THE BURR DISTRIBUTION AS A GENERAL PARAMETRIC FAMILY IN SURVIVORSHIP AND RELIABILITY THEORY APPLICATIONS

by

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ABSTRACT

ALCINDA WANGSNESS LEWIS. The Burr Distribution as a General Parametric Family in Survivorship and Reliability Theory Applications. (Under the direction of MICHAEL J. SYMONS)

In the analysis of survival time studies a parametric form of the survival distribution is commonly assumed. The choice of the parametric form of the survival distribution is an important issue which should be given careful consideration. The Burr Type XII distribution has as special limiting cases of its parameter values the Weibull and the exponential distributions, two commonly used survival distributions.

A generalized likelihood ratio test, based upon maximum likelihood procedures, is applied to assess the goodness-of-fit of the two special cases relative to that of the more comprehensive model. The true value of the parameter vector under the Weibull (or exponential) null distribution lies on the boundary of the Burr XII parameter space. This boundary condition violates one of the sufficient conditions of Wilks' [1938] theorem regarding the asymptotic chi-square approximation for the distribution of $-2\ln\Lambda$, where Λ represents the likelihood ratio and the number of degrees of freedom is the number of components of the parameter space that are specified under the null distribution. The empirical distribution of the generalized likelihood ratio for testing the standard exponential distribution versus the Burr XII distribution was obtained by means of a simulation. This distribution is compared to a number of chi-square distributions.

Special procedures are given for the accommodation of censored observations and concomitant variables. In the context of choosing an underlying distribution for accelerated life test data, two competing

models are discussed which correspond to two different methods for handling covariables representing the accelerated conditions. The performance of the Burr Type XII distribution is contrasted with that of the log-generalized gamma distribution, another very general family of parametric models, with respect to the reassessment of the Weibull distribution as a model for acute leukemia survival data.

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CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

1.1 Introduction

In a statistical analysis an investigator frequently is faced with choosing an underlying parametric model. In some instances physical considerations determine the distribution to be employed. However, numerous statistical problems arise from physical processes which cannot be theoretically linked to a particular parametric distribution. The lack of a general theory for the choice of an underlying parametric distribution has resulted in the usage of some distributions which are not necessarily robust to viable alternative distributions and which may lead to incorrect conclusions. The authors of the RADC Reliability Notebook [1961], a government research report intended to provide the necessary techniques to be utilized by design and reliability engineers working with complex electronic equipment, succinctly describe the problem that investigators in many different disciplines must overcome as they practice the art of model selection. The RADC authors state that "...the heart of the reliability problem is that no distribution can be safely assumed."

The emphasis of this research is placed upon the choice of an underlying parametric model in survival analyses or reliability studies. Since the reliability of a piece of equipment can be modeled by a probability statement concerning its lifetime operation, there is a close connection between reliability theory and survival theory, where it is desired to make probability statements about survival or remission times of patients. Due to its simplicity and its inherent association with the well-developed theory of Poisson processes, the exponential distribution has historically played a role of importance in survival analyses and reliability studies, analogous to that of the normal distribution in other areas of statistics. However, investigations concerning the robustness of the predicted time-to-failure or time-to-death under the assumed exponential distribution when the true underlying distribution belongs to the Weibull family have shown that these widely used statistical techniques are very sensitive to departures from exponentiality and may result in substantially increasing the probability of accepting equipment with poor mean time-to-failure or of over-estimating the mean time-to-death (Hager, Bain, Antle [1971] and Zelen and Dannemiller [1961]). More recently the Weibull distribution has emerged as the most popular parametric family of failure distributions. Its applicability to a wide variety of failure situations was discussed by Weibull [1951]. However, Canfield and Borgman [1975] have pointed out that due to a scarcity of statistical methodology dealing with the choice of a failure time distribution, even the very flexible Weibull distribution has possibly been overworked.

Early work by D. R. Cox [1961] served as a catalyst for theoretical research into the area of model selection. Cox recognized that it is rarely sufficient for an investigator to find evidence against the distribution in question since the problem is one of model selection. Hence, Cox formulated the problem in terms of a significance test to measure the departure of the observed data from the null distribution in the direction of an alternative distribution. Much of Cox's research was directed towards discriminating between two families of distributions, separate in the sense that a simple hypothesis in the null family cannot be obtained as a limit of simple hypotheses in the alternative family. However, in a more recent paper, Cox [1977] states that formulating alternative distributions representing the direction of a departure from the null distribution is weaker than embedding the null distribution in a family of competing models, all of which are candidates for representation of the observed data. When the null distribution is embedded in a general family of parametric models, testing the null hypothesis coincides with the estimation of parameters in the richer family. Hence, if none of the competing models are appropriate, the general distribution may provide an adequate fit. This approach assumes that the family of models is general enough.

Cox's discussion suggests that one approach to the model selection problem in survival analyses or reliability studies is to initially fit a very general family of parametric distributions, which incorporates as special cases of its parameters distributions commonly used in life time studies. The aim of the present research is to evaluate the appropriateness of the Burr system of distributions developed by Irving Burr [1942], and in particular, the Burr Type XII family, as a general family of parametric distributions to be fit to failure data. The two most common survival or failure time distributions, the exponential and Weibull distributions, are both special limiting cases of the Burr XII family of distributions. Emphasis will be placed upon the evaluation of the adequacy of the fit of these two distributions within the parametric framework of the Burr system of distributions. Other distributions which are members of the Burr family are the normal, extreme value, logistic, generalized logistic, Gompertz, and Pareto distributions. The gamma and lognormal distributions are two distributions frequently fit to life time data which are not special limiting or approximating cases of the Burr XII family. However, it has been shown that much of the region covered by the lognormal and gamma distributions in the skewness-kurtosis plane is also covered by the Burr XII distribution (Rodriguez [1979]). (Skewness, symbolized by α_3 is a measure of density curve asymmetry and is defined by $\mu_3/\mu_2^{3/2}$, where μ_k is the k-th central sample moment. Kurtosis, denoted by α_4, is a measure of density curve peakedness and is defined by μ_4/μ_2^{2}.)

A review of the literature dealing with the Burr system of distributions is presented in Section 1.2. This is followed in Section 1.3 by an overview of methods for model selection based upon hypothesis tests with specified alternatives. Special attention is directed to the generalized gamma distribution, another very general family of distributions whose usefulness in survivorship and reliability applications will be compared to that of the Burr Type XII family. Section 1.4 contains an outline of the material presented in subsequent chapters.

1.2 Burr System of Distributions

The Burr system of distributions includes twelve types of cumulative distribution functions which yield a variety of density shapes. The attractiveness of this relatively unknown family of distributions for model fitting is that it combines a simple mathematical expression for the cumulative frequency function with wide coverage in the skewness-kurtosis plane. Many standard theoretical distributions, including the

Weibull, exponential, logistic, generalized logistic, Gompertz, normal, extreme value, and uniform distributions are special cases or limiting cases of the Burr system of distributions. The simple closed form of these distributions has been applied to studies in simulation, quantal bioassay, non-normal control charts, the approximation of theoretical distributions, and the fitting of smooth curves to histograms. Their utility in survivorship applications will be of special interest in this effort.

Irving Burr [1942] developed the system of Burr distributions in 1941 as an outgrowth of his research into methods for fitting cumulative frequency functions rather than probability density functions to frequency data. Burr viewed the cumulative frequency function as being theoretically better adapted to model fitting since expected frequencies in any given range are found by differencing the cumulative function $F(x)$ rather than by integrating the density function $f(x)$. Burr also considered the process of differentiating $F(x)$ to find $f(x)$ theoretically much simpler than the converse.

Analogous to the Pearson system of distributions, the Burr distributions are solutions to a differential equation, which has the form

$$dy/dx = y(1-y)g(x,y) \ , \tag{1.1}$$

where y equal to $F(x)$ and $g(x,y)$ must be positive for y in the unit interval and x in the support of $F(x)$. Different functional forms of $g(x,y)$ result in different solutions $F(x)$, which define the families of the Burr system. For instance, if $g(x,y)$ is taken to be $g(x)$, then

$$F(x) = [\exp\{-\int_{-\infty}^{x} g(u)du\}+1]^{-1} \ . \tag{1.2}$$

Burr identified twelve families of distributions which are given in Table 1.1.

TABLE 1.1

TWELVE DISTRIBUTIONS OF THE BURR SYSTEM

Type	F(x)	Range of x
I	x	$(0 < x < 1)$
II	$(1+e^{-x})^{-k}$	$(-\infty < x < \infty)$
III	$(1+x^{-c})^{-k}$	$(0 < x)$
IV	$[1+\{(c-x)/x\}^{1/c}]^{-k}$	$(0 < x < c)$
V	$(1+ce^{-\tan x})^{-k}$	$(-\pi/2 < x < \pi/2)$
VI	$(1+ce^{-r\sinh x})^{-k}$	$(-\infty < x < \infty)$
VII	$2^{-k}(1+\tanh x)^{k}$	$(-\infty < x < \infty)$
VIII	$\{(2/\pi)\tan^{-1}e^{x}\}^{k}$	$(-\infty < x < \infty)$
IX	$1 - \dfrac{2}{c\{(1+e^{x})^{k}-1\} + 2}$	$(-\infty < x < \infty)$
X	$(1+e^{-x^2})^{k}$	$(0 < x)$
XI	$\{x - \sin(2\pi x)/2\pi\}^{k}$	$(0 < x < 1)$
XII	$1 - (1+x^{c})^{-k}$	$(0 < x)$

The parameters c, k, and r are positive. When fitting these distributions to data, a location parameter ξ and a scale parameter β are incorporated by replacing X with $(X-\xi)/\beta$. In cases where the location parameter may be assumed to be known, the parameter ξ may be excluded by substituting X' for $(X-\xi)$.

Some of the above forms of the Burr distributions are related by simple transformations. For example, the Burr Type III distributions can be obtained from the Burr Type II distributions by replacing X with $c\ell n(X)$, where $\ell n(\cdot)$ denotes the natural logarithm (Johnson and Kotz [1970]). Similarly, the Burr Type XII distribution can be derived from the Burr Type III distribution by replacing X with X^{-1} (Burr and Cislak [1968]).

The Burr I family is more commonly known as the uniform distribution. The Burr III, IV, V, IX, and XII families have a variety of density shapes and incorporate four parameters, the maximum number that can be efficiently estimated with the method of moments (Rodriguez [1979]). Types III and XII are the simplest functionally and, therefore, the most desirable for statistical modeling.

Burr investigated only the Type XII family in depth. The density functions corresponding to the Burr XII family are defined by

$$f(x;c,k) = ckx^{c-1}/(1+x^c)^{-(k+1)} , \tag{1.3}$$

where x is positive. Initially, Burr [1942] specified that both shape parameters, c and k, must be at least as large as one. Burr [1967a] later relaxed this restriction to require both parameters to be positive. Rodriguez [1977] has suggested that the reason Burr first required c and k to be greater than or equal to one is that the Burr XII density functions are unimodal at

$$x = \{(c-1)/(ck+1)\}^{1/c} , \tag{1.4}$$

if c exceeds one and L-shaped if c is less than or equal to one. Rodriguez modified the parameter constraints to require that c and k are both positive and that their product, ck, exceeds four to insure the existence of the first four moments. The r-th moment about the origin of a Burr Type XII distribution is

$$\mu'_r = kB(r/c+1,k-r/c) \ , \tag{1.5}$$

where r is less than the product ck and B(a,b) is the standard Beta function.

Drane, Owen, and Seibert [1978] identified the special approximating and limiting forms of the Burr XII distributions which correspond to well-known standard distributions. The Burr shape parameters and cumulative distribution functions associated with the special cases are displayed in Table 1.2.

TABLE 1.2

SPECIAL CASES OF THE PARAMETERS OF THE BURR XII DISTRIBUTION

c	k	Distribution	C.D.F.
c	k	Burr XII	$1-(1+x^c)^{-k}$
4.874	6.158	Approximate Normal	$1-(1+x^{4.874})^{-6.158}$
c	∞	Weibull	$1-\exp(-x^c)$
1	∞	Exponential	$1-\exp(-x)$
∞	k	Generalized Logistic	$1-(1+e^x)^{-k}$
∞	1	Logistic	$1-(1+e^x)^{-1}$
∞	∞	Gompertz	$1-\exp(-e^x)$
1	1	Pareto	$1-(1+x)^{-1}$

Numerous moment-ratio diagrams of the coverage of the Burr XII distributions appear in the literature. Hatke [1949] was the first to present a moment-ratio diagram. She used the (α_3^2,δ) coordinate system, where δ is a function of skewness (α_3) and kurtosis (α_4) given by

$$\delta = \frac{(2\alpha_4 - 3\alpha_3^2 - 6)}{(\alpha_4 + 3)} \quad , \quad \alpha_3^2 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2} \, . \tag{1.6}$$

In this coordinate system, which was developed by Craig [1936], the mappings for the Pearson system of frequency curves take a strikingly simple form. Therefore, use of this coordinate system facilitates comparisons between the coverage of the Burr versus the Pearson system of distributions. Burr and Cislak [1968] showed that the coverage of the Burr XII distributions in the (α_3^2,δ) plane is much greater than that first reported by Hatke [1949]. The Burr Type XII region covers area corresponding to the Pearson I, IV, and VI distributions (Burr and Cislak [1968]). The section of the Pearson Type III (gamma) line joining the normal and exponential points is also included. Part of the Pearson II line segment falls in the Burr region, but not the rectangular end-point (α_3=0, α_4=1.8) as claimed by Burr and Cislak (Rodriguez [1977]). Hatke [1949] recognized that the single functional form of the Burr XII family of distributions covers the same moment-ratio area for which the Pearson curves require several different functional forms.

Rodriguez [1977] discusses the inappropriateness of the (α_3^2,δ) coordinate system for displaying the moment-ratio coverage of the Burr XII distributions. He contends that the upper and lower bounds are not apparent in the diagram, nor are they derived analytically. With regard to the (α_3^2, α_4) version of the moment-ratio chart given by Ord [1972], Rodriguez found that it also fails to clarify the region of coverage and he questioned the validity of the shape of the lower bound. A final criticism Rodriguez had of the former moment-ratio diagrams was that they each failed to show the effect of modifying c and k on the shape characteristics of the Burr XII density functions.

Rodriguez [1977] found that an (α_3,α_4) coordinate system formerly used by Johnson and Kotz [1972] is preferable for displaying the moment-ratio diagram for the Burr XII distributions. Figure 1.1, reproduced from Rodriguez [1977], compares the coverage of the Burr XII distribution in the moment-ratio plane to that of the Pearsonian system. Rodriguez described the region occupied by the Burr XII distributions as 'the prow of an ancient Roman galley'. The boundaries of the Burr XII area are defined by limiting forms of the Burr Type XII distributions. The northern boundary consists of (α_3,α_4) points belonging to the Weibull distributions. Rodriguez [1977, 1979] showed this analytically by the following argument. If X follows the Burr XII distribution, then for fixed c,

$$\lim_{k \to \infty} \Pr\{X \leq (1/k)^{1/c} x\} = \lim_{k \to \infty}\{1-(1+x^c/k)^{-k}\} = 1-\exp(-x^c). \qquad (1.7)$$

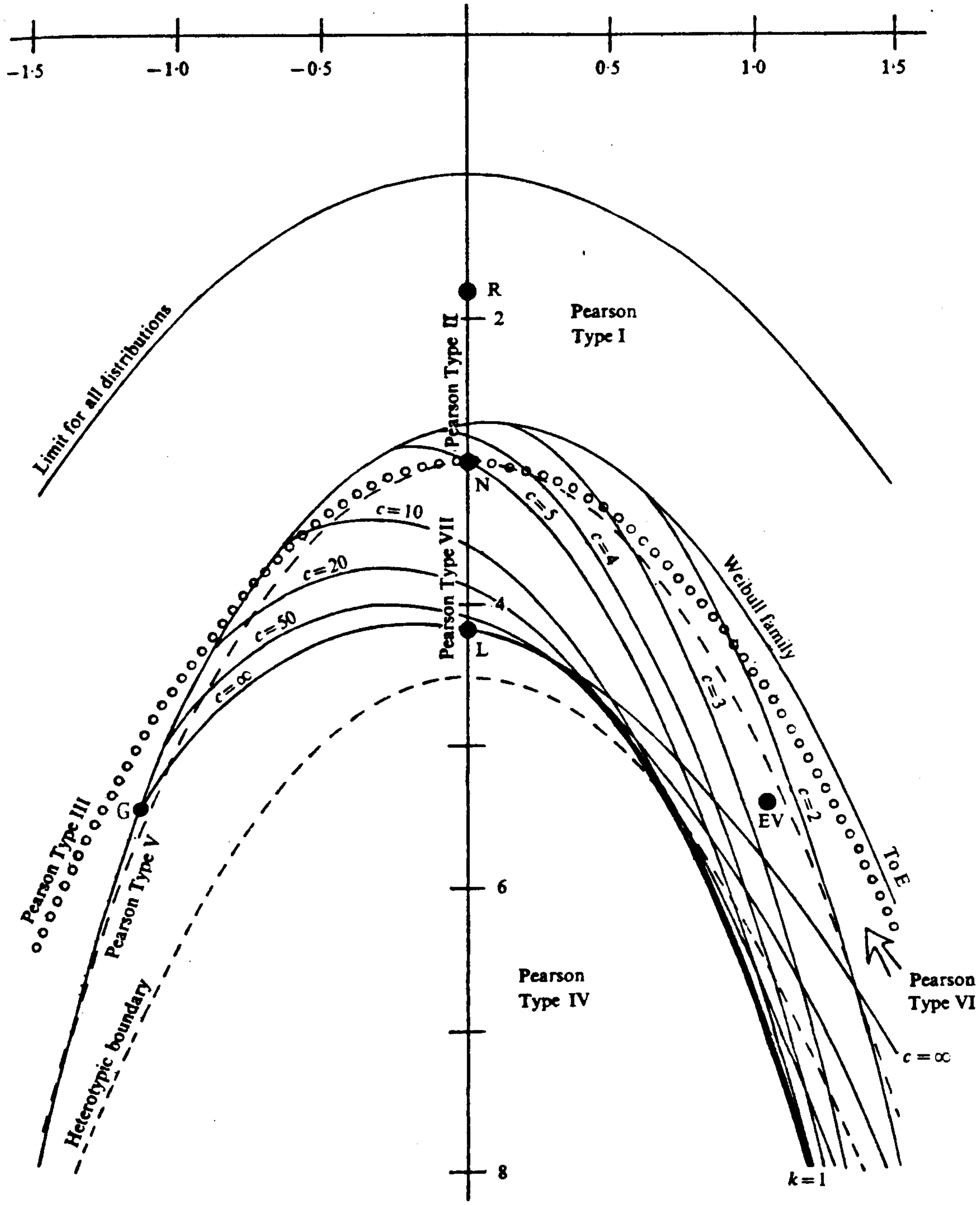
In the moment-ratio diagram as k increases from 4/c to infinity, the c-constant curve approaches the Weibull end-points in a counter-clockwise direction. The Weibull curve passes through the exponential point $(\alpha_3=2,\ \alpha_4=9)$, which lies on the c-constant Burr curve when c is equal to one. The southwest boundary is formed by part of the generalized logistic curve defined by

$$1 - (1+e^x)^{-k} . \qquad (1.8)$$

This family represents the limiting form of the Burr XII distribution as c approaches infinity for fixed k. The generalized logistic curve passes through the logistic point when k is equal to one. As both c and k tend to infinity the Burr constant curves approach a point on the Weibull curve. Rodriguez [1977] determined that this limiting point corresponds to the intersection of the Weibull and generalized logistic

FIGURE 1.1

MOMENT RATIO DIAGRAM OF THE BURR TYPE XII DISTRIBUTION
RODRIGUEZ [1977]



curves. Evaluation of the set of parametric equations

$$\lim_{\substack{c \to \infty \\ k \to \infty}} \left[\alpha_3 = \frac{\Gamma^2(k)\lambda_3 - 3\Gamma(k)\lambda_2\lambda_1 + 2\lambda_1^3}{\{\Gamma(k)\lambda_2 - \lambda_1^2\}^{3/2}} \right]$$

and

$$\lim_{\substack{c \to \infty \\ k \to \infty}} \left[\alpha_4 = \frac{\Gamma^3(k)\lambda_4 - 4\Gamma^2(k)\lambda_3\lambda_1 + 6\Gamma(k)\lambda_2\lambda_1{}^2 - 3\lambda_1^4}{\{\Gamma(k)\lambda_2 - \lambda_1^2\}^{3/2}} \right], \tag{1.9}$$

where

$$\lambda_j = \Gamma(j/c+1)\Gamma(k-j/c) \tag{1.10}$$

for integer values of j ranging from one to four allowed Rodriguez to determine that the coordinates of this limiting point in the skewness-kurtosis plane are (-1.14, 5.35). This corresponds to the Gompertz point, denoted by G in Figure 1.1 and by -EV in Figure 1.2. Johnson and Kotz [1970] state that although not generally regarded as such, the Gompertz distribution is an extreme value Type I distribution truncated at zero. The southeast boundary is generated by the Burr XII distribution with k equal to one. Rodriguez proved this by applying the method of Lagrange multipliers to minimize α_3 subject to the constraint that α_4 is equal to a constant greater than or equal to 4.2 (the minimum kurtosis value on the southern boundary in the positive skewness half-plane which corresponds to the logistic point, where the southwest boundary with c equal to infinity intersects the line α_3=0).

In addition to the special limiting forms which have been mentioned in the discussion of the moment-ratio coverage of the Burr Type XII distributions and which include the Weibull, exponential, logistic, generalized logistic, and Gompertz distributions, Rodriguez [1977, 1979] has shown that the extreme value Type I point, extreme value Type II curve, and parts of the lognormal curve (falling between the Pearson III and V curves) are also included in the moment-ratio coverage of the Burr XII family. Furthermore, Rodriguez states that all three families of the translation (Johnson) family are also covered by this very general family of distributions.

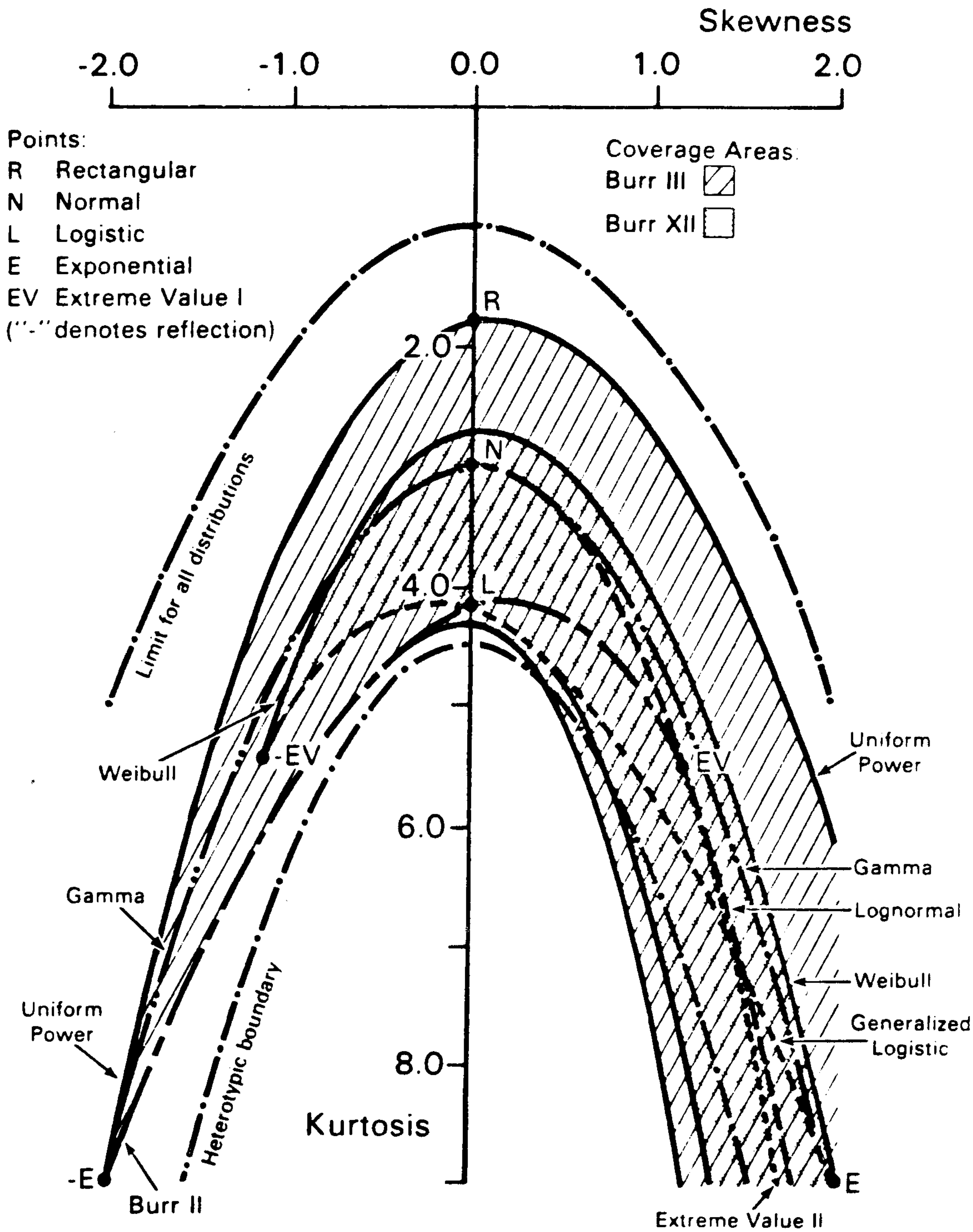
The main lack of coverage of the Burr XII family is towards low kurtosis (α_4) values for fixed skewness (α_3). Burr and Cislak [1968] were the first to mention this. Burr [1973] later found that by using the reciprocal transformation $Y = X^{-1}$, yielding the distribution function

$$G(y) = (1+y^{-c})^{-k} \tag{1.11}$$

for positive values of y, the coverage is greatly extended in this region. Burr noted that this extension covers all of the bell-shaped and J-shaped regions of the beta (Pearson Type I) area. Rodriguez [1979] further investigated this transformed distribution function, which corresponds to the Burr III family. This is the only other Burr family of curves to be discussed in the literature. Burr III densities are unimodal if ck exceeds one and twisted L-shaped if ck is less than or equal to one. Rodriguez presents a moment-ratio diagram, reproduced in Figure 1.2, which shows the coverage of both the Burr XII and Burr III distributions. The northern boundary of the Burr III distribution in the moment-ratio plane corresponds to that of the four-parameter generalized lambda distribution discussed by Ramburg _et al_. [1979]. The usefulness of the generalized lambda distribution has been recognized for cases where the underlying distribution is unknown since its coverage in the moment-ratio plane includes the uniform, exponential, lognormal, gamma, Weibull, and beta regions. The Burr III distribution covers these areas as well.

FIGURE 1.2

A COMPARISON OF THE MOMENT-RATIO COVERAGE OF THE BURR TYPE III AND BURR TYPE XII DISTRIBUTIONS
RODRIGUEZ [1979]



There is a region of double occupancy in the Burr XII coverage area. That is to say, in the area south of the generalized logistic curve (defined by c equal to infinity) there are two Type XII distributions for each (α_3,α_4) point, one with c greater than 3.6 and one with c less than 3.6 (Rodriguez [1977]). North of the c-constant curve each (α_3,α_4) point corresponds to a unique pair of shape parameters (c,k). There is a region of double occupancy in the Burr Type III coverage area as well. Such regions pose difficulties when model fitting is carried out by the method of moments.

In 1965 Dubey [1968] produced a new derivation of the Burr Type XII distribution. Takahasi [1965] appears to have obtained the same result independently. Both researchers considered a conditional random variable X which is distributed according to the Weibull (parental) distribution given by

$$f(x|\beta) = \alpha\beta x^{\alpha-1}\exp\{-\beta x^{\alpha}\} \quad , \tag{1.12}$$

where x is positive and the parameters α and β are both greater than zero. The parameter β is not regarded as being constant, but rather is assumed to behave as a gamma (mixing) random variable whose probability density function is given by

$$f(\beta) = \delta^{\gamma}\beta^{\gamma-1}\exp\{-\delta\beta\}/\Gamma(\gamma) \quad , \tag{1.13}$$

where β is non-negative and the parameters γ and δ are both positive. Then the probability density function for X corresponds to the Compound Weibull distribution and is given by

$$f(X) = \int_0^{\infty} f(X|\beta)f(\beta)d\beta = \alpha\gamma\delta^{\gamma}X^{\alpha-1}(X^{\alpha}+\delta)^{-(\gamma+1)} \quad , \tag{1.14}$$

where X and the three parameters α, γ, and δ are all positive. For δ

equal to one the Compound Weibull distribution reduces to the Burr Type XII distribution, and therefore, Dubey has suggested that the Compound Weibull distribution be considered as a generalized Burr distribution.

Takahasi [1965] derived a multivariate Burr Type XII distribution using the theory of compound distributions and showed that the distribution has two nice multivariate properties. In particular, he proved that marginal and conditional distributions of a multivariate Burr XII distribution are also distributed as multivariate Burr XII distributions. For the bivariate case Durling [1975] showed that the correlation coefficient is restricted to a point defined by a function of the shape parameters c and k. To remove this limitation Durling generalized the bivariate Burr distribution so that the correlation coefficient varies according to an additional parameter.

Few examples appear in the literature in which the Burr XII distribution has been fit to actual data. Burr [1942] developed a cumulative moment technique (which requires the integration of the cumulative distribution function instead of the probability density function) for estimating the shape parameters c and k from the computed sample skewness and kurtosis. This approach necessitates the use of tables (Burr [1973] and Burr and Cislak [1968]), relating the values of the parameters c and k to the first four moments μ, σ, α_3, and α_4 of the Burr XII distribution. Interpolation from these tables is difficult because of the marked nonlinearity of α_3 and α_4 as functions of c and k (Hatke [1949] and Rodriguez [1979]). Once the appropriate estimated values of c and k have been determined based on the computed sample skewness and kurtosis, estimates of the location parameter ξ and the scale parameter β are obtained from

$$\frac{X-\mu}{\sigma} = \frac{x-\bar{x}}{s_x} , \tag{1.15}$$

where X represents a Burr random variate with mean μ and standard deviation σ, x represents the observed data to be modeled, and $\bar{x}$ and s_x represent the sample mean and sample standard deviation, respectively. Estimates of ξ and β are determined by the equations

$$\tilde{\xi} = \bar{x} - \frac{\mu s_x}{\sigma} \quad \text{and} \quad \tilde{\beta} = \frac{s_x}{\sigma} . \tag{1.16}$$

Burr [1942] illustrated the method of moments procedure by fitting the four-parameter Burr XII distribution to a highly normal distribution of heights.

As an alternative approach to this table look-up method, Rodriguez [1980a] has developed an interactive graphics package for fitting Burr Type III curves to histograms. The program 'locates' the histogram's shape on a skewness-kurtosis diagram and then performs an iterative estimation procedure of the shape parameters, c and k, to determine a curve whose skewness and kurtosis match those of the histogram. Rodriguez suggests that other methods of fitting which can be used in conjunction with Burr XII distributions are maximum likelihood estimation and the method of percentiles. However, no examples of the use of such methods exist in the literature.

In addition to fitting smooth curves to histograms, other useful applications of the Burr system of distributions have been cited in the literature. Prior mention was made of the suitability of the Burr XII distribution for approximating the normal distribution. When the shape parameters c and k assume the approximating values of 4.874 and 6.158, respectively, it is possible for the Burr XII distribution to match the first four moments of the normal distribution. Burr [1967a] investigated the appropriateness of substituting the Burr cumulative distribution function with these values of the shape parameters for the standardized normal distribution function $\Phi(u)$ when solving for random normal deivates. His results showed that the largest discrepancy between $\Phi(u)$ and the Burr approximation was 0.00046, occurring at $u = \pm 0.6$.

In view of its wide coverage in the skewness-kurtosis plane, the Burr XII family clearly incorporates highly non-normal distributions as well. Burr [1967b] used non-normal Burr XII distributions to investigate the robustness of the control chart technique used in industrial processes to check the reliability of results or a process to non-normal data. Another use of Burr XII distributions in non-normal regions has been to study the exact distribution of the median and the range for small samples from non-normal populations (Burr [1968] and Burr and Cislak [1968]). Burr and Cislak solved for the exact distribution of the median and all existing moments as linear expressions of beta functions. They determined that the efficiency of the median relative to the mean, defined by the usual expression

$$E_n = \sigma_{\bar{X}}^2 / \sigma_{\tilde{X}}^2 \tag{1.17}$$

improves for greater non-normality in the direction of higher skewness and kurtosis. Specifically, at just beyond the exponential point in the skewness-kurtosis plane, the median is a more efficient estimator relative to the mean.

In a general approach to quantal response bioassay applications, Drane, Owen, and Seibert [1978] used the Burr XII family of distributions. Under the assumption that the shape parameters c and k are known, these authors applied an approximate minimum chi-square criterion for fitting the distribution to binomial responses ordered by the doses of the stimulus. Estimation of the location parameter ξ and the scale parameter β was accomplished by weighted linear least squares.

1.3 Distributional Hypothesis Tests with Specified Alternatives

Two general types of distributional tests which incorporate alternative hypotheses are classified according to the relationship between the null and the alternate hypotheses. With regard to the first type, which is referred to as tests of nested families, the null distribution is embedded (or nested) as a special case of the distribution parameters in a more general alternative distribution. The null family is usually more simple mathematically with a fewer number of parameters. In the second type of test, which is referred to as a test of separate families, a simple hypothesis in the null family cannot be obtained as a limit of simple hypotheses in the alternative family.

1.3.1 Nested Families

When modeling data it is desirable to use the simplest model which adequately describes the data. Under the assumption that a general alternative distribution correctly models the failure times, tests of hypotheses may be performed on the distribution parameters to determine if a simpler special case (the null distribution) of the general distribution is appropriate for a given application. For example, a test of the null hypothesis that the shape parameter of the gamma distribution is equal to one with an unspecified scale parameter is equivalent to a test of the adequacy of the one-parameter exponential distribution versus a two-parameter gamma alternative.

Since the parameter space of the null distribution is a subspace of the parameter space of the alternative distribution, the generalized likelihood ratio test (see Mood, Graybill, and Boes [1963, p. 419]) is appropriate for discriminating between nested families of distributions. While it can sometimes be difficult to find the distribution of the likelihood ratio (denoted by Λ), under certain regularity conditions $-2\ln\Lambda$ is asymptotically distributed as a chi-square random variable with r degrees of freedom, where r is the number of components of the parameter space that are specified by the null distribution.

Mood, Graybill, and Boes state that in general, a generalized likelihood ratio test will be a good test. Moran [1951] has shown that the generalized likelihood ratio test obtained when one tests the adequacy of the one-parameter exponential distribution nested in the two-parameter gamma distribution is locally (about the null hypothesis) the asymptotically most powerful test. Shorack [1972] has subsequently proven, by means of a lemma for scale type statistics, that Moran's statistic given by

$$M = n\left(\prod_{i=1}^{n} x_i\right)^{1/n} \Big/ \sum_{i=1}^{n} x_i \qquad (1.18)$$

is the uniformly (in the shape parameter) most powerful scale invariant test of exponentiality versus one-sided gamma alternatives.

Similar to the case of the exponential versus the gamma distribution, the tests based upon the maximum likelihood estimates of the shape parameter of the two-parameter Weibull distribution appear to be considerably better than other tests of exponentiality against more general Weibull alternatives. Thoman, Bain, and Antle [1969] developed a test based upon the pivotal function $\hat{c}/c$, when c and $\hat{c}$ denote the shape parameter and its maximum likelihood estimator, respectively, of the Weibull distribution. By showing that $\hat{c}/c$ has the same distribution as the maximum likelihood estimator of the Weibull shape parameter when the sampling is from a standard exponential function, these authors employed Monte Carlo methods to obtain the exact distribution of $\hat{c}/c$ for sample sizes 5(1)20, 22(2)80, 85(5)100, 110, and 120. The asymptotic distribution of $\hat{c}/c$ was also derived. Based upon the percentage points of this distribution, confidence intervals and equivalent tests of hypotheses may be constructed for the Weibull shape parameter c. Klimko, Antle, Rademaker, and Rockette [1975] showed that the test for exponentiality (testing whether c is equal to one) nearly achieves the upper bounds for the power of any scale invariant test for exponentiality against Weibull alternatives. The upper bounds were constructed by simulating the power (at c equal to 1.25, 1.5, and 2.0) of Shorack's [1972] most powerful invariant (with respect to scale) test of exponentiality versus one-sided Weibull alternatives where c is greater than one. Klimko _et al_. [1975] also obtained the upper bounds for the power of an invariant (under change of location and scale) test for discriminating between the two-parameter exponential and the three-parameter Weibull distributions, each of which include a location parameter. Since it is the intent of this research to focus on applications in survival and reliability analyses, for which the location parameter is usually assumed to be zero, the problem of choosing between the two-parameter exponential distribution and the three-parameter Weibull distribution will not be stressed, except to note that Antle and Klimko [1973], and Engelhardt and Bain [1973] have proposed tests for this case.

1.3.2 A Special Generalized (Nested) Family of Distributions-- Generalized Gamma Distribution

The three-parameter generalized gamma distribution is a general family of distributions which includes the Weibull, exponential, lognormal, and gamma distributions among others as special cases of the distribution parameters. Although generalized gamma distributions were first discussed by Amoroso [1925], who fitted the distribution to observed income rates, little attention was directed to this family of distributions until relatively recently. The current interest in reliability and survival time studies, for which the Weibull, exponential, normal, lognormal, and gamma distributions have been shown to have special merit, has served to stimulate renewed interest in the generalized gamma distribution. Since this family of distributions incorporates all of the competing distributions mentioned above except for the normal distribution, it provides a general parametric framework for discriminating between important failure time distributions. Moreover, since it includes some special cases which are also special cases of the Burr Type XII family, it provides an opportunity to compare and contrast the results obtained through fitting the Burr distribution with those of another generalized family.

The generalized gamma distribution is defined by

$$f(x;a,k,b) = ba^{-bk}x^{bk-1}\exp\{-(x/a)^b\}/\Gamma(k) \tag{1.19}$$

for positive values of x and parameters a, k, and b. Stacy [1962] derived the generalization by supplying a positive parameter b as an exponent in the exponential factor of the gamma distribution. Lienhard and Meyer [1967] provided a derivation of the generalized gamma distribution based on physical considerations. In their formulation they consider the occurrence of an event which depends on some variable X such as stress or the time a component has been subjected to a given level of stress. The parameter product bk represents the number of ways the event can occur in this physical framework.

Modifications to Stacy's form of the distribution have been introduced. Harter [1967] developed a four-parameter generalized gamma distribution by including a location parameter. Stacy and Mihram [1965] allowed the power parameter to assume negative values, thereby including the distribution of the reciprocal X^{-1} of a generalized gamma random variable X. However, the three-parameter generalized gamma distributions, as first defined by Stacy, have provided adequate flexibility for the reliability and survivorship applications cited in the literature.

The regions of the parameter space which define the distributions nested in the family of generalized gamma distributions are identified in Table 1.3. The generalized gamma distribution has had its greatest

TABLE 1.3

SPECIAL CASES OF THE PARAMETERS OF THE GENERALIZED GAMMA DISTRIBUTION

a	k	b	Distribution
a	1	1	Exponential
a	k	1	Gamma
a	1	b	Weibull
a	∞	b	Lognormal
2	n/2	1	Chi-square (n degrees of freedom)
$\sqrt{2}$	n/2	2	Chi (n degrees of freedom)
$\sqrt{2}$	1/2	2	Half-normal
$\sqrt{2}$	3/2	2	Circular normal
$c\sqrt{2}$	1	2	Rayleigh (c > 0)

use in the assessment of the adequacy of the Weibull ($k=1$) and the exponential ($k=1$, $a=1$) models relative to the more generalized model. Parr and Webster [1965] derived the joint marginal asymptotic density of the maximum likelihood estimators of k and a. Based on the joint asymptotic density these authors outlined a method for rejecting (with a known probability of false rejection) the Weibull and the exponential distributions when they do not fit the empirical distribution of a large number of failure times. Hager and Bain [1970] further investigated the properties of the asymptotic distribution of the maximum likelihood estimators as first derived by Parr and Webster. After demonstrating that $\hat{k}$ is distributed independently of the nuisance parameters, a and b, Hager and Bain proposed that the asymptotic distribution of $\hat{k}$ be used for testing the adequacy of the Weibull model for large samples. However, subsequent Monte Carlo results indicated that the asymptotic distribution of $\hat{k}$ may still not be in effect for samples as large as 400 when k was set equal to 1.0 and 2.0. Hager and Bain also reported basic difficulties associated with solutions to the maximum likelihood equations caused by the computational complexity of the generalized gamma distribution. The Newton-Raphson iterative solution procedure did not appear to work well and in some cases these authors were unable to obtain meaningful maximum likelihood estimates.

Alternative test procedures for assessing the suitability of the Weibull model as a special case of the generalized gamma model have been suggested by Hager and Bain. Two of these tests are based upon the statistic first developed by Moran [1951], which as discussed in Section 1.3.1, is the asymptotically most powerful test for testing exponentiality against gamma alternatives. This test has applicability to model selection within the generalized gamma distributional framework since the test of H_0: X~Generalized Gamma (a,k=1,b) versus H_1: X~Generalized Gamma (a,k>1,b) for known b reduces to a test of H_0: Y~Exponential (a) versus H_1: Y ~ Gamma (a,k) by the data transformation $Y = X^b$. For the case of unknown b, Hager and Bain proposed the modified (1.18) statistic

$$M_{\hat{b}} = n(\prod_{i=1}^{n} x_i^{\hat{b}})^{1/n} / \sum_{i=1}^{n} x_i^{\hat{b}} \quad , \tag{1.20}$$

which is distributed independently of a and b. While Monte Carlo results suggest that $M_{\hat{b}}$ behaves better than $\hat{k}$ for large samples, both of these statistics are based upon maximum likelihood estimates of the generalized gamma distribution parameters.

Hager and Bain discussed a second modification of the optimal statistic of Moran which avoids the computational difficulties and disadvantages of the maximum likelihood estimators for the general distribution. Consider the estimator $\hat{b}_1$ which corresponds to the maximum likelihood estimator of the Weibull power parameter when k is taken to be equal to one. The modified (1.18) statistic based upon $\hat{b}_1$ is given by

$$M_{\hat{b}_1} = n(\prod_{i=1}^{n} x_i^{\hat{b}_1})^{1/n} / \sum_{i=1}^{n} x_i^{\hat{b}_1} \quad . \tag{1.21}$$

Use of $M_{\hat{b}_1}$ has the advantage that existing Weibull programs can be used to calculate the statistic. Hager and Bain presented percentage points for $M_{\hat{b}_1}$ simulated under the null hypothesis that k is equal to one for sample sizes of 10, 20, 30, 50, 100, 200, and 400.

The final approach to hypotheses testing under the assumption of the generalized gamma distribution is a likelihood ratio test procedure. To test H_0: k=1 against H_1: k≠1, Hager and Bain used the statistic

$$R = \frac{\prod_{i=1}^{n} f(x_i;\hat{a},k=1,\hat{b})}{\prod_{i=1}^{n} f(x_i;\hat{a},\hat{k},\hat{b})} , \tag{1.22}$$

which they showed to be independent of the nuisance parameters a and b.

Hager and Bain compared the three statistics $\hat{k}$, $M_{\hat{b}_1}$, and R with regard to the power of the tests of the Weibull null hypothesis for k equal to one against the generalized gamma alternatives at k equal to two and three. The numerical results indicate that $M_{\hat{b}_1}$, which is the simplest statistic to compute, has approximately the same power as the other statistics at least for samples of size 200 to 400. $M_{\hat{b}_1}$ and R are superior to $\hat{k}$. Since the likelihood ratio test is under certain conditions asymptotically most powerful, Hager and Bain attributed the relatively low powers of 0.27 and 0.77 for sample sizes of 200 and 400 at k equal to two to the fact that the Weibull distribution and the generalized gamma distribution are quite similar. In a later paper Hager, Bain, and Antle [1971] considered the robustness of the Weibull model with respect to the generalized gamma model for reliability applications. The authors examined the adequacy of a Weibull estimation procedure of the reliability or survival function when the correct model is a generalized gamma distribution. The results of the simulation study suggested that there is little advantage in assuming the generalized model for sample sizes under 200. This is strong evidence for fitting the Weibull model since it avoids the computational difficulties associated with the generalized gamma model for smaller sample sizes.

Prentice [1974] also used a likelihood ratio type approach to the model selection problem in the generalized gamma framework. Unlike the

other researchers who have investigated the generalized gamma distribution, Prentice was primarily interested in discriminating between the Weibull and lognormal distributions. The lognormal distribution is a limiting case of the general distribution which is realized as the shape parameter k approaches infinity. Prentice preferred to model a log-generalized gamma variate which follows a location-scale distribution. He derived a parameter transformation $q=k^{-1/2}$ which allows the limiting normal model on the logarithmic scale to be mapped to the origin. After further parameter transformations on a and b, Prentice obtained analytic results which show that the resulting model likelihood is twice differentiable with respect to its three parameters at q equal to zero.
He then proceeded to use the asymptotic chi-square approximation for the distribution of the likelihood ratio test statistic. Monte Carlo results show that the empirical and asymptotic distributions of the maximum likelihood estimator $\hat{q}$ under the null lognormal distribution are equivalent for sample sizes of 25 and 50. Similarly, the empirical and asymptotic distributions of the maximum likelihood estimator $\hat{q}$ when the Weibull distribution is the null distribution are equivalent for sample sizes of 100. Farewell and Prentice [1977] applied the log-generalized gamma distribution to several data sets from the industrial and medical literature. They extended the model to incorporate censored observations as well as regression variables.

1.3.3 Separate Families

Distributional tests of separate families are characterized by a set of hypotheses for which a simple hypothesis in the null family cannot be obtained as a limit of simple hypotheses in the alternative family. Irwin [1942] was among the first to address the problem of discriminating between separate parametric families. His work was motivated by results of Withell [1942], who showed that the logarithms of the survival times for a wide range of microorganisms and poisons are approximately normally distributed even when the number of survivors is adequately represented by an exponential function of time. Irwin's results showed that although the normal approximation almost always slightly underestimates the true percentage of survivors calculated from the distribution of the logarithm of survival times under the exponential law, it would require very good data (subject to few sampling fluctuations) to distinguish between the two separate families.

Almost twenty years after Irwin's research, Cox [1961] proposed the Neyman-Pearson likelihood ratio

$$L_{fg} = \ell n \left\{ \frac{\sup\limits_{\underset{\sim}{\alpha} \varepsilon \Omega_\alpha} f(y,\underset{\sim}{\alpha})}{\sup\limits_{\underset{\sim}{\beta} \varepsilon \Omega_\beta} g(y,\underset{\sim}{\beta})} \right\} , \tag{1.23}$$

where $\ell n(\cdot)$ denotes the natural logarithm, as a method for selecting between two separate families $f(y,\underset{\sim}{\alpha})$ and $g(y,\underset{\sim}{\beta})$. H_f denotes the null hypothesis and H_g denotes the alternative hypothesis. Because $\underset{\sim}{\alpha}$ may contain more adjustable parameters than $\underset{\sim}{\beta}$ and, hence, $f(y,\underset{\sim}{\alpha})$ would provide a better fit, Cox pointed out that it is not sufficient to use the observed value of L_{fg} as a measure of evidence in favor of H_f versus H_g. Cox proposed as a test statistic

$$T_f = L_{fg} - E_{\underset{\sim}{\hat{\alpha}}}(L_{fg}) , \tag{1.24}$$

which represents a comparison of L_{fg} with the best estimate of the value expected under H_f. A small value would lead to the rejection of H_f.

Cox derived the asymptotic variance of T_f based upon the asymptotic distribution of $(\hat{\alpha},\hat{\beta})$ and advised that H_f be tested by referring to the standard normal distribution.

Cox applied these results to obtain a solution to the problem identified by Irwin of discriminating between the exponential and lognormal distributions. In a subsequent paper Cox [1962] derived the test statistics for choosing between the geometric and Poisson distributions. Jackson [1968] applied the likelihood ratio test to the case of selecting between the lognormal and gamma distributions.

Cox's approach to the model selection problem is well-suited to the case where either H_f or H_g is the true distribution or when high sensitivity is required against a particular alternative distribution. A contrasting approach treats the competing models on an equal basis and tries to assess the extent to which each provides an adequate fit. Atkinson [1970] employed the latter approach to investigate a suggestion made by Cox [1961] of combining the null and the alternative hypotheses into a general model of which they would both be a special case. Atkinson [1970] derived the new general distribution

$$H \propto \{f(y,\underset{\sim}{\alpha})\}^{\lambda}\{g(y,\underset{\sim}{\beta})\}^{1-\lambda} \quad , \tag{1.25}$$

which has the nice property of being additive in the log-likelihood. A test of the hypothesis that addresses the question of whether the models give significantly different fits to the data corresponds to testing λ equal to one-half. This test procedure may also be used to evaluate departures from one family in the direction of the other by testing the hypothesis that the value of λ is zero or one. However, there is difficulty when applying the asymptotic chi-square approximation to the distribution of the generalized likelihood ratio statistic in this case since $\lambda = \{0,1\}$ forms the boundary of the parameter space.

More recently Cox and Atkinson have investigated the design of experiments for discriminating among separate families of distributions. Since the present research is directed to observational data, these papers are only mentioned for the purpose of completeness. In a joint paper Atkinson and Cox [1974] discuss experimental designs for discriminating between several regression models. Atkinson and Fedorov [1975a, 1975b] present sequential experimental designs for choosing between rival regression models which are not required to be linear in the parameters.

The likelihood ratio test holds certain advantages for the selection of models from separate location-scale families of the form

$$f(x;a,b) = (1/b)g\{(x-a)/b\} , \tag{1.26}$$

where the location parameter a may be any real number, the scale parameter b is restricted to the positive real line, and x is bounded below by a. Under the following formulation of the model selection problem

$$\begin{aligned} H_0:&\quad X \sim f_0(x;a,b) = (1/b)g_0\{(x-a)/b\} \\ H_1:&\quad X \sim f_1(x;a,b) = (1/b)g_1\{(x-a)/b\} , \end{aligned} \tag{1.27}$$

Dumonceaux, Antle, and Haas [1973] determined that the distribution of the likelihood ratio statistic

$$R = \frac{\max\limits_{a,b} \prod\limits_{i=1}^{n} f_1(x_i;a,b)}{\max\limits_{a,b} \prod\limits_{i=1}^{n} f_0(x_i;a,b)} \tag{1.28}$$

is independent of unknown location and scale parameters. This result, which follows from a problem given by Lehmann [1959, p. 252], makes it feasible to construct the exact distribution of R under various combinations of null and alternative location-scale distributional hypotheses. Dumonceaux et al. [1973] used Monte Carlo methods to obtain the distribution of R for discriminating between the normal and the Cauchy distributions, the normal and the exponential distributions, and the normal and the double-exponential distributions. Dumonceaux and Antle [1973] showed that the lognormal and the Weibull distributions are both location-scale models on the logarithmic scale. Therefore, they constructed for moderate sample sizes the test for discriminating between these distributions and they note that the same exact tables may be applied to the problem of selecting between the normal and the Type I extreme value distributions as well, since the ratio of maximized likelihoods is invariant under transformations of the observations.

Bain and Engelhardt [1980] chose the likelihood ratio test as a method for discriminating between the Weibull and gamma distributions, two distributions which are both rather flexible and which overlap in the sense that the exponential distribution is a special case of both models. Since the parameters of the gamma distribution are not related to location-scale parameters, the likelihood ratio statistic is not independent of unknown parameters. As an alternative approach, the authors specified zero as the critical value of the natural logarithm of the likelihood ratio and then simulated the probabilities of correct selection (PCS) for various sample sizes and values of the parameters under each model. For standard exponential data the PCS is approximately 0.5 which is desirable since both models are correct. As the data move away from common exponential cases the PCS increases.

With regard to power considerations, Dumonceaux et al. [1973] pointed out that for the normal versus the exponential and the normal versus the uniform model selection problems, the likelihood ratio test is identical with the most powerful invariant test for discriminating between location-scale models given by Lehmann [1959, p. 248], which is of the form

$$LRI = \frac{\int_{-\infty}^{\infty}\int_{0}^{\infty} v^{n-2} f_1(vx_1+u_1,\ldots,vx_n+u_n)\,dv\,du}{\int_{-\infty}^{\infty}\int_{0}^{\infty} v^{n-2} f_0(vx_1+u_1,\ldots,vx_n+u_n)\,dv\,du}. \tag{1.29}$$

For a sample of 20 and Type I error of 0.05, Lilliefors [1967] reported the power of the Kolmogorov-Smirnov test for normality to be 0.61 when the sample is from the exponential distribution and the power of the chi-square test to be about 0.29. The likelihood ratio statistic has power of 0.96. For the selection problem defined by the normal versus the Cauchy distributions, Dumonceaux et al. [1973] showed by means of simulation that the likelihood ratio test is nearly as powerful as Lehmann's best invariant test. Dumonceaux and Antle [1973] evaluated, again by simulation, the power of the likelihood ratio test for log-normal alternatives given that the sampling is from the Weibull distribution, and vice versa, for moderate sample sizes. Their results indicate that in this case for a sample size of 20 and Type I error of 0.05 the power is only 0.43 when the true distribution is lognormal and 0.48 when the true distribution is Weibull. The authors acknowledged that the difficulty in distinguishing between two distributions depends upon the degree of similarity of their shapes. In the case of the Weibull versus the lognormal distributions, each is a positively skewed distribution for a positive random variable. It is noted that in a more recent application Farewell and Prentice [1977] were unable to discriminate using the large sample properties of the generalized likelihood ratio test between the Weibull and lognormal families nested in the generalized gamma distribution.

Gehan and Siddiqui [1973] considered the problem of choosing between parametric models for grouped survival data. For each model that they investigated, which included the exponential, linear hazard function, Gompertz, and Weibull models, either the hazard function or its logarithmic transform is a linear function of the parameters and the random variable representing time or the logarithm of time. This suggested to the authors that upon obtaining non-parametric estimates of the hazard function for each interval, they might estimate the parameters of each model by least squares. They then based their model selection procedures on the likelihood of the observed data under the various models. In particular, because the exponential distribution is a special case of each of the other models, the authors used the large sample chi-square approximation for the distribution of likelihood ratio test statistic to determine whether the data are exponentially distributed. To select among the remaining three models the authors employed an "informal" procedure based on Cox's [1961] test for separate families. Because each of the three models incorporates two unknown parameters, they based their selection on the actual values of L_{fg} rather than T_f as proposed by Cox. To decide if the best fitting model yields an adequate fit the authors considered twice the difference between the log-likelihood under the chosen model and the multinomial model as an approximate chi-square variate with (s-1-k) degrees of freedom where k is the number of parameters fitted in the model and (s-1) is the number of fixed finite intervals in the data.

1.4 Outline of Subsequent Chapters

In the following chapter maximum likelihood methods for testing the fit of Weibull and exponential distributions within the parametric framework of the Burr Type XII distribution are proposed. Chapter III focuses on special considerations for fitting the Burr XII distribution to survival or reliability data using maximum likelihood techniques. The choice of initial estimates for iterative solutions, alternative parameterization schemes, and the treatment of concomitant variables and censored observations are discussed.

In Chapters IV and V the methods of the previous chapters are applied to reliability and survival data. The Burr XII distribution is fitted to accelerated life test data in Chapter IV. Two competing models are discussed which correspond to two different methods for handling covariables representing the accelerated conditions. Chapter V contrasts the performance of the Burr XII and the generalized gamma distributions with respect to the reassessment of the Weibull fit for acute leukemia survival data.

The results of a simulation to investigate the exact distribution of the generalized likelihood ratio for testing the standard exponential distribution versus the Burr XII distribution are presented in Chapter VI. Finally, in Chapter VII, suggestions for further research on related topics are made.

CHAPTER II

THE DERIVATION OF GENERALIZED LIKELIHOOD RATIO TESTS FOR THE EVALUATION OF THE FIT OF THE WEIBULL AND EXPONENTIAL DISTRIBUTIONS AGAINST BURR TYPE XII ALTERNATIVES

2.1 Introduction

One approach to the model selection problem in survival analysis or reliability studies is to initially fit a very general family of parametric distributions which incorporates as special cases of its parameters distributions commonly used in life time studies. The Burr Type XII distribution, which includes the Weibull and exponential distributions as special cases, is one such general family. The intent of this chapter is the development of a method for assessing the adequacy of the fit of the Weibull and exponential distributions within the parametric framework of the Burr Type XII distribution. It will be assumed that this distribution is general enough to provide a suitable fit. Three specific cases will be considered: (i) discrimination between the Weibull distribution with unknown shape parameter and the Burr XII distribution; (ii) discrimination between the Weibull distribution with known shape parameter and the Burr XII distribution; and (iii) discrimination between the exponential distribution and the Burr XII distribution. It will be shown that a power transformation of the data reduces case (ii) to case (iii).

2.2 Assessing the Fit of the Weibull Distribution with Unknown Shape Parameter

2.2.1 The Weibull Distribution as a Limiting Case of the Burr Type XII Distribution

Rodriguez [1977] used a Taylor series expansion to show that the cumulative distribution function of the Weibull distribution can be obtained as a limit of cumulative distribution functions belonging to the Burr Type XII family. Given that T is a Burr random variate of standard form, specifically with the scale parameter β equal to one, the location parameter ξ equal to zero, and shape parameters c and k,

$$\begin{aligned} P\{T \le (1/k)^{1/c} t\} &= 1 - \{1+(t^c/k)\}^{-k} \\ &= 1 - \exp[-k\ell n\{1+(t^c/k)\}] \\ &= 1 - \exp[-k\{(t^c/k)-(t^c/k)^2/2+\ldots\}]. \end{aligned} \tag{2.1}$$

Then the limit as k approaches infinity of this cumulative distribution function is given by

$$\lim_{k \to \infty} P\{T \le (1/k)^{1/c} t\} = 1 - \exp(-t^c) \ . \tag{2.2}$$

This limiting expression is the cumulative distribution function of the standard Weibull family with shape parameter c.

For the non-standard case scale parameters β and θ_w are included in the cumulative distribution functions for the Burr Type XII and Weibull distributions, respectively. The location parameter ξ for each distribution may be assumed to equal zero in survivorship or reliability theory applications. In some cases ξ may be equal to a known non-zero threshold or guarantee time. However, a transformation of the data, $T_i' = T_i - \xi$, will reduce this to the simpler case of ξ equal to zero. Under the Weibull model

$$P\{T \le t\} = 1 - \exp\{-(t/\theta_w)^c\}$$

$$= 1 - \lim_{k \to \infty} \left\{1 + \frac{(t/\theta_w)^c}{k}\right\}^{-k}$$

$$= 1 - \lim_{k \to \infty} \{1 + (t/\theta_w k^{1/c})^c\}^{-k}$$

$$\doteq 1 - \{1+(t/\beta)^c\}^{-k} , \qquad (2.3)$$

where $\beta = \theta_w k^{1/c}$ and k is sufficiently large. Equation (2.3) is the cumulative distribution function for the Burr XII distribution. Therefore, in the Weibull region the scale parameter β of the Burr Type XII distribution obeys the functional relationship $\beta = \theta_w k^{1/c}$, where θ_w denotes the Weibull scale parameter, and c and k are the Burr XII shape parameters.

The results of research by Greenwood [1946] are useful for identifying how quickly the Burr distribution approaches the Weibull distribution as k increases. Greenwood fit the Pearsonian type curve defined by the cumulative distribution function

$$F(x;n,L) = 1 - \{1-(x/L)^n\} , \qquad (2.4)$$

for positive n and L, to the distribution of n points taken at random in a line of length L. He noted that the limiting form of the cumulative distribution function is exponential as n and L approach infinity, such that n/L remains finite. That is,

$$\lim_{\substack{n \to \infty \\ L \to \infty}} F(x;n,L) = 1 - \lim_{\substack{n \to \infty \\ L \to \infty}} \{1 + \frac{(-n/L)x}{n}\}^n$$

$$= 1 - \exp\{-(n/L)x\} . \qquad (2.5)$$

The derivation of the limiting form of the Burr Type XII distribution as k increases without bound, such that $\theta_w = \beta k^{-(1/c)}$ remains finite, may be modified to show that the Burr XII and Pearsonian type distributions approach their respective limiting forms in an identical manner. Specifically, for the Burr XII distribution we have

$$\lim_{k \to \infty} P\{T \le (1/k)^{1/c}(t/\theta_w)\} = 1 - \lim_{k \to \infty} \left\{1 + \frac{(t/\theta_w)^c}{k}\right\}^{-k}$$

$$= 1 - \exp\{-(t/\theta_w)^c\} \ . \qquad (2.6)$$

Greenwood found that for L larger than 100 and n larger than 10 in equation (2.4) there is little difference between the true distribution and its limiting form. Clearly, the implication is that for k exceeding 10 the Burr Type XII distribution is in the Weibull region.

2.2.2 The Burr XII Hazard Function

The hazard function is often considered in the choice of an underlying distribution for survival or reliability data. In this section some properties of the Burr Type XII hazard function are presented. In addition, the nature of the Burr hazard function as it approaches the Weibull hazard function is investigated.

The hazard function, denoted by $\lambda(t)dt$, represents the probability that an individual dies in the time interval $(t, t+dt)$, given that he survived to time t. It is defined mathematically as

$$\lambda(t) = \lim_{\Delta t \to 0} \left\{\frac{P(t<T<t+\Delta t \mid T>t)}{\Delta t}\right\} \ . \qquad (2.7)$$

The hazard function $\lambda(t)$ is related to the density function $f(t)$, the cumulative distribution function $F(t)$, and the survivorship function $S(t)$ according to the equations

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{S(t)} \quad . \tag{2.8}$$

The survivorship function, important to survival analysis and reliability theory, represents the probability that an individual or test unit survives until at least time t.

The Burr XII hazard function, derived from equation (2.8), is given by

$$\lambda(t) = (ck/\beta)(t/\beta)^{c-1}\{1+(t/\beta)^c\}^{-1} \quad . \tag{2.9}$$

The evaluation of $\lambda(0)$ shows that the initial Burr XII hazard function offers no more flexibility than does the initial hazard under the Weibull model. There are three cases to consider:

$$\begin{aligned} &\text{(i)} \quad c > 1 \ ; \qquad \lambda(0) = 0 \\ &\text{(ii)} \quad c < 1 \ ; \qquad \lambda(0) = \infty \\ &\text{(iii)} \quad c = 1 \ ; \qquad \lambda(0) = k/\beta \quad . \end{aligned} \tag{2.10}$$

For Burr Type XII distributions with c larger than one, $\lambda(0)$ is equal to zero. Several of the more common survival distributions for which the hazard function increases with age have hazard functions which are restricted to zero at time zero. Among these are the normal, lognormal, and gamma distributions, as well as the Weibull distribution with shape parameter c greater than one. Flehinger and Lewis [1959] and Hjorth [1980] have developed reliability models for which $\lambda(0)$ is not limited to zero. With regard to Burr XII distributions for which c is less than one, $\lambda(t)$ approaches infinity as t approaches zero. This is a property shared by decreasing Weibull hazard functions. Finally, for c equal to one the initial Burr hazard function is equal to the parameter ratio k/β. For the exponential limiting case of the Burr XII family when c is equal to one and k is sufficiently large, $\lambda(t)$ is identically equal to $\theta_e^{-1} = k/\beta$ for all t, where θ_e denotes the exponential scale parameter.

The shape of the hazard function $\lambda(t)$ reflects the type of risk to which the population under study is exposed as a function of time. To examine the shape of $\lambda(t)$ under the Burr XII model, where k is assumed to be finite, equation (2.9) is differentiated with respect to t, giving

$$d\{\lambda(t)\}/dt = [(ck/\beta^2)(t/\beta)^{c-2}\{1+(t/\beta)^c\}^{-2}]\{c-1-(t/\beta)^c\} \ . \quad (2.11)$$

The first factor of the product in (2.11) is everywhere positive for positive t and may therefore be disregarded in the determination of the sign of $d\{\lambda(t)\}/dt$. For c not exceeding one the second factor of the product in (2.11) is negative for positive t. Hence, in this case the hazard function decreases with increasing t and ultimately approaches zero. An example of a population exhibiting a decreasing hazard function is children who have undergone an operative procedure to correct some type of congenital defect.

For c larger than one, the second factor in (2.11) has a single root at $\beta(c-1)^{1/c}$. Thus, $\lambda(t)$ has a single mode when t is equal to $\beta(c-1)^{1/c}$. For t less than $\beta(c-1)^{1/c}$ the hazard function is increasing and for t larger than $\beta(c-1)^{1/c}$ the hazard function is decreasing. An application for which a hazard function with a single mode may be appropriate is the distribution of latency times for a particular cancer which has a period of highest risk, preceded and followed by periods of lesser risk.

It has been shown that for finite k, regardless of the value of c, the Burr hazard function approaches zero as t tends to infinity. However, as k approaches infinity the Burr scale parameter β approaches $\theta_w k^{1/c}$. When the asymptotic formula for β is substituted into equation (2.9), the Burr XII hazard function becomes

$$\lambda(t) = (c/\theta_w)(t/\theta_w)^{c-1}\{1+(1/k)(t/\theta_w)^c\}^{-1} , \tag{2.12}$$

where

$$\lim_{k \to \infty} \lambda(t) = (c/\theta_w)(t/\theta_w)^{c-1} . \tag{2.13}$$

The expression in (2.13) is the hazard function corresponding to the Weibull distribution. Therefore, for large k and c larger than one, the Burr hazard function like the Weibull hazard function increases as $(t/\theta_w)^{c-1}$. It can be shown alternatively that the Burr XII hazard function is increasing over all t when c is larger than one and k is tending to infinity. The asymptotic formula for the mode of $\lambda(t)$ is equal to $\theta_w k^{1/c}(c-1)^{1/c}$, where β has been replaced by $\theta_w k^{1/c}$. The mode approaches infinity as k approaches infinity and, hence, $\lambda(t)$ is everywhere increasing. The hazard function in this case is appropriate for a population in which there is an aging process that increases the rate of death.

2.2.3 Testing the Adequacy of the Fit of the Weibull Model Against Burr XII Alternatives

The model selection problem is formulated in terms of a hypothesis test. Under the null hypothesis the Weibull distribution is assumed and under the alternative hypothesis the more general Burr XII distribution is assumed. That is,

$$H_0: \quad P\{T \le t\} = F_0(t;c_w,\theta_w) = 1-\exp\{-(t/\theta_w)^{c_w}\}$$

$$H_1: \quad P\{T \le t\} = F_1(t;c,k,\beta) = 1-\{1+(t/\beta)^c\}^{-k} , \tag{2.14}$$

where the Weibull parameters c_w and θ_w and the Burr parameters c, k, and β are positive and t represents the non-negative failure or survival time to be modeled. Rodriguez [1977] modified the Burr parametric constraints to require that c and k are both non-negative and that their product, ck, exceeds four to insure that the first four moments are finite. Unlike the method of moments used by Rodriguez, the method of maximum likelihood will not require such restrictions in later applications.

It has been shown in Section 2.2.1 that the Weibull distribution is nested within the Burr Type XII family as a special limiting case of the shape parameter k. A standard approach for testing the fit of nested families of distributions is the generalized likelihood ratio test utilizing maximum likelihood estimates of the parameters. The family embedded in the more general distribution may be defined in terms of a subspace ω of the overall parameter space Ω to which p constraints have been applied. If $\hat{\underset{\sim}{\theta}}_{\Omega}$ and $\hat{\underset{\sim}{\theta}}_{\omega}$ are maximum likelihood estimators of the parameter vector $\underset{\sim}{\theta}$ in Ω and ω, respectively, then Wilks [1938] has shown that under certain regularity conditions the statistic

$$\lambda = -2\ln\{L(\hat{\underset{\sim}{\theta}}_{\omega})/L(\hat{\underset{\sim}{\theta}}_{\Omega})\} \tag{2.15}$$

is asymptotically distributed as a chi-square random variable with p degrees of freedom, where $L(\cdot)$ is the likelihood function and $\ln(\cdot)$ denotes the natural logarithm.

The question arises whether the statistic λ obeys the chi-square law for large samples when discriminating between the limiting Weibull and Burr distributions. The difficulty is that the subspace ω, corresponding to the Weibull distribution and defined by the limit value of k being infinite, forms part of the boundary of the parameter space Ω corresponding to the Burr distribution. This question is considered further in the next section.

2.2.4 The Log-Generalized Gamma Distribution: A Parallel Case

Prentice [1974] considered the fit of the lognormal distribution embedded in the more general class of generalized gamma distribution. As discussed in Section 1.3.2 the lognormal model is a limiting special case as k approaches infinity of the generalized gamma distributions. Prentice preferred to model the natural logarithm of a generalized gamma variate since it follows a location-scale distribution. He derived the unique parametric transformation $q = k^{-1/2}$ which maps the limiting normal model (on the logarithmic scale) to the origin and extended the log-generalized gamma model to include negative q in a smooth manner by reflecting the error probability density function at fixed q about the origin. Analytic results concerning the mean and the variance of the log-generalized gamma variate enabled Prentice to express the model in terms of a standardized error quantity. The model expressed in final form is

$$f(y=\ell nt;\alpha,\sigma,q) = \begin{cases} |q|\{\sigma\Gamma(q^{-2})\}^{-1}\exp(wq^{-2}-e^{w}) \;;\; q \neq 0 \\ (2\pi\sigma^{2})^{-1/2}\exp\{-(y-\alpha)^{2}/2\sigma^{2}\} \;;\; q = 0 \,, \end{cases} \quad (2.16)$$

where $w = (y-\alpha)\sigma^{-1}q+\mu^*$ and μ^* is the digamma function, $\psi(k) = d\ell n\Gamma(k)/dk$.

Farewell and Prentice [1977] applied a test of the adequacy of the fit of the lognormal model to a number of data sets from the medical and industrial literature, which is based on the generalized likelihood ratio given by

$$R = \frac{L(y;\hat{\alpha},\hat{\sigma},q=0)}{L(y;\hat{\alpha},\hat{\sigma},\hat{q})} = \frac{\prod_{i=1}^{n} (2\pi\hat{\sigma}^2)^{-1/2}\exp\{-(y_i-\hat{\alpha})^2/2\hat{\sigma}^2\}}{\prod_{i=1}^{n} |\hat{q}|\{\hat{\sigma}\Gamma(\hat{q}^{-2})\}^{-1}\exp(\hat{w}\hat{q}^{-2}-e^{\hat{w}})}, \qquad (2.17)$$

where $\hat{w} = (y_i-\hat{\alpha})\hat{\sigma}^{-1}\hat{q} + \psi(\hat{q}^{-2})$.

As in the case of discriminating between the Weibull and Burr distributions, the parameter 'subspace' corresponding to the normal model $(\alpha,\sigma,q=0)$ lies on the boundary of the parameter space corresponding to the log-generalized gamma model. In support of the application of the chi-square distribution with one degree of freedom as the limiting distribution of the test statistic $-2\ln R$ at q equal to zero, Prentice [1974] investigated whether the first and second partial derivatives of $\ln f(y;\alpha,\sigma,q)$ exist when q is equal to zero.

The log-density for y from (2.16) for non-zero q is given by

$$\ln f(y;\alpha,\sigma,q) = \ln|q|-\ln\sigma-\ln\Gamma(q^{-2}) + wq^{-2}-e^{w} \qquad (2.18)$$

from which

$$\partial\ln f/\partial\alpha = q\sigma^{-1}(e^{w}-q^{-2}) , \qquad (2.19)$$

$$\partial\ln f/\partial\sigma = \sigma^{-1}(e^{w}-q^{-2})(w-\mu^*) - \sigma^{-1}, \qquad (2.20)$$

$$\partial\ln f/\partial q = q^{-1}-2q^{-3}(w-\mu^*)-(e^{w}-q^{-2})\{(w-\mu^*)q^{-1}-2\sigma^{*2}q^{-3}\}, \qquad (2.21)$$

where $\mu^* = \psi(q^{-2})$ and $\sigma^{*2} = \psi'(q^{-2})$, that is, the digamma and trigamma functions, respectively. As k approaches infinity, the asymptotic formulae for μ^* and σ^{*2} are given by

$$\mu^* = \psi(q^{-2}) \doteq \ln(q^{-2}) - q^2/2 - q^4/12 \qquad (2.22)$$

and

$$\sigma^{*2} = \psi'(q^{-2}) \doteq q^2 + q^4/2 + q^6/6 . \qquad (2.23)$$

Substitution of the asymptotic formulae for μ^* and σ^{*2} and $(y-\alpha)\sigma^{-1}q+\mu^*$ for w in (2.18) through (2.21) leads to the following limits as q approaches zero:

$$\lim_{q \to 0} [\partial \ell nf/\partial\alpha] = (y-\alpha)\sigma^{-2} \tag{2.24}$$

$$\lim_{q \to 0} [\partial \ell nf/\partial\sigma] = (y-\alpha)^2\sigma^{-3} - \sigma^{-1} \tag{2.25}$$

$$\lim_{q \to 0} [\partial \ell nf/\partial q] = (y-\alpha)/2\sigma - (y-\alpha)^3/6\sigma^3 . \tag{2.26}$$

Since (2.24) and (2.25) can also be obtained by differentiating $\ell nf(y;\alpha,\sigma,q=0)$ with respect to α and σ, respectively, it is concluded that the first partials of ℓnf exist at q equal to zero. Similarly, the second partials of the likelihood function corresponding to the log-generalized gamma model can be shown to exist at q equal to zero.

2.2.5 Maximum Likelihood Estimation Under the Burr Type XII Model

The density function of the Burr Type XII distribution is given by

$$f(t;c,k,\beta) = (ck/\beta)(t/\beta)^{c-1}\{1+(t/\beta)^c\}^{-(k+1)} , \tag{2.27}$$

for positive c, k, and β and non-negative t. The corresponding log-likelihood equation for a random sample of size n is given by

$$\begin{aligned} \ell nL(c,k,\beta) &= \sum_{i=1}^{n} \ell nf(t_i;c,k,\beta) \\ &= n\ell nc+n\ell nk-nc\ell n\beta+(c-1)\sum_{i=1}^{n}\ell n(t_i)-(k+1)\sum_{i=1}^{n}\ell n\{1+(t_i/\beta)^c\}. \end{aligned} \tag{2.28}$$

It was shown in Section 2.2.1 that under the Weibull model the scale parameter β of the Burr Type XII distribution obeys the functional relationship

$$\beta = \theta_w k^{1/c} , \tag{2.29}$$

where θ_w denotes the Weibull scale parameter and c and k are the Burr shape parameters. If this constraint on the Burr scale parameter β is substituted in the log-likelihood equation (2.28), the following equation is appropriate in the neighborhood of the null hypothesis:

$$\ell n L_{H_0}(c,k,\beta=\theta_w k^{1/c}) = n\ell nc - nc\ell n\theta_w + (c-1)\sum_{i=1}^{n}\ell n(t_i)$$

$$-(k+1)\sum_{i=1}^{n}\ell n\{1+(1/k)(t_i/\theta_w)^c\} \ . \qquad (2.30)$$

Differentiation of (2.30) with respect to c, k, and θ_w gives rise to the first partial derivatives:

$$\partial\ell nL_{H_0}/\partial c = n/c - n\ell n\theta_w + \sum_{i=1}^{n}\ell n(t_i) - \{(k+1)/k\}\sum_{i=1}^{n}\frac{(t_i/\theta_w)^c\ell n(t_i/\theta_w)}{\{1+(1/k)(t_i/\theta_w)^c\}} \qquad (2.31)$$

$$\partial\ell nL_{H_0}/\partial k = -\sum_{i=1}^{n}\ell n\{1+(1/k)(t_i/\theta_w)^c\} + \{(k+1)/k\}\sum_{i=1}^{n}\frac{(t_i/\theta_w)^c}{\{k+(t_i/\theta_w)^c\}} \qquad (2.32)$$

$$\partial\ell nL_{H_0}/\partial\theta_w = -nc/\theta_w + \{c(k+1)/k\theta_w\}\sum_{i=1}^{n}\frac{(t_i/\theta_w)^{c-1}}{\{1+(1/k)(t_i/\theta_w)^c\}} \ . \qquad (2.33)$$

The evaluation of the limit of each of the first partial derivatives (2.31) through (2.33) as k approaches infinity results in the equations:

$$\lim_{k\to\infty}[\partial\ell nL_{H_0}/\partial c] = n/c - n\ell n\theta_w + \sum_{i=1}^{n}\ell n(t_i) - \sum_{i=1}^{n}(t_i/\theta_w)^c\ell n(t_i/\theta_w) \qquad (2.34)$$

$$\lim_{k\to\infty}[\partial\ell nL_{H_0}/\partial k] = 0 \qquad (2.35)$$

$$\lim_{k\to\infty}[\partial\ell nL_{H_0}/\partial\theta_w] = -nc/\theta_w + (c/\theta_w)\sum_{i=1}^{n}(t_i/\theta_w)^c \ . \qquad (2.36)$$

These results show that in the neighborhood of the null hypothesis the likelihood equations for the Burr distribution approach those of the

Weibull distribution since equations (2.34) and (2.36) are the first partial derivatives of the Weibull model with respect to c_w and θ_w. Moreover, the first partial derivative associated with the parameter k approaches zero as k approaches infinity. This tends to suggest that the Burr system of likelihood equation does in fact approach a limit which corresponds to the Weibull system of likelihood equations.

The second partial derivatives of (2.30) with respect to c, k, and θ_w are given by:

$$\partial^2 \ell n L_{H_0}/\partial c^2 = -n/c^2 - \{(k+1)/k\} \sum_{i=1}^{n} \frac{(t_i/\theta_w)^c\{\ell n(t_i/\theta_w)\}^2}{\{1+(1/k)(t_i/\theta_w)^c\}^2} \tag{2.37}$$

$$\partial^2 \ell n L_{H_0}/\partial k^2 = \sum_{i=1}^{n} \frac{(t_i/\theta_w)^c}{\{k^2+k(t_i/\theta_w)^c\}} - \sum_{i=1}^{n} \frac{(t_i/\theta_w)\{k^2+2k+(t_i/\theta_w)^c\}}{\{k^2+k(t_i/\theta_w)^c\}^2} \tag{2.38}$$

$$\partial^2 \ell n L_{H_0}/\partial \theta_w^2 = nc/\theta_w^2 - \{c(k+1)/k\theta_w^2\} \sum_{i=1}^{n} \frac{(t_i/\theta_w)^c\{1+c+(1/k)(t_i/\theta_w)^c\}}{\{1+(1/k)(t_i/\theta_w)^c\}^2} . \tag{2.39}$$

Then the evaluation of the limit of each of these second partial derivatives as k becomes unbounded results in the following equations:

$$\lim_{k \to \infty} [\partial^2 \ell n L_{H_0}/\partial c^2] = -n/c^2 - \sum_{i=1}^{n} (t_i/\theta_w)^c\{\ell n(t_i/\theta_w)\}^2 \tag{2.40}$$

$$\lim_{k \to \infty} [\partial^2 \ell n L_{H_0}/\partial k^2] = - \sum_{i=1}^{n} (t_i/\theta_w) \tag{2.41}$$

$$\lim_{k \to \infty} [\partial^2 \ell n L_{H_0}/\partial \theta_w^2] = nc/\theta_w^2 - \{c(c+1)/\theta_w^2\} \sum_{i=1}^{n} (t_i/\theta_w)^c . \tag{2.42}$$

Equations (2.40) and (2.42) are identical to the second partial derivatives with respect to c_w and θ_w of the log-likelihood function for the Weibull model.

2.2.6 Generalized Likelihood Ratio Test for Discriminating Between the Weibull and Burr XII Distributions

In view of the results of Section 2.2.5 two forms for the generalized likelihood ratio test are proposed for discriminating between the Weibull and Burr Type XII distributions. Analogous to the approach taken by Prentice [1974] the adequacy of the Weibull model may be evaluated by computing the ratio of maximized likelihoods given by

$$R_w = \frac{L_w(t;\hat{c}_w,\hat{\theta}_w)}{L_B(t;\hat{c},\hat{k},\hat{\beta})} , \tag{2.43}$$

where $L_w(\cdot)$ and $L_B(\cdot)$ denote the likelihood functions for the Weibull and Burr distributions, respectively. The results of Section 2.2.5 suggest that the test statistic $\lambda_w = -2\ln(R_w)$ may behave as a chi-square random variable with one degree of freedom for sufficiently large samples.

An alternative form of the generalized likelihood ratio test depends upon the likelihood functions under a restricted Burr model and the usual unrestricted Burr model. By imposing the constraint

$$\beta = \theta_w k^{1/c} , \tag{2.44}$$

where θ_w is the scale parameter under the Weibull model, β is the scale parameter, and c and k are the shape parameters under the Burr model, the Burr distribution may be restricted to the Weibull region. Since the true value of θ_w is normally unknown, it may be estimated by $\hat{\theta}_w$. Then the ratio of maximized likelihoods is specified by

$$R'_w = \frac{L_{RB}(\hat{c},\hat{k},\beta=\hat{\theta}_w\hat{k}^{1/\hat{c}})}{L_B(\hat{c},\hat{k},\hat{\beta})} , \tag{2.45}$$

where $L_{RB}(\cdot)$ and $L_B(\cdot)$ denote the likelihood functions for the restricted Burr and unrestricted Burr distributions, respectively. The test statistic $\lambda_w' = -2\ln(R_w')$ may be distributed asymptotically as a chi-square random variable with one degree of freedom. In the context of testing standard exponential data, this question is investigated further by simulation in Chapter VI.

2.3 Assessing the Fit of the Weibull Distribution with Known Shape Parameter

In some applications an investigator may find it useful to be able to test whether a set of survival data obeys the Weibull probability law with known shape parameter c_{w0} versus the more general Burr XII distribution. The null cumulative distribution function is given by

$$H_0: \quad P\{T \le t\} = F_0(t;c_{w0},\theta_w) = 1-\exp\{-(t/\theta_w)^{c_{w0}}\} \tag{2.46}$$

for positive c_{w0} and θ_w, and the alternative cumulative distribution function is

$$H_1: \quad P\{T \le t\} = F_1(t;c,k,\beta) = 1-(1+(t/\beta)^c)^{-k} , \tag{2.47}$$

for positive c, k, and β, and non-negative failure or survival time t. Under the null hypothesis the transformed random variable $T' = T^{c_{w0}}$ has the exponential distribution with scale parameter θ_w (see Johnson and Kotz [1970, p. 250]). Alternatively, if the Burr XII distribution with shape parameters c and k and scale parameter β is the true underlying distribution for T, then it can be shown that the transformed variable T' has the cumulative distribution function

$$P\{T' \le t\} = 1-\{1+(t/\beta^{c_{w0}})^{c/c_{w0}}\}^{-k} , \tag{2.48}$$

where c/c_{w0}, k, and $\beta^{c_{w0}}$ are each positive. The expression in (2.48) is the cumulative distribution function for the Burr XII distribution with shape parameters c/c_{w0} and k, and scale parameter $\beta^{c_{w0}}$. Therefore, through a power transformation of the data, namely $t' = t^{c_{w0}}$, the test for discriminating between the Weibull distribution with known shape parameter c_{w0} and the Burr XII distribution may be reduced to a test for discriminating between the exponential distribution and the Burr XII distribution. This case is addressed in Section 2.4.

2.4 Assessing the Fit of the Exponential Distribution

The test is formulated in terms of the set of hypotheses:

$$H_0: \quad P\{T \le t\} = F_0(t;\theta_e) = 1-\exp\{-(t/\theta_e)\}$$

$$H_1: \quad P\{T \le t\} = F_1(t;c,k,\beta) = 1-\{1+(t/\beta)^c\}^{-k}\,, \qquad (2.49)$$

where θ_e, c, k, and β are positive model parameters, and t represents the survival or failure time to be modeled. As indicated in Table 1.2 the exponential distribution is the limiting case of the Burr XII distribution as k approaches infinity with c equal to one. The proof for the standard Weibull case, which was given in the series of equations labeled (2.1), generalizes to the standard exponential case by setting the Burr shape parameter c equal to one. For the non-standard case, with location parameters still assumed to be zero, the Burr XII distribution has the exponential distribution as its limiting case when the following two constraints are placed on the Burr parameters:

$$\begin{aligned} &\text{(i)} \quad c = 1 \\ &\text{(ii)} \quad k \to \infty \text{ and } \beta \to \infty \text{ such that } \theta_e = \beta/k \text{ remains finite,} \end{aligned} \qquad (2.50)$$

where θ_e denotes the exponential scale parameter. The second restriction is obtained from (2.29) by replacing c with one.

When these two restrictions are placed on c and β in the Burr XII log-likelihood equation shown in (2.28), we obtain the equation

$$\ell n L_{H_0}(c=1,k,\beta=\theta_e k) = -n\ell n\theta_e - (k+1)\sum_{i=1}^{n} \ell n\{1+(t_i/\theta_e k)\} \ , \qquad (2.51)$$

which is appropriate in the neighborhood of the null hypothesis. Differentiation of (2.51) with respect to k and θ_e results in the first partial derivatives:

$$\ell n L_{H_0}/\partial k = -\sum_{i=1}^{n} \ell n\{1+(t_i/\theta_e k)\} + (k+1)\sum_{i=1}^{n} \frac{(t_i/\theta_e)}{\{k^2+k(t_i/\theta_e)\}} \qquad (2.52)$$

$$\ell n L_{H_0}/\partial\theta_e = -n/\theta_e + (k+1)\sum_{i=1}^{n} \frac{(t_i/\theta_e^2)}{\{k+(t_i/\theta_e)\}} \ . \qquad (2.53)$$

The evaluation of the limits of (2.52) and (2.53) using L'Hospital's rule as k approaches infinity gives rise to the equations:

$$\lim_{k\to\infty} [\partial\ell n L_{H_0}/\partial k] = 0 \qquad (2.54)$$

$$\lim_{k\to\infty} [\partial\ell n L_{H_0}/\partial\theta_e] = -n/\theta_e + \sum_{i=1}^{n} (t_i/\theta_e^2) \ . \qquad (2.55)$$

The limiting expression in (2.55) is the likelihood equation for the exponential model. This result combined with the result in (2.54) which shows that the first partial derivative associated with the parameter k approaches zero as k approaches infinity, suggests that the Burr system of likelihood equations subject to the constraints in (2.50) does approach the exponential likelihood equation. It may also be shown that the second partial derivative of (2.51) with respect to θ_e approaches the second derivative of the exponential log-likelihood function.

Hence, it has been demonstrated that there is sufficient regularity for the log-likelihood function for the Burr XII distribution to be twice differentiable at the exponential boundary of Ω defined by c equal to one and unbounded k. It is anticipated that the test statistic $\lambda_e = -2\ln(R_e)$, where

$$R_e = \frac{L_e(\hat{\theta}_e)}{L_B(\hat{c},\hat{k},\hat{\beta})} \tag{2.56}$$

and $L_e(\cdot)$ and $L_B(\cdot)$ denote the likelihood functions for the exponential and Burr distributions, respectively, behaves as a chi-square random variable with two degrees of freedom for sufficiently large samples.

An alternate form of the generalized likelihood ratio test for the exponential distribution versus the alternative Burr XII distribution utilizes the maximized likelihood functions under a restricted Burr model and the usual unrestricted Burr model. By imposing the two parameter constraints c equal to unity and β equal to $\theta_e k$, the Burr XII distribution is restricted to the exponential region. When the true value of θ_e is unknown, it may be estimated by the maximum likelihood estimator

$$\hat{\theta}_e = n^{-1} \sum_{i=1}^{n} t_i \ , \tag{2.57}$$

with uncensored samples (see Gross and Clarke [1975, p. 52]) and by

$$\hat{\theta}_e = n^{-1}[\sum_{i=1}^{n_1} t_i + \sum_{i=1}^{n-n_1} T_i] \tag{2.58}$$

with censored samples for which there are n_1 failure times represented by t_i and $(n-n_1)$ censoring times represented by T_i (see Gross and Clarke [1975, p. 55]). Then the test statistic $\lambda_e' = -2\ln(R_e')$, where

$$R_e' = \frac{L_{RB}(c=1,\hat{k},\beta=\hat{\theta}_e\hat{k})}{L_B(\hat{c},\hat{k},\hat{\beta})} \tag{2.59}$$

and $L_{RB}(\cdot)$ and $L_B(\cdot)$ represent the likelihood functions for the restricted

Burr and unrestricted Burr distributions, respectively, may behave asymptotically as a chi-square random variable with two degrees of freedom corresponding to the two parametric restrictions. The distribution of λ'_e is examined by means of a simulation in Chapter VI.

Thoman, Bain, and Antle [1969] developed an exact test for the power parameter c_w of the Weibull distribution based upon the pivotal function $\hat{c}_w/c_w$, which was briefly discussed in Section 1.3.1. Monte Carlo methods were employed to obtain the exact distribution of $\hat{c}_w/c_w$ for sample sizes 5(1)20, 22(2)80, 85(5)100, 110, and 120. Klimko, Antle, Rademaker, and Rockette [1975] showed that the test for exponentiality (c=1) nearly achieves the upper bounds for the power of any scale invariant test for exponentiality against Weibull alternatives. Therefore, once it has been established by the methods of Section 2.2.5 that a set of survival or failure data are distributed according to the Weibull distribution an alternative way of testing for exponentiality is to use the exact test based on $\hat{c}_w$. It should be recognized that the exact test will provide greater power since the alternate distribution is the Weibull rather than the more general Burr XII distribution. Thoman *et al.* [1969] present the power of their exact test which depends on the ratio c_{wA}/c_{w0} (c_{wi}, i=A, 0, represents the Weibull power parameter under H_i), the significance level α, and the sample size n.

CHAPTER III

SPECIAL CONSIDERATIONS FOR FITTING THE BURR TYPE XII DISTRIBUTION BY MAXIMUM LIKELIHOOD TECHNIQUES WHEN THERE ARE COVARIABLES AND CENSORED OBSERVATIONS

3.1 Introduction

A solution for the Burr XII system of likelihood equations must be obtained using some iterative technique. In the examples of subsequent chapters the FORTRAN subroutine MAXLIK, developed by Kaplan and Elston [1972], has been used to perform a direct search of the likelihood surface by successively fitting parabolas to points on the likelihood surface. In some analyses the direct search method is followed up by a numerically based Newton-Raphson iterative procedure to confirm the maximum likelihood estimates. MAXLIK incorporates the option of specified functional relationships among the parameters which are to be maximized. Therefore, it is possible to restrict the Burr XII distribution to the Weibull or exponential regions by imposing the functional relationships of (2.29) or (2.50), respectively.

In this chapter attention will be directed to special topics which arise when fitting the Burr Type XII distribution using MAXLIK. Specifically, a reparameterization of some of the distributional parameters, which reduces the number of iterations required for convergence, is presented in Section 3.2. Section 3.3 contains a discussion of how to obtain initial estimates of the model parameters. Finally, methods for incorporating concomitant variables as well as censored observations are discussed in Sections 3.4 and 3.5, respectively.

3.2 Reparameterization of the Burr XII Parameters

It has been shown theoretically that as the Burr XII distribution approaches the Weibull or exponential regions, the shape parameter k and the scale parameter β each approach infinity. In the actual fitting of the Burr XII distribution to data the likelihood surface in each of these regions has been observed to be extremely flat. Moreover, k and β are highly correlated, which is to be expected since β approaches infinity according to the parameter constraint $\beta = \theta_w k^{1/c}$ or $\beta = \theta_e k$, depending on whether the data are Weibull or exponential, respectively.

Two reparameterization schemes have been considered which reduce the number of iterations required for convergence. Transformation of the parameters k and β to

$$k' = \frac{1}{k} \quad \text{and} \quad \beta' = \frac{1}{\beta}, \tag{3.1}$$

implies that in the Weibull region k' and β' approach the finite point, zero, rather than infinity. Consequently, it has been observed that in general (but not always) when the data appear to behave as Weibull or exponential data, the redefined parameters in (3.1) are estimated with fewer iterations.

A second reparameterization, which involves a transformation of the parameter regions for k and β from $(0,\infty)$ onto $(0,1)$ by the reparameterization

$$k^* = \frac{k}{1+k} \quad \text{and} \quad \beta^* = \frac{\beta}{1+\beta}, \tag{3.2}$$

is even more effective in reducing the number of iterations required for convergence. However, it has also been found that in some cases this reparameterization is not precise enough, giving maximum likelihood estimates and maximized log-likelihood values which differ from those computed with k and β. In the examples of Chapter IV this was found to be so and, hence, the parameters expressed in (3.1) were used. But in the analyses of Chapter V the parameters k^* and β^* of (3.2) were found to work well. Hjorth [1980] suggested the reparameterization of (3.2) as a means for handling highly correlated parameters which tend to compensate each other. In the case of the Burr XII distribution, it appears that k^* and β^* are still highly correlated in the Weibull and exponential regions.

To demonstrate the effectiveness of these transformations a comparison between the number of iterations required for convergence under the three parameterizations for 200 simulated standard exponential observations is summarized in Table 3.1. Although this is only one data set, these results are typical of those of other analyses.

3.3 Initial Estimates

The subroutine MAXLIK, like most iterative procedures, requires that initial estimates of the model parameters be supplied. When the data sets are large or when there are many covariables under investigation, each iteration can be quite costly. Therefore, savings can be realized by having good initial estimates.

Since it is of primary interest to test the suitability of the Weibull or exponential distributions, it is advantageous to have initial estimates for the Burr XII parameters which are in the Weibull or exponential regions. Due to the inclusion of the exponential distribution in the Weibull family, no loss of generality is introduced by choosing initial estimates in the Weibull region. Hence, maximum likelihood estimates of the Weibull parameters, c_w and θ_w, provide initial estimates for the Burr parameters.

TABLE 3.1

THREE PARAMETERIZATION SCHEMES FOR THE BURR XII DISTRIBUTION

Standard Exponential Data: $c=1,\ k=\infty,\ \beta=\infty$

Parameterization	(c,k,β)	$(c,\frac{1}{k},\frac{1}{\beta})$	$(c,\frac{k}{1+k},\frac{\beta}{1+\beta})$
Initial Estimates	$c_0=1.0$ $k_0=10.0$ $\beta_0=10.0$	$c_0=1.0$ $k_0'=0.1$ $\beta_0'=0.1$	$c_0=1.0$ $k_0^*=.91$ $\beta_0^*=.91$
Initial Log-Likelihood	-209.4758	-209.4758	-209.4557
Maximized Log-Likelihood	-208.3385	-208.2622	-208.6589
Maximum Likelihood Estimates (s.e.)	$\hat{c}=0.9537$ (0.0553) $\hat{k}=97.0495$ (123.4947) $\hat{\beta}=121.5552$ (164.5528)	$\hat{c}=0.9512$ (0.0869) $\hat{k}'=0.0023$ (0.1295) $\hat{\beta}'=0.0016$ (0.0986)	$\hat{c}=0.9743$ (0.1217) $\hat{k}^*=0.9571$ (0.1812) $\hat{\beta}^*=0.9599$ (0.1938)
Number of Iterations	82	79	32

Mann, Schafer, and Singpurwalla [1974] recommend the use of two-order-statistics developed by Dubey [1967] as initial values in the iterative solution of the Weibull likelihood equations. Dubey's statistics, which estimate the Weibull shape and scale parameters, respectively, are given by

$$c_w^* = 2.99[\ell n\{X_{([0.9737n]+1)}\} - \ell n\{X_{([0.1673n]+1)}\}]^{-1} \qquad (3.3)$$

and

$$\theta_w^* = \exp[0.441\ \ell n\{X_{([0.3978n]+1)}\} + 0.559\ \ell n\{X_{([0.8211n]+1)}\}], \qquad (3.4)$$

where $X_{(1)},\ldots,X_{(n)}$ are the ordered sample observations, [pn] denotes the largest integer not exceeding pn, and n is the sample size. These statistics are simple to compute, but are not as efficient as maximum likelihood and best linear invariant statistics. Dubey determined that c_w^* is asymptotically 66% efficient when compared with the maximum likelihood estimator, and it is asymptotically superior to the moment estimator proposed by Menon [1963], based on all n observations of the sample. The estimator θ_w^* of the Weibull scale parameter is 82% efficient when compared with the maximum likelihood estimator.

In the case of progressively censored samples, Mann, Schafer, and Singpurwalla [1974] again recommend the use of Dubey's two-order-statistics as long as the sample size is nearly complete and is at least as large as 80. In the examples of Chapter V, which include censored observations, the sample sizes all exceed 140 and the smallest proportion of failures among failure and censored observations is 0.81. Therefore, for progressively censored data the censored observations will be disregarded in the computation of Weibull initial estimates based on (3.3) and (3.4).

The initial estimates for the iterative solution to the Weibull likelihood equations, based on Dubey's two-order-statistics, are presented in the first column of Table 3.2. It is reasonable to assume at the outset that the vector of parameters for the covariables is equal to the null vector. Depending upon the outcome of the Weibull fit, and specifically on $\hat{c}_w$, it may be desirable to test whether the data are exponentially distributed. Since it is necessary to obtain $\hat{\theta}_e$ iteratively (in the presence of explanatory variables) in order to specify the Burr scale parameter constraint, $\beta = \hat{\theta}_e k$, which defines the restricted Burr log-likelihood equation in (2.51), an optional exponential model may be fit. The initial estimates for the parameters of the exponential model are based on the Weibull maximum likelihood estimates, as illustrated in the second column of Table 3.2.

It was shown in Section 2.2.1 that in the Weibull region the Burr shape parameter, denoted by c, corresponds to the Weibull power parameter c_w, and the Burr scale parameter β is equal to $\theta_w k^{1/c}$, where θ_w is the Weibull scale parameter and c and k are the Burr shape parameters. Hence, the Burr initial estimates c_0 and β_0 are set equal to $\hat{c}_w$ and $\hat{\theta}_w k^{1/\hat{c}_w}$, respectively. Major Greenwood's [1946] results in working with a Pearsonian curve (see Section 2.2.1) suggest that for k larger than 10 the Burr XII distribution is very closely approximated by the Weibull distribution. The resultant initial estimates for the Burr parameters which correspond to the Weibull region are given in the third column of Table 3.2.

The initial estimates for the parameters of the restricted Burr models, which are constrained to the Weibull and exponential regions according to the log-likelihood functions in (2.30) and (2.51), are based on the maximum likelihood estimates obtained from the unrestricted Burr,

TABLE 3.2

INITIAL ESTIMATES FOR THE WEIBULL, EXPONENTIAL, AND BURR XII DISTRIBUTIONAL PARAMETERS

Weibull[a]	Optional Exponential[b]	Unrestricted Burr[b]	Restricted Burr (Weibull Region)[c]	Restricted Burr (Exponential Region)[d]
$c_{w0} = c_w^*$		$c_0 = \hat{c}_w$	$c_0 = \hat{c}$	$c \equiv 1$ (fixed)
$\theta_{w0} = \theta_w^*$	$\theta_{e0} = \hat{\theta}_w$	$k_0 = 10$	$k_0 = \hat{k}$	$k_0 = \hat{k}$
$\underset{\sim}{\gamma}_{w0} = \underset{\sim}{0}$	$\underset{\sim}{\gamma}_{e0} = \hat{\underset{\sim}{\gamma}}_w$	$\beta_0 = \hat{\theta}_w k_0^{1/\hat{c}_w}$	$\beta_0 = \hat{\theta}_w \hat{k}^{1/\hat{c}}$	$\beta_0 = \hat{\theta}_e \hat{k}$
		$\underset{\sim}{\gamma}_0 = \hat{\underset{\sim}{\gamma}}_w$	$\underset{\sim}{\gamma}_0 = \hat{\underset{\sim}{\gamma}}$	$\underset{\sim}{\gamma}_0 = \hat{\underset{\sim}{\gamma}}$

[a] The notation c_w^* and θ_w^* denotes Dubey's [1967] two-order-statistic estimators specified in (3.3) and (3.4).

[b] The notation $\hat{c}_w$, $\hat{\theta}_w$, $\hat{\underset{\sim}{\gamma}}_w$ denotes the maximum likelihood estimates of the Weibull model parameters.

[c] The notation $\hat{c}$, $\hat{k}$, $\hat{\underset{\sim}{\gamma}}$ denotes the maximum likelihood estimates of the unrestricted Burr model parameters.

[d] The notation $\hat{\theta}_e$ denotes the maximum likelihood estimate of the exponential scale parameter.

Weibull, and exponential models. These initial estimates are given in the fourth and fifth columns of Table 3.2.

3.4 Incorporating Covariables in the Burr Type XII Hazard Function

In survival analyses it is often desirable to compare the mortality experience of individuals who are known not to be comparable with respect to a covariable such as the age at first exposure to a known carcinogen, the amount of exposure, or the duration of the exposure. One method for handling such situations would be to categorize individuals based on their values of the covariable and then to make comparisons of interest within these categories. Small sample sizes would require wide categories which might not fully adjust for the covariable. Furthermore, the sample sizes often met in practice would not facilitate the simultaneous categorization and analysis of more than just a few covariables. Therefore, for non-homogeneous populations, it may be more beneficial to incorporate covariables directly into the model.

A regression-type approach toward the inclusion of covariables in a survival model is based upon the following form of the hazard function defined in (2.7):

$$\lambda(t_i, \underset{\sim}{z}_i) = \lambda_0(t_i)h(\underset{\sim}{z}_i, \underset{\sim}{\gamma}) , \tag{3.5}$$

where $\underset{\sim}{z}_i$ represents the column vector of covariable values for individual i, $\underset{\sim}{\gamma}$ is a row vector of unknown model parameters, $h(\cdot,\cdot)$ is some function of $\underset{\sim}{z}_i$ and $\underset{\sim}{\gamma}$, and $\lambda_0(t)$ is a function of time corresponding to the hazard function for the standard set of conditions $\underset{\sim}{z}_i=\underset{\sim}{0}$ and thus known as the underlying hazard function.

Fiegl and Zelen [1965] first proposed this method of adjustment of the hazard function for covariables. Of the three functional forms considered by these authors for $h(\underset{\sim}{z}_i, \underset{\sim}{\gamma})$, namely $e^{\underset{\sim}{\gamma}\underset{\sim}{z}_i}$, $(1+\underset{\sim}{\gamma}\underset{\sim}{z}_i)$ and $(1+\underset{\sim}{\gamma}\underset{\sim}{z}_i)^{-1}$, Kalbfleisch [1974] recommended $e^{\underset{\sim}{\gamma}\underset{\sim}{z}_i}$ since it allows $\underset{\sim}{\gamma}$ to assume any value, whereas in the other two functions $\underset{\sim}{\gamma}$ must be restricted to ensure that $\lambda(t_i, \underset{\sim}{z}_i)$ is non-negative. A weakness of the regression-type models, as they are commonly used with $\underset{\sim}{z}_i$ not dependent on time, is that they require hazards for individuals with different covariable values to be in constant ratio over time regardless of the form of $\lambda_0(t)$ (see Taulbee [1979], page 440). Hence, they are usually referred to as proportional hazard models.

Glasser [1967], like most investigators, assumed $\lambda_0(t)$ to be constant, which corresponds to taking the survival time for each individual to be exponentially distributed. The mean survival time can differ among individuals. Glasser found this set of assumptions to be reasonable when the aging effect in a population is of negligible importance, such as when the mortality rates are high, when the population is young or "ageless", or the follow-up time is relatively short. However, in the case of chronic diseases this is usually not a reasonable assumption. Although preliminary results on advanced lung cancer patients suggested an exponential survival function, Prentice [1973] initially considered $\lambda_0(t)$ to be the more general Weibull hazard function and then tested for exponentiality using methods discussed in Section 1.3.1. Cox [1972] developed a very robust method of estimating survival times based on (3.5) which allows $\lambda_0(t)$ to be unspecified. This method is not affected by transformations on t_i since it uses the ranks $t_{(i)}$.

In order to accommodate covariables in the Burr XII model, it will be assumed that the hazard function has the form of equation (3.5), where $h(\underset{\sim}{z}_i, \underset{\sim}{\gamma})$ is taken to be $e^{\underset{\sim}{\gamma}\underset{\sim}{z}_i}$. Chapter IV considers an alternative method for incorporating a vector of covariables in a model for accelerated life tests. Replacement of the underlying hazard function $\lambda_0(t)$ in (3.5) by the Burr XII hazard function given in (2.9) results in the following form for the proportional hazard function:

$$\lambda(t_i,\underset{\sim}{z}_i) = \frac{ck(t_i/\beta)^{c-1}}{\beta\{1+(t_i/\beta)^c\}} \exp(\underset{\sim}{\gamma}\underset{\sim}{z}_i) \ . \tag{3.6}$$

Using equation (2.8), which relates the hazard function to the density function f(t) and the survival function S(t), the survival function can be expressed in terms of the hazard function according to the fundamental relationship

$$S(t_i,\underset{\sim}{z}_i) = \exp\{ - \int_0^{t_i} \lambda(u,\underset{\sim}{z}_i)du\} \ . \tag{3.7}$$

Thus, the survival function corresponding to the Burr Type XII distribution is

$$S(t_i,\underset{\sim}{z}_i) = \exp[-k \exp(\underset{\sim}{\gamma}\underset{\sim}{z}_i)\ell n\{1+(t_i/\beta)^c\}] \ . \tag{3.8}$$

The density function may be obtained by differentiating (3.8) and changing the sign, which in the case of the Burr XII model becomes

$$f(t_i,\underset{\sim}{z}_i) = \frac{ckt_i^{c-1}\exp(\underset{\sim}{\gamma}\underset{\sim}{z}_i)}{\beta^c\{1+(t_i/\beta)^c\}} \exp[-k \exp(\underset{\sim}{\gamma}\underset{\sim}{z}_i)\ell n\{1+(t_i/\beta)^c\}], \tag{3.9}$$

for positive c, k, and β, real vectors $\underset{\sim}{\gamma}$ and $\underset{\sim}{z}_i$, and non-negative t. The corresponding log-likelihood equation for a random sample of size n is given by

$$\begin{aligned} \ell nL(c,k,\beta,\underset{\sim}{\gamma}) &= \sum_{i=1}^{n} \ell n\ f(t_i,\underset{\sim}{z}_i) \\ &= n\ell nk + n\ell nc - nc\ell n\beta + (c-1)\sum_{i=1}^{n}\ell n(t_i) - \sum_{i=1}^{n}\ell n\{1+(t_i/\beta)^c\} \\ &\quad + \sum_{i=1}^{n}\underset{\sim}{\gamma}\underset{\sim}{z}_i - k\sum_{i=1}^{n}\exp(\underset{\sim}{\gamma}\underset{\sim}{z}_i)\ell n\{1+(t_i/\beta)^c\} \ . \end{aligned} \tag{3.10}$$

In fitting the above likelihood equation it is necessary for numerical reasons to adjust the components of the vector $\underset{\sim}{z}_i$ for their respective sample means. That is, if $\underset{\sim}{z}_{ij}$ represents the value of the jth covariable for the ith individual, in equation (3.10), z_{ij} should be replaced by

$$z^*_{ij} = z_{ij} - z_{\cdot j} = z_{ij} - \sum_{i=1}^{n} z_{ij} \ . \qquad (3.11)$$

Fiegl and Zelen [1965] and Cox [1972] suggested this as a convenient reparameterization, while Glasser [1967] defined the components of $\underset{\sim}{z}_i$ to be adjusted for the respective sample means in the proportional hazard function specified in (3.5). When fitting the log-likelihood equation in (3.10), it has been found that <u>unless</u> $\underset{\sim}{z}_i$ is replaced by $\underset{\sim}{z}^{*\prime}_i = [z^*_{i1}, z^*_{i2}, \ldots, z^*_{ir}]$, computational difficulties result in the last term of (3.10).

3.5 Censored Observations

In survival studies it is common for individuals to enter a study at different times depending, for instance, on the time of medical diagnosis or environmental exposure. Therefore, the length of follow-up after entry to the study is usually different for each patient. Similarly, in industrial applications when the date of installation of equipment varies, the follow-up periods differ. Also, it frequently becomes necessary to terminate a survival or reliability study before all of the patients or test units have exhibited the condition under study. Thus, progressively censored samples are frequently encountered in survival analyses or reliability studies, and methods of analysis must accommodate censored observations.

Based on Bartholomew's [1957] development for the progressively censored case, the contribution to the likelihood function for the ith individual is given by

$$\ell_i(t_i) = \begin{cases} f(t_i) \ ; \ 0 \le t_i \le T_i \\ S(t_i) \ ; \ t_i > T_i \end{cases} , \tag{3.12}$$

where $f(\cdot)$ and $S(\cdot)$ denote the probability density and survival functions, respectively; t_i is the failure time and T_i is the maximum period of observation for patient i, namely,

$$T_i = (\text{study termination date - entry date}) \ . \tag{3.13}$$

Then the likelihood function corresponds to

$$L(\cdot) = \prod_{i=1}^{n} [f(t_i)]^{\delta_i} [S(t_i)]^{1-\delta_i} , \tag{3.14}$$

where δ_i is an indicator (of observed failure) variable defined by

$$\delta_i = \begin{cases} 1 \ ; \ 0 \le t_i \le T_i \\ 0 \ ; \ t_i > T_i \end{cases} . \tag{3.15}$$

In the case of the Burr Type XII distribution where there are no explanatory variables, the log-likelihood function derived from (3.14) becomes

$$\ell n\{L(c,k,\beta)\} = \sum_{i=1}^{n} \delta_i \ell n\{f(t_i;c,k,\beta)\}$$
$$+ \sum_{i=1}^{n} (1-\delta_i)\ell n\{S(t_i;c,k,\beta)\} , \tag{3.16}$$

where $f(t;c,k,\beta)$ is given in (2.27) and $S(t;c,k,\beta)$ may be shown to equal

$$S(t;c,k,\beta) = \{1+(t/\beta)^c\}^{-k} \ . \tag{3.17}$$

When there is a vector of covariables associated with each observation, equations (3.9) and (3.8) may be substituted in (3.16) for $f(t_i,\underset{\sim}{z}_i;c,k,\beta,\underset{\sim}{\gamma})$ and $S(t_i,\underset{\sim}{z}_i;c,k,\beta,\underset{\sim}{\gamma})$, respectively.

CHAPTER IV

APPLICATION OF THE BURR XII DISTRIBUTION TO ACCELERATED LIFE TESTS

4.1 Introduction

It is often not feasible to measure the reliability of a product or material under normal operating conditions due to the length of time required to observe some minimum number of failures. Even if it were possible to conduct a life test over a sufficiently long time interval, the rate of technical advance is frequently so great that the product would be obsolete before its reliability had been measured. Therefore, accelerated life tests are used to gain reliability information quickly by subjecting test units to conditions that are more severe than normal ones. These accelerated conditions are typically produced by increasing the voltage, temperature, vibration, pressure, cycling rate or load, or some combination of these. The goal of accelerated life tests is to make inferences for the life distribution of the product under normal conditions from results obtained under extreme testing conditions.

In order to make extrapolations it is necessary to have a model which relates the underlying lifetime distribution, determined by the basic physical process of failure, to the testing environment. The validity of such a model will depend strongly on the validity of the underlying distribution of failure times under constant environmental conditions. In this chapter the Burr XII distribution will be used to determine an appropriate underlying distribution for accelerated life data. Two competing Burr models will be discussed which correspond to two different methods for incorporating covariables representing the accelerated conditions into a failure distribution. Data from an accelerated life test, performed to measure the time to breakdown of an insulating fluid subjected to various elevated voltage stresses, will serve as an illustration. These data were first published by Nelson [1970].

4.2 Two Contrasting Models for Incorporating Covariables (Accelerated Conditions) in the Underlying Failure Distribution

4.2.1 A General Description of the Models

In order that the distribution of failure times at one level of the accelerated condition (stress) may be related to that at another level, it is necessary to assume that the basic physical process of failure which determines the underlying failure distribution is the same regardless of the degree of acceleration. This is likely to happen when there is a single predominant mode of failure. While both models under consideration make this assumption, they differ with regard to the assumed relationship between the accelerated condition and the failure process. Cox [1972] considered the proportional hazards model which assumes that the aging process is independent of stress. In contrast, the inverse power law model discussed by Nelson [1970] and Mann, Schafer, and Singpurwalla [1974] assumes that the accelerated conditions influence the values of the parameters of the underlying failure distribution in a known way. This model is closely related to the accelerated failure time model of Kalbfleisch and Prentice [1980] which assumes that the effect of the accelerated condition is multiplicative on time t, altering the rate at which an individual or testing unit proceeds along the time axis. Cox [1972] discussed this model as well.

For validity purposes both models require that the stress value falls within a certain range. For stress levels beyond these limits different models may have to be assumed. In addition, both models require that the underlying distribution is correctly specified. Initially, it will be assumed that the underlying failure distribution belongs to the Burr Type XII family. Then the appropriateness of the Weibull or exponential distributions will be investigated. It will be shown that for the Weibull and exponential special cases, the proportional hazards model and the inverse power law model are equivalent. Throughout the development of each model it will be assumed that only one type of stress has been applied. However, the results may be easily generalized to the case of a vector of accelerated conditions.

4.2.2 Model 1: The Proportional Hazards Model

Cox [1972] developed this model by assuming that the process of aging in the test units is independent of the applied stress. He further assumed that the stress acts multiplicatively on the hazard function. Then the hazard function for the ith unit is given by

$$\lambda(t_i, s_i) = \lambda_0(t_i) h(s_i) \, , \tag{4.1}$$

where $\lambda_0(t)$ is the hazard function for the underlying failure distribution and $h(s_i)$ is some function of stress denoted by s. If it is further assumed that

$$h(s_i) = s_i^{\gamma} \, , \tag{4.2}$$

that is, that the conditional probability of failure varies as a power of stress, the hazard function becomes

$$\lambda(t_i,z_i) = \lambda_0(t_i)e^{\gamma z_i} , \tag{4.3}$$

where $z=\ell n(s)$ and $\lambda_0(t)$ represents the underlying hazard function at $s=1$ or $z=0$. This is just the form of the hazard function with covariates which was discussed in Section 3.4. This model assumes that there are no cumulative stress effects.

If the underlying failure distribution is assumed to belong to the Burr XII family, the form of the underlying function $\lambda_0(t)$ in (4.3) is given by equation (2.9). When this is substituted in (4.3), the hazard function becomes

$$\lambda(t_i,z_i) = \frac{ck(t_i/\beta)^{c-1}}{\beta\{1+(t_i/\beta)^c\}} \exp(\gamma z_i) . \tag{4.4}$$

As discussed in Section 3.4, in actual model fitting $z_i = \ell n(s_i)$ is replaced by $z_i^* = z_i - \sum_{i=1}^{n} z_{ij}$, the value of the logged covariate which has been adjusted for the sample mean on the logarithmic scale.

The log-likelihood equation corresponding to the hazard function given in (4.4) is displayed as equation (3.10). Generalized likelihood ratio tests defined in (2.45) and (2.59) may be performed to evaluate the appropriateness of the Weibull and exponential distributions, respectively, for the underlying failure distribution.

4.2.3 Model 2: Inverse Power Law Model

This model has been satisfactorily applied to accelerated life test data on insulating fluids, capacitors, bearings, electronic devices, and other products for which voltage is the stress variable. Nelson [1970] discusses the inverse power law model used in conjunction with an assumed underlying Weibull failure distribution. Mann, Schafer, and

Singpurwalla [1974] discuss the model when the exponential distribution is taken to be the underlying distribution. The basic assumptions of the model are that:

(i) The severity of the stress variable does not change the type of lifetime distribution, but the stress levels have an influence on the values of the parameters of the underlying distribution.

(ii) The relationship between the stress level, s_i, and the scale parameter θ (θ under the Weibull distribution represents the characteristic life; θ under the exponential distribution represents the mean life) is given by

$$\theta_i = r s_i^{-p} , \tag{4.5}$$

where r and p are unknown model parameters, r is positive, and s_i is within a certain acceptable range.

When the inverse power law model is used in conjunction with an underlying Weibull distribution, a further assumption is needed:

(iii) The shape parameter c of the Weibull distribution is independent of the stress level s_i.

Mann, Schafer, and Singpurwalla [1974] determined that to obtain estimates of r and p that are asymptotically independent, the inverse power law in (4.5) must be amended slightly to

$$\theta_i = r(s_i/\dot{s})^{-p} , \tag{4.6}$$

where $\dot{s}$ is a weighted geometric mean given by

$$\dot{s} = \prod_{i=1}^{k} (s_i)^{u_i/\sum_{i=1}^{k} u_i} , \tag{4.7}$$

when there are u_i test units assigned to the ith stress level.

Kalbfleisch and Prentice [1980] generalized the inverse power law model to cases where the underlying failure distribution is not restricted to the Weibull family. By means of a log-linear model they derived the following form of the hazard function with covariable z_i:

$$\lambda(t_i, z_i) = \lambda_0(t_i e^{\gamma z_i}) e^{\gamma z_i} . \tag{4.8}$$

Cox [1972] suggested that this model is one of the simplest proposed models for the effect of stress on the distribution of failure time. If the covariable z_i represents $\ln(s_i)$, then (4.8) may be rewritten as

$$\lambda(t_i, s_i) = \lambda_0(t_i s_i^{\gamma})(s_i^{\gamma}) , \tag{4.9}$$

which is equivalent to the inverse power law model of Nelson [1970] and Mann, Schafer, and Singpurwalla [1974], when it is assumed that $\lambda_0(\cdot)$ is the Weibull or exponential hazard function. Kalbfleisch and Prentice [1980] point out that in contrast to the proportional hazards model of Cox [1972], which assumes that the covariable z_i affects the hazard rate multiplicatively, in the inverse power law model the covariable acts multiplicatively on the failure time t, serving to accelerate (or decelerate) the time to failure. Hence, these authors refer to this model as the accelerated failure time model.

In order to fit the Burr Type XII distribution as the underlying failure distribution in the inverse power law model, it is assumed that the relationship between the stress level, s_i, and the scale parameter β of the Burr XII distribution is identical to the Weibull and exponential cases. Then the Burr scale parameter is redefined as

$$\beta_i = r_B(s_i/\dot{s})^{-p_B} , \tag{4.10}$$

where r_B and p_B are unknown model parameters with the subscript "B" denoting the values of the parameters under the Burr model, r_B is positive, p_B is real-valued, and s_i is within a certain acceptable range.

In terms of Kalbfleisch and Prentice's [1980] form of the model, this is equivalent to assuming that the underlying hazard in expression (4.8) is the Burr hazard function given in (2.9). The hazard function for the inverse power law model then is of the form

$$\lambda(t_i, s_i) = \frac{\frac{ck}{r_B} (s_i/\dot{s})^{p_B} \left\{\frac{t_i (s_i/\dot{s})^{p_B}}{r_B}\right\}^{(c-1)}}{\left[1 + \left\{\frac{t_i (s_i/\dot{s})^{p_B}}{r_B}\right\}^{c}\right]} \quad . \tag{4.11}$$

In the Weibull region the scale parameter, r_B, of the inverse power law used in conjunction with the Burr distribution is related to the scale parameter, r_W, of the Weibull model according to the equation

$$r_B = r_W k^{1/c} \quad . \tag{4.12}$$

The parameter r_W may be replaced by its maximum likelihood estimator, $\hat{r}_W$, in the above expression when the restricted Burr model is fit. Then, the ratio of maximized likelihoods for discriminating between the Weibull and the Burr XII distributions, used in conjunction with the inverse power law model is given by

$$R'_W = \frac{L_{RB}(\hat{c}, \hat{k}, r_B = \hat{r}_W \hat{k}^{1/\hat{c}}, \hat{p}_B)}{L_B(\hat{c}, \hat{k}, \hat{r}_B, \hat{p}_B)} \quad , \tag{4.13}$$

where $L_{RB}(\cdot)$ and $L_B(\cdot)$ denote the likelihood functions for the restricted Burr and unrestricted Burr distributions, respectively. As an extension of the development in Section 2.2.5 it may be shown that in the Weibull region the likelihood equations for the restricted Burr distribution, used in conjunction with the inverse power law model, approach those of the Weibull distribution. This supports the assumption that the statistic $\lambda_w' = -2\ell n(R_w')$ behaves asymptotically as a chi-square random variable with one degree of freedom.

In order to test whether the underlying failure distribution is exponential, the two Burr XII parameter constraints displayed in (2.50) when incorporated in the inverse power law model become

$$\text{(i)} \quad c = 1$$

$$\text{(ii)} \quad r_B = r_e k \ , \tag{4.14}$$

where r_e denotes the exponential inverse power law parameter defined in (4.6), and c, k, and r_B are the Burr XII parameters for the inverse power law model. When the Burr XII distribution is restricted to the exponential region, the unknown parameter r_e is replaced by its maximum likelihood estimator, $\hat{r}_e$, under the exponential model. The ratio of maximized likelihood functions for choosing between the exponential and Burr XII distributions for the underlying failure distribution in the inverse power law model is specified by

$$R_e' = \frac{L_{RB}(c=1, \hat{k}, r_B = \hat{r}_e \hat{k}, \hat{p}_B)}{L_B(\hat{c}, \hat{k}, \hat{r}_B, \hat{p}_B)} \ , \tag{4.15}$$

where $L_{RB}(\cdot)$ and $L_B(\cdot)$ denote the likelihood functions for the restricted Burr and unrestricted Burr distributions, respectively. Based on the argument in Section 2.2.5, it is assumed that the statistic $\lambda_e' = -2\ell n(R_e')$ behaves asymptotically as a chi-square random variable with two degrees of freedom. This point is examined further by simulation in Chapter VI.

4.2.4 The Equivalence of the Proportional Hazards and Inverse Power Law Models when the True Underlying Failure Distribution Belongs to the Weibull Family

Kalbfleisch and Prentice [1980] show that when the underlying hazard, $\lambda_0(t)$, of the accelerated life test model is of the form

$$\lambda_0(t) = \frac{c}{\theta}\left(\frac{t}{\theta}\right)^{c-1} , \tag{4.16}$$

the proportional hazards model defined in (4.3) and the inverse power law model defined in (4.8), or equivalently in (4.9), are the same. Their method of proof consists of solving for the subset of inverse power law models for which the regression variable z acts multiplicatively on the hazard function. Denoting the proportional hazards and inverse power law models by the subscripts 1 and 2, respectively, it is required that

$$\lambda_{01}(t)\exp(\gamma_1 z) \equiv \lambda_{02}\{t \exp(\gamma_2 z)\}\exp(\gamma_2 z) , \tag{4.17}$$

for all (t,z). The equation in (4.17), when evaluated at z equal to zero, becomes

$$\lambda_{01}(t) \equiv \lambda_{02}(t) \tag{4.18}$$

for all t. Hence, $\lambda_{01}(\cdot)$ and $\lambda_{02}(\cdot)$ in (4.17) may be replaced by the common underlying hazard function $\lambda_0(\cdot)$. Then when z is set equal to $-\ln(t)/\gamma_2$, the evaluation of equation (4.17) at that t gives

$$\lambda_0(t)t^{-(\gamma_1/\gamma_2)} = \lambda_0(1)t^{-1} . \tag{4.19}$$

Then for all t

$$\lambda_0(t) = \frac{c}{\theta}\left(\frac{t}{\theta}\right)^{c-1} , \tag{4.20}$$

where

$$c = \gamma_1\gamma_2^{-1} \quad \text{and} \quad \theta = \left\{\frac{\lambda_0(1)}{c}\right\}^{-c^{-1}} . \tag{4.21}$$

Hence, the proportional hazards and inverse power law models are equivalent only when the underlying failure distribution belongs to the Weibull family, with the exponential distribution being a special case.

In the discussion of the Burr XII hazard function of Section 2.2.2 it was shown that when the Burr scale parameter β is replaced by $\theta_w k^{1/c}$, the functional form that it assumes in the Weibull region, the limiting form of the Burr hazard function as k approaches infinity is that of expression (4.16). Therefore, for large values of k when the underlying failure distribution is assumed to be the restricted Burr distribution constrained to the Weibull ($\beta=\theta_w k^{1/c}$) or exponential ($\beta=\theta_e k$) regions, the proportional hazards and inverse power law models are equivalent.

The first equation of (4.21) describes the relationship between the covariate parameters, γ_1 and γ_2, for the proportional hazards and inverse power law models, respectively. It should be noted that γ_1 corresponds to γ of equation (4.3), and γ_2 corresponds to p of equation (4.5). When the Weibull distribution is assumed to be the underlying failure distribution

$$\gamma_w = cp_w \ , \tag{4.22}$$

where c is the Weibull shape parameter and the subscript "w" denotes the values of the covariate parameters under the Weibull model. As a special case, when the underlying hazard is assumed to be the constant exponential hazard and the Weibull shape parameter c is equal to 1, equation (4.22) becomes

$$\gamma_e = p_e \ , \tag{4.23}$$

where the subscript "e" denotes the values of the covariate parameters for the exponential distribution. These relationships between γ_w and p_w and γ_e and p_e may be noted in the two accelerated life test examples of Section 4.3.3.

4.3 Analysis of Accelerated Life Test Data Representing Times to Breakdown of an Insulating Fluid

4.3.1 Description of the Data

Under normal conditions the voltages to which insulating fluids are subjected are so low that the average time to a breakdown runs millions of years (Nelson, [1970]). Therefore, in a reliability study conducted by the General Electric Company, elevated test voltages were used to save time in observing the breakdown of insulating fluids. All test units were run to failure and the voltages were held constant throughout the testing period. The life test was conducted in a laboratory with a pair of parallel plate electrodes of a fixed area and separation. There are two sets of data which correspond to separate runs with different purposes and experimental conditions. The separation between the electrodes was constant but different for the two sets of data. The purpose of the Run 1 data was to assess whether the underlying distribution of time to breakdown at a constant voltage is exponential. These data consist of 220 times to breakdown, measured at three voltage stresses ranging from 34 to 36 kilovolts. The Run 2 data were intended to be used to determine the relationship between the distribution of time to breakdown and voltage stress. These data consist of 76 times measured at seven voltage levels ranging from 26 to 38 kilovolts. The four failure times equal to 0.00 in the Run 1 data were set equal to 0.005 in the analysis.

4.3.2 Discussion of Prior Analyses

Nelson [1970] fit the inverse power law model to these data. Under the assumption that the underlying failure distribution belongs to the Weibull family and using Nelson's parameterization, the fraction of the test units failing by positive time t under a constant positive stress, v_i, is

$$F(t_i, v_i; k, m, n) = 1 - \exp\{-(kv_i^n t_i)^m\} . \tag{4.24}$$

The methods of fitting which he employed are probability plots, least squares, maximum likelihood, and linear estimates, which are discussed in Hahn and Nelson [1970]. A weakness of Nelson's [1970] analysis is that he obtained model parameter estimates for each voltage level, v_i, separately, and then solved for combined estimates. Through the use of a number of graphical and analytical procedures, Nelson assessed the validity of the power law model. In particular, he considered (i) the assumed linear relationship between the mean of the extreme value distribution of the log-failure times and the logarithmic stress; (ii) the non-dependence of the Weibull shape parameter, m, on the level of v_i; and (iii) the goodness-of-fit of the Weibull distribution for constant v_i.

With regard to the Run 1 data, Nelson found the exponential distribution to provide an adequate overall fit to the data when the three test voltages (34kv, 35kv, 36kv) are analyzed separately. However, there appeared to be an excess of failure times at the lower tail when 35 kilovolts were applied. Nelson attributed this to difficulties associated with measuring very short failure times. Four failure times were recorded at 0.00, when in fact they were not zero, but less than 0.01 minute.

For the Run 2 data the combined maximum likelihood estimator for the Weibull shape parameter is 0.752 with a corresponding 95% confidence interval of (0.611, 0.924). Hence, an underlying exponential failure distribution would be rejected for the Run 2 data set on the basis of these findings. The maximum likelihood estimators for the power law parameters, k and n, were

$$\hat{k} = 2.0730 \times 10^{-28}$$

and (4.25)

$$\hat{n} = 17.4188 ,$$

respectively. Although a chi-square test gave no statistical evidence that the Weibull shape parameter for the Run 2 data differs from one voltage level to another, Weibull probability plots in Figure 5.4 of Nelson [1970] suggest that the shape parameter for 32 kilovolts is smaller than that for the other test voltages, indicating a possible faulty experimental procedure. Finally, the assumed Weibull distribution was found to adequately describe the underlying failure distribution, based on a plot of the standardized model residuals shown in Figure 7.2 of Nelson [1970].

Farewell and Prentice [1977] fit a log-generalized gamma regression model, which is described in Section 1.3.2, to the logarithms of the fluid breakdown times, with log-voltage serving as a regressor. Their model, which assumes log-voltage, z_i, to be linear in log-failure times, is

$$y = \ell n(t_i) = \alpha + \beta z_i + \sigma w , \tag{4.26}$$

where β is the parameter for the log-voltage covariable and w is the error random variable. The probability density function for the log-failure time, y, is given in equation (2.16). The corresponding model for t is given by (4.8), where the underlying hazard function, $\lambda_0(t)$, is the generalized gamma hazard function.

As discussed in Section 4.2.1, Kalbfleisch and Prentice's [1980] log-linear model defined in (4.26) incorporates some aspects of the inverse power law model of Nelson [1970]. Specifically, both models assume that there is a linear relationship between log-voltage levels and log-failure times. Consequently there is a close correspondence between the 95% confidence intervals for the covariate parameter under the two models. Farewell and Prentice [1977] reported the interval to be (-20.77, -14.50) and Nelson [1970] reported it as (-20.86, -13.98).

A difference between the two analyses is that Nelson [1970] assumed that the underlying failure distribution belongs to the Weibull family, while Farewell and Prentice [1977] initially fit the richer log-generalized gamma distribution. The latter authors subsequently tested the appropriateness of the Weibull and exponential distributions as special cases, using the generalized likelihood ratio tests discussed in Section 2.2.4. It is noted that the parameter subspaces for the Weibull and exponential distributions lie in the interior of the log-generalized gamma parameter space. For the Run 1 data, Farewell and Prentice rejected the Weibull distribution at the $\alpha = 0.05$ significance level. They attributed the inability of the Weibull distribution to model these data adequately to the excess of small failure times. However, they found no evidence against the Weibull model for the Run 2 data.

4.3.3 Determination of the Distributional Shape of the Run 1 and Run 2 Data Using the Burr Type XII Distribution

Under the assumption that the underlying failure distribution belongs to the Burr XII family, the proportional hazard function of expression (4.4) and the inverse power law hazard function of expression (4.11) were each fit to the Run 1 and Run 2 data. Summaries of the iterative maximum likelihood solutions to the proportional hazards and inverse power law models are presented in Tables 4.1 and 4.2 for the Run 1 data and in Tables 4.4 and 4.5 for the Run 2 data. As discussed in Section 3.3, the ordering of the analyses for a particular data set is to initially fit the Weibull model and optionally the exponential model, the unrestricted Burr, the Burr restricted to the Weibull region, and then optionally the Burr restricted to the exponential region. Each summary table displays the initial estimates of the model parameters, the initial value of the log-likelihood function, the number of iterations required for convergence, the maximum likelihood estimates and corresponding value of the maximized log-likelihood function, specification of the constraint placed upon the Burr scale parameter in the restricted Burr models, and the value of the log-likelihood function when the parameter for the covariate representing the accelerated condition is set equal to zero under the best fitting model.

A comparison of Tables 4.1 and 4.2 shows that for every one of the succession of models fit to the Run 1 data, the maximized log-likelihood values are identical for the proportional hazards and inverse power law models. The equivalence of these two accelerated life time models for the Weibull, exponential, and restricted Burr XII distributions was demonstrated in Section 4.2.4. However, the proportional hazards and inverse power law models are not necessarily equivalent when the unrestricted Burr XII distribution is taken to be the underlying failure distribution. The close agreement for these data can be explained by the fact that for each of the accelerated

TABLE 4.1

PROPORTIONAL HAZARDS MODEL FIT TO RUN 1 DATA

Underlying Failure Distribution	Weibull	Optional Exponential	Unrestricted Burr XII*	Restricted Burr* (Weibull Region)	Restricted Burr (Exponential Region)
Initial Estimates	$c_{w0} = 0.97$ $\theta_{w0} = 3.78$ $\gamma_{w0} = 0.00$	$\theta_{e0} = 3.58$ $\gamma_{e0} = 10.96$	$c_0 = 0.98$ $k'_0 = 1.0\times10^{-9}$ $\beta'_0 = 2.0\times10^{-10}$ $\gamma_0 = 10.96$	$c_0 = 0.98$ $k'_0 = 0.003$ $\beta'_0 = 0.0007$ $\gamma_0 = 10.96$	$c \equiv 1$ (fixed) $k'_0 = 0.003$ $\beta'_0 = 0.0007$ $\gamma_0 = 10.96$
Initial Log-Likelihood	-508.6810	-502.1274	-502.0514	-502.0956	-502.1556
Number of Iterations	16	10	5	13	10
Maximized Log-Likelihood	-502.0514	-502.1179	-502.0514	-502.0514	-502.1178
Maximum Likelihood Estimates (s.e.)	$\hat{c}_w = 0.98$ (0.05) $\hat{\theta}_w = 3.58$ (0.26) $\hat{\gamma}_w = 10.96$ (3.04)	$\hat{\theta}_e = 3.60$ (0.24) $\hat{\gamma}_e = 11.18$ (2.98)	$\hat{c} = 0.98$ $\hat{k}' = 1.0\times10^{-9}$ $\hat{\beta}' = 2.0\times10^{-10}$ $\hat{\gamma} = 10.96$	$\hat{c} = 0.98$ (0.05) $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b. $\hat{\gamma} = 10.96$(3.04)	$c \equiv 1$ $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b. $\hat{\gamma} = 11.18$(2.98)
Burr Scale Parameter Constraint	Not Applicable	Not Applicable	Not Applicable	$\beta'=(3.58)^{-1}(k')^{1/c}$	$\beta' = (3.60)^{-1}k'$
Log-Likelihood When Covariate Parameter is Set Equal to Zero Under the Best Fitting Model	Not Applicable	-509.0700	Not Applicable	Not Applicable	Not Applicable

*Computational difficulties arise when k' and β' converge to their respective lower bounds (l.b.) of 0.00+. To ensure that the maximized log-likelihood value under the unrestricted Burr model is at least as large as that under the Burr model restricted to the Weibull region, the final estimates of the restricted Burr model (Weibull region) are used as initial estimates in the unrestricted Burr model. Standard errors could not be computed.

TABLE 4.2

INVERSE POWER LAW MODEL FIT TO RUN 1 DATA

Underlying Failure Distribution	Weibull	Optional Exponential	Unrestricted Burr XII*	Restricted Burr* (Weibull Region)	Restricted Burr (Exponential Region)
Initial Estimates	$c_{w0} = 0.97$ $r_{w0} = 3.78$ $p_{w0} = 0.00$	$r_{e0} = 3.58$ $p_{e0} = 11.18$	$c_0 = 0.98$ $k'_0 = 1.0\times10^{-9}$ $r'_{B0} = 2.0\times10^{-10}$ $p_{B0} = 11.18$	$c_0 = 0.99$ $k'_0 = 0.0004$ $r'_{B0} = 0.00009$ $p_{B0} = 11.18$	$c \equiv 1$ (fixed) $k'_0 = 0.0004$ $r'_{B0} = 0.0001$ $p_{B0} = 11.18$
Initial Log-Likelihood	-508.6810	-502.1247	-502.0514	-502.0619	-502.1222
Number of Iterations	14	8	5	6	4
Maximized Log-Likelihood	-502.0514	-502.1179	-502.0514	-502.0514	-502.1178
Maximum Likelihood Estimates (s.e.)	$\hat{c}_w = 0.98$ (0.05) $\hat{r}_w = 3.58$ (0.26) $\hat{p}_w = 11.18$ (3.04)	$\hat{r}_e = 3.60$ (0.24) $\hat{p}_e = 11.18$ (2.98)	$\hat{c} = 0.98$ $\hat{k}'$ converged to l.b. $\hat{r}'_B$ converged to l.b. $\hat{p}_B = 11.18$	$\hat{c} = 0.98$ (0.05) $\hat{k}'$ converged to l.b. $\hat{r}'_B$ converged to l.b. $\hat{p}_B = 11.18$ (2.10)	$c \equiv 1$ $\hat{k}'$ converged to l.b. $\hat{r}'_B$ converged to l.b. $\hat{p}_B = 11.18$ (0.77)
Burr Scale Parameter Constraint	Not Applicable	Not Applicable	Not Applicable	$r'_B = (3.58)^{-1}(k')^{1/c}$	$r'_B = (3.60)^{-1}k'$
Log-Likelihood When Covariate Parameter is Set Equal to Zero Under the Best Fitting Model	Not Applicable	-509.0111	Not Applicable	Not Applicable	Not Applicable

*Computational difficulties arise when k' and r'_B converge to their respective lower bounds (l.b.) of 0.00+. To ensure that the maximized log-likelihood value under the unrestricted Burr model is at least as large as that under the Burr model restricted to the Weibull region, the final estimates of the restricted Burr model (Weibull region) are used as initial estimates in the unrestricted Burr model. Standard errors could not be computed.

life time models, the results suggest that the true underlying distribution belongs to the Weibull, and in particular, to the exponential family. Hence, the unrestricted Burr maximum likelihood estimates lie in the exponential region where the proportional hazards and inverse power law models are equivalent, as seen in the right-most columns of Tables 4.1 and 4.2.

In Table 4.3 the generalized likelihood ratio test statistics corresponding to the hypotheses of interest for the Run 1 data are presented. By initially fitting the Burr XII distribution it is concluded that an exponential distribution with a constant hazard adequately describes the underlying failure distribution for the Run 1 data. The large sample 95% confidence interval for the parameter corresponding to the applied stress under either accelerated life test model, when the underlying failure distribution is the exponential, is (-17.02, -5.33).

These findings are in agreement with those of Nelson [1970], who determined by means of probability plots that the exponential distribution provides a good fit for these data. However, Farewell and Prentice [1977] rejected the Weibull distribution as a special case of the log-generalized gamma distribution for the Run 1 data. These authors attributed the inappropriateness of the Weibull distribution to an excess of small failure times. Nelson [1970] also noted the predominance of small failure times, especially at 35 kilovolts. When the log-generalized gamma distribution is fit to the data at each stress level separately, the exponential distribution is accepted at the $\alpha = 0.01$ level of significance, based on generalized likelihood ratio tests, for the 34 kilovolt and 36 kilovolt data. Hence, the Weibull distribution is rejected at the $\alpha = 0.05$ significance level only for the 35 kilovolt

TABLE 4.3

GENERALIZED LIKELIHOOD RATIO TEST STATISTICS AND P-VALUES FOR HYPOTHESES OF INTEREST FOR RUN 1 DATA

Model	Null Distribution	Alternative Distribution	Test Statistic*	Chi-Square Value	Degrees of Freedom	P-Value
Proportional Hazards Model	Weibull	Burr XII	λ'_w	0.0000	1	1.00
	Exponential	Burr XII	λ'_e	0.1328	2	0.94
Inverse Power Law Model	Weibull	Burr XII	λ'_w	0.0000	1	1.00
	Exponential	Burr XII	λ'_e	0.1328	2	0.94

* λ'_w is defined in Section 2.26; λ'_e is defined in Section 2.4.

failure times, when it is assumed that the log-generalized gamma distribution adequately describes the underlying failure distribution. In contrast, when it is assumed that the underlying failure distribution belongs to the Burr Type XII family, the exponential distribution is accepted at the $\alpha = 0.01$ level of significance for each value of the accelerated stress. The question is raised whether the sample sizes of 80, 80, and 60 at 34, 35, and 36 kilovolts, respectively, are large enough to justify a chi-square distribution as an approximation to the asymptotic distribution of the generalized likelihood ratio test statistic. Otherwise one can conclude that the log-generalized gamma distribution may be more sensitive than is the Burr XII distribution to deviations from the Weibull distribution in the lower tail.

The analyses for the Run 2 data based on the proportional hazards and inverse power law models are summarized in Tables 4.4 and 4.5, respectively. The two accelerated life time models give identical fits with regard to the values of the maximized Weibull and exponential log-likelihood functions, and the Weibull and exponential distributional parameters, which is also in agreement with the analytical results of Section 4.2.4. Table 4.6 displays the results of generalized likelihood ratio tests for testing the appropriateness of the Weibull and exponential models against Burr XII alternatives. The Weibull distribution is accepted at the $\alpha = 0.01$ level of significance, while the exponential distribution is rejected at the $\alpha = 0.05$ significance level by both accelerated life test models. It is noted that the Burr XII distribution restricted to the Weibull region has maximum likelihood estimates which are more similar to the Weibull maximum likelihood estimates when the proportional hazards rather than the inverse power law model is assumed.

TABLE 4.4

PROPORTIONAL HAZARDS MODEL FIT TO RUN 2 DATA

Underlying Failure Distribution	Weibull	Optional Exponential	Unrestricted Burr XII*	Restricted Burr* (Weibull Region)	Restricted Burr (Exponential Region)
Initial Estimates	$c_{w0} = 0.39$ $\theta_{w0} = 21.6$ $\gamma_{w0} = 0.00$	$\theta_{e0} = 17.57$ $\gamma_{e0} = 13.77$	$c_0 = 0.78$ $k'_0 = 1.0\times10^{-9}$ $\beta'_0 = 1.0\times10^{-13}$ $\gamma_0 = 13.77$	$c_0 = 0.78$ $k'_0 = 0.005$ $\beta'_0 = 0.00006$ $\gamma_0 = 13.69$	$c \equiv 1$ (fixed) $k'_0 = 0.005$ $\beta'_0 = 0.0002$ $\gamma_0 = 13.69$
Initial Log-Likelihood	-340.5101	-312.6569	-300.8174	-300.9501	-310.0283
Number of Iterations	15	12	5	8	11
Maximized Log-Likelihood	-300.8174	-305.5373	-300.8174	-300.8174	-305.5372
Maximum Likelihood Estimates (s.e.)	$\hat{c}_w = 0.78$ (0.07) $\hat{\theta}_w = 17.57$ (2.74) $\hat{\gamma}_w = 13.77$ (1.74)	$\hat{\theta}_e = 20.50$ (2.35) $\hat{\gamma}_e = 17.70$ (1.29)	$\hat{c} = 0.78$ $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b. $\hat{\gamma} = 13.77$	$\hat{c} = 0.78$ (0.06) $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b. $\hat{\gamma} = 13.77$ (1.70)	$\hat{c} \equiv 1$ $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b. $\hat{\gamma} = 17.70$ (1.29)
Burr Scale Parameter Constraint	Not Applicable	Not Applicable	Not Applicable	$\beta' = (17.57)^{-1}(k')^{1/c}$	$\beta' = (20.50)^{-1}(k')$
Log-Likelihood When Covariate Parameter is Set Equal to Zero Under the Best Fitting Model	-339.6543	Not Applicable	Not Applicable	Not Applicable	Not Applicable

*Computational difficulties arise when k' and β' converge to their respective lower bounds (l.b.) of 0.00+. In order to ensure that the maximized log-likelihood value under the unrestricted Burr model is at least as large as that under the Burr model restricted to the Weibull region, the final estimates of the restricted Burr model (Weibull region) are used as initial estimates in the unrestricted Burr model. Standard errors could not be computed.

TABLE 4.5

INVERSE POWER LAW MODEL FIT TO RUN 2 DATA

Underlying Failure Distribution	Weibull	Optional Exponential	Unrestricted Burr XII	Restricted Burr (Weibull Region)	Restricted Burr (Exponential Region)
Initial Estimates	$c_{w0} = 0.39$ $r_{w0} = 21.6$ $p_{w0} = 0.00$	$r_{e0} = 17.57$ $p_{e0} = 17.73$	$c_0 = 0.78$ $k'_0 = 0.10$ $r'_{B0} = 0.003$ $p_{B0} = 17.73$	$c_0 = 0.85$ $k'_0 = 0.15$ $r'_{B0} = 0.006$ $p_{B0} = 17.85$	$c \equiv 1$ (fixed) $k'_0 = 0.15$ $r'_{B0} = 0.007$ $p_{B0} = 17.85$
Initial Log-Likelihood	-340.5101	-306.4865	-570.5167	-301.0350	-303.7856
Number of Iterations	16	7	75	13	4
Maximized Log-Likelihood	-300.8174	-305.5373	-300.6386	-300.7637	-303.7824
Maximum Likelihood Estimates (s.e.)	$\hat{c}_w = 0.78$ (0.07) $\hat{r}_w = 17.57$ (2.74) $\hat{p}_w = 17.73$ (1.61)	$\hat{r}_e = 20.50$ (2.35) $\hat{p}_e = 17.70$ (1.29)	$\hat{c} = 0.85$ (0.15) $\hat{k}' = 0.15$ (0.28) $\hat{r}'_B = 0.007$ (0.02) $\hat{p}_B = 17.85$ (1.62)	$\hat{c} = 0.80$ (0.10) $\hat{k}' = 0.04$ (0.14) $\hat{r}'_B = 0.001$ (0.00) $\hat{p}_B = 17.78$ (1.63)	$c \equiv 1$ $\hat{k}' = 0.16$ (0.11) $\hat{r}'_B = 0.008$ (0.005) $\hat{p}_B = 17.93$ (1.51)
Burr Scale Parameter Constraint	Not Applicable	Not Applicable	Not Applicable	$r'_B = (17.57)^{-1}(k')^{1/c}$	$r'_B = (20.50)^{-1}k'$
Log-Likelihood When Covariate Parameter is Set Equal to Zero Under the Best Fitting Model	-339.6543	Not Applicable	Not Applicable	Not Applicable	Not Applicable

TABLE 4.6

GENERALIZED LIKELIHOOD RATIO TEST STATISTICS AND P-VALUES FOR HYPOTHESES OF INTEREST FOR RUN 2 DATA

Model	Null Distribution	Alternative Distribution	Test Statistic*	Chi-Square Value	Degrees of Freedom	P-Value
Proportional Hazards Model	Weibull	Burr XII	λ'_w	0.0000	1	1.00
	Exponential	Burr XII	λ'_e	9.4396	2	0.01
Inverse Power Law Model	Weibull	Burr XII	λ'_w	0.2502	1	0.62
	Exponential	Burr XII	λ'_e	6.2876	2	0.04

* λ'_w is defined in Section 2.26; λ'_e is defined in Section 2.4.

The findings with regard to the Run 2 data are in agreement with those of Farewell and Prentice [1977] as well as Nelson [1970]. The combined (each stress level was analyzed separately) maximum likelihood estimator derived by Nelson for the Weibull shape parameter is equal to 0.752 with a 95% confidence interval of (0.611, 0.924). The maximum likelihood estimator of the shape parameter c for the underlying Burr XII distribution restricted to the Weibull region in the proportional hazards model is equal to 0.777 with a large sample 95% confidence interval of (0.650, 0.904). Since the shape parameter c is less than one, it is concluded that the rate of breakdown of the insulating fluid decreases with increasing time in the Run 2 data. With regard to the power parameter p of the inverse power law model, Farewell and Prentice [1977] reported the 95% confidence interval to be (-20.77, -14.50) based on the log-generalized gamma distribution and Nelson reported it to be (-20.86, -13.98) based on the Weibull distribution where each stress level was analyzed separately. When the underlying failure distribution in the inverse power law model belongs to the Burr XII family restricted to the Weibull region the 95% confidence interval is (-20.97, -14.59), whereas when the underlying failure distribution is the Weibull distribution the 95% confidence interval is (-20.89, -14.58). These results are all in very good agreement. In this example then the Burr XII distribution and the log-generalized gamma distribution are quite comparable in their inferences.

CHAPTER V

A COMPARISON BETWEEN TWO GENERAL SURVIVAL DISTRIBUTIONS

5.1 Introduction

A general family of parametric distributions, which includes distributions commonly used in survival analyses as special cases with various choices of parameter values, is useful not only for determining, but also for confirming, the choice of an underlying distribution. Two such general families of distributions were discussed in Chapter I. The Burr Type XII distribution was shown to combine a simple mathematical expression for the survival function with wide coverage in the moment-ratio plane. The generalized gamma distribution, which also incorporates as special cases of its parameter values several of the distributions contained in the Burr XII family, has been previously used to confirm the choice of underlying distributions in the literature.

Adult acute leukemia survival data collected by the Southwest Oncology Group has been analyzed by Gehan, Dyer, and Sampson [1980]. There are five separate data sets which correspond to five remission induction treatments. The application of a model selection procedure for grouped data developed by Gehan and Siddiqui [1973] suggested that the two-parameter Weibull distribution is the best fitting distribution among four competing models for each of the data sets.

In this chapter a reassessment of the fit of the Weibull distribution for these data will be made using the Burr XII and generalized gamma models. Section 5.2 contains a comparison of some properties of the two general distributions. Section 5.3 is directed to maximum likelihood considerations for fitting the two models in the presence of censored observations. A brief description of the five leukemia data sets is given in Section 5.4, and a review of the analysis performed by Gehan et al. [1980] is contained in Section 5.5. Section 5.6 addresses the fit of the Weibull distribution to the adult acute leukemia data within the parametric framework of the Burr Type XII and generalized gamma distributions. Finally, Section 5.7 contrasts the performance of the two general distributions with regard to computer cost considerations.

5.2 A Comparison of Some General Properties of the Burr Type XII and the Generalized Gamma Distributions

The Burr XII distribution and the generalized gamma distribution are three-parameter distributions. The shape of each distribution is determined by two of the parameters, while the third parameter adjusts the scale. A fourth parameter, which is a location parameter, may be incorporated into both distributions. However, for survival or reliability data the location parameter is usually assumed to equal zero.

The probability density function corresponding to the Burr XII distribution is given by

$$f(t;c,k,\beta) = (ck/\beta)(t/\beta)^{c-1}\{1+(t/\beta)^{c}\}^{-(k+1)} , \qquad (5.1)$$

where the shape parameters c and k and the scale parameter β are positive, and the time to failure, t, is non-negative.

The generalized gamma probability density function, in its most common form, is equal to

$$g(t;a,k,b) = ba^{-bk}t^{bk-1}\exp\{-(t/a)^{b}\}/\Gamma(k) , \qquad (5.2)$$

where b and k are positive shape parameters, and a is a positive scale parameter. Again, the non-negative variable t represents time to failure.

A comparison of Tables 1.2 and 1.3 shows that the generalized gamma distribution contains two common survival distributions, the gamma and the lognormal, which are not special cases of the Burr Type XII distribution. However, as was discussed in Section 1.2, the coverage of the Burr distribution in the moment-ratio plane includes parts of each of these distributions.

Among the sufficient conditions for the asymptotic chi-square approximation for the distribution of the generalized likelihood ratio test statistic is that the parameter space of the null distribution must lie in the interior of the parameter space of the alternative distribution. Otherwise, the approximation may not hold. Therefore, an advantage of the generalized gamma distribution is that of the four common failure distributions contained in this family, namely, the exponential, the Weibull, the lognormal, and the gamma, only one of these, the lognormal, is a limiting case of the parameters. In contrast, of the distributions nested in the Burr XII family which are commonly used in survival analyses, only the normal distribution is a non-limiting case. The exponential distribution, the Weibull distribution, and the Gompertz distribution are all defined by parameter subspaces which lie on the boundary of the Burr XII parameter space.

In other respects the Burr XII distribution has advantages over the generalized gamma distribution, especially in regard to the incorporation of censored observations into the likelihood function. The survival function under the Burr model exists in simple closed form.

In contrast, the survival function under the generalized gamma model has no closed form. A numerical integration technique must be used to evaluate the contribution of each censored observation to the generalized gamma likelihood function.

There have been some computational difficulties associated with the generalized gamma distribution which have been cited in the literature. These were discussed in detail in Section 1.3.2. Hager and Bain [1970] reported basic difficulties when solving the maximum likelihood equations, which they attributed to the computational complexity of the distribution. The Newton-Raphson iterative solution procedure did not perform well and in some cases it was not possible to obtain meaningful maximum likelihood estimates. When the parameter k of the generalized gamma distribution, specified in equation (5.2), is equal to one, the general distribution reduces to the Weibull special case. Hager and Bain investigated the appropriateness of using the asymptotic distribution of $\hat{k}$ for testing the adequacy of the Weibull model for large samples. However, Monte Carlo results indicated that the asymptotic distribution of $\hat{k}$ was not in effect for samples as large as 400, when k was set equal to 1.0 and 2.0.

There have apparently been no reported cases where the Burr Type XII distribution has been fit by maximum likelihood techniques. For this reason, the acute leukemia data provide a good opportunity for comparing its performance with that of the generalized gamma distribution. Practical considerations which pertain to fitting these generalized distributions by maximum likelihood techniques are discussed in the next section.

5.3 The Models

The decision was made to use the log-generalized gamma distribution developed by Prentice [1974] rather than the usual parameterization of the generalized gamma function, as derived by Stacy [1962] and given in equation (5.2). This choice was influenced by the fact that Farewell and Prentice [1977] had previously used the log-generalized gamma distribution to investigate the distributional shape of some survival data sets. It was of interest to see how the Burr XII distribution compared to this alternative family of survival distributions. Another reason why the log-generalized gamma distribution was selected is that there have been reported computational difficulties when fitting the generalized gamma distribution by maximum likelihood techniques, as just noted. In contrast, Farewell and Prentice found by means of a Monte Carlo study that under the Weibull null hypothesis, the empirical and asymptotic distributions of the transformed maximum likelihood estimator $\hat{q} = \hat{k}^{-1/2}$ of the log-generalized gamma model agree for sample sizes of 100.

A discussion in Section 2.2.4 outlines the method by which Prentice [1974] obtained the log-generalized gamma distribution from the generalized gamma distribution. The probability density function corresponding to the log-generalized gamma model is equal to

$$g(y=\ell n(t);\alpha,\sigma,q) = \begin{cases} |q|\{\sigma\Gamma(q^{-2})\}^{-1}\exp(wq^{-2}-e^{w}); & q \neq 0 \\ (2\pi\sigma^{2})^{-1/2}\exp\{-(y-\alpha)^{2}/2\sigma^{2}\}\ ; & q = 0, \end{cases} \tag{5.3}$$

where $w = (y-\alpha)\sigma^{-1}q + \mu^*$ and μ^* is the digamma function, $\psi(k) = d\ell n\Gamma(k)/dk$. The parameters α, σ, and q are related to the parameters a, b, and k of the generalized gamma model in (5.2) by the following equations:

$$\alpha = \ell n(a)+\mu^* b^{-1}, \quad \sigma = qb^{-1}, \quad \text{and} \quad q = k^{-1/2} , \tag{5.4}$$

where α and σ are required to be positive, and q is defined for all real values.

The emphasis in the analysis of the adult acute leukemia data is to reassess the fit of the Weibull model. Because the log-generalized gamma distribution reduces to the extreme minimum value distribution (or the Weibull for the untransformed failure times) when q is equal to unity, consideration will be restricted to the region of the parameter space where q is positive. In this region of the parameter space the log-generalized gamma model corresponds to the generalized gamma model. Then the probability density function in (5.3) is replaced by

$$g(y=\ell n(t);\alpha,\sigma,q) = q\{\sigma\Gamma(q^{-2})\}^{-1}\exp(wq^{-2}-e^{w}) , \tag{5.5}$$

where α, σ, and q are positive parameters, and y is a real-valued random variable representing the natural logarithm of the failure time.

Each of the adult acute leukemia data sets contains right-censored observations. Bartholomew's [1957] methodology for the incorporation of censored observations into the likelihood function was discussed in Section 3.5. Specifically, by defining an indicator variable of observed failure, δ_i, which assumes the value one if t_i represents a failure time and the value zero if t_i is right-censored, the likelihood function may be expressed as

$$L(\cdot) = \prod_{i=1}^{n} \{f(t_i)\}^{\delta_i}\{S(t_i)\}^{1-\delta_i} , \tag{5.6}$$

where $f(\cdot)$ and $S(\cdot)$ denote the probability density function and the survival function, respectively.

The survival function for the log-generalized gamma distribution, which does not exist in closed form, is equal to

$$S(y=\ell n(t);\alpha,\sigma,q) = \{\Gamma(q^{-2})\}^{-1} \int_{e^v}^{\infty} x^{(q^{-2}-1)} e^{x} dx \ , \qquad (5.7)$$

where

$$v = q\{(y-\alpha)/\sigma\} - 2\ell n(q) \ .$$

The FORTRAN subroutine DCADRE of the International Mathematical Subroutine Library was used to perform the numerical integration of expression (5.7). This routine uses a scheme whereby the interval of integration is divided into suitably small subintervals over which the integral is estimated. The final estimate is then obtained by summing up these subinterval estimates. Seven digits of accuracy were achieved in the approximation of the integrals required.

In order to solve the log-generalized gamma likelihood equations arising from (5.6), Farewell and Prentice [1977] used a partial rather than a full Newton-Raphson iterative procedure. For each of a number of q-values, the likelihood function was maximized with respect to α and σ, via Newton-Raphson. The resulting maximized relative likelihood function was then scanned to determine the maximum likelihood estimator, $\hat{q}$. Since computational difficulties have been reported when the generalized gamma likelihood equations are solved by a Newton-Raphson procedure, in this research a direct search of the likelihood surface was performed using MAXLIK (Kaplan and Elston [1972]). A Newton-Raphson iteration based on numerical derivatives was then performed to confirm the final maximum likelihood estimates.

Prentice [1974] and Farewell and Prentice [1977] did not discuss initial estimates for the three parameters of the log-generalized gamma model. Due to the computational complexity and cost associated with the numerical integration required when censored observations occur, a preliminary analysis was performed on the failures only, where the initial parameter estimates were taken to be

$$\alpha_0 = 0.0, \quad \sigma_0 = 1.0, \quad \text{and} \quad q_0 = 1.0 \ . \tag{5.8}$$

These parameter values correspond to an exponential distribution with scale parameter equal to -0.5772, which is the negative of Euler's constant. The resulting maximum likelihood estimates were then used as initial estimates for an analysis of the entire data set, including both failures and right-censored observations.

When the Burr XII distribution was fit to the adult acute leukemia data, the probability density function $f(\cdot)$ and the survival function $S(\cdot)$ in (5.6) were replaced by

$$f(t;c,k^*,\beta^*) = \frac{ck^*(1-\beta^*)}{(1-k^*)\beta^*}\{t(1-\beta^*)/\beta^*\}^{c-1}[1+\{t(1-\beta^*)/\beta^*\}^c]^{-(1-k^*)^{-1}} \tag{5.9}$$

and

$$S(t;c,k^*,\beta^*) = [1+\{t(1-\beta^*)/\beta^*\}^c]^{-\{k^*/(1-k^*)\}} , \tag{5.10}$$

respectively, where c is positive and k^* and β^* are contained by the unit interval. The reparameterization

$$k^* = \frac{k}{1+k} \quad \text{and} \quad \beta^* = \frac{\beta}{1+\beta} , \tag{5.11}$$

which was discussed in Section 3.2, was used in equations (5.9) and (5.10) to transform the positive real line onto the unit interval.

The likelihood equations under the Burr XII model were solved initially using the direct search algorithm of MAXLIK. A numerically based Newton-Raphson iterative procedure was then performed to confirm the maximum likelihood estimates. It was found that for these data the Newton-Raphson solution usually increased the value of the log-likelihood function. In addition, when the initial estimates of the unrestricted

Burr parameters were based on the Weibull maximum likelihood estimates, the Newton-Raphson procedure converged at a local maximum in the Weibull region. However, when Dubey's [1967] two-order statistics were used to derive initial estimates for the unrestricted Burr model, the Newton-Raphson procedure sometimes yielded a substantially larger maximum value elsewhere. In Section 5.6 the reported maximum likelihood estimates of the Burr XII parameters correspond to Newton-Raphson solutions, unless stated otherwise.

5.4 Description of the Data

The Southwest Oncology Group conducted a series of studies in adult acute leukemia over the period June, 1967 to December, 1979. The patients who were selected for the studies had received no prior treatment. There are five separate data sets which correspond to the five remission induction treatments specified in Table 5.1.

5.5 Prior Analyses

Gehan *et al.* [1980] utilized the methodology presented in Gehan and Siddiqui [1973], which was discussed in Section 1.3.3, to select an underlying parametric survival distribution for each of the data sets. The model selection technique pertains to grouped survival data only. In each of the competing models, the exponential, the linear hazard function, the Gompertz, and Weibull distributions, either the hazard function or its logarithmic transform is a linear function of the parameters and the random variable representing time or the logarithm of time. Therefore, the authors developed a weighted least squares method of estimating the parameters, based on non-parametric estimates of the hazard function for each interval. Three alternate weighting

TABLE 5.1

A DESCRIPTION OF THE ADULT LEUKEMIA STUDIES

Study Number	Dates	Induction Treatments	Sample Size	Number of Deaths	Number of Censored Observations
427	06/01/67-11/04/69	Cytarabine by continuous infusion (48 vs. 120 hours)	176	170	6
555	10/01/69-02/04/71	COAP vs. Cytarabine	149	140	9
557*	02/04/71-06/24/71	COAP vs. OAP	202	185	17
560*	06/04/71-10/06/72	COAP vs. OAP vs. DOAP			
7315	07/16/73-01/01/75	10-day OAP	179	155	24
7416	01/01/75-01/19/77	VP and simultaneous or sequential Ad-OAP	543	441	102

*Studies 557 and 560 were combined by the Southwest Oncology Group for analysis purposes.

schemes were proposed where the weight, w_i, for the i-th time interval is given by:

(i) 1, corresponding to regular least squares.

(ii) $1/\hat{v}_i$, where $\hat{v}_i$ is the estimated variance of the hazard in the i-th interval,

(iii) $n_i h_i$, the product of the number of patients entering the interval and the interval width, a measure of the total exposure to risk in the i-th interval.

The best fitting distribution and weighting scheme combination was defined to be that which has the largest log-likelihood value for the observed data. Since the exponential distribution is a special case of each of the other models, the large sample chi-square approximation for the distribution of the likelihood ratio was proposed to test whether the likelihood of the data under the linear hazard, the Gompertz, and the Weibull models is significantly larger than under the exponential model. However, the remaining three models belong to separate families. As was discussed in Section 1.3.3, Cox [1961] cautioned that it does not suffice to use the observed log-likelihood value as a measure of evidence in favor of one distributional family over a separate distributional family. One reason for this is that one set of distributional parameters may be more adjustable than another set. Although Gehan and Siddiqui [1973] proposed a direct comparison of the log-likelihood values under the linear hazard, the Gompertz, and the Weibull models, they stressed that confirmation of the chosen model should be sought.

They proposed a goodness-of-fit test to determine whether the best fitting model adequately fits the data. The test statistic is equal to twice the difference between the log-likelihoods under the chosen model and the sample data. Under the assumption that the number of deaths in each interval is a multinomially distributed random variable, the statistic is an approximate chi-square variate with (s-1-r) degrees of freedom where r is the number of independent parameters fitted in the model and (s-1) is the number of fixed finite intervals in the data.

The adult acute leukemia data consist of times to death or times to censoring, measured in weeks. Since the chosen model selection technique requires grouped data, it was necessary for Gehan _et al_. [1980] to group the failure and censoring times. The authors defined the following time intervals, measured in weeks, for each of the five studies: [0,4), [4,7), [7,13), [13,26), [26,39), [39,52), [52,65), [65,78), [78,91), [91,104), [104,130), [130,156), [156,182). Beyond 182 weeks, the intervals varied among the five studies.

A summary of the analysis results obtained by Gehan _et al_. is presented in Table 5.2. Included in the table for each study are the estimated parameters and standard errors for the best fitting model, an indication of the best weighting scheme, the chi-square goodness-of-fit statistic, and the corresponding degrees of freedom and p-value. In every case the Weibull distribution provided the best fit as determined by the maximized log-likelihood value. However, based on the chi-square goodness-of-fit test, the Weibull model was rejected at the 0.01 significance level in every study except Study 555. In Study 555 the Weibull fit was questionable with a p-value of 0.07. These contradictory results suggest that although the model selection criterion of Gehan and Siddiqui [1973] supported the Weibull distribution with a decreasing hazard function as the best fitting underlying distribution among those tested for the adult acute leukemia data, there is some question as to how well this distribution fits the data.

TABLE 5.2

A SUMMARY OF THE RESULTS OBTAINED BY GEHAN *ET AL.* [1980]
IN THE ANALYSIS OF THE ACUTE LEUKEMIA DATA

Study Number	Estimated Weibull Parameters*		Best Weighting Scheme	χ^2 G-O-F	D.F.	P-Value
	$\hat{\lambda}_0$(s.e.)	$\hat{\lambda}_1$(s.e.)				
427	0.4636 (0.2439)	0.3384 (0.0931)	$w_i = 1$	40.003	14	0.00
555	0.1910 (0.0804)	0.4872 (0.0834)	$w_i = n_i h_i$	20.031	12	0.07
557 and 560	0.5124 (0.1339)	0.3264 (0.0467)	$w_i = 1/\hat{v}_i$	84.001	15	0.00
7315	0.2680 (0.0772)	0.3999 (0.0545)	$w_i = 1/\hat{v}_i$	30.398	12	0.00
7416	0.0924 (0.0187)	0.5761 (0.0017)	$w_i = n_i h_i$	34.070	11	0.00

*Gehan *et al.* used the following parameterization for the Weibull survival function: $S(t) = \exp\{-(\lambda_0 t)^{\lambda_1}\}$.

5.6 A Reassessment of the Fit of the Weibull Distribution

The fit of the Weibull distribution is evaluated against log-generalized gamma and Burr XII alternatives, by means of generalized likelihood ratio statistics. The log-generalized gamma distribution reduces to the Weibull distribution for the untransformed failure times when the shape parameter q is equal to one. Hence, the generalized likelihood ratio statistic for comparing the fit of the Weibull model versus the log-generalized gamma model has the form $\tau = -2\ell nR$, where

$$R = \frac{L_{GG}(\hat{\alpha},\hat{\sigma},q=1)}{L_{GG}(\hat{\alpha},\hat{\sigma},\hat{q})} \tag{5.12}$$

and $L_{GG}(\cdot)$ denotes the likelihood function for the log-generalized gamma distribution.

As shown in Section 2.2.1, the Burr Type XII distribution is restricted to the Weibull distribution by the parameter constraint

$$\beta^*/(1-\beta^*) = \theta_w\{k^*/(1-k^*)\}^{1/c} , \tag{5.13}$$

as k^* approaches one. In the above expression c denotes the Burr shape parameter, k^* the reparameterized Burr shape parameter, β^* the reparameterized Burr scale parameter, and θ_w the Weibull scale parameter. The generalized likelihood ratio statistic for comparing the fit of the Burr distribution restricted to the Weibull region versus the fit of the unrestricted Burr distribution is equal to $\lambda_w' = -2\ell nR_w'$, where

$$R_w' = \frac{L_{RB}\left\{\hat{c},\hat{k}^*,\left(\frac{\beta^*}{1-\beta^*}\right) = \hat{\theta}_w\left(\frac{\hat{k}^*}{1-\hat{k}^*}\right)^{1/\hat{c}}\right\}}{L_B(\hat{c},\hat{k}^*,\hat{\beta}^*)}, \tag{5.14}$$

and $L_{RB}(\cdot)$ and $L_B(\cdot)$ denote the restricted Burr and unrestricted Burr likelihood functions, respectively.

Tables 5.3 through 5.7 present the maximum likelihood solutions to the log-generalized gamma and the Burr XII likelihood equations. Table 5.8 is a summary table of the values of the likelihood ratio test statistics, the degrees of freedom for the approximating chi-square distribution, and the corresponding p-values for testing the fit of the Weibull distribution against the log-generalized gamma distribution, the Burr XII distribution, or the multinomial distribution for grouped data.

The inferences regarding distributional shape for the Study 427 data based on the three generalized distributions are in agreement. The Weibull distribution was rejected when compared with the log-generalized gamma, the Burr XII, and the multinomial (for grouped data) alternatives at the 0.01 significance level. With regard to the Study 555 data, the log-generalized gamma and Burr XII distributions both provided a significantly better fit than did the Weibull model, where the significance level attained was 0.01. However, when Gehan et al. [1980] grouped the data, they found that the Weibull distribution adequately described the grouped data when compared to the more general multinomial model. The p-value of 0.07 for the likelihood ratio test comparing the two models for grouped data suggested, however, that the Weibull fit was questionable. In addition, there may have been a loss of power associated with the grouping of the data. The three methods for assessing the fit of the Weibull distribution for the data from the combined Studies 557 and 560 each rejected the Weibull model. For the 7315 data the two general parametric distributions led to different conclusions. The fit of the Weibull model was marginally acceptable (p=0.07) when compared to the fit of the more general log-generalized gamma distribution. However, the Weibull distribution was rejected at the 0.05 significance level

TABLE 5.3

ANALYSIS OF THE STUDY 427 ACUTE LEUKEMIA DATA

Underlying Failure Distribution	Log-Generalized Gamma Distribution			Burr Type XII Distribution		
	Failures Only	Failures and Censorings	Restricted LGG (Weibull)	Weibull	Unrestricted Burr	Restricted Burr (Weibull)
Initial Estimates	$\alpha_0 = 0.0$ $\sigma_0 = 1.0$ $q_0 = 1.0$	$\alpha_0 = 2.5$ $\sigma_0 = 2.0$ $q_0 = 0.75$	$\alpha_0 = 2.506$ $\sigma_0 = 2.307$ $q \equiv 1$	$c_{w0} = 0.555$ $\theta_{w0} = 14.3$	$c_0 = 0.555$ $k_0^* = 0.909$ $\beta_0^* = 0.999$	$c_0 = 0.680$ $k_0^* = 0.568$ $\beta_0^* = 0.968$
Initial Log-Likelihood	-5268.1967	-407.0982	-409.4969	-706.6504	-705.2068	-700.0732
Number of Iterations	20	8	10	5	20	4
Maximized Log-Likelihood	-378.4259	-401.8809	-405.3378	-693.1912	-687.1636	-691.5579
Maximum Likelihood Estimates (s.e.)	$\hat{\alpha}$=2.532 (0.242) $\hat{\sigma}$=2.043 (0.131) $\hat{q}$=0.752 (0.172)	$\hat{\alpha}$=2.506 (0.022) $\hat{\sigma}$=2.307 (-0.000) $\hat{q}$=0.536 (-0.000)	$\hat{\alpha}$=3.017 (0.173) $\hat{\sigma}$=2.168 (0.127) $q \equiv 1$	$\hat{c}_w$=0.4614 (0.0270) $\hat{\theta}_w$=20.43 (3.54)	$\hat{c}$ =0.6805 (0.0870) $\hat{k}^*$=0.5685 (0.0916) $\hat{\beta}^*$=0.9256 (0.0575)	$\hat{c}$ =0.5207 (0.0444) $\hat{k}^*$=0.8516 (0.0789) $\hat{\beta}^*$=0.9983 (0.0024)
Estimates of k and β based on $\hat{k}^*$ and $\hat{\beta}^*$	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$k \approx 1.316$ $\beta \approx 12.44$	$k \approx 5.738$ $\beta \approx 587.2$
Parameter Constraints	Not Applicable	Not Applicable	$q = 1$	Not Applicable	Not Applicable	$\left(\frac{\beta^*}{1-\beta^*}\right)=20.43\left(\frac{k^*}{1-k^*}\right)^{1/c}$
Iterative Procedure	Direct Search	Direct Search	Direct Search	Newton-Raphson	Newton-Raphson	Newton-Raphson

TABLE 5.4

ANALYSIS OF THE STUDY 555 ACUTE LEUKEMIA DATA

Underlying Failure Distribution	Log-Generalized Gamma Distribution			Burr Type XII Distribution		
	Failures Only	Failures and Censorings	Restricted LGG (Weibull)	Weibull	Unrestricted Burr	Restricted Burr (Weibull)
Initial Estimates	$\alpha_0 = 0.0$ $\sigma_0 = 1.0$ $q_0 = 1.0$	$\alpha_0 = 3.3$ $\sigma_0 = 1.5$ $q_0 = 0.97$	$\alpha_0 = 3.120$ $\sigma_0 = 2.063$ $q = 1$	$c_{w0} = 0.700$ $\theta_{w0} = 24.6$	$c_0 = 0.700$ $k_0^* = 0.909$ $\beta_0^* = 0.999$	$c_0 = 0.790$ $k_0^* = 0.533$ $\beta_0^* = 0.979$
Initial Log-Likelihood	-4900.2881	-339.6136	-328.3019	-693.6369	-667.6673	-669.3232
Number of Iterations	16	12	10	11	18	4
Maximized Log-Likelihood	-280.6849	-318.1931	-322.6210	-663.3482	-656.9335	-661.4363
Maximum Likelihood Estimates (s.e.)	$\hat{\alpha}=3.294$ (0.248) $\hat{\sigma}=1.547$ (0.137) $\hat{q}=0.970$ (0.255)	$\hat{\alpha}=3.120$ (0.250) $\hat{\sigma}=2.063$ (0.141) $\hat{q}=0.400$ (0.184)	$\hat{\alpha}=3.694$ (0.166) $\hat{\sigma}=1.896$ (0.125) $q=1$	$\hat{c}_w=0.5276$ (0.0347) $\hat{\theta}_w=40.21$ (6.66)	$\hat{c}=0.7896$ (0.1076) $\hat{k}^*=0.5332$ (0.1006) $\hat{\beta}^*=0.9524$ (0.0356)	$\hat{c}=0.6045$ (0.0553) $\hat{k}^*=0.8231$ (0.0856) $\hat{\beta}^*=0.9980$ (0.0022)
Estimates of k and β based on k^* and β^*	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$k \approx 1.142$ $\beta \approx 20.03$	$k \approx 4.653$ $\beta \approx 499.0$
Parameter Constraints	Not Applicable	Not Applicable	$q = 1$	Not Applicable	Not Applicable	$\left(\frac{\beta^*}{1-\beta^*}\right)=40.21\left(\frac{k^*}{1-k^*}\right)^{1/c}$
Iterative Procedure	Direct Search	Direct Search	Direct Search	Direct Search	Newton-Raphson	Newton-Raphson

TABLE 5.5

ANALYSIS OF THE STUDIES 557 AND 560 ACUTE LEUKEMIA DATA

Underlying Failure Distribution	Log-Generalized Gamma Distribution			Burr Type XII Distribution		
	Failures Only	Failures and Censorings	Restricted LGG (Weibull)	Weibull	Unrestricted Burr	Restricted Burr (Weibull)
Initial Estimates	$\alpha_0 = 0.0$ $\sigma_0 = 1.0$ $q_0 = 1.0$	$\alpha_0 = 2.8$ $\sigma_0 = 1.9$ $q_0 = 0.77$	$\alpha_0 = 2.858$ $\sigma_0 = 2.374$ $q \equiv 1$	$c_{w0} = 0.590$ $\theta_{w0} = 19.3$	$c_0 = 0.590$ $k_0^* = 0.909$ $\beta_0^* = 0.999$	$c_0 = 0.766$ $k_0^* = 0.461$ $\beta_0^* = 0.965$
Initial Log-Likelihood	-6281.9716	-466.9536	-464.0348	-855.8437	-835.1260	-842.4307
Number of Iterations	15	11	7	12	31	4
Maximized Log-Likelihood	-399.0199	-451.1725	-456.7148	-825.7833	-816.0842	-823.8385
Maximum Likelihood Estimates (s.e.)	$\hat{\alpha}$=2.793 (0.218) $\hat{\sigma}$=1.900 (0.118) $\hat{q}$=0.768 (0.168)	$\hat{\alpha}$=2.858 (0.229) $\hat{\sigma}$=2.374 (0.140) $\hat{q}$=0.410 (0.135)	$\hat{\alpha}$=3.501 (0.164) $\hat{\sigma}$=2.169 (0.125) $q \equiv 1$	$\hat{c}_w$=0.4611 (0.0269) $\hat{\theta}_w$=33.16 (5.44)	$\hat{c}$ =0.7657 (0.0990) $\hat{k}^*$=0.4614 (0.0758) $\hat{\beta}^*$=0.8898 (0.0605)	$\hat{c}$ =0.5194 (0.0416) $\hat{k}^*$=0.8315 (0.0803) $\hat{\beta}^*$=0.9986 (0.0018)
Estimates of k and β based on k^* and β^*	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$k \approx 0.8567$ $\beta \approx 8.074$	$k \approx 4.935$ $\beta \approx 713.3$
Parameter Constraints	Not Applicable	Not Applicable	$q = 1$	Not Applicable	Not Applicable	$\left(\frac{\beta^*}{1-\beta^*}\right)=33.16\left(\frac{k^*}{1-k^*}\right)^{1/c}$
Iterative Procedure	Direct Search	Direct Search	Direct Search	Direct Search	Newton-Raphson	Newton-Raphson

TABLE 5.6

ANALYSIS OF THE STUDY 7315 ACUTE LEUKEMIA DATA

Underlying Failure Distribution	Log-Generalized Gamma Distribution: Failures Only	Log-Generalized Gamma Distribution: Failures and Censorings	Log-Generalized Gamma Distribution: Restricted LGG (Weibull)	Burr Type XII Distribution: Weibull	Burr Type XII Distribution: Unrestricted Burr	Burr Type XII Distribution: Restricted Burr (Weibull)
Initial Estimates	$\alpha_0 = 0.0$ $\sigma_0 = 1.0$ $q_0 = 1.0$	$\alpha_0 = 3.4$ $\sigma_0 = 1.6$ $q_0 = 1.2$	$\alpha_0 = 3.528$ $\sigma_0 = 2.246$ $q_0 \equiv 1$	$c_{w0} = 0.724$ $\theta_{w0} = 24.8$	$c_0 = 0.724$ $k_0^* = 0.909$ $\beta_0^* = 0.999$	$c_0 = 0.662$ $k_0^* = 0.559$ $\beta_0^* = 0.986$
Initial Log-Likelihood	-5241.2016	-420.1057	-394.1546	-801.5016	-756.3502	-746.0829
Number of Iterations	18	18	10	14	19	4
Maximized Log-Likelihood	-323.2971	-389.7182	-391.3962	-739.2296	-735.6851	-738.5665
Maximum Likelihood Estimates (s.e.)	$\hat{\alpha}=3.376$ (0.243) $\hat{\sigma}=1.567$ (0.142) $\hat{q}=1.195$ (0.250)	$\hat{\alpha}=3.528$ (0.263) $\hat{\sigma}=2.246$ (0.175) $\hat{q}=0.624$ (0.191)	$\hat{\alpha}=3.899$ (0.167) $\hat{\sigma}=2.052$ (0.134) $q \equiv 1$	$\hat{c}_w=0.4874$ (0.0319) $\hat{\theta}_w=49.34$ (8.23)	$\hat{c}=0.6624$ (0.0865) $\hat{k}^*=0.5595$ (0.1100) $\hat{\beta}^*=0.9684$ (0.0306)	$\hat{c}=0.5209$ (0.0445) $\hat{k}^*=0.8790$ (0.1002) $\hat{\beta}^*=0.9996$ (0.0009)
Estimates of k and β based on k^* and β^*	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$k \approx 1.270$ $\beta \approx 30.64$	$k \approx 7.264$ $\beta \approx 2499.$
Parameter Constraints	Not Applicable	Not Applicable	$q = 1$	Not Applicable	Not Applicable	$\left(\frac{\beta^*}{1-\beta^*}\right)=49.34\left(\frac{k^*}{1-k^*}\right)^{1/c}$
Iterative Procedure	Direct Search	Direct Search	Direct Search	Direct Search	Newton-Raphson	Newton-Raphson

TABLE 5.7

ANALYSIS OF THE STUDY 7416 ACUTE LEUKEMIA DATA

Underlying Failure Distribution	Log-Generalized Gamma Distribution			Burr Type XII Distribution		
	Failures Only	Failures and Censorings	Restricted LGG (Weibull)	Weibull	Unrestricted Burr*	Restricted Burr* (Weibull)
Initial Estimates	-	-	-	$c_{w0} = 0.683$ $\theta_{w0} = 28.8$	$c_0 = 0.567$ $k'_0 = 1.0\ 10^{-9}$ $\beta'_0 = 2.0\ 10^{-18}$	$c_0 = 0.577$ $k'_0 = 0.054$ $\beta'_0 = 0.0001$
Initial Log-Likelihood	-	-	-	-2293.8343	-2202.8326	-2203.2566
Number of Iterations	-	-	-	11	5	11
Maximized Log-Likelihood	-	-	-	-2202.8326	-2202.8326	-2202.8326
Maximum Likelihood Estimates (s.e.)	-	-	-	$\hat{c}_w$=0.5671 (0.0228) $\hat{\theta}_w$=61.53 (5.18)	$\hat{c}$ = 0.5671 $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b.	$\hat{c}$ = 0.5671 $\hat{k}'$ converged to l.b. $\hat{\beta}'$ converged to l.b.
Estimates of k and β based on k^* and β^*	-	-	-	Not Applicable	Not Applicable	Not Applicable
Parameter Constraints	-	-	-	Not Applicable	Not Applicable	$(1/\beta')=61.53(1/k')^{1/c}$
Iterative Procedure	-	-	-	Direct Search	Direct Search	Direct Search

*Computational difficulties arise when k' and β' converge to their respective lower bounds (l.b.) of 0.00+. To ensure that the maximized log-likelihood value under the unrestricted Burr model is at least as large as that under the Burr model restricted to the Weibull region, the final estimates of the restricted Burr model (Weibull region) are used as initial estimates in the unrestricted Burr model. Standard errors could not be computed.

TABLE 5.8

GENERALIZED LIKELIHOOD RATIO TEST STATISTICS AND P-VALUES FOR THE EVALUATION OF THE FIT OF THE WEIBULL DISTRIBUTION TO THE ACUTE LEUKEMIA DATA

Study Number	Null Distribution	Alternative Distribution	Test Statistic*	Chi-Square Value	Degrees of Freedom	P-Value
427	Weibull	Burr XII	λ_W'	8.789	1	0.00
	Weibull	Log-Generalized Gamma	τ	6.914	1	0.01
	Weibull (Grouped Data)	Multinomial	G-S χ^2	40.003	14	0.00
555	Weibull	Burr XII	λ_W'	9.006	1	0.00
	Weibull	Log-Generalized Gamma	τ	8.856	1	0.00
	Weibull (Grouped Data)	Multinomial	G-S χ^2	20.031	12	0.07
557 and 560	Weibull	Burr XII	λ_W'	15.509	1	0.00
	Weibull	Log-Generalized Gamma	τ	11.085	1	0.00
	Weibull (Grouped Data)	Multinomial	G-S χ^2	84.001	15	0.00
7315	Weibull	Burr XII	λ_W'	5.763	1	0.02
	Weibull	Log-Generalized Gamma	τ	3.356	1	0.07
	Weibull (Grouped Data)	Multinomial	G-S χ^2	30.398	12	0.00
7416+	Weibull	Burr XII	λ_W'	0.000	1	1.00
	Weibull	Log-Generalized Gamma	τ	--	--	--
	Weibull (Grouped Data)	Multinomial	G-S χ^2	34.070	11	0.00

*The Gehan and Siddiqui [1973] goodness-of-fit χ^2 statistic is denoted by G-S χ^2.

+The numerical integration subroutine DCADRE failed for the log-generalized gamma analysis of the 7416 data due to a reported ill-behaved integrand. An unsuccessful attempt to avoid the computational difficulties was made by lowering the specified degree of accuracy.

against the Burr XII alternative. The results of the Burr XII analysis are supported by the significant ($p=0.00$) likelihood ratio test comparing the fit of the Weibull versus the multinomial model for the grouped data.

It was not possible to fit the log-generalized gamma distribution to the Study 7416 data because of computational difficulties. The numerical integration subroutine DCADRE failed due to a reported ill-behaved integrand. This data set had the highest proportion of censored observations among the five acute leukemia data sets. It is probable that the numerical difficulties are related to this. An unsuccessful attempt to avoid the computational difficulties was made by lowering the specified degree of accuracy.

Based on the analysis using the Burr Type XII distribution, the data of Study 7416 were found to behave as Weibull data. The maximized log-likelihood values, as well as the maximum likelihood estimates of the Weibull power parameter, were identical under the Weibull, the unrestricted Burr, and the restricted Burr models (see Table 5.7). Moreover, the maximum likelihood estimate of the Weibull power parameter for the grouped analysis performed by Gehan _et al_. (see Table 5.2) agreed very closely with the power parameter estimates under the Weibull, the unrestricted Burr, and the restricted Burr models for the exact survival times. There was agreement between estimators for the grouped and ungrouped analyses for this data set only. While the Weibull distribution was the best fitting model among those considered by Gehan _et al_. for the grouped survival times, the chi-square goodness-of-fit test rejected the Weibull distribution with a p-value of 0.00. Plots of the survival times versus the fitted Weibull and Burr XII survival functions, which will be discussed subsequently, tend to confirm the comparability of the fits of the Weibull and Burr XII models for the ungrouped data.

In summary, it was found that the three methods of assessing distributional shape were in agreement for the data of Study 427 and the combined Studies 557 and 560. The three data sets for which the model selection procedures pointed to different conclusions are Study 555, Study 7315, and Study 7416. Plots of the observed survival or censoring times versus the fitted Weibull, Burr Type XII, and log-generalized gamma models have been generated to determine visually which of these three distributions provided the best fit.

The plots in Figures 5.1 through 5.3 display the observed survival and censoring times, measured in weeks, of the Study 555 data, as well as the fitted survival functions corresponding to the Weibull, Burr Type XII, and log-generalized gamma distributions, respectively. Based on the generalized likelihood ratio tests, the Weibull distribution was rejected for these data against the Burr XII and log-generalized gamma alternatives and marginally accepted over the multinomial model for grouped data. A visual comparison of the fit of the Weibull distribution with that of the two generalized distributions shows that the log-generalized gamma model provides a superior fit. There appears to be little if any advantage associated with fitting the Burr XII distribution. While the fitted Burr distribution seems to be a better predictor of survival times in the 50 to 100 week range, the Weibull distribution is superior in the 30 to 50 week interval. These results suggest that the generalized likelihood ratio tests are more sensitive to departures from the Weibull distribution than are the visual impressions from plots such as these.

FIGURE 5.1
STUDY 555 SURVIVAL TIMES AND FITTED WEIBULL DISTRIBUTION

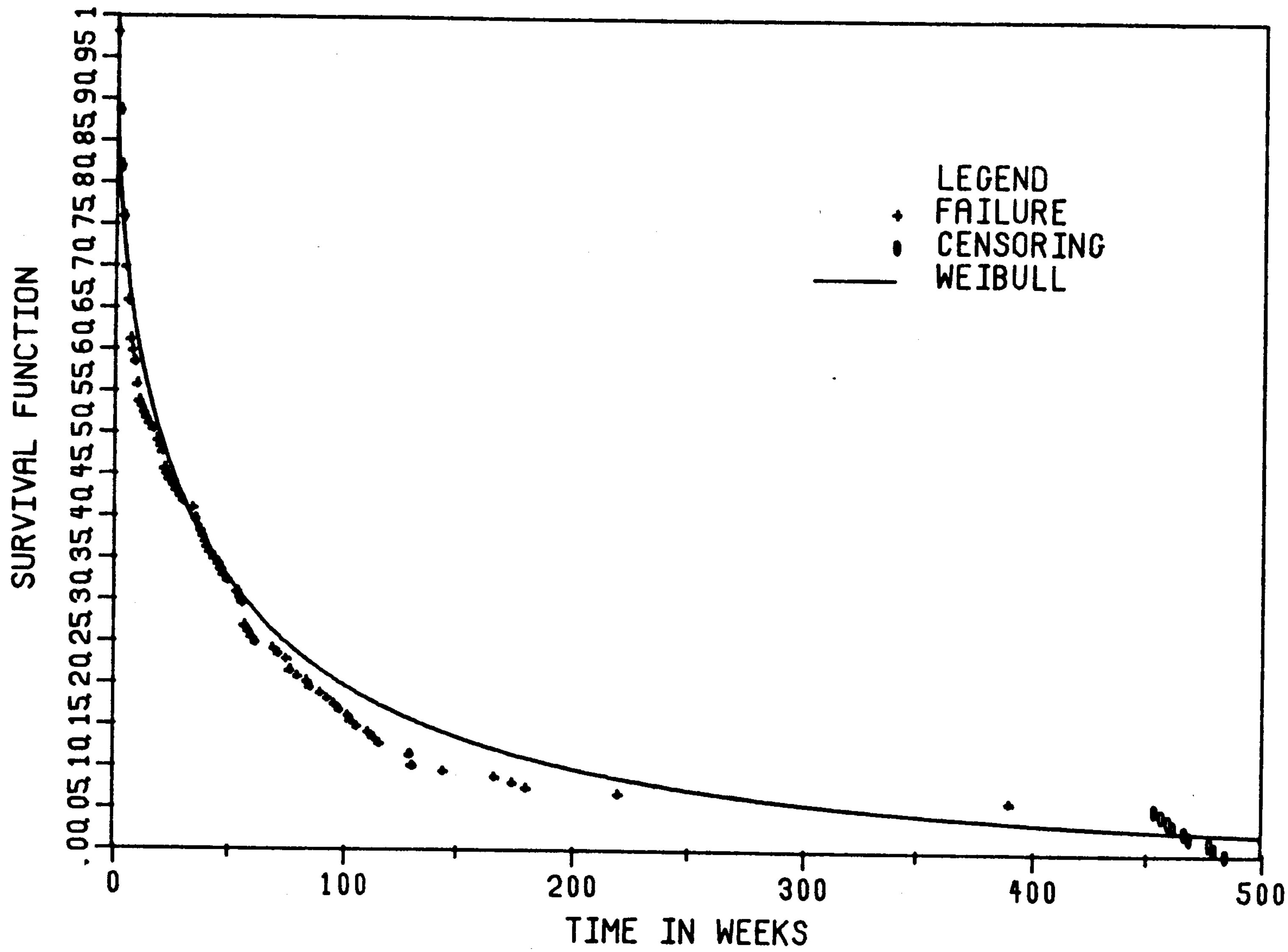


FIGURE 5.2
STUDY 555 SURVIVAL TIMES AND FITTED BURR XII DISTRIBUTION

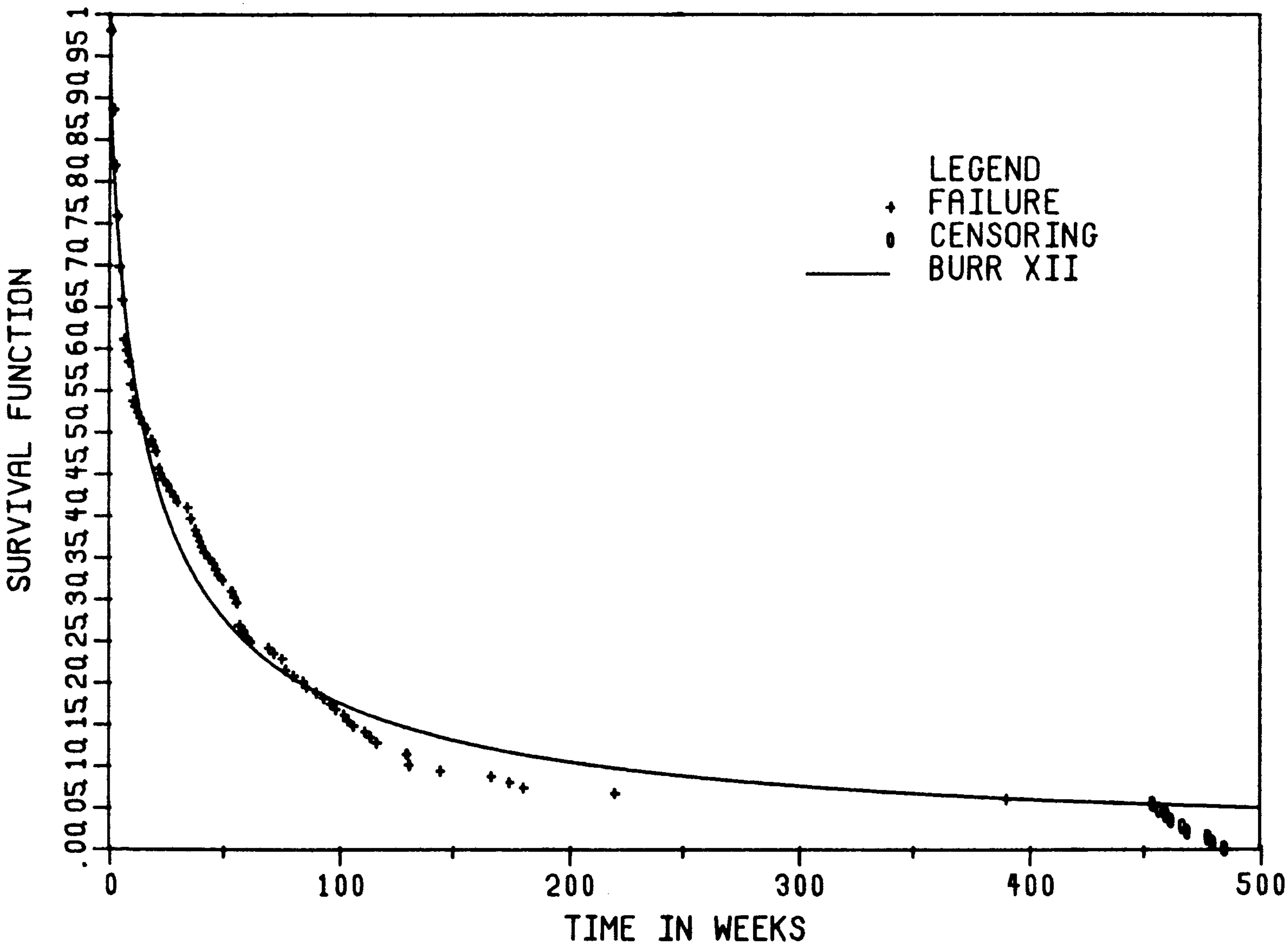
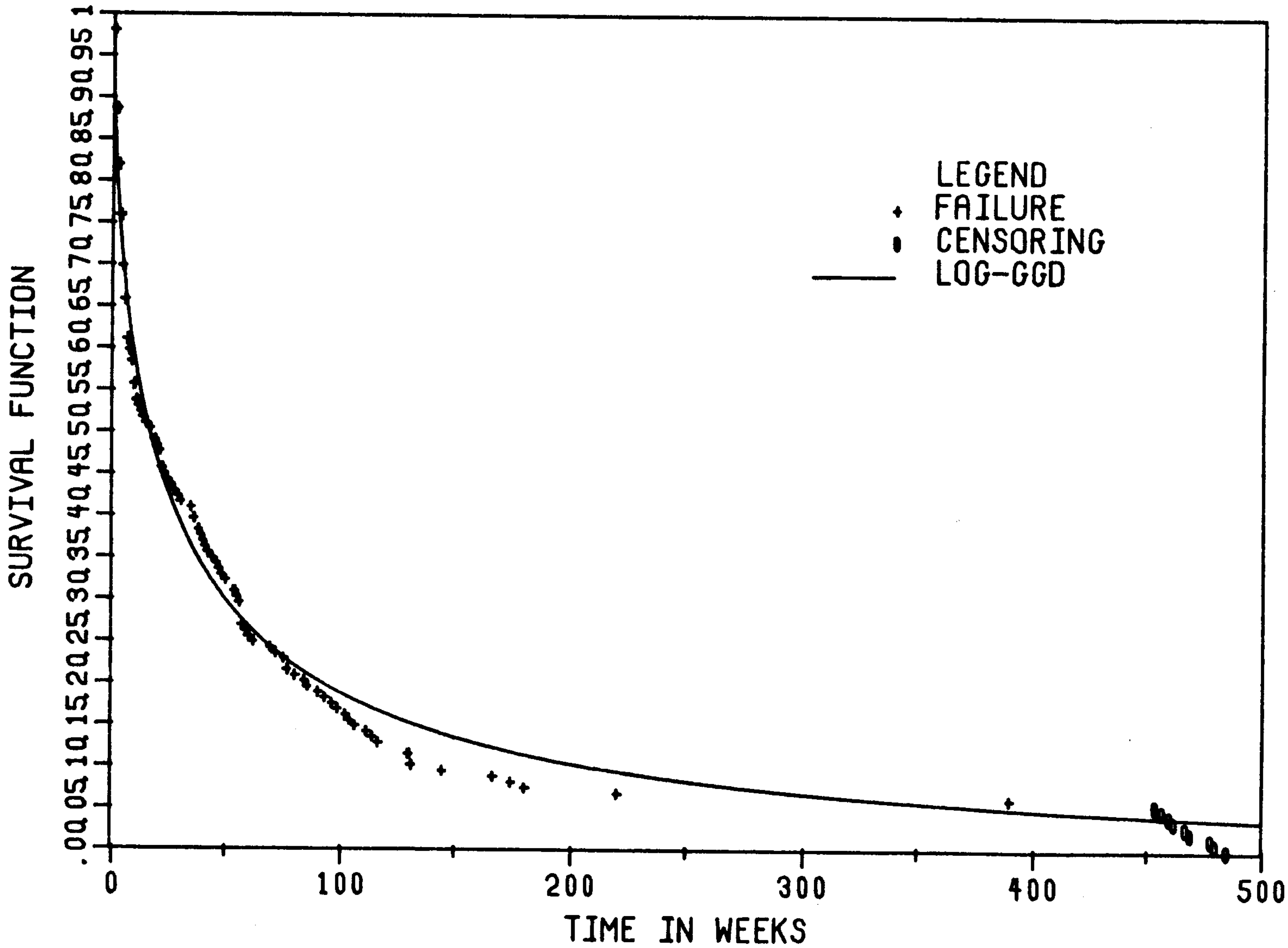


FIGURE 5.3
STUDY 555 SURVIVAL TIMES AND LOG-GENERALIZED GAMMA DISTRIBUTION



Figures 5.4 through 5.6 contain plots of the observed 7315 data and the fitted survival curves of the Weibull, Burr XII, and log-generalized gamma distributions, respectively. On the basis of the generalized likelihood ratio tests, the Burr XII distribution provided a significantly better fit than did the Weibull distribution, in contrast to the log-generalized gamma distribution which did not provide a better fit. The chi-square goodness-of-fit test for grouped data rejected the fit of the Weibull model against the multinomial alternative. Based on the plots it is clearly seen that the Weibull distribution provides an inferior fit when compared to the two generalized distributions. Moreover, the Burr XII distribution, which appeared to be more sensitive to departures from the Weibull model, appears to fit the data slightly better than does the log-generalized gamma distribution. The plots for the 7315 data support the conclusions of the generalized likelihood ratio tests discussed previously.

The survival times of Study 7416 as well as the fitted survival curves of the Weibull and Burr XII distributions are plotted in Figures 5.7 and 5.8, respectively. These plots confirm the conclusion of the generalized likelihood ratio test that the Weibull distribution provides a satisfactory fit when compared to the more general Burr XII distribution.

Whereas in the examples of Chapter IV the data were found to behave as Weibull data, four of the five adult acute leukemia data sets provide the opportunity of observing the characteristics of the unrestricted and restricted Burr XII parameters when the underlying distribution does not belong to the Weibull family. Tables 5.3 through 5.6 give the maximum likelihood estimates of c, k, and β for the unrestricted and restricted (Weibull region) Burr distributions fit to the Study 427, 555, 557 and 560 (combined), and 7315 data sets, respectively.

FIGURE 5.4
STUDY 7315 SURVIVAL TIMES AND FITTED WEIBULL DISTRIBUTION

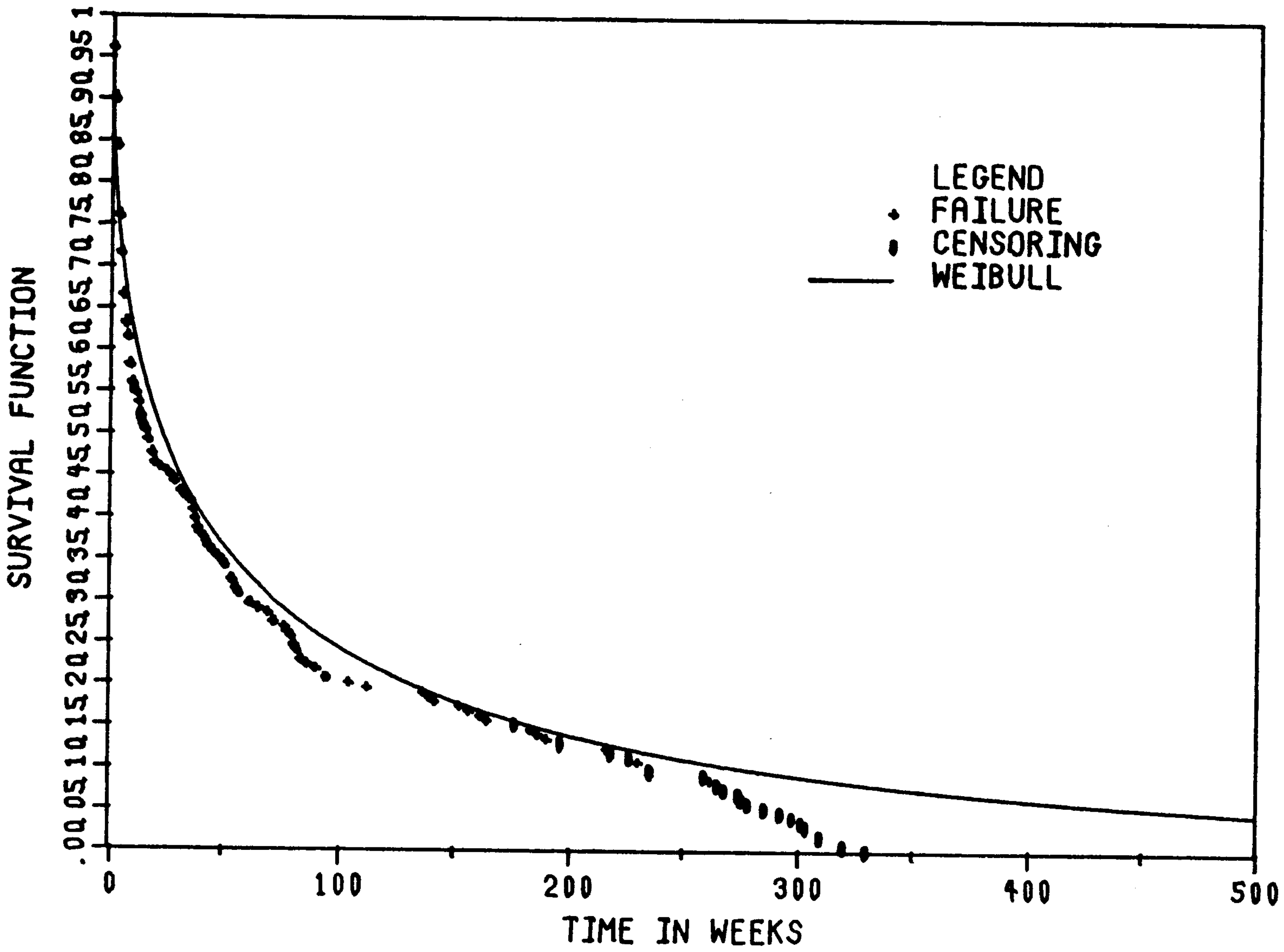


FIGURE 5.5
STUDY 7315 SURVIVAL TIMES AND FITTED BURR XII DISTRIBUTION

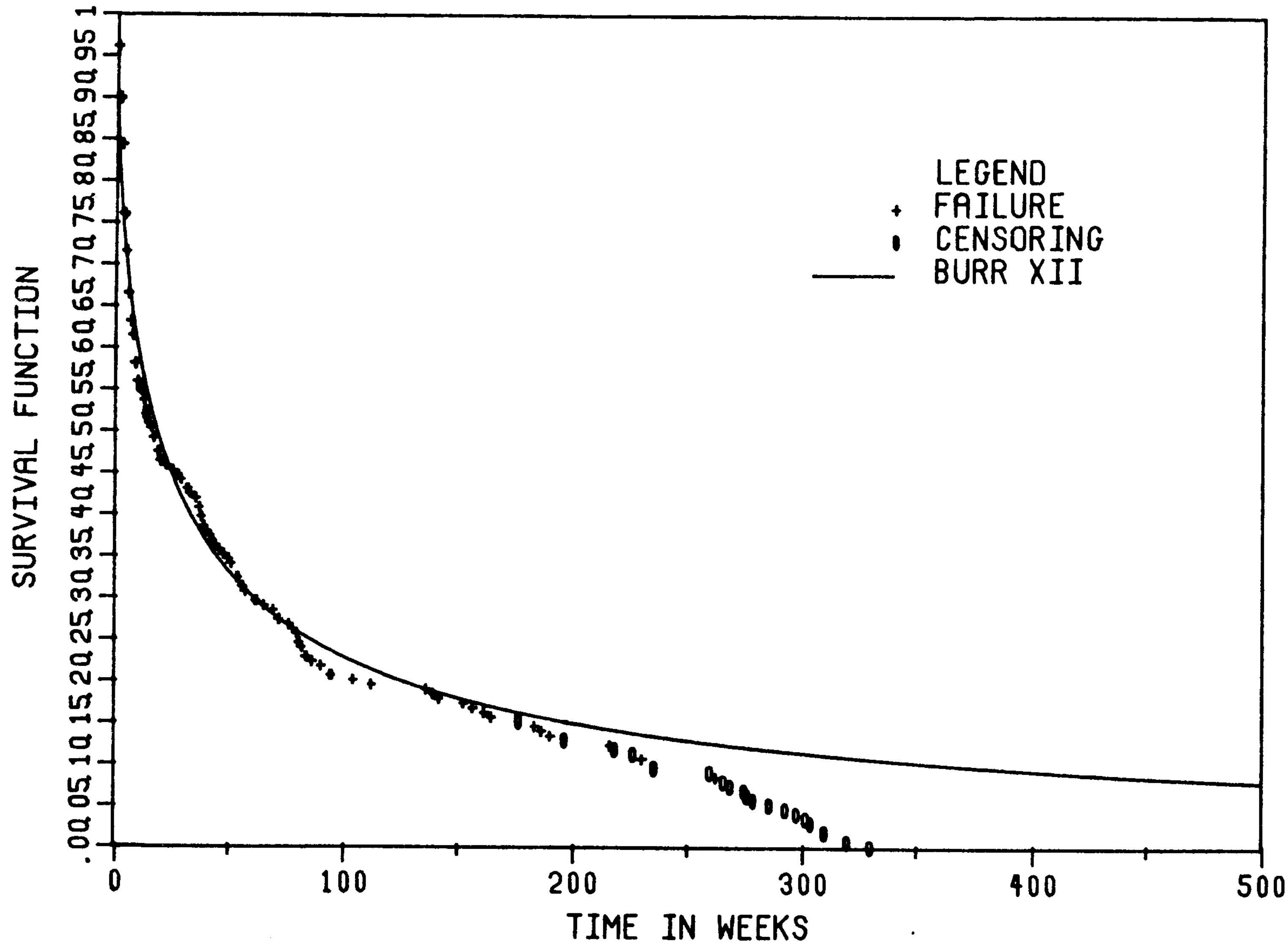


FIGURE 5.6
STUDY 7315 SURVIVAL TIMES AND LOG-GENERALIZED GAMMA DISTRIBUTION

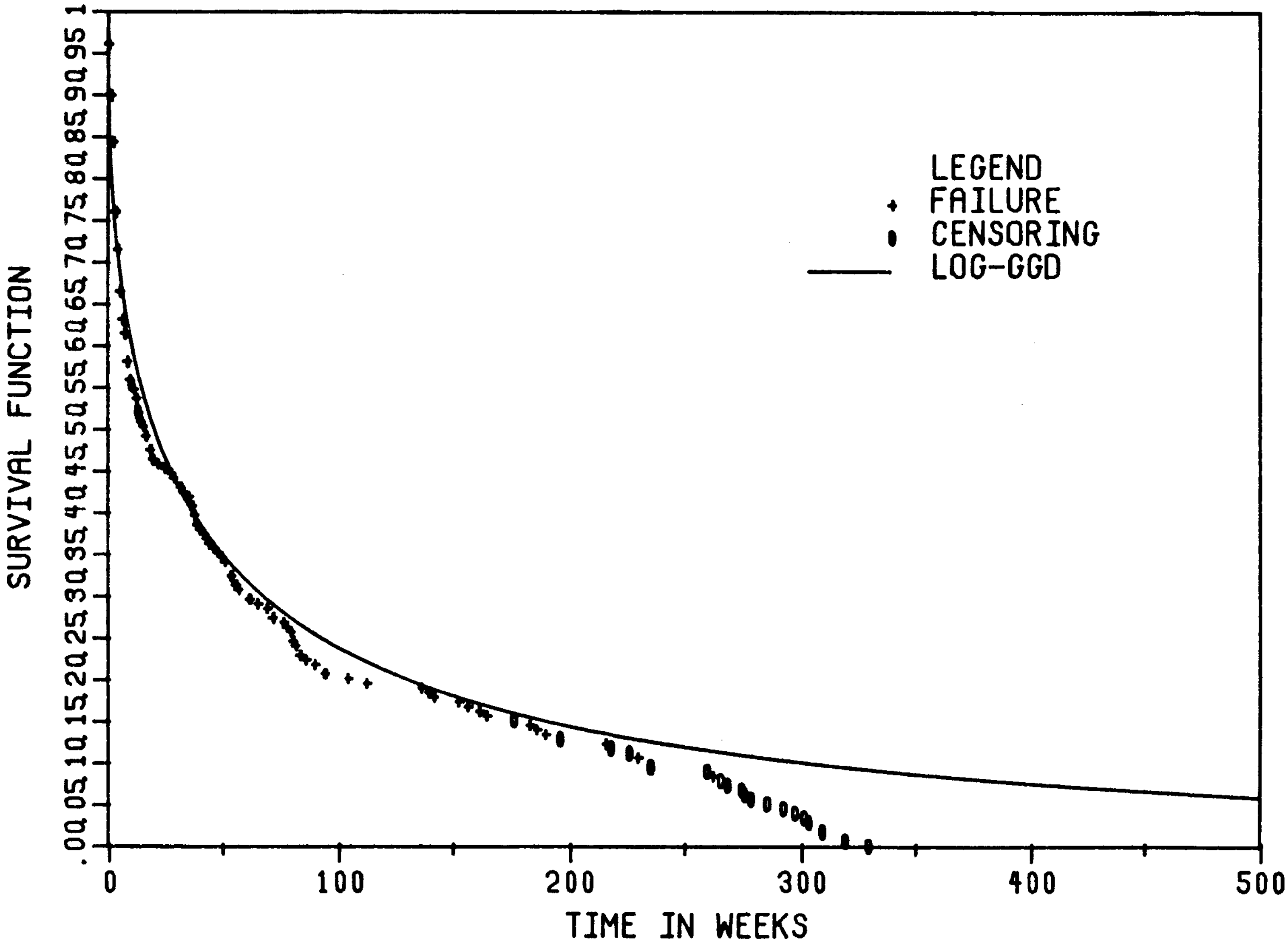


FIGURE 5.7
STUDY 7416 SURVIVAL TIMES AND FITTED WEIBULL DISTRIBUTION

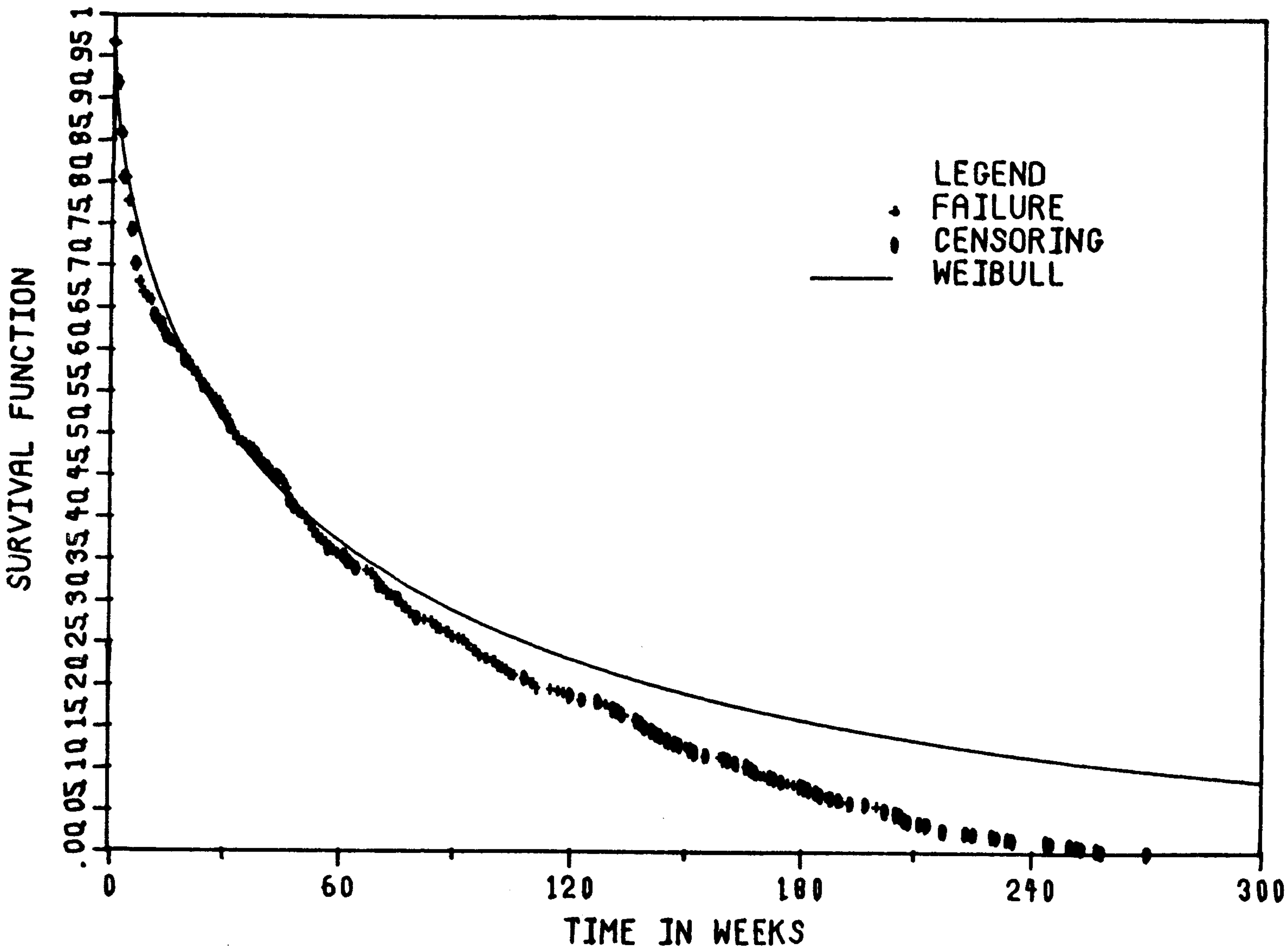
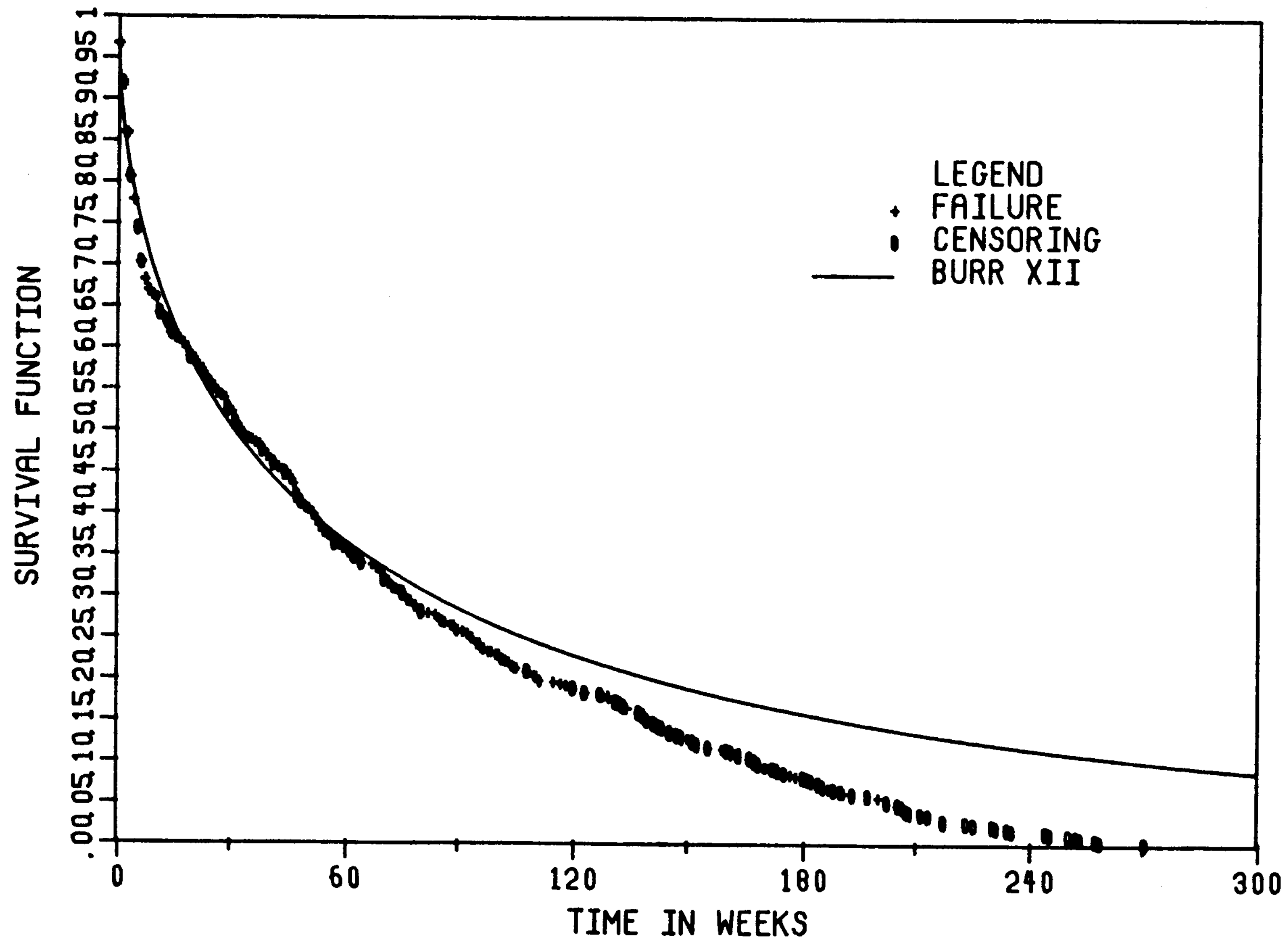


FIGURE 5.8
STUDY 7416 SURVIVAL TIMES AND FITTED BURR XII DISTRIBUTION



For the purpose of illustration, if we direct our attention to the Study 555 data set, the maximum likelihood estimates under the three investigative models are as follows:

Weibull	Unrestricted Burr	Restricted Burr (Weibull Region)
$\hat{c}_w = 0.5276$	$\hat{c} = 0.7896$	$\hat{c} = 0.6045$
$\hat{\theta}_w = 40.21$	$\hat{k}^* = 0.5332$	$\hat{k}^* = 0.8231$
	$\hat{\beta}^* = 0.9524$	$\hat{\beta}^* = 0.9980$.

Several points are noteworthy. In Section 2.2.1 it was shown that as the Burr Type XII distribution approaches the Weibull limiting case, the shape parameter k approaches infinity, the scale parameter β approaches infinity according to the functional constraint $\beta = \theta_w k^{1/c}$, and the shape parameter c approaches the value of the Weibull power parameter c_w. With regard to the unrestricted Burr maximum likelihood estimators for the 555 data, $\hat{k}^*$ equal to 0.5332 (or $k \approx 1.142$) is not large enough to suggest that these data are Weibull data. In the restricted Burr model, where β is replaced by $\hat{\theta}_w k^{1/c}$, $\hat{k}^*$ is larger. Moreover, $\hat{c}$ under the restricted Burr model is more similar in value to the maximum likelihood estimate of the Weibull shape parameter c_w than is $\hat{c}$ under the unrestricted Burr model. Under the null hypothesis that the data follow the Weibull distribution, the Weibull scale parameter θ_w may be estimated from the unrestricted Burr maximum likelihood estimators according to the equation

$$\theta_w \approx \hat{\beta}/\hat{k}^{1/\hat{c}} \quad . \tag{5.15}$$

For the Study 555 data the estimate of θ_w based on equation (5.15) is equal to 16.95. In contrast, under the Weibull model, the maximum likelihood estimate of θ_w is 40.21. When the restricted Burr maximum likelihood estimators are substituted in equation (5.15) the corresponding estimate of θ_w is equal to 39.22. These findings suggest that while the results of an unrestricted Burr analysis may indicate that the data are non-Weibull, when the scale parameter β is replaced by $\hat{\theta}_w k^{1/c}$, the solutions to the Burr system of likelihood equations tend to converge to estimates which are closer to the Weibull region. The analyses of the data from Studies 427, 557 and 560 (combined), and 7315 all exhibited similar findings.

A final observation regarding the maximum likelihood estimates of the unrestricted Burr XII parameters for Studies 427, 555, 557 and 560 (combined), and 7315 is that in every case the product $\hat{c}\hat{k}$ is less than one. The rth moment about the origin for the Burr XII distribution is equal to

$$\mu_r' = kB(r/c+1,k-r/c), \tag{5.16}$$

where r is less than the product ck and B(a,b) denotes the standard beta function. Therefore, no moments exist for the fitted unrestricted Burr distributions. However, the plots of Figures 5.2 and 5.5 show that, despite this, the fitted Burr XII distributions provide reasonable fits.

5.7 A Cost Efficiency Comparison of the Log-Generalized Gamma and Burr XII Distributions

An evaluation of the usefulness of a general family of parametric distributions for a survival analysis or reliability study should take into account how costly it is to fit the distribution to actual data. When the log-generalized gamma and Burr XII distributions are fit by the method of maximum likelihood, an iterative procedure must be used to solve both systems of likelihood equations. An important cost consideration then is the number of iterations required for convergence.

The cost is also dependent upon the computational complexity of the log-likelihood function which must be evaluated in each iteration. For example, the numerical integration subroutine which is required to evaluate the contribution of a right-censored observation to the log-generalized gamma likelihood function inflates the time and cost of performing each iteration.

Table 5.9 tabulates the number of iterations required for convergence, the computer time required to fit the models, and the total computer costs for the analysis of four of the adult acute leukemia data sets using the log-generalized gamma and Burr XII distributions. In each analysis the Burr parameters were reparameterized according to (3.2). If the original parameterization had been used, the number of iterations and the cost associated with each Burr analysis would have been substantially greater. Since an analysis performed by either of the two general distributions requires three separate models to be fit, the reported values in Table 5.9 represent sums taken over the three models. For example, the log-generalized gamma model fit to the failures only, the unrestricted log-generalized gamma model, and the restricted log-generalized gamma model together required thirty-eight iterations to solve for the maximum likelihood solutions necessary for the computation of the generalized likelihood ratio statistic τ for the Study 427 data.

For the analyses of Studies 427, 555, and 557 and 560 combined, the log-generalized gamma and Burr Type XII distributions performed equally well with respect to computer cost considerations. However, the subroutine DCADRE required an excessive amount of time to evaluate the integral associated with each censored observation of the 7315 data

TABLE 5.9

A COMPARISON OF THE PERFORMANCE OF THE LOG-GENERALIZED GAMMA AND BURR XII DISTRIBUTIONS WITH REGARD TO COMPUTER TIME AND COSTS

Study	Criterion	Log-Generalized Gamma	Burr Type XII
427	Iterations*	38	29
	Time*	1:31.4	1:25.1
	Cost*	$5.93	$6.07
555	Iterations	38	33
	Time	1:46.0	1:09.1
	Cost	$6.68	$4.43
557 and 560	Iterations	33	47
	Time	2:18.7	2:03.5
	Cost	$8.72	$8.74
7315†	Iterations	46	37
	Time	18:58.8	1:19.0
	Cost	--	$5.05

*Each reported value for the log-generalized gamma distribution represents a sum over the three models required for a log-generalized analysis; each reported value for the Burr XII distribution represents a sum over the three models required for a Burr XII analysis.

†The subroutine DCADRE required an excessive amount of time to evaluate the integral associated with each censored observation under the log-generalized gamma model. It was necessary to run this job at a lower priority to reduce the cost. Therefore the cost factor is not comparable.

under the log-generalized gamma model. The computer job which performed this analysis had to be run using a low priority in order to reduce the cost. Hence, a major computational advantage offered by the Burr XII distribution over the log-generalized gamma distribution is that the Burr survival function exists in closed form. This savings becomes more substantial as the proportion of censored observations increases.

CHAPTER VI

A SIMULATION STUDY OF THE DISTRIBUTION OF THE LIKELIHOOD RATIO FOR TESTING THE EXPONENTIALITY OF DATA AGAINST BURR XII ALTERNATIVES

6.1. Introduction

This chapter deals with an investigation of the asymptotic behavior of a generalized likelihood ratio statistic when the true value of the parameter vector under the null distribution lies on the boundary of the parameter space of the alternative distribution. The Weibull and exponential distributions, which correspond to special limiting cases of the Burr Type XII distributional parameters, are the null distributions in question. The two-dimensional Weibull parameter subspace, given by

$$\{(c,\theta_w):\ 0<c<\infty,\ k\to\infty,\text{ and }\beta\to\infty\text{ such that }0<\theta_w=\beta k^{-(1/c)}<\infty\}, \tag{6.1}$$

and the one-dimensional exponential parameter subspace, given by

$$\{\theta_e: c=1,\ k\to\infty,\text{ and }\beta\to\infty\text{ such that }0<\theta_e=\beta/k<\infty\}, \tag{6.2}$$

lie on the boundary of the three-dimensional Burr XII parameter space

$$\{(c,k,\beta):\ 0<c<\infty,\quad 0<k<\infty,\quad 0<\beta<\infty\}\ . \tag{6.3}$$

This boundary condition does not provide a neighborhood in the Burr parameter space for the hypothesized parameter subspace, thereby violating one of the sufficient conditions of Wilks' [1938] theorem regarding the asymptotic chi-square approximation for the distribution of $-2\ell n\Lambda$. Recall, Λ represents the likelihood ratio, and the number of degrees of freedom is the number of components of the parameter space that are specified by the null distribution. However, analytical results of Sections 2.2.5 and 2.4 suggest that the generalized likelihood ratio statistics $\lambda'_w = -2\ln R'_w$ and $\lambda'_e = -2\ln R'_e$, where R'_w and R'_e are defined in (2.45) and (2.59), respectively, are distributed as chi-square random variables. Specifically, it was shown that the Burr XII system of likelihood equations, when restricted to the parameter subspaces defined in (6.1) and (6.2), does approach the limiting Weibull and exponential systems of likelihood equations, respectively.

Wolfe [1971] empirically investigated the sampling distribution of a likelihood ratio statistic R for testing the null hypothesis that the true "mixture" of multinormal distributions contains only one component versus the alternative hypothesis that the mixture consists of two multinormal distributions. Wilks' [1938] theorem is not in effect for this mixture problem since the null hypothesis can be viewed as the mixing proportion π being equal to zero, which is at the boundary of the closed set [0,1]. The simulations of Wolfe [1971] suggest that even in reasonably large samples the usual chi-square approximation does not provide an adequate fit for the distribution of $-2\ln R$.

A simulation was performed to investigate the nature of the distribution of the likelihood ratio statistic λ'_e for large samples. It was of interest to determine not only whether the chi-square distribution provided an adequate fit for λ'_e, but also whether the degrees of freedom differed from the predicted number of degrees of freedom based on the chi-square approximation when the parameter space under the null hypothesis lies in the interior of the full parameter space.

6.2 Special Considerations for the Simulation of the Distribution of λ_e' when the Data are Sampled from the Standard Exponential Distribution

The sampling distribution of the likelihood ratio statistic λ_e' was simulated for the following set of distributional hypotheses:

$$H_0: \quad P\{T \le t\} = F_0(t) = 1-\exp(-t)$$

$$H_1: \quad P\{T \le t\} = F_1(t;c,k,\beta) = 1-\{1+(t/\beta)^c\}^{-k} \ . \qquad (6.4)$$

Since the null distribution corresponds to the standard exponential distribution, the likelihood ratio statistic $-2\ell nR_e'$ will be denoted by λ_{se}' throughout the remainder of the chapter. The decision was made to generate 1000 "random" samples of size 200, in the range of the sample sizes for the examples of Chapters IV and V. The FORTRAN subroutine GGEXN of the International Mathematical Subroutine Library was used to generate the exponential random deviates. This subroutine incorporates a basic uniform pseudo-random number generator which is discussed by Learmonth and Lewis [1973].

In order to minimize the number of iterations to be performed by MAXLIK (Kaplan and Elston [1972]) to obtain the maximized log-likelihood values for the Burr XII model restricted to the standard exponential region as well as for the unrestricted Burr XII model, the following measures were taken. Hjorth's [1980] parameterization scheme was used to map the range of k and β from the positive real line onto the unit interval, as discussed in Section 3.2. The restricted Burr XII likelihood equations always attain convergence in fewer iterations than do the unrestricted Burr XII likelihood equations. Therefore, for each standard exponential random sample, the Burr distribution subject to the restrictions

$$\text{(i)} \quad c = 1$$
$$\text{(ii)} \quad \beta = k \,, \tag{6.5}$$

was fit using the initial estimates

$$c_0 = 1.0, \quad k_0^* = 0.909, \quad \text{and} \quad \beta_0^* = 0.909 \,. \tag{6.6}$$

These initial estimates lie in the standard exponential region, defined by (6.2) of the Burr parameter space. Then the maximum likelihood estimates of the restricted Burr XII parameters were used as initial estimates of the unrestricted Burr parameters.

The total computer time required to generate 1000 observations of λ'_{se} was seven hours and fifty-five minutes and the cost was $197. The computer jobs were run under a low priority, which minimizes charges by executing the jobs when the usage of the computer system is relatively low.

6.3 The Simulated Distribution of the Likelihood Ratio Test Statistic λ'_{se}

Selected percentiles η_α for the empirical distribution of λ'_{se} are presented in Table 6.1. Based on the usual asymptotic chi-square

TABLE 6.1

PERCENTILES η_α FOR THE EMPIRICAL DISTRIBUTION OF λ'_{se}
SUCH THAT $P\{\lambda'_{se} \le \eta_\alpha\} \le \alpha$

α	0.02	0.05	0.10	0.25	0.40	0.50	0.60	0.70
$\hat{\eta}_\alpha$	0.0000	0.0003	0.0014	0.0142	0.2373	0.5394	0.8447	1.2204

α	0.75	0.80	0.85	0.90	0.95	0.975	0.99
$\hat{\eta}_\alpha$	1.4455	1.8035	2.2238	2.8747	4.0565	4.7290	6.7467

approximation for the distribution of the likelihood ratio test statistic $-2\ell n\Lambda$ under standard conditions, λ'_{se} should behave as a chi-square random variable with three degrees of freedom for sufficiently large samples. The number of degrees of freedom corresponds to the two parameter constraints in (6.5) and the additional constraint of the exponential scale parameter taken to be equal to one.

However, use of the maximum likelihood estimates of the restricted Burr parameters as the initial estimates of the unrestricted Burr parameters served to reduce the number of degrees of freedom. There is a local maximum in the Weibull region (which includes the standard exponential region) of the Burr XII parameter space, where the Burr parameters obey the following functional relationships:

$$\begin{aligned} k &> k_0 \\ \beta &\approx \theta_w k^{1/c} \quad . \end{aligned} \tag{6.7}$$

Due to the widely recognized flexibility of the Weibull family of distributions and the choice of initial estimates in the Weibull region, the Burr likelihood equations tend to converge at the Weibull local maximum, regardless of whether a global maximum exists in another region of the Burr parameter space. This difficulty was encountered in the analysis of the adult leukemia data, which was presented at the end of Section 5.3.

In the simulation study when the initial estimates of the unrestricted Burr parameters were taken to be equal to the maximum likelihood estimates for the restricted (standard exponential region) Burr model, three initial constraints were imposed:

$$c = 1$$

$$\beta = \theta_w k^{1/c}$$

and (6.8)

$$\theta_w = 1 \ .$$

With regard to the final unrestricted Burr maximum likelihood estimates, although $\hat{c}$ varied from one, for every simulated sample $\hat{\beta}$ was approximately equal to $\hat{k}^{1/\hat{c}}$. Therefore, it appears that the choice of initial estimates effectively imposed two constraints on the unrestricted Burr model:

$$\beta = \theta_w k^{1/c}$$

and (6.9)

$$\theta_w = 1 \ ,$$

which restricted the model to a local maximum in the Weibull region. Consequently, the sampling distribution of the likelihood ratio statistic λ'_{se} corresponds to the pair of distributional hypotheses:

$$\begin{aligned} H_0&: P\{T \le t\} = F_0(t) = 1-\exp(-t) \\ H_1&: P\{T \le t\} = F_1(t;c) = 1-\exp(-t^c) \ , \end{aligned} \tag{6.10}$$

which compares the special case of the standard exponential distribution to the Weibull distribution with scale parameter equal to one. For the test specified in (6.10) the chi-square distribution with one degree of freedom is the usual asymptotic approximation for the distribution of the likelihood ratio statistic λ'_{se}. Under alternative initial value strategies away from the Weibull region, the unrestricted Burr equations may have converged at a global maximum elsewhere, thereby introducing more variation into the simulated distribution of λ'_{se}. It is possible that the true distribution of λ'_{se} behaves asymptotically as a chi-square random variable with three degrees of freedom, which would be consistent with Wilks' [1938] theorem. Further research regarding this issue is necessary.

The hypothesis that the choice of initial values effectively reduced the number of degrees of freedom to one was investigated graphically. The step-function for the empirical cumulative distribution function for λ'_{se} and the cumulative distribution function for the chi-square distribution with one degree of freedom are plotted in Figure 6.1. This plot shows that the one degree of freedom chi-square distribution approximates the empirical cumulative distribution for λ'_{se} reasonably well. The fit is best for those sample values which are greater than the 0.95 quantile, upon which inferences regarding the fit of the standard exponential distribution are based.

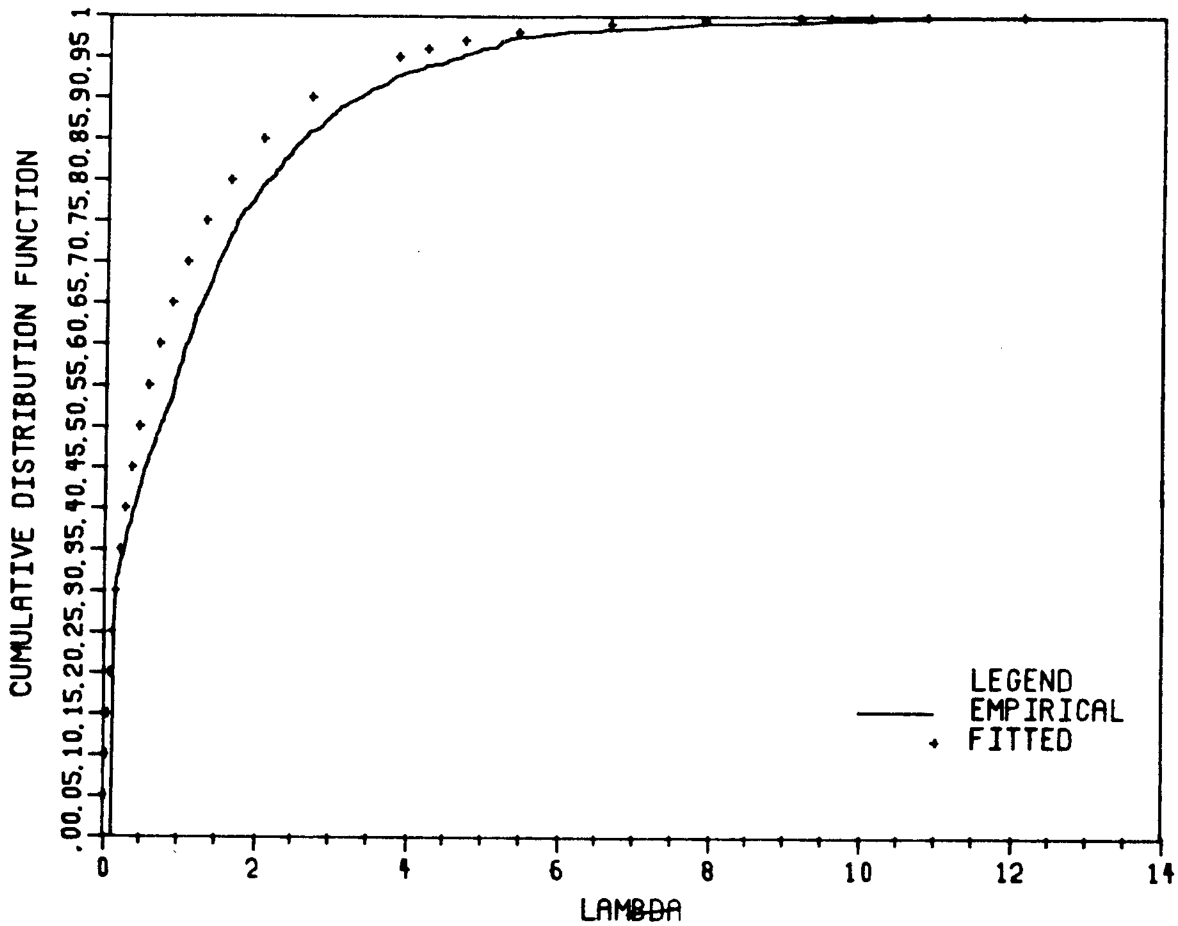
Alternative chi-square approximations for the simulated distribution of λ'_{se} were investigated. Since published tables of the percentiles for the chi-square distribution include integer degrees of freedom, chi-square distributions with integer degrees of freedom were considered first.

The central moments for the chi-square distribution are integer multiples of the number of degrees of freedom. For example, the first three central moments for a chi-square distribution with C degrees of freedom are equal to the following:

$$M_1 = C, \quad M_2 = 2C, \quad \text{and} \quad M_3 = 8C \; . \tag{6.11}$$

The corresponding sample moments based on the 1000 simulated values of λ'_{se} are equal to

FIGURE 6.1
EMPIRICAL CDF AND FITTED CHI-SQUARE (1 D.F.) CDF OF LRT STATISTIC



$$m_1 = 1.054, \quad m_2 = 2.136, \quad \text{and} \quad m_3 = 7.703 . \tag{6.12}$$

By equating the first three moments, the following estimates C_i, where i denotes the estimate based on the i-th moment, are obtained for the number of degrees of freedom C:

$$\tilde{C}_1 = 1.054, \quad \tilde{C}_2 = 1.068, \quad \text{and } \tilde{C}_3 = 0.963 . \tag{6.13}$$

These results suggest that among the chi-square distributions with integer degrees of freedom, the one degree of freedom chi-square distribution is most appropriate for approximating the empirical distribution of λ'_{se}.

A method used by Quade [1979] was employed to determine whether some linear transformation of λ'_{se} of the form, $A + B\lambda'_{se}$, behaves as a chi-square random variate. This is equivalent to investigating whether λ'_{se} follows a gamma distribution. Estimates of A, B, and C may be obtained by equating the moments for the chi-square distribution given in (6.11) to functions of the sample moments in (6.12). The parameters A and B of the linear transformation, as well as the number of degrees of freedom C are determined by equating functions of the first three moments. This results in the following system of equations:

$$A + B(1.054) = C, \quad B^2(2.136) = 2C, \text{ and } B^3(7.703) = 8C. \tag{6.14}$$

The unique solution of these three equations in three unknowns is

$$\tilde{A} = 0.144, \quad \tilde{B} = 1.109, \text{ and } \quad \tilde{C} = 1.313 . \tag{6.15}$$

Thus, the chi-square distribution with 1.313 degrees of freedom provides an approximation for the transformed likelihood ratio statistic $0.144 + (1.109)\lambda'_{se}$, which fits the first three moments exactly.

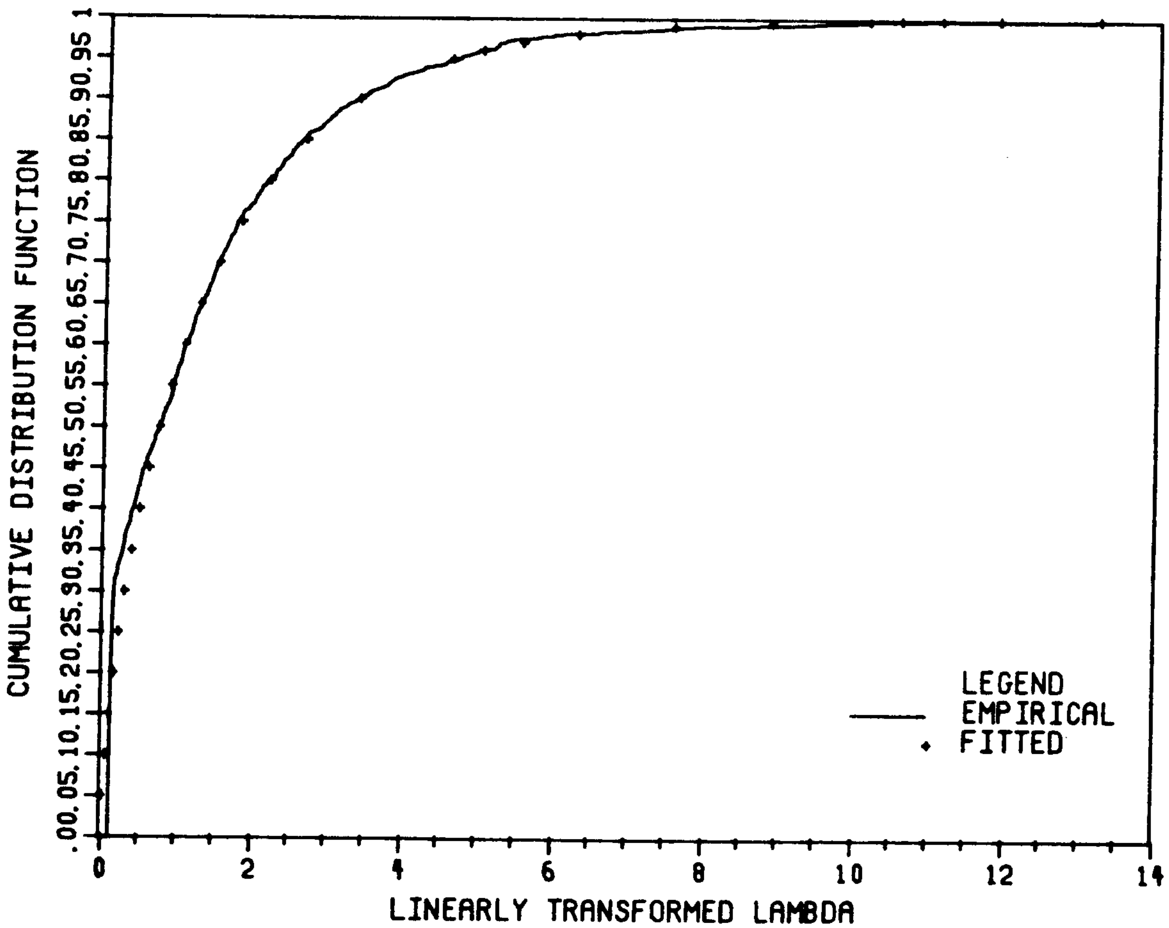
In order to assess the exactness of the gamma approximation, the empirical cumulative distribution step-function for the transformed test statistic was plotted against the cumulative function of the approximating chi-square distribution with 1.313 degrees of freedom. As shown in Figure 6.2, the fit is extremely good for the linearly transformed values exceeding the 0.50 quantile.

A comparison of the plots of the chi-square approximation in Figure 6.1 and the gamma approximation in Figure 6.2 suggests that there are only very slight differences, based on visual impressions, between the fits of the chi-square distribution with one degree of freedom and the gamma distribution. Clearly, the gamma distribution should provide the best fit since this approximation matched three moments exactly. However, unlike the chi-square approximation, the gamma approximation obtained by fitting a $\chi^2(C)$ distribution to the transformed likelihood statistic, $A + B\lambda'_{se}$, requires the specialized constants given in (6.15). Consideration must be given to the stability of the estimated parameters $\tilde{A}$ and $\tilde{B}$, as well as the stability of the estimated number of degrees of freedom $\tilde{C}$. Variance estimates of these parameters based on the jackknife technique (see Miller [1974]) could be computed from the 1000 simulated values of λ'_{se}. Alternatively, the simulation could be repeated a number of times to obtain some indication of the amount of variability that is associated with each of these estimators. The estimates of these coefficients are expected to be fairly stable.

6.4 The Consideration of Other Null Hypotheses

By means of various data transformations the empirical distribution of λ'_{se}, simulated for the pair of distributional hypotheses in (6.4), could be used for testing a number of null distributions, for

FIGURE 6.2
EMPIRICAL CDF AND CHI-SQUARE (1.313 D.F.) CDF OF LRT STATISTIC



which all of the parameters are specified, against Burr XII alternatives. For example, if it is desired to test whether a set of data represented by the variable T are exponentially distributed with mean equal to θ_{e0} rather than equal to one, as in the standard exponential case, it is possible to test whether the transformed variable

$$T' = T/\theta_{e0} \tag{6.16}$$

has the standard exponential distribution. Similarly, the transformation

$$T^* = (T/\theta_{w0})^{c_{w0}} \tag{6.17}$$

enables the testing of whether the random variable T has the Weibull distribution with parameters equal to c_{w0} and θ_{w0} by testing whether the variable T* follows the standard exponential distribution. Under the alternative hypothesis that T is distributed according to the Burr XII probability law, the transformed variables T' and T* behave according to the Burr XII distribution with shape and scale parameters which are functions of θ_{e0}, c_{w0}, or θ_{w0}.

These methods are only appropriate for simple null hypotheses, where it is desired to test whether a distribution corresponds to a particular member of the Weibull or exponential families. However, the simulated distribution of λ'_{se} is not appropriate for testing whether failure times follow the Weibull or exponential distributions with unspecified parameters. The reason for this is that if the data transformations specified in (6.16) and (6.17) are applied using maximum likelihood estimates $\hat{\theta}_e$, $\hat{c}_w$, and $\hat{\theta}_w$, there would be a substantial loss of power associated with performing a transformation which is based upon the data.

It was the intent of this chapter to investigate the distribution of the generalized likelihood ratio statistic when the null parameter space corresponds to the standard exponential limiting boundary point of the Burr XII parameter space. However, due to the choice of initial parameter estimates in the maximization procedure, the distribution of the generalized likelihood ratio statistic was effectively simulated for the null exponential distribution as a special case of a family of Weibull distributions. If it were possible to simulate the likelihood ratio distribution for the standard exponential versus the Burr XII distributional hypotheses (using alternative initial value strategies), the results could be generalized to the case for which the null hypothesis specifies alternative limiting points on the boundary of the Burr parameter space when k is unbounded. However, further work would be necessary to determine the nature of the distributions of the likelihood ratio statistics λ'_w and λ'_e when the null parameter space with unspecified nuisance parameters corresponds to other boundary regions, defined by k approaching infinity, of the Burr parameter space.

CHAPTER VII

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

7.1 An Overview

The research presented in the preceding chapters has dealt with the choice of an underlying parametric model in survival analyses or reliability studies. Emphasis was placed upon the Burr Type XII distribution which incorporates as limiting cases of one of its shape parameters, the Weibull and exponential distributions. Generalized likelihood ratio tests were developed to assess the adequacy of the fits of these two limiting special cases within the parametric framework of the more comprehensive model. The examples in Chapters IV and V demonstrated that the Burr XII distribution can be effectively used to evaluate the fit of Weibull or exponential distributions for life test and survival data. The conclusions regarding distributional shape based upon the Burr XII distribution were supported by results based upon the log-generalized gamma distribution, another general family of parametric distributions, as well as by plots of the data versus the fitted distributions.

It was the intent of Chapter VI to investigate the empirical distribution of the likelihood ratio statistic for testing the null standard exponential distribution against the alternative Burr XII distribution by means of a simulation. Because the true value of the parameter vector under the null hypothesis does not lie in the interior of the Burr parameter space, the usual asymptotic chi-square approximation could not be assumed without further investigation. However, due to the choice of initial parameter estimates in the maximization procedure, the distribution of the generalized likelihood ratio statistic was simulated for the null exponential distribution as a special case of a family of Weibull distributions rather than as a special case of the Burr XII distribution.

7.2 Distributional and Power Considerations for the Likelihood Ratio Tests Defined by λ_e' and λ_w'

As was discussed in the concluding remarks of Chapter VI, further investigative work is needed with regard to the nature of the distributions of the likelihood ratio statistics λ_e' and λ_w' when the nuisance parameters θ_e, and c_w, and θ_w for the exponential and Weibull distributions, respectively, are known as well as when they are not known. In addition, it needs to be determined how variations in the values of these nuisance parameters affect the distributions of the test statistics.

Although in the examples the likelihood ratio tests based on λ_e'. and λ_w' exhibited sensitivity and specificity which appeared to be comparable to that of the likelihood ratio tests for the log-generalized gamma analyses, the power of each of these tests should be established. Thoman, Bain, and Antle [1969] determined the power of an exact test of exponentiality versus Weibull alternatives as a function of specific values of the Weibull power parameter. For comparison purposes, it would be of interest to consider these same Weibull alternatives when consideration is given to the power of the likelihood ratio test for discriminating between the standard exponential and the Burr XII distributions.

7.3 The Burr III Distribution

Rodriguez [1979] determined that the coverage of the Burr III distribution is much larger than that of the Burr XII distribution in the moment-ratio plane. For this reason he has concentrated his more recent research efforts on this member of the Burr system (see Rodriguez [1980b]). The cumulative distribution function of the Burr III distribution is given by

$$F(t;c,k,\beta) = \{1+(t/\beta)^{-c}\}^{-k} , \tag{7.1}$$

where c and k are positive shape parameters, β is a positive scale parameter, and t represents the positive random variable. Rodriguez [1979] has identified the extreme value Type II distribution

$$F(t;c,\theta) = \exp\{-(t/\theta)^{-c}\}, \tag{7.2}$$

for positive c, θ, and t to be the limiting form of the Burr III distribution in (7.1) as the shape parameter k tends to infinity.

It can be shown that the generalized likelihood ratio test for discriminating between the Weibull and Burr Type XII distributions is equivalent to a generalized likelihood ratio test for discriminating between the extreme value Type II distribution in (7.2) and the Burr Type III distribution in (7.1). If X has a two-parameter Weibull distribution, then Y equal to X^{-1} has an extreme value Type II distribution. Similarly, if X has the Burr XII distribution, then Y equal to X^{-1} has the Burr III distribution. Since the ratio of maximized likelihood is invariant to transformations of the data, it follows that the two discrimination problems are equivalent. This equivalence serves to increase the importance and usefulness of the generalized likelihood ratio tests developed in Chapter II for other areas of application.

BIBLIOGRAPHY

1. Amoroso, L. (1925). Ricerche Intorno alla Curva dei Redditi, *Annali di Mathematica*, Serie IV, Tomo II, 123-159.

2. Antle, C.E., and Klimko, L.A. (1973). Choice of Model for Reliability Studies and Related Topics III, *Report Number ARL 75-0133*, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

3. Atkinson, A.C. (1970). A Method for Discriminating Between Models (with Discussion), *Journal of the Royal Statistical Society Series B 32*, 323-353.

4. Atkinson, A.C., and Cox, D.R. (1974). Planning Experiments for Discriminating Between Models (with Discussion), *Journal of the Royal Statistical Society Series B 36*, 321-348.

5. Atkinson, A.C., and Fedorov, V.V. (1975a). The Design of Experiments for Discriminating Between Two Rival Models, *Biometrika 62*, 57-78.

6. Atkinson, A.C., and Fedorov, V.V. (1975b). Optimal Design: Experiments for Discriminating Between Several Models, *Biometrika 62*, 289-303.

7. Bain, L.J., and Engelhardt, M. (1980). Probability of Correct Selection of Weibull Versus Gamma Based on Likelihood Ratio, *Communications in Statistics A9*, 375-381.

8. Bartholomew, D.J. (1957). A Problem in Life Testing, *Journal of the American Statistical Association 52*, 350-355.

9. Burr, Irving W. (1942). Cumulative Frequency Functions, *Annals of Mathematical Statistics 13*, 215-232.

10. Burr, Irving W. (1967a). A Useful Approximation to the Normal Distribution Function with Application to Simulation, *Technometrics 9*, 647-651.

11. Burr, Irving W. (1967b). The Effect of Non-Normality on Constants for $\bar{X}$ and R Charts, *Industrial Quality Control 23*, 563-569.

12. Burr, Irving W. (1968). On a General System of Distributions III. The Sample Range, *Journal of the American Statistical Association 63*, 636-643.

13. Burr, Irving W. (1973). Parameters for a General System of Distributions to Match a Grid of α_3 and α_4, *Communications in Statistics 2*, 1-21.

14. Burr, Irving W., and Cislak, P.J. (1968). On a General System of Distributions: I. It's Curve-Shape Characteristics; II. The Sample Median, *Journal of the American Statistical Association 63*, 627-635.

15. Canfield, Robert V., and Borgman, Leon E. (1975). Some Distributions of Time to Failure for Reliability Applications, *Technometrics 17*, 263-268.

16. Cox, D.R. (1961). Tests of Separate Families of Hypotheses, *Proceedings of the Berkeley Symposium 1*, 105-123.

17. Cox, D.R. (1962). Further Results on Tests of Separate Families of Hypotheses, *Journal of the Royal Statistical Society Series B 24*, 406-424.

18. Cox, D.R. (1972). Regression Models and Life-Tables, *Journal of the Royal Statistical Society Series B 34*, 187-202.

19. Cox, D.R. (1977). The Role of Significance Tests, *Scandanavian Journal of Statistics 4*, 49-70.

20. Craig, C.C. (1936). A New Exposition and Chart for the Pearson System of Frequency Curves, *Annals of Mathematical Statistics 7*, 16-28.

21. Drane, S.W., Owen, D.B., and Seibert, G.B., Jr. (1978). The Burr Distribution and Quantal Responses, *Statistische Hefte 19*, 204-210.

22. Dubey, Satya D. (1967). Some Percentile Estimators of Weibull Parameters, *Technometrics 9*, 119-129.

23. Dubey, Satya D. (1968). A Compound Weibull Distribution, *Naval Research Logistic Quarterly 15*, 179-188.

24. Dumonceaux, R. and Antle, C.E. (1973). Discrimination Between the Lognormal and the Weibull Distributions, *Technometrics 15*, 923-926.

25. Dumonceaux, R., Antle, C.E., and Haas, G. (1973). Likelihood Ratio Test for Discriminating Between Two Models with Unknown Location and Scale Parameters (with Discussion), *Technometrics 15*, 19-31.

26. Durling, F.C. (1975). The Bivariate Burr Distribution, *Statistical Distributions in Scientific Work 1*, D. Reidel Publishing Company, Dordrecht, Holland, 329-335.

27. Engelhardt, M., and Bain, L.J. (1973). Tests of Two-Parameter Exponentiality Against Three-Parameter Weibull Alternatives, *Technometrics 17*, 353-356.

28. Farewell, V.T., and Prentice, R.L. (1977). A Study of Distributional Shape in Life Testing, *Technometrics 19*, 69-75.

29. Fiegl, P., and Zelen, M. (1965). Estimation of Exponential Survival Probabilities with Concomitant Information, *Biometrics 21*, 826-838.

30. Flehinger, B., and Lewis, P. (1959). Two Parameter Lifetime Distribution for Reliability Studies of Renewal Processes, *IBM Journal of Research and Development 3*, 58-73.

31. Gehan, E.A., Dyer, T., and Sampson, S. (1980). Long-term Survival Analysis of Patients with Acute Leukemia, Paper Presented at the Southwest Oncology Group Meeting.

32. Gehan, E.A., and Siddiqui, M.M. (1973). Simple Regression Methods for Survival Time Studies, *Journal of the American Statistical Association 68*, 848-856.

33. Glasser, M. (1967). Exponential Survival with Covariance, *Journal of the American Statistical Association 62*, 561-568.

34. Greenwood, Major (1946). The Statistical Study of Infectious Diseases, *Journal of the Royal Statistical Society (Part II) 109*, 85-109.

35. Gross, A.J., and Clarke, V.B. (1975). *Survival Distributions: Reliability Applications in the Biomedical Sciences*, John Wiley and Sons, New York.

36. Hager, H.W., and Bain, L.J. (1970). Inferential Procedures for the Generalized Gamma Distribution, *Journal of the American Statistical Association 65*, 1601-1609.

37. Hager, H.W., Bain, L.J., and Antle, C.E. (1971). Reliability Estimation for the Generalized Gamma Distribution and Robustness of the Weibull Model, *Technometrics 13*, 547-557.

38. Harter, L.H. (1967). Maximum Likelihood Estimation of the Parameters of a Four-Parameter Generalized Gamma Population from Complete and Censored Samples, *Technometrics 9*, 159-165.

39. Hatke, M.A. (1949). A Certain Cumulative Probability Function, *Annals of Mathematical Statistics 20*, 461-463.

40. Hjorth, Urban (1980). A Reliability Distribution with Increasing, Decreasing, Constant and Bathtub-Shaped Failure Rates, *Technometrics 22*, 99-107.

41. Irwin, J.O. (1942). The Distribution of the Logarithm of Survival Times when the True Law is Exponential, *Journal of Hygiene 42*, 328-333.

42. Jackson, O.A.Y. (1968). Some Results on Tests of Separate Families of Hypotheses, *Biometrika 55*, 355-363.

43. Johnson, N.L., and Kotz, S. (1970). *Continuous Univariate Distributions 1*, Houghton Mifflin, Boston.

44. Johnson, N.L., and Kotz, S. (1972). Power Transformations of Gamma Variables, *Biometrika 54*, 226-229.

45. Kalbfleisch, J.D. (1974). Some Extensions and Applications of Cox's Regression and Life Model, Paper Presented at the Tallahassee, Florida Meetings of the Biometric Society, Institute for Mathematical Statistics, and American Statistical Association, March 20-22, 1974.

46. Kalbfleisch, J.D., and Prentice, R.L. (1980). *Statistical Analysis of Failure Time Data*, John Wiley and Sons, New York.

47. Kaplan, E.B., and Elston, R.C. (1972). A Subroutine Package for Maximum Likelihood Estimation (MAXLIK), *Institute of Statistics Mimeo Series No. 823*, Department of Biostatistics, School of Public Health, University of North Carolina, Chapel Hill, N.C.

48. Klimko, L.A., Antle, C.E., Rademaker, A.W., and Rockette, H.E. (1975). Upper Bounds for the Power of Invariant Tests for the Exponential Distribution with Weibull Alternatives, *Technometrics 17*, 357-360.

49. Learmonth, G.P., and Lewis, P.A.W. (1973). Naval Postgraduate School Random Number Generator Package LLRANDOM, *NPS55LW73061A*, Naval Postgraduate School, Monterey, California.

50. Lehmann, E.L. (1959). *Testing Statistical Hypotheses*, John Wiley and Sons, New York.

51. Lienhard, J.H., and Meyer, P.L. (1967). A Physical Basis for the Generalized Gamma Distribution, *Quarterly of Applied Mathematics 25*, 330-334.

52. Lilliefors, H.W. (1967). On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown, *Journal of the American Statistical Association 62*, 399-402.

53. Mann, N.R., Schafer, R.E., and Singpurwalla, N.D. (1974). *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley and Sons, New York.

54. Menon, M.V. (1963). Estimation of the Shape and Scale Parameters of the Weibull Distribution, *Technometrics 5*, 175-182.

55. Miller, R.G. (1974). The Jackknife - a Review, *Biometrika 61*, 1-15.

56. Mood, A.M., Graybill, F.A., and Boes, D.C. (1963). *Introduction to the Theory of Statistics*, McGraw-Hill, New York.

57. Moran, P.A.P. (1951). The Random Division of an Interval-Part II, *Journal of the Royal Statistical Society Series B 13*, 147-150.

58. Nelson, W.B. (1970). Statistical Methods for Accelerated Life Test Data - The Inverse Power Law Model, *TIS Report 71-C-011*, General Electric Corporate Research and Development.

59. Ord, J.K. (1972). *Families of Frequency Distributions*, Griffin, London.

60. Parr, V.B., and Webster, J.T. (1965). A Method for Discriminating Between Failure Density Functions Used in Reliability Predictions, *Technometrics 7*, 1-10.

61. Prentice, R.L. (1973). Exponential Survivals with Censoring and Explanatory Variables, *Biometrika 60*, 279-288.

62. Prentice, R.L. (1974). A Log Gamma Model and Its Maximum Likelihood Estimation, *Biometrika 61*, 539-544.

63. Quade, D. (1979). Regression Analysis Based on the Signs of the Residuals, *Journal of the American Statistical Association 74*, 411-423.

64. Ramberg, J.S., Dudewicz, E.J., Tadikamalla, P.R., and Mykytka, E.F. (1979). A Probability Distribution and Its Uses in Fitting Data, *Technometrics 21*, 201-214.

65. Rodriguez, R.N. (1977). A Guide to the Burr Type XII Distributions, *Biometrika 64*, 129-134.

66. Rodriguez, R.N. (1979). Burr Distributions. *Research Publication GMR-2954*, General Motors Research Laboratories, Warren, Michigan.

67. Rodriguez, R.N. (1980a). Shape: An Iteractive Graphics SAS Procedure for Fitting Histograms, *Research Publication GMR-3215*, General Motors Research Laboratories, Warren, Michigan.

68. Rogriguez, R.N. (1980b). Multivariate Burr III Distributions. Part I. Theoretical Properties, *Research Publication GMR-3232*, General Motors Research Laboratories, Warren, Michigan.

69. Shorack, G.R. (1972). The Best Test of Exponentiality Against Gamma Alternatives, *Journal of the American Statistical Association 67*, 213-214.

70. Stacy, E.W. (1962). A Generalization of the Gamma Distribution, *Annals of Mathematical Statistics 33*, 1187-1194.

71. Stacy, E.W., and Mihram, G.A. (1965). Parameter Estimation for a Generalized Gamma Distribution, *Technometrics 7*, 349-358.

72. Takahasi, K. (1965). Note on the Multivariate Burr's Distribution, *Annals of the Institute of Statistical Mathematics Toyko 17*, 257-260.

73. Taulbee, J.D. (1979). A General Model for the Hazard Rate with Covariables, *Biometrics 35*, 439-449.

74. Thoman, D.R., Bain, L.J., and Antle, C.E. (1969). Inferences on the Parameters of the Weibull Distribution, *Technometrics II*, 445-460.

75. U.S. Department of Commerce (1961). *RADC Reliability Notebook.*

76. Weibull, W. (1951). A Statistical Distribution Function of Wide Applicability, *Journal of Applied Mechanics 18*, 293-297.

77. Wilks, S.S. (1938). The Large-Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses, *Annals of Mathematical Statistics 9*, 60-62.

78. Withell, E.R. (1942). The Significance of the Variation in Shape of Time-Survivor Curves, *The Journal of Hygiene 42*, 124-183.

79. Wolfe, J.H. (1971). A Monte Carlo Study of the Sampling Distribution of the Likelihood Ratio for Mixtures of Multinormal Distributions, *Research Memorandum 72-2*, Naval Personnel and Training Research Laboratory, San Diego, California.

80. Zelen, M., and Dannemiller, M. (1961). The Robustness of Life Testing Procedures Derived from the Exponential Distribution, *Technometrics 3*, 29-49.