The general solutions of toroidal shells and curved tubes

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1 INTRODUCTION

Because of the wide application of thin-walled toroidal shells and curved tubes in industry the stress and deformation analysis for them attracts the attention of many engineers and researchers. For example, there are more than 10,000 bends used in one nuclear reactor unit and more than million bellows per year are produced in some industrial countries. In the present paper the term toroidal shell is used for those which are closed in ϕ-direction but unclosed in x-direction while the curved tubes are closed in x-direction and unclosed in ϕ-direction. (Fig.1) For the toroidal shells Zhang Wei (1948), Clark (1950) gave asymptotic solutions of symmetric loadings. Chernykh (1972) obtained an asymptotic solution of wind-type loadings. For the curved tubes a detailed survey was given in AMR by Axelrad and Emmerling (1984). But the solutions of nonsymmetric loadings and with complex boundary conditions are relatively less studied.

In the present paper the general solution of toroidal shells and curved tubes has been obtained based on Novozhilov's thin shell equations.(1962) They can be used in stress and deformation analysis of those structures with various boundary conditions and subjected to symmetric or nonsymmetric loadings.

2 BASIC EQUATIONS

The governing equations for shells of revolution in the form of function of complex variables are

$$G(\bar{U})-[1-ic(\frac{1}{R_1}-\frac{1}{R_2})\frac{1}{\cos^2 x}]\frac{\partial^2\bar{T}}{\partial\phi^2}=\bar{f}(x,\phi)$$

$$-icG(\bar{T})+\bar{T}+(\frac{1}{R_1}-\frac{1}{R_2})\frac{1}{\cos^2 x}\bar{U}=R_2\bar{q}_n' \qquad (1)$$

For the toroidal shells or curved tubes we have

$$R_1=r_0 \quad R_2=R_0\eta/\cos x \quad \eta=1+\alpha\cos x \quad \alpha=r_0/R_0 \qquad (2)$$

Eliminating $\bar{U}$ from eqns.(1) a partial differential eqn. of fourth order for function $\bar{T}$ is obtained.

$$\eta^4\frac{\partial^4\bar{T}}{\partial x^4}-8a\eta^3\sin x\frac{\partial^3\bar{T}}{\partial x^3}+[i\frac{ar_0}{c}\eta^3\cos x+$$

$$\eta^2(1+14a^2-4a\cos x-19a^2\cos^2x)]\frac{\partial^2\bar{T}}{\partial x^2}-[i\frac{ar_0}{c}\eta^2\sin x(3+7a\cos x)$$

$$+a\eta\sin x(2+4a^2-6a\cos x-12a^2\cos^2x)]\frac{\partial\bar{T}}{\partial x}+a^4\frac{\partial^4\bar{T}}{\partial\phi^4}+2a^2\eta^2\frac{\partial^4\bar{T}}{\partial x^2\partial\phi^2}$$

$$-4a^3\eta\sin x\frac{\partial^3\bar{T}}{\partial x\partial\phi^2}+[i\frac{a^2r_0}{c}\eta^2+a^3(a-2\cos x-2a\cos^2x)]\frac{\partial^2\bar{T}}{\partial\phi^2}+$$

$$i\frac{ar_0}{c}\eta[7a+(9a^2-2)\cos x-12a\cos^2x-12a^2\cos^3x]\bar{T}=\bar{F}(x,\phi) \qquad (3)$$

All internal force components can be expressed by complex function $\bar{T}$

$$T_1=-\frac{c}{r_0}\eta\,\mathrm{Im}\frac{\partial^2\bar{T}}{\partial x^2}+\frac{ac}{r_0}\sin x\,\mathrm{Im}\frac{\partial\bar{T}}{\partial x}-a\cos x\,\mathrm{Re}\bar{T}-\frac{a^2c}{r_0\eta}\mathrm{Im}\frac{\partial^2\bar{T}}{\partial\phi^2}+q_nr_0\eta \qquad (4)$$

...

Then the displacement components can be derived by integrating the elastic relations.

$$\epsilon_1=\frac{1}{r_0}(\frac{\partial u}{\partial x}+w)=\frac{1}{Eh}(T_1-\nu T_2) \qquad (5)$$

...

3 THE GENERAL SOLUTION OF TOROIDAL SHELLS

Note that the toroidal shells are closed in ϕ-direction with period of 2π and all the coefficients of eqn.(3) are merely functions of x, so it is appropriate to assume

$$\bar{T}(x,\phi)=\sum_{m=0}^{\infty}[\sum_{k=0}^{\infty}(A_k+iB_k)x^k]e^{im\phi} \qquad (6)$$

All hamonics of $\bar{T}$ are uncoupled and the recursion formula for the constants A_k, B_k can be obtained after substituting (6) into (3). For every harmonic of $\bar{T}$ there are eight arbitrary constants A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 which will be determined from the boundary conditions.

Three cases may be distinguished:

1.1 m=0, symmetric loadings. There are only 5 independent boundary forces at both boundaries of a toroidal shell. To decide the 8 arbitaray constants in the solution 3 additional conditions are required. When integrating the elastic relations an arbitrary constant representing rigid-body motion will be obtained.

1.2 m=1, wind-type loadings. There are only 6 independent forces at both boundaries and 2 additional conditions are required. Two arbitrary constants representing rigid-body motion will be appeared when integrating the elastic relations.

1.3 m≥2. The 8 boundary forces at both ends are all in self-equilibrium. No additional condition is required and there is no rigid-body motion in the case.

4 THE GENERAL SOLUTION OF CURVED TUBES

The curved tubes are closed in x-direction with period of 2π, so let

$$\bar{T}(x,\phi)=\sum_{m=0}^{\infty} \bar{T}_m(\phi)\cos mx+\sum_{m=1}^{\infty} \bar{T}_m(\phi)\sin mx \qquad (7)$$

The two parts of the Fourier series in (7) are called symmetric (or in-plane) part and antisymmetric (or out-of-plane) part, respectively. They are uncoupled after substituting (7) into (3) and an ordinary differential matrix equation of biquadratic order is obtained

$$a^4(I)\{\bar{T}^{(4)}(\phi)\}+(\bar{B})\{\bar{T}^{(2)}(\phi)\}+(\bar{C})\{\bar{T}(\phi)\}=\{\bar{F}(\phi)\} \qquad (8)$$

where

$$\{\bar{T}(\phi)\}=[\bar{T}_0(\phi),\bar{T}_1(\phi),\ldots]^T \quad \{\bar{F}(\phi)\}=[\bar{F}_0(\phi),\bar{F}_1(\phi),\ldots]^T \qquad (9)$$

Let

$$\{\bar{T}(\phi)\}=\{\bar{A}\}e^{\bar{k}\phi} \qquad (10)$$

The $\bar{K}$ and $\{\bar{A}\}$ are eigenvalues and corresponding eigenvectors of eqn.(11)

$$[\bar{K}^4a^4(I)+\bar{K}^2(\bar{B})+(\bar{C})]\{\bar{A}\}=\{0\} \qquad (11)$$

If the first L iterms are taken in the Fourier expansion, 2L eigenvalues and corresponding eigenvectors will then be solved. There is a zero eigenvalue and a negative unit eigenvalue but all other eigenvalues are complex and the solution can be written as:

$$\{\bar{T}(\phi)\}=\sum_{j=1}^{2L} [\bar{G}_j e^{a_j(\phi-\phi_2)}(\cos\beta_j\phi+i\sin\beta_j\phi)+$$

$$\bar{H}_j e^{-a_j(\phi-\phi_1)}(\cos\beta_j\phi-i\sin\beta_j\phi)]\{\bar{A}\}_j \qquad (12)$$

where ϕ_1 and ϕ_2 are end values of ϕ, G_j and H_j are complex constants which will be determined from the boundary conditions while a_j and β_j are real and imaginal part of the eigenvalue $\bar{K}_j$.

5 SOME NUMERICAL AND EXPERIMENTAL RESULTS

5.1 A model as shown in Fig.2 was skillfully made by special process. The shell is made of epoxy resin and the cylinders stuck to the shell at the boundaries of $x=\pi$ are made of steel. Strains of the shell were measured with the resistance gauges and the displacements were measured by laser interferometry. Comparision

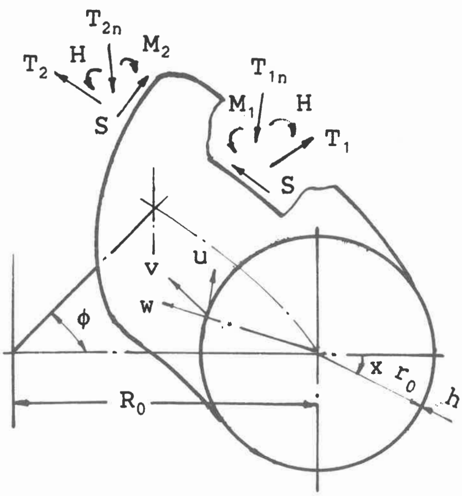

Fig.1 Geometry & internal forces

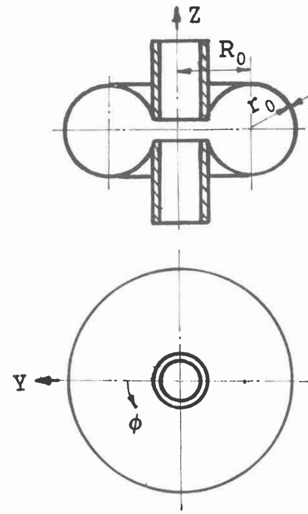

Fig.2 Toroidal shell subjected to bending moment

between the test and the calculated results is illustrated in Fig.3 and Fig.4, where scattered symbols are test values. It is clear that the two results are in good agreement.

5.2 A closed toroidal shell subjected to local linear distributed load is shown in Fig.5. The $q(\phi)$ should be expanded into Fourier series and then the solution obtained by adding all harmonic solutions together. It is found that the error would be less than 5% by taking 18 terms in the Fourier expansion.

5.3 A bend subjected to an in-plane bending moment with warping ringframes at both ends is shown in Fig.6. The same problem has been calculated by Ory and Wilczek (1983) based on the semimembrane shell theory. The distribution of longitudinal stress versus x at $\phi=L/2R$ is shown in Fig.6. The semimembrane theory disregards the longitudinal internal moments M_2 and H, so it presents only one curve of σ_2. But our result has revealed that there are considerable longitudinal moments and the stress curves of the external and inner surfaces of the bend differ from eath other.

6 CONCLUSION

The general solution of toroidal shells and curved tubes is presented. A series of calculated results and experimental results have justified their reliability. It is expected that the method presented in the paper will provide reliable solutions for versatile boundary-value problems with more complex loading conditions.

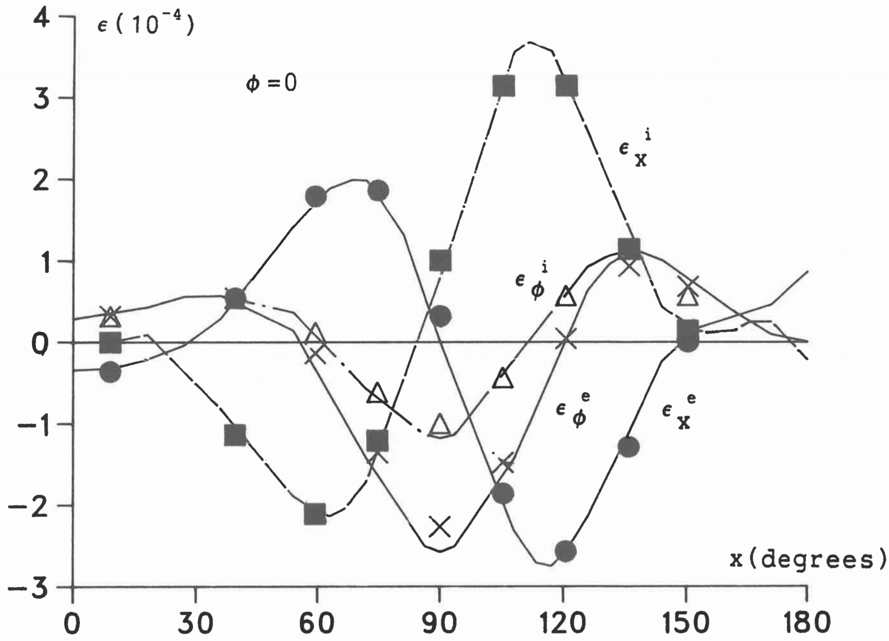

Fig.3 Strains vs x at external & inner surfaces

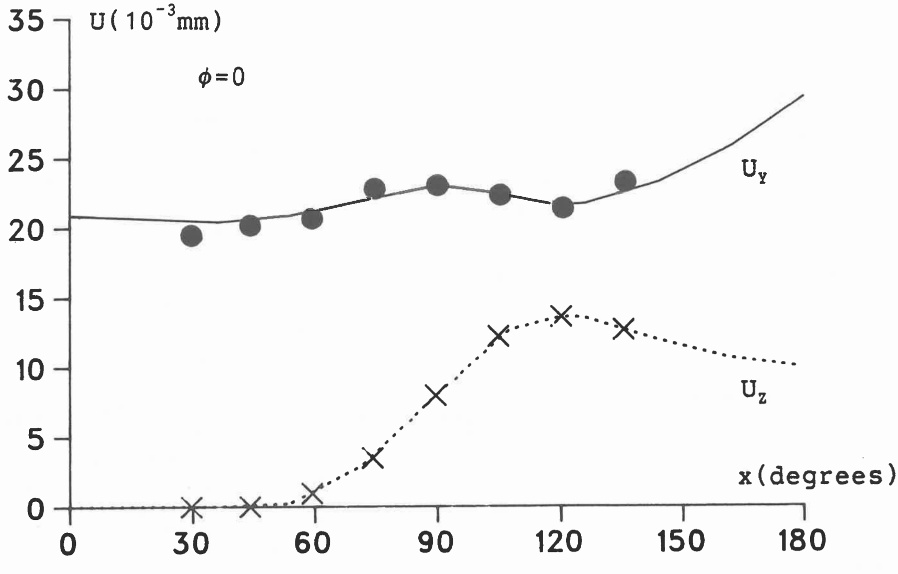

Fig.4 Displacements vs x at external surface

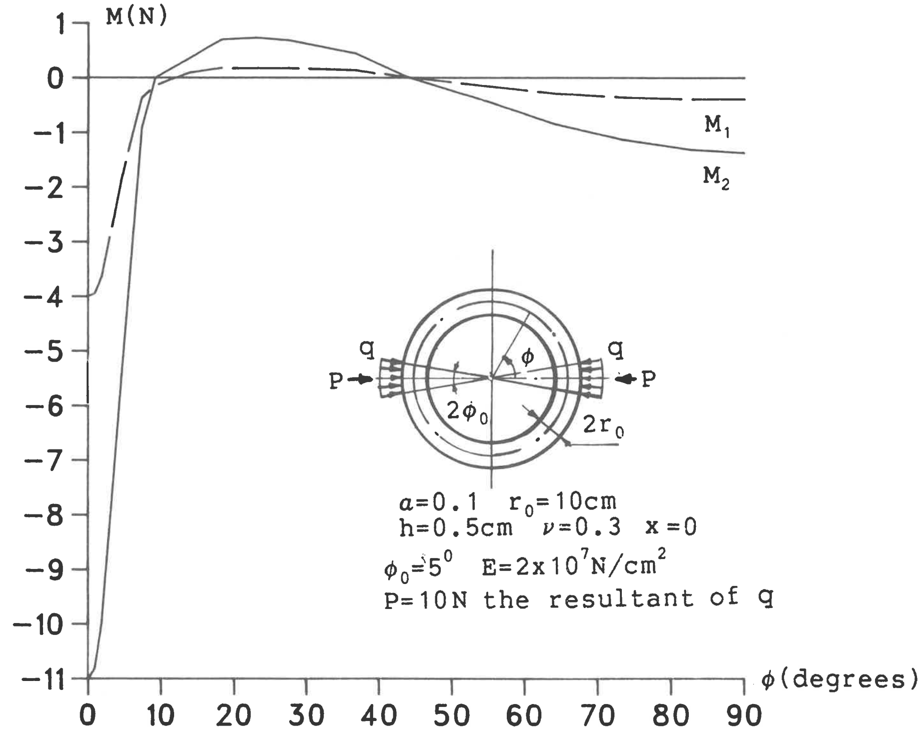

Fig.5 Closed toroidal shell subjected to local linear distributed load

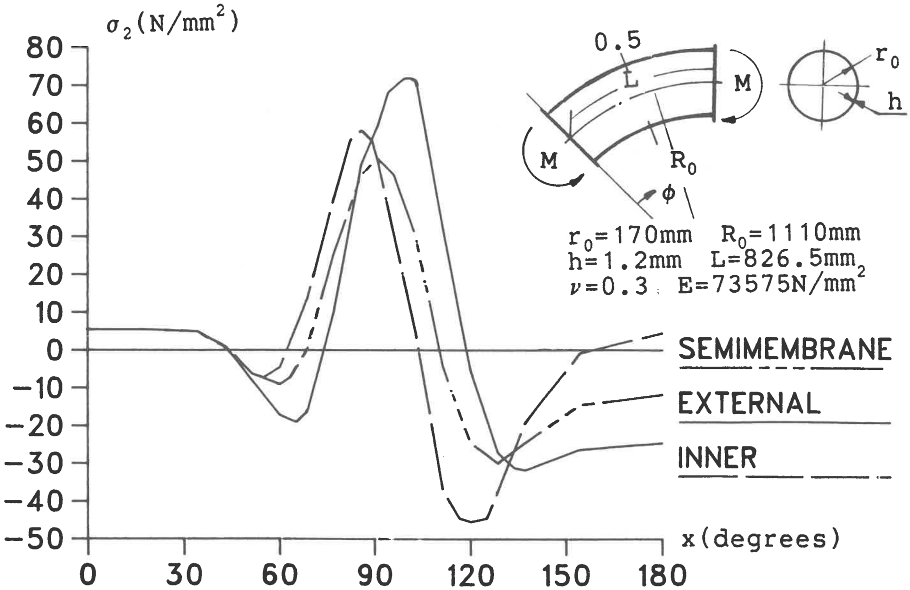

Fig.6 Bend with warping ringframes at both ends

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