Stochastic Response of Systems with Uncertain Properties

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Abstract

First-order reliability methods are used to investigate the failure probability of structural systems with uncertain properties subjected to stochastic loads. The properties of the system, such as stiffnesses, masses, and damping characteristics are uncertain and are modeled as random variables. It is shown that for certain systems, such as primary-secondary systems, uncertainties in properties of the system can have significant effect on the failure probability, even when the input is a wide-band stochastic process.

1. Introduction

It is generally assumed that uncertainties in the parameters of a structural system, such as stiffness, mass, and damping, have second-order effects on the response of the system to stochastic inputs, particularly for wide-band inputs such as earthquake ground motions. This notion is the basis for using deterministic models of structures in stochastic dynamic analysis [9]. However, there are systems for which the uncertainty in the parameters has an effect on the response which is of the same order as the effect of the uncertainty in the input.

In this paper, first-order reliability methods are used to investigate the probability of failure of a structural system with random parameters subjected to stochastic excitation at its base. Concepts from reliability theory, random vibrations, and matrix computational algebra are used to formulate and solve the reliability problem. By specifying the base excitation through its power spectral density or its mean response spectrum, the conditional distribution of the peak response of the system for a given set of parameters is obtained. By assigning distributions to the parameters of the the system, a first-order reliability analysis is carried out to determine the well-known reliability index, β, and the probability of failure. The approach requires evaluation of partial derivatives relative to the uncertain parameters of the system. Closed-form expressions for the derivatives of eigenvalues and eigenvectors are derived which are used to increase efficiency in the reliability analysis. The influence of the uncertain parameters on the reliability of the system is measured by the normalized gradient of β.

The method is applied to two example systems: A 4-story shear building and a 7-degree-of-freedom (7-DOF) primary-secondary system. The stiffnesses, masses, and damping ratios of the two systems are uncertain parameters and the effects of these uncertainties on the system responses are investigated.

2. First-Order Reliability Analysis

Consider an N-DOF linear, viscously damped system with stiffness, damping, and mass matrices given by $\mathbf{K}(\mathbf{X})$, $\mathbf{C}(\mathbf{X})$, and $\mathbf{M}(\mathbf{X})$, respectively, which are functions of a set of parameters $\mathbf{X}=[X_1, \ldots, X_n]^T$. For each set of values for the parameters $\mathbf{X}$, the corresponding mode shapes, φ_i, natural frequencies, ω_i, and damping ratios, ζ_i, can be determined by eigenvalue analysis. If the system is nonclassically damped, i.e., the matrix of mode shapes does not diagonalize the damping matrix, then the mode shapes are complex-valued and it is convenient to use a state-space approach to solve the eigenvalue problem [5]. Once the modal properties are determined, the spectral moments of the response to a stationary excitation, $\lambda_m = \int_0^\infty \omega^m G(\omega) d\omega$, $m = 0, 1, 2, \cdots$, where $G(\omega)$ is the one-sided power spectral density, are obtained by using a modal combination rule such as in Ref. 2 for classically damped systems, and Ref. 8 for non-classically damped systems. This analysis can also be applied when the input excitation is specified in terms of a mean response spectrum. In that case, the spectral moments are approximated in terms of ordinates of the response spectrum at the modal frequencies and damping ratios of the system. The procedure is described in detail in Refs. 3 and 6.

For any given set of the system parameters $\mathbf{X} = \mathbf{x}$, the conditional distribution of the peak response may be obtained directly in terms of the first three spectral moments from [10],

$$F_{R\,|\,\mathbf{X}}(r,\mathbf{x}) = \left[1 - \exp\left(\frac{-r^2}{2\lambda_0}\right)\right] \exp\left[-\nu\tau \frac{1-\exp\left[-\sqrt{\frac{\pi}{2\lambda_0}}\, q_e r\right]}{\exp\left(\frac{r^2}{2\lambda_0}\right) - 1}\right] \tag{1}$$

in which $\nu = \sqrt{\lambda_2/\lambda_0}/\pi$ is the mean zero-crossing rate, τ is the duration of excitation, and $q_e = q^{1.2}$, where $q = (1-\lambda_1^2/\lambda_0\lambda_2)^{1/2}$. Let $f_{\mathbf{X}}(\mathbf{x})$ represent the joint probability density function of $\mathbf{X}$. The unconditional distribution of the peak response is given by the multi-fold integral

$$F_R(r) = \int F_{R\,|\,\mathbf{x}}(r,\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{2}$$

If the number of uncertain parameters is greater than 2, the numerical evaluation of the above integral is impractical. Herein, this integral is computed using first-order reliability techniques for dependent, non-normal variables [4].

The procedure for determining the probability of the peak response exceeding a given threshold r_0 is summarized as follows:

1. The set of basic variables are considered to be R and $\mathbf{X}$. In terms of these variables, the performance function is formulated as: $g(R,\mathbf{X}) = r_0 - R$.
2. The set of basic variables are transformed into the standard normal space:

$$\mathbf{Y} = \mathbf{T}(R,\mathbf{X}) \tag{3}$$

Since the conditional distribution of R given $\mathbf{X}$ is known, it is convenient to express the above transformation in a partitioned form:

$$\mathbf{Y} = \begin{Bmatrix} Y_R \\ \mathbf{Y}_{\mathbf{X}} \end{Bmatrix} = \begin{Bmatrix} T_R(R\,|\,\mathbf{X}) \\ \mathbf{T}_{\mathbf{X}}(\mathbf{X}) \end{Bmatrix} \tag{4}$$

The transformation $T_R(R\,|\,\mathbf{X})$ is given by: $T_R(R\,|\,\mathbf{X}) = \Phi^{-1}[F_{R|\mathbf{X}}(R,\mathbf{X})]$ where $\Phi(.)$ is the standard normal cumulative probability. The transformation $\mathbf{T}_{\mathbf{X}}(\mathbf{X})$ depends on the distribution of $\mathbf{X}$. Using the Rosenblatt transformation [4], the i-th component of $\mathbf{T}_{\mathbf{X}}(\mathbf{X})$ is in the form: Y_i =

$\Phi^{-1}[F_{X_i|X_{i-1}\cdots X_1}(x_i|x_{i-1}\cdots x_1)]$ for $i = 1, \ldots, n$, in which $F_{X_i|X_{i-1}\cdots X_1}(x_i|x_{i-1}\cdots x_1)$ is the conditional cumulative distribution function which can be obtained in terms of $f_{\mathbf{X}}(\mathbf{x})$.

3. The performance function is transformed into standard normal space:

$$g(R,\mathbf{X}) = r_0 - R = r_0 - T_R^{-1}(Y_R,\mathbf{X}) = r_0 - T_R^{-1}(Y_R,\mathbf{T}_{\mathbf{X}}^{-1}(\mathbf{Y}_{\mathbf{X}})) = G(\mathbf{Y}) \tag{5}$$

where a superscript -1 indicates the inverse transformation.

4. In the standard normal space, the nearest point $\mathbf{y}^*$ to the origin on the failure surface $G(\mathbf{y}) = 0$ is determined. The reliability index is computed as the distance $\beta = \sqrt{\mathbf{y}^{*T}\mathbf{y}^*}$. In terms of this distance, first-order approximation of the failure probability (i.e., exceeding threshold r_0) is given by:

$$Prob(R \geq r_0) \doteq \Phi(-\beta) \tag{6}$$

5. In addition to the failure probability, it is informative to obtain a measure, γ, of the sensitivity of the reliability of the system with respect to standard variations in the system parameters. This measure can be represented by the gradient of the reliability index with respect to mean values of the the system parameters, $\nabla_{\bar{\mathbf{X}}}\beta$, normalized by the standard deviations of the parameters. It can be shown that γ is given by

$$\gamma = \nabla_{\mathbf{Y}}g\, \mathbf{J}^{-1}\mathbf{J}_{\bar{\mathbf{X}}}\mathbf{D} \tag{7}$$

where $\mathbf{J}$ is the Jacobian matrix of $\mathbf{T}$ whose elements are given by $\{\partial Y_i / \partial X_j\}$, $\mathbf{J}_{\bar{\mathbf{X}}}$ is the Jacobian matrix of $\mathbf{T}_{\mathbf{X}}$ (treated as a function of R, X_j, and $\bar{X}_j$) whose elements are given by $\{\partial Y_i / \partial \bar{X}_j\}$, and $\mathbf{D}$ is the diagonal matrix of standard deviations σ_{X_i}. The i-th component of γ is the change in the reliability index, β, resulting when the mean value of the parameter X_i is increased by one standard deviation.

The determination of the nearest point $\mathbf{y}^*$, in general, requires a numerical procedure. For the present application, the iterative procedure described in Ref. 4 is used. This procedure requires, at each iteration, the evaluation of the Jacobian matrix, $\mathbf{J}$. Most of the elements of this matrix can be easily derived using standard formulas, however, the last row of this matrix requires calculation of the n partial derivatives of the conditional distribution $F_{R|\mathbf{X}}$ with respect to X_i. If simple two-point finite difference methods are used, $n+1$ numerical eigenvalue analyses are required since an analysis is required to evaluate $F_{R|\mathbf{X}}$ at each point. However, using results from matrix algebra, closed-form expressions for the partial derivatives of the eigenvalues and eigenvectors can be derived which do not entail any numerical eigenvalue analyses.

To obtain general expressions for the partial derivatives of the mode shapes, natural frequencies, and damping ratios for both nonclassically and classically damped systems, an additional complex parameter, $s_i = -\omega_i\zeta_i + i\omega_i\sqrt{1-\zeta_i^2}$, is used where $i = \sqrt{-1}$. The following expressions are derived following Ref. 1:

$$\frac{\partial s_i}{\partial X_j} = -\frac{1}{m_i}\left[\varphi_i^T\left(\frac{\partial \mathbf{K}}{\partial X_j} + s_i\frac{\partial \mathbf{C}}{\partial X_j} + s_i^2\frac{\partial \mathbf{M}}{\partial X_j}\right)\varphi_i\right] \tag{8}$$

where $m_i = \varphi_i^T(\mathbf{C} + 2s_i\mathbf{M})\varphi_i$ and

$$\frac{\partial \varphi_i}{\partial X_j} = \sum_{k=1}^{n}\alpha_{ijk}\varphi_k \tag{9}$$

where

$$\alpha_{ijk} = -\frac{1-\delta_{kj}}{m_j(s_j - s_k)}\left[\varphi_k^T\left(\frac{\partial \mathbf{K}}{\partial X_j} + s_j\frac{\partial \mathbf{C}}{\partial X_j} + s_j^2\frac{\partial \mathbf{M}}{\partial X_j}\right)\varphi_j\right] \tag{10}$$

and δ_{kj} is the Kronecker delta.

3. Example Applications

Two example systems are considered to illustrate the method and to investigate the effects of the uncertainties in the systems on their response. The first example is the 4-story shear building shown in Fig. 1. The building is modeled with five uncertain parameters: the stiffnesses and masses of the upper two floors are given by K_1 and M_1, respectively; the same parameters for the lower two floors are given by K_2 and M_2, respectively; and the modal damping ratios have the common value ζ. The second example is the 7-DOF primary-secondary system shown in Fig. 2. The system is modeled with six uncertain parameters: the stiffnesses, masses, and modal damping ratios of the primary subsystem are given by K_p, M_p, and ζ_p, respectively; the same parameters for the secondary subsystem are given by K_s, M_s, and ζ_s, respectively. The means and coefficients of variation of these parameters are listed in Table 1. For each system, the correlation between the stiffnesses as well as the correlation between the masses are assigned a common value ρ, and all other pairs of variables are uncorrelated. A filtered, white-noise process with 11 second duration, 2.5 Hz central frequency, and 0.60 damping parameter is used as acceleration input at the base of the systems. The response quantities that are examined are the peak interstory displacement of the lowest floor of the shear building and the peak relative displacement between the lower two secondary masses of the primary-secondary system. The P-exceedance threshold response levels are evaluated for three failure probabilities, P = 0.05, 0.10, 0.20, for three levels of correlation, ρ = 0.0, 0.3, 0.6, as well as for a system with deterministic parameters fixed at their mean values. The sensitivity parameter, γ, for P = 0.10 is listed in Table 1 for each level of correlation, and the P-exceedance threshold levels are listed in Table 2.

For the shear building, the sensitivity parameter indicates that the damping is the most sensitive parameter for determining the system reliability. However, for the primary-secondary system, the secondary stiffnesses and masses are considerably more sensitive than the damping. This can be partially explained by the resonance or tuning effect: The response of the secondary subsystem is amplified when the spacing between the primary and secondary frequencies become small. Thus, changes in the secondary properties which change the spacing between the subsystem frequencies result in an amplification (or attenuation) of peak response levels and a corresponding increase (or decrease) of the failure probability. In addition to tuning, there are other dynamic characteristics of primary-secondary systems such as non-classical damping, interaction, and spatial coupling [7] which have significant influence on the system response and may also affect the sensitivity of the system parameters.

The P-exceedance levels for the shear building is higher for uncertain parameters as compared with the levels for deterministic parameters. Consequently, the reliability estimated from a building with deterministic parameters will underestimate the true reliability. For instance, if the building is designed for 0.05 failure probability using the deterministic model, the actual failure probability, assuming correlation ρ = 0.3, would be slightly above 0.10 (see Table 2, columns 4 and 6). This can be partially explained in terms of the damping parameter: For a building with uncertain parameters, there is a high likelihood that the damping is lower than its mean value, resulting in higher threshold values. For the primary-secondary system, the difference between the uncertain and deterministic models is even greater. A system designed for 0.05 failure probability using the deterministic model would actually have nearly 0.20 failure probability (for $\rho = 0.3$). For this system, in addition to the damping effect, tuning will also influence the threshold values: For a primary-secondary system with uncertain parameters, there is a high likelihood that the subsystem frequencies are closer than their mean values, resulting in higher threshold values. The other aforementioned dynamic characteristics of primary-secondary systems may also contribute to the decreased reliability of such uncertain systems.

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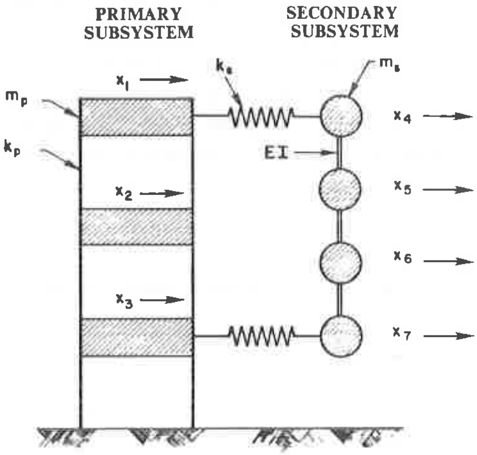

Figure 2. 7-DOF Primary-Secondary System

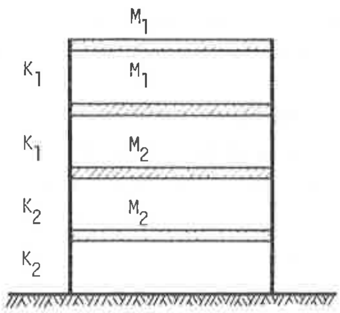

Figure 1. 4-story Shear Building

Table 1. System Parameters and Sensitivities

System	Parameter	Mean	Coefficient of Variation	Sensitivity, γ		
				ρ = 0.0	ρ = 0.3	ρ = 0.6
4-DOF Shear Building	K_1	1.97×10^7 N/m	0.1	-0.087	-0.088	-0.088
	K_2	3.94×10^7 N/m	0.1	-0.331	-0.329	-0.333
	M_1	0.67×10^6 kg	0.1	0.178	0.178	0.185
	M_2	1.00×10^6 kg	0.1	0.215	0.213	0.216
	ζ	0.05	0.2	-0.497	-0.491	-0.500
7-DOF Primary-Secondary System	K_p	2.50×10^8 N/m	0.1	0.062	0.055	-0.004
	K_s	4.21×10^5 N/m	0.1	-0.597	-0.622	-0.638
	M_p	5.00×10^6 kg	0.1	-0.031	-0.027	0.019
	M_s	0.10×10^6 kg	0.1	0.489	0.527	0.575
	ζ_p	0.02	0.2	-0.297	-0.321	-0.376
	ζ_s	0.005	0.2	-0.099	-0.101	-0.098

Table 2. Normalized P-Exceedance Threshold Levels

System	Probability of Exceedance	Correlation between Parameters			Deterministic Parameters
		ρ = 0.0	ρ = 0.3	ρ = 0.6	
4-DOF Shear Building	0.20	2.12	2.11	2.08	1.92
	0.10	2.55	2.54	2.48	2.17
	0.05	2.95	2.93	2.85	2.38
7-DOF Primary-Secondary System	0.20	1.29	1.27	1.20	1.06
	0.10	1.62	1.58	1.43	1.18
	0.05	1.97	1.91	1.65	1.29