# Combination of Responses from Closely Spaced Modes

N.C. Tsai
*NCT Engineering, Inc., 3650 Mt. Diablo Blvd., Suite 190, Lafayette, California 94549, U.S.A.*

ABSTRACT

In the response spectrum method of analysis, modes having their frequencies apart from each other by 10% or less are commonly defined as closely spaced. The common practice is to combine the responses of closely spaced modes on an absolute sum basis, such as the grouping method. The double sum method, however, need not differentiate closely spaced from non-closely spaced modes because it utilizes statistically derived modal coupling coefficients, and it appears to combine very closely spaced modes on an algebraic sum basis. One major drawback of the double sum method is that the modal coupling coefficient does not converge to what should be analytically anticipated at both very low and high frequency. A new method is presented here which also uses modal coupling coefficient but such coefficient is derived empirically from an extensive parametric study and hence excludes the drawback in the double sum method. Nevertheless, it confirms that responses from very closely spaced modes should be algebraically combined, as one would anticipate from the analytical point of view.

As an illustration, the proposed method is applied to the response computation of a two-mass system that represents a light secondary system coupled to a primary system. Closely spaced modes take place when the secondary system frequency is tuned to the primary system and the mass ratio between the two is small. In this particular case, one finds that the square-root-of-sum-of-squares (SRSS) method also substantially overestimates the response of the secondary system although it underestimates the primary system response. This well known dilemma is now resolved using the method proposed here.

## 1. Introduction

For the response spectrum method of analysis of a system, the several commonly used methods for the modal response combination can be expressed in the following common form:

$$R = \sqrt{\sum_j \sum_k c_{jk} R_j R_k} \qquad (1)$$

The coefficient $c_{jk}$ is a positive parameter equal to or less than 1.0, and it is equal to 1.0 for $j=k$. For $j \neq k$, $c_{jk}$ is equal to 0.0 for the square-root-of-sum-of-squares (SRSS) method of modal combination. There has been concern that the SRSS method may not be adequate for closely spaced modes, which are conventionally defined as modes having their frequencies apart from each other by 10% or less. Thus methods such as the grouping method and double sum method are also widely used [1,2]. For the grouping method, $c_{jk}$ is also equal to zero except for closely spaced modes, for which $c_{jk}$ is equal to 1.0 and both $R_j$ and $R_k$ are taken to be positive such that the closely spaced modes are combined on the absolute sum basis. For the double sum method, $c_{jk}$ is a function of the modal frequency, modal damping, difference in modal frequency, and the earthquake duration, T. In the author's opinions, the double sum method is by far the most appropriate method because it need not differentiate closely spaced from non-closely spaced modes in accordance with the 10% differential frequency practice. The drawback is that $c_{jk}$ is not well defined at very low frequency and it approaches a constant at high frequency rather than, as should be analytically expected, converges to 1.0. Furthermore, taking the parameter T to be the entire earthquake duration may not be appropriate because earthquake motions are rarely a stationary random process as was assumed in the statistical derivation of $c_{jk}$. Its effect, however, is significant only for low damping and very closely spaced modes. In view of the above, we propose another response spectrum method of analysis here.

The response spectrum method of analysis presented here assumes the same form as eq.(1). Similar to the double sum method, $c_{jk}$ is also a function of the modal frequency, modal damping, and modal frequency differential, but is derived empirically from the deterministic approach. In doing so, four earthquake motions having different duration and frequency contents were included in the parametric study. In general, $c_{jk}$ from the double sum method agreed well with the empirical one for frequency lower than, say, about 5 Hz; beyond 5 Hz, the empirical $c_{jk}$ asymtotically converges to 1.0, as should be expected. The new method is applied to the response calculation for a two-mass system containing a small mass (secondary system) and large mass (primary system). The closeness between the two modes can be controlled by adjusting the mass ratio and tuning the frequencies of the two subsystems. When the two modes are closely spaced, it is a well known dilemma that the SRSS method would substantially overestimate the secondary system response while still underestimates the primary system response [3]. This dilemma can now be resolved with the application of the method presented here.

2. Determination of $c_{jk}$

To deterministically estimate $c_{jk}$, we performed the following parametric time history response analyses:

$$\ddot{y}_j(t) + 2d_j f_j \dot{y}_j(t) + f_j^2 y_j(t) = -\ddot{u}_g(t), \qquad j = 1,2 \tag{2}$$

Let:

$$\begin{aligned} (f_1 + f_2)/2 &= \overline{f} = \text{mean frequency for } f_1 \text{ and } f_2, \\ (f_2 - f_1)/\overline{f} &= \Delta = \text{dimensionless frequency differential}, \\ |f_j^2 y_j|_{max} &= S_j = \text{spectral acceleration response, } j = 1,2. \end{aligned} \tag{3}$$

We next computed $c'_{12}$ and $c''_{12}$ from the following definitions:

$$\begin{aligned} |f_1^2 y_1 + f_2^2 y_2|_{max} &= \sqrt{S_1^2 + S_2^2 + 2c'_{12}S_1S_2} = R' \\ |f_1^2 y_1 - f_2^2 y_2|_{max} &= \sqrt{S_1^2 + S_2^2 - 2c''_{12}S_1S_2} = R'' \end{aligned} \tag{4}$$

Four earthquake ground motions were the input motion $\ddot{u}_g(t)$. Their durations are 16, 14, 10 and 27 seconds, respectively, and their peak accelerations are all 0.45g except for the 16-second motion which has a 0.25g peak acceleration. The 5% damping acceleration response spectra of these motions are shown in Figure 1. Beside the input motion, we considered the following parameters:

(a) 19 values of $\overline{f}$ : from 0.1 to 30 Hz.

(b) 6 values of $\Delta$ : 0.01, 0.02, 0.05, 0.1, 0.15, 0.2.

(c) 3 values of modal damping $d_j$ : 1%, 2%, 5%.

All together, 1368 pairs of $c'_{12}$ and $c''_{12}$ were computed. For $c'_{12}$, a positive computed value denotes that the SRSS method underestimated R'. About 92% of the time we found that SRSS alone underestimated R' by more than 5%. On the contrary, a positive c" indicates that the SRSS method alone overestimated R". For the study, about only 8% of the time when SRSS combination underestimated the response R" by more than 5%, which occurred at frequency between 0.5 and 6.0 Hz and for a frequency differential of 0.10 or more.

For each combination of $\overline{f}$, d and $\Delta$, $c'_{12}$and $c''_{12}$from the four earthquakes were first averaged. The mean of the averaged $c'_{12}$and $c''_{12}$ was then calculated:

$$c_{jk} = c_{12} = (c'_{12} + c''_{12})/2 \tag{5}$$

Figures 2 and 3 illustrate the computed $c_{jk}$ as a function of $\overline{f}$ and $\Delta$, for a damping of 1% and 5%, respectively. It can be seen that all $c_{jk}$ would converge to 1.0 at high frequencies beyond about 30 Hz. For purpose of application, we simplified Figures 2 and 3 to those shown in Figures 4 and 5. The simplified $c_{jk}$ can be expressed as follows:

$$c_{jk} = 1 - \left[\frac{\Delta^2}{\Delta^2 + 4(d+0.01)^2}\right] H(\overline{f}) \tag{6}$$

in which the function H is idealized as that illustrated in Figure 6.

As a comparison, the $c_{jk}$ from the double sum method is also shown in Figures 4 and 5 in dashed lines. It was computed using T = 10 seconds. It can be observed that the two methods compare well for frequency below 5 Hz.

The significant observation is that, at both very low and high frequency, the empirically derived $c_{jk}$ converges to 1.0 as should be anticipated. This means that the modal responses should be combined on the algebraic sum basis at both very low and high frequency. Because responses were not computed for $\Delta$ exceeding 0.2 in this study, we recommend using $c_{jk} = 0$ for $\Delta > 0.2$, i.e., using the SRSS combination for lack of additional information at this time.

## 3. Application to A Two-Mass System Having Closely Spaced Modes

The system shown in Figure 7 represents a secondary system ($m_1$) coupled to the primary system ($m_2$). Let us examine the tuned condition in which the secondary system frequency, f', is equal to the primary system frequency, f. The two modes of the coupled system can then be made arbitrarily close to each other by simply decreasing the mass ratio, $R = m_1/m_2$, so that [3],

$$f_{1,2} \cong ( 1 \mp \sqrt{R}/2 )f \tag{7}$$

Thus, the frequency differential is $\Delta \cong \sqrt{R}$ and hence a mass ratio of 0.01 or smaller will result in closely spaced modes according to the conventional definition. In this example, we chose the following parameters: f'=f= 3.18 Hz (20 rad/sec); d' = secondary system damping = 1%; d= primary system damping = 5%; and R = 0.001 to 0.01. The earthquake input is the one having the 0.25g peak acceleration, of which the response spectrum is given in Figure 1.

We first computed the secondary system acceleration with the floor spectrum approach. That is, a floor spectrum was generated from the time hsitory analysis of the primary system alone, without the secondary system's presence; the secondary system response can then be obtained from the floor spectrum. This gave the acceleration response of $m_1$ equal to $a_1$ = 9.2g, as shown in the dashed line in Figure 8. It should represent the limiting solution for $a_1$ for the coupled system with the mass ratio approaching zero.

We next performed a response spectrum analysis of the coupled system. It is important to properly determine the modal damping for the analysis, and we have found that, for very small value of the mass ratio,

$$\begin{aligned} d_1 = d_2 &\cong \sqrt{d \times d'} = 2.24\% \text{ ..... for computing } a_1 \\ d_1 = d_2 &\cong d = 5\% \text{ ............. for computing } a_2 \end{aligned} \tag{8}$$

It can then be established that [3]:

$$\begin{aligned} (a_1)_{1,2} &\cong \underline{S}_{1,2}(1 \pm 1/\sqrt{R})/2 \qquad (\underline{S}_{1,2} \text{ for 2.24\% damping}) \\ (a_2)_{1,2} &\cong \bar{S}_{1,2}(1 \pm \sqrt{R})/2, \qquad (\bar{S}_{1,2} \text{ for 5\% damping}) \end{aligned} \tag{9}$$

The results for the secondary system response, $a_1$, from both the SRSS and the new method are shown in Figure 8. It is evident that the SRSS solution is approaching unbounded as R decreases whereas the solution from the new method correctly approaches the 9.2g solution with the floor spectrum approach. On the other hand, for small values of R the SRSS method gives $a_2 = \bar{S}_1/\sqrt{2}$ while the current method gives $a_2 = \bar{S}_1$ which is what should be expected. In summary, in the above example the SRSS method overestimated the acceleration of the secondary system but underestimated the primary system response, a well known dilemma that undermines the belief that the SRSS method is always unconservative for closely spaced modes. The method proposed here properly resolves the dilemma.

## 4. Conclusion

The new method presented here takes the form of eq.(1), with the coefficient $c_{jk}$ empirically derived from four earthquake motions. Significant conclusions are:

(a) $c_{jk}$ approaches 1.0 at both very low and high frequency, as one should expect from the analytical point of view.

(b) In eq.(1), the signs of the modal response, $R_j$ and $R_k$, should be retained. This is also the case with the original double sum method, and is contrary to the conventional practice of taking only the absolute values of $R_j$ and $R_k$. This suggests that, when $c_{jk}= 1$, the modal responses should be combined on the algebraic sum basis.

(c) The essential difference between the double sum method and the currently proposed one is in the $c_{jk}$ value for frequency higher than about 5 Hz. The $c_{jk}$ in the double sum method approaches a constant at the higher frequencies.

(d) Parametric computations were not made for $c_{jk}$ for frequency differential exceeding 0.2. As a substitution, we propose that $c_{jk}= 0.$ be used for such condition.

A tuned two-mass system representing a coupled secondary-primary system was utilized as an example for illustration. The modal frequencies were made close to each other by decreasing the mass ratio such that a mass ratio of 0.01 or smaller would produce a modal frequency differential of 10% or less. The results indicated that the SRSS method, while underestimated the primary system response, substantially overestimated the secondary system response in the presence of closely spaced modes. The current method, however, properly produced the response solution that corresponded to what should be expected from the analytical point of view.

## REFERENCE


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3 Tsai, N. C., "Spectral Response Analysis of A Coupled System," presented at Pressure Vessels & Piping Conf., Orlando, Florida, U.S.A., June 27 to July2, 1982, PVP Vol. 65.

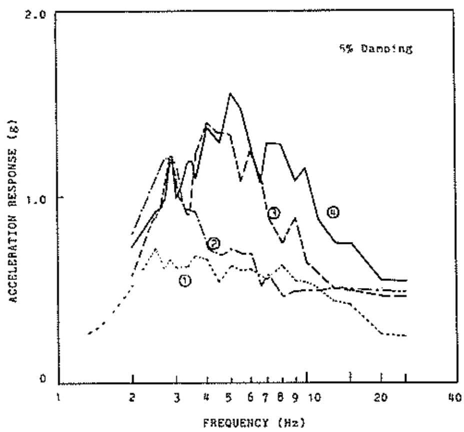


Fig. 1 5% Damping Acceleration Response Spectrum of the Four Earthquake Motions

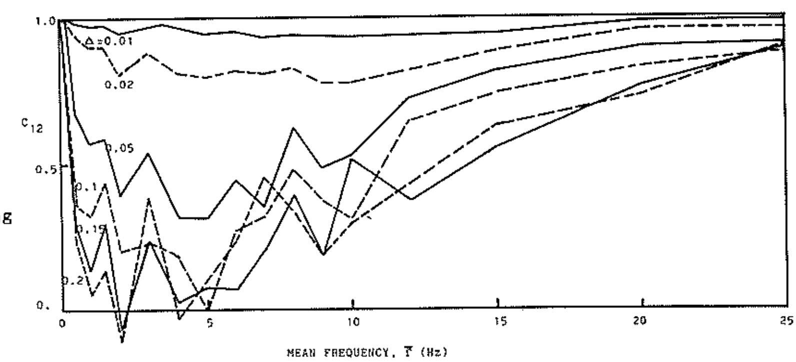


Fig. 2
Averaged Modal Coupling Coefficient for 1% Modal Damping

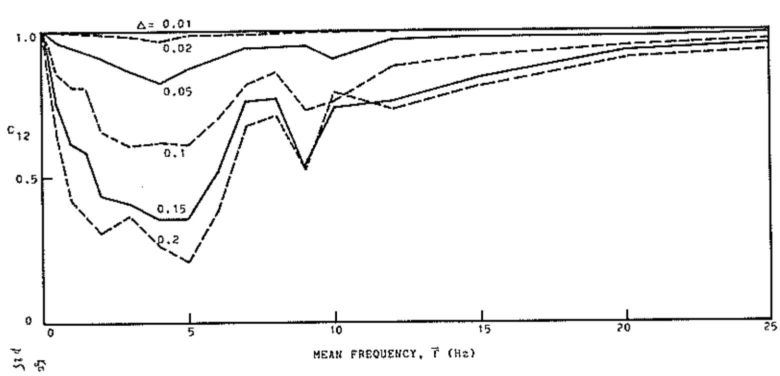


Fig. 3
Averaged Modal Coupling Coefficient for 5% Modal Damping

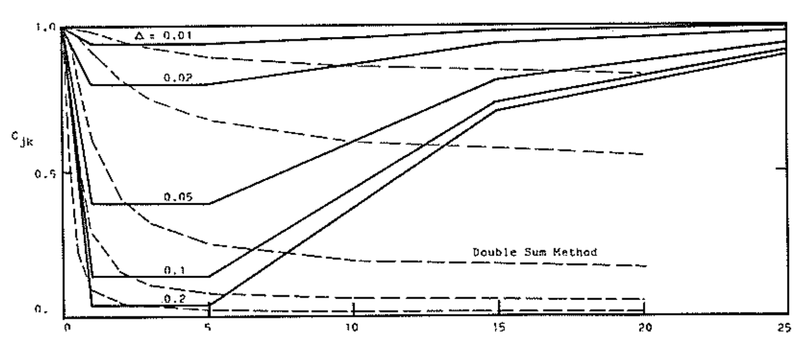


Fig. 4 Idealized Modal Coupling Coefficient for 1% Modal Damping

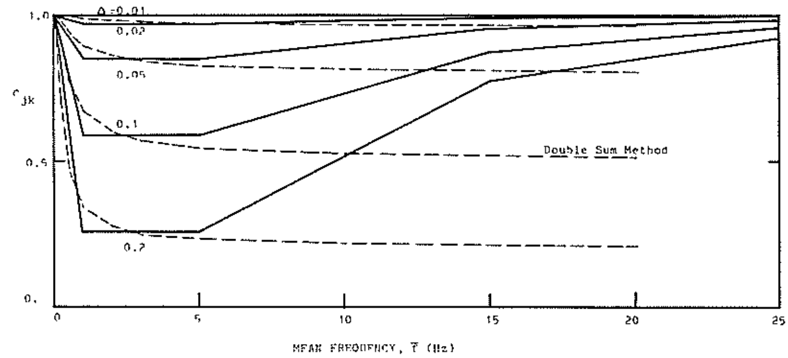


Fig. 5 Idealized Modal Coupling Coefficient for 5% Modal Damping

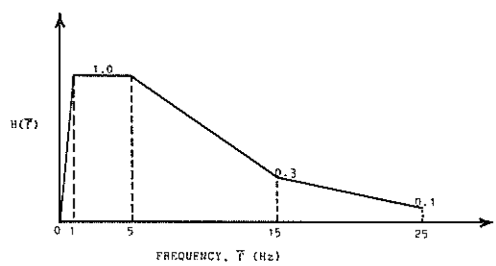


Fig. 6 Function $H(\bar{f})$ for the Modal Coupling Coefficient

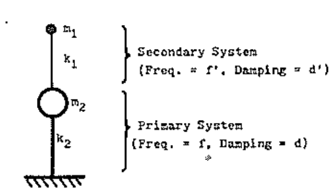


Fig. 7 The Secondary-Primary Coupled System

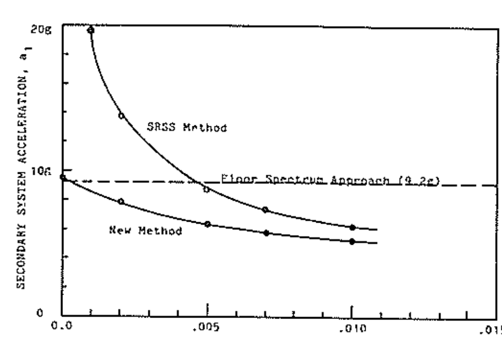


Fig. 8 Secondary System Acceleration Response, $a_1$