A Numerical Method for Analysis of Free Vibrations of Rotational Shells Partially Filled with Liquid

Takashi Mikami, Jin Yoshimura
Hokkaido University, Sapporo, Japan

INTRODUCTION

In the dynamic analysis of the liquid-rotational shell interaction problems, a fundamental step is the determination of the natural frequencies and mode shapes of the free vibration. The boundary element method and finite element method have been used in recent years. However, the former requires the existence of appropriate fundamental solutions, and the latter leads to a large system of equations resulting from spatial discretizations.

The objectives of this paper are: (1) To present a simple and effective solution procedure, based on the collocation method, for the free vibration of rotational shells partially filled with liquid; and (2) to demonstrate the high accuracy of the procedure developed by means of numerical examples.

SHELL GEOMETRY AND COORDINATE SYSTEM

Figure 1 shows a shell of revolution partially filled with a liquid to be height, H. The shell under consideration is of uniform thickness h, base radius a, made of homogeneous, isotropic material with elasticity modulus E, Poisson's ratio ν and mass density ρ_s. The points on the middle surface of the shell will be referred to the coordinate system (s, θ, ζ).

With the liquid density denoted by ρ_l, and the points in the liquid with the origin at the surface are specified by the coordinate system (r, θ, z), as shown in Fig. 1, where r is the distance of point on the middle surface of the shell from the axis of revolution, and z is the axial distance measured from the liquid free surface.

EQUATIONS GOVERNING LIQUID MOTION

The liquid is assumed to be homogeneous, incompressible, invisid and irrotational. Under these conditions the liquid motion is governed by the Laplace equation

$$\partial^2\Phi/\partial r^2+1/r\cdot\partial\Phi/\partial\theta+1/r^2\cdot\partial^2\Phi/\partial\theta^2+\partial^2\Phi/\partial z^2=0 \quad .....(1)$$

in which Φ is a velocity potential function. The liquid-shell boundary conditions for the velocity potential are as follows:

$$\partial \Phi / \partial z = 0; \text{ at } z = H \quad \text{..........(2)} \qquad \Phi = 0; \text{ at } z = 0 \quad \text{..........(3)}$$

$$\partial \Phi / \partial n = \partial W / \partial t; \text{ at } r = r \quad \text{.........(4)}$$

in which W is the displacement of the middle surface of the shell in normal direction ζ, and $\partial(\cdot)/\partial n$ is the directional derivative along the outward unit normal. The pressure distribution, p, can be determined from the linearized Bernoulli equation and is given by

$$p = \rho_l \cdot \partial \Phi / \partial t \qquad \text{............................(5)}$$

All variables used herein will be expanded into Fourier series with appropriate normalization to provide nondimensional Fourier coefficients. For each circumferential wave number n, the displacement W, the velocity potential Φ, and the hydrodynamic pressure p can be written in the form as

$$W = \frac{\sigma a^2}{Eh} \sum_n w \cos n\theta e^{i\omega t}, \quad \Phi = i\omega \frac{\sigma a^3}{Eh} \sum_n \phi \cos n\theta e^{i\omega t} \quad \text{......(6a, b)}$$

$$p = \rho_l \frac{\omega^2 \sigma a^3}{Eh} \sum_n p \cos n\theta e^{i\omega t} \qquad \text{........................(7)}$$

in which σ is a reference stress level, and ω is the circular natural frequency.

A problem such as described by Eq. 1 to Eq. 3 is called a Sturm-Liouville problem. Using the method of separation of variables, the solution ϕ can be expressed as

$$\phi = \sum_i A_i I_n(\lambda_i \rho) f_i(\eta) \qquad \text{.........................(8)}$$

in which $\rho = r/a$, $\eta = z/H$, A_i are unknown coefficients, I_n is the modified Bessel function of the first kind, and the eigenvalues λ_i and corresponding eigenfunctions $f_i(\eta)$ are given by

$$\lambda_i = (2j-1)\pi a/2H, \quad f_i(\eta) = \sin(\lambda_i H \eta / a) \qquad \text{...................(9)}$$

The unknown coefficients A_i in Eq. 8 can be determined by satisfying the boundary condition (Eq. 4) at the liquid-shell interface. After the appropriate algebraic manipulations of the boundary condition, the coefficients A_i can be obtained from the following equation:

$$[a]\{A\} = \{b\} \qquad \text{.........................(10)}$$

in which $\{A\}^T = (A_1, A_2, \ldots A_l, \ldots)$, and the elements of [a] and {b} are given by

$$a_{ls} = \int_0^1 g_s(\eta) \sin(\lambda_l H \eta / a) d\eta \quad (l, s = 1, 2, \ldots) \qquad \text{...............(11)}$$

$$b_l = \int_0^1 w \sin(\lambda_l H \eta / a) d\eta \quad (l = 1, 2, \ldots) \qquad \text{.....................(12)}$$

where $g_s(\eta)$ are functions, which can be obtained by substituting Eq. 6 into Eq. 4.

Since matrix [a] in Eq. 10 possesses an inverse, therefore

$$\{A\} = [a]^{-1}\{b\} \qquad \text{........................(13)}$$

Using the above equation and substituting Eq. 8 into Eq. 5, the hydrodynamic pressure acting on the inner surface of the shell can be evaluated, and its expression contains the integral (Eq. 12).

EQUATIONS GOVERNING SHELL MOTION

The analytical formulation is based on an improved theory (Magrab 1979) with the effects of shear deformation and rotary inertia. The system of equations for the analysis of shells of revolution and boundary conditions can be formulated with the middle surface displacements as the dependent variables.

Considering the hydrodynamic pressure exerted on the wall of the shell and substituting the stress resultant-displacement relations into equilibrium equations, the governing equations are described by a system of integro-differential equations for harmonic amplitudes of the displacement variables. These equations can be written in matrix form:

$$[C]\{X''\}+[D]\{X'\}+[E]\{X\}=\Omega^2([F]\{X\}+\{p\}) \qquad \text{...........(14)}$$

in which $(\;)'=d(\;)/d\xi$, ξ is the nondimensional independent variable, where ξ takes the values 0 to 1; $\Omega^2=a^2\rho_s(1-\nu^2)/E$ is a frequency parameter; and the displacement vector $\{X\}$ is given by

$$\{X\}^T=(X_1,\ X_2,\ X_3,\ X_4,\ X_5)=(u,\ v,\ w,\ \beta_x,\ \beta_\theta) \qquad \text{........(15)}$$

in which u, v= tangental displacements; w= normal displacement; β_x, β_θ= rotation angles. Moreover, [C], [D], [E] and [F] are the 5x5 coefficient matrices whose elements are functions of elastic coefficients, harmonic number, and geometric properties of the shell. The formulas (Mikami et al, 1986) for the elements of these matrices have been omitted here for brevity. {p} is the hydrodynamic pressure vector of the element which is given by

$$\{p\}^T=(0,\ 0,\ p_w,\ 0,\ 0) \qquad \text{........................(16)}$$

where $$p_w=(\rho_l/\rho_s)(a/h)\sum_{s=1}^{\infty}A_s I_n(\lambda_s\rho)\sin(\lambda_s H\eta/a) \qquad \text{.......(17)}$$

The right hand side {p} in Eq. 14 contains the integral (Eq. 12), therefore the problem under consideration leads to integro-differential equations.

The quantities which appear in the boundary conditions on the edges of the shell are the displacement components, the stress resultants N_x, $N_{x\theta}$, θ_x and the moment resultants M_x and $M_{x\theta}$. The Fourier coefficients for the stress and moment resultants can be expressed in terms of the displacements. These can be written in the form:

$$\{T\}=[G]\{X'\}+[H]\{X\} \qquad \text{..........................(18)}$$

in which $\{T\}^T=(n_x,\ n_{x\theta},\ q_x,\ m_x,\ m_{x\theta})$ is the vector which are composed of the Fourier coefficients, and [G] and [H] are the 5x5 matrices (Mikami et al, 1986). For the boundary conditions, the following appropriate five of the quantities are to be prescribed at each boundary:

$$u \text{ or } n_x,\ v \text{ or } n_{x\theta},\ w \text{ or } q_x,\ \phi_x \text{ or } m_x,\ \phi_\theta \text{ or } m_{x\theta} \qquad \text{......(19)}$$

SOLUTION PROCEDURE

The displacement functions in Eq. 14 are approximated by the following form:

$$X_i = \sum_{s=1}^{M+2} N_s(\xi) X_{is}^{*} \qquad (i=1\sim5) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

in which $N_s(\xi)$ are interpolation functions; and X_{is}^{*} are the values of unknowns at the (M+2)-points, and these are to be solved for.

The following comments regarding the solution procedure seem to be in order:
1. The derivative approximation is based on the collocation method. The discrete points consist of the interior collocation points $\xi_k (k=1\sim M)$ and the two end-points $\xi_0=0$, $\xi_{M+1}=1$. In this paper the interior collocation points are selected to be zeros of the M-th shifted Legendre polynomial $P_M(\xi)$ defined on $0<\xi<1$.
2. To decrease the computational effort required, the following two matrices [A] and [B] are used to approximate the first and second derivatives of Eq. 14:

$$\{X_i'\}=[A]\{X_i^{*}\},\ \{X_i''\}=[B]\{X_i^{*}\} \qquad (i=1\sim5) \qquad \ldots\ldots\ldots\ldots(21)$$

in which [A] and [B] are the (M+2)x(M+2) matrices; and the vectors $\{X_i^{*}\}$, etc. are $\{X_i^{*}\}^T=\{X_i^{*}(\xi_0),\ X_i^{*}(\xi_1),\ \ldots,\ X_i^{*}(\xi_{M+1})\}$, etc.
3. In this formulation the complete set of displacement vector $\{X_i^{*}\}$ is partitioned into the set associated with the interior collocation points $\{X_{i,c}^{*}\}$ and the set associated with the end-points $\{X_{i,e}^{*}\}$; thus

$$\{X_{i,c}^{*}\}^T=\{X_i(\xi_1),\ X_i(\xi_2),\ \ldots,\ X_i(\xi_M)\}$$

$$\{X_{i,e}^{*}\}^T=\{X_i(\xi_0),\ X_i(\xi_{M+1})\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

in which the subscripts c and e refer to the interior collocation points and the end-points, respectively.
4. Making the change of variable $(\eta\rightarrow\xi)$ and choosing the distinct points of the interval [0, 1] to be coincident with the interior collocation points, the integration involved in the evaluation of {p} (Eq. 16) is carried out approximately by Gauss-Legendre rule. This integral gives rise to the definition of an added mass matrix associated with the normal displacement vector $\{X_{3,c}^{*}\}=\{W_c\}$.

Let us consider the application of the method to a completely filled shell. In this problem there are 5M+10 unknowns.

Following the approximations for derivatives and integrals, Eq. 14 can be written in the matrix form as

$$[\alpha_c]\{\delta_c\}+[\alpha_e]\{\delta_e\}=\Omega^2([M_c^s]+[M_c^l])\{\delta_c\} \qquad \ldots\ldots\ldots(23)$$

in which $[M_c^s]$ is the 5Mx5M mass matrix of the shell, the 5Mx5M matrix $[\alpha_c]$ and the 5Mx10 matrix $[\alpha_e]$ are composed of the elements of matrices [A] and [B] shown in Eq. 21, $[M_c^l]$ is the 5Mx5M added mass matrix of the liquid, and the 5M and 10 dimensional vectors, $\{\delta_c\}$ and $\{\delta_e\}$, are

$$\{\delta_c\}^T=(\{u_c\}^T,\ \{v_c\}^T,\ \{w_c\}^T,\ \{\beta_{xc}\}^T,\ \{\beta_{\theta c}\}^T)$$

$$\{\delta_e\}^T=(\{u_e\}^T,\ \{v_e\}^T,\ \{w_e\}^T,\ \{\beta_{xe}\}^T,\ \{\beta_{\theta e}\}^T) \qquad \ldots\ldots\ldots(24)$$

The remaining 10-equations are obtained from the boundary conditions on the edges of the shell. Using Eqs. 19 and 21, the boundary conditions lead to the matrix form

$$[\gamma_c]\{\delta_c\}+[\gamma_e]\{\delta_e\}=\{0\} \qquad \text{..................(25)}$$

in which $[\gamma_c]$ and $[\gamma_e]$ are the 10x5M and 10x10 matrices, respectively.

From two sets of equations (Eqs. 23 and 25), $\{\delta_e\}$ can be eliminated, resulting in

$$([\alpha_c]-[\alpha_e][\gamma_e]^{-1}[\gamma_c])\{\delta_c\}=\Omega^2([M_c{}^s]+[M_c{}^l])\{\delta_c\} \qquad \text{.........(26)}$$

Eq. 26 represents a 5Mx5M matrix eigenvalue problem.

In the case when the shell is partly-filled with liquid, two different equations similar to Eq. 23 are obtained for the lower (wet) and the upper (dry) portions of the shell. The eigenvalue equation similar to Eq. 26 is obtained by satisfying the boundary conditions at the bottom and top of the shell and the compatibility equations at the junction of the wet and dry parts of the shell.

NUMERICAL STUDIES

In order to test the validity of the present method, two types of shells are employed as illustrative examples; i.e., (1) a cylindrical shell; (2) a spherical shell.

Fixed-Free Cylindrical Shell Completely Filled with Liquid. — The dimensions and properties are a (radius)=7.32m, h (thickness)=1.09cm, L (height)=21.96m, ν (Poisson's ratio)=0.3, E (Young's modulus)=2.1×10^6kg/cm^2, ρ_s(shell density) =8.00×10^{-6}kg•sec^2/cm^4, and ρ_l(liquid density)=1.02×10^{-6}kg•sec^2/cm^4. The analysis was carried out with M=11. The computed natural frequencies are presented in Table 1 with those calculated by the finite element method (Haroun et al, 1982). It should be noted that the results by the present method are considerably accurate.

Fixed Hemispherical Shell Partially Filled with Liquid. — The shell of the thickness-to-radius ratio h/a=0.01 and liquid depth-to-radius ratio H/a=0.5 was considered. The material properties are the same values as in the first example. The shell partially filled with the liquid was modeled with 2 elements (portions), and the number of discrete points was taken to be M=11 per element. The convergence of solution is illustrated in Table 2 by computing the frequency parameters using 8, 10 (standard), and 12 terms in the series expansion of the velocity potential ϕ (Eq. 8). It can be seen that the results converge rapidly. Table 3 presents the computed results for the circumferential wave numbers (n=1~4), together with results obtained by the Rayleigh-Ritz method. It may be observed from the table that good accuracy is obtained with a relatively small number of discrete points.

CONCLUSION

A simple and computationally effective method for analyzing the free vibrations of the shells of revolution filled with liquid is presented. The liquid region is treated analytically and the shell is discretized by the collocation method. From the example problems studied, it is observed that the results of the method are in excellent agreement with those obtained from other numerical methods.

REFERENCES

Magrab, E. B. (1979). Vibrations of Elastic Structural Members. Sijhoff & Noodhoff.

Mikami, T. and Yoshimura, J. (1986). Application of the Collocation Method to Stress Wave Problems of Shells of Revolution. Proceedings of Japan Society of Civil Engineers (JSCE), No. 374/I-6, pp. 319-328 (in Japanese).

Haroun, M. M. and Housner, G. W. (1982). Dynamic Characteristics of Liquid Storage Tanks. Journal of the Engineering Mechanics Division (ASCE), Vol. 108, pp.783-800.

Table 1. Comparative analysis of natural frequencies f_{mn} for cylindrical shell

Solution procedure	m	n=1	n=2	n=3	n=4	n=5
Present F.E.M.	1	3.548 3.559	1.638 1.65	0.934 0.95	0.632 0.65	0.531 0.55
Present F.E.M.	2	10.338 10.450	6.550 6.66	4.401 4.52	3.162 3.28	2.397 2.52

n = number of circumferential waves
m = number of meridional waves

Table 2. Convergence of frequency parameters Ω_{mn} for spherical shell

I	n	m=1	m=2	m=3	m=4	m=5
8 10 12	1	0.366 0.365 0.365	0.633 0.632 0.632	0.732 0.731 0.731	0.896 0.895 0.895	0.939 0.938 0.937
8 10 12	2	0.596 0.596 0.596	0.750 0.749 0.748	0.946 0.945 0.944	0.985 0.983 0.983	1.048 1.047 1.047

Table 3. Comparative analysis of frequency parameters Ω_{mn} for spherical shell

n	m	Present	R.R.M.
1	1	0.365	0.368
2	1	0.484	0.494
3	1	0.535	0.546
4	1	0.569	0.575

R.R.M. = Rayleigh-Ritz solution using the first five modes in air

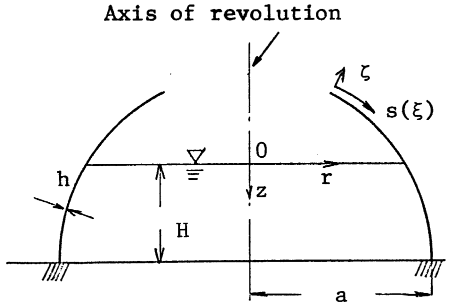

Fig. 1 Shell geometry and coordinate system