G_1\left(t_s+\frac{1}{w_1}\right)\hat{g}_2\left(\bar{\mu}\omega_0 a w_1\left(\bar{\epsilon}_1\boldsymbol{X}-\frac{\bar{c}\bar{\epsilon}Z\bar{\epsilon}_1\bar{\mu}w_1\boldsymbol{x}_s}{\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2}}\right)\right)$$

$$\frac{\bar{c}\bar{\epsilon}\omega_0^4\left[A_4\left(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|\right)^4+\bar{\epsilon}_1\bar{\mu}A_2\left(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|\right)^2+\bar{\epsilon}_1^2\bar{\mu}^2A_0\right]}{8\bar{\epsilon}_1\left(\bar{\epsilon}_1\bar{\mu}-\left(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|\right)^2\right)\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2}}$$

$$\frac{\bar{\epsilon}^2\left(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|\right)^2+\frac{\epsilon(0)^2}{\bar{c}^2}\left(1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2\right)}{\left(1+\frac{\omega_0^2\bar{\epsilon}\left(z_s+\frac{\bar{c}\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2|\boldsymbol{x}_s|^2}}{w_1}\right)\left[A_4(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|)^4+\bar{\epsilon}_1\bar{\mu}A_2(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|)^2+\bar{\epsilon}_1^2\bar{\mu}^2A_0\right]}{8\bar{\epsilon}_1\left(\bar{\epsilon}_1\bar{\mu}-(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|)^2\right)}\right)^2}$$

$$\left(\bar{\epsilon}\left(\bar{\epsilon}_1\bar{\mu}w_1|\boldsymbol{x}_s|\right)^2\hat{f}_{0,z}(.)-\frac{\epsilon(z_s)}{\bar{c}}\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2\left|\boldsymbol{x}_s\right|^2}\bar{\epsilon}_1\bar{\mu}w_1\boldsymbol{x}_s\cdot\hat{\boldsymbol{f}}_{0,t}(.)\right)\Bigg|_{\frac{1}{Tw}\left(-T+\bar{\epsilon}_1\bar{\mu}w_1\boldsymbol{x}_s\cdot\boldsymbol{X}+\frac{1}{\bar{c}}Z\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2|\boldsymbol{x}_s|^2}\right)}$$

$$e^{-i\omega_0\left(\bar{\epsilon}_1\bar{\mu}w_1\boldsymbol{x}_s\cdot\boldsymbol{X}+\frac{1}{\bar{c}}Z\sqrt{1-\bar{\epsilon}_1\bar{\mu}w_1^2|\boldsymbol{x}_s|^2}\right)}dw_1 e^{i\omega_0 T}+O(a^3) \tag{223}$$

We first consider the case $x_s > 0$, $x_s = |\boldsymbol{x}_s|$. Change $w_1 \mapsto p = |\boldsymbol{x}_s| w_1 \sqrt{\epsilon_1 \bar{\mu}}$ gives

$$\mathbb{E}E_{TR,z}(t,z,\boldsymbol{x}) = -\frac{\bar{\epsilon}_1^2 \bar{\mu}^3 a^2 \omega_0^4 e^{i\omega_0 T}}{32(2\pi)^3 \epsilon(0)^2 \epsilon(z_s)^2 x_s} \int_{\cos\left(\theta_1^{(1)}\right)}^{\cos\left(\theta_2^{(1)}\right)} G_1\left(t_s + \frac{x_s\sqrt{\epsilon_1\bar{\mu}}}{p}\right) \hat{g}_2\left(\frac{\omega_0 a p}{\bar{c} x_s}\left(\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\boldsymbol{X} - \frac{pZ\boldsymbol{x}_s}{x_s\sqrt{1-p^2}}\right)\right)$$
$$\frac{\left(A_4p^4 + A_2p^2 + A_0\right)\left(\bar{\epsilon}\epsilon_1 p^2 + \epsilon(0)^2(1-p^2)\right) p \left[\sqrt{\bar{\epsilon}\epsilon_1} p f_{0,z}(.) - \epsilon(z_s)\sqrt{1-p^2}\frac{\boldsymbol{x}_s}{x_s}\cdot \boldsymbol{f}_{0,t}(.)\right]\left(-\frac{T}{T_w} + \frac{p\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + Z\sqrt{1-p^2}}{\bar{c}T_w}\right)}{(1-p^2)^{\frac{3}{2}}\left(1 + \frac{\omega_0^2(A_4p^4+A_2p^2+A_0)(pz_s + \sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}x_s\sqrt{1-p^2})}{8\bar{c}^2 p(1-p^2)}\right)^2}$$
$$e^{-\frac{i\omega_0}{\bar{c}}\left(p\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + \sqrt{1-p^2}Z\right)} dp + O(a^3) + (\omega_0 \mapsto -\omega_0) \tag{224}$$

where

$$\cos\left(\theta_j^{(1)}\right) = \frac{x_s\sqrt{\epsilon_1\bar{\mu}}}{T_j - t_s} \tag{225}$$

For $\boldsymbol{E}_{TR,t}$ we obtain

$$\mathbb{E}\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = \frac{\boldsymbol{x}_s}{x_s}\frac{\bar{\epsilon}_1^{\frac{3}{2}} \bar{\mu}^3 a^2 \omega_0^4 e^{i\omega_0 T}}{32(2\pi)^3 \sqrt{\bar{\epsilon}}\epsilon(0)^2 \epsilon(z_s) x_s} \int_{\cos\left(\theta_1^{(1)}\right)}^{\cos\left(\theta_2^{(1)}\right)} G_1\left(t_s + \frac{x_s\sqrt{\bar{\epsilon}_1\bar{\mu}}}{p}\right) \hat{g}_2\left(\frac{\omega_0 a p}{\bar{c} x_s}\left(\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\boldsymbol{X} - \frac{pZ\boldsymbol{x}_s}{x_s\sqrt{1-p^2}}\right)\right)$$
$$\frac{\left(A_4p^4 + A_2p^2 + A_0\right)\left(\bar{\epsilon}\epsilon_1 p^2 + \epsilon(0)^2(1-p^2)\right) \left[\sqrt{\bar{\epsilon}\epsilon_1} p f_{0,z}(.) - \epsilon(z_s)\sqrt{1-p^2}\frac{\boldsymbol{x}_s}{x_s}\cdot \boldsymbol{f}_{0,t}(.)\right]\left(-\frac{T}{T_w} + \frac{p\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + Z\sqrt{1-p^2}}{\bar{c}T_w}\right)}{(1-p^2)\left(1 + \frac{\omega_0^2(A_4p^4+A_2p^2+A_0)(pz_s + \sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}x_s\sqrt{1-p^2})}{8\bar{c}^2 p(1-p^2)}\right)^2}$$
$$e^{-\frac{i\omega_0}{\bar{c}}\left(p\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + \sqrt{1-p^2}Z\right)} dp$$
$$+ \frac{\boldsymbol{x}_s^\perp}{x_s}\frac{(\bar{\mu}\bar{\mu}_1)^{\frac{3}{2}}\bar{\epsilon} a^2 \omega_0^4 t e^{i\omega_0 T}}{32(2\pi)^3 x_s} \int_{\cos\left(\theta_1^{(2)}\right)}^{\cos\left(\theta_2^{(2)}\right)} G_1\left(t_s + \frac{x_s\sqrt{\bar{\epsilon}\bar{\mu}_1}}{p}\right) \hat{g}_2\left(\frac{\omega_0 a p}{\bar{c} x_s}\left(\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\boldsymbol{X} - \frac{pZ\boldsymbol{x}_s}{x_s\sqrt{1-p^2}}\right)\right)$$
$$\frac{\left(B_4p^4 + B_2p^2 + B_0\right)\left[\frac{1}{x_s}\boldsymbol{x}_s\cdot \boldsymbol{f}_{0,t}^\perp(.)\right]\left(-\frac{T}{T_w} + \frac{p\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + Z\sqrt{1-p^2}}{\bar{c}T_w}\right)}{(1-p^2)^{\frac{3}{2}}\left(1 + \frac{\omega_0^2(B_4p^4+B_2p^2+B_0)(pz_s + \sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}x_s\sqrt{1-p^2})}{8\bar{c}^2 p(1-p^2)}\right)^2} e^{-\frac{i\omega_0}{\bar{c}}\left(p\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + \sqrt{1-p^2}Z\right)} dp$$
$$+ O(a^3) + (\omega_0 \mapsto -\omega_0) \tag{226}$$

where

$$B_4 = \gamma_\nu + \gamma_{\nu_1} - 2\gamma_{\nu_1\nu} \tag{227}$$

$$B_2 = \gamma_{\nu\eta} + \gamma_{\eta\nu} + 2\gamma_{\nu_1\nu} - 2\gamma_\nu - \gamma_{\eta\nu_1} - \gamma_{\nu_1\eta} \tag{228}$$

$$B_0 = \gamma_\nu + \gamma_\eta - \gamma_{\nu\eta} - \gamma_{\eta\nu} \tag{229}$$

and

$$\cos\left(\theta_j^{(2)}\right) = \frac{x_s\sqrt{\bar{\epsilon}\bar{\mu}_1}}{T_j - t_s} \tag{230}$$

To perform integration we fix the argument to be its center value $\cos(\bar{\theta})$ and linearize the exponent around it:

$$\bar{\theta}^{(i)} = \frac{1}{2}\left(\theta_1^{(i)} + \theta_2^{(i)}\right) \tag{231}$$

$$p \mapsto p = \cos(\bar{\theta}^{(i)}) - \sin(\bar{\theta}^{(i)})\xi \tag{232}$$

Also to the slow terms in the product we apply the middle point theorem from calculus: $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$ for some c. Then the $(\boldsymbol{X}, Z)$ dependence is given by the products

$$\begin{aligned}
|E_{TR,z}(T,\boldsymbol{X},Z)| \simeq & \left|\cos\left(\omega_0\left(T - \frac{1}{\bar{c}}\left(\cos(\bar{\theta}^{(1)})\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + \sin(\bar{\theta}^{(1)})Z\right)\right)\right)\right| \\
& \left|\Re\hat{g}_2\left(\frac{\omega_0 a}{\bar{c}x_s\tan(\bar{\theta}^{(1)})}\left(\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\boldsymbol{X}\sin(\bar{\theta}^{(1)}) - \frac{Z\boldsymbol{x}_s}{x_s}\cos(\bar{\theta}^{(1)})\right)\right)\right| \\
& \left|\mathrm{sinc}\left(\frac{\omega_0\Delta\theta^{(1)}}{2\bar{c}}\left(\sin(\bar{\theta}^{(1)})\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} - \cos(\bar{\theta}^{(1)})Z\right)\right)\right| \\
& \left|\left[\sqrt{\bar{\epsilon}\bar{\epsilon}_1}\cos(\bar{\theta}^{(1)})f_{0,z}(.) - \epsilon(z_s)\sin(\bar{\theta}^{(1)})\frac{\boldsymbol{x}_s}{x_s}\cdot\boldsymbol{f}_{0,t}(.)\right]\left(-\frac{T}{T_w} + \frac{\cos(\bar{\theta}^{(1)})\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s} + Z\sin(\bar{\theta}^{(1)})}{\bar{c}T_w}\right)\right| \\
= & \left|\cos\left(\omega_0\left(T - \frac{1}{\bar{c}}(\boldsymbol{X},Z)\cdot\boldsymbol{w}_3\right)\right)\right|\left|\Re\hat{g}_2\left(\frac{\omega_0 a}{\bar{c}x_s\tan(\bar{\theta}^{(1)})}\left((\boldsymbol{X},Z)\cdot\boldsymbol{w}_1\frac{\boldsymbol{x}_s}{x_s} + (\boldsymbol{X},Z)\cdot\boldsymbol{w}_2\frac{\boldsymbol{x}_s^\perp}{x_s}\right)\right)\right|
\end{aligned}$$

$$
\left|\mathrm{sinc}\left(\frac{\omega_0\Delta\theta^{(1)}}{2\bar{c}}(\boldsymbol{X},Z)\cdot\boldsymbol{w}_1\right)\right|
$$
$$
\left|\left[\sqrt{\bar{\epsilon}\bar{\epsilon}_1}\cos(\bar{\theta}^{(1)})f_{0,z}(.)-\epsilon(z_s)\sin(\bar{\theta}^{(1)})\frac{\boldsymbol{x}_s}{x_s}\cdot\boldsymbol{f}_{0,t}(.)\right]\left(-\frac{T}{T_w}+\frac{(\boldsymbol{X},Z)\cdot\boldsymbol{w}_3}{\bar{c}T_w}\right)\right| \tag{233}
$$

$$
\left|\boldsymbol{E}_{TR,t}(T,\boldsymbol{X},Z)\cdot\frac{\boldsymbol{x}_s}{x_s}\right|\simeq|E_{TR,z}(T,\boldsymbol{X},Z)| \tag{234}
$$

$$
\begin{aligned}
\left|\boldsymbol{E}_{TR,t}(T,\boldsymbol{X},Z)\cdot\frac{\boldsymbol{x}_s^{\perp}}{x_s}\right|\simeq&\left|\cos\left(\omega_0\left(T-\frac{1}{\bar{c}}\left(\cos(\bar{\theta}^{(2)})\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s}+\sin(\bar{\theta}^{(2)})Z\right)\right)\right)\right|\\
&\left|\Re\hat{g}_2\left(\frac{\omega_0 a}{\bar{c}x_s\tan(\bar{\theta}^{(2)})}\left(\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\boldsymbol{X}\sin(\bar{\theta}^{(2)})-\frac{Z\boldsymbol{x}_s}{x_s}\cos(\bar{\theta}^{(2)})\right)\right)\right|\\
&\left|\mathrm{sinc}\left(\frac{\omega_0\Delta\theta^{(2)}}{2\bar{c}}\left(\sin(\bar{\theta}^{(2)})\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s}-\cos(\bar{\theta}^{(2)})Z\right)\right)\right|\\
&\left|\left[\frac{1}{x_s}\boldsymbol{x}_s\cdot\boldsymbol{f}_{0,t}(.)\right]\left(-\frac{T}{T_w}+\frac{\cos(\bar{\theta}^{(2)})\sqrt{\frac{\bar{\mu}_1}{\bar{\mu}}}\frac{\boldsymbol{x}_s\cdot\boldsymbol{X}}{x_s}+Z\sin(\bar{\theta}^{(2)})}{\bar{c}T_w}\right)\right|
\end{aligned} \tag{235}
$$

where

$$
\boldsymbol{w}_1=\left(\sin(\bar{\theta}^{(1)})\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s}{x_s},-\cos(\bar{\theta}^{(1)})\right) \tag{236}
$$
$$
\boldsymbol{w}_2=\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s^{\perp}}{x_s} \tag{237}
$$
$$
\boldsymbol{w}_3=\left(\cos(\bar{\theta}^{(1)})\sqrt{\frac{\bar{\epsilon}_1}{\bar{\epsilon}}}\frac{\boldsymbol{x}_s}{x_s},\sin(\bar{\theta}^{(1)})\right) \tag{238}
$$

We also can write (235) in terms of characteristic directions similar to $\boldsymbol{w}_i$'s. Note that in general $\boldsymbol{w}_1$ is not orthogonal to $\boldsymbol{w}_3$. Here the arguments of sinc function do not depend on a and the ***OS*** dependence can be compensated for by time window function width. This demonstrates the super resolution effect: the effective aperture at the focus point can be made of order one and independent of ***OS***.

5.4 Focal spots comparison

We plot the refocused spots for the time reversal experiment in homogeneous and random media assuming the same geometry of the mirror and its location with respect to the source, the only difference is the presence of the inhomogeneous medium and the part of the received signal which is being re-radiated. It shows the difference in the shape of the refocusing spot and demonstrates that in the inhomogeneous case the effective aperture gets enhanced significantly.

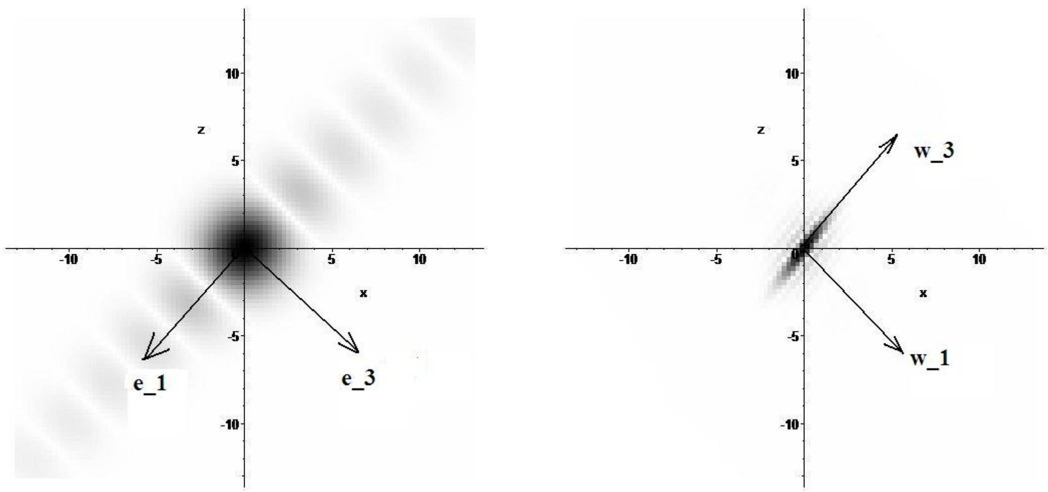

Figure 2: Refocus spot in homogeneous and random media

5.5 Statistical stability of the refocused pulse

To show the statistical stability of the pulse we consider the variance of the pulse amplitude. We prove stability only for $E_{TR,z}$ since other components can be treated similarly. The square of the amplitude is a sum of several terms. We consider one involving product of four generalized transmission coefficients. Expanding generalized coefficients in serieses we will have a sum of terms like

$$\begin{aligned}U_{n_{11}n_{12}n_{21}n_{22}} = &\overline{T_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, z_s, 0) R_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, z_s, 0)}^{n_{11}} \overline{R_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, -L, z_s)}^{n_{11}}\\
&T_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, z_s, 0) R_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, z_s, 0)^{n_{12}} R_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, -L, z_s)^{n_{12}}\\
&\overline{T_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, z_s, 0) R_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, z_s, 0)}^{n_{21}} \overline{R_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, -L, z_s)}^{n_{21}}\\
&T_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, z_s, 0) R_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, z_s, 0)^{n_{22}} R_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, -L, z_s)^{n_{22}}\end{aligned} \tag{239}$$

By independence of propagators of the two slabs $[-L, z_s]$ and $[z_s, 0]$ we need to study the expectations of

$$\begin{aligned}&\overline{T_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, z_s, 0) R_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, z_s, 0)}^{p_1}\\
&T_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, z_s, 0) R_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, z_s, 0)^{q_1}\\
&\overline{T_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, z_s, 0) R_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, z_s, 0)}^{p_2}\\
&T_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, z_s, 0) R_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, z_s, 0)^{q_2}\end{aligned} \tag{240}$$

and

$$\begin{aligned}&R_1(\omega_1 + \frac{\varepsilon h_1}{2}, \kappa_1 + \frac{\varepsilon l_1}{2}, -L, z_s)^{p_1} \overline{R_1(\omega_1 - \frac{\varepsilon h_1}{2}, \kappa_1 - \frac{\varepsilon l_1}{2}, -L, z_s)}^{q_1}\\
&R_1(\omega_2 + \frac{\varepsilon h_2}{2}, \kappa_2 + \frac{\varepsilon l_2}{2}, -L, z_s)^{p_2} \overline{R_1(\omega_2 - \frac{\varepsilon h_2}{2}, \kappa_2 - \frac{\varepsilon l_2}{2}, -L, z_s)}^{q_2}\end{aligned} \tag{241}$$

Using the same method as in [1] Sec. 9.2.4 we obtain that

$$
\begin{aligned}
&\mathbb{E}\Bigg(\overline{T_1(\omega_1-\frac{\varepsilon h_1}{2},\kappa_1-\frac{\varepsilon l_1}{2},z_s,0)R_1(\omega_1-\frac{\varepsilon h_1}{2},\kappa_1-\frac{\varepsilon l_1}{2},z_s,0)}^{p_1}\\
&\quad T_1(\omega_1+\frac{\varepsilon h_1}{2},\kappa_1+\frac{\varepsilon l_1}{2},z_s,0)R_1(\omega_1+\frac{\varepsilon h_1}{2},\kappa_1+\frac{\varepsilon l_1}{2},z_s,0)^{q_1}\\
&\quad \overline{T_1(\omega_2-\frac{\varepsilon h_2}{2},\kappa_2-\frac{\varepsilon l_2}{2},z_s,0)R_1(\omega_2-\frac{\varepsilon h_2}{2},\kappa_2-\frac{\varepsilon l_2}{2},z_s,0)}^{p_2}\\
&\quad T_1(\omega_2+\frac{\varepsilon h_2}{2},\kappa_2+\frac{\varepsilon l_2}{2},z_s,0)R_1(\omega_2+\frac{\varepsilon h_2}{2},\kappa_2+\frac{\varepsilon l_2}{2},z_s,0)^{q_2}\Bigg)\\
&\overset{\varepsilon\to 0}{\longrightarrow}\mathbb{E}\Bigg(\overline{T_1(\omega_1-\frac{\varepsilon h_1}{2},\kappa_1-\frac{\varepsilon l_1}{2},z_s,0)R_1(\omega_1-\frac{\varepsilon h_1}{2},\kappa_1-\frac{\varepsilon l_1}{2},z_s,0)}^{p_1}\\
&\quad T_1(\omega_1+\frac{\varepsilon h_1}{2},\kappa_1+\frac{\varepsilon l_1}{2},z_s,0)R_1(\omega_1+\frac{\varepsilon h_1}{2},\kappa_1+\frac{\varepsilon l_1}{2},z_s,0)^{q_1}\Bigg)\\
&\mathbb{E}\Bigg(\overline{T_1(\omega_2-\frac{\varepsilon h_2}{2},\kappa_2-\frac{\varepsilon l_2}{2},z_s,0)R_1(\omega_2-\frac{\varepsilon h_2}{2},\kappa_2-\frac{\varepsilon l_2}{2},z_s,0)}^{p_2}\\
&\quad T_1(\omega_2+\frac{\varepsilon h_2}{2},\kappa_2+\frac{\varepsilon l_2}{2},z_s,0)R_1(\omega_2+\frac{\varepsilon h_2}{2},\kappa_2+\frac{\varepsilon l_2}{2},z_s,0)^{q_2}\Bigg)
\end{aligned}
\tag{242}
$$

if $p_1 = q_1$, $p_2 = q_2$ and the limit is zero otherwise and similarly for the products (241). This proves that

$$\mathbb{E}E_{TR,z}^2 = (\mathbb{E}E_{TR,z})^2 \tag{243}$$

which is equivalent to the fact that the variance of the refocused waveform is zero:

$$\mathbb{E}\left[E_{TR,z} - \mathbb{E}E_{TR,z}\right]^2 = 0. \tag{244}$$

6 Conclusion

We have studied the the refocusing of the pulse obtained in time reversal experiment of electromagnetic waves. The propagation of electromagnetic waves is described by two systems which is the main difference from the acoustic case. On the phonomenological level this leads to anisotropy of the effective medium which in particular demonstrates itself in different pulse orientation of two transverse components.

7 Appendices

8 Appendix with long formulas

The formulas in this appendix are too large to be put in the main course of the thesis. They were obtained using software written in *Maple*, exported in *LaTeX* and post processed with scripts in *emacs* text editor.

$$
\begin{aligned}
\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) =& \frac{1}{2}\frac{1}{(2\pi\varepsilon)^6}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})} e^{\frac{i\omega_2}{\varepsilon}(t_s-\boldsymbol{\kappa}_2\cdot\boldsymbol{x}_s)} \overline{\hat{G}_1\left(\frac{\omega_1+\omega_2}{\varepsilon}\right)} \hat{G}_2\left(\frac{-\omega_2\boldsymbol{\kappa}_2+\omega_1\boldsymbol{\kappa}_1}{\varepsilon}\right)\omega_2^2\omega_1^2 \\
&\Bigg[\Bigg(1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2} \\
&-1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2} \\
&-1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0} \\
&+1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0} \\
&+1/2\,T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp} \\
&-1/2\,R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp} \\
&-1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}
\end{aligned}
$$

$$+\,1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$+\,1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$-\,1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$-\,1/2\,T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp}$$

$$+\,1/2\,R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\left.\times\, S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp}\right)\boldsymbol{\kappa}_{1,0}$$

$$-\left(1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\right.$$

$$\times\, S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}{\boldsymbol{\kappa}_{1,0}}^{\perp}\boldsymbol{\kappa}_{2,0}$$

$$-\,1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}{\boldsymbol{\kappa}_{1,0}}^{\perp}\boldsymbol{\kappa}_{2,0}$$

$$-\,1/2\,T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}{\boldsymbol{\kappa}_{1,0}}^{\perp}{\boldsymbol{\kappa}_{2,0}}^{\perp}$$

$$+\,1/2\,R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times\, S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}{\boldsymbol{\kappa}_{1,0}}^{\perp}{\boldsymbol{\kappa}_{2,0}}^{\perp}$$

$$-\,1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$
\begin{aligned}
&\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(-\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp}\\
&-1/2\,R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(-\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\left.\left.\times S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp}\right)\boldsymbol{\kappa}_{1,0}{}^{\perp}\right]\\
&d\omega_1 d\kappa_1 d\omega_2 d\kappa_2 \qquad (245)
\end{aligned}
$$

$$
\begin{aligned}
E_{TR,z}(t,z,\boldsymbol{x}) = &-\frac{1}{2\epsilon(z)}\frac{1}{(2\pi\varepsilon)^6}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}e^{\frac{i\omega_2}{\varepsilon}(t_s-\boldsymbol{\kappa}_2\cdot\boldsymbol{x}_s)}\overline{\hat{G}_1\left(\frac{\omega_1+\omega_2}{\varepsilon}\right)}\hat{G}_2\left(\frac{-\omega_2\boldsymbol{\kappa}_2+\omega_1\boldsymbol{\kappa}_1}{\varepsilon}\right)\omega_2^2\omega_1^2\\
&\times\left[1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\right.\\
&\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\times S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}
\end{aligned}
$$

$$
\begin{aligned}
&- 1/2\, R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp}\\
&+ 1/2\, T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&- 1/2\, R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&- 1/2\, T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+ 1/2\, R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+ 1/2\, T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp}\\
&- 1/2\, R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right) T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{\boldsymbol{\kappa}_{2,0}}^{\perp}\Bigg]\\
&d\omega_1 d\kappa_1 d\omega_2 d\kappa_2
\end{aligned}
\tag{246}
$$

$$
\begin{aligned}
\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = \frac{1}{2}\frac{1}{(2\pi\varepsilon)^6}\int & e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}e^{-\frac{i\omega_2}{\varepsilon}(t_s-\boldsymbol{\kappa}_2\cdot\boldsymbol{x}_s)}\overline{\hat{G}_1\left(\frac{\omega_1-\omega_2}{\varepsilon}\right)}\hat{G}_2\left(\frac{\omega_2\boldsymbol{\kappa}_2+\omega_1\boldsymbol{\kappa}_1}{\varepsilon}\right)\omega_2^2\omega_1^2\\
&\Bigg[\Bigg(1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}\\
&\quad\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&- 1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right) e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}
\end{aligned}
$$

$$\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$-1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$+1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$+1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}$$

$$-1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}$$

$$-1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$+1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1||\boldsymbol{\kappa}_2|\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$+1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$-1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$-1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$\times \overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}$$

$$+1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)}$$

$$
\begin{aligned}
&\times \overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_1\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}\Bigg)\boldsymbol{\kappa}_{1,0} \\
&-\Bigg(1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0} \\
&-1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0} \\
&-1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp} \\
&+1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp} \\
&-1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0} \\
&+1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0} \\
&+1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp} \\
&-1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,2}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}\left(\lambda_2(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s\right)} \\
&\quad\times \overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\xi_2\left(\boldsymbol{\kappa}_1\right)\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}{}^{\perp}\boldsymbol{\kappa}_{2,0}{}^{\perp}\Bigg)\boldsymbol{\kappa}_{1,0}{}^{\perp}\Bigg]d\omega_1 d\boldsymbol{\kappa}_1 d\omega_2 d\boldsymbol{\kappa}_2
\end{aligned} \tag{247}
$$

$$
E_{TR,z}(t,z,\boldsymbol{x}) = -\frac{1}{2\epsilon(z)}\frac{1}{(2\pi\varepsilon)^6}\int e^{-\frac{i\omega_1}{\varepsilon}(t-\boldsymbol{\kappa}_1\cdot\boldsymbol{x})}e^{-\frac{i\omega_2}{\varepsilon}(t_s-\boldsymbol{\kappa}_2\cdot\boldsymbol{x}_s)}\overline{\hat{G}_1\left(\frac{\omega_1-\omega_2}{\varepsilon}\right)}\hat{G}_2\left(\frac{\omega_2\boldsymbol{\kappa}_2+\omega_1\boldsymbol{\kappa}_1}{\varepsilon}\right)\omega_2^2\omega_1^2
$$

$$
\begin{aligned}
&\Big[1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}\\
&-1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}R_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}\\
&+1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}\left(|\boldsymbol{\kappa}_1|\right)^2|\boldsymbol{\kappa}_2|\left(\xi_1\left(\boldsymbol{\kappa}_1\right)\right)^{-1}\frac{1}{\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
&-1/2\,\overline{T_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z-\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}\\
&\quad\times\overline{S_{a,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}\\
&+1/2\,\overline{R_{g,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_1(\boldsymbol{\kappa}_2)\omega_2 z_s)}
\end{aligned}
$$

$$\times \overline{S_{b,1}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_1\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}$$

$$+ 1/2\,\overline{T_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{i\varepsilon^{-1}(-\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s)}$$

$$\times \overline{S_{a,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}$$

$$- 1/2\,\overline{R_{g,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}T_{g,1}\left(z,\omega_1,\boldsymbol{\kappa}_1\right)e^{-i\varepsilon^{-1}(\lambda_1(\boldsymbol{\kappa}_1)\omega_1 z+\lambda_2(\boldsymbol{\kappa}_2)\omega_2 z_s)}$$

$$\left.\times \overline{S_{b,2}\left(z_s,\omega_2,\boldsymbol{\kappa}_2\right)}|\boldsymbol{\kappa}_1|\sqrt{\xi_2\left(\boldsymbol{\kappa}_2\right)}\boldsymbol{\kappa}_{1,0}\boldsymbol{\kappa}_{2,0}{}^{\perp}\right]d\omega_1 d\boldsymbol{\kappa}_1 d\omega_2 d\boldsymbol{\kappa}_2 \tag{248}$$

$$\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x}) = \frac{1}{2}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{1/4ih\boldsymbol{l}\cdot(\boldsymbol{x}-\boldsymbol{x}_s)\varepsilon}e^{1/2ih(t_s-t+(\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})+1/2i\omega\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s)}e^{\frac{i\omega}{\varepsilon}(-(t+t_s)+(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})}$$

$$\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)(\omega-\varepsilon h/2)^2(\omega+\varepsilon h/2)^2$$

$$\Bigg[\Bigg(\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)$$

$$\times e^{\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)+i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}$$

$$\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\left|-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)$$

$$\times e^{\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)+i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}$$

$$\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\left|-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}}\left(\epsilon\left(0\right)\right)^{-2}$$

$$-\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)$$

$$\times e^{\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)+i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}$$

$$\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}$$

$$\times\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0$$

$$+\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)$$

$$
\begin{aligned}
&\times e^{\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\\
+\,&\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)+i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)}\\
&\times \overline{S_{a,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\\
-\,&\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)+i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)}\\
&\times \overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\\
-\,&\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)-i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\left|-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}}\left(\epsilon\left(0\right)\right)^{-2}\\
+\,&\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)-i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\left|-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}}\left(\epsilon\left(0\right)\right)^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}\\
&\quad\times\overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{\frac{1}{2}ih\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}\\
&\quad\times\overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&-\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)}\\
&\quad\times\overline{S_{a,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&+\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)}\\
&\quad\times\overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\Bigg)\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&-\Bigg(\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times e^{\frac{1}{2}\,ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\cdot\left(\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\\
-&\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\cdot\left(\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\\
-&\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)+i\omega\,\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)}\\
&\times \overline{S_{a,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\\
+&\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,2}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)+i\omega\,\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z_s\right)}\\
&\times \overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\\
-&\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)-i\omega\,\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}
\end{aligned}
$$

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\begin{aligned}
&\times \left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0 \cdot \left(\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp \\
&+\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right) \\
&\times e^{-\frac{1}{2}ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)-i\omega\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)} \\
&\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)} \\
&\times \left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0 \cdot \left(\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp \\
&+\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right) \\
&\times e^{-\frac{1}{2}ih\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)+i\omega\varepsilon^{-1}\left(-\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)} \\
&\times \overline{S_{a,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)} \\
&\times \left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp \cdot \left(\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp \\
&-\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,2}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right) \\
&\times e^{\frac{1}{2}ih\left(-\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)-i\omega\varepsilon^{-1}\left(\lambda_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z_s\right)} \\
&\times \overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\xi_2\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)} \\
&\times \left(\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp \cdot \left(\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp\right)\left(\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^\perp\Bigg] d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh
\end{aligned}
\tag{249}
$$

$$
\begin{aligned}
E_{TR,z}(t,z,\boldsymbol{x}) = &-\frac{1}{2\epsilon(z)}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{1/4ih\boldsymbol{l}\cdot(\boldsymbol{x}-\boldsymbol{x}_s)\varepsilon}e^{1/2ih(t_s-t+(\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})+1/2i\omega\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s)}e^{\frac{i\omega}{\varepsilon}(-(t+t_s)+(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})} \\
&\overline{\hat{G}_1(h)}\hat{G}_2(h\boldsymbol{\kappa}+\omega\boldsymbol{l})(\omega-\varepsilon h/2)^2(\omega+\varepsilon h/2)^2 \\
&\Bigg[\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right) \\
&\times e^{\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)+i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \overline{S_{a,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \left(\left|\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right|\right)^2 \left|-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right| \\
&\times \frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)}} \left(\xi_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)^{-1} (\epsilon(0))^{-2} \\
- &\overline{R_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\times e^{\frac{1}{2} ih\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z + z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right) + i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z - z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)} \\
&\times \overline{S_{b,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \left(\left|\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right|\right)^2 \left|-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right| \\
&\times \frac{1}{\sqrt{\xi_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)}} \left(\xi_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)^{-1} (\epsilon(0))^{-2} \\
- &\overline{T_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\times e^{\frac{1}{2} ih\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z - z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right) + i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z + z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)} \\
&\times \overline{S_{a,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \left|\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right| \sqrt{\xi_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \\
&\times \left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0 \cdot \left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0 \\
+ &\overline{R_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\times e^{\frac{1}{2} ih\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z + z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right) + i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z - z_s\lambda_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)} \\
&\times \overline{S_{b,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \left|\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right| \sqrt{\xi_1\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \\
&\times \left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0 \cdot \left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)_0 \\
+ &\overline{T_{g,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\times e^{\frac{1}{2} ih\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z - z_s\lambda_2\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right) + i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)z + z_s\lambda_2\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)\right)} \\
&\times \overline{S_{a,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} \left|\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right| \sqrt{\xi_2\left(-\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\\
&-\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\\
&+\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left(\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\right)^{2}\left|-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\\
&\times \frac{1}{\overline{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}}}\left(\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)^{-1}\left(\epsilon\left(0\right)\right)^{-2}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)-i\omega\,\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left(\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\right)^{2}\left|-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\\
&\times \frac{1}{\overline{\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}}}\left(\xi_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)^{-1}\left(\epsilon\left(0\right)\right)^{-2}\\
&-\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\,\varepsilon\,h,\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}\,ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)+i\omega\,\varepsilon^{-1}\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)\right)}\\
&\times \overline{S_{a,1}\left(z_s,\omega-\frac{1}{2}\,\varepsilon\,h,-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right|\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)}\\
&\times \left(-\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\,\varepsilon\,\boldsymbol{l}\right)_0
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z-z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}\\
&\quad\times\overline{S_{b,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\sqrt{\xi_1\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&+\overline{T_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{-\frac{1}{2}ih\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)+i\omega\varepsilon^{-1}\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}\\
&\quad\times\overline{S_{a,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\\
&-\overline{R_{g,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\quad\times e^{\frac{1}{2}ih\left(-\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)-i\omega\varepsilon^{-1}\left(\lambda_1\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)z+z_s\lambda_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right)}\\
&\quad\times\overline{S_{b,2}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\left|\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right|\sqrt{\xi_2\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}\\
&\quad\times\left(\left(-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\right)^{\perp}\cdot\left(\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)_0\Bigg]\\
&d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh
\end{aligned}
\tag{250}
$$

$$
\begin{aligned}
\boldsymbol{E}_{TR,t}(t,z,\boldsymbol{x})=&\frac{1}{2}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{1/2ih(t_s-t+(\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})+1/2i\omega\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s)}e^{\frac{i\omega}{\varepsilon}(-(t+t_s)+(\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})}\\
&\overline{\hat{G}_1\left(h\right)}\hat{G}_2\left(h\boldsymbol{\kappa}+\omega\boldsymbol{l}\right)\omega^4\\
&\left[\left\{\kappa^2\frac{1}{\sqrt{\xi_1\left(\kappa\right)}}\epsilon\left(0\right)^{-2}+\xi_1\left(\kappa\right)^{3/2}\right\}\right.\\
&\left(\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\right.\\
&\quad\times e^{\frac{1}{2}ih\lambda_1(\kappa)(z-z_s)+\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z+z_s)-\frac{i\omega}{2}\frac{\bar{\epsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z-z_s)}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&- \overline{R_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{\frac{1}{2} ih\lambda_1(\kappa)(z+z_s) + \frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z-z_s) - \frac{i\omega}{2}\frac{\bar{\epsilon}}{\epsilon_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)} \overline{S_{b,1}(z_s, \omega, -\boldsymbol{\kappa})} \\
&- \overline{T_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} T_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{-\frac{1}{2} ih\lambda_1(\kappa)(z+z_s) - \frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z-z_s) + \frac{i\omega}{2}\frac{\bar{\epsilon}}{\epsilon_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)} \overline{S_{a,1}(z_s, \omega, -\boldsymbol{\kappa})} \\
&+ \overline{R_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} T_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{-\frac{1}{2} ih\lambda_1(\kappa)(z-z_s) - \frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z+z_s) + \frac{i\omega}{2}\frac{\bar{\epsilon}}{\epsilon_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z-z_s)} \overline{S_{b,1}(z_s, \omega, -\boldsymbol{\kappa})}\Bigg) \boldsymbol{\kappa}_0 \\
&- \xi_2(\kappa)^{3/2}\Bigg(- \overline{T_{g,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,2}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{\frac{1}{2} ih\lambda_2(\kappa)(z-z_s) + \frac{i\omega}{\varepsilon}\lambda_2(\kappa)(z+z_s) - \frac{i\omega}{2}\frac{\bar{\mu}}{\mu_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}(z-z_s)} \overline{S_{a,2}(z_s, \omega, -\boldsymbol{\kappa})} \\
&+ \overline{R_{g,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,2}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{\frac{1}{2} ih\lambda_2(\kappa)(z+z_s) + \frac{i\omega}{\varepsilon}\lambda_2(\kappa)(z-z_s) - \frac{i\omega}{2}\frac{\bar{\mu}}{\mu_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}(z+z_s)} \overline{S_{b,2}(z_s, \omega, -\boldsymbol{\kappa})} \\
&+ \overline{T_{g,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} T_{g,2}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{-\frac{1}{2} ih\lambda_2(\kappa)(z+z_s) - \frac{i\omega}{\varepsilon}\lambda_2(\kappa)(z-z_s) + \frac{i\omega}{2}\frac{\bar{\mu}}{\mu_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}(z+z_s)} \overline{S_{a,2}(z_s, \omega, -\boldsymbol{\kappa})} \\
&- \overline{R_{g,2}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} T_{g,2}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right) \\
&\quad \times e^{-\frac{1}{2} ih\lambda_2(\kappa)(z-z_s) - \frac{i\omega}{\varepsilon}\lambda_2(\kappa)(z+z_s) + \frac{i\omega}{2}\frac{\bar{\mu}}{\mu_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_2(\kappa)}(z-z_s)} \overline{S_{b,2}(z_s, \omega, -\boldsymbol{\kappa})}\Bigg) \boldsymbol{\kappa}_0{}^{\perp}\Bigg] d\boldsymbol{\kappa} d\boldsymbol{l} d\omega dh
\end{aligned}
\tag{251}
$$

$$
\begin{aligned}
E_{TR,z}(t, z, \boldsymbol{x}) = &- \frac{1}{2\epsilon(z)}\frac{1}{(2\pi)^6\varepsilon^3}\int e^{1/2ih(t_s - t + (\boldsymbol{x}+\boldsymbol{x}_s)\cdot\boldsymbol{\kappa}) + 1/2i\omega\boldsymbol{l}\cdot(\boldsymbol{x}+\boldsymbol{x}_s)} e^{\frac{i\omega}{\varepsilon}(-(t+t_s) + (\boldsymbol{x}-\boldsymbol{x}_s)\cdot\boldsymbol{\kappa})} \\
&\overline{\hat{G}_1(h)}\hat{G}_2(h\boldsymbol{\kappa} + \omega\boldsymbol{l})\omega^4\left\{\kappa^3\xi_1(\kappa)^{-3/2}(\epsilon(0))^{-2} + \kappa\sqrt{\xi_1(\kappa)}\right\} \\
&\Bigg[\overline{T_{g,1}\left(z_s, \omega - \frac{1}{2}\varepsilon h, -\boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)} R_{g,1}\left(z, \omega + \frac{1}{2}\varepsilon h, \boldsymbol{\kappa} + \frac{1}{2}\varepsilon \boldsymbol{l}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times e^{\frac{1}{2}ih\lambda_1(\kappa)(z-z_s)+\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z+z_s)-\frac{i\omega}{2}\frac{\bar{\varepsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z-z_s)}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}R_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\times e^{\frac{1}{2}ih\lambda_1(\kappa)(z+z_s)+\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z-z_s)-\frac{i\omega}{2}\frac{\bar{\varepsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&+\overline{T_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\times e^{-\frac{1}{2}ih\lambda_1(\kappa)(z+z_s)-\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z-z_s)+\frac{i\omega}{2}\frac{\bar{\varepsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z+z_s)}\overline{S_{a,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\\
&-\overline{R_{g,1}\left(z_s,\omega-\frac{1}{2}\varepsilon h,-\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)}T_{g,1}\left(z,\omega+\frac{1}{2}\varepsilon h,\boldsymbol{\kappa}+\frac{1}{2}\varepsilon\boldsymbol{l}\right)\\
&\left.\times e^{-\frac{1}{2}ih\lambda_1(\kappa)(z-z_s)-\frac{i\omega}{\varepsilon}\lambda_1(\kappa)(z+z_s)+\frac{i\omega}{2}\frac{\bar{\varepsilon}}{\bar{\epsilon}_1}\frac{\boldsymbol{\kappa}\cdot\boldsymbol{l}}{\lambda_1(\kappa)}(z-z_s)}\overline{S_{b,1}\left(z_s,\omega,-\boldsymbol{\kappa}\right)}\right]d\boldsymbol{\kappa}d\boldsymbol{l}d\omega dh
\end{aligned}
\tag{252}
$$

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