

THE UNIQUENESS OF THE L_2 ASSOCIATION SCHEME

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1. Summary. The L_2 association scheme for a class of partially balanced incomplete block design determines the parameters of the second kind. This paper considers the converse problem whether or not these parameters imply the L_2 association scheme. Necessary conditions for the existence of such designs are also obtained.

2. Introduction. A partially balanced incomplete block design [1] with two associate classes is said to have L_2 association scheme [2], if the number of treatments is s^2 where s is a positive integer and the treatments can be arranged in a $s \times s$ square such that any two treatments in the same row or the same column are first associates, whereas any two treatments not in the same row and not in the same column are second associates. The following relations are easily seen to hold in this case.

(1) The number of first associates of any treatment is $n_1 = 2s - 2$.

(2) With respect to any two treatments θ_1 and θ_2 which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is

$$p_{11}^1(\theta_1, \theta_2) = s - 2.$$

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(3) With respect to any two treatments θ_3 and θ_4 which are second associates, the number of treatments which are first associates of both θ_3 and θ_4 is $p_{11}^2(\theta_3, \theta_4) = 2$.

We examine the converse problem, i.e., whether or not the relations (1), (2) and (3) imply that the association scheme is of the L_2 type. We show that this converse is true for $s \geq 3$ excepting possibly $s = 4$. Necessary conditions for the existence of such designs are also obtained.

It is worthwhile to recall what is known about other partially balanced designs. It is known [6] that the converse problem is true in the case of group divisible designs. Recently Connor [4] has shown that the same is true in the case of triangular designs for $n(n-1)/2$ treatments if $n \geq 9$. In an unpublished thesis [3] Mesner has given results for the case L_g , $g \geq 2$. The proof presented here for L_2 is much simpler than that given by him. It is also shown that for case $s \cong 4$, that there are only two types of association schemes possible.

3. Statement and proof of a lemma.

Lemma. If $s = 3$ or $s > 4$, and if the 1-associates of any treatment θ are $\phi_1, \phi_2, \dots, \phi_{s-1}, \psi_1, \psi_2, \dots, \psi_{s-1}$ where the set $(\phi_2, \dots, \phi_{s-1})$ is the set of common 1-associates of both θ and ϕ_1 and the set $(\psi_1, \dots, \psi_{s-1})$ is the set of 1-associates of θ and 2-associates of ϕ_1 , then any two treatments from the set $(\phi_1, \dots, \phi_{s-1})$ are 1-associates. Similarly, any two treatments from the set $(\psi_1, \dots, \psi_{s-1})$ are 1-associates, while any treatment ϕ_i is a 2-associate of any treatment ψ_j , $i, j = 1, 2, \dots, s-1$.

Proof. We will use the notation $(\theta, \phi) = i$ to denote that θ and ϕ are i -associates $i = 1, 2$. The lemma is trivially true for $s = 3$. We now

consider $s > 4$. For convenience replace θ by $1, \phi_1, \phi_2, \dots, \phi_{s-1}$ by $2, 3, \dots, s$ and $\psi_1, \psi_2, \dots, \psi_{s-1}$ by $s+1, s+2, \dots, (2s-1)$ respectively.

We then have the treatments $2, 3, \dots, s, s+1, \dots, (2s-1)$ for 1-associates of 1, of which the set $T_1 = (3, 4, \dots, s)$ is the set of common 1-associates of both 1 and 2, whereas the set $T_2 = (s+1, s+2, \dots, (2s-1))$ is the set of 1-associates of 1 and 2 associates of 2. Let α be any treatment of T_2 . Then $(2, \alpha) = 2$. Since $p_{11}^2(2, \alpha) = 2$, and 1 is one of the common 1-associates of both 2 and α , therefore, α has at most one 1-associate in T_1 . Since $p_{11}^1(1, \alpha) = s - 2$, α has at least $(s-3)$ 1-associates in T_2 . But T_2 contains besides α only $s-2$ treatments. Hence α has at most one 2-associate in T_2 . Hence, we have the following two possibilities.

Either (i) with respect to any treatment of T_2 every other treatment of T_2 is 1-associate, in which case any two treatments of T_2 form a 1-associate pair, or, (ii) there exists a treatment α of T_2 such that there is a treatment β of T_2 where $(\alpha, \beta) = 2$ and every other treatment of T_2 besides α and β is 1-associate of α . Put $T_2^1 = T_2 - (\alpha, \beta)$. Consider the treatment β . Since it can have at most one 2-associate in T_2 and this is α , the set T_2^1 is the set of 1-associates of β . Thus the set T_2^1 is the set of common 1-associates of both α and β where $(\alpha, \beta) = 2$. Treatment 1 is also 1-associate of both α and β . The set T_2^1 and the treatment 1 give a set of $(s-2)$ treatments which are 1-associates of both α and β . But $s-2 > 2$. This contradicts the fact that $p_{11}^2(\alpha, \beta) = 2$. Thus this case is impossible. Hence we are left with case (i) only.

From (i), for every α of T_2^1 , the $s-2$ treatments of T_2 excepting α are the $p_{11}^1 = s-2$ treatments which are 1-associates of both 1 and α .

Hence the treatment 2 and all the treatments of T_1 are the $(s-1)$ treatments which are 2-associates of α . Let γ be any treatment of T_1 , then $(1, \gamma) = 1$ and the $(s-1)$ treatments of T_2 are 2-associates of γ . Thus the treatment of T_1 are all 1-associates of γ . Hence any two treatments from the set 2 and T_1 are 1-associates. This completes the proof of the lemma.

4. Statement and proof of the main theorem.

Theorem 1. If the parameters of the second kind for a partially balanced incomplete block design with s^2 treatments are given by

$$n_1 = 2s - 2, \quad p_{11}^1 = s - 2, \quad p_{11}^2 = 2,$$

then the design has L_2 association scheme if $s = 3$ or $s > 4$.

Proof. From the above lemma, we can write down the 1-associates of 1 in the following scheme.

$$\begin{array}{ccccccc} \theta & \phi_1 & \phi_2 & \dots & \phi_{s-1} & & \\ \psi_1 & & & & & & \\ \psi_2 & & & & & & \\ \vdots & & & & & & \\ \psi_{s-1} & & & & & & \end{array}$$

where any two treatments in the first row or in the first column are 1-associates, and any treatment ϕ is a 2-associate of any treatment ψ . Let δ be any 2-associate of θ . We have $p_{11}^2(\theta, \delta) = 2$. Hence δ cannot have more than two 1-associates from the set $\phi_1, \phi_2, \dots, \phi_{s-1}$. Similarly, it cannot have more than two 1-associates from the set $\psi_1, \dots, \psi_{s-1}$ and the number of 1-associates of δ from the set of ϕ_i and ψ_j is exactly 2. Suppose δ has two 1-associates ϕ_i and ϕ_j ; then ϕ_i and ϕ_j have the $s-2$

remaining treatments of the first row and δ for their common 1-associates. But this makes the number $p_{11}^1(\phi_i, \phi_j) = s-1 > s-2$ which is the value of p_{11}^1 . We thus get a contradiction. Similarly, if δ has no 1-associate from the set $(\phi_1, \dots, \phi_{s-1})$, then both these 1-associates of δ must come from the set $\psi_1, \dots, \psi_{s-1}$, which will again give a contradiction. Thus δ has exactly one 1-associate from the set of ϕ_i 's and exactly one 1-associate from the set of ψ_j 's. Hence any δ , where $(\theta, \delta) = 2$, determines uniquely a pair (ϕ_i, ψ_j) such that $(\phi_i, \delta) = 1, (\psi_j, \delta) = 1$. Conversely we show that any pair (ϕ_i, ψ_j) uniquely determine a δ such that $(\theta, \delta) = 2$ and $(\phi_i, \delta) = (\psi_j, \delta) = 1$. For suppose there are two such δ 's, say δ_1 and δ_2 . Then we have the following relations.

$$\begin{aligned} (\phi_i, \psi_j) &= 2 \\ (\phi_i, \theta) &= (\psi_j, \theta) = 1 \\ (\phi_i, \delta_1) &= (\psi_j, \delta_1) = (\phi_i, \delta_2) = (\psi_j, \delta_2) = 1. \end{aligned}$$

This gives the value $p_{11}^2(\phi_i, \psi_j) = 3$ which is a contradiction. Thus the correspondence between δ and the pair (ϕ_i, ψ_j) is 1 - 1. We can, therefore, put δ in the position determined by the column of ϕ_i and row of ψ_j . Thus the $(s-1)^2$ positions can be uniquely filled by utilizing the $(s-1)^2$ 2-associates of θ . We thus get the following scheme.

θ	ϕ_1	ϕ_2	\dots	ϕ_{s-1}
ψ_1	δ_1	δ_2	\dots	δ_{s-1}
ψ_2	δ_s	δ_{s+1}	\dots	$\delta_{2(s-1)}$
\vdots	\vdots	\vdots	\dots	\vdots
ψ_{s-1}	δ_{s^2-3s+s}	δ_{s^2-3s+4}	\dots	$\delta_{(s-1)^2}$

Then all the 1-associates of ϕ_1 are exactly the treatments in the row and column corresponding to it. A similar result is true for any ψ_j . Now consider ψ_1 . Its 1-associates are contained in the second row and first column. Among these 1-associates the elements $\psi_2, \dots, \psi_{s-1}$ are the common 1-associates of ψ_1 and θ , whereas $\delta_1, \delta_2, \dots, \delta_{s-1}$ are the 1-associates of ψ_1 and 2-associates of θ . Hence the application of the lemma gives the result that any two treatments in the second row are 1-associates. Similarly, we get the result that any two treatments in the second column are 1-associates. A similar result is obviously true for any other row or any other column. Thus for any treatment whatsoever, all its 1-associates are obtained by taking the treatments in the row and column corresponding to that treatment. Hence any two treatments which are neither in the same row nor in the same column are 2-associates. This completes the proof of the theorem.

5. Some known results on rational equivalence of matrices and Hilbert norm-residue symbol. Let A and B be two symmetric matrices of order n with elements in the rational field. The matrices A and B are rationally equivalent, written $A \sim B$, if there exists a nonsingular C with elements in the same field, such that $A = C'BC$. The congruence of matrices satisfies the usual requirements of an "equals" relationship.

If A is an integral symmetric matrix of order and rank n, we can always construct an integral diagonal matrix $D = (d_1, \dots, d_n)$, $d_i \neq 0$, $i = 1, 2, \dots, n$, such that $D \sim A$. The number of negative terms i , called the index of A, is an invariant of A by Sylvester's Law.

Define $d = (-1)^i \delta$, where δ is the square-free positive part of $|A|$. Then since $|B| = |C|^2 A$, d is another invariant of A .

Now let A be a nonsingular and symmetric integral matrix of order n . Let D_r be the leading principal minor determinant of r and suppose that $D_r \neq 0$, $r = 1, 2, \dots, n$. Define

$$(5.1) \quad c_p(A) = (-1, -D_n)_p \prod_{j=1}^{n-1} (D_j, -D_{j+1})_p$$

for every odd prime p where $(m, m')_p$ is the Hilbert norm residue symbol. Then we have the following results [5, 6].

Theorem (A). If m and m' are integers not divisible by the odd prime p , then

$$(5.2) \quad (m, m')_p = +1$$

$$(5.3) \quad (m, p)_p = (p, m)_p = (m/p)$$

where (m/p) is the Legendre symbol. Moreover if $m \equiv m' \not\equiv 0 \pmod{p}$, then

$$(5.4) \quad (m, p)_p = (m', p)_p.$$

Theorem (B). For arbitrary non-zero integers m, m', n, n' and every prime p ,

$$(5.5) \quad (-m, m)_p = +1$$

$$(5.6) \quad (m, n)_p = (n, m)_p$$

$$(5.7) \quad (mm', n)_p = (m, n)_p (m', n)_p$$

$$(5.8) \quad (mm', m-m')_p = (m, -m')_p.$$

From the above it is easy to verify that for p an odd prime and every positive integer m

$$(5.9) \quad (m, m+1)_p = (-1, m+1)_p$$

$$(5.10) \quad \prod_{j=1}^m (j, j+1)_p = ((m+1)!, -1)_p.$$

The fundamental Minkowski-Hasse Theorem states :

Theorem (C). Let A and B be two integral symmetric matrices of order and rank n. Suppose that the leading principal minor determinants of A and B are all different from zero. Then $A \sim B$ if and only if A and B have the same invariants i, d and c_p for every odd prime p.

In the rest of the paper p stands for an odd prime and will generally be omitted in writing the Hilbert norm-residue symbol.

6. Necessary conditions for the existence of symmetrical P.B.I.B.

designs with $v = s^2$, $n_1 = 2s-2$, $p_{11}^1 = s-2$, $p_{11}^2 = 2$, when $s \geq 3$ and $s \neq 4$.

Consider the symmetrical design with parameter

$$(6.1) \quad \begin{aligned} v = b = s^2, \quad r = k, \quad \lambda_1, \lambda_2, \quad n_1 = 2s-2, \quad n_2 = (s-1)^2 \\ p_{11}^1 = s-2, \quad p_{12}^1 = s-1, \quad p_{22}^1 = (s-1)(s-2) \\ p_{11}^2 = 2, \quad p_{12}^2 = 2s-4, \quad p_{22}^2 = (s-2)^2, \quad s \geq 3, \quad s \neq 4. \end{aligned}$$

Then we have

$$(6.2) \quad \begin{aligned} r(r-1) &= 2(s-1)\lambda_1 + (s-1)^2 \lambda_2 \quad \text{or} \\ r^2 &= [r + (s-1)\lambda_1] + (s-1) [\lambda_1 + (s-1)\lambda_2]. \end{aligned}$$

Let $N = (n_{ij})$ be the incidence matrix of the design where

$$\begin{aligned} n_{ij} &= 1 \text{ if treatment } i \text{ occurs in block } j \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then by renumbering the treatments, if necessary, and using Theorem

1, we have

$$(6.3) \quad NN' = \begin{pmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \dots & \vdots \\ B & B & \dots & A \end{pmatrix}$$

where A is an $s \times s$ symmetric matrix with r in the main diagonal and λ_1 elsewhere and B is another $s \times s$ symmetric matrix with λ_1 in the main diagonal and λ_2 elsewhere. By a succession of elementary transformations on rows of NN' considered as a partitioned matrix and the same elementary transformation on columns of NN' and using only the rational numbers it is easy to verify that

$$(6.4) \quad NN' \sim T = \begin{pmatrix} 1 \cdot 2(A-B) & & & \\ & 2 \cdot 3(A-B) & & \\ & & (s-1)s(A-B) & \\ & & & s(A+(s-1)B) \end{pmatrix}$$

Put

$$(6.5) \quad P = (r - \lambda_1) + (s-1)(\lambda_1 - \lambda_2)$$

$$(6.6) \quad Q = r - 2\lambda_1 + \lambda_2$$

$$(6.7) \quad \lambda = \lambda_1 - \lambda_2, \quad \lambda' = \lambda_1 + (s-1)\lambda_2.$$

Then it is easy to verify that

$$(6.8) \quad |A-B| = Q^{s-1}P$$

$$(6.9) \quad |A+(s-1)B| = r^2 p^{s-1}.$$

Hence

$$(6.10) \quad |T| = r^2 (s!)^{2s} Q^{(s-1)^2} P^{2(s-1)}.$$

Since NN' is semipositive defined, so is T . Hence we have $P \geq 0$ and $Q \geq 0$. Further $|NN'| = |N|^2$ is a perfect square. Hence $|T|$ is a perfect square. Thus, if $P > 0$ and $Q > 0$, which means that N is non-singular, a necessary condition for existence of the design when s is even is that Q must be a perfect square. In what follows we assume that $P > 0$ and $Q > 0$. This result can also be obtained by using the results of Connor and Clatworthy [7].

Let

$$(6.11) \quad T_1 = \begin{pmatrix} 1 \cdot 2(A-B) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & (s-1)s(A-B) \end{pmatrix}$$

and

$$(6.12) \quad T_2 = s(A + (s-1)B).$$

Then

$$(6.13) \quad T = \begin{pmatrix} T_1 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & T_2 \end{pmatrix}$$

Further, if R is the $(s-1)(s-1)$ diagonal matrix

$$(6.14) \quad R = \text{diag} \{1 \cdot 2, 2 \cdot 3, \dots, (s-1)s\}.$$

Then

$$(6.15) \quad T_1 = R \times (A-B)$$

when \times denotes the Kronecker product of the matrices. It is easily verified, using the results of section 5, that

$$(6.16) \quad |R| = (s-1)! s \quad \text{and}$$

$$(6.17) \quad c(R) = 1.$$

We now evaluate the values of $c(A-B)$ and $c(A+(s-1)B)$.

Following [6, p. 379] we get

$$c(A-B) = (Q, -1) \frac{s(s-1)}{2} (PQ, \lambda) (P, Q)^s.$$

Now, since $P > 0$, $Q > 0$ and

$$P - Q = s \lambda \neq 0$$

we get from (5.8)

$$\begin{aligned} (PQ, \lambda) &= (PQ, P-Q) (PQ, s) \\ &= (P, -1) (P, Q) (P, s) (Q, s) \end{aligned}$$

$$\begin{aligned}
 \therefore c(A-B) &= (Q, -1)^{s(s-1)/2} (P, Q)^{s-1} (P, -1) (P, s) (Q, s) \\
 &= (P, -1) (Q, -1)^{s(s-1)/2} (-P, Q)^{s-1} (-1, Q)^{s-1} (P, s) (Q, s) \\
 (6.18) \quad &= (P, -1) (Q, -1)^{(s-1)(s-2)/2} (-P, Q)^{s-1} (P, s) (Q, s) .
 \end{aligned}$$

Again following [6, p. 379] we get

$$c(A + (s-1)B) = (P, -1)^{s(s-1)/2} (P, \lambda') (r^2, \lambda') .$$

Since

$$r^2 - P = s \lambda' \neq 0$$

$$\begin{aligned}
 c(A + (s-1)B) &= (P, -1)^{s(s-1)/2} (r^2 P, \lambda') \\
 &= (P, -1)^{s(s-1)/2} (r^2 P, r^2 - P) (r^2 P, s) \\
 &= (P, -1)^{s(s-1)/2} (r^2 P, r^2 - P) (r^2 P, s) \\
 &= (P, -1)^{s(s-1)/2} (r^2, -P) (r^2, s) (P, s) \\
 (6.19) \quad &= (P-1)^{s(s-1)/2} (P, s) .
 \end{aligned}$$

Since

$$T_1 = R \times (A-B)$$

from [8] we have

$$\begin{aligned}
 c(T_1) &= [c(R)]^s [c(A-B)]^{s-1} (|A-B|, -1)^{(s-1)(s-2)/2} (|R|, -1)^{s(s-1)/2} \\
 &\quad (|R|, |s|)^{s(s-1)-1} .
 \end{aligned}$$

Substituting the values obtained above we get after some simplification

$$(6.20) \quad c(T_1) = (P, -1)^{s(s-1)/2} (-P, Q)^{s-1} (P, s)^s (s, -1)^{s(s-1)/2} .$$

Similarly from [9] we have

$$\begin{aligned}
 c(T_2) &= c(A+(s-1)B) (s, -1)^{s(s+1)/2} (s, |A+(s-1)B|)^{s-1} \\
 (6.21) \quad &= (P, -1)^{s(s-1)/2} (P, s)^s (s, -1)^{s(s+1)/2}
 \end{aligned}$$

after some simplifications. Also we have

$$(6.22) \quad |T_1| = (s-1)!^{2s} s^s P^{s-1} Q^{(s-1)^2}$$

$$(6.23) \quad |T_2| = r^2 s^s P^{s-1} .$$

Since $T = \begin{pmatrix} T_1 & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & T_2 \end{pmatrix}$

is the direct sum of T_1 and T_2 , we have [6]

$$c(T) = c(T_1) c(T_2) (|T_1|, |T_2|) .$$

Substituting the values already found out it is easy to verify that

$$(6.24) \quad c(T) = (PQ, -1)^{s-1} .$$

Since $I \sim NN' \sim T$ and $c(I) = +1$ for all odd prime p , we must have

$$(PQ, -1)_p^{s-1} = 1 \text{ for all odd prime } p .$$

If s is odd the above relation is always true. For s even we have the relation

$$(PQ, -1) = 1$$

or $(P, -1) (Q, -1) = 1 .$

But when s is even a necessary condition for existence is that Q be a perfect square. Hence we get the further necessary condition for existence.

$$(P, -1)_p = +1 \text{ for all odd prime } p .$$

We can thus state the following theorem.

Theorem 2. A necessary condition for existence for existence of the symmetric partially balanced incomplete block design satisfying (6.1).

(i) $P \geq 0, Q \geq 0,$ and

(ii) if s is even and $P \neq 0$, $Q \neq 0$, then Q must be a perfect square, and $(P, -1)_p = 1$ for all odd prime p .

I understand that M. N. Vartak has submitted a paper to the Annals of Mathematical Statistics which considers a similar problem for a 3-associate class of partially balanced designs.

7. Association scheme for the case $s = 4$. Consider the partially balanced incomplete design with the following parameters

$$(7.1) \quad \begin{array}{lll} v = 16, & n_1 = 6, & n_2 = 9 \\ p_{11}^1 = 2, & p_{12}^1 = 3, & p_{22}^1 = 6 \\ p_{11}^2 = 2, & p_{12}^2 = 4, & p_{22}^2 = 4. \end{array}$$

Let $(\alpha_1, \alpha_2) = 1$ and α_3, α_4 be the common 1-associate of both α_1 and α_2 . Then we have either

- case (i) $(\alpha_3, \alpha_4) = 1$, or
 case (ii) $(\alpha_3, \alpha_4) = 2$.

Consider case (i). Let $\alpha_5, \alpha_6, \alpha_7$ be the remaining 1-associates of α_1 ; then these are obviously 2-associates of α_2 , giving the following scheme:

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_5 & & & \\ \alpha_6 & & & \\ \alpha_7 & & & \end{array}$$

Now any two treatments of the first row are 1-associates and hence $\alpha_5, \alpha_6, \alpha_7$ will be 2-associates of α_3 and α_4 . Since $\alpha_2, \alpha_3, \alpha_4$ are 2-associates of α_5 ; α_6, α_7 must be 1-associates of α_5 . Similarly α_5, α_7 are 1-associates of α_6 . Hence any two treatments in the first column are 1-associates. It now follows, as in the proof of Theorem 1, that the association scheme is

of L_2 type. Hence if β_1 and β_2 are any two treatments which are 1-associates and β_3, β_4 are common 1-associates of them both then $(\beta_3, \beta_4) = 1$. i.e. if case (i) holds for any one pair of 1-associates, it must hold for all such pairs.

We now consider case (ii). Replace treatments $\alpha_1, \alpha_2, \dots, \alpha_7$ by $1, 2, \dots, 7$ for sake of convenience, giving the scheme

1	2	3	4
5			
6			
7			

Considering the pair $(1,3)$ and the value $p_{11}^1(1,3) = 2$, we see that 3 has just one 1-associate from the set $(5,6,7)$. Without loss of generality assume that $(3,6) = 1$ and hence $(3,5) = (3,7) = 2$. Consider the pair $(3,4)$. Here 1 and 2 are 1-associates of both 3 and 4, accounting for the value $p_{11}^2(3,4) = 2$. Hence since $(3,6) = 1$, $(4,6) = 2$. Now $(6,2) = (6,4) = 2$, and $(6,3) = 1$. Hence from the values $p_{11}^1(1,6)$ and $p_{12}^1(1,6)$ we see that 6 has just one 1-associate and one 2-associate from the set $(6,7)$. Let $(6,5) = 1$ and $(6,7) = 2$. Consider the pair $(2,7)$ where $(2,7) = 2$. Since 1 is common 1-associate of both, 7 has at most one 1-associate from the set $(3,4)$. Hence considering the pair $(1,7) = 1$, 7 has at least one 1-associate from the set $(5,6)$. But $(7,6) = 2$, hence $(7,5) = 1$. Now $(1,5) = 1$ and 6,7 are common 1-associates of both. Hence $(5,2) = (5,3) = (5,4) = 2$. Consider the pair $(1,3) = 1$. Here 1-associates of 3 are 2 and 6. Hence 4,5,7 are 2-associates of 3 giving in particular $(3,5) = (3,7) = 2$. Since $(7,5) = 1$ and $(7,6) = (7,2) = 2$, 7 has just one 1-associate from the set $(3,4)$ and since $(7,3) = 2$ we must have $(7,4) = 1$. Thus with respect to treatment 2, 1-associates of 1 can be written down in the following scheme.

	1	2	<u>3</u>	<u>4</u>
$S_1:$	5			
	6			
	7			

The explanation of the scheme is as follows. Any two treatments in the first row are 1-associates excepting the two treatments which are underlined, and these form a 2-associate pair. A similar result is true for the first column where the only 2-associate pair is (6,7), which is indicated by a line on the left. We write these pairs i.e., (3,4) and (6,7) in the third and fourth position respectively. Further the third and fourth treatments in the column are 1-associates respectively of the third and fourth treatment in the row and 2-associates respectively of the fourth and third treatment in the row. Also second, third and fourth treatment in the column (row) are 2-associates of the second treatment in the row (column). We will adopt the convention of writing down the 1-associates of any treatment β_1 (here 1) with respect to any 1-associate treatment β_2 (here 2) in the scheme of the above type which will immediately bring out the association relationship amongst the various treatments. For the sake of convenience and completeness we write down the association relationship implied by the scheme S_1 .

$$\begin{aligned}
 (1,2) &= (1,3) = (1,4) = (1,5) = (1,6) = (1,7) = 1 \\
 (2,1) &= (2,3) = (2,4) = 1, (2,5) = (2,6) = (2,7) = 2 \\
 (3,1) &= (3,2) = (3,6) = 1, (3,4) = (3,5) = (3,7) = 2 \\
 (4,1) &= (4,2) = (4,7) = 1, (4,3) = (4,5) = (4,6) = 2 \\
 (5,1) &= (5,6) = (5,7) = 1, (5,2) = (5,3) = (5,4) = 2 \\
 (6,1) &= (6,3) = (6,5) = 1, (6,2) = (6,4) = (6,7) = 2 \\
 (7,1) &= (7,4) = (7,5) = 1, (7,2) = (7,3) = (7,6) = 2
 \end{aligned}
 \tag{7.2}$$

Now amongst the treatments 1,2,...,7, only the treatments 1,3,4 are 1-associates of 2. Let the remaining 1-associates of 2 be 8,9,10. Then writing the row

$$2 \quad 1 \quad \underline{3} \quad 4$$

we see that only one of the treatments 8,9,10 is 1-associate of 3. Without loss of generality let $(3,9) = 1$. Then 9 has just one associate from the set $(8,10)$. Let $(9,8) = 1$ and hence $(9,10) = 2$. Hence referring to S_1 for comparison we can write down the scheme

$$S_2: \begin{array}{c} 2 \quad 1 \quad \underline{3} \quad 4 \\ 8 \\ 9 \\ 10. \end{array}$$

We then have the following relations

$$(7.3) \quad \begin{aligned} (1,8) = (1,9) &= (1,10) = 2 \\ (2,8) = (2,9) &= (2,10) = 1 \\ (3,9) = 1, (3,8) = (3,10) &= 2 \\ (4,10) = 1, (4,8) = (4,9) &= 2 \\ (8,9) = (8,10) &= 1 \\ (9,10) &= 2. \end{aligned}$$

We now consider the association relationship of any treatment from the set $(5,6,7)$ with any treatment from the set $(8,9,10)$.

Consider $(2,6) = 2$. Treatments 1,3 are common 1-associates of both 2 and 6 and $p_{11}^2(2,6) = 2$. Hence the remaining 1-associates of 2 i.e., 4,8,9,10 are 2-associates of 6. Thus if we combine schemes S_1 and S_2 into a new scheme

$$S_3: \begin{array}{c} 1 \quad 2 \quad \underline{3} \quad 4 \\ 5 \quad 8 \\ 6 \quad 9 \\ 7 \quad 10 \end{array}$$

We see that all the treatments of the second column are 2-associates of 6. Similarly $(2,7) = 2$ and 1 and 4 are common 1-associates of both. Hence 3,8,9,10 are 2-associates of 7. Hence again all the elements in the second column are 2-associates of 7. Again $(1,9) = 2$ and 2,3 are common 1-associates of both. Hence the remaining 1-associates of 1 are 2-associates of 9. Hence all the treatments in the first column are 2-associates of 9. Similarly they are 2-associates of 10. Now $(1,8) = 2$ and 3,4,6,7 are 2-associates of 8, giving $p_{12}^2(1,8) = 4$. Hence the remaining treatment i.e., 5 must be 1-associate of 8. We summarize these relations below.

$$\begin{aligned}
 (5,8) &= 1, & (5,9) &= (5,10) = 2 \\
 (7.4) \quad (6,8) &= (6,9) &= (6,10) &= 2 \\
 (7,8) &= (7,9) &= (7,10) &= 2 .
 \end{aligned}$$

We can verbally state the result as follows. Amongst the pairs which can be formed by taking one treatment from the last three positions in the first column and one treatment from the last three positions in the second column, the only pair of 1-associates is that from the pair in the second row. We will utilize this method of combining two schemes (here S_1 and S_2) to get new relations.

Now among the treatments 1,2,...,10, the treatments 1,2,6,9 are 1-associates whereas 4,5,7,8,10 are 2-associates of 3. Let the remaining 1-associates of 3 be 11 and 12. The common 1-associates of 1 and 3 are 2 and 6, where $(2,6) = 2$. Hence we write down the row

$$3 \quad 1 \quad \underline{2 \quad 6} .$$

Of the remaining treatment 9,11,12, we know that $(2,9) = 1$. Hence 9 is placed in the third position in the column for 3. Again let $(9,11) = 1$ and $(9,12) = 2$. Then we have the scheme

$$S_4: \begin{array}{cccc} & 3 & 1 & \underline{2} \quad 6 \\ & 11 & & \\ & | & 9 & \\ & | & 12 & . \end{array}$$

Similarly completing the scheme for 1 3 2 6 and utilizing the relations already obtained we have

$$S_5: \begin{array}{cccc} & 1 & 3 & \underline{2} \quad 6 \\ & 7 & & \\ & | & 4 & \\ & | & 5 & . \end{array}$$

S_4 and S_5 can be combined into

$$S_6: \begin{array}{cccc} & 3 & 1 & 2 & 6 \\ & 11 & 7 & & \\ & | & 9 & | & 4 \\ & | & 12 & | & 5 . \end{array}$$

From S_4 , S_5 , S_6 we get the following relations.

$$(7.5) \quad \begin{aligned} (1,11) &= (1,12) &= 2 \\ (2,11) &= (2,12) &= 2 \\ (3,11) &= (3,12) &= 1 \\ (4,11) &= (4,12) &= 2 \\ (5,11) &= (5,12) &= 2 \\ (6,12) &= 1, (6,11) &= 2 \\ (7,11) &= 1, (7,12) &= 2 \\ (9,11) &= 1, (9,12) &= 2 \\ (11,12) &= 1 . \end{aligned}$$

Now common 1-associates of 2 and 3 are 1,9 where $(1,9) = 2$. Hence utilizing the previous relations we have the scheme

$$S_7: \begin{array}{ccc} 3 & 2 & \underline{1 \quad 9} \\ 12 & & \\ | & 6 & \\ | & 11 & . \end{array}$$

Similarly we have S_8 and S_9 by combining S_7 and S_8 .

$$S_8: \begin{array}{ccc} 2 & 3 & \underline{1 \quad 9} \\ 10 & & \\ | & 4 & \\ | & 8 & . \end{array}$$

$$S_9: \begin{array}{ccc} 3 & 2 & \underline{1 \quad 9} \\ 12 & 10 & \\ | & 6 & | 4 \\ | & 11 & | 8 . \end{array}$$

S_7, S_8, S_9 give rise to the following relations.

$$(7.6) \quad \begin{aligned} (8,11) &= (8,12) = 2 \\ (10,12) &= 1, (10,11) = 2 . \end{aligned}$$

Now among the treatments 1,2,...,12, the 1-associates of 4 are 1,2,7,10. Let the remaining two 1-associates of 4 be 13 and 14. The common 1-associates of 4 and 1 are 2,7, where $(2,7) = 2$. Writing the row

$$4 \quad 1 \quad \underline{2 \quad 7}$$

we see that of the remaining 1-associates of 4 i.e., 13,10,14, the treatment 10 is 1-associate of 2. Without loss of generality assume that $(10,13) = 1$ and $(10,14) = 2$. Then we have the scheme

$$S_{10}: \begin{array}{ccc} 4 & 1 & \underline{2 \quad 7} \\ & 13 & \\ & | 10 & \\ & | 14 & . \end{array}$$

We have similarly

$$S_{11}: \begin{array}{ccc} & 1 & 4 \quad \underline{2 \quad 7} \\ & 6 & \\ & | 3 & \\ & | 5 & \end{array}$$

and by combining S_{10} and S_{11}

$$S_{12}: \begin{array}{ccc} & 4 & 1 \quad \underline{2 \quad 7} \\ & 13 & 6 \\ & | 10 & | 3 \\ & | 14 & | 5 . \end{array}$$

From S_{10} , S_{11} , S_{12} we get the relations

$$(7.7) \quad \begin{aligned} (1,13) &= (1,14) &= 2 \\ (2,13) &= (2,14) &= 2 \\ (3,13) &= (3,14) &= 2 \\ (4,13) &= (4,14) &= 1 \\ (5,13) &= (5,14) &= 2 \\ (6,13) &= 1, (6,14) &= 2 \\ (7,14) &= 1, (7,13) &= 2 \\ (10,13) &= 1, (10,14) &= 2 \\ (13,14) &= 1 . \end{aligned}$$

Again we can verify that only possible schemes for rows

$$4 \quad 2 \quad \underline{1 \quad 10} \quad \text{and} \quad 2 \quad 4 \quad \underline{1 \quad 10}$$

are

$$\begin{array}{cccc} 4 & 2 & \underline{1} & \underline{10} \\ 14 & & & \\ \left| \begin{array}{c} 7 \\ 13 \end{array} \right. & & & \end{array}$$

and

$$\begin{array}{cccc} 2 & 4 & \underline{1} & \underline{10} \\ 9 & & & \\ \left| \begin{array}{c} 3 \\ 8 \end{array} \right. & & & \end{array}$$

Combining these into the scheme

$$S_{13}: \begin{array}{cccc} 4 & 2 & \underline{1} & \underline{10} \\ 14 & 9 & & \\ \left| \begin{array}{c} 7 \\ 13 \end{array} \right| & \left| \begin{array}{c} 3 \\ 8 \end{array} \right| & & \end{array}$$

we have the relations

$$\begin{aligned} (8,13) &= (8,14) = 2 \\ (9,14) &= 1, (9,13) = 2 \end{aligned}$$

Now the 1-associates of 5 amongst the treatment 1,2,...,14 are 1,6,7,8. Hence 15 and 16 are the remaining two 1-associates of 5. The common 1-associates of 5 and 1 are 6,7 where $(6,7) = 2$. Now 8 is known to be 2-associate of 6 and 7. Hence 8 occupies the second position in the column for 5. Let 15 be 1-associate of 6 ; hence 16, 2-associate of 6.

Then we have

$$\begin{array}{cccc} 5 & 1 & \underline{6} & \underline{7} \\ 8 & & & \\ \left| \begin{array}{c} 15 \\ 16 \end{array} \right. & & & \end{array}$$

We also have

$$\begin{array}{ccc}
 1 & 5 & \underline{6 \quad 7} \\
 2 & & \\
 \left| \begin{array}{c} 3 \\ 4 \end{array} \right. & &
 \end{array}$$

and hence combining these two we get

$$\begin{array}{ccc}
 5 & 1 & \underline{6 \quad 7} \\
 8 & 2 & \\
 s_{14}: \left| \begin{array}{c} 15 \\ 16 \end{array} \right. & \left| \begin{array}{c} 3 \\ 4 \end{array} \right. &
 \end{array}$$

and we get the following relations.

$$\begin{array}{l}
 (1,15) = (1,16) = 2 \\
 (2,15) = (2,16) = 2 \\
 (3,15) = (3,16) = 2 \\
 (4,15) = (4,16) = 2 \\
 (7.8) \quad (5,15) = (5,16) = 1 \\
 (6,15) = 1, (6,16) = 2 \\
 (7,16) = 1, (7,15) = 2 \\
 (8,15) = (8,16) = 1 \\
 (15,16) = 2 .
 \end{array}$$

Consistent with the previous relations, it is easy to verify that we have the only possible schemes

$$\begin{array}{ccc}
 5 & 6 & \underline{1 \quad 15} \\
 16 & 12 & \\
 s_{15}: \left| \begin{array}{c} 7 \\ 8 \end{array} \right. & \left| \begin{array}{c} 3 \\ 13 \end{array} \right. &
 \end{array}$$

giving the relations

$$(12,13) = (12,16) = 1, (12,15) = 2$$

$$(13,15) = 1, (13,16) = 2$$

and

$$S_{16}: \begin{array}{cc} 4 & 7 \\ 10 & 16 \\ \left| \begin{array}{c} 2 \\ 13 \end{array} \right| & \left| \begin{array}{c} 5 \\ 11 \end{array} \right| \\ \hline & \underline{1 \quad 14} \end{array}$$

giving the relations

$$(10,16) = 1$$

(7.9)

$$(11,14) = (11,16) = 1, (11,13) = 2$$

$$(13,16) = 2$$

$$(14,16) = 2 .$$

Now counting the 1-associates and 2-associates of 12 in the previous relations we get

$$(7.10) \quad (12,14) = 2 .$$

Similarly counting the 1-associates and 2-associates of 9 in the previous relations we see that the 1-associates of 9 are 2,3,8,11,14 and either 15 or 16. Now $(7,9) = 2$ and 1-associates of 7 are 1,4,5,11,14 and 16. Hence from the value $p_{11}^2(7,9) = 2$ it is easy to see that

$$(7.11) \quad (9,15) = 1$$

$$(9,16) = 2 .$$

Again counting the 1-associates and 2-associates of 10 in the previous relations we see that

$$(7.12) \quad (10,15) = 2 .$$

Similarly we can verify that

(7.13)

$$(11,15) = 2$$

$$(14,15) = 1.$$

The relations (7.2) to (7.13) give the following table of 1-associates.

<u>Treatment</u>	<u>1-associates</u>
1	2, 3, 4, 5, 6, 7
2	1, 3, 4, 8, 9, 10
3	1, 2, 6, 9, 11, 12
4	1, 2, 7, 10, 13, 14
5	1, 6, 7, 8, 15, 16
6	1, 3, 5, 12, 13, 15
7	1, 4, 5, 11, 14, 16
8	2, 5, 9, 10, 15, 16
9	2, 3, 8, 11, 14, 15
10	2, 4, 8, 12, 13, 16
11	3, 7, 9, 12, 14, 16
12	3, 6, 10, 11, 13, 16
13	4, 6, 10, 12, 14, 15
14	4, 7, 9, 13, 11, 15
15	5, 6, 8, 9, 13, 14
15	5, 7, 8, 10, 11, 12

It is obvious that the association scheme for this case is unique and that the two common 1-associates of any two treatments, which are 1-associates, must be 2-associate, for otherwise the association scheme is of L_2 type from case (i). Mesner [3] has known that for $s = 4$, if we interchange the first and second associates in L_3 we get a design with parameters (7.1). The association scheme for case (ii) must therefore be of the same type as obtained from Mesner's result.

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