ABSTRACT

JOHNSON, GREGORY LEWIS. The Universal Location of the Yang Lee Edge Singularity for O(N) Universality Classes. (Under the direction of Dr. Vladimir Skokov).

A large body of work within the field of critical phenomenon has been devoted to the determination of various aspects which are universal. This includes critical exponents, certain ratios of critical amplitudes, and more importantly, the universal equation of state. However, there is a relevant universal quantity which had remained undetermined for the most ubiquitous class of phase transitions, the three dimensional $O(N)$ universality classes. The universal equation of state has a complex conjugate pair of singularities in the complex plane corresponding the another distinct critical point, the Yang-Lee edge singularity. As described, this location has an obvious universality being that it is a property of a universal function. Through various theorems of complex analysis, knowledge of the location of the Yang-Lee edge singularity, a branch point singularity, yields powerful information on the thermodynamics for ordinary real values.

In this thesis, we give an overview of the background information of the Yang-Lee theory of phase transitions which offers the most fundamental understanding of the origins of the Yang-Lee edge singularity. It is our perspective that this location is a basic conceptual quantity, and as such, a chapter has been devoted to studying aspects of the Yang-Lee edge singularity in the context of the van der Waals equation, as could be done in a textbook chapter on phase transitions. We follow the discussions of Lee-Yang theory with a chapter on the renormalization group which contains the notations used for the quantities we study. Finally, we then give an overview of the analytic and semi-analytic results for the universality associated with the Yang-Lee edge singularity. These consist of mean field, the large N approximation, the ε expansion (which is of limited use for three dimensions), the exact 1D Ising model result, and a range of results for various N in two dimensions.

As the main results of this thesis, we employ the functional renormalization group (FRG) with the derivative expansion to determine the universal location of the Yang-Lee edge singularity (YLE) for $O(N)$ universality classes with emphasis on the cases of $N = 1 - 4$. The leading order of the derivative expansion, known as the local potential approximation (LPA), include in the effective action a standard kinetic term and an effective potential. Our original study presented the first results for the location for the $O(N)$ classes in three dimensions. It was performed at LPA' which additionally includes a

running wavefunction renormalization and featured an extrapolation scheme to determine the location of the YLE. We followed this up with another implementation of FRG which goes to next-to-leading order in the derivative expansion utilizing an alternate method which is capable of determining the location without need for extrapolation.

The Universal Location of the Yang Lee Edge
Singularity for O(N) Universality Classes

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Physics

Raleigh, North Carolina
2023

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BIOGRAPHY

Gregory Lewis Johnson was raised on a farm in northeastern North Carolina roughly 10 miles from a small one stoplight town named Scotland Neck. Gregory is the youngest of three brothers as well as all of his first cousins. He attended a "pre-K through 12th" school in small town of Hobgood before moving counties to live on Lake Gaston, a somewhat large lake surrounded by a rural area, and attend Halifax Academy, another pre-K through 12th school. Although somewhat isolated, living on the lake gave him many great experiences being in and around his cousin's wakeboard school on the lake as well as being near an uncle who exposed him to guitar. Both wakeboarding and guitar would become lifelong (at least up to present day) interests.

Gregory was interested in most subjects, though a first significant interest was the structure of grammar, though this wasn't pursued very far. He wasn't thoroughly exposed to many academic subjects, particularly science, philosophy, or music, until the later years of his undergrad experience pursuing an electrical engineering degree at the University of North Carolina in Wilmington (UNCW). Finding these subjects somewhat due to chance, together with inspiration from one of his brothers who had become an electrical engineer, he began to change course from an electrical engineering degree to a double major tagging on a degree in physics with a minor in mathematics.

As a part of this double major program, he was due to transfer to North Carolina State University (NCSU) to finish off the engineering degree after completing the physics and math degrees at UNCW. The summer before entering into NCSU, he had an internship with his brother at a power engineering consultant firm. Throughout the summer he realized his true interests lie more in physics. As such, he dropped the engineering degree and enrolled last minute in a masters program in mathematics at UNCW with the support and encouragement of his undergrad advisor, Russell Herman. This would serve as a stepping stone to his enrollment in a Ph.D. program in physics at North Carolina State University in Raleigh, North Carolina.

ACKNOWLEDGEMENTS

I've been looking forward to this opportunity to share my gratitude and appreciation. I am grateful to be putting together my thoughts at this time as I've recently turned 30 and received an adorable niece into the family. She may indeed be my last "nibling," for a while at least. As for other recent life experiences which have deepened my empathy and love for those in my life, I had the pleasant realization from first hand experience that the adage "left loosey, right tighty" also works in Germany. Exposition aside, I am grateful for the many people that have supported me, in one way or another, throughout the years as I worked towards this eventual degree. I have restricted my attention here to a core group though I plan on extending my acknowledgements in other media once I am free to be more pensive and feel I am better able to express my appreciation and gratitude.

My mother had a way of helping me navigate personal difficulties, of which hindsight only serves to improve my awareness, which encouraged the formation and growth of some of my personal traits to which I attribute a good deal of my overall well-being. In particular, I think of my abilities to endure the challenges and stress of my academic pursuits. My father, and his father (my grandaddy) as well, have particular ways of doing things which were hard to appreciate if you weren't aware of the same subtitles they were. For instance, my grandaddy sat me down when I was very young because he noticed the way I drew the number four was too similar to the number nine - there was a better way. This sort of habit, and my awareness of it, plays a supporting role in my physics.

Now for my brothers. My oldest brother, Byron, exposed me very early on to physics, amongst many other things such as the problem of free will. I remember he would bring home physics and math problems from college that he was excited about. I think I was at least partially interested in these ideas, but mainly I looked up to him enough that I automatically placed value on this type of thinking and retroactively tried to figure out why anyone would find such things valuable. Fields, our middle brother, is closer in age to me though still older. He has had a long going habit of randomly calling me after he has been entertaining some idea - physics, philosophy, business, or whatever - and he wants to discuss. I believe this does a good job capturing the role he has played in me getting me to this Ph.D. However, he may respond by saying that this acknowledgement is a really odd thing for him to have said to himself.

Next, I thank my whole (extended) family - aunts and uncles, cousins, and especially my grandparents. The atmosphere they engender is a creative and curious one with a great deal of humor and mutual respect. We are a family of characters, as all families probably are, to each I owe a particular appreciation. My steps many steps towards obtaining this degree were always met positively with support, and a good deal of curiosity, though never any concern or doubts.

Moving again outwards, I think of my partner, Megan Franklin, whose support and friendship has been incredibly powerful. There has not been a challenge, frustration, joy, achievement, or any life event small or large which she has not found a way to either directly or indirectly give me just what I needed. This is amazing considering she is also working towards her Ph.D in plant biology, just a year or so behind me.

Over the years, I have been privileged to have a diverse support network of many friends and mentors. Since this degree has drawn rather deeply on me, I find it hard to separate my identity as a "physicist" from their contributions. I will try to list them chronologically. I have had many mentors whom I am incredibly grateful for: Allison Daniel, TW Chambliss, Amy Poole, Tom Merritt, Dr. Amy Reamer, Dr. Brian Davis, Dr. Dylan McNamara, Dr. Russell Herman, Dr. Gabriel Lugo, Dr. Dijana Jakelic, Dr. Mark Lammers, Dr. David Aspnes, Rhonda Bennett, Dr. Yuye Zhang, and Dr. Danny Doucette. Of my many friends along the way - Abby Daniel, Forrest Alston, Justin Lambert, Brad Silver, Andrew Daniel, Natalie Reid, Kyle Kosciuszko, James Cobb, John Champlain, Steve Murphy, Sage Jordan, Dan Raitz, Dylan Ebbs, Allison Jamison, Youssef Dris, Jacob Koile, Jason King, Tanner Gordon, Kyle Bowman, Damien Wright, Tyler Burger, Brittany Whitener, Kelsey Beyer, Akhil Francis, Robert Valdillez, Haowu Duan, Nick Gray, Ashkan Ghaan, Sam Sridhar, Brent Boland, Jin Choi, Keith Hillaire, William Lo, Mohamed Moshtohry, Andres Anders, Somdutta Ghosh, Deepasika Dayananda, Camen Royse, Faith Thomas, Lauren Klaasmeyer, McKayla Wieczorek, Bill and Monica Lovelace, Jim Lovelace, Anna Dye, Kyle Stahler, Victoria Yell, Andy Andis, Kelsey Lund, Mendel Nguyen, Kevin Do, Shaswat Tiwari, Efekan Kokcu, Anjali Agrawal, Justin Middleton, and Tania Proksch.

My committee members who advised me through my doctoral research, I want to thank separately. I am grateful to my advisor, Dr. Vladimir Skokov, for both his professional and personal support and guidance throughout my research. His support allowed for freedom on my end though plenty of patient, thoughtful guidance when I was stuck. I also think of the many conversations which may have started on physics though often turned to various other topics. His patience and mindfulness have had a lasting impact. To Dr. Fabian Rennecke, I am grateful for many long and winding conversations. In these conversations, I am grateful for the enthusiasm he showed while entertaining my recent findings, excitements, or confusions in research all the while validating me in my endeavours. To Dr. Thomas Schäfer and Dr. Alexander Kemper, I am grateful for the feedback and questions at various stages of my research which proved to deepen my understanding and extend the scope of my perspective. To Dr. Radmila Sazdanovic, I am grateful for her support and encouragement at various times through my research.

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INTRODUCTION

1.1 Critical phenomenon in the complex plane

anyThe study of critical phenomenon and phase transitions has long been a central topic of study in physics. The first notion likely to come to mind is the boiling or freezing of water. A possibly more subtle direction would be ferromagnets and their demagnetization phase transition due to increasing temperature beyond the Curie point. Even further in that direction, we have superconductors, deconfinement and chiral symmetry breaking in quantum chromodynamics (QCD), and so on into possibly more exotic realms such as topological phase transitions or dynamical phase transitions.

A remarkable feature of certain types of phase transitions known as second order or continuous is that the large scale, macroscopic properties of huge swathes of seemingly different phenomenon collapse into a single so called universality class. For instance, the liquid-gas critical point, the Curie point of magnetic materials as described by the Ising model, scalar ϕ^4 field theory, and the critical end point of the phase diagram of quantum chromodynamics (QCD), which are all defined by wildly different microscopic theories all belong to the so called Ising universality class. This is a powerful concept. By universality, through the study of one of the most basic physical models, the Ising model, or the ordinary liquid-gas critical point, we can infer an understanding of the complex behavior of hot matter found from smashing together heavy nuclei at relativistic speeds - the relevant physics of the QCD critical endpoint.

This is not to say that every conceivable property of one member of a class agrees with every other member. So then, what all corresponds to these universality classes? In the most basic sense, universality classes can be thought of as a set of numbers which correspond to critical exponents and critical amplitude ratios. These quantities capture the behavior of the system's thermodynamic properties as the critical point is approached. Typical variables governing this approach, known as relevant variables, are the deviation from a critical temperature, $t \propto T - T_c$, and/or the deviation from a critical external field, $H - H_c$, where H is analogous to a magnetic field. While for ordinary critical points T_c is finite and H_c is often zero, one can have a purely imaginary H_c as in the case of the Yang-Lee edge singularity (YLE).

The emergence of universality is intimately related to the divergence of the correlation length. This divergence implies that liquid water very near its critical point, see Fig. 2.1, will experience the sudden appearance of pockets of gas whose size is only limited by the correlation length, see [1]. In other words, fluctuations are very likely essential to fully illuminating the critical point. (This divergence of the correlation length and the ensuing fluctuations on all length scales then effectively implies scale invariance.) Lastly, if one then tried to describe these large scale fluctuations which occur near the critical point, one is inevitably lead to writing down a quantum/statistical field theory, though an implicit reference to the microscopic scale of the original description is still present, typically in terms of an ultraviolet cutoff, Λ. Directly on top of the critical point, in the scaling limit, one has a true continuum field theory.

The original methods developed to study the behavior of systems near a critical point were commonly of a form which can be collectively referred to as mean field theories. As the name suggests, in one way or another the quantities at hand were approximated by a mean value about which fluctuations were not fully accounted for. As a result, the predictions of these theories differed from the observed behavior in three dimensional systems. The discrepancy proves to be worse as one looks to lower dimensional theories where exact results are known such as for the Ising model in one and two dimensions. The one dimensional case shows a second order transition only at $T = 0$ [2], whereas for a two dimensional case a result was obtained in 1944 by Lars Onsager [3] which shows it *does* have a finite temperature phase transition.

There was a major breakthrough in the late 60s and early 70s attributed to Kadanoff [4] and Wilson [5, 6, 7]. The ideas spawned the concept of the Renormalization Group (RG) whereby one naturally accounts for the large scale fluctuations associated with a diverging correlation length. As an inspiration, Kadanoff describes an earlier paper by Widom [8] whereby the author seemingly guessed the appropriate form that systems should abide near their critical points, now referred to as the Widom scaling form. The predictions which followed from the ideas of RG proved to substantially improve upon the mean field results when compared with the experimental observations of three dimensional systems. A modern implementation of Wilson's version of RG emerged around mid 90s [9, 10, 11] known as the functional, exact, or non-perturbative renormalization group (FRG) which allows one to surmount the difficulties that plagued the older implementations. *In this thesis we use FRG to study a deeper universal notion of certain systems near criticality known as the Yang-Lee edge singularity, the study of which is particularly suited for FRG.*

Let us move to discuss these deeper notions captured by the universality classes. There are properties associated with the continuum theory emerging at the critical point, such as the anomalous dimension (the exponent) and the central charge, a conformal field theory (CFT) concept. Performing small perturbations about the critical point yields an eigenvalue equation. The eigenvalues corresponding to perturbations in the relevant variables, themselves universal, produce all of the universal critical exponents. There are other variables which come into play, though these represent attractive directions in the space of coupling constants. As such, they are known as irrelevant and have eigenvalues, again universal, of differing sign from the relevant ones. There are in addition variables with zero eigenvalue, known as marginal, though true marginal variables are rare [12]. Typically, these only occur before fluctuations were taken into account. The eigenfunctions of this eigenvalue equation are in general, non-linear functions of the basic variables such as t and H and whatever irrelevant variables there are in play. The eigenfunctions are known as scaling fields. Only very near the critical point can one approximate these as linear functions of the relevant variables [12].

Now one can go further by perturbing about the critical point and asking what is the corresponding large scale physics. Take the water phase diagram, Fig. 2.1. Stepping towards slightly lower temperatures puts one in either the liquid or gas phase whereas slightly higher temperatures corresponds to the supercritical fluid. In the magnetic analogue, we'd respectively have finite magnetization or zero magnetization. While the temperature dependence of the magnetization or the density difference between liquid and gas in this scenario corresponds to a universal exponent, the overall scales, known as critical amplitudes, are not universal and so differ between the two members of the Ising universality class. Surprisingly enough, ratios of certain critical amplitudes *are* universal. These ratios can capture the interplay between the two "sides" of the critical point.

Instead of moving into the phases mentioned above, which were implicitly assumed to be the *stable* states, one could also move into a *metastable* state. Examples of metastable states are those which are obtained by supercooling liquid water below its freezing point or superheating liquid above its boiling point. Given strong enough kicks, these states will decay into a mixture of the stable states. For Ising-like theories, the metastable physics in the low temperature regime, $T < T_c$, features another type of critical point, the spinodal point, see Section 2.2.2. Through this consideration one has access to a larger set of universal ratios which capture the interplay between stable and metastable states.

More exotically, the metastable states in some sense move through the origin and into the complex plane as the temperature is raised through the critical temperature. These "states" in the complex plane also yield a suite of universal amplitude ratios detailing the interplay between themselves and stable states. The generalized notion of state here is an extrema of the effective potential which may occur at real or complex values. These extrema in the complex plane are related to a distinct critical point known as the Yang-Lee edge singularity, in much the same way the spinodal point is related to the metastable states. The most basic exponent and universal amplitude ratios associated with the spinodal point are identical to those of the Yang-Lee edge singularity. While the equality of the exponents has been known [13], the equality of the universal amplitude ratios is a novel product of this thesis. *We emphasize that this thesis is the first place in which this discussion explicitly appears - our earlier papers only discussed this universality in terms of the universal location Yang-Lee edge singularity.*

Underlying the physics in the complex plane associated with the Yang-Lee edge singularity is a more fundamental understanding of phase transitions dating back to 1952 in the works of Lee and Yang [14, 15]. They demonstrated that the phase structure is intimately related to behavior of zeros of the partition function in finite volume. They proved that the zeros of the partition function of the ferromagnetic Ising model lie at purely imaginary values of the magnetic field, independent of the lattice structure and its dimension. In the infinite volume limit, these zeros coalesces onto lines along the purely imaginary axis forming branch cuts described by a density, $g(H, T)$. Importantly, the density of zeros depends on the temperature as well. For $T < T_c$ the line of zeros cross through the origin whereas for $T > T_c$, a gap opens about the origin in the density of zeros,

$g(H_c < H < H_c = 0, T) = 0$, producing two edges of the density occurring at $H = \pm H_c$. At the edges, it was seen by Kortman and Griffiths that the density of zeros scales as $g(H) \propto (H - H_c)^{\sigma}$ [16] . Fisher later showed [17] that the edge of the distribution of zeros corresponds to a distinct critical point described by a cubic field theory with a purely imaginary coupling, so deemed the Yang-Lee edge singularity (YLE). The exponent σ is related to the eigenvalue of the single relevant perturbation of the YLE [17].

The connection we have described is then that both the spinodal point and the YLE yield an identical set of universal amplitude ratios. While the YLE and spinodal point both exist for a wide-range of temperatures respectively above and below the critical temperature, this universality only applies within the critical region of the critical point. One can extract these universal numbers from the known results of the 1D, the 2D Ising model results beyond Onsager's solution [18, 19], mean field for $d \geq 4$, and the spherical model/large N limit [20]. *However, none of these numbers were computed for the 3D Ising class or any other $O(N)$ class before the work of this thesis.*

Let us give an alternative description of these "new" universal amplitude ratios as being the location of the YLE (and actually our original understanding). To do so, we need to cycle back to universality classes. More generally, universality classes also contain universal functions of the scaling variables. Variables denoted as scaling variables are simply those which do not depend on scale. For Ising-like theories, the scaling variables are functions of rescaled versions of t, H, and the order parameter M which is the generalization of the magnetization. See Section 4.4.3 for the explicit definitions of the rescaled variables denoted $\bar{t}$ and h. Explicitly, there are two independent scaling variables which one can define as $z = \bar{t}\, h^{-1/\beta\delta}$ and $\omega = M\, h^{-1/\delta}$ where β and δ are two critical exponents. Via a basic proof, one can see the values of the scaling variables at the location of the YLE, $z_c = \bar{t}\, h_c^{-1/\beta\delta}$ and $\omega_c = M_c\, h_c^{-1/\delta}$, are universal. The value M_c is the value of the magnetization at the YLE, $M_c = M(h = h_c)$. There is also a third quantity R_e, though we will discuss it later in Section 4.4.3. *The works that went into this thesis were only concerned with the value of z_c.*

The YLE itself has a distinct notion of universality associated to it related to the fact that it occurs at finite external fields. Instead of the Ising model, one could take the spins to have some integer number, N, of components; $N = 1$ is the Ising model, $N = 2$ gives the XY model, $N = 3$ gives the Heisenberg model. The $O(4)$ class is also notable being that it describes the chiral phase transition of QCD. We can collectively refer to them as $O(N)$ models, and so the $O(N)$ universality classes, where the $O(N)$ reflects the fact that these models have an $O(N)$ symmetry where $O(N)$ is the N dimensional orthogonal group. Fascinatingly, one can describe polymer physics using $N = 0$ [21] and loop erased random walks at $N = -2$ [22, 23]. One can perform a similar analysis of their zeros and find the corresponding edge of the distribution of zeros. The YLE still occurs at a finite purely imaginary magnetic field H_c which explicitly breaks the $O(N)$ symmetry reducing it down to $O(N = 1) = \mathbb{Z}_2$, or so just an Ising symmetry. This statement is the content of the Lee-Yang theorems and has been proven in the literature for $1 \leq N \leq 3$ and the large N limit, see Section 3.1; however, through our study we have found the Lee-Yang theorem to hold at least for all $N \geq -2$. Thus, all $O(N)$ classes have a YLE appearing in their high temperature regions. In other words, the exponent σ is independent of N, as was realized by Fisher [17].

So far, we have glossed over some important aspects of the discussion of critical phenomenon. As was likely gathered, universality classes can be classified by an underlying symmetry of the transition, $\mathbb{Z}_2$ and $O(2)$ for instance imply different universality classes. The dimension of the system also has a powerful impact on the physics. For standard critical points, there is an upper critical dimension at which fluctuations become unimportant and mean field theory accurately describes the transition. On the lower end of dimensions, there is the Mermin-Wagner-Coleman theorem [24, 25] which states that continuous symmetries cannot be broken at finite temperature in two dimensions or less. Thus, while the discrete Ising symmetry allows for a finite temperature phase transition, models with $N \geq 2$ do not see a finite temperature phase transition at $d = 2$, at least not an ordinary one. The case of $N = 2$ corresponds to the BKT transition [26, 27, 28] which is of topological origin. There is then an interesting question - what is the fate of the location of YLE in 2d $O(N \geq 2)$ models? A similar question - what about for $-2 \leq N \leq 2$ for $1 < d < 3$?

Now, though the YLE critical point itself is independent of N, however, it should be emphasized that z_c, ω_c, and R_e are all still N-dependent quantities. The exponent σ has been well studied, see [17, 29, 30, 31, 32] for the ferromagnetic zeros, [33] for an antiferromagnetic study, [34, 35] for CFT results, [36, 37, 38, 39, 40] for multicritical studies, and [41, 42, 43, 44] for FRG studies. *However, before the work of this thesis none of z_c, ω_c, nor R_e were computed for the 3D $O(N)$ classes. The work of this thesis focused on the computation of z_c for three dimensional $O(N)$ classes with $N \geq 1$.*

1.2 Scope

In this thesis, we use the functional renormalization group to compute the universal location of the Yang-Lee edge singularity for the three dimensional static $O(N)$ universality classes. Being a basic conceptual quantity on the same level of importance as the critical exponents, this quantity could be discussed alongside basic textbook material on the subject. As such, we discuss the Yang-Lee edge singularity and its universal aspects in the context of the phase diagram of water and the van der Waals equation, a mean field description, as the opening chapter of this thesis, Chapter 2. To clarify the understanding, we follow in Chapter 3 with a review of the Lee-Yang theory of phase transitions. An overview of the Lee-Yang theorem and its extensions is given before introducing the framework which utilizes the density of zeros to reconstruct various thermodynamic quantities. We finish off the Chapter with a Section 3.3 where we discuss the Yang-Lee edge singularity (YLE) as both the edge of the density of zeros and as a $i\,g\phi^3$ field theory. Many discussions of literature on the YLE can be found in this section.

To fully appreciate the results of this thesis, more background on critical phenomenon and RG is needed. As such, in Chapter 4 we give a basic overview of the theories we will be studying followed by a brief sketch of the loop expansion with a focus on when mean field descriptions are valid. In Section 4.4 we introduce the convention/notation for the objects we will be studying - the universal location of the Yang-Lee edge singularity. Importantly, the end of this section contains the explanation for the universality of various aspects of the YLE (and spinodal point).

After all the notation and background has been introduced, we devote Chapter 5 to an overview of the analytic and semi-analytic results of the universal aspects of the Yang-Lee edge singularity. The beginning section on mean field theory is a brief pedagogical approach to understanding and obtaining the mean field predictions on the Yang-Lee edge singularity and the spinodal point.

The main results of this thesis are contained in Chapter 6. After a brief introduction of the functional renormalization group, we overview our computations of the universal location of the YLE for $O(N)$ universality classes. Section 6.2 contains an expanded discussion on the methods and results of our original LPA' study as given in Ref. [45]. Similarly, in Section 6.3 we cover Ref. [44]. We describe the new computational method used which we pushed to higher orders in the derivative expansion and significantly improved the precision of our initial results.

We finish this thesis in Chapter 7 with a brief overview of this thesis as well as some concluding remarks on our results and their use. We further discuss other conceptual aspects associated with the Yang-Lee edge singularity, particularly its generalizations which found no central place for discussion in the main body of this text.

CHAPTER

2

UNIVERSALITY OF THE YANG-LEE EDGE: A MEAN FIELD TREATMENT

2.1 The critical phenomenon of water

The study of critical phenomenon and phase transitions has long been a central topic of study of physics. The first notion likely to come to mind is the of boiling or freezing water. A possibly more subtle direction would be ferromagnets and their demagnetization phase transition due to increasing temperature beyond the so-called Curie point. Even further in that direction, we have superconductors, deconfinement and chiral symmetry breaking in quantum chromodynamics (QCD), and so on into possibly more exotic realms such as topological phases transitions or dynamical phase transitions.

For the sake of discussion, let us have the phase diagram of water in mind, see Fig. 2.1. The solid lines in the figure depict the first order, discontinuous phases transitions. Say we keep the pressure constant at 1 atm and vary temperature to cross the liquid-gas line. Across this line, the density of the system experiences the discontinuity due the finite difference between the gas and liquid densities of water at these pressures.

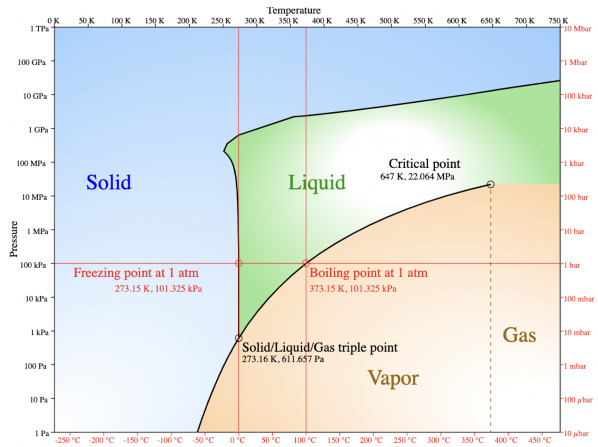


Figure 2.1: The phase diagram of water.

Increasing the pressure has the effect of decreasing the difference in densities of the two phases. The end result is that at some pressure there is a critical point (the end point of the first order line) which is a second order, or continuous, phase transition - the density being the continuous quantity in mind. Beyond this point, the distinction between the two phases no longer exists. Near this point, due to the very small difference in densities of the liquid and gas phases, large bubbles of gas can form in the liquid regions and vice versa since the free energy difference is small. Typically these fluctuations occur on microscopic levels, so what is meant by "large", is to say this length scale is much larger than the average inter-particle spacing, or so macroscopic. The underlying physical notion to the size of these fluctuations is the correlation length. At the critical point itself, these fluctuations can occur on any length scale available – all the way up to infinity for infinite volume systems. In other words, we are simply motivating the common remark that at second order phases transitions, the correlation length diverges.

There is an observable phenomenon associated with these large scale density fluctuations. Once the fluctuations become on the nanometer length scale, the slight difference in densities amounts to a slight difference in the refractive indices, and so visible light is scattered - critical opalescence.

Of course, there is also the notion of supercooling or superheating which is not captured in this figure. Let us focus on superheating as the other notions are conceptually identical. For very pure samples in the liquid phase at pressures say near 1 atm, the temperature can increased to some extent beyond the boiling point. The superheated fluid is to some degree stable, but a light disturbance will cause the sample to spontaneously undergo

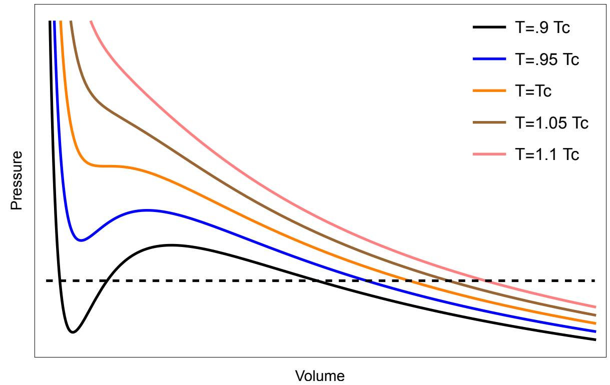


Figure 2.2: The van der Waals equation of state for various temperatures.

phase separation during which some amount transitions into the gaseous phase. These states of matter are then reasonably referred to as metastable. Though water at 1 atm has a boiling point of approixmately 373 Kelvin, it can be superheated up to 604 Kelvin (and supercooled to around 317 Kelvin)! [46] These endpoints of stability, beyond which the metastable phase becomes unstable, are generally referred to as the spinodal points. Beyond spinodal points, the system becomes unstable and experiences spontaneous phase separation known as the spinodal decomposition.

2.2 The universal van der Waals equation of state

A standard textbook introduction to the study of the liquid-gas phase transition of water is via the van der Waals equation of state:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \tag{2.1}$$

where a and b are parameters specific to the material and R is the ideal gas constant [1, 47]. This model extends the ideal gas law to account for the repulsive interactions approximated as an excluded volume, b, and for a modification of the pressure due to attractive interactions. Surprisingly, this basic relation has all the ingredients needed to understand the main results of this thesis.

In Fig. 2.2, we show a typical plot of the pressure vs. volume for a range of temperatures - isotherms. The feature to notice in this graph is the existence and disappearance of extrema as the temperature is varied. Furthermore, there are the multiple volumes corresponding to a single pressure for certain temperatures, the smallest volume, V_1,

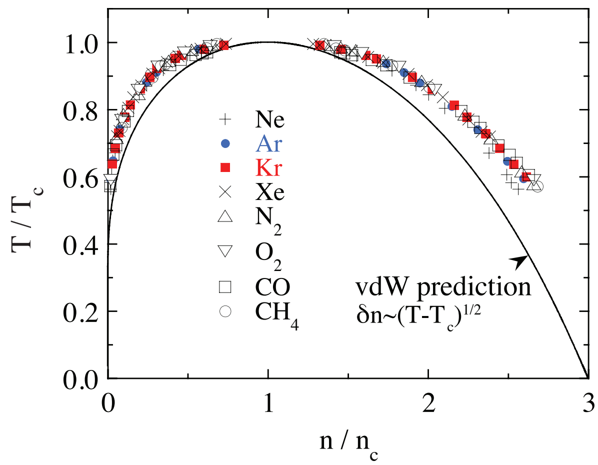


Figure 2.3: The Guggenheim plot.

corresponds to a liquid phase while the largest volume, V_2, corresponds to the gas phase. The middle solution is decided to be unphysical by noticing that the positive slope implies an increase in pressure as the volume increases. As the temperature is increased, the two physical solutions join producing an inflection point. Since at the transition/critical point, we also expect the liquid and gas volumes to coincide, we identify this temperature T_c with the critical temperature. See Fig. 2.2.

Let us denote P_c, V_c, and T_c as the values of the given parameters at the inflection point. We can re-express the van der Waals equation of state (2.1) in a form independent of the model dependent parameters using rescaled variables $\bar{P} = \frac{P}{P_c}$, $\bar{V} = \frac{V}{V_c}$, and $\bar{T} = \frac{T}{T_c}$:

$$\left(\bar{P} + \frac{3}{\bar{V}}\right)\left(\bar{V} - \frac{1}{3}\right) = \frac{8}{3}\bar{T}\,. \tag{2.2}$$

Regardless of the physical differences which produce different critical temperatures, pressures, and volumes, every system can be described via the above equation of state. Here, all critical values are unity, $\bar{P}_c = \bar{T}_c = \bar{V}_c = 1$. Pushing this observation further, it was predicted that the expression in Eq. (2.2) was a *universal* equation of state for all fluids - the law of corresponding states. Though the van der Waals equation does capture qualitative features of this phase transition, it does not do so well quantitatively. In defense of this idea though - that there is an underlying universality for all fluids - there is a famous plot dating back to 1945, the Guggenheim plot [48], see Fig. 2.3. This plot compares various gases near the critical point by plotting the rescaled temperature vs the rescaled density (denoted n in the plot). Amazingly, one does indeed see that all the data falls on the same, seemingly universal, curve. However, there is a clear discrepancy between the data and the van der Waals (vdW in the plot) prediction although it does a surprisingly good job. Lastly, this idea of universality is central to this thesis and can be set on a rigorous footing using the concept of the renormalization group, but more on this later.

2.2.1 Basic universal data

Common to the discussion of universality is a set of universal critical exponents which detail how properties of the system near criticality *scale* as we deviate from the critical point. Below we introduce some of the quantities:

$$\begin{aligned} M &\approx B\,(-t)^{\beta} \quad \text{at} \quad H=0 \quad \text{and} \quad t<0\,, && (2.3)\\ M &\approx B_c H^{1/\delta} \quad \text{at} \quad t=0\,, && (2.4)\\ \chi &\approx C_2^{+}\,t^{-\gamma} \quad \text{at} \quad H=0 \quad \text{and} \quad t>0\,. && (2.5) \end{aligned}$$

We have introduced the notion of a real valued *order parameter*, $M = V_2 - V_1 \geq 0$, which is non-zero below the transition, or $T < T_c$, and zero for $T > T_c$. Recall that V_1 is the liquid volume, the leftmost solution whereas V_2 is the gas volume, the rightmost solution [1, 47]. We have conjugate variable $H = \bar{P} - \bar{P}_c$ (note that $P_2 - P_1 = 0$), as well as the reduced temperature t which is defined in general as $t = \frac{T-T_c}{T_c}$. Lastly, with an abuse of notation to ease a transition to magnetic systems later on, we define $\chi \equiv \frac{\partial M}{\partial H}$, which near the critical point we can write $\chi = 2\frac{\partial V}{\partial P}$ where the two arises from the equality of the magnitudes of $\frac{\partial V_1}{\partial P}$ and $\frac{\partial V_2}{\partial P}$. In the above definitions, we have also introduced some *critical amplitudes* by the factors B, B_c, and C_2^+ from above. In general, unlike the critical exponents, the critical amplitudes are *non-universal*.

To determine B and β in Eq. (2.3), one may use Eq. (2.2) to obtain an implicit expression for $V_1 - V_2$ being that they both have the same temperature and pressure [1, 47]. Expanding this expression about small M yields

$$M \approx 4\,(t)^{\frac{1}{2}}\ . \tag{2.6}$$

The expression for δ and B_c in Eq. (2.4) is had simply by expanding the van der Waals pressure about criticality in temperature and volume, $\bar{V} = \bar{T} = 1$, and noting that near

T_c we can take $V_2 - V_1 = 2(V - V_c)$,

$$M \approx 2\left(\frac{2}{3}\right)^{\frac{1}{3}} H^{\frac{1}{3}} . \tag{2.7}$$

Lastly, the expression for γ and C_2^+ in Eq. (2.5) is had by the same expansion,

$$\chi \approx \left(\frac{1}{3}\right) t^{-1} . \tag{2.8}$$

We thus have some universal data, $\beta = \frac{1}{2}$, $\delta = 3$, and $\gamma = 1$. These predictions agree with a class of models that fall under the guise of a *mean field* approximation. The simplest mean field mode, the Landau-Ginzburg approach, will be discussed later. As for the amplitudes, while they are not universal, it turns out there do exist *universal amplitude ratios*. Again, this is another concept to be understood via the renormalization group. One of these ratios we may also go ahead and collect at the mean field level as it will be useful later on,

$$R_\chi = \frac{C_2^+ \, B^{\delta-1}}{B_c^\delta} . \tag{2.9}$$

At mean field, we find that $R_\chi = 1$.

2.2.2 Basic spinodal physics

So far, we have hastily glossed over a somewhat subtle discussion related to discarded the unphysical state occurring at temperatures less than T_c which had the positive slope, see Fig. 2.2. To address this, there is the *Maxwell construction* which accommodates for the *coexistence* of the liquid and gas phases at a particular pressure. We roughly follow Refs. [1, 47]. Imagining we have a system below the critical temperature and at a high enough pressure that we are sure to be in the liquid phase, if we decrease the pressure, at some point the fluid will begin to boil. At this pressure, we should have a situation similar to a boiling pot of water, some water remains liquid while other water has vaporized and escapes as steam. This mixture of gas and liquid can then have any volume ranging between the gas and liquid volume. Hence, this pressure corresponds to many volumes, see Fig. 2.4, and we have a flat portion of the equation of state. This is characteristic of a first order transition. Technically, the claim of the Maxwell construction is that the shaded areas in the plot should be equal, see [1, 47]. A deeper notion here, too, is that

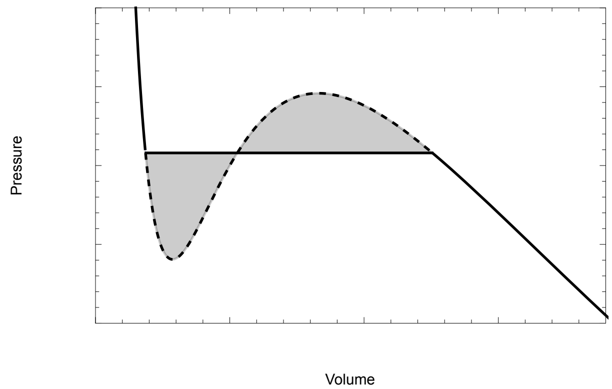


Figure 2.4: The Maxwell construction: the solid black line is the modified equation of state obtained by making the shaded areas equal.

the Maxwell construction results in a monotonic equation of state which yields a convex free energy, something not to be taken lightly.

Below we can begin introducing some unorthodox universal quantities, and now the real fun is about to be afoot: what happens to the states with volumes along the shaded curve which still seem to be physical since their slope is negative? These states correspond to the metastable states obtained by superheating and supercooling. The limit to their stability is given by the spinodals point which occur at the extrema of the isotherms. For instance, the leftmost extrema (the minimum) corresponds to the limit to which liquid can be expanded before phase separating into a mixture of liquid and gas phases.

As the temperature approaches T_c from below, these spinodal points tend to a critical value coinciding with P_c, (or $H = 0$). This set of points constructs the so called spinodal curve as depicted in Fig. 2.5. Within the spinodal curve, the states are unstable and spontaneously phase separate into a mixture of liquid and gas. Very near the critical point, the spinodal curve becomes symmetric in the volume and we have a nice rendition of a feature ubiquitous to phase transitions, power law behavior.

A typical aspect of a phase transition, or critical point, is a diverging susceptibility. In this sense, the spinodal point itself is also a critical point - at the extrema, χ must diverge as it is defined graphically as the inverse slope, $\chi = 2\left(\frac{\partial P}{\partial V}\right)^{-1}$. A point to be distinguished later is that the spinodal critical point is distinct from the liquid-gas critical point. The two critical points have an interesting interplay when the two are close together, or so when t is sufficiently small (and negative). This notion will be the topic of the following sections.

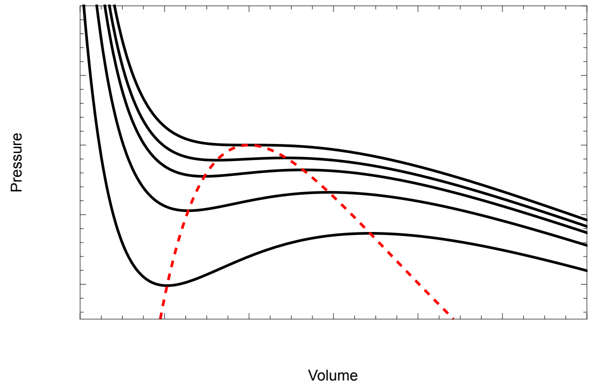


Figure 2.5: The spinodal curve.

2.2.3 New universality of the spinodal point

Let us denote volumes at the spinodal points as $V_{i,\mathrm{sp}}$ with $V_{1,\mathrm{sp}}$ being the leftmost and $V_{2,\mathrm{sp}}$ the rightmost. Let us also define the order parameter at the spinodal as $M_{\mathrm{sp}} = V_{2,\mathrm{sp}} - V_{1,\mathrm{sp}}$ and its conjugate H in terms of P at the spinodal point as $H_{\mathrm{sp}} = P_{1,\mathrm{sp}} - P_c$ and a set of related critical data,

$$
\begin{aligned}
M_{\mathrm{sp}} &= B_y \, (-t)^{\beta'} \, , && (2.10)\\
H_{\mathrm{sp}} &= D_y \, (-t)^{\Delta} \, , && (2.11)\\
M - M_{\mathrm{sp}} &= \bar{E}_y \, (H - H_{\mathrm{sp}})^{\sigma} \, , \qquad t = \mathrm{const} < 0 \, . && (2.12)
\end{aligned}
$$

We have introduced the exponent β' anticipating is equality to β for reasons that will be made clear once RG has been discussed. The last equation introduces a distinct exponent, σ which relates the deviation of M from M_{sp} to the analogous deviation for its conjugate variable - this form is functionally equivalent to the δ scaling form in Eq. (2.4) and some references *mistakenly* claim that $\delta = \frac{1}{\sigma}$.

We should also note that from the freedom to define $M_{\mathrm{sp}} = V_{2,\mathrm{sp}} - V_{1,\mathrm{sp}}$ or $M = V_{1,\mathrm{sp}} - V_{2,\mathrm{sp}}$, the two of which differing by a sign, we also have this freedom in defining H_{sp} to yield either a positive or negative value depending on the choice $H_{\mathrm{sp}} = P_{1,\mathrm{sp}} - P_c < 0$ versus $H_{\mathrm{sp}} = P_{2,\mathrm{sp}} - P_c > 0$. This is ultimately a symmetry of the liquid-gas system - the underlying Ising symmetry. As such, we expect two solutions to each scaling form in Eqs. (2.10)-(2.12), differing only in sign yet equal in magnitude. Taking the definition which gives $H_{\mathrm{sp}} < 0$ aligns with following the liquid phase, which is stable for the pressure range $P_c > P > P_{1,\mathrm{sp}}$, and so $H < 0$. (This also aligns with the magnetic notion of metastability for $H < 0$ and $M > 0$).

Something should be said as well about the case of $H < H_{\text{sp}}$, or equivalently $P < P_{\text{sp}}$, in Eq. (2.12) which yields a complex M. This is actually as we should expect since for fixed $t < 0$, there is no *real* volume along the isotherm which corresponds to $P < P_{1,\text{sp}}$ (for our other solution, $P > P_{2,\text{sp}}$). Interestingly, the exponent σ determines the angle which the order parameter takes into complex plane once the liquid phase becomes unstable (or the gas phase since there is this symmetry near the critical point) .

Now let us use the universal van der Waals equation of state, Eq. (2.2), to compute the term in the scaling forms associated with the spinodal critical point. First, we look to compute Eq (2.10). Again, the volumes at the spinodal points $V_{i,\text{sp}}$ are simply the locations of the two extrema, $V_{1,\text{sp}}$ being the left most and $V_{2,\text{sp}}$ the rightmost. The two individually tend to one, whereas their difference expanded about $T = T_c = 1$ yields

$$M_{\text{sp}} \;=\; \frac{4}{\sqrt{3}}\,(1-T)^{\frac{1}{2}}\ . \tag{2.13}$$

We see our $\beta' = \frac{1}{2}$ which does indeed equal our β. While this might been seen to be just an artifact of any mean field treatment, we will see this is a genuine equality via RG. Now we can compute Δ from Eq. (2.11) by expanding the difference about $T = T_c = 1$ and $V = V_c = 1$,

$$H_{\text{sp}} \;=\; \frac{8}{\sqrt{3}}\,(1-T)^{\frac{3}{2}}\ . \tag{2.14}$$

Thus, we get the mean field result that $\Delta = \frac{3}{2}$, as it should, since one might know of the scaling relation $\Delta = \beta\delta$. Lastly, we look to the scaling form in Eq. (2.12). First, we note that in expanding the pressure about the spinodals, it is obviously at least second order in $(V - V_{\text{sp}})$. The expansion then looks like

$$P \;=\; P_{\text{sp}} + \left.\frac{\partial^2 P}{\partial V^2}\right|_{V=V_{\text{sp}},|t|\ll 1} (V - V_{\text{sp}})^2\ , \tag{2.15}$$

assuming the second derivative is finite. We then expand the second derivative about $T = T_c$ as we are interested in the properties near the liquid-gas critical point,

$$\left.\frac{\partial^2 P}{\partial V^2}\right|_{V=V_{\text{sp}},t\ll 1} \;\approx\; 6\sqrt{3}\,\sqrt{-t}\,. \tag{2.16}$$

Secondly, we have $M - M_{\rm sp} \approx\, -2(V_1 - V_{1,\rm sp}))$ and $H - H_{\rm sp} = P - P_{\rm sp}$. The end result is

$$P = P_{\rm sp} + 3\sqrt{3}\,\sqrt{-t}\left(\frac{M - M_{\rm sp}}{2}\right)^2 \tag{2.17}$$

and so

$$M - M_{\rm sp} \;\; = \;\; \pm\frac{2}{3^{3/4}}\,(-t)^{-1/4}\,(H - H_{\rm sp})^{1/2}\;. \tag{2.18}$$

First, let us address the choice of sign here. Given that $M - M_{\rm sp} \approx\, -2(V_1 - V_{1,\rm sp}))$, for pressures greater than $P_{\rm sp}$, or $H > H_{\rm sp}$, we expect $V_1 < V_{1,\rm sp}$ and so $M - M_{\rm sp} > 0$. We should therefore choose the solution

$$M - M_{\rm sp} \;\; = \;\; \frac{2}{3^{3/4}}\,(-t)^{-1/4}\,(H - H_{\rm sp})^{1/2}\;. \tag{2.19}$$

We finally read off the mean field spinodal exponent, $\sigma = \frac{1}{2}$. A subtle feature of the above scaling is that the amplitude actually depends on the deviation from the critical temperature t. Thankfully, we can redefine our amplitude as $\bar{E}_y = E_y\,(-t)^e$ where we have found that the exponent e takes on $e = -\frac{1}{4}$ at mean field. As likely guessed, e is related to a combination of σ, β, and δ. We leave this point for later discussion.

Again, we also have the non-universal critical amplitudes $D_y = \frac{8}{\sqrt{3}}$, $B_y = \frac{4}{\sqrt{3}}$, and $E_y = \frac{2}{3^{3/4}}$ from above which can be put into ratios which are actually universal. To do so, we do need our previously computed amplitudes from the standard critical point, B, B_c and C_2^+. Let us define some new amplitude ratios now which will later be proven to be universal - part of the crux of this thesis.

To obtain the so called universal form of the van der Waals equation of state, we needed to rescale our parameters by the critical values. To obtain universal ratios, one may take a similar route by rescaling t and H to absorb the amplitudes B and B_c into a redefinition of t and H:

$$\bar{t} \;\; = \;\; B^{1/\beta}\,t\,, \tag{2.20}$$

$$h \;\; = \;\; B_c^{\delta}\,H\,. \tag{2.21}$$

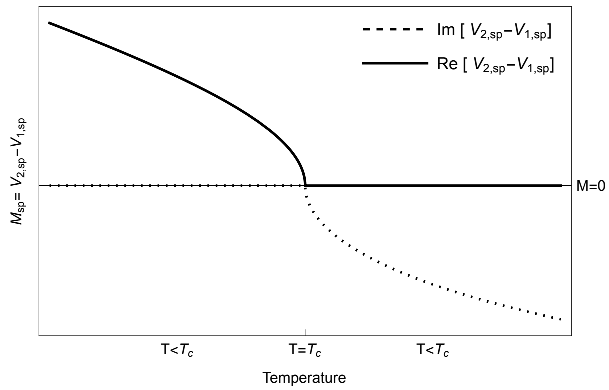


Figure 2.6: The order parameter at the spinodal point as the temperature varies through the transition.

In terms of the rescaled variables $\bar{t}$ and h, we have nicely

$$M \approx (-\bar{t})^{\beta} \quad \text{at} \quad h = 0 \quad \text{and} \quad t < 0\,, \tag{2.22}$$

$$M \approx h^{1/\delta} \quad \text{at} \quad \bar{t} = 0\,, \tag{2.23}$$

$$\bar{\chi} \approx \frac{C_2^+ B^{\gamma/\beta}}{B_c^{\delta}}\,\bar{t}^{-\gamma} \quad \text{at} \quad H = 0 \quad \text{and} \quad t > 0\,. \tag{2.24}$$

where $\bar{\chi} = \frac{\partial M}{\partial h}$. From the above rescaling, we could make the heuristic claim that R_χ is universal by recognizing it as the amplitude for the rescaled $\bar{\chi}$.

Now, for the spinodal scaling forms we have in general,

$$M_{\rm sp} = \frac{B_y}{B^{\beta/\beta'}}\,(-\bar{t})^{\beta'}\,, \tag{2.25}$$

$$h_{\rm sp} = \frac{B_c^{\delta} D_y}{B^{\frac{-\Delta}{\beta}}}\,(-\bar{t})^{\Delta} \tag{2.26}$$

where upon using the scaling relation $\beta = \beta'$ and $\Delta = \beta\delta$ we obtain

$$M_{\rm sp} = \frac{B_y}{B}\,(-\bar{t})^{\beta}\,, \tag{2.27}$$

$$h_{\rm sp} = \frac{B_c^{\delta} D_y}{B^{\delta}}\,(-\bar{t})^{\Delta}\,, \tag{2.28}$$

$$M - M_{\rm sp} = \frac{E_y}{B^{1-\delta\sigma}\,B_c^{\delta\,\sigma}}\,(-\bar{t})^{e}\,(h - h_{\rm sp})^{\sigma}\,. \tag{2.29}$$

It is a claim, though as yet unsubstantiated, of this thesis that the above amplitude ratios are universal and were unknown beyond mean field and large N for three dimensional systems. Let us define them:

$$R_m = \frac{B_y}{B}\,, \tag{2.30}$$

$$R_h = \frac{B_c^\delta\, D_y}{B^\delta}\,, \tag{2.31}$$

$$R_e \equiv \frac{E_y}{B^{1-\delta\sigma}\, B_c^{\delta\,\sigma}}\,, \tag{2.32}$$

$$|z_c| = \frac{B^{1/\beta}}{B_c^{1/\beta}\, D_y^{1/\beta\delta}}\,. \tag{2.33}$$

See Section 4.4.3 for the reasoning behind the introduction of z_c. In summary, the picture is still a bit obscure and the above ratios are discussed in a slightly different context that the title of this thesis suggests. This will be remedied by the discussion of scaling functions and RG in Chapter 4.

Explicitly, the universal van der Waals expressions yield the amplitudes $B = 4$, $B_c = 2\left(\frac{2}{3}\right)^{1/3}$, $B_y = \frac{4}{\sqrt{3}}$, $D_y = \frac{8}{\sqrt{3}}$, and $E_y = \frac{2}{3^{3/4}}$, and

$$R_m = \frac{1}{\sqrt{3}}\,, \tag{2.34}$$

$$R_h = \frac{2}{3^{3/2}}\,, \tag{2.35}$$

$$R_e = \frac{1}{3^{1/4}}\,, \tag{2.36}$$

$$|z_c| = \frac{3}{2^{2/3}}\,. \tag{2.37}$$

2.2.4 Mapping the spinodal to the Yang-Lee edge singularity

In this context, it may seem a bit strange to consider this, but given the plots we have seen, it seems the spinodals disappear for $T > T_c$; however, we can still find similar critical points by looking at complex pressures! These critical points in the high temperature region will be seen to be the Yang-Lee edge singularities. Neatly, we will be able to get a basic look at them by a less abstract discussion of the spinodal physics. Though there is a crutch in this approach in that we need to discuss a mapping between the two critical points, the spinodal point and the Yang-Lee edge singularity.

To begin, notice that in the mean field expressions, Eq. (2.13), Eq. (2.14), and Eq. (2.19), it seems we could naively take $t > 0$, thus having the minus sign produce a factor

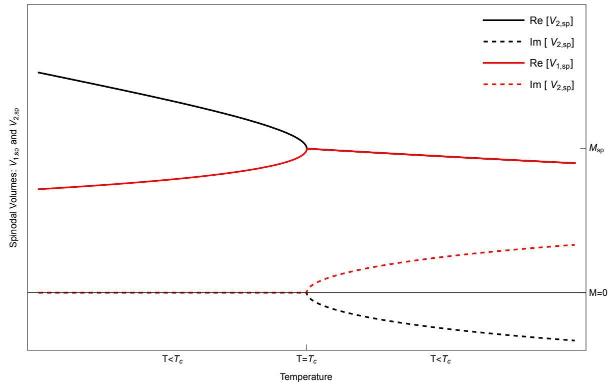


Figure 2.7: The spinodal volumes as the temperature varies through the transition.

of i in the related expressions. While this is instructive in the mean field analysis, it is a bad general practice because the exponents may not be integers which obscures some results. In general, this analytic continuation needs a bit more care, see Section 4.4.3.

Let us denote the values of the order parameter and its conjugate under our naive mapping as $M_{\text{sp}} \to M_c$ and $H_{\text{sp}} \to H_c$. The naive mean field result is that M_c and h_c are purely imaginary whereas $M - M_c$ and $H - H_c$ have a relative phase difference of $-\frac{\pi}{4}$. Thus, if H is purely imaginary and $|H| < |H_c|$, the factor $(H - H_c)^{1/2}$ has a phase factor $(-i)^{1/2} = e^{-i\pi/4}$. Combining this with the overall phase yields an overall factor of $-i$. In summary, if H is purely imaginary and $|H| < |H_c|$, then $M - M_c = -i\,\#$ where $\# \in \mathbb{R}^+$. Surprisingly, this is exactly what we expect from the result of the Lee-Yang theory of phase transitions in Chapter 3. In general, this overall phase can be seen to be $\Delta\phi = -\frac{\pi}{2}(1 - \sigma)$, see Section 4.4.3.

Furthermore, in the vicinity of the critical point, we need to do no more work to find the analogous exponents and universal ratios associated with the Yang-Lee edge singularity. As will be proven, the two sets are identical! The set in Eqs. (2.30)-(2.33) together with σ describe both the spinodal point and the Yang-Lee edge singularity! The mean field results of Eqs. (2.34)-(2.37) together with $\sigma = 1/2$, of course, capture both as well. Let us write this below with the correct generalization of Eq. (2.29):

$$M_c = i\,\frac{B_y}{B}\,\bar{t}^{\beta}\,, \tag{2.38}$$

$$h_c = i\,\frac{B_c^{\delta} D_y}{B^{\delta}}\,\bar{t}^{\Delta}\,, \tag{2.39}$$

$$M - M_c = \frac{E_y}{B^{1-\delta\sigma}\,B_c^{\delta\,\sigma}}\,\bar{t}^{e}\,e^{-i\frac{\pi}{2}(1-\sigma)}\,(h - h_c)^{\sigma}\,. \tag{2.40}$$

Let's turn back to the original equation of state, Eq. (2.2), and see what's happening. First, we plot the real *and* imaginary parts of the order parameter evaluated at the spinodal as the temperature is varied through the transition in Fig. 2.6 (we now allow the order parameter to take on complex values). We see that, indeed, M becomes purely imaginary in the high temperature region. This fact is not an artifact of the mean field analysis, but is instead due to the Lee-Yang theorem, see [15]. This plot also makes clear the equality of $\beta = \beta'$, at the mean field level at least. A last observation - the similarity of the plot for $T < T_c$ in Fig. 2.6 to the plot of the magnetization in the Ising model is no accident. The Ising model near its critical point can be mapped on the liquid gas system, which is exactly what we have done. Thus the notation of M, the magnetization, and H being the magnetic field.

Let us also compare the behavior of the spinodal volumes $V_{1,\text{sp}}$ and $V_{2,\text{sp}}$ as function of the temperature. The question is how do they behave for $T > T_c$, see Fig. 2.7. We see that below the transition, the volumes are real whereas above the critical temperature they become complex, though *not purely imaginary*. It is a property of the proper order parameter and its conjugate at the Yang-Lee edge singularity that they are purely imaginary.

At the (ordinary) critical point, the spinodals merge as the critical temperature is approach from below; however, one may equally well say that the Yang-Lee edge singularities merge as the critical temperature is approach *from above*. This interpretation of events will surprisingly be the more fundamental perspective which is related to the behavior of the partition function zeros.

A final point is in order in regards to results beyond mean field. As we will see, the spinodal point does not actually remain real once one obtains results beyond mean field. While the YLE occurring at a purely imaginary external field H_c is ensured by the Lee-Yang theorem, not such relation holds for the spinodal point. Beyond mean field, the critical magnetic field H_s at which the spinodal point occurs is shifted off the negative real axis by a phase $\varphi = \pi\left(\frac{3}{2} - \beta\delta\right)$. See also Ref. [13], Section 4.4.3, and Fig. 6.3.

In summary, starting from the van der Waals equation which gives a basic qualitative, though not quantitative, description of the liquid-gas phase transition, we argued for a crude sense of universality. This was the statement of the law of corresponding states - that all fluids near their critical points are described the universal van der Waals equation of state which due to the use of rescaled variables had no reference to the model dependent quantities such as T_c, P_c, and V_c. Furthermore, this equation of state yields predictions for a set of universal critical exponents and amplitude ratios governing the liquid-gas critical point. Near enough to the critical point, aspects of the spinodal point - a critical point in its own right - become universal. We were also able to see that the spinodal point is intimately related to another critical point occurring at complex pressures and volumes, though purely imaginary in order parameter M and external field H, the Yang-Lee edge singularity.

While the exponent σ governing the critical behavior near the spinodal point and, fascinatingly enough, the Yang-Lee edge as well, has been known for decades, other aspects of these critical points are indeed universal which were not as well studied before the works involved in this thesis. The extended set of universal quantities includes three universal ratios, R_m, R_h, and R_e, which govern the scaling properties of both the spinodal and Yang-Lee edge singularities *near* the ordinary critical point. In the following sections, we will elaborate more on the details of these universal quantities as well as their values for the physical relevant case of three dimensions as well as (semi-)analytic results in Chapter 5.

CHAPTER

3

REVIEW OF THE LEE-YANG THEORY OF PHASE TRANSITIONS

Although it is intimately related to the spinodal point, the most fundamental understanding of the Yang-Lee edge singularity is given in terms of what has become known as the Lee-Yang theory of phase transitions. Given that the partition function contains all the physical information of a given system with typical arguments being the temperature and fugacity-type variables z, it is natural to ask what are its properties over its complexified domain. For instance, if one is looking at Taylor expanding about the vanishing baryon chemical potential $\mu = 0$, a temperature like variable, the radius of convergence of this expansion is limited by the nearest singularity to the origin. As we will see, this singularity turns out to be the Yang-Lee edge singularity, see [49].

Many approaches to physics have profited well from extending their notions into the complex plane. Firstly, dispersion relations demonstrate the connection between the real and imaginary parts of response functions. There is Reggge theory which sees resonances, bounds states, and scattering states as intimately related under analytic continuation of the systems parameters into the complex plane. When studying asymptotic series, there is the study of Stokes lines which represent a sort of branch cut which upon crossing one must reshuffle their solutions. Some studies have also found the curious observation that Lee-Yang zeros fall on top of the Stokes lines, [50, 51, 52, 53].

For the case of phase transitions, as mentioned in the intro, the study of the analytic structure of the partition function in complex plane, particularly its zeros, was initiated by Lee and Yang in 1952, [14, 15]. Many works have followed which have extended considerably the original ideas introduced in [14, 15]. In the following sections, we will discuss the Lee-Yang theorems as well as some of its extensions, the edge of the distribution of zeros - the Yang-Lee edge singularity, and some more general extensions the ideas.

3.1 The Lee-Yang theorems

Let us review the ideas in the context of the Ising model, the most basic system one could think of which exhibits a phase transition. Consider a set of spins $\{s_i\}$ on a d-dimensional lattice which can take the values ± 1 which interact with an external magnetic field H_i which can take on any real, or complex, number. The spins interact with each other at a finite temperature T via an exchange interaction J which we will take to be nearest neighbor. The Hamiltonian for a given configuration of N spins $\{s_i\}$ is

$$\mathcal{H}_N\left[\{s_i\}\right] \;=\; -\sum_{\langle ij\rangle} J\, s_i\, s_j - \sum_{i=1}^{N} H_i\, s_i \tag{3.1}$$

where $\langle ij\rangle$ denotes the sum over nearest neighbors. We check the signs by the understanding that for a ferromagnetic system, $J > 0$, aligning spins with their neighbors and with H minimizes the energy. Unless otherwise stated, it will be assumed that we are discussing the ferromagnetic Ising model.

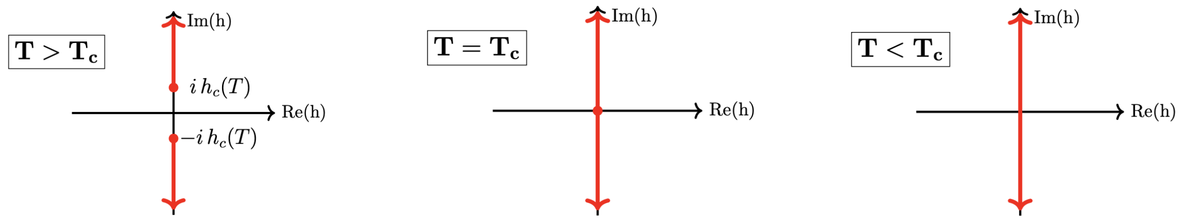
The partition function is given by the sum over the spin configurations as

$$\mathcal{Z}_N \;=\; \sum_{\{s_i\}} \exp\left(-\beta\, \mathcal{H}_N\left[\{s_i\}\right]\right)\,. \tag{3.2}$$

Following Mattis [54], first recognize that

$$e^{\beta H(s_i-1)} = \begin{cases} 1 & s_i = 1 \\ z & s_i = -1 \end{cases}\,. \tag{3.3}$$

By factoring out N factor of $e^{\beta H}$ allows one to see that the partition function for the



(a) $T > T_c$ (b) $T = T_c$ (c) $T < T_c$

Figure 3.1: Complex H zeros for systems following the Lee-Yang Theorem. In the limit of $T \to T_c$, the zeros pinch the axis whereas for $T < T_c$, the zeros smoothly cross the axis.

Ising model can be rewritten as a polynomial in the fugacity $z = e^{-2\beta H}$ (up to a term):

$$\mathcal{Z}_N \;=\; e^{N\beta H} \sum_{n=0}^{N} a_n \, z^n \,. \tag{3.4}$$

The term z^m corresponds to the configurations with m spins down ($s_i = -1$). If $J = 0$, the one would have simply that the coefficients a_n are the binomial coefficients. By turning on the interactions, the a_n's are no longer binomial coefficients, but sums of terms of $n\, e^{\pm m\, \beta J}$ where n and m are both positive integers. For instance, here is $\sum a_n z^n$ for $N = 5$ with non-zero J

$$e^{5\beta \mathrm{J}} z^5 + 5e^{\beta \mathrm{J}} z^4 + \left(5e^{-3\beta \mathrm{J}} + 5e^{\beta \mathrm{J}}\right) z^3 + \left(5e^{-3\beta \mathrm{J}} + 5e^{\beta \mathrm{J}}\right) z^2 + 5e^{\beta \mathrm{J}} z + e^{5\beta \mathrm{J}} \tag{3.5}$$

versus with $J = 0$

$$z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1 \,.$$

Notice also that $a_n = a_{N-n}$. By factoring out $e^{\beta N K}$ one could simplify the analysis further [54].

The first paper of Lee and Yang [14] proves the existence and continuity of the infinite volume free energy

$$\mathcal{F} \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_N \tag{3.6}$$

for any positive real value of z. Secondly, noting that $\mathcal{Z}_N$ has N zeros $\{z_n\}$ occurring in general in the complex plane, they proved that $\mathcal{F}$ and all its derivatives are analytic functions *in regions free of zeros*. This nicely introduces the notion that the occurrence of zeros, and thus non-analyticities, indicate the possibility of a phase transition.

Referring back to Eq. (3.4), the coefficients a_n are positive functions of β and J. It follows then that none of the zeros of $\mathcal{Z}_N$ in terms of z could occur at real values and so must lie in the complex z-plane, off the real axis. This is the statement that no phase transition can occur at finite volume. It is only in the thermodynamic limit that the zeros could approach the real axis and produce the non-analyticities observed at phase transition.

In general, in the thermodynamic limit the zeros could form areas, lines, or tend towards isolated points. Strikingly, Lee and Yang proved in their second paper [15] that all the zeros form continuous lines along the purely imaginary values of h, or so the unit circle in the complex z-plane - the famous Lee-Yang circle theorem. Remarkably, the theorem holds regardless of the lattice type or dimension, the only condition being that the system is ferromagnetic, $J > 0$.

Before we relate the zeros to the phase structure, let us address the features and generalizations of the Lee-Yang theorem. the following discussion of the literature is by no means complete, as the number of papers on the subject is rather immense. But, we will try to focus on the result most relevant to this thesis - the $O(N)$ universality classes. For a nice discussion, see [55] as well as Kurtze's thesis [56].

A reasonable first question may be to ask whether this depends on the spin dimension or if it is specific to nearest neighbor. The original Lee-Yang theorem only requires a ferromagnetic exchange interaction, so the theorem holds beyond nearest neighbor. It did not address arbitrary spin Ising models. In 1968, Asano and Suzuki in separate works proved the theorem for spin 1 and spin 1, 3/2 respectively [57, 58]. Later that same year, Griffiths proved that the Lee-Yang theorem holds for arbitrary and even continuous spin ferromagnetic Ising models [59].

As another generalization of the Ising model, one may allow for the spins to have more components and ask whether the theorem stills holds. For two components, we have the so-called XY model whereas for three components, the Heisenberg model. The Lee-Yang theorems pertaining to the Heisenberg model was first produced by Asano [60] and Suzuki [61]. Interestingly, Suzuki provides a method for representing higher spin Heisenberg systems in terms of a spin $\frac{1}{2}$ system. Suzuki used this later in generalizing the theorems to higher order spin interactions, [62, 63]. However, the most general proof for the fully anisotropic ferromagnetic (quantum) Heisenberg model was given by Suzuki and Fisher [64] wherein Suzuki's spin reduction method was also used. They additionally showed that the theorem holds for many-spin interactions in the Ising model.

For the question of fields theories, the original relation was extended to a single component ϕ^4 Euclidean quantum field theory stemming from the relation between the classical Ising model and ϕ^4 theory by Simon and Griffiths [65]. The theorem was later extended to the case of classical XY and Heisenberg models ($N = 2$ and $N = 3$ component field theories) by Dunlop and Newman [66]. A separate technique was used by Lieb and Skokal [67] to prove the theorem for 1 and 2 component classical ferromagnets. They also include a proof for $N \geq 3$ for sufficiently anisotropic ferromagnetic interactions.

Lastly, the limit of $N \to \infty$ yields the so called spherical model which is exactly solvable. This model corresponds to a field theory in as many components. Here it was demonstrated that the Lee-Yang theorem holds in arbitrary dimension by Kurzte and Fisher [29]. It is also notable that the theorem has not been proven for N component ϕ^4 theories with $N > 3$, although the results of this thesis indicate that it does, at least for regions near enough to the origin.

Note that the Lee-Yang theorem was not proved to hold for *quantum* XY models. For this case, the zeros can take much more interesting behaviors, see [68, 69]. There are regions where the Lee-Yang theorem does hold, and in fact one can see a crossover behavior between two types of distributions of the zeros, [70]. See also [71].

For a final note on the ferromagnetic zeros, we remark that for more exotic like that of the Blume-Capel model [72, 73], there are results due to Suzuki [62, 63] that imply while the theorem does not hold in general, it does hold in regions where the system is sufficiently "ferromagnetic." The transition between these regions yields a bifurcation in the distribution of zeros which leads to very interesting implications for the Yang-Lee edge singularity and its generalizations. The original works exploring this situation are [74, 37, 38, 39, 40]. See also [75, 76, 77] as well as [78, 79].

While everything we discussed above is for the case of ferromagnetic zeros, one may be curious about anti-ferromagnetic models. The situation here is nowhere near as nice as for the ferromagnetic systems - there are no general theorems for the location of anti-ferromagnetic zeros. Although, there are results of the type which prove regions which should be free of zeros, see [80]. For a strikingly display of the transition from ferromagnetic zeros to anti-ferromagnetic, see Figures 4 and 5 in [68]. See also [81] for antiferromagnetic Ising as well as [82] ,[71] for antiferromagnetic XY zeros, [83] for antiferromagnetic Heisenberg, and [84, 33, 85, 86] for antiferromagnetic Ising with nearest-neighbor and next-to-nearest-neighbor interactions, the antiferromagnetic Blume-Capel model, and others. We leave the rest to the references.

There is additionally another huge field of study concerning the zeros in the complex temperature plane, initially studied by Fisher in [87] in which he relates the logarithmic singularity of the 2D Ising model to the behavior of the density of zeros. Furthermore, it has been show that while the complex temperature zeros fall on lines for the isotropic Ising model, the zeros fall onto areas for the anisotrpoic case, see [55]. They conclude that rather than forming lines, in general the complex temperature zeros fall onto areas as opposed to behaviors seen in studies such as [88]. Particularly relevant are also [50, 51] in their relation to the QCD phase diagram.

Finally, we mention some reviews which discuss some of the materials mentioned above as well as others, [89, 90].

3.2 Thermodynamics from the density of zeros

In the thermodynamic limit the zeros for the systems we are interested in are distributed along lines on the purely imaginary H axis forming branch cuts which are symmetric about the real axis due to the reality of $\mathcal{Z}_N$. These branch cuts represent the non-analyticities that we observe near phase transitions. For instance, at low temperatures we expect to have two solutions to the magnetization for vanishing field $M(H=0)=\pm M$ thus implying a Riemann sheet structure of $M(H)$. In this case, the zeros must cross the real axis at $H=0$. Varying H through zero amounts to crossing the branch cut thus forcing the solution to jump between the Riemann sheets - something very non-analytic - a manifestation of the first order phase transition.

More interestingly, the location of the zeros depends on the temperature and one might imagine that for certain temperatures, say $T>T_c$, the zeros stay away from the real axis. In this case, the branch cuts do not intersect the real axis and there is no phase transition for a *real value* of the magnetic field H. See Fig. 3.1 for a general depiction

For the Ising model, the relevant variable is the temperature. Here one finds that for $T>T_c$ there is a gap in the distribution of zeros which closes as T approaches T_c from above. This gap means that one can smoothly interpolate from positive to negative values of the magnetic field - no phase transition. Specifically at $T=T_c$, tuning to $H=0$ amounts to tuning to a second order phase transition; therefore the gap must close only in the limit of $T\to T_c$ from above. Finally, for $T<T_c$ the gap is closed and there is a first order phase transition at $H=0$ due to the broken $\mathcal{Z}_2$ symmetry. We highlight these features in Fig. 3.1.

Let us now turn to deriving expressions for the thermodynamics in terms of the zeros themselves, in particular the density of zeros $g(H, t)$, which will clarify the discussion above. The original derivation of these results seems to be due to Suzuki [91, 92, 58] and Abe [93]. Note that the years align with the appearance of the Lee-Yang theorem extensions. In what follows, we will improve upon the results of Suzuki and Abe to better clarify features of the Yang-Lee edge singularity.

Since the partition function of Eq. (3.2) is a simple polynomial in z, we can rewrite this as a product over its N zeros $\{z_n\}$

$$\mathcal{Z}_N = \mathcal{N} e^{N\beta H} \prod_{n=1}^{N} (z - z_n) \tag{3.7}$$

where $\mathcal{N}$ is some unimportant constant. In the thermodynamic limit, the zeros can be described by a density function $g(H, t)$ where we denote the dependence on $t \propto T - T_c$. The free energy in the thermodynamic limit is given by

$$-\beta\mathcal{F} = \lim_{N\to\infty} \frac{1}{N} \mathcal{Z}_N = \beta\, H + \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log\left(z - z_n\right) \tag{3.8}$$

where we've ignored the constant term.

Next, we introduce the density of zeros, $g(z, t)$ which has explicit dependence on both the fugacity z and the reduced temperature $t = \frac{T - T_c}{T_c}$. The density has the interpretation that $Ng(\omega, t)d\omega$ depicts the number of zeros lying between ω and $\omega + d\omega$. This motivates the normalization condition:

$$\int g(\omega, t) d\omega = 1\,. \tag{3.9}$$

Additionally, due to the reality of $\mathcal{Z}$ the density is symmetric about the real axis: $g(z^*, t) = g(z, t)$. In terms of $g(z, t)$ we then have

$$-\beta\mathcal{F}\left(H, t\right) = \beta H + \int g(\omega, t) \log\left(z - \omega\right) d\omega\,. \tag{3.10}$$

For systems obeying the Lee-Yang circle theorem, $\omega = e^{i\,\theta}$ and we can write

$$-\beta\mathcal{F}\left(H, t\right) = \beta H + \int_0^{2\pi} g(\theta, t) \log\left(z - e^{i\,\theta}\right) d\theta\,. \tag{3.11}$$

Furthermore, using $g(\theta,t)=g(-\theta,t)$ we can simply the integral as follows:

$$\begin{aligned}
&= \frac{1}{2}\left[\int_0^{2\pi} g(\theta,t)\log\left(z-e^{i\theta}\right)d\theta+\int_0^{2\pi} g(\theta,t)\log\left(z-e^{i\theta}\right)d\theta\right], && (3.12)\\
&= \frac{1}{2}\left[\int_0^{2\pi} g(\theta,t)\log\left(z-e^{i\theta}\right)d\theta-\int_0^{-2\pi} g(-\theta,t)\log\left(z-e^{-i\theta}\right)d\theta\right], && (3.13)\\
&= \frac{1}{2}\left[\int_0^{2\pi} g(\theta,t)\log\left(z-e^{i\theta}\right)d\theta+\int_{-2\pi}^{0} g(\theta,t)\log\left(z-e^{-i\theta}\right)d\theta\right], && (3.14)\\
&= \frac{1}{2}\left[\int_0^{2\pi} g(\theta,t)\log\left(z-e^{i\theta}\right)d\theta+\int_0^{2\pi} g(\theta,t)\log\left(z-e^{-i\theta}\right)d\theta\right], && (3.15)\\
&= \frac{1}{2}\int_0^{2\pi} g(\theta,t)\log\left(z^2-2z\cos\theta+1\right)d\theta\,, && (3.16)\\
&= \int_0^{\pi} g(\theta,t)\log\left(z^2-2z\cos\theta+1\right)d\theta\,. && (3.17)
\end{aligned}$$

Let us rewrite our expression in terms of H:

$$\begin{aligned}
-\beta\mathcal{F} &= \beta H+\int_0^{\pi} g(\theta,t)\log\left[e^{-2\beta H}\left(e^{-2\beta H}-2\cos\theta+e^{2\beta H}\right)\right], && (3.18)\\
-\beta\mathcal{F} &= \int_0^{\pi} g(\theta,t)\log\left[2\left(\cosh 2\beta H-\cos\theta\right)\right]. && (3.19)
\end{aligned}$$

We have the magnetization $M=-\frac{\partial\mathcal{F}}{\partial H}$ given by

$$M=2\sinh\beta H\int_0^{\pi}\frac{g\left(\theta,t\right)}{\cosh\beta H-\cos\theta}d\theta\,. \tag{3.20}$$

Thus, given the density of zeros, one can construct the magnetization and the free energy. We can also deduce some properies of $\mathcal{F}$ and M. First, $M(-H)=-M(H)$, which holds even for complex H! Furthermore, we have $M(H^*)=M^*(H)$. The free energy satisfies similar relations: $\mathcal{F}(-H)=\mathcal{F}(H)$ and $\mathcal{F}(H^*)=\mathcal{F}(H)$. However, while knowledge of the density of zeros is clearly very powerful, obtaining expressions for the density is in general very difficult. Outside of the 1D Ising model and mean field, one typically has to resort numerical methods.

Let us quote the results for the density for the 1D Ising model and for mean field theory. For the 1D Ising model, the density obeys [15, 94]

$$g(\theta,t) = \frac{1}{2\pi}\begin{cases} \frac{\sin\frac{\theta}{2}}{\sqrt{\sin^2\frac{\theta}{2}-\sin^2\frac{\theta_c}{2}}} & |\theta| > \theta_c \\ \\ 0 & |\theta| < \theta_c \end{cases} \tag{3.21}$$

where $\theta_c = \arccos\left(1-2e^{-2\beta J}\right)$. Alternatively, see Ref. [15] for the expression for the zeros themselves. See Fig. 3.2 for the behavior of the zeros of the 1D Ising model. We quote below the implicit equation for mean field density of Ref. [92]:

$$\theta^2 \;=\; \left(\frac{4}{3}\right)^2\left(t+\frac{1}{3}\frac{g^2}{4\pi^2}\right)\left(t+\frac{4}{3}\frac{g^2}{4\pi^2}\right)^2 . \tag{3.22}$$

The above expressions are plotted in Fig. 3.3.

Let us now derive a basic relation between the density of zeros and the discontinuities experienced at phase transitions. First, we recall that the density of zeros for temperatures $t > 0$ is expected to vanish on the interval $\theta \in (-\theta_c, \theta_c)$. Whereas for $t < 0$, the density is smooth and finite about the origin.

It will prove convenient to introduce the variable $h = 2\beta H$. Let us suppose we are near the transition (though possibly the first order transition) and expand M about small h:

$$M \;=\; 4h\int_0^\pi d\theta\frac{g\left(\theta,t\right)}{h^2+\theta^2} \tag{3.23}$$

where we also expand in θ the reason for which will become clear. Let us focus on the case $M(h=0)$. Inspecting the above expression, we recognize the appearance of the delta function

$$\delta(\theta) \;=\; \frac{1}{\pi}\lim_{h\to 0}\frac{h}{h^2+\theta^2} \tag{3.24}$$

so that we have the relation

$$M(h=0) \;=\; 4\pi\int_0^\pi \delta(\theta)g(\theta,t)d\theta\,, \tag{3.25}$$

$$M(h=0) \;=\; 2\pi\, g(0,t)\,. \tag{3.26}$$

Thus, the value of the magnetization at vanishing field is given by the density of zeros!

For $t < 0$, we expect M to be finite and thus that the density of zeros is non-zero at the origin. On the other hand, for $t > 0$, we expect M to vanish at $H = 0$ and so there must be a gap in the distribution of zeros.

Noting that had we really have the relation $M(0^{\pm}) = \pm 2\pi\, g(0,t)$ so that

$$\lim_{\epsilon\to 0} \frac{1}{2}\left(M(\epsilon) - M(-\epsilon)\right) = 2\pi g(0,t) \tag{3.27}$$

which is just the more explicit statement that the density of zeros gives the discontinuity of M at the first order phase transition.

We can derive a more general expression between the magnetization and the density of zeros. Consider the discontinuity in the magnetization from two opposing sides on the imaginary axis. Taking $h = a + i\,b$, one obtains the expression

$$\frac{M(a+i\,b) - M(-a+i\,b)}{2} \;=\; 2\int_0^{\pi} d\theta\, g(\theta,t)\left\{\frac{a}{a^2+(b+\theta)^2} + \frac{a}{a^2+(b-\theta)^2}\right\}.$$

Furthermore, we have that the discontinuity in the magnetization about the cut is given by the real part of the magnetization. This is due to the relation $M(-H) = -M(H)$ together with $M^*(H) = M(H^*)$:

$$\begin{aligned}
\operatorname{Re} M(i\,b) &= \lim_{a\to 0}\frac{M(a+i\,b) - M(-a+i\,b)}{2}\,, && (3.28)\\
&= \lim_{a\to 0}\frac{M(a+i\,b) + M(a-i\,b)}{2}\,, && (3.29)\\
&= \lim_{a\to 0}\frac{M(a+i\,b) + M^*(a+i\,b)}{2}\,, && (3.30)\\
&= 2\pi\int_0^{\pi} d\theta\, g(\theta,t)\left\{\delta(b-\theta) + \delta(b+\theta)\right\}. && (3.31)
\end{aligned}$$

Extending the integral's lower bound to $-\pi$ then yields the relation

$$\operatorname{Re} M(i\,b) \;=\; 2\pi\, g(b,t)\,. \tag{3.32}$$

What the above expressions tells us is that for $t > 0$, the magnetization is purely imaginary for *purely imaginary fields below the edge*! For purely imaginary fields beyond H_c, the magnetization picks up an additional real component thus moving off into the complex plane. We also write the full expression for $\mathrm{Re}(M)$ for complex H:

$$\mathrm{Re}(M)(H) \;=\; 2\int_0^{\pi} d\theta\, g(\theta,t) \left\{ \frac{a}{a^2+(b+\theta)^2} + \frac{a}{a^2+(b-\theta)^2} \right\} . \tag{3.33}$$

One may ask what information is contained in the imaginary part of the magnetization.

$$\mathrm{Im}\, M(a+i\,b) \;=\; \frac{1}{2i}\left(M(a+i\,b) - M^*(a+i\,b)\right) , \tag{3.34}$$

$$\;=\; \frac{1}{2i}\left(M(a+i\,b) - M(a-i\,b)\right) . \tag{3.35}$$

Given the second expression, the imaginary part can be interpreted as the discontinuity in the magnetization over the real axis. So, not particularly informative, at least in this setting, since we expect that for real h, the magnetization is also real - no discontinuity. However, this notion could be useful for studying discontinuities on the real axis.

Using similar methods as for $\mathrm{Re}(M)$, one finds similar expressions for $\mathrm{Im}(M)$:

$$\mathrm{Im}\, M(H) \;=\; 2\int_0^{\pi} d\theta\, g(\theta,t) \left\{ \frac{(\theta-b)}{a^2+(\theta-b)^2} - \frac{(\theta+b)}{a^2+(\theta+b)^2} \right\} . \tag{3.36}$$

First, note that for $b=0$, $\mathrm{Im}(M)=0$, as we expect. Next, for purely imaginary H, the kernel over which $g(\theta,t)$ is integrated has simple poles at $\theta=b$. Thus, in order to have a a finite $\mathrm{Im}(M)$, the density must combat these poles. As we will see, it does indeed behave in such a way as to keep the magnetization continuous. More on $\mathrm{Im}(M)$ we be discussed later.

Similar expressions can be derived for the susceptibility above and below the transition. Let us start with expressions for vanishing fields:

$$\chi_+ \;=\; 4\beta \int_0^{\pi} d\theta\, g(\theta,t) \left\{ \frac{\cosh\, h}{\cosh\, h - \cos\,\theta} - \frac{\sinh^2\, h}{(\cosh\, h - \cos\,\theta)^2} \right\} , \tag{3.37}$$

$$\chi_+ \;=\; 8\beta \int_0^{\pi} d\theta\, \frac{g(\theta,t)}{\theta^2} \tag{3.38}$$

where in the second expression we have again expanded in small h and θ. The expression for χ_+ would have had a divergence at $\theta=0$ except this was thankfully avoided since the density vanishes well before the origin at $\theta_c = |H_c|$. For the expressions below T_c, we must subtract out this divergence as the density is finite at $\theta=0$, or use the alternative definition, see [58]. One obtains

$$
\begin{aligned}
\chi_- &= \lim_{H\to 0} \frac{M(H)-M(0)}{H}\,, && (3.39)\\
&= \lim_{H\to 0} \frac{2\beta}{h}\left(2h\int_0^\pi d\theta\,\frac{g(\theta,t)}{\cosh\,h-\cos\,\theta} - 2\pi g(0,t)\right)\,, && (3.40)\\
\chi_- &= 8\beta\int_0^\pi d\theta\,\frac{g(\theta,t)-g(0,t)}{\theta^2} && (3.41)
\end{aligned}
$$

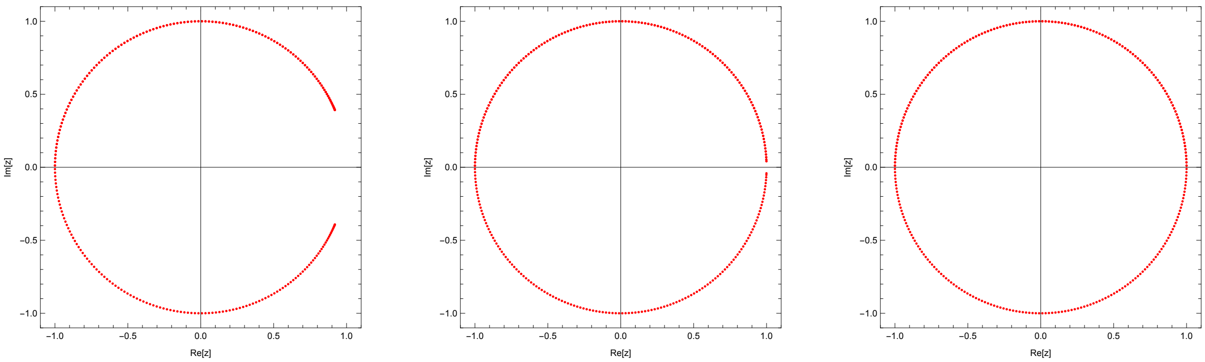
where in the third step we inserted the delta function representation.

One can also inspect the real and imaginary parts of the susceptibility χ_+. We do this in anticipation of probing the edge of the distribution of zeros. Taking $h = a + i\,b$

$$
\begin{aligned}
\mathrm{Re}\,\chi_+ &= \int_0^\pi d\theta\,g(\theta,t)\left(4a^2\left(-\frac{1}{(a^2+(b+\theta)^2)^2}-\frac{1}{(a^2+(b-\theta)^2)^2}\right)\right.\\
&+ \left.\frac{a^2+b(b-\theta)+2}{a^2+(b-\theta)^2}+\frac{a^2+b(b+\theta)+2}{a^2+(b+\theta)^2}\right)\,, && (3.42)\\
\mathrm{Im}\,\chi_+ &= \int_0^\pi d\theta\,g(\theta,t)\left(4a\left(\frac{b-\theta}{(a^2+(b-\theta)^2)^2}+\frac{b+\theta}{(a^2+(b+\theta)^2)^2}\right)\right.\\
&+ \left.\frac{a\theta}{a^2+(b-\theta)^2}-\frac{a\theta}{a^2+(b+\theta)^2}\right)\,. && (3.43)
\end{aligned}
$$

In the above expressions, one should notice the delta function-like structures emerging, such as the last two terms in $\mathrm{Im}\chi_+$ which become true delta functions $\theta\delta(b\pm\theta)$ at $a=0$. Similarly, the first two terms in $\mathrm{Im}\chi_+$ and $\mathrm{Re}\chi_+$ resemble delta function derivatives $\delta'(b\pm\theta)$. Doing an integration by parts to put the derivative on $g(\theta,t)$ reveals this structure:

$$
\begin{aligned}
\mathrm{Re}\,\chi_+ &= -\int_{\theta_c}^\pi d\theta\,\frac{\partial g(\theta,t)}{\partial\theta}\left(\frac{2(b-\theta)}{a^2+(b-\theta)^2}-\frac{2(b+\theta)}{a^2+(b+\theta)^2}\right.\\
&+ \frac{1}{2}\left(b\log\left(a^2+(b+\theta)^2\right)-b\log\left(a^2+(b-\theta)^2\right)\right)\\
&- \left.a\tan^{-1}\left(\frac{b-\theta}{a}\right)+a\tan^{-1}\left(\frac{b+\theta}{a}\right)\right)+\text{boundary terms}\,, && (3.44)\\
\mathrm{Im}\,\chi_+ &= -\int_{\theta_c}^\pi d\theta\,\frac{\partial g(\theta,t)}{\partial\theta}\left(\frac{2a}{a^2+(b-\theta)^2}-\frac{2a}{a^2+(b+\theta)^2}\right.\\
&+ \frac{1}{2}a\left(\log\left(a^2+(b-\theta)^2\right)-\log\left(a^2+(b+\theta)^2\right)\right)\\
&- \left.b\tan^{-1}\left(\frac{b-\theta}{a}\right)+b\tan^{-1}\left(\frac{b+\theta}{a}\right)\right)+\text{boundary terms}\,. && (3.45)
\end{aligned}
$$



(a) $T > T_c$ (b) $T \approx T_c$ (c) $T = T_c = 0$

Figure 3.2: The zeros of the 1D Ising Model in the complex fugacity plane, $z = \mathrm{Exp}\,(-2\beta H)$. Notice that for $T > T_c$, the zeros pile up at the edge which results in the exponent σ being negative. For positive σ, the zeros would become sparse near the edge. For $T = 0$, the zeros become uniformly distributed which can be shown as a general property for $T = 0$.

The boundary terms at $\theta = \theta_c$ vanish for $g(\theta_c, t) = 0$ whereas the term evaluated at $\theta = \pi$ yields a divergence at $\mathrm{Im}\, H = \pi$. However, we expanded about small h and θ, so this is likely an artifact of this expansion. We will explore this more in the next section.

A last note before we move to discuss the details on the edge of the density of zeros and the Yang-Lee edge singularity. We expect that the free energy and magnetization are analytic functions (away from any singularities that is). As such, we list the relevant Cauchy-Riemann equations below which relate derivatives of the magnetization to the appropriate parts of the susceptibility:

$$\begin{aligned} \frac{\partial \mathrm{Re}\, M}{\partial\, \mathrm{Re\, H}} &= \mathrm{Re}\, \chi\,, & \frac{\partial \mathrm{Im}\, M}{\partial\, \mathrm{Re\, H}} &= \mathrm{Im}\, \chi\,, \\ \frac{\partial \mathrm{Re}\, M}{\partial\, \mathrm{Im\, H}} &= -\mathrm{Im}\, \chi\,, & \frac{\partial \mathrm{Im}\, M}{\partial\, \mathrm{Im\, H}} &= \mathrm{Re}\, \chi\,. \end{aligned} \tag{3.46}$$

The above expressions will be of use later. We note in closing on this section that one may use the expressions derived for the magnetization and susceptibility to determine how to scaling behavior of the density is related to the critical data, though we will not pursue this here, see [92, 93].

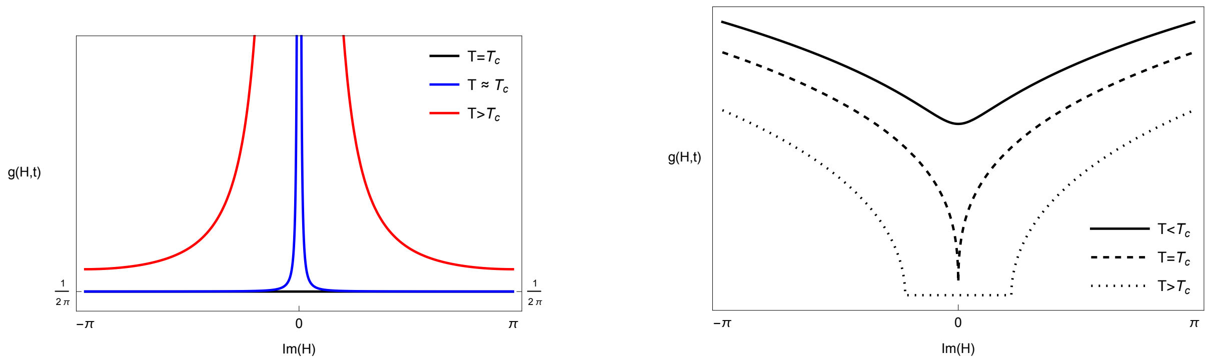


Figure 3.3: Expressions for density of zeros are plotted for the 1D Ising model (left) and the mean field model (right). Notice that while both models yield a gap in the zeros, the density diverges for the one dimensional model whereas it vanishes for the mean field model.

3.3 The Yang-Lee edge singularity

While in the previous section we discussed the basic properties of the density of zeros and its relation to thermodynamic functions, we reserved until now further discussion of the typical features present at the gap of density of zeros which occurs for $T > T_c$. As first pointed out by Kortman and Griffiths [16], the density of zeros satisfies as a scaling form about the edge

$$g(\theta, t > 0) = g_0 \left(\theta - \theta_c\right)^{\sigma} \tag{3.47}$$

for $\theta > \theta_c$. They collected the results of 1D Ising model where $\sigma = -\frac{1}{2}$ (see Section 5.4), mean field yields $\sigma = \frac{1}{2}$ (see Section 5.1), and their studies indicated that for 2D lattices $\sigma < 0$. It was later shown by Cardy [34] via CFT methods that in two dimensions, $\sigma = -\frac{1}{6}$. See Fig. 3.2 and Fig. 3.3 for a visual display of the difference between positive and negative σ. It is interesting to wonder what might happen if the density were instead to approach a constant at θ_c before dropping to zero. This could be investigated by simply inserting the ansatz for $g(\theta, t)$ in the expressions of Chapter 3.

While it was realized that the susceptibility diverges at the edge in [16], Fisher was the first to realize the edge itself behaves as a critical point which he deemed the Yang-Lee edge singularity (YLE) [17]. He followed by showing that is could be described by a $i\, g\, \phi^3$

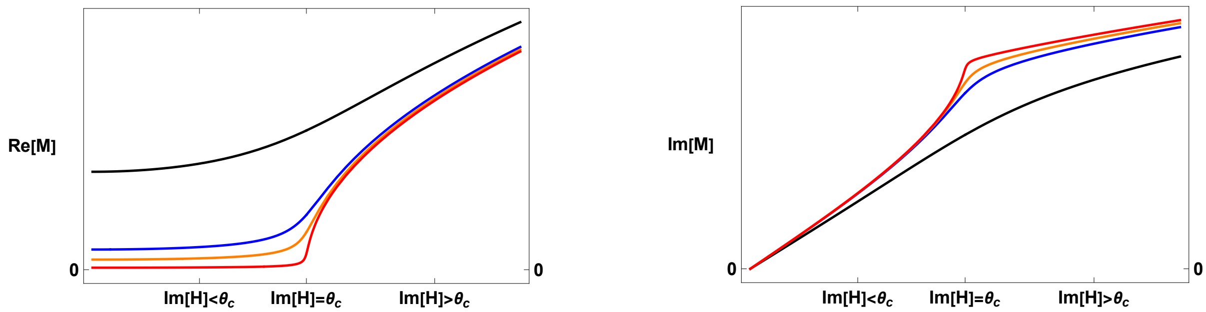


Figure 3.4: Taking t small and using the expression for the density of zeros $g(\theta, t) = (\theta - \theta_c)^\sigma$ with $\sigma = \frac{1}{2}$, one can integrate the expression in Eqs. (3.33,3.36) to obtain expressions for Re(M) and Im(M) for complex H. As the real part of the field decreases, the critical point can be seen by the kinks in the magnetization which occur at $\mathrm{Im}(H) = \theta_c$.

theory given by

$$S[\phi] \;=\; \int_{\mathbf{x}} \left\{ (\partial\phi)^2 + i\, g\, \phi^3 + (H - H_c)\phi \right\} \,. \tag{3.48}$$

Furthermore, the magnetization inherits the scaling of the density of zeros:

$$M - M_c \propto (H - H_c)^\sigma \,. \tag{3.49}$$

Since the YLE occurs at finite external fields, it was argued in Ref. [17] that σ is independent of N and is thus the same for all $O(N)$ models.

Fisher and Kurtze [29] found in other studies that $\sigma = \frac{1}{2}$ for spherical models which correspond to the large N limit. This result is independent of dimension $2 < d < 4$ which seems to violate the independence of σ from N. In Ref. [29], they worked with finite N. They found in three dimensions $\sigma \approx 0.086$ while for two dimensions that $\sigma \approx -.163$. Their two dimensional result is strikingly close to the Cardy's CFT result of $\sigma = -\frac{1}{6}$. A few years after, the large N paradox was resolved by showing that the YLE is sub-dominant in N by Bander and Itzykson [95]. It should be noted that Kurtze also worked to resolve this issue, [32]. More recently, the YLE as an isolated critical point has been studied via the functional renormalization group [41, 42] and CFT techniques [35, 96], and ϵ expansions [97, 98].

While the YLE is seen to be independent of N, and so the universal edge of a sequence of universality classes, it has also been shown to be identical for anti-ferromagnetic systems,

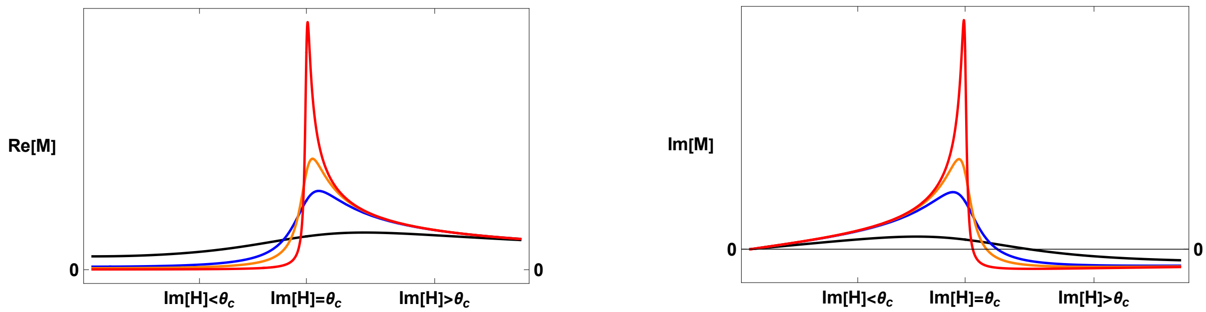


Figure 3.5: Taking t small and using the expression for the density of zeros $g(\theta, t) = (\theta - \theta_c)^\sigma$ with $\sigma = -\frac{1}{2}$ as in the 1D Ising model, one can integrate the expression in Eqs. (3.33,3.36) to obtain expressions for $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ for complex H. The plots for decreasing but fixed $\mathrm{Re}(H)$ demonstrate the divergence of both $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ at purely imaginary field $H = i\,\theta_c$.

albeit one-dimensional, as well as for arbitrary spins, [33, 99]. *One may even conjecture on the behavior once fermions are added leading the author to conjecture that the same edge singularity controls the critical points with fermionic degrees of freedom as well since the external fields gives mass to the fermions, thus causing them to drop out.*

In another direction mentioned earlier, one could study the behavior of the zeros near the tricritcal point in the Blume-Capel model or its relatives. The original works exploring this situation are Refs. [36, 37, 38, 39, 40]. See also Ref.s [75, 76, 77] as well as Refs. [78, 79]. One work stands out as (likely) the first to notice that the tricritical point likely yields a new type of Yang-Lee edge singualirty, see Ref. [74]. Most works above were able to see that near a tricritical point, the edge of the density of zeros exhibits a new type of scaling, which for one dimension is $\sigma = -\frac{2}{3}$. A commonality of the density of zeros near tricritical points seems to be a bifurcation of the zeros. This new edge theory was also conjectured on and explored in Refs. [78, 79, 42, 100, 101]. It seems the critical theory should be an $ig\phi^5$ theory. The main idea for the connection between CFTs and edge singularities is simply stated that alongside the sequence of $O(N)$ multi-critical points is a sequence of edge singularities corresponding to a sequence of minimal models. Thereby one could obtain the field theories corresponding to these minimal models.

Finally, we note that there has been studies whereby one can experimentally probe both the Lee-Yang zeros and the Yang-Lee edge singularity [102, 103, 104, 70]

Now let us turn back to the expressions relating the density of zeros to the thermodynamic functions. Using these, we would like to understand what happens near the YLE.

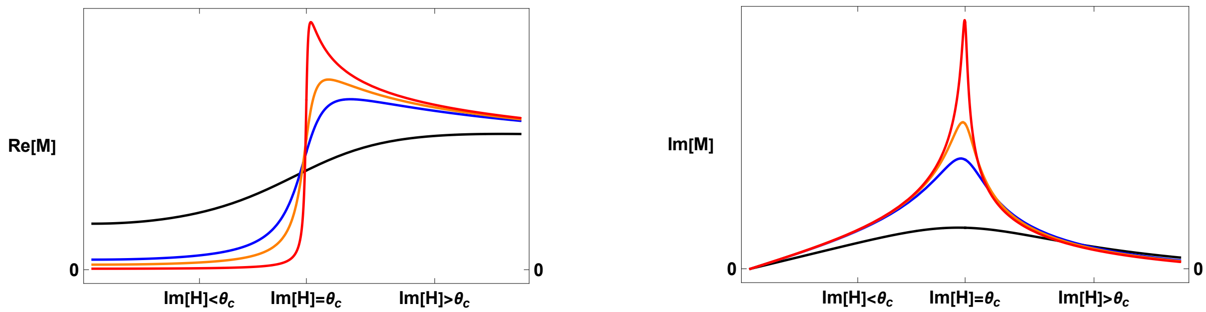


Figure 3.6: Taking t small and using the expression for the density of zeros $g(\theta, t) = (\theta - \theta_c)^\sigma$ with $\sigma = -\frac{1}{6}$ as in the 2D Ising model, one can integrate the expression in Eqs. (3.33,3.36) to obtain expressions for $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ for complex H. The plots for decreasing but fixed $\mathrm{Re}(H)$ demonstrate the divergence of both $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ at purely imaginary field $H = i\,\theta_c$.

Firstly, let us derive the expression in Eq. (3.49). Let us start by naively expanding the real and imaginary parts separately for $H \in \mathbb{C}$:

$$
\begin{aligned}
M(H) &= \mathrm{Re}\, M(H) + i\, \mathrm{Im}\, M(H)\,, && (3.50)\\
&= \mathrm{Re}\, M(i\, b) + \left.\frac{\partial \mathrm{Re}\, M}{\partial \mathrm{Re}\, H}\right|_{\mathrm{Re}\, H=0} \mathrm{Re}\, H \\
&+ \left(\mathrm{Im}\, M(i\, b) + \left.\frac{\partial \mathrm{Im}\, M}{\partial \mathrm{Re}\, H}\right|_{\mathrm{Re}\, H=0} \mathrm{Re}\, H\right)\,. && (3.51)
\end{aligned}
$$

Using the exact expression for the ReM along the imaginary axis,

$$
\begin{aligned}
M(H) &= 2\pi(b-\theta_c)^\sigma \Theta(b-\theta_c) \\
&+ \mathrm{Re}\, H \left(\left.\frac{\partial \mathrm{Re}\, M}{\partial \mathrm{Re}\, H}\right|_{\mathrm{Re}\, H=0} + i \left.\frac{\partial \mathrm{Im}\, M}{\partial \mathrm{Re}\, H}\right|_{\mathrm{Re}\, H=0}\right) \\
&+ i\left(M_c + \left.\frac{\partial \mathrm{Im}\, M}{\partial \mathrm{Im}\, H}\right|_{H=i\,\theta_c} (\mathrm{Im}\, H - \theta_c)\right) && (3.52)
\end{aligned}
$$

where we have written

$$
M_c = 2\int_{\theta_c}^{\pi} d\theta\, g(\theta, t)\left\{\frac{1}{\theta-\theta_c} - \frac{1}{\theta+\theta_c}\right\}\,. \qquad (3.53)
$$

Taking $g(\theta, t) \sim (\theta - \theta_c)^\sigma$, we see that M_c behaves roughly as

$$M_c \sim 2 \int_{\theta_c}^{\pi} d\theta \, (\theta - \theta_c)^{\sigma - 1} \,, \tag{3.54}$$

$$\sim (\theta - \theta_c)^\sigma \Big|_{\theta_c}^{\pi} \tag{3.55}$$

which for $\sigma > 0$ is finite and given by $M_c \propto (\pi - \theta_c)^\sigma$. We also note that for purely imaginary fields, $\mathrm{Im}(M)$ is still finite given by

$$\mathrm{Im}\, M = 2 \int_{\theta_c}^{\pi} d\theta \, g(\theta, t) \left\{ \frac{1}{\theta - b} - \frac{1}{\theta + b} \right\} \tag{3.56}$$

from which one can infer that $\mathrm{Im}(M)$ increases from zero to M_c on the interval $b \in [0, \theta_c]$. Using $g(\theta, t) \sim (\theta - \theta_c)^\sigma$, one can obtain an exact result for the magnetization, see Figs. 3.4, 3.5 and 3.6.

Returning to the full expression of $M(H)$ for complex fields near $H = i\,\theta_c$, one can use the Cauchy-Riemann equations in Eq. (3.46) to obtain

$$\begin{aligned} M(H) &= i\, M_c + 2\pi (b - \theta_c)^\sigma \Theta(b - \theta_c) \\ &+ \mathrm{Re}\, H \left(\mathrm{Re}\, \chi\big|_{\mathrm{Re}\, H = 0} + i\, \mathrm{Im}\, \chi\big|_{\mathrm{Re}\, H = 0} \right) \\ &+ i\, \mathrm{Re}\, \chi\big|_{H = i\,\theta_c} \left(\mathrm{Im}\, H - \theta_c \right) \,. \end{aligned} \tag{3.57}$$

Given the expressions for χ in terms of complex H in Eqs. (3.44-3.45), we can write

$$\mathrm{Re}\, \chi_+ \approx 2 \int_{\theta_c}^{\pi} d\theta \, \frac{\partial g(\theta, t)}{\partial \theta} \left(\frac{1}{b - \theta} - \frac{1}{b + \theta} \right) \,. \tag{3.58}$$

For all values of b except the bounds of the integral, the result is finite. At $H = i\,\theta_c$, $\mathrm{Re}\chi$ diverges as

$$\mathrm{Re}\, \chi_+ \propto (H - i\,\theta_c)^{\sigma - 1} \,. \tag{3.59}$$

Curiously, one also sees a divergence at $\mathrm{Im}(H) = \pi$ coming from the boundary terms, though it could either be an artifact of the small h truncation or be cancelled the density vanishing by as an appropriate power, see Fig. 3.8. We do not explore this further here. A similar line leads to $\mathrm{Im}\chi$ having the same divergences. The analysis is a bit cleaner here as one can see the simple appearance of $\delta(\theta - b)$ which picks out the divergence in

$\partial_\theta \, g(\theta, t)$ when $b = \theta_c$.

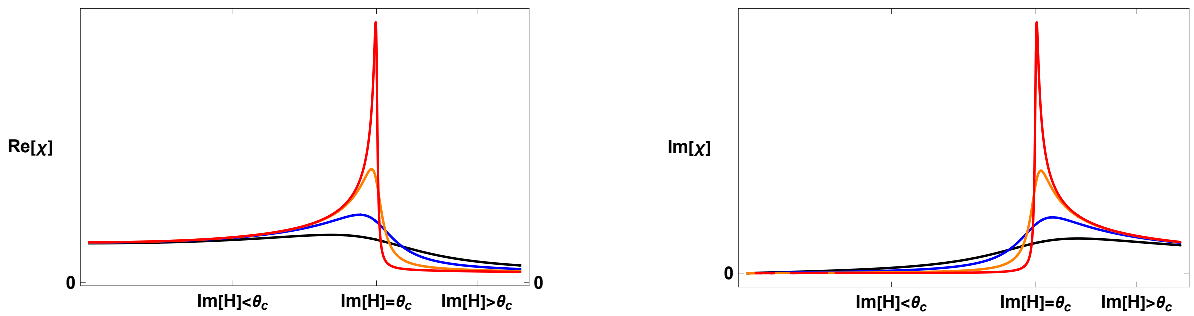


Figure 3.7: Taking t small and using the expression for the density of zeros $g(\theta, t) = (\theta - \theta_c)^\sigma$ with $\sigma = \frac{1}{2}$ as in mean field and large N, one can integrate the expression in Eqs. (3.42,3.43) to obtain expressions for $\mathrm{Re}(\chi)$ and $\mathrm{Im}(\chi)$ for complex H. As the real part of the field decreases, the critical point can be seen by the emergence of a true divergence at $\mathrm{Im}(H) = \theta_c$.

One can also obtain an expression for the susceptibility as was done for the magnetization. See Fig. 3.7. Since $\mathrm{Re}(M) = 0$ for purely imaginary fields below the edge, one also expects that $\mathrm{Re}(\chi) = 0$ here also. We do indeed recover this result with the rough approximation of the density as $g(\theta, t) = (\theta - \theta_c)^\sigma$. See Fig. 3.8 for the case of larger t (and so larger θ_c) and the full range of $\mathrm{Im}(H)$.

In summary, we do indeed obtain the expression in Eq. (3.49), as well as the expected divergence in χ as expected for an ordinary critical point. Further aspects of the Yang-Lee edge near criticality are left to the relevant sections in Section 4.4.3, Chapter 5, and Chapter 6.

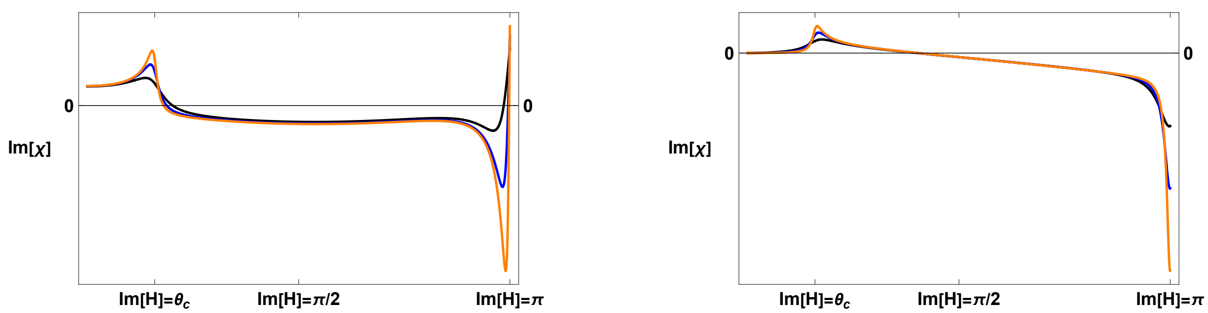


Figure 3.8: Using the expressions for χ as in Fig. 3.7, one can take a larger t to see both the divergence at the edge and the suspected divergence at $\mathrm{Im}(M) = \pi$. However, we cannot fully trust the existence of this divergence since we both expanded about small h and θ and only input the behavior of the density near the edge. The full expression may be finite at $\theta = \pi$ for finite $g(\pi, t)$, or the divergence is still present yet density may behave in a way to cancel this divergence.

CHAPTER

4

THE RENORMALIZATION GROUP

4.1 Scalar and $O(N)$ field theories

In Chapter 2 we discussed critical physics as derived from the van der Waals equation which falls into a class of descriptions know as mean field. What wasn't discussed was the validity of the statements made, meaning, do the results actually describe the three dimensional systems in which we are interested. The answer is no, but the description does do a good job qualitatively. To understand the limitations of mean field approximations and how to improve upon them, we need to discuss some concepts of the renormalization group (RG). Through RG one can see there is an upper critical dimension d_c, which for ϕ^4 $d_c = 4$, above which mean-field results are valid. However, below d_c one cannot trust mean-field and more work needs to be done to account for the fluctuations ignored by mean field. Additionally, for ϕ^3 theories the upper critical dimensions is $d_c = 6$. The difference of these upper critical dimensions is an important consideration for perturbative approaches, see Chapter 5.

Let us start again with the Ising model and for ease of reading, we carry down the expression for the Hamiltonian

$$\mathcal{H}_N\left[\{s_i\}\right] \;=\; -\sum_{\langle ij\rangle} J\, s_i\, s_j - \sum_{i=1}^{N} H_i\, s_i \tag{4.1}$$

with the partition function given by

$$\mathcal{Z}_N \;=\; \sum_{\{s_i\}} \exp\left(-\beta\, \mathcal{H}_N\left[\{s_i\}\right]\right)\,. \tag{4.2}$$

Near the critical point ($T = T_c$ and $H = 0$), one can map the partition function to a scalar φ^4 Euclidean field theory

$$\mathcal{Z} \;=\; \int \mathcal{D}\varphi_\Lambda \exp\left(-S_\Lambda[\varphi]\right) \tag{4.3}$$

where the microscopic/bare action is given by

$$S_\Lambda[\varphi] = \int_{\mathbf{x}} \left\{ \frac{1}{2} Z(\partial\varphi(\mathbf{x}))^2 + \frac{1}{2} r_0\, \varphi^2(\mathbf{x}) + \frac{1}{4} u_0\, \varphi^4(\mathbf{x}) - H(\mathbf{x})\varphi(\mathbf{x}) \right\} \tag{4.4}$$

and the integral is over d dimensional Euclidean space. The theory has an underlying cutoff $\Lambda = \frac{1}{a}$ where a is the original lattice spacing at which the action is defined. The path integral measure is then defined to only include fields which fluctuate on scales larger than a, or equivalently have momentum $p \leq \Lambda$.

Let us review the object appearing in the action. The field φ is a continuous function of space $\mathbf{x}$. The coefficient Z of the kinetic energy term is the field renormalization (or wave-function renormalization). The linear term, H is still the external magnetic field. The quadratic coefficient r_0 is related to the squared mass of particle-like excitations. Furthermore, r_0 carries the temperature dependence as $r_0 \propto T - T_c^{\mathrm{MF}}$ where T_c^{MF} is the mean-field critical temperature obtained by equating the bare classical action with the free energy. Whereas mean field theory predicts that the critical point lies at $r_0 = h = 0$, properly taking fluctuations into account leads to criticality at a negative critical value of r_0 given by $r_c < 0$. The understanding behind this is that the bosonic fluctuations tend to destroy order or, equivalently, drive the quadratic term towards more positive values. In order to combat this, one needs a sufficiently negative r which is driven exactly to zero by the fluctuations. For $r_0 < r_c$, the field takes on a non-zero expectation value $M = \langle\varphi\rangle$ which we can interpret as the magnetization or the order parameter. The existence of this expectation value reflects the spontaneous breaking of the Ising $\mathcal{Z}_2$ symmetry. Thus, we refer to this as the symmetry broken phase. On the other hand, for $r_0 > r_c$ the expectation value vanishes and the symmetry is still present in the ground state. Thus, we refer to the phase with $M = 0$ as the symmetric phase.

If one had started instead with a system of N component spins, the resulting action would take a similar form with the modification that the field φ is an N component vector:

$$S[\boldsymbol{\varphi}] = \int_{\mathbf{x}} \left\{ \frac{1}{2} Z(\partial \boldsymbol{\varphi}(\mathbf{x}))^2 + \frac{1}{2} r_0 \, \boldsymbol{\varphi}^2(\mathbf{x}) + \frac{1}{4} u_0 \, \boldsymbol{\varphi}^4(\mathbf{x}) - \boldsymbol{H}(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) \right\} \tag{4.5}$$

where bold denotes vector quantities. This modification extends the discrete Ising symmetry $\mathcal{Z}_2$ to continuous $O(N)$ symmetries for $N \geq 2$. The notation in the action is slightly abusive in that by $\boldsymbol{\varphi}^4$ we mean $\boldsymbol{\varphi}^4 = (\varphi_i \varphi^i)^2$ where we have used summation notation and sum over repeated indices.

The development of a non-zero expectation value for systems with continuous symmetries creates $N - 1$ massless excitations called Goldstone bosons. In the broken symmetry phase, the Goldstone modes dominate the physics. Furthermore, they cause severe divergences in three dimensional perturbation theory when one looks to large-scale/infrared (IR) physics. On the other hand, the external field explicitly breaks this $O(N)$ symmetry by producing a non-zero expectation value. In this scenario, the Goldstone modes acquire a mass which is controlled by h.

4.2 The effective action

Given the Euclidean action, $S_\Lambda[\boldsymbol{\varphi}]$, of some N-component scalar field $\boldsymbol{\varphi}$ defined at some microscopic scale $a = \Lambda^{-1}$, we write the partition function in the presence of an N-component external field $\mathbf{J}$ in the path integral formalism as

$$\mathcal{Z}[J] = \int \mathcal{D}\boldsymbol{\varphi}_\Lambda \, e^{-S_\Lambda[\boldsymbol{\varphi}] + \mathbf{J} \cdot \boldsymbol{\varphi}} \tag{4.6}$$

where $\mathbf{J} \cdot \boldsymbol{\varphi} = \int d^d x \, \mathbf{J}(x) \cdot \boldsymbol{\varphi}(x)$ and the integral measure is only over modes with variations on scales larger than Λ^{-1}. In the above, $\mathbf{J}$ takes on the notion of the external magnetic field. Some basic physical quantities of interest would be the n-point functions which we can compute at zero $\mathbf{J}$ as

$$\langle \varphi_{i_1} ... \varphi_{i_n} \rangle = \frac{1}{\mathcal{Z}[0]} \int \mathcal{D}\boldsymbol{\varphi} \, \varphi_{i_1} ... \varphi_{i_n} \, e^{-S_\Lambda[\boldsymbol{\varphi}] + \mathbf{J}\cdot\boldsymbol{\varphi}} \Big|_{\mathbf{J}=0} , \tag{4.7}$$

$$= \partial_{J_{i_1}} ... \partial_{J_{i_n}} \frac{\mathcal{Z}[\mathbf{J}]}{\mathcal{Z}[0]} \Big|_{\mathbf{J}=0} . \tag{4.8}$$

The above correlation functions are analogous to moments of the distribution $e^{-S_\Lambda[\boldsymbol{\varphi}]+\mathbf{J}\cdot\boldsymbol{\varphi}}$, assuming it has such an interpretation. If one were interested instead in the cummulants, the so-called connected correlation functions, we would use the following expression:

$$\langle \varphi_{i_1} ... \varphi_{i_n} \rangle_c = \partial_{J_{i_1}} ... \partial_{J_{i_n}} \ln \mathcal{Z}[\mathbf{J}] \Big|_{\mathbf{J}=0} . \tag{4.9}$$

The first few connected correlation functions are given as

$$\langle \varphi_i \rangle_c = \langle \varphi_i \rangle , \tag{4.10}$$

$$\langle \varphi_i \, \varphi_j \rangle_c = \langle \varphi_i \, \varphi_j \rangle - \langle \varphi_i \rangle \, \langle \varphi_j \rangle , \tag{4.11}$$

$$\begin{aligned} \langle \varphi_i \, \varphi_j \, \varphi_k \rangle_c &= \langle \varphi_i \, \varphi_j \, \varphi_k \rangle - \langle \varphi_i \rangle \, \langle \varphi_j \, \varphi_k \rangle - \langle \varphi_j \rangle \, \langle \varphi_k \, \varphi_i \rangle - \langle \varphi_k \rangle \, \langle \varphi_j \, \varphi_i \rangle \\ &+ 2\langle \varphi_i \rangle \, \langle \varphi_j \rangle \, \langle \varphi_k \rangle . \end{aligned} \tag{4.12}$$

To characterize the connected correlation functions, it is useful to construct their generating function, $W[\mathbf{J}]$:

$$W[\mathbf{J}] = \ln \mathcal{Z}[\mathbf{J}] \tag{4.13}$$

which gives the definition of the connected correlation functions as

$$\langle \varphi_{i_1} ... \varphi_{i_n} \rangle_c = \partial_{J_{i_1}} ... \partial_{J_{i_n}} W[\mathbf{J}] \Big|_{\mathbf{J}=0} . \tag{4.14}$$

In statistical mechanics, $W[\mathbf{J}]$ corresponds to the Helmholtz free energy. In the free energy language, the action itself $S_\Lambda[\varphi]$ acquires the interpretation of the (Wilsonian) free energy of the microscopic field modes capturing physics on length scales smaller the a.

For a free (non-interacting) theory, the partition function is simply

$$\mathcal{Z}_0[J] = \left(\frac{2\pi}{\det(-\partial^2 + r_0)} \right)^{N/2} e^{-\frac{1}{2} J_i \cdot (-\partial^2 + r_0)^{-1} \cdot J^i} \tag{4.15}$$

since the integral is a product of N Gaussians. The term in the exponent yields the free propagator as we have the general relation that the propagator is identical to the connected two-point correlation function - the Green's function. Using Eq. (4.14) we obtain

$$\begin{aligned} G_{c,0,ij}(x,y) &= \delta_{ij}\langle\varphi_i\,\varphi_j\rangle_{c,0}\,, &(4.16)\\ &= \delta_{ij}\int\frac{d^dq}{(2\pi)^d}\frac{e^{iq\cdot(x-y)}}{q^2+r_0}\,. &(4.17)\end{aligned}$$

By translation invariance, we can write $G_{c,0,ij}(r)$ since it can only depend on the separation $r=|x-y|$:

$$G_{c,0,ij}(r) \;=\; \delta_{ij}\int\frac{d^dq}{(2\pi)^d}\frac{e^{iq\cdot r}}{q^2+r_0}\,. \tag{4.18}$$

We can write the above in momentum space as well:

$$G_{c,0,ij}(q,p) \;=\; \frac{\delta_{ij}}{q^2+r_0}(2\pi)^d\delta(q+p) \tag{4.19}$$

where the delta function appears from translational invariance.

With some effort, one can perform the momentum integral to obtain an expression for the real space Green's function [12, 105]

$$G_{c,0,ii}(r)\sim\begin{cases}\dfrac{e^{-|x-y|/\xi}}{r^{(d-1)/2}} & r\gg\xi\\[2ex] \dfrac{1}{r^{d-2}} & r\ll\xi\end{cases} \tag{4.20}$$

where $\xi=\frac{1}{\sqrt{r_0}}$ is the correlation length - the range over which correlations are significant. The above expression for the correlation function is known as the Ornstein-Zernicke form. Importantly, wee see two things. A correlation length can be interpreted as an inverse mass which implies $m^2=r_0$. Secondly, we see that tuning $m^2=r_0\to0$ yields a diverging correlation length, for the free theory.

In certain regimes, the field may develop a non-zero expectation value $\langle\boldsymbol{\varphi}\rangle$. The expectation value of φ is obtained via a simple relation to $W[\mathbf{J}]$. Let us define $\boldsymbol{\phi}_{\mathbf{J}}(x)=\langle\boldsymbol{\varphi}(x)\rangle_{\mathbf{J}}$ where we make explicit that the expectation value depends on space and is to be taken in the presence of the external field $\mathbf{J}$. Then,

$$\begin{aligned}\boldsymbol{\phi}_{\mathbf{J}}(\mathbf{x}) &= \frac{1}{\mathcal{Z}[J]}\int \mathcal{D}\varphi\, \boldsymbol{\varphi}(\mathbf{x})\, e^{-S_\Lambda[\boldsymbol{\varphi}]+J\cdot\varphi}\,, \\ \boldsymbol{\phi}_{\mathbf{J}}(\mathbf{x}) &= \frac{\delta W[\mathbf{J}]}{\delta \mathbf{J}(\mathbf{x})}\,. \qquad (4.21)\end{aligned}$$

Another useful quantity is the effective action or Gibbs free energy, $\Gamma[\boldsymbol{\phi}]$. This quantity is obtained by a Legendre transform of the free energy $W[\mathbf{J}]$:

$$\Gamma[\boldsymbol{\phi}] = -W[\mathbf{J}] + \mathbf{J}\cdot\boldsymbol{\phi}\,. \qquad (4.22)$$

Above, $\mathbf{J}$ is taken as a function of $\boldsymbol{\phi}$ (assumed to be invertible) by equation (4.21). In this way, Γ remains as a function of only the expectation value $\boldsymbol{\phi}$ which is now taken as the independent variable - hence we dropped the $\mathbf{J}$ subscript. The utility in $\Gamma[\boldsymbol{\phi}]$ stems from the following:

$$\begin{aligned}\frac{\delta\Gamma[\boldsymbol{\phi}]}{\delta\boldsymbol{\phi}(\mathbf{x})} &= -\frac{\delta W}{\delta \mathbf{J}}\cdot\frac{\delta \mathbf{J}}{\delta\boldsymbol{\phi}} + \frac{\delta \mathbf{J}}{\delta\boldsymbol{\phi}}\cdot\boldsymbol{\phi} + \mathbf{J}(\mathbf{x})\,, \qquad (4.23) \\ &= \mathbf{J}(\mathbf{x})\,. \qquad (4.24)\end{aligned}$$

Thus, at vanishing external fields we need to minimize $\Gamma[\boldsymbol{\phi}]$ to find the full equations of motion for $\boldsymbol{\phi}$. This is just the same as one obtains the classical equations of motion by minimizing the classical action.

Furthermore, while $W[\mathbf{J}]$ is the generating function of the connected correlations functions, $\Gamma[\boldsymbol{\phi}]$ turns out to be the generating function of the *vertex functions*. For instance, the three-point correlator $G_c^{(3)}$ of a scalar theory $N = 1$ satisfies the relation [106]

$$G_c^{(3)}(p_1,p_2,p_3) = -\int\prod_{i=1}^{3}\frac{d^dq_i}{(2\pi)^d}G_c^{(2)}(p_1,q_1)G_c^{(2)}(p_2,q_2)G_c^{(2)}(p_3,q_3)\Gamma^{(3)}(q_1,q_2,q_3)$$

(4.25)

where

$$\Gamma^{(3)}(q_1, q_2, q_3) = \frac{\partial^3 \Gamma[\phi]}{\partial\phi(p_1)\partial\phi(p_2)\partial\phi(p_3)} \,. \tag{4.26}$$

For a free theory, only the first two vertex functions are non-zero.

$$\Gamma_i^{(1)}[\boldsymbol{\phi}] \;=\; J_i \,, \tag{4.27}$$

$$\Gamma_{ij}^{(2)}[\boldsymbol{\phi}] \;=\; \frac{\partial J_i}{\partial \phi_j} \,, \tag{4.28}$$

$$\;=\; \left(\frac{\partial \phi_j}{\partial J_i}\right)^{-1} \,. \tag{4.29}$$

The first result we already had from the Legendre transform. For the two-point function, we can realize ϕ_j as $\frac{\partial W[\mathbf{J}]}{\partial J_j}$:

$$\Gamma_{ij}^{(2)}[\boldsymbol{\phi}] \;=\; \left(\frac{\partial W[\mathbf{J}]}{\partial J_i \partial J_j}\right)^{-1} = G_{c,0,ij}^{-1} \tag{4.30}$$

and so we have the nice relation that $\Gamma^{(2)}$ is simply the inverse propagator. In momentum space we have

$$\Gamma_{0,ij}^{(2)}[\boldsymbol{\phi}; q, p] \;=\; \delta_{ij}\left(p^2 + r_0\right)(2\pi)^d \delta(q+p) \,. \tag{4.31}$$

While the above holds for free theories, turning on interactions will not only turn on higher order correlations, it will also modify the mass.

In closing off this section, let us briefly discuss some subtleties of the Legendre transform. Firstly, the transform is well-defined only for convex functions. An alternate definition of the transform is given by

$$\Gamma[\boldsymbol{\phi}] \;=\; \sup_{\phi}\left(-W[\mathbf{J}] + \mathbf{J}\cdot\boldsymbol{\phi}\right) \tag{4.32}$$

where the supremum of ϕ amounts to identifying $\boldsymbol{\phi} = \boldsymbol{\phi}_{\mathbf{J}}$. But, this relation is only well-defined if $W[\mathbf{J}]$ is convex, or else the equation $W'[\mathbf{J}] = \boldsymbol{\phi}$ necessarily has multiple solutions and hence $\boldsymbol{\phi} = \boldsymbol{\phi}_{\mathbf{J}}$ would not be invertible unless we restrict the domain. However, it can be shown that $W[\mathbf{J}]$ is convex, see below.

One might ask though how $W[\mathbf{J}]$ could be convex, especially considering spontaneous symmetry breaking implies multiple ground states. The more natural quantity to realize as convex is the effective action $\Gamma[\boldsymbol{\phi}]$. In the case of spontaneously broken symmetry, the non-zero expectation value $\boldsymbol{\phi}$ seems to imply a negative curvature in the region about the origin, at least for the tree-level potential. This is reminiscent of the Maxwell construction in statistical mechanics which corrects for an un-physical regime which develops in the coexistence region and predicts a pressure which increases as the volume increases. The reconciliation is that within the coexistence region (which is non-convex) the system is equally likely to be in any combination of phases - say liquid/gas or negative and positive magnetization which leads to a convex free energy, see [1, 107]. Furthermore, the full effective action along with $W[\mathbf{J}]$ can be proven to be convex - see [108]. See also [109, 110, 111, 112] for related discussions.

4.3 The loop expansion

To motivate a technique for tackling the path integral, let us recourse to a scalar theory. We start by using the relation between the effective action $\Gamma[\phi]$ and the Helmholtz free energy $W[J]$ in Eq. (4.22) to obtain a path integral expression for Γ:

$$
\begin{aligned}
e^{-\Gamma[\phi]+J\phi} &= \int \mathcal{D}\phi \; e^{-S[\varphi]+J\varphi}, \\
e^{-\Gamma[\phi]} &= \int \mathcal{D}\varphi \; e^{-S[\varphi]+J(\varphi-\varphi)}, \\
&= \int \mathcal{D}\varphi \; e^{-S[\phi+\varphi]+J\varphi}
\end{aligned} \tag{4.33}
$$

where in the last line we have shifted the variable φ. One could interpret this expression as integrating out the fluctuations about the expectation value ϕ. The above expression also intuits an expansion about ϕ which we are tempted to identify with the so-called classical field ϕ_{cl} which minimizes the classical action $S'[\phi_{\text{cl}}] = J$. This amounts to appealing to the saddle point method of solving integrals.

Say we have such a solution ϕ_{cl} and want to expand the action to leading order. For our purposes, ϕ_{cl} will be taken to be spatially uniform. Firstly, the path integral is invariant under shifts in the field φ

$$
\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-S[\varphi]+J\cdot\varphi}\,, \tag{4.34}
$$

$$
= \int \mathcal{D}\varphi \; e^{-S[\varphi+\phi_{\text{cl}}]+J\cdot(\varphi+\phi_{\text{cl}})}\,. \tag{4.35}
$$

We now expand the action about $\varphi = \phi_{\rm cl}$

$$\mathcal{Z} \;\approx\; e^{-S[\phi_{\rm cl}]+J\cdot\phi_{\rm cl}} \int \mathcal{D}\varphi \; e^{-S'[\phi_{\rm cl}]\cdot(\varphi-\phi_{\rm cl})+J\cdot(\varphi-\phi_{\rm cl})-\frac{1}{2}(\varphi-\phi_{\rm cl})\cdot S''[\phi_{\rm cl}]\cdot(\varphi-\phi_{\rm cl})} \tag{4.36}$$

where we only include the leading order corrections. The term linear in ϕ cancels since the classical field $\phi_{\rm cl}$ satisfies $S'[\phi_{\rm cl}] = J$ which leaves

$$\mathcal{Z} \;=\; e^{-S[\phi_{\rm cl}]+\phi_{\rm cl}} \int \mathcal{D}\varphi \; e^{-\frac{1}{2}(\varphi-\phi_{\rm cl})\cdot\, S''[\phi_{\rm cl}]\,\cdot(\varphi-\phi_{\rm cl})} \,. \tag{4.37}$$

Lastly, recognizing the integral as Gaussian we can further write

$$\mathcal{Z} \;=\; e^{-S[\phi_{\rm cl}]+\phi_{\rm cl}} \;\det\left(\frac{S''[\phi_{\rm cl}]}{2\pi}\right)^{-1/2} \,, \tag{4.38}$$

$$\;=\; e^{-S[\phi_{\rm cl}]+J\phi_{\rm cl}-\frac{1}{2}\ln\det\left(\frac{S''[\phi_{\rm cl}]}{2\pi}\right)} \,. \tag{4.39}$$

Using the definition of $\Gamma[\phi]$, we have

$$e^{-\Gamma[\phi]+J\phi} = e^{-S[\phi_{\rm cl}]+J\phi_{\rm cl}-\frac{1}{2}\ln\det\left(\frac{S''[\phi_{\rm cl}]}{2\pi}\right)} . \tag{4.40}$$

We then see that to this order in the approximation,

$$\phi \;=\; \phi_{\rm cl}\,, \tag{4.41}$$

$$\Gamma[\phi] \;=\; S[\phi] + \frac{1}{2}\ln\det\left(\frac{S''[\phi]}{2\pi}\right) . \tag{4.42}$$

The above result for the effective action is commonly referred to as the one-loop expression. So, where are the loops? We will see they are buried in the second term. One could approximate the effective action by ignoring these loops. The resulting action is referred to as tree-level, or more generally, the mean-field action. We will see in Section 5.1 that this action produces the same critical physics as the van der Waals equation in Chapter 2.

4.3.1 Scalar ϕ^4 theory

Taking the scalar ϕ^4 theory action,

$$S_\Lambda[\phi] = \int_{\mathbf{x}} \left\{ \frac{1}{2}(\partial\phi(\mathbf{x}))^2 + \frac{1}{2} r_0\, \phi^2(\mathbf{x}) + \frac{1}{4!} u_0\, \phi^4(\mathbf{x}) \right\} \tag{4.43}$$

where we introduced the 4! for easier counting. Before tackling the loop contribution, first let us write down the first few tree level vertices $\Gamma^{(n)}_{\mathrm{cl}}$ given by simple derivatives of the bare action $S[\phi]$ evaluated on our classical background field ϕ_{cl} which minimizes $S[\phi]$. The background field is given by the solution to

$$0 = r_0 \phi_{\mathrm{cl}} + \frac{1}{3!} u_0 \phi_{\mathrm{cl}}^3 \,. \tag{4.44}$$

The above yields different minimal solutions depending on the sign of r_0

$$\phi_{\mathrm{cl}} = \begin{cases} 0 & r_0 > 0 \\ \pm\sqrt{\dfrac{-6r_0}{u_0}} & r_0 < 0 \end{cases} \,. \tag{4.45}$$

Given that we are computing at vanishing external field, $\Gamma^{(1)}_{\mathrm{cl}} = 0$. The remaining non-zero vertices are

$$\begin{aligned} \Gamma^{(2)}_{\mathrm{cl}}[\phi_{\mathrm{cl}}; p] &= p^2 + r_0 + \frac{1}{2} u_0 \phi_{\mathrm{cl}}^2 \,, && (4.46) \\ \Gamma^{(3)}_{\mathrm{cl}}[\phi_{\mathrm{cl}}; p_1, p_2] &= u_0 \phi_{\mathrm{cl}} \,, && (4.47) \\ \Gamma^{(4)}_{\mathrm{cl}}[\phi_{\mathrm{cl}}; p_1, p_2, p_3] &= u_0 && (4.48) \end{aligned}$$

where the above are reduced vertices in that they have been stripped of their delta functions and had their arguments reduced accordingly. From above, we identify the tree-level mass as

$$m^2 = r_0 + \frac{1}{2} u_0 \phi_{\mathrm{cl}}^2 = \begin{cases} r_0 & r_0 > 0 \\ -2r_0 & r_0 < 0 \end{cases} \,. \tag{4.49}$$

Thus, mean field predicts criticality at $r_0 = r_c = 0$. Recalling that $r_0 = \bar{r_0}(T - T_c^{\mathrm{MF}})$, the relation above can be rewritten as

$$m^2 \propto |t| \tag{4.50}$$

where we have the mean field relation $t \propto T - T_c^{\mathrm{MF}}$. The above is the prediction at *mean field* for the mass dependence on the temperature.

Now let us move on to the one-loop expression. We write

$$\Gamma[\phi] \quad = \quad S[\phi] + \frac{1}{2}\ln\det\left(-\partial^2 + r_0 + 3\,u_0\phi^2\right)\,. \tag{4.51}$$

Using the operator relation $\ln\det A = \mathrm{Tr}\,\ln A$, we can rewrite the correction term as [106]

$$\ln\det\left(-\partial^2 + r_0 + \frac{1}{2}\,u_0\right) \quad = \quad \mathrm{Tr}\,\ln\left(\left(-\partial^2 + r_0\right)\frac{-\partial^2 + r_0 + \frac{1}{2}u_0\phi^2}{-\partial^2 + r_0}\right)\,, \tag{4.52}$$

$$= \quad \mathrm{Tr}\,\ln\left(-\partial^2 + r_0\right) + \mathrm{Tr}\,\ln\left(\frac{-\partial^2 + r_0 + \frac{1}{2}u_0\phi^2}{-\partial^2 + r_0}\right) \tag{4.53}$$

The first term in the last line serves to normalize the partition function when $u_0 = 0$. Furthermore, we can treat the log perturbatively in u_0 [106]:

$$\mathrm{Tr}\,\ln\left(\frac{-\partial^2 + r_0 + \frac{1}{2}u_0\phi^2}{-\partial^2 + r_0}\right) \quad \sim \quad \mathrm{Tr}\,\ln\left(1 + \frac{\frac{1}{2}u_0\phi^2}{-\partial^2 + r_0}\right)\,, \tag{4.54}$$

$$= \quad \mathrm{Tr}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\frac{1}{2}u_0)^n}{n}\left(\frac{\phi^2}{-\partial^2 + r_0}\right)^n\right)\,, \tag{4.55}$$

$$= \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\frac{1}{2}u_0)^n}{n}\,\mathrm{Tr}\left(\left(\frac{\phi^2}{-\partial^2 + r_0}\right)^n\right)\,, \tag{4.56}$$

$$= \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\frac{1}{2}u_0)^n}{n}\,\mathrm{Tr}\left(\left(\frac{\phi^2}{-\partial^2 + r_0}\right)^n\right) \tag{4.57}$$

where we ignored the free theory contribution. If we are only interested in the momentum independent portion, we can pull ϕ^2 out of the trace:

$$\mathrm{Tr}\,\ln\left(\frac{-\partial^2 + r_0 + \frac{1}{2}u_0\phi^2}{-\partial^2 + r_0}\right) \quad \approx \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\frac{1}{2}u_0)^n}{n}\,\phi^2\,\mathrm{Tr}\left(\left(\frac{1}{-\partial^2 + r_0}\right)^n\right)\,. \tag{4.58}$$

Now here the loops arise. The first two terms of the above expansion yield

$$\Gamma[\phi] \quad = \quad S[\phi] + \frac{1}{4}u_0\phi^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2 + r_0} - \frac{(\frac{1}{2}u_0)^2}{4}\phi^4\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2 + r_0\right)^2} + \ldots \tag{4.59}$$

where we have taken the field ϕ to be spatially uniform. Let us take for granted for the moment that these integrals can be made finite, even at $r_0 = 0$, for dimensions $d > 2$.

These so-called loop corrections modify the two-point and four-point functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$:

$$\begin{aligned}\Gamma^{(2)}[\phi;p] &= \frac{\partial^2\Gamma[\phi]}{\partial\phi(p)\partial\phi(-p)}\,, &(4.60)\\ \Gamma^{(2)}[\phi;p]|_{p=0} &= r_0+\frac{1}{2}u_0\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+r_0}\,, &(4.61)\end{aligned}$$

$$\begin{aligned}\Gamma^{(4)}[\phi;p_1,p_2,p_3] &= \frac{\partial^4\Gamma[\phi]}{\partial\phi(p_1)\partial\phi(p_2)\partial\phi(p_3)\partial\phi(-(p_1+p_2+p_3))}\,, &(4.62)\\ \Gamma^{(4)}[\phi;p_1,p_2,p_3]|_{p_i=0} &= u_0-\frac{3u_0^2}{2}\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+r_0)^2} &(4.63)\end{aligned}$$

where we have brought back out our cutoff Λ which sets an upper bound on the momentum integrals. We now use a renormalization prescription to redefine the renormalized mass and the four-point coupling:

$$\begin{aligned}\Gamma_R^{(2)}[\phi;p]|_{p=0} &= m^2\,, &(4.64)\\ \Gamma_R^{(4)}[\phi;p_1,p_2,p_3]|_{p_i=0} &= u\,. &(4.65)\end{aligned}$$

This entails a relation between the bare mass r_0 and four-point coupling u_0 [106]:

$$\begin{aligned}m^2 &= r_0+\frac{1}{2}u_0\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+r_0}\,, &(4.66)\\ u &= u_0-\frac{3u_0^2}{2}\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+r_0)^2}\,. &(4.67)\end{aligned}$$

Up to one-loop order precision, we are free to replace r_0 with m^2 inside the loops:

$$\begin{aligned}m^2 &= r_0+\frac{1}{2}u_0\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2}\,, &(4.68)\\ u &= u_0-\frac{3u_0^2}{2}\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+m^2)^2}\,. &(4.69)\end{aligned}$$

Whereas the tree-level action predicted a diverging correlation length at $m^2=r_0=0$, the one-loop prediction for $m^2=0$ is

$$r_c = -\frac{1}{2}u_0\int_\Lambda\frac{d^dq}{(2\pi)^d}\frac{1}{q^2}\,. \qquad (4.70)$$

Thus, provided the integral can be made finite, criticality occurs at a negative value of r_0! Given that we had $r_0 = \bar{r}_0(T - T_c^{\mathrm{MF}})$ where $\bar{r}_0$ is a proportionality constant and T_c^{MF} the mean-field critical temperature, we can see that from

$$T_c \;=\; T_c^{\mathrm{MF}} - \frac{1}{2}u_0 \int_\Lambda \frac{d^d q}{(2\pi)^d}\frac{1}{q^2} \tag{4.71}$$

the fluctuations decrease the critical temperature. This also allows us to solve for T_c^{MF} and write

$$\begin{aligned} r_0 &= r_c + \bar{r}_0(T - T_c)\,, && (4.72)\\ &= r_c + t && (4.73) \end{aligned}$$

where t is the reduced temperature.

Analyzing the loop integrals, we see that dimension has a significant impact on the importance of the corrections. One can derive the following result via Γ functions using the concepts of dimensional regularization where the cutoff is not needed, see [113]:

$$\int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+m^2)^n} \;=\; \frac{1}{2}\frac{\Omega_d}{(2\pi)^d} m^{d-2n}\frac{\Gamma(\frac{2n-d}{2})\Gamma(\frac{d}{2})}{\Gamma(n)}\,. \tag{4.74}$$

Specifically, the correction for the two-point and four-point functions behaves

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+m^2)} &\propto m^{d-2}\,, && (4.75)\\ \int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+m^2)^2} &\propto m^{d-4}\,. && (4.76) \end{aligned}$$

To gauge the importance of these integrals, we should compare them to the predictions of the bare theory (mean field/tree-level). Ultimately, we are interested in the small m^2 behavior which takes us near criticality. Let us then massage Eq. (4.68) using Eq. (4.70) [20, 114] :

$$\begin{aligned} m^2 &= r_0 + \frac{1}{2}u_0 \int_\Lambda \frac{d^d q}{(2\pi)^d}\frac{1}{q^2+m^2}\,, && (4.77)\\ &= r_0 - r_c + \frac{1}{2}u_0 \int_\Lambda \frac{d^d q}{(2\pi)^d}\left(\frac{1}{q^2+m^2} - \frac{1}{q^2}\right)\,. && (4.78) \end{aligned}$$

For $d > 4$, the integrals are convergent and we can write

$$m^2 \;=\; t - \# m^{d-2}\, u_0 \tag{4.79}$$

which still agrees with the tree-level result Eq (4.50) as $m^2 \to 0$. Thus, for $d > 4$ the fluctuations do not modify the critical behavior. For specifically $d = 4$, the tree-level behavior will be modified by logarithmic corrections. Thus, for dimensions $d > 4$, mean field offers an exact description of the critical theory ($d = 4$ suffers from logarithmic corrections to mean field.). In Section 5.1, we will acquire a very solid understanding of the Yang-Lee edge singularity using only mean field theory.

On the other hand, for $d < 4$ the integrals diverge as $m \to 0$! However, this divergence can be tamed and the m^2 dependence can be extracted as for the above except that now as $m^2 \to 0$ the leading order behavior is

$$\# m^{d-2}\, u_0 \;=\; t\,. \tag{4.80}$$

Thus, for $d < 4$ one needs to account for the fluctuations and figure out how to manage these divergent integrals. To do so, a common approach is the ε expansion. In this approach, one expands the integrals by analytically continuing *in dimension* via a "small" parameter ϵ defined by $d = 4 - \epsilon$. The general notion is to expand about the given theory's upper critical dimension, where we've found for ϕ^4, $d_c = 4$. The quotation marks on "small" are because one wants to look at $d = 3$, or $\epsilon = 1$, and even $d = 2$, where $\epsilon = 2$. These are certainly not small numbers. For now we will leave further discussion of the ε expansion to Section 5.3.

4.3.2 ϕ^3 theory

We mentioned in Section 3.3 that the Yang-Lee edge singularity corresponds to a $ig\phi^3$ theory. Let us repeat the above study for the YLE action (3.48). We can skip some steps now though. The question is when can we ignore loops? The integrals we should look at then are those which contribute to the two and three point functions. We find respectively

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 + r_0\right)^2} \propto r_0^{\frac{d-4}{2}}\,, \tag{4.81}$$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 + r_0\right)^3} \propto r_0^{\frac{d-6}{2}}\,. \tag{4.82}$$

Repeating the arguments as for ϕ^4, we see that one can ignore fluctuations for a cubic theory not for $d > 4$, but $d > 6$. Hence, the upper critical dimension for cubic theories is $d_c = 6$. Below $d = 6$, the fluctuations are necessary for capturing the physics at critical points. This difference in upper critical dimensions will be important when discussing the application of the ε expansion to determine the location of YLE, see Section 5.3.

4.4 Conceptual RG and scaling

While the previous section was rooted in field theory and saw RG as a means by which one can perturb about a theory's upper critical dimension, this is not the only take on RG. Unfortunately, there are many. Likely the most central approach to renormalization dates back to the late 60s and early 70s and is attributed to Kadanoff [4] and Wilson [5, 6, 7]. The concepts were derived not from high-energy physics, but critical physics of models like the Ising model and resulted in much more intuitive pictures.

Emanating from a paper by Widom [8], a central idea was that at critical points thermodynamic functions obey *scaling laws*. In other words, the functions are *homogeneous functions* of their arguments which are typically the reduced temperature t and external (magnetic) field H. More specifically, it is the *singular* portions of the thermodynamics that obey scaling laws. For example, near the critical point, the free energy $f(t, H)$ can be decomposed into a singular piece $f_s(t, H)$ and a regular piece $f_r(t, H)$ which is analytic about the critical point. The expression of scaling is that by appropriately rescaling by t and H by some length scale b, the free energy behaves as

$$f_s(t, H) \quad = \quad b^{-d} \, f_s(tb^{y_t}, H \, b^{y_h}) \, . \tag{4.83}$$

The exponents y_t and y_h correspond to the RG eigenvalues about the critical point. From these two numbers, all basic critical exponents can be derived.

This idea of scaling goes under another name - the scaling hypothesis; however, it is not a hypothesis anymore since it can be derived from RG. We could have seen this structure emerge from our loop expansion in Section 4.3 by deriving the β functions which characterizes how the couplings *flow* as we vary the cutoff Λ. Finding the critical point amounts to finding a point which exhibits scale invariance. At such points, the theory must look the same no matter Λ. And so, zeros of the β function correspond to critical points. Furthermore, zeros of the β functions are fixed points. So lastly, critical points can be realized as fixed points of β functions. For an excellent and concise review, see

Ref. [12] and also Ref. [115].

There is another subtle feature associated with the scale invariance that arises near the critical points - *universality*. We got a taste of this from the universal van der Waals equation from which was surmised to characterize all fluids. The Guggenheim plot [48], see Fig. 2.3, furthered our suspicion that something of this sort must be going on. Historically, this picture was not fully cleared up until the advent of RG. Near these critical points or continuous phase transitions, the large scale fluctuations one encounters in whatever the observable may be - magnetization, density, or what have you - can be described by a field theory. Surprisingly, descriptions of this sort are very restrictive and one finds the same theory characterizes many different physical systems. Here is a common phrase for this phenomenon: The large scale/long wavelength fluctuations near critical points are insensitive to the microscopic details of the theory which leads to many different microscopic theories yielding the same macroscopic physics.

Once one has the field theory, one can compute the β functions. Linearizing the β functions about the fixed points then yields a stability analysis, a matrix eigenvalue equation with vectors consisting of all the operators of the given theory with which one can perturb about the fixed point. The eigenvectors denote directions emanating from the fixed point which are either repulsive/unstable, attractive/stable, or have no effect/marginal. These directions correspond, in general, to non-linear functions of the theory's parameters. They're referred to as scaling fields, extensions of directions corresponding to thermal t and magnetic h perturbations. The eigenvalues for a given scaling field are like the exponents y_t and y_h. For a given operator, if the eigenvalue is positive, turning on this perturbation pushes one away from the fixed point - an unstable or *relevant* direction. Others have negative eigenvalues and correspond to stable or *irrelevant* perturbations which die out under successive rescalings. Lastly, there are marginal operators with zero eigenvalues.

Near enough to the critical point, one can restrict attention to the *relevant* variables t and H, though away from the critical point, the non-linearity of the scaling fields is important. Here one should generalize to

$$u_t = t_0\, t + t^2 g_{11} + H^2 g_{21} + O(t^3,\, H^2,\, H^4)\,, \tag{4.84}$$

$$u_h = H\left(h_0 + t\, g_{12} + H^2\, g_{22} + O(t^2,\, t\, H^2,\, H^4)\right)\,, \tag{4.85}$$

$$u_3 = g_{13} + t\, g_{23} + H^2\, g_{23} + O(t^2,\, t\, H^2,\, H^4) \tag{4.86}$$

where u_t and u_h take the place of t and h [116]. We also note that we included the non-universal proportionality constants t_0 and h_0 [12]. The new addition u_3 is the leading irrelevant variable given by $g_{12}(\lambda) = u_0$ to leading order. Including the effects of the nonlinearties produces what are known as corrections to scaling.

In terms of the scaling fields above, the expression for f_s is then

$$f_s(u_t, u_h, u_3) \quad = \quad b^{-d} \, f_s(\, u_t \, b^{y_t}, \; u_h \, b^{y_h}, \; u_3 \, b^{y_3}) \tag{4.87}$$

where $y_3 < 0$. One can then chose the scaling parameter b such that $|u_t \, b^{y_t}| = 1$ or so $b = |u_t|^{-1/y_t}$ where we account for the two possible signs of u_t:

$$f_s(u_t, u_h, u_3) \quad = \quad |u_t|^{d/y_t} \, f_s \left(\pm 1, \; \frac{u_h}{|u_t|^{y_h/y_t}}, \; \frac{u_3}{|u_t|^{y_3/y_t}} \right) . \tag{4.88}$$

Alternatively, one could have set $|u_h \, b^{y_h}| = 1$. The expressions obtained from either route will be of use in the following section to derive expressions between the RG eigenvalues and the exponents and to introduce universal scaling functions which demonstrate the universality of the YLE.

4.4.1 Critical Data - $O(N)$ and YLE

Now to apply this formalism, let us introduce the suite of critical data we will be interested in for $O(N)$ universality classes. Before we begin, we need to motivate the occurrence of some of these exponents and scaling forms that appear. What follows will be minimal and brief.

Let us start by doing some dimensional analysis and inspect the action in Eq. (4.4). We can set the units by either specifying the dimension of the quadratic term $[r_0] = 2$ or the coordinates $[x] = -1$. Since the action must be dimensionless, the field must have an *engineering dimension* of $[\phi] = \dfrac{d-2}{2}$. Likewise, one can determine the engineering dimensions of the various couplings.

However, let us use the above result and naively determine the spatial dependence of the Green's function. In the following we reproduce an argument from Ref. [12]. Given that

$$G(r) \sim \langle \phi(0)\phi(r) \rangle \, , \tag{4.89}$$

we might expect that $G(r) \sim r^{-(d-2)}$. However, there is another length scale hidden in the system - the lattice spacing $a \sim \Lambda^{-1}$. It would be perfectly acceptable for a to appear in appropriate dimensionless ratios which would alter the scaling of $G(r)$ while maintaining its engineering dimension:

$$G(r) \sim \frac{a^\eta}{r^{d-2+\eta}}\,. \tag{4.90}$$

It is simple to check that $G(r)$ has the correct engineering dimension since (a/r) is dimensionless. However, now $G(r)$ has a different *scaling dimension* given by $[G] = d-2+\eta$. The correction η is somewhat appropriately named the anomalous dimension. This exponent gauges the difference between the scaling dimension and the engineering dimension. Lastly, we can backtrack the scaling dimension of $G(r)$ to obtain the scaling dimension of the field as $[\phi] = \frac{1}{2}(d-2+\eta)$.

Other quantities to inspect near the critical point include the static susceptibility $\chi = \frac{\partial M}{\partial H}$, the correlation length, and the magnetization M itself. We expect the susceptibility χ and correlation length ξ to diverge as some power of t or H. On the other hand, the magnetization M should scale about zero as powers of t and H. There are, of course, other quantities though will we not be interested in them here. We introduce the relevant items below:

$$\begin{aligned}
M &= B\,(-t)^\beta \quad \text{at} \;\; H=0,\;\; t<0\,, & \chi &= C_2^{+,-}\,|t|^{-\gamma} \quad \text{at} \;\; H=0\,,\\
M &= B_c\,|H|^{1/\delta} \quad \text{at} \;\; t=0\,, & \xi &= f^c\,|H|^{-\nu_c} \quad \text{at} \;\; t=0\,,\\
G(r) &\sim r^{d-2+\eta} \quad \text{at} \;\; H=0,\;\; t=0\,, & \xi &= f^{+,-}\,|t|^{-\nu} \quad \text{at} \;\; H=0\,.
\end{aligned} \tag{4.91}$$

The collection above consists of universal critical exponents $\beta,\delta,\gamma,\nu,\eta$ and *non-universal* critical amplitudes. In the above, χ and ξ take on different amplitudes above (+) and below (-) the transition. The exponents satisfy the following scaling (and hyperscaling) relations

$$\begin{aligned}
\gamma &= \beta(\delta-1)\,, & \gamma &= \nu(2-\eta)\,,\\
\nu_c &= \frac{\nu}{\beta\delta}\,, & \delta &= \frac{d+2-\eta}{d-2+\eta}\,,\\
\beta &= \frac{\nu}{2}(d-2+\eta)\,, & \Delta &= \beta\delta\,.
\end{aligned} \tag{4.92}$$

The bottom right relation defines another exponent Δ. The relations between the exponents and the RG eigenvalues are given below.

$$\begin{aligned} y_t &= \frac{1}{\nu}\,, & \delta &= \frac{y_h}{d-y_h}\,, \\ y_h &= \frac{1}{\nu_c}\,, & \beta &= \frac{d-y_h}{y_t}\,, \\ \frac{y_h}{y_t} &= \Delta\,, & \eta &= d+2-2y_h\,. \end{aligned} \tag{4.93}$$

All other relations follow from the scaling relations.

Interestingly, the hyperscaling relations (those where the dimension d appears) are no longer satisfied above the critical dimension $d > 4$ which implies $\eta = 0$. For $d > 4$, mean field exponents apply, see Section 5.1:

$$\begin{aligned} \gamma &= 1\,, & \beta &= \frac{1}{2}\,, & & \\ \nu &= \frac{1}{2}\,, & \delta &= 3\,, & \eta &= 0\,. \\ \nu_c &= \frac{1}{3}\,, & \Delta &= \frac{3}{2}\,, & & \end{aligned} \tag{4.94}$$

There is another set of exact exponents which can be obtained for $O(N)$ models by considering the large N limit, see Ref. [20] and Section 5.2. The results below are valid for $2 < d < 4$. Note that they agree with mean-field at $d = 4$.

$$\begin{aligned} \gamma &= \frac{2}{d-2}\,, & \beta &= \frac{1}{2}\,, & & \\ \nu &= \frac{1}{d-2}\,, & \delta &= \frac{d+2}{d-2}\,, & \eta &= 0\,. \\ \nu_c &= \frac{2}{d+2}\,, & \Delta &= \frac{1}{2}\left(\frac{d+2}{d-2}\right)\,, & & \end{aligned} \tag{4.95}$$

While the amplitudes in Eq. (4.91) are by themselves non-universal, certain combinations of them are indeed universal, see Ref. [116] for more universal ratios as well collections of results for the exponents and other critical data for $O(N)$ models. For our purposes, we define below the only standard universal ratio we will be interested in:

$$R_\chi \;=\; \frac{C_2^+\, B^{\gamma/\beta}}{B_c^\delta}\,. \tag{4.96}$$

At the level of mean field, $R_\chi = 1$, which also agrees with large N for $2 < d < 4$.

For completeness, we also introduce here the suite of exponents and universal amplitude ratios related to the edge singularity:

$$\begin{aligned} M_c &= i\, B_y\, t^{\beta} \quad \text{at } t > 0\,, & M - M_c &= e^{-i\frac{\pi}{2}(1-\sigma)}\, E_y\, t^{e}\, (H - H_c)^{\sigma} \quad \text{at } t > 0\,, \\ H_c &= i\, D_y\, t^{\Delta} \quad \text{at } t > 0\,, & e &= -\beta(\delta\sigma - 1)\,. \end{aligned} \tag{4.97}$$

As usual, the amplitudes above are non-universal whereas the exponents are universal. The exponents above are the $O(N)$ exponents with the addition of the YLE exponent σ. The above expressions for M_c and $M - M_c$ hold for $\sigma > 0$. For $\sigma < 0$, M_c diverges at the YLE (see Figs. 3.5 and 3.6). We then have a reduced set for $\sigma < 0$:

$$H_c = i\, D_y\, t^{\Delta} \quad \text{at } t > 0\,, \qquad M = e^{i\frac{\pi}{2}(1+\sigma)}\, E_y\, t^{e}\, (H - H_c)^{\sigma} \quad \text{at } t > 0\,. \tag{4.98}$$

The phase $e^{-i\,\frac{\pi}{2}(1-\sigma)}$ for $\sigma > 0$ serves to yield $M - M_c \in -i\,\mathbb{R}^+$ for $H \in i\,\mathbb{R}$ with $|H| < |H_c|$. Similarly, the phase $e^{i\,\frac{\pi}{2}(1+\sigma)}$ serves to produce $M \in i\,\mathbb{R}^+$ for $H \in i\,\mathbb{R}$ with $|H| < |H_c|$. This feature corresponds to Eq. (3.33) and $\mathrm{Im}(M)$ increasing on the interval $H \in (0, H_c)$. See Sections 3.2 and 3.3.

There is a similar set for the spinodal point which occurs in the low temperature phase $t < 0$. Here we only consider for $\sigma > 0$. The expressions below can be obtained by continuing $t \to -t$ through the *negative half-plane* which takes one over the appropriate branch cut. All forms below have $t < 0$:

$$\begin{aligned} M_s &= e^{i\,\pi\left(\frac{1}{2}-\beta\right)}\, B_s\, (-t)^{\beta}\,, & M - M_s &= e^{-i\,\pi\left(\frac{1}{2}(1-\sigma)+e\right)}\, E_s\, (-t)^{e}\, (H - H_c)^{\sigma}\,, \\ H_s &= e^{i\,\pi\left(\frac{1}{2}-\Delta\right)}\, D_s\, (-t)^{\Delta}\,, & e &= -\beta(\delta\sigma - 1)\,. \end{aligned} \tag{4.99}$$

Given that σ is critical data of the YLE whose upper critical dimension is 6, the mean field result of this exponent applies at $d = 6$. In spite of this, the large N result $\sigma = \frac{1}{2}$ holds for all dimensions $d < 6$, see Ref. [95]. For a similar discussion of the relation between the $O(N)$ mean field result and σ, see Ref. [13]. Other exact results for σ are $\sigma_{2D} = -\frac{1}{6}$ [34], and $\sigma_{1D} = -\frac{1}{2}$ which is obtained from the exact 1D Ising solution, see Ref. [15], Appendix A, and Section 5.4. For a collection of results for σ in dimensions $2 \leq d \leq 6$ from various methods, see Refs. [41, 42].

Although the amplitudes for the spinodal turn out to be equal to their YLE counterparts, we allow them to differ for the time being. Amongst these YLE related amplitudes, we can define a set of universal ratios as in Section 2.2.3. Their universality along with the equality of spinodal and YLE amplitudes will be demonstrated in Section 4.4.3:

$$\begin{aligned} R_m &= \frac{B_y}{B}\,, & R_e &= \frac{E_y\, B^{\delta\sigma-1}}{B_c^{\delta\sigma}}\,, \\ R_h &= \frac{B_c^{\delta}\, D_y}{B^{\delta}}\,, & |z_c| &= R_h^{-1/\Delta} \end{aligned} \tag{4.100}$$

where $|z_c|$ is related to the YLE as a singularity of the scaling function $f_G(z)$. There is an identical set from the spinodal point. For later purposes, we introduce another universal quantity

$$\zeta_c = \frac{z_c}{R_\chi^{1/\gamma}}\,. \tag{4.101}$$

The calculation of the quantity z_c (R_h) and/or ζ_c for three dimensional $O(N)$ models is the main work of this thesis. See Chapter 6. We quote below the mean field and large N results for these quantities:

$$\begin{aligned} R_m &= \frac{1}{\sqrt{3}}\,, & R_e &= \frac{1}{3^{1/4}}\,, \\ R_h &= \frac{2}{3\sqrt{3}}\,, & |z_c| &= \frac{3}{2^{2/3}}\,, \\ R_\chi &= 1\,, & \zeta_c &= z_c\,, \end{aligned} \tag{4.102}$$

and for large N we have

$$\begin{aligned} R_m &= \left(\frac{d-2}{d+2}\right)^{\frac{1}{2}}\,, & R_e &= 4\,(d-2)^{\frac{1}{4}}\,(d+2)^{-\frac{5}{4}}\left(\frac{d+2}{4}\right)^{\frac{d}{2(d-2)}}\,, \\ R_h &= \left(\frac{d-2}{d+2}\right)^{\frac{1}{2}}\left(\frac{4}{d+2}\right)^{\frac{2}{d-2}}\,, & |z_c| &= 4^{-\frac{4}{d+2}}(d+2)^{\frac{2(d-2)}{d+2}}\left(\frac{d+2}{d-2}\right)^{\frac{d-2}{d+2}}\,, \\ R_\chi &= 1\,, & \zeta_c &= z_c\,. \end{aligned} \tag{4.103}$$

Note that $z_c(d \to 2) = 1$. See Section 6.2 for a plot of z_c for $3 < d < 4$.

4.4.2 Scaling: leading order and corrections

Now that we have introduced the notations and definitions of the critical data we are interested in, let us review the application of the scaling forms and follow this with a discussion of the extended universality of the Yang-Lee edge singularity. Stating from Eq. (4.88), the idea is that remaining function

$$F_{\pm}\left(\frac{u_h}{|u_t|^{y_h/y_t}}, \frac{u_3}{|u_t|^{y_3/y_t}}\right) \quad = \quad f_s\left(\pm 1, \frac{u_h}{|u_t|^{y_h/y_t}}, \frac{u_3}{|u_t|^{y_3/y_t}}\right) \tag{4.104}$$

is universal. Deriving the expression for f_s for $u_t < 0$ by H yields a scaling form for the magnetization

$$M(t,H) \quad = \quad |u_t|^{\frac{d-y_h}{y_t}} F_{-}^{(1,0)}\left(\frac{u_h}{|u_t|^{y_h/y_t}}, \frac{u_3}{|u_t|^{y_3/y_t}}\right) \frac{\partial u_h}{\partial H} \tag{4.105}$$

where $F_{-}^{(1,0)}$ denotes a derivative with respect to the first argument. Let us expand u_h to next-to-leading order and u_t to leading order in Eq. (4.84). Now, Taylor expand about $F_{-}^{(1,0)}$ about $(0,0)$:

$$\begin{aligned} M(t,H) \quad &= \quad |t_0\, t|^{\frac{d-y_h}{y_t}} \left(1 + t_0\, t\, g_{12}\right) \sum_{n,m} \frac{1}{n!m!} F_{-}^{(1+n,m)} \left(\frac{u_h}{|t_0\, t|^{y_h/y_t}}\right)^n \left(\frac{u_3}{|t_0\, t|^{y_3/y_t}}\right)^m , \\ &\approx \quad (-t_0\, t)^{\frac{d-y_h}{y_t}} \left(1 + t_0\, t\, g_{12}\right) \left(F_{-}^{(1,0)}(0,0) + F_{-}^{(2,0)}(0,0) \left(\frac{u_h}{|t_0\, t|^{y_h/y_t}}\right)\right. \\ &+ \quad \left.F^{(1,1)}(0,0) \left(\frac{u_3}{|t_0\, t|^{y_3/y_t}}\right)\right) . \end{aligned} \tag{4.106}$$

Let us truncate $u_h \to h_0\, H$ and $u_3 \to g_{13}$ in the expansion. Since the irrelevant eigenvalue, y_3 is negative, we define the *correction to scaling* exponent ω as $y_3 = -\omega$. Lumping together constant terms

$$M(t,H) = B\,(-t)^{\frac{d-y_h}{y_t}} \left(1 + t\, c_{12}\right) \left(a_{10} + a_{20}\; H\, |t|^{-y_h/y_t} + a_{11}|t|^{\omega/y_t}\right) . \tag{4.107}$$

From the above expression, we recognize that $\beta = \dfrac{d-y_h}{y_t}$. Other relations between exponents and the RG eigenvalue follow from similar steps. Rewriting the above in terms of the critical exponents and lumping the constant terms together gives

$$M(t,H) = B\,(-t)^{\beta} \left(1 + t\, g_{12}\right) \left(1 + a_{20}\; H\, |t|^{-\Delta} + a_{11}|t|^{\omega\nu}\right) . \tag{4.108}$$

The above expression illuminates a handful of features. The leading scaling as in Eq. (4.91) only applies for t sufficiently small. Stepping away from leading scaling, there are a few notions of *correction to scaling*. The first term in parenthesis is due to the higher-order contribution of non-linear scaling field - it yields an analytic correction which can be ignored for $O(N)$ models as its correction is typically smaller than others [12]. In the second set of parenthesis, there are non-analytic corrections stemming from non-zero H (and the nonlinearities of the scaling fields) as well as the presence of an irrelevant variable.

Similar expressions for the corrections to scaling can be obtained for any scaling expression. For instance, at $t = 0$ and $|H| \ll 1$ one obtains

$$M(t, H) = B_c |H|^{1/\delta} \left(1 + b_{11} |H|^{\omega\nu_c}\right) . \tag{4.109}$$

In our work, we will find both of these corrections to be both important for precision purposes, but also useful for optimizing procedures where computing at precisely $H = 0$ is computationally expensive. The exponents extracted from the corrections to scaling also give a further check of the numerical accuracy.

4.4.3 Scaling functions and YLE/spinodal universality

Let us now look back to Eq. (4.88). The quantity on the RHS must reproduce all the non-universal critical amplitudes in Section 4.4.1. Being that the function on the RHS is universal, the model dependence must then come from the scaling variables u_t, u_h, and u_3 [12]. Similar to what was done to obtain the universal van der Waals equation, we can rescale the thermal and magnetic variables t and H to express the scaling forms in a universal way. Doing so eliminates the appearance of these non-universal quantities and makes the universal aspects apparent.

For instance, there is the Widom scaling form

$$h = M^\delta f\left(x \equiv \bar{t}\, M^{-1/\beta}\right) \tag{4.110}$$

where $h = B_c^\delta\, H$ and $\bar{t} = B^{1/\beta}\, t$. There are normalization conditions on the scaling function $f(x)$:

$$f(0) = 1\,, \tag{4.111}$$

$$f(-1) = 0 \tag{4.112}$$

which ensure the forms $M = h^{1/\delta}$ and $M = (-\bar{t})^\beta$ are reproduced in appropriate limits.

However, regardless of the rescaling, $f(x)$ is universal. One could write

$$B_c^\delta\, H = M^\delta\, f(x \equiv t\, M^{-1/\beta}) \tag{4.113}$$

with the normalization conditions

$$f(0) = 1\,, \tag{4.114}$$
$$f(-B) = 0 \tag{4.115}$$

which is explicitly non-universal due to the appearance of the amplitude B. However, by rescaling we are able to make apparent the universal behavior of $f(x)$.

An alternate scaling form is the so called universal magnetic scaling equation of state:

$$M = h^{1/\delta}\, f_G(z \equiv \bar{t}\, h^{-1/\beta\delta}) \tag{4.116}$$

where $\bar{t}$ and h have the same definitions as for the Widom scaling form. The normalization conditions on $f_G(z)$ are

$$f_G(0) = 1\,, \tag{4.117}$$
$$\lim_{z\to\infty} \frac{f_G(z)}{(-z)^\beta} = 1 \tag{4.118}$$

which again ensure the forms $M = h^{1/\delta}$ and $M = (-\bar{t})^\beta$ are reproduced in appropriate limits.

Given these universal functions, we can readily see the universality of certain amplitude ratios. For instance, after some algebra one finds that for large z behaves as

$$f_G(z \to \infty) \approx R_\chi z^{-\gamma}\,, \tag{4.119}$$

thus demonstrating the universality of R_χ.

Universality of the YLE

More importantly for us, from the universality of the scaling functions, we can see the universality of the quantities discussed in Section 2.2.3 and Section 4.4.1. Given that near the YLE we expect $M - M_c \propto (H - H_c)^\sigma$ where $\sigma < 1$, and so the presence of a branch point at $H = \pm H_c$ with the cut along the imaginary H axis beyond $\pm H_c$. The scaling function $f_G(z)$ must then have the information of this branch cut. The location of the branch point is then automatically universal when expressed in terms of the scaling variable z. Evaluating at $t > 0$ and $h = h_c$ yields

$$f_G(z_c) = M_c\, h_c^{-1/\delta}\,, \tag{4.120}$$

$$z_c = \bar{t}\, h_c^{-1/\beta\delta} \tag{4.121}$$

where z_c is the universal location of the YLE. Furthermore, re-expressing in terms of the basic amplitudes B, B_c, and D_y as in Eq. (4.97), one has after some algebra

$$z_c \;=\; \frac{B^{1/\beta}}{B_c^{1/\beta} D_y^{1/\beta\delta}}\, e^{\pm\frac{\pi i}{2\beta\delta}}\,, \tag{4.122}$$

$$f_G(z_c) \;=\; \frac{B_y}{B_c\, D_y^{1/\delta}}\, e^{\pm\, i\frac{\pi}{2}(1-\frac{1}{\delta})} \tag{4.123}$$

where we used the scaling form of H_c as in Eq. (4.97). Thus, we have that the universal location is indeed expressible as a universal ratio! So, one can either study this scaling function or the scaling of the gap H_c where one would compute R_h as in Eq. (4.100).

From the above, we also have the universality of R_m as in Eq. (4.100) by noting that

$$f_G(z_c) \;=\; i\,\frac{B_y}{B}\, z_c^{\beta}\,, \tag{4.124}$$

$$\;=\; i\, R_m\, z_c^{\beta}\,. \tag{4.125}$$

Lastly, we have to show the universality of the ratio R_e as defined in Eq. (4.100). First, we note that in terms of the rescaled variables

$$M - M_c \;=\; e^{-i\frac{\pi}{2}(1-\sigma)}\, R_e\, \bar{t}^{e}\, (h-h_c)^{\sigma}\,. \tag{4.126}$$

Let us both motivate the appearance of t^e along with the universality of R_e. To begin, let us write

$$M - M_c = \bar{E}_e\, (h-h)^{\sigma}\,. \tag{4.127}$$

Using the definition of $f_G(z)$

$$f_G(z) - f_G(z_c) = \bar{E}_e\, z^{\beta}\, \bar{t}^{\beta(\delta\sigma-1)} \left(z^{-\beta\delta} - z_c^{-\beta\delta}\right)^{\sigma} , \tag{4.128}$$

$$= \bar{E}_e\, z^{\beta(1-\delta\sigma)}\, \bar{t}^{\beta(\delta\sigma-1)} \left(1 - \left(\frac{z}{z_c}\right)^{\beta\delta}\right)^{\sigma} . \tag{4.129}$$

Given that the above expression is universal and so should be independent of $\bar{t}$, the amplitude $\bar{E}_e$ must contain a power of $\bar{t}$:

$$\bar{E}_e = E_e\, \bar{t}^{-\beta(\delta\sigma-1)} \tag{4.130}$$

which yields

$$f_G(z) - f_G(z_c) = E_e\, z^{e} \left(1 - \left(\frac{z}{z_c}\right)^{\beta\delta}\right)^{\sigma} \tag{4.131}$$

where $e = -\beta(\delta\sigma - 1)$. Here, we again have a trivial sort of proof that E_e is universal since it is specific value of $f_G(z)$. We also provide another expression obtained by expanding about z_c:

$$\frac{f_G(z) - f_G(z_c)}{(z - z_c)^{\sigma}} = R_e\, z_c^{e-\sigma} \left(-\beta\delta\right)^{\sigma} . \tag{4.132}$$

Universality of the spinodal

Now let us relate the above discussion to the spinodal point. The idea to motivate is that the YLE branch point and the spinodal point are identical in the eyes of $f_G(z)$ - they are the same branch point as seen from different Riemann sheets.

As an alternative object, one could see z_c as determining the critical h as a function of t, or so

$$h = \left(\frac{t}{z_c}\right)^{\beta\delta} , \tag{4.133}$$

$$= e^{i\left(\frac{\pi}{2} + \beta\delta \operatorname{Arg} t\right)} |z_c|^{-\beta\delta}\, |t|^{\beta\delta} \tag{4.134}$$

where one can check that for $t > 0$, one recovers $h_c = i\,\#$. To find the spinodal, we need to analytically continue $t \to -t$; however, we have two options. One could continue through the lower half plane branch cut or the upper half plane branch cut. The question at hand then is which sheet corresponds to the metastable physics of $M > 0$ and $H < 0$ occurring at $t < 0$?

If we take $t < 0$ and look for the spinodal, we find

$$h_s \;=\; e^{i\,\pi\left(\frac{1}{2}\pm\beta\delta\right)}|z_c|^{-\beta\delta}\,|t|^{\beta\delta} \tag{4.135}$$

where we've allowed for both paths of continuation. Given mean field exponents $\beta\delta = \frac{3}{2}$, one can see that the proper choice is continuation through the lower half plane which yields the mean field result Arg$H = -i\,\pi$ or so $H < 0$. Also, continuing through the lower half complex t plane takes one from h_c along a clockwise circular arc through the branch cut along the negative axis before ending at the spinodal point on the other sheet. Going under the branch cut along the positive imaginary h axis would have apparently lead us to the sheet describing the physics of $M < 0$ and $t < 0$. (Ultimately, this point scales with the same exponent and amplitude however with a different phase.)

Thus, in general

$$h_s \;=\; e^{i\,\pi\left(\frac{1}{2}-\beta\delta\right)}|z_c|^{-\beta\delta}\,|t|^{\beta\delta}\,. \tag{4.136}$$

The above result allows for a handful of observations. First, beyond mean field the spinodal point is shifted off the negative real axis by an angle $\Delta\phi = \pi\left(\frac{3}{2} - \beta\delta\right)$, see Ref. [13] and Fig. 6.3. Next, it makes clear the equality of the amplitudes D_y and D_s appearing in Eqs. (4.97) and (4.99) as well as the t dependence of H_s. Following this line of reasoning, one also easily shows that $B_s = B_y$ and $E_s = E_e$ along with the appropriate phases and identical exponents. Finally, due to their equivalence as a branch point of scaling functions, the location of YLE and spinodal along with the associated universal amplitudes can be described by the three universal amplitude ratios R_m, R_h, and R_e as defined in Eq. (4.100).

CHAPTER

5

(SEMI-)ANALYTICAL RESULTS FOR THE UNIVERSALITY OF THE YANG-LEE EDGE SINGULARITY

In this chapter, we collect the analytic and semi-analytic results for the extended universality associated with the Yang-Lee edge singularity as discussed in Sections 4.4.1 and 4.4.3 and Chapter 2. For the Ising universality classes, one has access to enough critical data to extract the universal YLE location $|z_c|$ for $d = 1$, $d = 2$, and $d = 4$ (mean field) while $d = 3$ offers no such analytic result. For the $O(N)$ universality classes, there are results for the large N limit (for $2 < d < 4$) and via the ϵ expansion about $d = 4$. In the following sections, we overview these results for the location. We determine the additional universal quantity R_m and R_e only for mean field and large N, although they are in principle calculable from the other results as well. Lastly, we utilize the mean field approximation to motivate further aspects of the YLE and spinodal aside from the location itself.

In Chapter 6, we formulate the FRG approach to locating the YLE singularity. As it is our primary objective, our truncation method in Ref. [44] was optimized to perform simulations in the symmetric phase. Calculations in the broken phase are possible in a different truncation scheme, such as in [45]; however, for Ref. [44] we want to extract all required quantities within one scheme to avoid introducing systematic errors by mixing different truncations in the simulations. We thus strive to avoid the broken phase. This motivates us to introduce another universal quantity, as discussed in Section 4.4.1,

$$\zeta = \frac{z}{R_\chi^{1/\gamma}} \tag{5.1}$$

where the universal ratio R_χ is defined by Eq. (4.96). In terms of the critical amplitudes and *unscaled* variables t and H, the expression for ζ is seen to be

$$\zeta = \left(\frac{B_c}{C_2^+}\right)^{1/\gamma} \frac{t}{H^{1/\Delta}} \tag{5.2}$$

which is independent of the amplitude B, thus explicitly demonstrating that in order to extract the location of the YLE singularity in ζ we do not need to perform simulations in the broken phase.

We stress that ζ and z are related through a universal number R_χ and universal critical exponent γ. On one hand, R_χ can be computed in the FRG approach but would require probing the broken phase and thus switching the FRG truncation scheme used in this thesis. The associated systematic error is difficult to assess. On the other hand, for applications to lattice QCD, $|z_c|$ is often considered. We thus will provide a separate set of results for $|z_c|$ using R_χ obtained in Ref. [117].

5.1 Mean field theory

A simple phenomenological approach to tackling the ϕ^4 theory devised above is given the Landau-Ginzburg action. The idea is to model the free energy on both the symmetry of the theory and build in all relevant interactions. Thus, one is left with the expression for the Landau-Ginzburg effective action $\Gamma_{\mathrm{LG}}[\phi]$ as

$$\Gamma_{\mathrm{LG}}[\phi] \;=\; \int_{\mathbf{x}} \left\{ \frac{1}{2} Z(\partial\phi(\mathbf{x}))^2 + \frac{1}{2} r_0\, \phi^2(\mathbf{x}) + \frac{1}{4!} u_0\, \phi^4(\mathbf{x}) - H\phi(\mathbf{x}) \right\} \tag{5.3}$$

where we have simplified to only include spatial uniform external fields. Note that the above form also follows from ignoring the loop contributions in Eq. (4.42) equating the effective action with the bare action. As discussed in Section 4.3, this is the mean field approximation and is only valid for $d \geq 4$. Furthermore, for $O(N)$ models with $d \geq 4$ the contribution of the Goldstone modes is also finite; thus, they produce no modification to the mean field expression. The result is that the same mean field expressions applies for all $O(N)$ models in dimensions $d \geq 4$.

Given that the effective action Γ is analogous to the Gibbs free energy, in what follows we will reduce Γ to a free energy density. Let us now take the approximation further in noting that upon minimizing the free energy, one may restrict attention to uniform fields $\phi(x) = M$. This does correspond to the physics we are interested in as well. The result is the expression for the mean field free energy density f_{MF}:

$$f_{\mathrm{MF}} = \frac{1}{\mathrm{Vol}} \Gamma \bigg|_{\phi(x)=M}, \tag{5.4}$$

$$= \frac{1}{2} t M^2 + \frac{1}{4!} u_0 M^4 - H\phi \tag{5.5}$$

where M takes on the role of our order parameter. In the above, we have used that at the level of mean field, we can identify $r_0 = t$ since $r_c = 0$. The equation of state (EOS) is given by

$$h = M \left(t + \frac{1}{6} u_0 M^2 \right) \tag{5.6}$$

with the effective mass or inverse susceptibility

$$\chi^{-1} = m^2 = t + \frac{1}{2} u_0 M^2 \,. \tag{5.7}$$

Solving the EOS at $H = 0$ gives (see Eq. (4.45))

$$M(0) = \begin{cases} 0 & r_0 > 0 \\ \pm\sqrt{\dfrac{-6r_0}{u_0}} & r_0 < 0 \end{cases} . \tag{5.8}$$

Thus, from the above we have the following results:

$$\begin{array}{lll} \beta = \dfrac{1}{2}\,, & \delta = 3\,, & \gamma = 1\,, \\ B = \sqrt{\dfrac{6}{u_0}}\,, & B_c = \left(\dfrac{6}{u_0}\right)^{\frac{1}{3}}\,, & C_2^+ = 1\,. \end{array} \tag{5.9}$$

To obtain a universal expression for the free energy, we rescale the thermal and magnetic variables $t \mapsto \bar{t}$ and $H \mapsto h$ so as to produce the scaling forms

$$M = (-\bar{t})^{\beta} \quad \text{at} \;\; h = 0,\;\; t < 0\,, \tag{5.10}$$

$$M = |h|^{1/\delta} \quad \text{at} \;\; \bar{t} = 0\,. \tag{5.11}$$

In terms of the rescaled variables, the free energy, EOS, and effective mass take the form

$$f(M) = \frac{1}{2}\bar{t}M^2 + \frac{1}{4}M^4 + hM\,, \tag{5.12}$$

$$h = M\left(\bar{t} + M^2\right)\,, \tag{5.13}$$

$$\bar{\chi}^{-1} = \bar{t} + 3M^2 \tag{5.14}$$

where $\bar{\chi} = \frac{\partial M}{\partial h}$. All expressions obtained from the rescaled free energy will be universal. For instance, the susceptibility in terms of the rescaled variables yields

$$\bar{\chi} = \frac{C_2^+\, B^{\gamma/\beta}}{B_c^{\delta}}\,\bar{t}^{-\gamma}\,, \tag{5.15}$$

$$= R_\chi\,\bar{t}^{-\gamma} \tag{5.16}$$

from which we recover the mean field prediction of $R_\chi = 1$.

To find the YLE, we set $t > 0$ and look for solutions to $f' = f'' = 0$ for *complex* fields h and M. We obtain M_c from $\chi^{-1} = 0$:

$$M_c = \pm i\,\frac{1}{\sqrt{3}}\,\bar{t}^{\frac{1}{2}} \tag{5.17}$$

which the EOS then yields

$$H_c = \pm\frac{2}{3\sqrt{3}}\,\bar{t}^{\frac{3}{2}}\,. \tag{5.18}$$

The mean field expressions for M_c and H_c do indeed recover the expectation of H_c and

M_c being purely imaginary, the exponents β and Δ as given in Eq. (4.97), as well as the mean field predictions for the universal amplitude ratios R_m and R_h as in Eq. (5.56).

Now to demonstrate the cubic theory associated with the YLE, we expand $f(M)$ about M_c:

$$f(M) \;\; = \;\; (h - h_c)\,(M - M_c) + M_c\,(M - M_c)^3 + \frac{1}{4}\,(M - M_c)^4 \tag{5.19}$$

where we dropped field independent terms. Thus we see that do to $M_c \in i\,\mathbb{R}$, the critical theory for the YLE is indeed an $i\,g\,\phi^3$ where $\phi = M - M_c$ as in Eq. (3.48). The relevant equation of state for this theory is given by

$$h - h_c \;\; = \;\; 3\,M_c\,(M - M_c)^2 + (M - M_c)^3 \tag{5.20}$$

which near to criticality ($H = H_c$) can be expanded as

$$\begin{aligned} M - M_c \;\; &= \;\; \frac{1}{\sqrt{3M_c}}\,(h - h_c)^{\frac{1}{2}}\;, &(5.21)\\ &= \;\; 3^{-\frac{1}{4}}\,e^{-i\,\frac{\pi}{4}}\,\bar{t}^{-\frac{1}{4}}\,(h - h_c)^{\frac{1}{2}}\;. &(5.22)\end{aligned}$$

The above yields that standard mean field result $\sigma = \frac{1}{2}$, the universal ratio $R_e = 3^{-\frac{1}{4}}$, and the exponent $e = -\frac{1}{4} = -\beta(\delta\sigma - 1)$.

Let us discuss the phase appearing above. Given the result of Eq. (3.33), we expect the magnetization to be a purely imaginary number below M_c for purely imaginary h below h_c. Thus, $M - M_c$ must be purely imaginary and negative. To produce this effect, the difference $h - h_c$ yields a factor of $(-i)^{1/2}$ which combines with the overall phase to yield $e^{-i\,\pi} = -1$. For general σ, some algebra produces the general phase $e^{-i\,\frac{1}{2}(1-\sigma)}$ as in Eq. (4.97). Lastly, for the mean field prediction of universal location, $|z_c|$, one easily recovers the result quoted in Eq. (5.56).

Lastly, let us repeat the above discussion for the spinodal. Looking to χ^{-1} as shown Eq (5.12), we see that for $\bar{t} < 0$, there is a solution to $\chi^{-1} = 0$ for real and positive M. Explicitly,

$$M_s = \pm\frac{1}{\sqrt{3}}\,(-\bar{t})^{\frac{1}{2}} \tag{5.23}$$

which we note is smaller than the minimum $M_0 = (-\bar{t})^{\frac{1}{2}}$. Reasonably then, we expect this solution to correspond to a negative h. Upon using the EOS to solve for h_s yields

$$h_s \;=\; \mp\frac{2}{3\sqrt{3}}\,(-\bar{t})^{\frac{3}{2}}\;. \tag{5.24}$$

Re-expanding the free energy now about $M = M_s$,

$$M - M_s = (h - h_s)(M - M_s) + M_s\,(M - M_s)^3 + \frac{1}{4}\,(M - M_s)^4 \tag{5.25}$$

which shows that the spinodal is governed by a real cubic field theory since M_s is real! Lastly, the relevant EOS for the spinodal yields

$$M - M_s \;=\; \frac{1}{\sqrt{3M_s}}\,(h - h_s)^{\frac{1}{2}}\;, \tag{5.26}$$

$$\;=\; 3^{-\frac{1}{4}}\,(-\bar{t})^{-\frac{1}{4}}\,(h - h_s)^{\sigma}\;. \tag{5.27}$$

The above relations for M_s, h_s, and $M - M_s \propto (h - h_s)^{\sigma}$ reproduce the expected results detailed in Eq. (4.99). Furthermore, the above form recovers the expectation that for $h > h_s$, the magnetization satisfies $M - M_s > 0$. In other words, the stable/metastable magnetization is an increasing function of h (on the appropriate Riemann sheet).

On a final note, one can derive the mean field expression for the Widom scaling form Eq. (4.110) by a simple re-writing of the EOS

$$h \;=\; M^3\left(1 + \bar{t}\,M^{-2}\right)\;. \tag{5.28}$$

Using $z = \bar{t}M^{-1/\beta}$ together with the mean field exponents yields the simple expression for the Widom scaling function f:

$$f(x) = 1 + x\,. \tag{5.29}$$

5.2 $N \to \infty$ limit

The large N scaling equation of state can be readily computed; for a review, see Ref. [20], where the Widom scaling relation and critical exponents were derived:

$$f = (1 + x)^{2/(d-2)} \tag{5.30}$$

and

$$\delta = \frac{d+2}{d-2}, \quad \beta = \frac{1}{2}\,. \tag{5.31}$$

The function f defines the value of $\gamma = 2/(d-2)$. Let us quote the result for the free energy as well,

$$f(M) \quad = \quad \frac{d-2}{2d}\left(M^2 + \bar{t}\right)^{\frac{d}{d-2}} \tag{5.32}$$

from which one can easily determine $R_\chi = 1$. To find the position of the YLE singularity, it is convenient to consider the inverse magnetic field susceptibility $\chi^{-1} = \frac{\partial h}{\partial M}\Big|_t$. Its zeroes define the values of x at the YLE, x_c. In terms of the function $f(x)$, we have

$$\beta\delta f(x_c) - x_c f'(x_c) = 0\,. \tag{5.33}$$

This leads to

$$x_c = -\frac{d+2}{d-2}\,, \tag{5.34}$$

$$f_c = \left(\frac{4}{2-d}\right)^{\frac{2}{d-2}}\,. \tag{5.35}$$

Now we can proceed with finding z_c. For this we express z_c in terms of x and f:

$$z_c = \frac{t_c}{h_c^{1/\beta\delta}} = \frac{x_c}{f_c^{1/\beta\delta}} \tag{5.36}$$

leading to

$$|z_c| = \frac{d+2}{2^{\frac{8}{d+2}}}(d-2)^{\frac{2-d}{2+d}}\,. \tag{5.37}$$

Since at large N, $R_\chi = 1$, thus $|\zeta_c| = |z_c|$. To compare to the result of the next section, we perform the expansion near four dimensions $d = 4 - \varepsilon$:

$$|z_c| \approx |z_c^{\mathrm{MF}}| \left(1 - \frac{\ln 2}{9}\varepsilon\right) \tag{5.38}$$

where we introduced z_c^{MF} as the value of z_c at the upper critical dimension of the $O(N)$ universality classes $d_c = 4$ and the corresponding absolute value $|z_c^{\mathrm{MF}}| = \frac{3}{2^{2/3}}$.

Near the lower critical dimension $d = 2 + \tilde{\varepsilon}$, we obtain

$$|z_c| \approx 1 + \frac{1}{4}\left(1 - \ln \frac{\tilde{\varepsilon}}{4}\right)\tilde{\varepsilon}\,. \tag{5.39}$$

Lastly, we note that one repeats identical steps as for mean field theory to demonstrate that large N predicts $\sigma = \frac{1}{2}$ and to furthermore compute R_m and R_e as in Eq. (4.103).

5.3 The epsilon expansion

Although the $N \to \infty$ limit provides an analytic result for any d in the range $2 < d < 4$, it is not well suited to describe phenomenologically relevant universality classes. Specifically for finite temperature QCD we are interested in the Ising universality class $N = 1$ and the Heisenberg model with $N = 2$ (due to lattice discretization artifacts, see e.g. Ref. [118]) and $N = 4$. There is another analytic limit in which one can perform the calculation – near the upper critical dimension $d = 4 - \varepsilon$. As we document below, as far as the position of the YLE singularity is concerned, the utility of this approach is somewhat restricted and it cannot be systematically improved to yield a reliable result in three dimensions. However, it provides some useful information on the location of the YLE singularity near four dimensions. It is also not limited to $N \to \infty$. Moreover, it serves an estimate of the value of N at which one can apply a large N approximation for the purpose of locating the YLE singularity.

In the conventional ε expansion, see e.g. Ref. [106], near the upper critical dimension $d = 4 - \varepsilon$, the critical exponents are

$$\gamma = 1 + \frac{N+2}{2(N+8)}\varepsilon + O\left(\varepsilon^2\right)\,, \tag{5.40}$$

$$\beta\delta = \frac{3}{2} + \frac{1}{2}\left(1 - \frac{9}{N+8}\right)\varepsilon + O\left(\varepsilon^2\right)\,. \tag{5.41}$$

The same method yields the universal amplitude ratio, see Ref. [119],

$$R_\chi = 1 + \frac{3}{2(N+8)}\ln\left(\frac{27}{4}\right)\varepsilon + O\left(\varepsilon^2\right)\,. \tag{5.42}$$

Note that, in the large N limit, this is consistent with the previous subsection, $R_\chi = 1$.

To the linear order in ε [108], the scaling function f has the following form

$$f(x) = 1 + x \\ + \varepsilon \frac{(N-1)(x+1)\ln(x+1) + 9(x+3)\ln(x+3) - 9\ln 3 + 3x \ln \frac{4}{27}}{2(N+8)} + \mathcal{O}(\varepsilon^2)\,. \tag{5.43}$$

We now turn to the location of the YLE singularity. At the leading order ε-expansion, we obtain $x_c^{(0)} = -3$ for the solution of Eq. (5.33). For our purpose, the exact expression for $\hat{x}(\varepsilon)$ is of no importance as will be demonstrated below. Moreover the first correction $\hat{x}(\varepsilon)$ to the leading order $x_c^{(0)}$ is already non-perturbative (See Appendix D for details).

To the first order in ε,

$$f_c = f(x_c) = -2 + \frac{\varepsilon}{2(N+8)}\left[-2(N-1)\ln(-2) + 18\ln\left(\frac{3}{2}\right)\right] + \hat{x}(\varepsilon)\,. \tag{5.44}$$

Here $\hat{x}(\varepsilon)$ is the leading correction to $\varepsilon = 0$ value of f_c. It seems that the presence of the non-perturbative contribution would prevent us from finding the ε order correction to the location of the YLE singularity. This is, however, not the case. Indeed, after expressing z in terms of x_c and f_c

$$z_c = \frac{t_c}{h_c^{1/\beta\delta}} = \frac{x_c}{f_c^{1/\beta\delta}} \tag{5.45}$$

we obtain

$$z_c \approx z_c^{\mathrm{MF}}\left[1 + \frac{27\ln\left(\frac{3}{2}\right) - (N-1)(\ln 2 - 5i\pi)}{9(N+8)}\epsilon\right] \tag{5.46}$$

where the terms proportional to $\hat{x}(\varepsilon)$ *cancel*. At higher orders of the ε expansion, this cancellation does not happen. This prevents us from extracting corrections beyond the linear order. For the special case of $N = 1$, Eq. (5.46) was previously derived in Ref. [43].

For the absolute value of z_c, we get

$$|z_c| \approx |z_c^{\mathrm{MF}}|\left[1 + \frac{27\ln\left(\frac{3}{2}\right) - (N-1)\ln 2}{9(N+8)}\epsilon\right]\,. \tag{5.47}$$

Note that, the slope of the function $|z_c(\varepsilon)|$ is negative for $N < 1 + 27\left(\frac{\ln 3}{\ln 2} - 1\right) \approx 16.8$. It demonstrates that, at least as long the slope of the d dependence is concerned, to reproduce the result of $N \to \infty$ limit one has to consider rather large values of $N \gg 17$. Note that the ε-expansion at any given ε leads to a monotonic dependence of the location on N. As we demonstrate in Section 6.3.8, for the physical point $d = 3$ or $\varepsilon = 1$, this dependence is non monotonic.

Using Eq. (5.42) for the universal ratio R_χ leads to

$$|\zeta_c| \approx |z_c^{\mathrm{MF}}| \left[1 - \frac{2(N-1)\ln 2 + 27\ln 3}{18(N+8)}\epsilon\right] . \tag{5.48}$$

In $N \to \infty$ limit, this results reproduces the leading order expansion near four dimension of Section 5.2 in $d = 4 - \varepsilon$, see Eq. (5.38).

5.4 1D Ising Model

Let us start with the expression for the free energy of the one dimensional Ising model as derived in Appendix A,

$$F = -\frac{1}{\beta}\ln\left(e^{\beta J}\cosh\beta H + \sqrt{e^{2\beta J}\cosh^2\beta H - 2\sinh 2\beta J}\right) . \tag{5.49}$$

Following Fisher [120] and Ref [43], we define the thermal scaling variable as $x = e^{-4\beta J}$. Additionally, we take $h = \beta H$ and $f = -\beta F$:

$$f = \ln\left(\cosh h + \sqrt{\sinh^2 h + x}\right) \tag{5.50}$$

where we have dropped terms independent of t and h. Expanding the above about $x = h = 0$ we obtain

$$f = x^{\frac{1}{2}}\left(1 + \frac{h^2}{2x}\right) . \tag{5.51}$$

Comparing the above expression to Eq. (4.88), we can identify the ratio in parenthesis with the argument of the scaling function. Thus, $\dfrac{y_h}{y_t} = \dfrac{1}{2}$ or so $\beta\delta = \frac{1}{2}$. Furthermore, $\dfrac{1}{y_t} = \dfrac{1}{2}$, or so $y_t = 2$. From the eigenvalues, we find $\gamma = \frac{1}{2}$ and, oddly enough, $\beta = 0$ and $\delta = \infty$ whose product equals $\beta\delta = \frac{1}{2}$.

Looking to the magnetization at small x and h,

$$M \approx \frac{h}{\sqrt{h^2 + x}} \tag{5.52}$$

from which one can see the possible source of the odd results for β and δ. One extracts $B = B_c = 1$ where $T = T_c = 0$ corresponds to $x = 0$.

We then find the susceptibility for small x and h

$$\chi \approx \frac{x}{(h^2 + x)^{3/2}} \tag{5.53}$$

from which one obtains $C_2^+ = 1$. One can also see a divergence for $h_c^2 + x = 0$, or

$$h_c = \pm i\, x^{\frac{1}{2}}\,. \tag{5.54}$$

The above location of h_c gives us the amplitude $D_y = 1$. We see this divergence also for M. Near the YLE h_c, the magnetization M behaves as

$$M \approx \frac{1}{\sqrt{2}} e^{i\,\frac{\pi}{4}}\, x^{\frac{1}{4}}\, (h - h_c)^{-\frac{1}{2}}\ . \tag{5.55}$$

Thus, for the one dimensional YLE, $\sigma = -\frac{1}{2}$. Also, $E_y = \dfrac{1}{\sqrt{2}}$ and x has the appropriate power. Lastly, the phase behaves as we expect by the discussion after Eq. (4.97).

All together, we have the following results for the universality of the YLE within the one dimensional Ising model:

$$\begin{aligned} R_m &= \mathrm{NA}\,, & R_e &= \frac{1}{\sqrt{2}}\,, \\ R_h &= 1\,, & |z_c| &= 1\,. \end{aligned} \tag{5.56}$$

On a final note, the thermal scaling scaling has an ambiguity in its definition, see [2]. This ambiguity amounts to the exponents γ, ν and $2 - \alpha$ being defined up to an overall constant. However, this ambiguity does not effect the above results. See [43] as well.

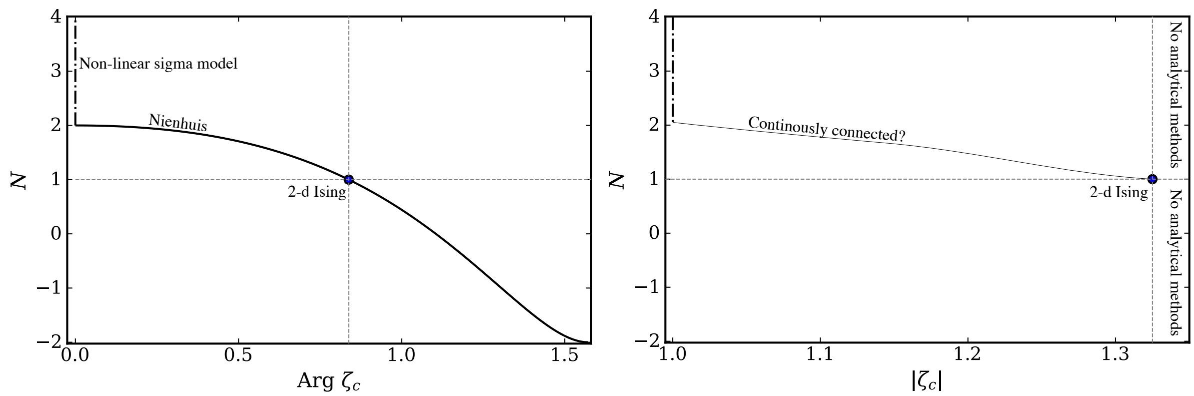


Figure 5.1: Argument and the absolute value of the Yang-Lee edge singularity at $d = 2^+$; the Ising model $N = 1$ is depicted by the dots. For $-2 < N < 2$, the argument of the YLE singularity is defined by the critical exponents of the underlying universality class, which are known exactly, see Ref. [121]. For $N > 2$, the non-linear sigma model predicts the argument of the singularity. There are no exact results for the location of the singularity in $d = 2$. For $N = 1$, Ref. [19] provides the location with high precision. For $N > 2$, we conjecture that $|\zeta_c| = 1$. We expect to have a continuous connection between the start of the dash-dotted line and $|\zeta_c|$ for $d = 2$.

5.5 Behaviour near/at $d = 2$

For the Ising universality class, $N = 1$, $d = 2$ is analytically treatable via Refs. [18, 19] for $d = 2$ where in Ref. [43] the two-dimensional results were also presented in the same normalization scheme as in this thesis. We quote the result for the YLE location of the 2D Ising model:

$$|z_c| =3.95509(12)\,, \tag{5.57}$$

$$|\zeta_c| =1.32504(4)\,. \tag{5.58}$$

The case of $d = 2$, $N \neq 1$ deserves a special attention.

- $N > 2$: The perturbative analysis of the non-linear sigma model concludes that the theory near its lower critical dimension $d = 2$ has a stable ultra-violet fixed point for $N > 2$ with $N = 2$ corresponding to Berezinskii-Kosterlitz-Thouless phase transition [26, 27, 28]. Complemented by the scaling relations this analysis also reveals the full set of critical exponents near two dimensions $d = 2 + \tilde{\varepsilon}$ [122, 123]

$$\eta = \frac{\tilde{\varepsilon}}{N-2} + \mathcal{O}(\tilde{\varepsilon}^0), \tag{5.59}$$

$$\beta = \frac{N-1}{2(N-2)} + \mathcal{O}(\tilde{\varepsilon}) \,. \tag{5.60}$$

Using the scaling reltion between η and δ Eq. (4.92), we can establish that to the leading order in ε

$$\Delta \approx \frac{2}{\tilde{\varepsilon}} \tag{5.61}$$

and thus Δ is N-independent (and coincides with the $N \to \infty$ result). This fixes the argument of the Yang-Lee edge singularity to $\operatorname{Arg} z_c = \pi\tilde{\varepsilon}/4 \to 0$ at $d \to 2^+$. By the analogy of the behaviour of the Ising model near its lower critical dimension and the result of $N \to \infty$ limit, see Eq. (5.38), we conjecture that for any $N \geq 2$, $|\zeta_c| = |z_c| = 1$. We checked by direct analytic calculations that the location of the singularity in the non-linear sigma model for the Heisenberg model ($N = 3$, see Ref. [124] for the equation of state) follows our conjecture.

- $-2 < N < 2$: For $-2 < N < 2$, the nonlinear sigma model does not have an ultraviolet fixed point (at least perturbatively). This, however, does not exclude the presence of the Ising-like fixed point. We expect that the behaviour of the fractional N smoothly interpolates between $N = 1$ and $N = 2^+$; for the location of the singularity, it means that $|z_c|$ changes from about 4 at $N = 1$ to 1 at $N = 2^-$. This plausible assumption is indirectly supported by the analytic results on the critical exponents, see Ref. [121], which behave smoothly as a function of N in the range from -2 to 2^-. (See also FRG studies at fractional N and $d = 2$ in Refs. [125, 126, 127]) Specifically, the argument of the location for the singularity

$$\operatorname{Arg} z_c = \frac{\pi}{2\beta\delta} = \frac{4\pi(2-v)v}{(v+1)(v+3)} \tag{5.62}$$

where $N = -2\cos\frac{2\pi}{v}$ and $1 < v < 2$, smoothly traverses through $N = 1$ (which corresponds to $v = 3/2$), where it accepts the two-dimensional Ising model's value $\operatorname{Arg}\ z_c = \frac{4\pi}{15}$. It then approaches 0 at $N = 2^-$ ($v = 2$), making a continuous connection to the result of the previous item, see Fig. 5.1.

- $N = -2n$ where $n \in \mathbb{Z}^+$: In the absence of the magnetic field, the theory with negative even integer number of components is Gaussian for arbitrary d. Direct calculations show that $\gamma = 1$ and $\eta = 0$ and independent of N [128, 129]. Using scaling relations, one thus finds that $\Delta = \frac{d+2}{4}$ and thus, in $d = 2$, $\operatorname{Arg} \zeta_c = \frac{\pi}{2}$.

CHAPTER

6

UNIVERSALITY OF THE YANG-LEE EDGE SINGULARITY FROM FRG

6.1 The Functional Renormalization Group

In this section, we give a basic review of the Function Renormalization Group. For reviews, see [130, 131, 132, 133]. The Function Renormalization Group (FRG) (Nonperturbative RG or also Exact RG) is a direct implementation of Wilson's renormalization group scheme applied to an scale dependent effective action, Γ_k, with the scale factor given by k. FRG accomplishes the integration of momentum shells by the inclusion of a scale-dependent mass-like regulator function R_k into the action. The effect is the suppression of the low-momentum, long-wavelength modes below the given scale k and so entails an implementation of Wilsonian RG whereby one integrates over momentum shells.

Given a microscopic Euclidean action S_Λ defined at at high energy (or ultraviolet UV) scale Λ which can be interpreted as the inverse lattice spacing for an associated lattice model, we take as an initial condition $\Gamma_{k=\Lambda} = S_\Lambda$. One can then chose R_k such that the differential $\partial_k \Gamma_k$ is supported on a finite momentum range around the scale k; i.e.

a momentum-shell. Following through, we will arrive at an equation on our microscopic theory under RG as a functional-integral differential equation:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{ \partial_t R_k \left(\Gamma_k^{(2)} + R_k \right)^{-1} \right\}. \tag{6.1}$$

The above is referred to as the Wetterich equation [9, 10, 11], or simply just the flow equation. It prescribes the behaviour of Γ_k between the classical tree-level action at an initial scale $k = \Lambda$ in the ultraviolet, $\Gamma_{k=\Lambda} = S$ – an initial condition, and the desired full quantum action at $k = 0$, $\Gamma_{k=0} = \Gamma$. The FRG equation provides a versatile realization of the Wilsonian RG and is as such well-suited to study critical physics. Both the scaling function and the critical exponents have been computed in great detail for $\mathrm{O}(N)$ theories for real external fields with the FRG, see, e.g., Refs. [134, 143, 149, 150, 151, 152, 153, 154, 155, 135, 136, 137, 138, 139, 126, 140, 125, 141, 142, 144, 145, 146, 147, 148]

While the flow equation is exact, it defines an infinite tower of coupled partial differential equations for the effective action and its functional derivatives. With few exceptions (like the $O(N)$ model in the large-N limit [156, 157]), truncations are therefore necessary in practice. There is often no obvious small parameter which can be used to define a systematic truncation scheme. Fortunately, this is not the case for critical physics, where the diverging correlation length facilitates a systematic expansion about vanishing momentum. Such a derivative expansion [158] has been shown to have a finite radius of convergence and a systematic error estimate is possible [159, 160, 161]. The leading order of the derivative expansion is referred to as the local potential approximation (LPA) which offers a cheap extension denoted LPA' whereby one additionally includes a running field independent wave function renormalization. Beyond this, one moves into the next-to-leading order of the derivative expansion including all possible momentum-squared order terms.

In our original FRG study of the location of the Yang-Lee edge singularity [45], the location of the YLE for $O(N \geq 1)$ universality classes was computed at LPA' using a finite temperature formalism. To do so, we developed an extrapolation method whereby one computes along lines in the complex-H plane parallel to the imaginary axis. Along these lines one can track the peak of the susceptibility as the real part of H is decreased thereby allowing for an extrapolation to purely imaginary H. This method sidesteps numerical difficulties encountered in the method of Taylor expanding about the running minimum for purely imaginary H.

An alternative method, denoted here the crossover flow, was developed in Refs. [43, 44] which allows one to avoid computations in the complex and instead follow purely real and/or purely imaginary fields. Within this approach, one is able to tune the system such that a crossover is seen between critical $O(N)$ physics and critical YLE physics. A further prospect of this approach is the elimination of error due to extrapolation allowing for more precision of the results. These computations were pushed to a truncated next-to-leading order scheme, denoted $\mathcal{O}\left(\partial^2\right)$, in the derivative expansion for the location of the YLE for $O(N \geq 1)$ universality classes. To further our error analysis, we accounted for the residual dependence on the regulator by performing a minimal sensitivity analysis [151] which was motivated by minimizing the sensitivity to nonphysical parameters in conventional perturbation theory with different renormalization schemes [162].

In the following sections, we give a discussion and derivation of the Wetterich equation Eq. (6.1). The following two sections cover our LPA' study via extrapolation (Section 6.2) and our $\mathcal{O}\left(\partial^2\right)$ study via the crossover flow.

6.1.1 The FRG Equation

Let us now derive the FRG equation [9, 10, 11]. For simplicity, let us start with a single component scalar field theory. The generalization to a N component field is straightforward. We begin by introducing to the action a mass-like term $\Delta S_k[\varphi]$ which depends on a parameter k:

$$\Delta S_k[\varphi] \;=\; \frac{1}{2}\int d^d x\, d^d y\, \varphi(\mathbf{x}) R_k(\mathbf{x},\mathbf{y})\, \varphi(\mathbf{y})\,. \tag{6.2}$$

The function R_k is typically referred to as the regulator and is constructed to suppress modes with momentum $p < k$. We choose R_k to be invariant under rotations and translations; hence,

$$\Delta S_k[\varphi] \;=\; \frac{1}{2}\int d^d x\, d^d y\, \varphi(\mathbf{x}) R_k(|\mathbf{x}-\mathbf{y}|)\, \varphi(\mathbf{y})\,. \tag{6.3}$$

In momentum space we have

$$\Delta S_k[\varphi] \;=\; \frac{1}{2}\int d^d q\, \varphi(-\mathbf{q}) R_k(\mathbf{q})\, \varphi(\mathbf{q}) \tag{6.4}$$

as we expect for a mass-like term. Furthermore, in order for R_k to suppress the low momentum modes, for $|\mathbf{q}| < k$, $R_k(\mathbf{q}) \sim k^2$. On the other hand, for modes with $p > k$ we want to have $R_k(\mathbf{q})$ such that it tends to zero so as to not affect the high momentum modes.

$$\lim_{\frac{|\mathbf{q}|}{k}\to 0} R_k(\mathbf{q}) \quad \sim \quad k^2\,, \tag{6.5}$$

$$\lim_{\frac{|\mathbf{q}|}{k}\to \infty} R_k(\mathbf{q}) \quad = \quad 0\,. \tag{6.6}$$

The second condition is consistent with the requirement that the original theory be reproduced in the limit $k \to 0$ where we should have have $R_{k=0}(\mathbf{q}) = 0$. From the above, we also expect that $\partial_k R_k(\mathbf{q})$ should be sharply peaked around $|\mathbf{q}| \approx k$. The encoding of the UV cutoff Λ into the regulator will be discussed in the next section.

By introducing the scale dependent regulator, the partition function itself depends on the scale k:

$$\mathcal{Z}_k[J] \quad = \quad \int \mathcal{D}\varphi e^{-S_\Lambda[\varphi]-\Delta S_k[\varphi]+J\cdot\varphi}\,. \tag{6.7}$$

We wish to understand the effect of integrating out more and more high momentum modes which in effect corresponds to lowering the scale k. The goal is to find the differential equation on Γ_k in Eq. (6.1). After some work, we find the derivative of the scale dependent action $W_k[J]$ as follows using $\partial_t = k\partial_k$:

$$\partial_t W_k[J] \quad = \quad -\frac{1}{\mathcal{Z}_k}\int \mathcal{D}\varphi\, \frac{1}{2}\varphi\cdot\partial_t R_k\cdot\varphi\, e^{-S_\Lambda-\Delta S_k+J\cdot\varphi}\,. \tag{6.8}$$

We can massage this into a more useful form by replacing fields with derivatives with respect to their conjugate variable J,

$$\partial_t W_k[J] \quad = \quad -\frac{1}{\mathcal{Z}_k}\frac{1}{2}\int_{x,y} \frac{\delta}{\delta J(\mathbf{x})}\,\partial_t R_k\,\frac{\delta}{\delta J(\mathbf{y})}\mathcal{Z}_k[J]\,, \tag{6.9}$$

$$\quad = \quad -\frac{1}{2}\int_{x,y} \partial_t R_k \left(\frac{\delta W_k}{\delta J(\mathbf{x})}\frac{\delta W_k}{\delta J(\mathbf{y})} + \frac{\delta^2 W_k}{\delta J(\mathbf{x})\delta J(\mathbf{y})}\right)\,. \tag{6.10}$$

Next, we also have that $\phi = \langle\varphi\rangle_J$ can also be taken as k dependent if we keep the external field J independent of k. For the Legendre transform in the presence of the regulator, it is useful to add an additional term in the transform:

$$\Gamma_k[\phi] \quad = \quad -W_k[J] + J\cdot\phi - \Delta S_k\,. \tag{6.11}$$

Note that the modifying term vanishes at $k = 0$ yielding the standard Legendre transform.

It is important to see that in the above expression for the scale dependent effective action Γ_k, whereas we originally kept J independent of k, it is natural to shift the k dependence into J through the relation $\phi = \langle\varphi\rangle_J$. For the differential of Γ_k, we then keep ϕ fixed.

$$\begin{aligned}\partial_t\Gamma_k[\phi] &= -\partial_t W_k[J] - \frac{\delta W_k}{\delta J}\cdot\partial_t J - \partial_t\Delta S_k[\phi] + \partial_t J\cdot\phi\,, &(6.12)\\ &= -\partial_t W_k[J] - \frac{1}{2}\phi\cdot\partial_t R_k\cdot\phi\,. &(6.13)\end{aligned}$$

The second and fourth terms in the first line nicely cancel due to the definition of ϕ. Inserting now the expression for $\partial_t W_k[J]$

$$\begin{aligned}\partial_t\Gamma_k[\phi] &= \frac{1}{2}\int_{x,y}\partial_t R_k\left(\frac{\delta W_k}{\delta J(\mathbf{x})}\frac{\delta W_k}{\delta J(\mathbf{y})} + \frac{\delta^2 W_k}{\delta J(\mathbf{x})\delta J(\mathbf{y})}\right) - \frac{1}{2}\int_{x,y}\partial_t R_k\,\frac{\delta W_k}{\delta J(\mathbf{x})}\frac{\delta W_k}{\delta J(\mathbf{y})}\,,\\ &= \frac{1}{2}\int_{x,y}\partial_t R_k\,\frac{\delta^2 W_k}{\delta J(\mathbf{x})\delta J(\mathbf{y})}\,. &(6.14)\end{aligned}$$

The last step is to express the above right hand side in terms of Γ_k.

$$\begin{aligned}\Gamma_k[\phi] &= -W_k[J] + J\cdot\phi - \Delta S_k\,, &(6.15)\\ \frac{\delta\Gamma_k}{\delta\phi(\mathbf{x})} &= J - \phi R_k\,, &(6.16)\\ \frac{\delta^2\Gamma_k}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{y})} &= \frac{\delta J}{\delta\phi} - R_k &(6.17)\end{aligned}$$

which upon rearranging yields the familiar relation

$$\Gamma_k^{(2)} + R_k = \left(\frac{\delta^2 W_k}{\delta J^2}\right)^{-1} \tag{6.18}$$

where we have used the notation $\Gamma_k^{(2)} = \frac{\delta^2\Gamma_k}{\delta\phi^2}$ Normally, the two point vertex is directly related to the two point function, but the presence of the regulator adds an additional mass to the propagator. Replacing W_k with the relation in terms of Γ_k yields [9, 10, 11]

$$\begin{aligned}\partial_t\Gamma_k[\phi] &= \frac{1}{2}\int_{x,y}\partial_t R_k\left(\Gamma_k^{(2)} + R_k\right)^{-1}\,, &(6.19)\\ &= \frac{1}{2}\,\mathrm{Tr}\left\{\partial_t R_k\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\}\,. &(6.20)\end{aligned}$$

The trace in position space is understood as contracting two ends of a propagator with an insertion of $\partial_t R_k$ - a one loop type expression. Extending to N component fields amounts to extending the trace to include all internal indices, and spinor indices if fermions are also included.

6.1.2 Recovering the Saddle Point

We reproduce here a derivation from [131]. Further constraints are placed on the regulator R_k in order to yield $\Gamma_{k=\Lambda} = S_\Lambda$, though these constraints are relaxed in practice. Given the defining relation for the scale dependent effective action,

$$\Gamma_k[\phi] \quad = \quad -W_k[J] + J \cdot \phi - \Delta S_k \tag{6.21}$$

we obtain the relation

$$\frac{\delta \Gamma_k}{\delta \phi} \quad = \quad J - \frac{\delta \, \Delta S_k}{\delta \phi} \,, \tag{6.22}$$

$$\quad = \quad J - R_k \cdot \phi \,. \tag{6.23}$$

With these two relations, we can now derive an expression relating the classical (Wilsonian) action and effective action

$$e^{-\Gamma_k[\phi]} \quad = \quad \int \mathcal{D}\varphi \, e^{-S_\Lambda + \frac{\delta \Gamma}{\delta \phi} \cdot (\varphi - \phi) - \frac{1}{2}(\varphi - \phi) \cdot R_k \cdot (\varphi - \phi)} \,. \tag{6.24}$$

$$\tag{6.25}$$

Now for $k = \Lambda$, we desire that $\Gamma_{k=\Lambda} = S_\Lambda$. From the above we see that this relation only holds for a divergent regulator which freezes all modes below the scale Λ, or the limit $R_{k \to \Lambda} \to \infty$. This amounts to the effective replacement in the above expression:

$$e^{-\Gamma_\Lambda[\phi]} \quad \approx \quad \int \mathcal{D}\varphi \, e^{-S_\Lambda + \frac{\delta \Gamma}{\delta \phi} \cdot (\varphi - \phi)} \, \delta(\varphi - \phi) \,, \tag{6.26}$$

$$\quad = \quad e^{-S_\Lambda} \,. \tag{6.27}$$

When studying critical phenomenon, relaxing the requirement that R_k diverge at $k = \Lambda$ offers no issues thanks to the occurrence of universality - independence from the features of a particular microscopic theory within the larger universality class. See [132] for a discussion.

6.2 The universal YLE location at LPA' via extrapolation

In this section we discuss the details of our method and results for our FRG determination of the universal location of the YLE at LPA' in the derivative expansion [45]. An overview of this study was given in the intro to this chapter.

6.2.1 FRG at LPA' wtih finite temperature

In this study, we use the first order derivative expansion of the scale-dependent effective action in Euclidean space,

$$\Gamma_k = \int d^{d+1}x \left\{ \frac{1}{2} Z_k(\rho)(\partial_\mu \phi)^2 + U_k(\rho) - h\sigma \right\}, \tag{6.28}$$

where d is the number of spatial dimensions, the field ρ is defined as $\rho = \phi^2/2 = (\sigma^2+\pi^2)/2$, $U_k(\rho)$ is the scale dependent potential, and h is the external field explicitly breaking the O(N) symmetry. σ denotes the radial excitation, and π the Goldstone modes. The extension to finite temperature is done by using the standard Matsubara formalism $\int \frac{dp_0}{2\pi} \to T \sum_n$ and $p_0 \to 2n\pi T$. Accordingly, we use the optimized regulator function R_k for the spatial momentum modes, $R_k(\vec{p}) = Z_k(k^2 - \vec{p}^{\,2})\Theta(k^2 - \vec{p}^{\,2})$, put forward in [163, 164] and refer to [165, 151, 166, 167, 168] for detailed discussions regarding the regularization scheme. Here we comment that the commonly applied local potential approximation (LPA) which neglects the scale-dependence of the wavefunction renormalization, $Z_{\phi,k}(\rho) = 1$, giving zero anomalous dimension η, is not appropriate to study the location of the edge singularity since η is expected to be of order one in its vicinity. This necessitates the inclusion of scale-dependent wavefunction renormalization, which we assume to be field-independent, $Z_k(\rho) = Z_k$. This approximation is referred to as LPA$'$. The anomalous dimension is related to the wavefunction renormalization via $\partial_t Z_k = -\eta_k Z_k$, where we introduced the RG time $t = \ln(k/\Lambda)$.

To describe critical phenomena, it is helpful to look at the renormalization flow of dimensionless observables rather than the dimensionful counterparts as their magnitudes can vary wildly. The flow for the dimensionless effective potential, $\bar{U}_k(\bar{\rho}) = k^{-d-1} U_k(\bar{\rho})$ where $\bar{\rho} = Z_k k^{1-d} \rho$ is the dimensionless renormalized field, can be obtained by evaluating

(6.1) on uniform field configurations at fixed $\bar{\rho}$ which yields [169, 170]

$$\begin{aligned}\partial_t \bar{U}_k(\bar{\rho}) &= -(d+1)\bar{U}_k + (d-1+\eta_k)\bar{\rho}\bar{U}_k' \\ &+ \frac{v_d}{2(2\pi)^d}\left(1-\frac{1}{d+2}\eta_k\right)\left[\frac{N-1}{\sqrt{1+\bar{m}_\pi^2}}\left(1+2n_\pi\right)+\frac{1}{\sqrt{1+\bar{m}_\sigma^2}}\left(1+2n_\sigma\right)\right],\end{aligned} \tag{6.29}$$

where $v_d = S_{d-1}/d$ with $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$, $n_{\sigma,\pi} = n_{\sigma,\pi}(T,k)$ are the standard Bose-Einstein distribution functions, and the primes denote derivatives with respect to $\bar{\rho}$. The dimensionless renormalized masses, $\bar{m}_{\sigma,\pi}$, are given by $\bar{m}_\pi^2 = \bar{U}'$ and $\bar{m}_\sigma^2 = \bar{U}' + 2\bar{\rho}\bar{U}''$. See Appendix B for the derivation of this expression.

To extract the anomalous dimension, one uses the projection

$$\eta_{\phi,k} = -\frac{1}{2Z_k}\lim_{p\to 0}\frac{\partial^2}{\partial p^2}\left[\frac{\delta^2\left(\partial_t\Gamma_k\right)}{\delta\pi_i(p)\delta\pi_i(-p)}\right], \tag{6.30}$$

where the choice of i is arbitrary. The result is

$$\begin{aligned}\eta_{\phi,k} = &\frac{4v_d}{(2\pi)^d}\bar{\rho}_0\bar{U}''(\bar{\rho}_0)^2 \\ &\times\frac{T}{k}\sum_{n=-\infty}^{\infty}\left(\frac{p_0^2}{k^2}+\bar{m}_\pi^2+1\right)^{-2}\left(\frac{p_0^2}{k^2}+\bar{m}_\sigma^2+1\right)^{-2}.\end{aligned} \tag{6.31}$$

The above sum, see Appendix B for a derivation, is evaluated to a closed form in practice, though its form its not insightful [171]. It is also important to note the occurrences of N and the dimension d in Eqs. (6.29) and (6.31). Clearly, one can easily change N with no other considerations as long as $N \geq 1$. As for the dimension d, we can safely analytically continue d to any non-integer value.

In order to numerically solve (6.29) we consider a sixth order Taylor expansion of the dimensionless effective potential about the scale dependent minimum $\bar{\rho}_{0,k}$: $\bar{U}_k = \sum_{i=0}^{6}\frac{a_{i,k}}{i!}(\bar{\rho}-\bar{\rho}_{0,k})^i$. We explicitly checked that the expansion is converged at this order. In terms of the Taylor expansion coefficients, one obtains a coupled set of ordinary differential equations:

$$\partial_t a_{n,k} = \left(-d+n(d-2\eta_k)\right)a_{n,k} + \frac{\partial^n}{\partial\bar{\rho}^n}\left(l_0(\bar{m}_\sigma^2)+(N-1)l_0(\bar{m}_\pi^2)\right) + a_{n+1,k}\partial_t\bar{\rho}_{0,k}\,, \tag{6.32}$$

$$\partial_t\bar{\rho}_{0,k} = -\left(d-2+\eta_k\right) - \frac{1}{a_{2,k}}\frac{\partial}{\partial\bar{\rho}}\left(l_0(\bar{m}_\sigma^2)+(N-1)l_0(\bar{m}_\pi^2)\right) \tag{6.33}$$

where we defined

$$l_0(\omega) = \frac{v_d}{2(2\pi)^d} \left(1 - \frac{1}{d+2}\eta_k\right) \frac{N-1}{\sqrt{1+\omega}} \Big(1 + 2\, n(\omega)\Big) \tag{6.34}$$

with $n(\omega) = \left(e^{\beta\omega} - 1\right)^{-1}$.

6.2.2 The extrpolation scheme

To extract the scaling function $f_G(z)$ for a given N and d, we first compute the universal critical exponents β and δ along with the non-universal critical amplitudes T_0 and H_0. In the following, we equate the order parameter M with the radial expectation value $M = \sigma$. The amplitudes are defined via the replacements $t \to t/T_0$ and $h \to h/H_0$ so as to reproduce the scaling forms $M = (-t)^\beta$ and $M = h^{1/\delta}$ at $h = 0$ and $t = 0$, respectively.

In the ultraviolet (UV), which thanks to universality we can simply define as $k = 1$, we start with the initial condition

$$\bar{U}_{k=1}(\bar{\rho}) = \frac{1}{2}\bar{r}\bar{\rho} + \frac{1}{4}\bar{\rho}^2 \tag{6.35}$$

where we fix $\bar{r} < 0$ sufficiently negative so as to have a finite magnetization at an arbitrary reference temperature. The condition of $\bar{\rho}_{0,k}$ being an extrema, $\left(\bar{U}'\left(\bar{\phi}\right) = \bar{H}\right)$, yields the following relation for a Taylor expansion

$$\sqrt{2\bar{\rho}_{0,k}}\, a_{1,k} = \bar{H}_k\,. \tag{6.36}$$

The flow of $\bar{H}_k$ is trivial since $H = \text{const}$. At vanishing $\bar{H}$, this implies either $\bar{\rho}_{0,k} = 0$ or $a_{1,k} = 0$. Clearly, for $T < T_c$ we must have $\bar{\rho}_{0,k=0} \neq 0$. However, one expects that T_c corresponds to an ultraviolet (UV) theory with broken symmetry, $\bar{r} < 0$. This is due to the symmetry restoring property of ordinary bosonic fluctuations. Starting with $\bar{\rho}_{0,k} \neq 0$, one can have two possibilities for $\bar{\rho}_{0,k=0}$. For $T < T_c$, the symmetry remains broken in the IR implying $\bar{\rho}_{0,k=0} \neq 0$. However, for $T > T_c$ the symmetry is restored in the IR implying that at some point along the flow $\bar{\rho}_{0,k} = 0$. Thus, we build in a algorithm which switches between the parametrization for $a_{1,k} = 0$ once/if $\bar{\rho}_{0,k} = 0$. Afterwards, the flows stays within the parametrization for $\bar{\rho}_{0,k} = 0$ and $a_{1,k} \neq 0$. Given this notion, one can find the critical temperature T_c using a bisection algorithm on $\bar{\rho}_{0,k=0}$. Once T_c is found, the computation of β and T_0 is straightforward via a linear fit of an appropriate log-log plot.

To find δ and H_0, one needs to turn on an external field. From Eq. (6.36) this entails one can eliminate reference to the variable $a_{1,k}$ in favor of $\bar{\rho}_{0,k}$ and $\bar{H}_k$. For the initial condition of $\bar{\rho}_{0,k=1}$, one simply picks the extrema which is continuously connected to the $\bar{H} = 0$ solution. Following this into the IR allows one to set $T = T_c$ and vary only H. Thus, again one obtains δ and H_0 via a linear fit of an appropriate log-log plot.

With the rescaled variables, we can ensure that the magnetic equation of state satisfies the following normalization conditions $f_G(z = 0) = 1$ and $f_G(z \to -\infty) = (-z)^\beta$. In terms of the FRG variables,

$$f_G(z = th^{-1/\Delta}) = h^{-1/\delta} \lim_{k\to 0} \sqrt{2\bar{\rho}_{0,k} k^{d-1}/Z_k}\,. \tag{6.37}$$

We checked that our numerical calculations reproduce the O(4) magnetic equation of state of Ref. [153, 172] for real values of the argument.

In contrast to lattice Monte-Carlo calculations, FRG does not have a sign problem; this enables us to evaluate $f_G(z)$ for any complex z. In practice, we fix a real value of t and numerically find $f_G(z)$ at a complex value $h = h(z)$.

Additional care needs to be taken when approaching the edge singularity using FRG with the Taylor expansion method. This is due to the locality of the expansion and its failure to capture first-order phase transitions. Furthermore, the edge occurs at purely imaginary values of the external field, another case for which the FRG equations become numerically challenging. To circumvent this issue, we note that the edge can be treated as an ordinary second-order phase transition, and we can determine its location by studying the peak of the magnetic susceptibility, $\chi_\sigma \propto 1/m_\sigma^2$, at a complex external field with just a small real part. Given that near the edge the order parameter σ behaves as $M - M_c \sim (h^2 - h_c^2)^{\sigma_{\rm YL}}$ [17], the susceptibility for the sigma field behaves as $\chi_\sigma \sim 2h\,\sigma_{\rm YL}\,(h^2 - h_c^2)^{\sigma_{\rm YL}-1}$. From this form, we see that for any sufficiently small $\mathrm{Re}(h)$, $|\chi_\sigma|$ will have a peak at some $\mathrm{Im}(h) = h_{\rm peak}$. Moreover, $h_{\rm peak}$ is a quadratic function of $\mathrm{Re}(h)$ for sufficiently small values of $\mathrm{Re}(h)$. Thus, we can scan the complex h-plane at fixed reduced temperature t along lines in the first quadrant parallel to the imaginary axis. Then h_c can be determined by incrementally decreasing $\mathrm{Re}(h)$, determining $h_{\rm peak}$ for each increment, performing a quadratic fit to the data, and extrapolating to purely imaginary external fields. From this, we find the location of the edge as $z_c = th_c^{-1/\Delta}$.

There are several sources of error in this work, presumably the largest being the systematic error which results from working in LPA$'$. One manifestation of the systematic error can be the violation of the hyperscaling relation between the exponents: $2 - \eta = d(\delta - 1)/(\delta + 1)$. For our work, we find this violation to be at the sub-permille level.

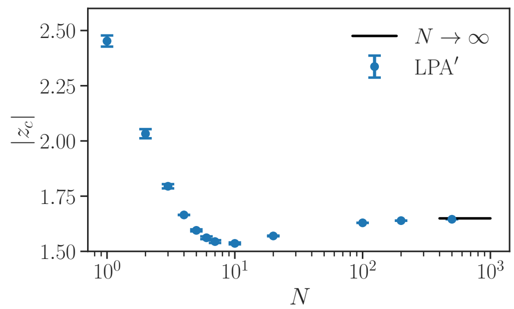


Figure 6.1: The magnitude of the location of the Yang-Lee edge singularity $|z_c|$ as a function of N in three dimensions. The large N result is indicated by the solid line. There is a non-monotonous behaviour in the approach to large N limit.

Regarding the critical exponents themselves, e.g. for $d = 3$ and $N = 1(2)$, the disagreement with well-known results from Ref. [173] for Δ is about 0.9% (1.9%). We note that a potentially large anomalous dimension close to the edge singularity indicates nontrivial momentum dependencies in the system. These can be captured by higher orders in the derivative expansion. A systematic improvement of our truncation in this direction, which will significantly reduce the systematic error, is deferred to future work.

Other error sources are the implementation of a finite infrared scale k_0 at which the RG flows are terminated and the fits to obtain the exponents and the value of h_c. One can choose a k_0 such that the continued running of the flows below this value contributes negligibly to the error. For this work, we chose a common k_0 for all N and d such that the error is negligibly small. The error from the fits for the exponents and extrapolation to purely imaginary values of h are much smaller than the truncation error and we do not report them here. The error bars seen in Figs. 6.1 and 6.2 are then given by the truncation error, which we measure as the difference between the result at 5^{th} and 6^{th} order Taylor expansions of the effective potential.

6.2.3 Results and discussions

Our results for the location of the Yang-Lee edge singularity, z_c, are shown in Figure 6.1 for $d = 3$ and numerous N, and in Figure 6.2 for $N = 1, 2, 4, 100$ and various d. Analytically, z_c is only known in two limiting cases: $N \to \infty$ and the mean-field approximation. For both, the magnetic equation of state can be written in the following form (see e.g. [174,

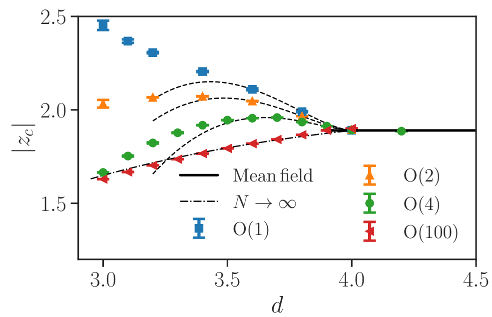


Figure 6.2: The magnitude of the location of the Yang-Lee edge singularity $|z_c|$ for $N = 1, 2, 4, 100$ as a function of the number of dimensions d. The solid line demonstrates the mean-field value. The dashed lines display a fit motivated by the parametrization of the equation of state for O(1) $|z_c| = |z_c^{\mathrm{MF}}| \{1 + (4-d)^2[p_1 \ln(4-d) + p_2]\}$, see Ref. [13]. Note that the parameter p_2 is non-perturbative and cannot be found using the ϵ-expansion. The dash-dotted line shows the analytical $N \to \infty$ result of Eq. (6.39) with $\gamma = 2/(d-2)$.

20, 175])

$$f_G(z)\left[z + f_G^2(z)\right]^\gamma = 1, \tag{6.38}$$

where $\gamma = 1$ ($\gamma = \frac{2}{d-2}$) for mean-field (large N in $d < 4$ dimensions). Due to its nature as a branch point, the location of the Yang-Lee edge singularity can be found from the condition that the derivative of the inverse function is zero, $z'(f_G) = 0$. This leads to

$$|z_c| = (2\gamma + 1)\,(2\gamma)^{-\frac{2\gamma}{2\gamma+1}}\,, \quad \mathrm{Arg}\, z_c = \pm\pi(2\gamma+1)^{-1}\,. \tag{6.39}$$

Thus in mean-field approximation (in the large N limit, $d = 3$) one gets $|z_c^{\mathrm{MF}}| \approx 1.8899$ ($|z_c^{\infty}| \approx 1.6494$).

The fact that our numerical calculations reproduce the analytical results for $|z_c|$ is highly non-trivial in that it provides a direct test of our FRG approach and the method of extracting the location of the singularity. Figure 6.1 demonstrates the converge of $|z_c|$ to the large N value. We note that $|z_c|$ displays some non-monotonous behavior as it dips below its large N value before approaching the limit from below. For the convergence of $|z_c|$ to its mean field result, see Figure 6.2. We also see here non-monotonous behavior as the dimension is varied. Our numerical calculation for $N = 100$ faithfully reproduces the infinite N result in arbitrary number of dimensions $3 < d < 4$. In three dimensions, the values of $|z_c|$ are 2.452 ± 0.025, 2.032 ± 0.021 and 1.665 ± 10^{-3} for N =1, 2, 4 correspondingly.

To give an explicit example for importance of the edge singularity, we determine the asymptotic form of the coefficients of the Taylor expansion of $f_G(z)$. This can be done solely based on its analytic structure in the complex plane. For the sake of simplicity, we consider the expansion near the origin. In this case, the theorem of Darboux states (see e.g. Ref. [176]) that the n-th expansion coefficient for sufficiently large n is

$$f_G^{(n)} \sim 2B_0 |z_c|^{-n} \frac{n^{\sigma_{\mathrm{YL}}-1}}{\Gamma(\sigma_{\mathrm{YL}})} \cos\left(\beta_0 - \frac{\pi n}{2\Delta}\right), \tag{6.40}$$

where B_0 and β_0 are defined as the value of the analytic part of the magnetization at the singularity $B_0 \exp(i\beta_0) = \lim_{z\to z_c} (1-z/z_c)^{-\sigma_{\mathrm{YL}}}(f_G(z) - f_G(z_c))$. This explicitly demonstrates that the expansion coefficients are uniquely defined by $|z_c|$, Δ and σ_{YL}.

Another immediate application of our results is to lattice QCD calculations [177]. As we alluded to before, due to the sign problem, lattice calculations can not be effectively performed at non-zero baryon chemical potential/density $\mu \neq 0$. Nevertheless, extrapolations to finite μ based on a Taylor series expansion of, e.g., the pressure about $\mu = 0$ are used. In the crossover region, this series expansion has a finite radius of convergence which is defined by z_c and a non-universal map of T, μ, and m to the scaling variable z,

$$z(T,\mu) = z_0 \left(\frac{m_{u,d}}{m_s}\right)^{-\frac{1}{\Delta}} \left[\frac{T - T_c^0}{T_c^0} + \kappa_2^B \left(\frac{\mu}{T_c^0}\right)^2 + \dots\right].$$

The non-universal parameters (T_c^0, κ_2^B, and z_0) for this map were computed in lattice QCD see Refs. [178, 179, 180, 177]. This, complemented by our result, shows that the radius of convergence $R = |\mu_c|$ defined by $z(T,\mu_c) = z_c$ is ≥ 420 MeV for the O(4) universality class and ≥ 460 MeV for the O(2) universality class (which is of relevance for staggered fermions). These numbers are consistent with the ratio test applied to the Taylor series expansion in Refs. [178]. More importantly, they are well below the latest results for the location of a critical endpoint in the QCD phase diagram [181, 182, 183]. This highlights that the edge singularity, not the critical endpoint, is the crucial quantity for explorations of the QCD phase structure by lattice QCD.

6.3 The universal YLE location at $\mathcal{O}\left(\partial^2\right)$ via crossover flow

In this section we discuss the details of our method and results for our FRG determination of the universal location of the YLE at $\mathcal{O}\left(\partial^2\right)$ in the derivative expansion from Ref. [44]. An overview of this study was given in the intro to this chapter.

6.3.1 First order derivative expansion

In order to solve the flow equation (6.1) numerically we will consider a constant field configuration around small momentum. The latter is only required to extract equations for the wave function renormalization. Specifically, we use the following anzatz for the scale-dependent effective action

$$\Gamma_k = \int d^d x \left[\frac{1}{2} Z_k(\rho)\, \partial^\mu \phi_a \partial_\mu \phi_a + \frac{1}{4} Y_k(\rho)\, \partial^\mu \rho \partial_\mu \rho + U_k(\rho)\right], \tag{6.41}$$

where $\rho = \frac{1}{2}\phi_a \phi^a$. The above expression contains all possible terms up to $\partial_\mu \partial^\mu$ in the derivative expansion. This approximation is appropriate for describing long-wavelength excitations in the critical region.

Consider small deviations around the homogeneous field background, chosen to be non zero for the $i = 1$ field component

$$\phi_i(x) = \phi \delta_{i,1} + \varphi_i(x) \tag{6.42}$$

then the wave-function renormalization for transverse and radial modes are

$$Z_k^{\perp}(\rho) = Z_k(\rho) = \lim_{p \to 0} \frac{\partial}{\partial p^2} \left. \frac{\delta \Gamma_k}{\delta(\varphi_2(p)\varphi_2(-p))} \right|_{\varphi_a = 0}, \tag{6.43}$$

$$Z_k^{\parallel}(\rho) = Z_k(\rho) + \rho Y_k(\rho) = \lim_{p \to 0} \frac{\partial}{\partial p^2} \left. \frac{\delta \Gamma_k}{\delta(\varphi_1(p)\varphi_1(-p))} \right|_{\varphi_a = 0}. \tag{6.44}$$

6.3.2 Flow equations for the next-to-leading order in the derivative expansion

The flow equations for the potential can be obtained by substituting the truncation Eq. (6.41) to Eq. (6.1) evaluated for a constant field configuration

$$\partial_t U_k(\rho) = \frac{1}{2} \int \bar{d}^d q \, \partial_t R_k \left(q^2\right) \left[G_k^{\parallel} + (N-1) G_k^{\perp}\right], \tag{6.45}$$

where

$$G_k^{\perp} = \frac{1}{Z_k^{\perp}(\phi) q^2 + U_k'(\phi)/\phi + R_k(q^2)} = \frac{1}{Z_k^{\perp}(\rho) q^2 + U_k'(\rho) + R_k(q^2)}, \tag{6.46}$$

$$G_k^{\parallel} = \frac{1}{Z_k^{\parallel}(\phi) q^2 + U_k''(\phi) + R_k(q^2)} = \frac{1}{Z_k^{\parallel}(\rho) q^2 + U_k'(\rho) + 2\rho U_k''(\rho) + R_k(q^2)}. \tag{6.47}$$

For the integral measure, here and below we use

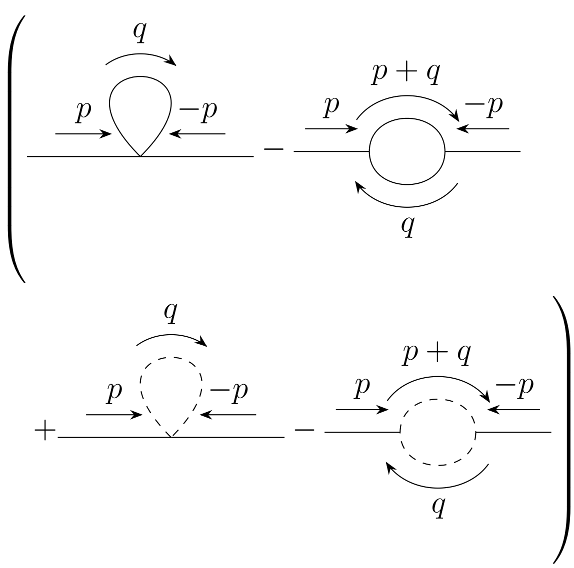
$$\int \bar{d}^d q \equiv \int \frac{d^d q}{(2\pi)^d}\,. \tag{6.48}$$

Introducing the tilde differential operator (see Ref. [130])

$$\tilde{\partial}_t = \int d^d l \, \partial_t R_k(l^2) \frac{\delta}{\delta R_k(l^2)} \tag{6.49}$$

allows us to write the equations in a succinct diagrammatic manner,

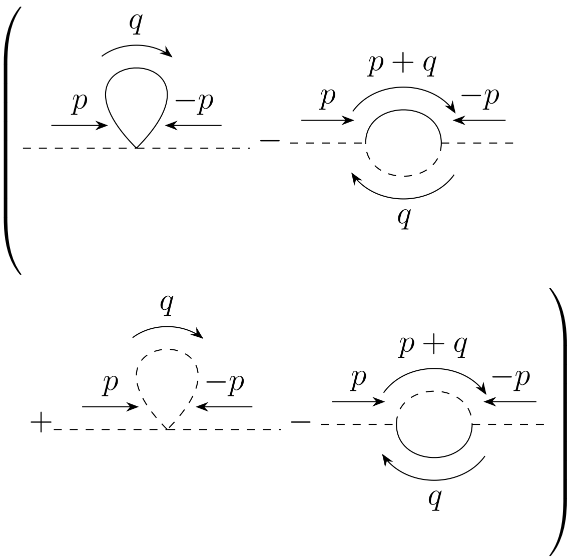
$$\partial_t \Gamma_k^{(2),\parallel}(p) = \frac{1}{2} \int \bar{d}^d q \, \tilde{\partial}_t$$



$$, \tag{6.50}$$

and

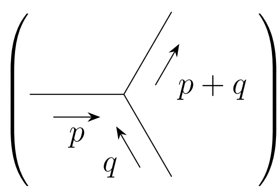
$$\partial_t \Gamma_k^{(2),\perp}(p) = \frac{1}{2}\int \bar{d}^d q\, \tilde{\partial}_t$$



(6.51)

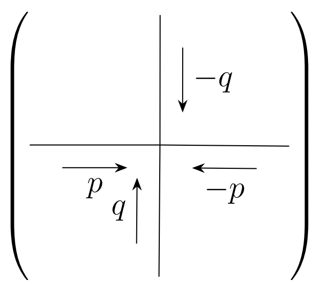
In the diagrams, the internal lines represent the scale-dependent Green functions for the transverse (solid line) and radial (dashed line) modes. The vertices describing interaction relevant for the above flow equations are

$$\Gamma_{111}^{(3)}(p,q) =$$



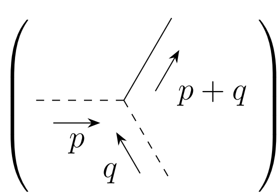
$$= (p^2 + q^2 + p\cdot q) Z_k^{\parallel\,\prime}(\phi) + U_k^{(3)}(\phi), \tag{6.52}$$

$$\Gamma_{1111}^{(4)}(p,-p,q) =$$



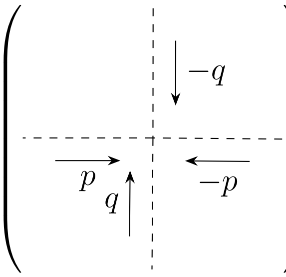
$$= (p^2 + q^2) Z_k^{\parallel\,\prime\prime}(\phi) + U_k^{(4)}(\phi), \tag{6.53}$$

$$\Gamma_{1ij}^{(3)}(p,q) =$$



$$= \left((p+q)^2 \frac{Z_k^{\parallel}(\phi) - Z_k^{\perp}(\phi)}{\phi} - p \cdot q Z_k^{\perp\prime}(\phi) + \left(\frac{1}{\phi} U_k'(\phi)\right)' \right) \delta_{ij}, \tag{6.54}$$

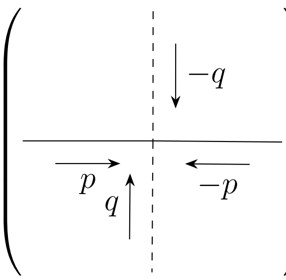
$$\Gamma_{ij}^{(4)}(p,-p,q) = \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right)$$

$$= (p^2+q^2) \left[2\delta_{ij} \frac{Z_k^{\parallel} - Z_k^{\perp}}{\phi^2} + \frac{1}{\phi} Z_k^{\perp\prime}(\phi) \right] + \frac{1}{\phi} \frac{\partial}{\partial \phi} \left[\frac{1}{\phi} U_k'(\phi) \right], \tag{6.55}$$

$$\Gamma_{\parallel ij}^{(4)}(p,-p,q) = \left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right)$$

$$= \left(q^2 Z_k^{\perp\prime\prime}(\phi) + p^2 \left[\frac{1}{\phi} Z_k^{\parallel\prime}(\phi) - 2 \frac{Z_k^{\parallel} - Z_k^{\perp}}{\phi^2} \right] + \left(\frac{1}{\phi} U_k'(\phi)\right)'' \right) \delta_{ij}\,. \tag{6.56}$$

Using these vertices and applying the tilde derivative on the resulting expressions one can easily derive the flow equations for $\Gamma_k^{(2),\parallel}(p)$ and $\Gamma_k^{(2),\perp}(p)$:

$$\begin{aligned} \partial_t \Gamma_k^{(2),\parallel}(p) = \int \bar{d}^d q \, \partial_t R_k(q^2) \times \Bigg[& \\ & [G^{\parallel}(q)]^2 \left(-\frac{1}{2} \Gamma_{\parallel\parallel}^{(4)}(p,-p,q) + [\Gamma_{\parallel\parallel}^{(3)}(p,q)]^2 G^{\parallel}(p+q) \right) + \\ & (N-1)[G^{\perp}(q)]^2 \left(-\frac{1}{2} \Gamma_{\parallel\perp}^{(4)}(p,-p,q) + [\Gamma_{\parallel\perp}^{(3)}(-(p+q),q)]^2 G^{\perp}(p+q) \right) \Bigg], \end{aligned} \tag{6.57}$$

and

$$\partial_t \Gamma_k^{(2),\perp}(p) = \int \bar{d}^d q\, \partial_t R_k(q^2) \times \Bigg[\\ [G^{\parallel}(q)]^2 \left(-\frac{1}{2}\Gamma^{(4)}_{\parallel\perp}(p,-p,q) + [\Gamma^{(3)}_{\parallel\perp}(p,-(p+q))]^2 G^{\perp}(p+q) \right) + \\ [G^{\perp}(q)]^2 \left(-\frac{1}{2}(N-1)\Gamma^{(4)}_{\perp\perp}(p,-p,q) + [\Gamma^{(3)}_{\parallel\perp}(-(p+q),q)]^2 G^{\parallel}(p+q) \right) \Bigg]. \tag{6.58}$$

Finally by taking the derivative of $\partial_t \Gamma_k^{(2),\parallel}(p)$ and $\partial_t \Gamma_k^{(2),\perp}(p)$ with respect to p^2 and evaluating at zero p^2 we arrive to the flow equations for the wave-function renormalizations:

$$\partial_t Z_{\parallel}(\phi) = \int \bar{d}^d q \partial_t R_k(q^2) \Bigg\{ G_{\parallel}^2 \Big[\gamma_{\parallel}^2 \big(G_{\parallel}' + 2G_{\parallel}''\frac{q^2}{d}\big) + 2\gamma_{\parallel} Z_{\parallel}'(\phi)\big(G_{\parallel} + 2G_{\parallel}'\frac{q^2}{d}\big) \\ + (Z_{\parallel}'(\phi))^2 G_{\parallel}\frac{q^2}{d} - \frac{1}{2}Z_{\parallel}''(\phi)\Big] \\ + (N-1)G_{\perp}^2 \Big[\gamma_{\perp}^2 \big(G_{\perp}' + 2G_{\perp}''\frac{q^2}{d}\big) + 4\gamma_{\perp} Z_{\perp}'(\phi) G_{\perp}'\frac{q^2}{d} + (Z_{\perp}'(\phi))^2 G_{\perp}\frac{q^2}{d} \\ + 2\frac{Z_{\parallel}(\phi) - Z_{\perp}(\phi)}{\phi}\gamma_{\perp} G_{\perp} - \frac{1}{2}\left(\frac{1}{\phi}Z_{\parallel}'(\phi) - \frac{2}{\phi^2}(Z_{\parallel} - Z_{\perp})\right)\Big]\Bigg\} \tag{6.59}$$

$$\partial_t Z_k^{\perp}(\phi) = \int \bar{d}^d q \partial_t R_k(q^2) \Bigg\{ G_{\parallel}^2 \Big[\bar{\gamma}_{\perp}^2 \big(G_{\perp}' + 2G_{\perp}''\frac{q^2}{d}\big) + 2\bar{\gamma}_{\perp} Z_{\perp}'(\phi)\big(G_{\perp} + 2G_{\perp}'\frac{q^2}{d}\big) \\ + (Z_{\perp}'(\phi))^2 G_{\perp}\frac{q^2}{d} - \frac{1}{2}Z_{\perp}''(\phi)\Big] \\ + G_{\perp}^2 \Big[\bar{\gamma}_{\perp}^2 \big(G_{\parallel}' + 2G_{\parallel}''\frac{q^2}{d}\big) + 4\bar{\gamma}_{\perp}\left(\frac{Z_{\parallel} - Z_{\perp}}{\phi} - Z_{\perp}'(\phi)\right) G_{\parallel}'\frac{q^2}{d} \\ + \left(2\frac{Z_{\parallel} - Z_{\perp}}{\phi} - Z_{\perp}'(\phi)\right)^2 G_{\parallel}\frac{q^2}{d} \\ + 2\frac{Z_{\parallel} - Z_{\perp}}{\phi}\bar{\gamma}_{\perp}\big(G_{\parallel} + 2G_{\parallel}'\frac{q^2}{d}\big) - \frac{Z_{\parallel} - Z_{\perp}}{\phi^2} - \frac{1}{2}(N-1)\frac{1}{\phi}Z_{\perp}'(\phi)\Big] \tag{6.60}$$

where we introduced short-hand notations for

$$\gamma_{\|} = q^2 Z'_{\|}(\phi) + U^{(3)}(\phi)\,, \tag{6.61}$$

$$\gamma_{\perp} = q^2 Z'_{\perp}(\phi) + \frac{\partial}{\partial\phi}\left(\frac{1}{\phi}U'(\phi)\right)\,, \tag{6.62}$$

$$\bar{\gamma}_{\|} = q^2 \frac{Z_{\|} - Z_{\perp}}{\phi} + U^{(3)}(\phi)\,, \tag{6.63}$$

$$\bar{\gamma}_{\perp} = q^2 \frac{Z_{\|} - Z_{\perp}}{\phi} + \frac{\partial}{\partial\phi}\left(\frac{1}{\phi}U'(\phi)\right)\,. \tag{6.64}$$

as well as to simplify the expression we denoted $G' = \frac{\partial G}{\partial q^2}$ and $G'' = \frac{\partial^2 G}{(\partial q^2)^2}$. In order to obtain Eqs. (6.59) and (6.60) we applied the identity

$$\int d^d q (p\cdot q)^2 f(q^2) = \frac{p^2}{d}\int d^d q q^2 f(q^2) \tag{6.65}$$

and used the expansion

$$f\big((p+q)^2\big) = f(q^2) + (p^2 + 2p\cdot q) f'(q^2) + 2(p\cdot q)^2 f''(q^2) + \mathcal{O}(p^3)\,. \tag{6.66}$$

In this study, we consider the so-called strict derivative expansion. The logic behind this approximation is straightforward, see Ref. [184]. Below we rephrase it for the truncation of interest. At the ∂^2-order, the momentum-dependent contributions of order q^4 to $\Gamma^{(4)}(p,q,q)$ are neglected; this justifies neglecting similar terms originating from the square of three-field vertex, $(\Gamma^{(3)}(p,q))^2$. Dropping the corresponding terms we end up with

$$\begin{aligned}
\partial_t Z_\parallel(\phi) = \int \bar{d}^d q \partial_t R_k(q^2) \Bigg\{ & G_\parallel^2 \Big[\gamma_\parallel^0(2\gamma_\parallel - \gamma_\parallel^0)\big(G'_\parallel + 2G''_\parallel \frac{q^2}{d}\big) + 2\gamma_\parallel^0 Z'_\parallel(\phi)\big(G_\parallel + 2G'_\parallel \frac{q^2}{d}\big) \\
& - \frac{1}{2} Z''_\parallel(\phi)\Big] \\
& + (N-1) G_\perp^2 \Big[\gamma_\perp^0(2\gamma_\perp - \gamma_\perp^0)\big(G'_\perp + 2G''_\perp \frac{q^2}{d}\big) + 4\gamma_\perp^0 Z'_\perp(\phi) G'_\perp \frac{q^2}{d} \\
& + 2\frac{Z_\parallel(\phi) - Z_\perp(\phi)}{\phi}\gamma_\perp^0 G_\perp - \frac{1}{2}\left(\frac{1}{\phi}Z'_\parallel(\phi) - \frac{2}{\phi^2}(Z_\parallel - Z_\perp)\right)\Big]\Bigg\}, \qquad (6.67) \\
\partial_t Z_k^\perp(\phi) = \int \bar{d}^d q \partial_t R_k(q^2) \Bigg\{ & G_\parallel^2 \Big[\bar{\gamma}_\perp^0(2\bar{\gamma}_\perp - \bar{\gamma}_\perp^0)\big(G'_\perp + 2G''_\perp \frac{q^2}{d}\big) + 2\bar{\gamma}_\perp^0 Z'_\perp(\phi)\big(G_\perp + 2G'_\perp \frac{q^2}{d}\big) \\
& - \frac{1}{2} Z''_\perp(\phi)\Big] \\
& + G_\perp^2 \Big[\bar{\gamma}_\perp^0(2\bar{\gamma}_\perp - \bar{\gamma}_\perp^0)\big(G'_\parallel + 2G''_\parallel \frac{q^2}{d}\big) + 4\bar{\gamma}_\perp^0 \left(\frac{Z_\parallel - Z_\perp}{\phi} - Z'_\perp(\phi)\right) G'_\parallel \frac{q^2}{d} \\
& + 2\frac{Z_\parallel - Z_\perp}{\phi}\bar{\gamma}_\perp^0\big(G_\parallel + 2G'_\parallel \frac{q^2}{d}\big) - \frac{Z_\parallel - Z_\perp}{\phi^2} - \frac{1}{2}(N-1)\frac{1}{\phi}Z'_\perp(\phi)\Big]\Bigg\} \qquad (6.68)
\end{aligned}$$

where $\gamma_i^0 = \gamma_i(q=0)$.

The final form of equations used for the flows of the expansion functions U_k, $Z_k^\perp$, and $Z_k^{||}$ in (6.45) and (6.67)-(6.68), while written above in terms of the fields ϕ, are re-expressed in terms of ρ when probing the Wilson-Fisher point. In this form, the regularity of the flows at $\rho = 0$ becomes apparent. With that said, the expressions in terms of ϕ are also necessary to probe the Yang-Lee edge singularity, as will be discussed.

6.3.3 Regulator and wave function renormalization

There are many different choices for the function $R_k(p)$. In this work, we consider Litim regulator [164]

$$R_k(q^2) = a Z_k^{||}(k^2 - q^2)\theta(k^2 - q^2)\,. \qquad (6.69)$$

Here a is a parameter to be varied and optimized under the principle of minimal sensitivity [151], we come back to it in Sect. 6.3.5. We note that we included the radial wave function renormalization at a given field background ϕ_0, $Z_k^{||} = Z_k^{||}(\phi = \phi_0)$, in the regulator. In this, we deviate from the conventional way when $Z_\perp(k)$ is introduced in the

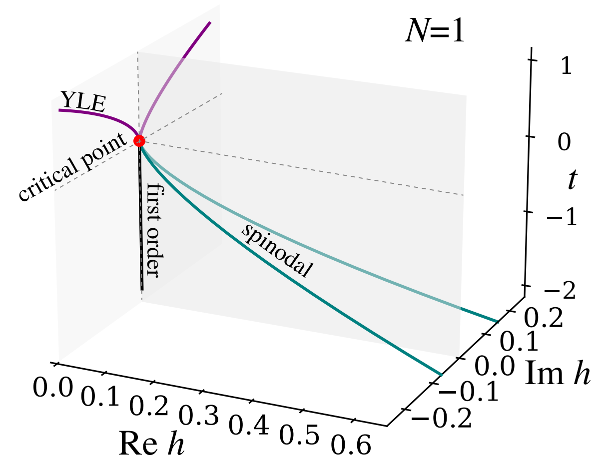


Figure 6.3: Analytic structure of the universal phase diagram for $N = 1$. Only branch points are displayed; the cuts are omitted for the clarity of the figure. To draw this figure, we used $h = (t/z_c)^{\beta\delta}$ with the realistic critical exponents and the location of the singularity obtained $|z_c|$ from Ref. [43] complemented by the value of $R_\chi = 1.72$ from Ref. [185] For the mean-field equation of state and for the large N limit in $d = 3$, the spinodals are located on the real h axis due to integer values of $2\beta\delta$. Only positive values for real values of h are shown.

regulator. Our choice is shaped by the problem we are solving: at the YLE, it is expected that the transverse degrees of freedom decouple (the YLE is at a finite imaginary value of the magnetic field, see Fig. 6.3) while the radial mode is massless and thus dominant. It is convenient to explicitly separate $Z_k^{\parallel}$ from the field-dependent $Z_k^{\parallel}(\phi)$. Moreover, we also normalize $Z_k^{\perp}(\phi)$ by the same factor, that is

$$Z_k^i(\phi) = Z_k^{\parallel} z_k^i(\phi) \tag{6.70}$$

where $i = \perp, \parallel$. From the definition of $Z_k^{\parallel}$, it follows that $z_k^{\parallel}(\phi_0) = 1$. At the same time, $z_k^{\perp}(\phi_0) \neq 1$ in general.

To simplify the equation, it is also convenient to introduce the "renormalized" field

$$\phi_r = \sqrt{Z_k^{\parallel}}\phi\,. \tag{6.71}$$

This enables us to rewrite the Green functions in the following form

$$G_{\parallel}^{-1}(q^2) = Z_k^{\parallel}[z_k^{\parallel}(\phi_r)q^2 + U''(\phi_r) + a(k^2 - q^2)\theta(k^2 - q^2)] \tag{6.72}$$

and

$$G_{\perp}^{-1}(q^2) = Z_k^{\parallel}[z_k^{\perp}(\phi_r)q^2 + U'(\phi_r)/\phi_r + a(k^2 - q^2)\theta(k^2 - q^2)]\,. \tag{6.73}$$

At the fixed points, the anomalous dimension is related to the wave-function normalization through

$$\partial_t Z_k^{\parallel} = -\eta_k Z_k^{\parallel}\,. \tag{6.74}$$

The anomalous dimension for the transverse component can be defined analogously, but it is not required for our needs.

Note that

$$Z_k^{\perp}(\phi) = Z_k^{\parallel}(\phi) - \frac{\phi^2}{2} Y_k(\phi) \tag{6.75}$$

and by analogy to Eq. (6.70), it is convenient to introduce

$$y_k(\phi) = \frac{Y_k(\phi)}{Z_k^{\parallel}}\,. \tag{6.76}$$

6.3.4 Truncation: notation and methodology

Equation (6.1) is a differential (in k) and functional-differential (in the field space) equation. Its solution is not known. Without introducing a truncation scheme, this equation cannot be treated numerically.

We thus perform a Taylor series expansion of the functions $z_{\parallel}^{(n)}(\phi)$, $y_{\parallel}^{(n)}(\phi)$ and $U(\phi)$ about a scale-dependent expansion point. In the vicinity of the Wilson-Fisher point, it is convenient to perform an expansion in terms of the "renormalized" field $\rho_r = \phi_r^2/2$, see Eq. (6.71). To simplify the notation, we omit the subscript and imply $\rho_r \to \rho$ unless indicated explicitly. We have

$$z^{\parallel}(\rho) = 1 + \sum_{n=1}^{N_Z} \frac{1}{n!} z_n (\rho - \rho_{0,k})^n\,, \tag{6.77}$$

$$y(\rho) = \sum_{n=0}^{N_Y} \frac{1}{n!} y_n (\rho - \rho_{0,k})^n\,, \tag{6.78}$$

$$U(\rho) = \sum_{n=0}^{N_U} \frac{1}{n!} u_n (\rho - \rho_{0,k})^n\,. \tag{6.79}$$

In order to find the location of the YLE singularity, we also will perform the expansion in ϕ, that is

$$z^{\parallel}(\phi) = 1 + \sum_{n=1}^{\underline{N_Z}} \frac{1}{n!} \underline{z}_n (\phi - \phi_{0,k})^n \,, \tag{6.80}$$

$$y(\phi) = \sum_{n=0}^{\underline{N_Y}} \frac{1}{n!} \underline{y}_n (\phi - \phi_{0,k})^n \,, \tag{6.81}$$

$$U(\phi) = \sum_{n=0}^{\underline{N_U}} \frac{1}{n!} \underline{u}_n (\phi - \phi_{0,k})^n \,. \tag{6.82}$$

We also omitted the subscript k, but emphasize that all parameters are running here. In this paper, we require the truncation orders to satisfy the requirement $(\underline{N_U}, \underline{N_{Z_\parallel}}, \underline{N_Y}) = 2(N_U, N_{Z_\parallel}, N_Y)$. This guarantees that the truncations are consistent and one can perform switching between the variables without losing information about the corresponding function.

The FRG flow equations for the coefficients can be readily derived starting from Eqs. (6.45), (6.67) and (6.68). For renormalized quantities, we get a set of equations

$$\dot{u}_n - u_{n+1}\dot{\rho}_0 = \eta_\parallel (u_{n+1}\rho_0 + n u_n) + \frac{d^n}{d\rho^n} \left(\partial_t U\right)\Big|_{\rho=\rho_0}, \tag{6.83}$$

$$\dot{z}^i_n - \eta_\parallel z^i_n - z^i_{n+1}\dot{\rho}_0 = \eta_\parallel (z^i_{n+1}\rho_0 + n z^i_n) + \frac{1}{Z_\parallel}\frac{d^n}{d\rho^n} \left(\partial_t Z^i\right)\Big|_{\rho=\rho_0} \tag{6.84}$$

where we introduced the polarization index $i = (\parallel, \perp)$. The coefficients $z^n_\perp$ are related to $y^{(n)}$ through $z^n_\perp = z^n_\parallel - n y^{(n-1)} - \rho_0 y^{(n)}$.

In order to find the fixed point solutions, we also need to determine the flow equations for the dimensionless renormalized coefficients. They can be obtained by computing the expansion coefficient of dimensionless quantities, e.g. U/k^d and $z^\parallel$, as functions of dimensionless $\tilde{\rho} = \rho_r / k^{d-2}$:

$$\tilde{u}_n = k^{n(d-2)-d} u_n \,, \tag{6.85}$$

$$\tilde{z}^i_n = k^{n(d-2)} z^i_n \,. \tag{6.86}$$

$$\tag{6.87}$$

We have

$$\dot{\tilde{u}}_n - \tilde{u}_{n+1}\dot{\tilde{\rho}}_0 = -d\tilde{u}_n + (d-2+\eta_\parallel)(\tilde{u}_{n+1}\tilde{\rho}_0 + n\tilde{u}_n) + \frac{1}{k^d}\frac{d^n}{d\tilde{\rho}^n}\left(\partial_t U\right)\Big|_{\tilde{\rho}=\tilde{\rho}_0}, \tag{6.88}$$

$$\dot{\tilde{z}}^i_n - \eta_\parallel \tilde{z}^i_n - \tilde{z}^i_{n+1}\dot{\tilde{\rho}}_0 = (d-2+\eta_\parallel)(\tilde{z}^i_{n+1}\tilde{\rho}_0 + n\tilde{z}^i_n) + \frac{1}{Z_\parallel}\frac{d^n}{d\tilde{\rho}^n}\left(\partial_t Z^i\right)\Big|_{\tilde{\rho}=\tilde{\rho}_0}. \tag{6.89}$$

These equations can be rewritten in terms of the expansion coefficients of the series of ϕ without many modifications. It amounts to replacing z^i_n, u_n and ρ_0 with $\underline{z}^i_n, \underline{u}_n$ and ϕ_0 and η with $\eta/2$ in the right hand sides of Eqs. (6.83), (6.84), (6.88), (6.89).

The equations above are to be supplemented by a choice of the expansion point. In this work we choose the expansion point ϕ_0 by fixing the radial excitation mass, $m_R = \sqrt{U'(\rho) + 2\rho\, U''(\rho)}$, to a constant, $\partial_t m_R = 0$. The conventional expansion scheme, which is used in the majority of applications, follows the physical point defined by the minimum of the effective potential. Since the YLE singularity is located at the (imaginary) magnetic field H_c where the radial mass at vanishes, finding it requires numerically expensive fine-tuning with this scheme to find the value of H_c.

The advantage of our expansion scheme is that we can directly follow the flow of the edge singularity by simply following the field value corresponding to vanishing mass. Within this setup, the magnetic field is then a function of the renormalized mass and the temperature defined through the relation $H_k \equiv U'(\phi_0) = \underline{u}_1$. For computations in the symmetric phase, the renormalized mass at the origin (at zero field) obtains a finite value in the IR. Expanding about fixed renormalized mass greater than this value corresponds to real fields away from the origin. Importantly, if one expands about fixed renormalized mass *less than* the value at the origin, the solution is necessarily pushed into the complex plane at a finite value of k. Thanks to the Lee-Yang theorem, the solution is pushed specifically along the imaginary ϕ axis which corresponds to $\rho < 0$. If one further takes t sufficiently small, the flow plateaus about the Wilson-Fisher point before being kicked off towards the YLE.

Thus, within this "crossover flow" scheme, computing at $m_R = 0$ in the symmetric phase the expansion point ρ_0 experiences the Wilson-Fisher plateau and before eventually being kicked off to negative values where it flows towards the YLE. Although the parametrization in terms of ρ naturally probes the Wilson-Fisher point and crosses through to negative values, in order to capture the criticality near the YLE, one needs to transform to a truncation in terms of ϕ. We perform this switch at small negatives values of η_k and check that the results are insensitive to the exact value of the switching point.

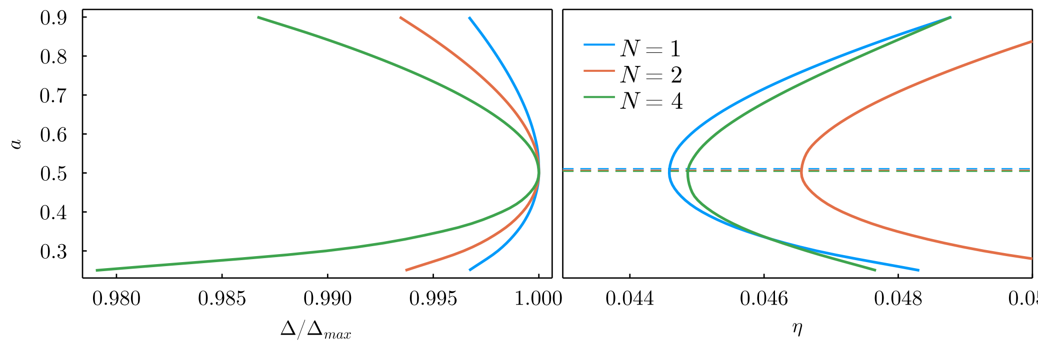


Figure 6.4: Regulator parameter dependence of the gap critical exponent Δ (normalized by its value at the maximum) and the anomalous dimension $\eta(a)$ for a set of N at the Wilson-Fisher fixed point. The maxima of Δ define the minimal sensitivity point, a_Δ. The dashed lines in the right panel show a_Δ. The figure demonstrates that the location of the minima of $\eta(a)$ is fairly close to a_Δ.

A downside of our expansion scheme is that numerical computations in the broken phase are more challenging. The reason is that in this case at any finite k our expansion point ϕ_0 lies on the real axis below the physical point. The FRG flow flattens the potential in this region since it has to be convex in the deep IR. This convexity-restoring flow is driven by near-singular propagators and therefore numerically challenging to resolve, see, e.g., Refs. [186, 171], as we have argued in the intro to Chapter 5, we do not need to compute in the broken phase.

6.3.5 O(N) fixed point and Minimal Sensitivity Analysis

The Wilson-Fisher fixed point can be found by solving the set of algebraic equations $\dot{\tilde{u}}_n = \dot{\tilde{z}}^i_n = \dot{\tilde{\rho}}_0 = 0$. There are several ways how we are going to use this solution. First, it defines the O(N) anomalous dimension; we use this critical exponent to apply the minimal sensitivity analysis via the regulator parameter a; see below. Second, we can use a slightly perturbed fixed point solution as the initial condition for the FRG evolution towards the IR to extract the metric factors, critical exponents, and finally the location of the YLE singularity.

The critical exponent η fully defines δ through the scaling relation in Eq. (4.92). Moreover, calculating the stability matrix at the fixed point solution allows one to find the critical exponent ν and thus the gap critical exponent through the relation $\Delta = \frac{\nu}{2}(d+2-\eta)$. This motivates our strategy in defining the parameter a as the extremum of the function

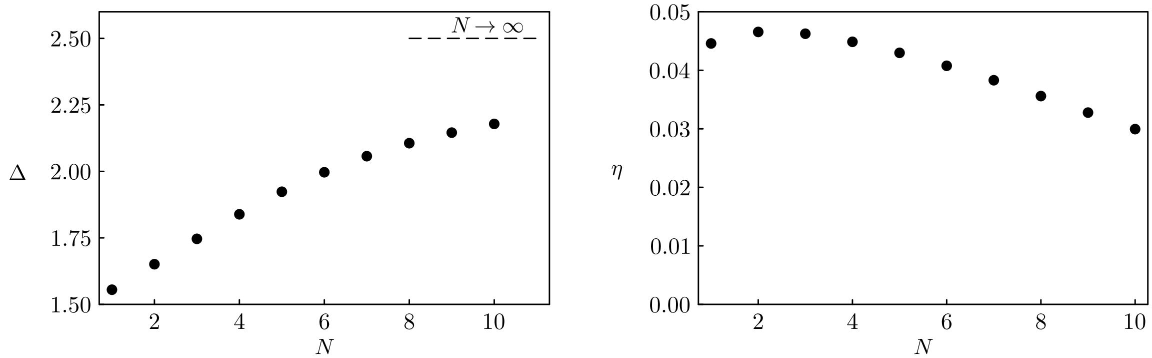


Figure 6.5: The results of the minimal sensitivity analysis at the Wilson-Fisher fixed point for the gap critical exponent $\Delta = \Delta(a_\Delta)$. The corresponding anomalous dimension is shown on the right panel $\eta = \eta(a_\Delta)$.

$\Delta(a)$ where a enters to the regulator through Eq. (6.69). This fixes the regulator parameter $a = a_\Delta$ that we use for calculating the metric factors (B_c and C_2^+), critical exponents (δ and γ), and the location of the YLE singularity. By choosing the extremum as a function of a, it is guaranteed that among the family of regulators defined in Eq. (6.69), we use the one where the regulator dependence, and hence the systematic error, is minimal at least for Δ.

We show the dependence of the gap critical exponent and the anomalous dimension on the regulator parameter a in Fig. 6.4 for a few values of N. These calculations were performed using the strict derivative expansion and the truncation scheme $(N_U, N_{Z_\parallel}, N_Y) = (6, 3, 2)$. As can be seen in the figure, for N of phenomenological interest, the location of the extrema of $\Delta(a)$ and $\eta(a)$ are fairly close at this truncation.

We obtained reasonable values of the anomalous dimension critical exponent displayed in Fig. 6.5. The non-monotonic dependence of η on the number of components N is expected from the ε-expansion.

The minimal sensitivity analysis for different quantities does not necessarily result in the same regulator parameter a. In order to estimate the corresponding systematic uncertainty we also performed the analysis for the value of the magnetic field at the YLE singularity, see Table 6.1.

6.3.6 Yang-Lee edge singularity fixed point solution

Near and at the YLE fixed point, due to explicit symmetry breaking $h \neq 0$, there is only one light degree of freedom – the radial mode. Therefore we expect our result for the critical exponent η_{YLE} (or σ_{YLE}) to trivially reproduce those of the one component theory. This also serves as a cross-check on our calculations as it provides a powerful constraint on all our equations for $N \neq 1$.

There is one subtle point related to how one approaches this fixed point. We remind the reader that at the Wilson-Fisher fixed point, the FRG equations are k-independent for properly scaled variables as we introduced in Sec. 6.3.4. The k independence implies that the fixed point can be reached at a finite value of k. To the contrary, when we start from the general equations for the multi-component field theory, one cannot expect k-independence for the YLE fixed point, as the complete separation of the transverse degrees of freedom is only possible at asymptotically small k. Thus, when finding the algebraic equation for the YLE fixed point, one additionally has to take the limit $k \to 0$ for the terms that involve transverse degrees of freedom. In effect, this amounts to taking the limit of the dimensionless renormalized Goldstone mass to infinity. By computing this limit, we were able to show explicitly that the fixed point equations of our theory reduced to those of the single component theory, that is, the N-dependent terms originating from Eqs. (6.59) and (6.45) drop out.

6.3.7 Critical exponents and amplitudes

The scaling variable ζ, defined in Eq. (6.94), requires the determination of the critical exponents and metric factors. They can be found by performing calculations near the critical point. Thus appropriate perturbations of the initial conditions near the Wilson-Fisher fixed point allow us to extract the required quantities. The numerical procedure, common for arbitrary N, essentially coincides with that performed in Ref. [43].

In order to solve the flows, one starts with an initial condition for the effective action. An easy set of initial conditions, and likely the most familiar, amount to a UV potential with a non-zero minima

$$U_{\mathrm{UV}}(\rho) = r\rho + \rho^2 \tag{6.90}$$

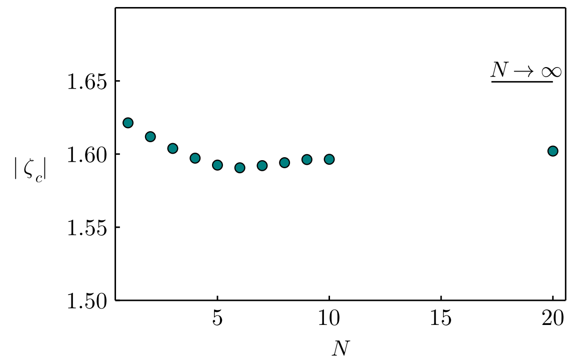


Figure 6.6: Location of the YLE singularity $|\zeta_c| = |z_c|/R_\chi^{1/\gamma}$ as a function of N. The estimated uncertainty is withing the marker size. The infinite N limit ($|\zeta_c| \approx 1.649$) is approached from below. As was demonstrated in Ref. [45], this approach is parametrically slow.

where $r = r_c + t$. The expansion point initial condition is defined via

$$m_R^2 = r + 6\rho_{0,\mathrm{UV}} \,. \tag{6.91}$$

In this setup, $Z_{\parallel,\mathrm{UV}}(\rho) = 1$ and $Y_k(\rho) = 0$. This amounts to setting $u_n = 0$ for $n \geq 3$, $z_n = 0$ for $n \geq 1$, and $y_n = 0$ for all n. However, there are two setbacks to this approach. One has to numerically determine r_c. On top of this, one has to compute the portion of the RG flows from the UV to the fixed point for all IR flows.

One can avoid both of these computational expenses by instead using thermal perturbations about the $O(N)$ fixed point solution as the initial condition for the flows. The fixed point potential naturally yields a value of r_c; however, now all the higher order Taylor coefficients will be generically non-zero. One still defines a thermal perturbation of the fixed point solution by adding a term $t\,\rho$ to the fixed point potential, $U_*(\rho)$. Let us denote this perturbed potential as δU_* which is defined as $\delta U_*(\rho) = U_*(\rho) + t\rho$. The initial condition for the expansion point is determined by solving for ρ in the expression $\delta U_*'(\rho) + 2\rho\,\delta U_*''(\rho) = m_R^2$. Lastly, the initial conditions for all of the Taylor coefficients are found by evaluating appropriate derivatives of the fixed point solutions $U_*(\rho)$, $Z_{\parallel,*}(\rho)$, and $Y_*(\rho)$ (as given by their truncated Taylor series approximations) at the expansion point as defined above.

Once the fixed point solutions are obtained, one can begin computing the critical data. Let us begin with computing δ and B_c as in Eq. (4.91). When computing at $t = 0$ the

Table 6.1: The regulator parameter as determined by minimal sensitivity analysis applied to the gap critical exponent, a_Δ, and the anomalous dimension, a_η, at the Wilson-Fisher fixed point as well as to the value of the magnetic field at YLE singularity, a_h. The numbers are quoted to the fourth digit.

N	1	2	3	4
a_Δ	0.5108	0.5069	0.5026	0.5044
a_η	0.5044	0.5075	0.5064	0.4906
a_h	0.6299	0.5921	0.5724	0.5617

mass at the origin is in theory zero, so fixing the mass to positive values pushes one to positive magnetization and magnetic field. In this regime, the magnetic field scales as m_R^2 to some power of the critical exponents. (At $t = 0$, the scaling of the correlation length $\xi \propto |H|^{-\nu_c}$ implies $m_R^2 \propto |H|^{\nu/\beta\delta}$. A simple rewriting yields the relation $H \propto (m_R^2)^{\beta\delta/\nu}$.) Thus, to obtain δ and B_c, we simply take $t = 0$ and vary m_R^2 within some sufficiently small range. To the data, we fit corrections to scaling similar to Eq. (4.109) to increase the precision of our predictions:

$$M = B_c H^{1/\delta} \left(1 + b_{11} H^{\omega\nu_c} + b_{12} H^{2\omega\nu_c}\right) \tag{6.92}$$

where the additional correction b_{12} is fitted if there is sufficient precision. In practice, the combination $\omega\nu_c$ is fitted as a single quantity.

Next, we move to the symmetric phase with $H = 0$ to compute γ and C_2^+ as in Eq. (4.91). For $t > 0$, the mass at the origin is positive and so one must tune m_R^2 to match the renormalized mass at the origin in order to reproduce $H = 0$ in the IR. To do so, we implement a root finding algorithm on the output of the flow to tune $H \to 0$. This is a large computational expense not present in the conventional Taylor expansion scheme about the running minimum. In practice to minimize this cost, it is easier to compute to some sufficiently small H and fit the relevant corrections to scaling

$$\chi = C_2^+ t^{-\gamma} \left(1 + a_H H^2 t^{-2\beta\delta} + a_t t^{\omega\nu}\right) \tag{6.93}$$

where we fit the combinations $\beta\delta = \Delta$ and $\omega\nu$ as a single quantity. In the above, we used that for $t > 0$, the magnetization is an odd function of H. It follows then that the non-zero H corrections to scaling for χ with $t > 0$ begins with H^2. As an aside, note that taking a derivative of Eq. (4.108) reproduces the relation $\gamma = \beta(\delta - 1)$.

Table 6.2: The location of the YLE singularity, $|\zeta_c| = |z_c|/R_\chi^{1/\gamma}$ for a representative number of components N. The numbers in the parentheses ($\Delta_{\rm tr}$), ($\Delta_{\rm reg}$) show the truncation error and the error due to residual regulator dependence. The uncertainty quoted in the text corresponds to the maximum of $\Delta_{\rm tr}$ and $\Delta_{\rm reg}$. In all considered cases, $\Delta_{\rm tr}$ is the largest. For the three-dimensional Ising universality class $N = 1$, the result of the current work is consistent with the previous study of Ref. [43].

N	1	2	3	4	5
$\lvert\zeta_c\rvert$	1.621(4)(1)	1.612(9)(0)	1.604(7)(0)	1.597(3)(0)	1.5925(2)(1)

As a check on the consistency of the method, we checked that the corrections to scaling reproduce the scaling relations $\Delta = \beta\delta$ and $\frac{\omega\nu}{\omega\nu_c} = \beta\delta = \frac{\gamma\delta}{\delta-1}$.

6.3.8 Location of Yang-Lee edge singularity

To locate the YLE singularity, we compute in the symmetric phase $t > 0$ and at vanishing renormalized mass of the radial excitations. Practically we use small but nonzero values $m_R = 0+$. As we are interested in extracting the universal location, we have to consider rather small positive t to minimize non-universal contributions. As described in Sect. 6.3.4, we perform the switch of the parametrization of the solution from ρ to ϕ at some small but non-zero negative value of ρ_0^s. We checked that our results are insensitive to the variation of the value of ρ_0^s.

Solving the flow numerically yields the determination of $H_c = U'(\phi_0)$ in the infrared. The result for ζ_c is then given by

$$\zeta_c = \left(\frac{B_c}{C_2^+}\right)^{1/\gamma} \frac{t}{H_c^{1/\Delta}}\,. \tag{6.94}$$

We obtained the results presented in Table 6.2 and illustrated in Fig. 6.6. The error was computed as follows. First, to estimate the error due to the truncation of the field dependence, $\Delta_{\rm tr}$, we compare the results in the (5, 2, 1) and (5, 3, 2) truncation schemes to the ones obtained for (6,3,2). We use the result of the highest truncation (6,3,2) for central points and maximal of the absolute values of the two differences $|\zeta_c|^{(6,3,2)} - |\zeta_c|^{(5,3,2)}$ and $|\zeta_c|^{(6,3,2)} - |\zeta_c|^{(5,2,1)}$ for the error estimate due to truncation. Second, to evaluate the uncertainty associated with the regulator dependence $\Delta_{\rm reg}$, we perform calculations of $|\zeta_c|$ at two values of a: a_Δ determined by the minimal sensitivity analysis applied to the critical exponent Δ at the Wilson-Fisher fixed point (see Sect. 6.3.5) and a_h determined by the dependence of the magnetic field at the YLE singularity H_c. The difference between the corresponding values of $|\zeta_c|$ determines Δ_{reg}. The numerical values for the regulators determined by both schemes are listed in Table 6.1. Both our errors are measures for the convergence of our truncation within the next-to-leading order of the derivative expansion. A meaningful estimate for the systematic error of the derivative expansion itself requires us to go to next-to-next-to-leading order [159, 184].(Since the anomalous dimension is always zero in the leading order derivative expansion, it is not a suitable truncation to describe the YLE singularity and therefore also not suitable for a meaningful error estimate.) While this is required for a precision calculation, this is beyond the scope of the present work.

The next step is to perform the transformation from $|\zeta_c|$ to $|z_c|$. This step requires the determination of R_χ. Our current set up does not allow us to compute this quantity as it necessitates solving the FRG equation in the broken phase which cannot be done using an expansion point defined by equation $\partial_t m_R = 0$ at finite k. Fortunately, high precision calculations of R_χ were performed recently in Ref. [117] for $N = 2, 3, 4$ and 5. We will use these results together with the value of R_χ computed for $N = 1$ in Ref. [187]. Reference [187] does not provide systematic uncertainty on the value of R_χ; we estimated it by comparing to earlier calculations of R_χ in the LPA' FRG of Ref. [134]. We list the results in Table 6.3, where to find $R_\chi^{1/\gamma}$ the value of γ was taken from Ref. [184]. With this we can perform the transformation to $|z_c|$. The result is presented in Table 6.4. Within the systematic uncertainty the values are consistent to our previous calculations in LPA', see Ref. [45].

Tables 6.2 and 6.4 constitute the main results of this study. Combining the input from our FRG results with the semi-exact two-dimensional Ising model result of Ref. [19], the epsilon expansion about $d = 4$ (see Sect. 5.3), and the behavior of $O(N > 2)$ systems near two dimensions (see Sect. 5.5), we perform a Padè approximation to capture the dependence of the YLE singularity location, $|z_c|$, on the number of spatial dimensions, d, for $N = 1 - 5$, see Fig. 6.7. Qualitatively this result is similar to our earlier calculation of Ref. [45].

Table 6.3: The combination required to map ζ_c to z_c. Critical amplitude R_χ and critical exponent γ are obtained from Ref. [134, 187] and precision calculations of Refs. [184, 117].

N	1	2	3	4	5
$R_\chi^{1/\gamma}$	1.497(22)	1.26(5)	1.140(34)	1.058(21)	0.974(26)

Table 6.4: Location of the YLE singularity, z_c, at different N. The uncertainty is dominated by the uncertainty in determination of R_χ.

N	1	2	3	4	5
$\lvert z_c\rvert$	2.43(4)	2.04(8)	1.83(6)	1.69(3)	1.55(4)

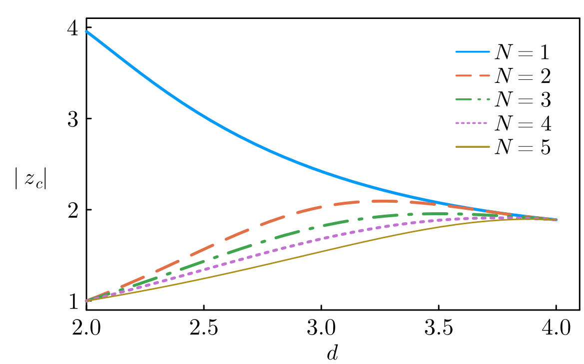


Figure 6.7: Four-parameter Padé approximation for the dependence of the YLE location on the number of spatial dimension, see text for details. The result for the location of the YLE singularity in two-dimensional Ising model is taken from Ref. [19], see also Ref. [43] for reparametrization to $|z_c|$. We expect that, at $d = 2$, going to fractional values of N would fill in the gap from $N = 1$ to $N = 2$. The line corresponding to $N = 1$ reaches maximum at some value of d in the range $1 < d < 2$ and then drops to 1 at the lower critical dimension $d = 1$.

CHAPTER

7

CONCLUSIONS AND OUTLOOK

In the high temperature regime of the $O(N)$ universality classes, we have seen the emergence of this *protocritical* point [120], the Yang-Lee edge singularity (YLE), whose origin is related to the scaling of the density of zeros, $g(H,t)$. More specifically, above the critical temperature a gap forms in the density of zeros resulting in a vanishing density, $g(H,t)=0$, within the gap, $|H|<H_c$. About the edge of this gap, H_c, the density of zeros scales as $g(H,t>0)\propto(\theta-\theta_c)^{\sigma}$ where $-\frac{1}{2}\leq\sigma\leq\frac{1}{2}$. The magnetization inherits this structure as a branch point: $M-M_c\propto(H-H_c)^{\sigma}$. This in turn implies that the susceptibility diverges as $\chi\propto(H-H_c)^{-|1-\sigma|}$, thus, the telltale sign of a critical point. Albeit, it is a complex conjugate pair of critical points occurring at purely imaginary values of the external field $\pm i|H_c|$.

Within the critical region of the $O(N)$ fixed points, one may re-express the branch point of the magnetization corresponding to the YLE as a branch point of the universal scaling function $f_G(z)$. The universal function $f_G(z)$ is given by the universal magnetic equation of state $M=h^{1/\delta}f_G(z)$. To define the scaling variable z, we use rescaled reduced temperature $\bar{t}$ and external field h, see Section 4.4.3: $z=\bar{t}h^{-1/\beta\delta}$. In terms of $f_G(z)$, the YLE appears as complex conjugate branch points z_c and z_c^*. As such, z_c is the *universal* location of the Yang-Lee edge singularity.

Furthermore, the YLE is the nearest singularity to the origin. Thus, the YLE has a powerful effect on the thermodynamics for ordinary real parameters. For example, it controls the radius of convergence and also determines the functional form of large order Taylor coefficients. This can be applied to the QCD phase diagram whereby from a combination of universal and non-universal input allows one to infer the radius of convergence of Taylor expansions about vanishing chemical potential and/or reconstruct the equation of state, see [51, 177, 49, 176, 188, 189].

Being that the location of the YLE z_c is such an important quantity, it is somewhat surprising that its value for any 3D $O(N)$ universality class was never computed before. We can understand why by analyzing the main tools used to study critical phenomenon - the ε-expansion and Monte Carlo simulations. For the ε-expansion, we discussed in Section 5.3 how the location of the YLE is non-perturbative beyond leading order in ϵ. Thus, this method is of little hope to characterize the value of z_c is three dimensions. This non-perturbative feature can be reasoned to occur due to the mismatch of the critical dimensions of ϕ^4 theories ($d_c = 4$) and ϕ^3 theories ($d_c = 6$). As for approaching the problem via Monte Carlo, the imaginary values of the magnetic field prohibit conventional applications of the sampling algorithm due to the sign problem. However, there has been work done in regards to studying the Lee-Yang zeros on the lattice, see e.g. [190, 191, 192, 193, 194, 195].

By using the functional renormalization group (FRG), we are not affected by either of these obstacles. FRG is inherently non-perturbative and naturally extends to the complex parameter plane. In our first study [45], we produced the first results of the YLE location z_c for three dimensional $O(N)$ universality classes, see Section 6.2. These results were obtained at low orders of the derivative expansion (LPA') via an extrapolation scheme. By expanding upon the approach of Ref. [43], we significantly improved upon these LPA' results in Ref. [44]. The scheme captures the criticality of both the $O(N)$ fixed point *and* the YLE. We denoted this scheme for this thesis as the crossover flow. These $O(N)$ results were accompanied with an error analysis utilizing the principle of minimal sensitivity alongside the truncation error due to the Taylor expansion. See Tables 6.2 and 6.4 and Section 6.3 for discussions. Complemented by recent ideas and methods of Refs. [177, 49, 188, 196, 194, 197, 189, 198] our findings might help to establish the existence and potentially the location of the QCD critical point.

Aside from our three dimensional results for z_c (and ζ_c), we have given an overview in Chapter 5 of the results for various N and dimensions. The two dimensional results of that Chapter, which were obtained from the literature, range from standard applications of the ε-expansion, large N techniques, analytic continuation in N, to conformal field theory approaches. At a more basic level, it was shown in Chapter 2 that the van der Waals equation rather amazingly captures all the qualitative features associated with the Yang-Lee edge singularity and its universal aspects. This discussion can easily be adapted to a textbook chapter on phase transitions.

Going further, there is a fascinating aspect of the branch point nature of the YLE which we mentioned a few times in this thesis. The branch points imply a Riemann structure to the equation of state where the different sheets correspond to different physical regions of parameter space. The principle sheet can be identified with the high temperature region which has two branch points at $\pm i|H_c|$. By analytically continuing over these branch cuts, one can access the low temperature region. Recall that at low temperature and vanishing fields there are two solutions for the magnetization $M = \pm\#$. There are then at least two sheets for low temperature physics corresponding to the full analytic domain of these solutions obtained by continuing H. At mean field, the universal equation of state, $h = M(\bar{t} + M^2)$, yields a cubic polynomial for M, and so there are three sheets. However, beyond mean field other sheets appear due to the more complicated nature of the equation of state, see e.g. [13].

A basic result of this thesis is illuminated by the above discussion - the equivalence of the universality of the YLE with the universality of the spinodal point. (This feature was essentially also observed in Ref. [13].) Being that this mapping can be seen as a high-low temperature duality, it is interesting to wonder if there is anything deeper to this equivalence.

From the discussion of the literature in Section 3.1, we can infer that this picture occurs for anti-ferromagnetic spin systems as well as generalizations of the Ising model exhibiting multicritical points. Generalizing this Riemann sheet picture to more complicated phase structures offers a very rich playground. Let us elaborate on the discussion and literature review of Section 3.1. For instance, due to the triple well nature of a tricritical point, one can reason that for a certain parameter region there should be two Yang-Lee edge singularities (most likely off the imaginary axis), see for instance [38]. In Suzuki's extension of the Lee-Yang theorem to these types of models [199], the theorem only holds under certain conditions, beyond which a bifurcation in the density of zeros likely occurs. A similar generalization is seen nicely in [101], a multicritical point of the YLE! There is reason to believe this multicritical YLE point is an $ig\phi^5$ theory, see [42, 100, 200]. It is a fundamental question to ask how the features of multicriticality are understood via the Lee-Yang theory and behaviors of the density of zeros.

Another basic question to ask is how the addition of fermions changes things. One can reason that since the edge singularity of these systems should occur at some finite external field value (though probably complex), the fermions will be massive and thus decouple from the physics about the critical point. This reasoning leads one to believe that the edge singularity for systems with fermions is still the Yang-Lee edge singularity. This idea is also supported by the large N result for the Gross-Neveu-Yakawa model computed in Ref. [20]. It is reasonable to then suppose that the YLE is a sort of *superuniversal* protocritical point characterizing all bi-critical points.

In summary, the study of critical phenomenon in the complex plane provides many powerful and fascinating results from basic yet abstract properties of the zeros of the partition function to applications in determining the features of the QCD phase diagram. As we have described the location of the Yang-Lee edge singularity as an important universal quantity, we expect to see studies which obtain the locations at levels of precision eventually comparable to the precision enjoyed by the critical exponents and amplitude ratios. Even further off, since the ideas described in this thesis are conceptually basic yet powerful, it is exciting to wonder when these results may show up in a textbook chapter on phase transitions.

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APPENDICES

APPENDIX

A

THE ONE DIMENSIONAL ISING MODEL

We can characterize the one dimensionsional classical Ising model by placing N classical spins s_i on a periodic lattice which take values of ± 1. While it is possible to study this theory with very general interactions between the spins, we will take the simplest approach of using the nearest neighbor interaction. We furthermore place the system in an external magnetic field oriented along the direction corresponding to $s_i = +1$. The Hamiltonian for this system of N spins is written as

$$\begin{aligned} \mathcal{H}_N[\{s_i\}] &= -J \sum_{i=0}^{N-1} s_i\, s_{i+1} - \sum_{i=0}^{N-1} H\, s_i\,, && \text{(A.1)} \\ &= -\sum_{i=0}^{N-1} \left[J\, s_i\, s_{i+1} + \frac{H}{2}(\, s_i + s_{i+1}) \right]. && \text{(A.2)} \end{aligned}$$

where we made explicit that the Hamiltonian is a function of the configuration of spins $\{s_i\}$. The partition follows as

$$\begin{aligned} \mathcal{Z}_N &= \sum_{\{s_i\}} \mathrm{e}^{-\beta\,\mathcal{H}_N}\,, && \text{(A.3)} \\ &= \sum_{s_0=\pm 1} \dots \sum_{s_{N-1}=\pm 1} \prod_{i=0}^{N-1} \mathrm{e}^{\beta\left(J\,s_i\,s_{i+1} + \frac{H}{2}(s_i+s_{i+1})\right)}\,. && \text{(A.4)} \end{aligned}$$

The trick we use to solve the above is by introducing the transfer matrix:

$$T_{s_i,s_j} = \mathrm{e}^{\beta\,(J\,s_i\,s_j + \frac{1}{2}H\,(s_i+s_j))}\,. \quad \text{(A.5)}$$

See Ref. [54]. With this notation, let's look to $\mathcal{Z}_1$:

$$\begin{aligned} \mathcal{Z}_1 &= \sum_{s=\pm 1} e^{\beta\,(J\,s^2 + H\,s)}\,, && \text{(A.6)} \\ &= e^{\beta\,J}\left(e^{\beta\,H} + e^{-\beta\,H}\right)\,, && \text{(A.7)} \\ &= 2\,e^{\beta\,J}\cosh\beta\,H\,. && \text{(A.8)} \end{aligned}$$

After some algebra, for $N = 2$ one would find

$$\mathcal{Z}_2 = \sum_{s_3=\pm 1}\sum_{s_2=\pm 1}\sum_{s_1=\pm 1} \delta_{s_3,s_1}\,T_{s_1,s_2}\,T_{s_2,s_3}\,. \quad \text{(A.9)}$$

This is likely enough to see the trend:

$$\mathcal{Z}_N = T_{s_1,s_2}\,T_{s_2,s_3}\dots T_{s_{N-1},s_N}\,T_{s_N,s_{N+1}}\,\delta_{s_{N+1},s_1}\,. \quad \text{(A.10)}$$

Alternatively, we can realize T as a matrix which gives

$$\mathcal{Z}_N = \mathrm{Tr}\,T^N \quad \text{(A.11)}$$

where

$$T = \begin{bmatrix} e^{\beta\,J+\beta\,H} & e^{-\beta\,J} \\ e^{-\beta\,J} & e^{\beta\,J-\beta\,H}\,. \end{bmatrix} \quad \text{(A.12)}$$

The matrix above is obtained by evaluating T_{s_i,s_j} with $s_{i,j} = \pm 1$. From a simple fact of operators

$$\begin{aligned}
\mathcal{Z}_N &= \text{Eigen}\left(T^N\right) , && \text{(A.13)}\\
&= \text{Eigen}\left(T\right)^N , && \text{(A.14)}\\
&= \lambda_+^N + \lambda_-^N . && \text{(A.15)}
\end{aligned}$$

The characteristic equation for the given eigenvalue problem is

$$\begin{aligned}
\det\left(T - \lambda\, I\right) &= \left(e^{\beta\, J - \beta\, H} - \lambda\right)\left(e^{\beta\, J + \beta H} - \lambda\right) - e^{-2\beta\, J} , && \text{(A.16)}\\
&= \lambda^2 - e^{\beta\, J}\,\lambda\left(e^{-\beta\, H} + e^{\beta\, H}\right) + e^{2\beta\, J} - e^{-2\beta\, J} , && \text{(A.17)}\\
&= \lambda^2 - 2\lambda\, e^{\beta\, J}\cosh\beta\, H + 2\sinh\beta\, J . && \text{(A.18)}
\end{aligned}$$

Further algebra yields the following solutions for the eigenvalues:

$$\lambda_\pm = e^{\beta\, J}\cosh\beta\, H \pm \sqrt{e^{2\beta\, J}\cosh^2\beta\, H - 2\sinh 2\beta\, J} . \qquad \text{(A.19)}$$

We note that the system in the thermodynamic limit $N \to \infty$, one can recognize that the larger of the two eigenvalues will dominate, λ_+.

$$\begin{aligned}
\mathcal{Z} &= \lim_{N\to\infty} \mathcal{Z}_N , && \text{(A.20)}\\
&= \lambda_+^N + \lambda_-^N , && \text{(A.21)}\\
&= \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right) , && \text{(A.22)}\\
&\approx \lambda_+^N . && \text{(A.23)}
\end{aligned}$$

From here we have the free energy, $\mathcal{F} = -\frac{1}{\beta\, N}\ln\mathcal{Z}$

$$\begin{aligned}
\mathcal{F} &= -\frac{1}{\beta}\ln\lambda_+ , && \text{(A.24)}\\
&= -\frac{1}{\beta}\ln\left(e^{\beta\, J}\cosh\beta\, H + \sqrt{e^{2\beta\, J}\cosh^2\beta\, H - 2\sinh 2\beta\, J}\right) . && \text{(A.25)}
\end{aligned}$$

and the spontaneous magnetization from $M = -\beta \, \frac{\partial \mathcal{F}}{\partial H}$,

$$M = \frac{e^{\beta J} \sinh(\beta \, H)}{\sqrt{e^{2\beta \, J} \cosh^2(\beta \, H) - 2 \sinh(2\beta \, J)}} \, . \tag{A.26}$$

APPENDIX

B

LPA' FLOWS AT FINITE TEMPERATURE

B.1 Scalar Theory

For LPA', we work with a truncation of the form

$$\Gamma_k[\phi] = \int_{\mathbf{x}} \left\{ U_k(\rho) + \frac{1}{2} Z_k (\partial\phi)^2 \right\}, \tag{B.1}$$

where the wave function renormalization Z_k is allowed to run. From the above ansatz we have

$$\begin{aligned} \Gamma_k^{(2)}[\phi, q, p]\big|_{\phi=\mathrm{uni}} &= \left\{ \frac{\partial^2 U_k(\rho)}{\partial \phi^2} + Z_k\, q^2 \right\} (2\pi)^d \delta^d(q+p), && \text{(B.2)} \\ &= \left\{ U_k'(\rho) + 2\rho U_k''(\rho) + Z_k\, q^2 \right\} (2\pi)^d \delta^d(q+p). && \text{(B.3)} \end{aligned}$$

To project the flow onto the effective potential, simply note that $\Gamma[\phi]\Big|_{\phi=\text{uni}} = V\, U_k(\rho)$. Evaluating Eq. (6.1) on a uniform field then gives

$$\partial_t U_k(\rho) = \frac{1}{V}\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{\partial_t R_k(q)}{U_k'(\rho)+2\rho U_k''(\rho)+Z_kq^2+R_k(q)}(2\pi)^d\delta^d(q-q), \tag{B.4}$$

$$= \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{\partial_t R_k(q)}{U_k'(\rho)+2\rho U_k''(\rho)+Z_kq^2+R_k(q)}, \tag{B.5}$$

where we have used that $(2\pi)^d\delta^d(0) = V$. We can take this further by using the Litum regulator $R_k(q) = Z_k\,(k^2-q^2)\Theta(k^2-q^2)$. Applying ∂_t to R_k and using $\partial_t Z_k = -\eta_k\, Z_k$,

$$\partial_t\left(Z_k\,(k^2-q^2)\Theta(k^2-q^2)\right) = \left(-\eta_k\,(k^2-q^2)+2k^2\right)Z_k\Theta(k^2-q^2)\,, \tag{B.6}$$

$$= \left(\eta_k q^2+(2-\eta_k)k^2\right)Z_k\Theta(k^2-q^2) \tag{B.7}$$

where we dropped a term $\sim x\,\delta(x)$.

$$\partial_t U_k(\rho) = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\frac{\left(-\eta_k Z_k\,(k^2-q^2)+2Z_kk^2\right)\Theta(k^2-q^2)}{U_k'(\rho)+2\rho U_k''(\rho)+Z_kk^2}\,, \tag{B.8}$$

$$= \frac{1}{2}\frac{Z_kS_{d-1}}{(2\pi)^d}\int_0^k dq\, q^{d-1}\,\frac{\eta_k q^2+(2-\eta_k)k^2}{U_k'(\rho)+2\rho_0U_k''(\rho)+Z_kk^2}\,, \tag{B.9}$$

$$\partial_t U_k(\rho) = \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)\frac{Z_k}{U_k'(\rho)+2\rho U_k''(\rho)+Z_kk^2}\,. \tag{B.10}$$

In the above, S_{d-1} is the volume of the $d-1$ dimension hyper-sphere. To include finite temperature, imagine that we left out the sum over the discrete time-like momentum $p_0 = 2\pi nT$, the Matsubara frequencies.

$$\partial_t U_k(\rho) = \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)T\sum_{n=-\infty}^{\infty}\frac{Z_k}{Z_kp_0^2+U_k'(\rho)+2\rho U_k''(\rho)+Z_kk^2}\,, \tag{B.11}$$

$$= \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)T\sum_{n=-\infty}^{\infty}\frac{1}{(2\pi nT)^2+\tilde{E}_k^2}\,, \tag{B.12}$$

$$= \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{2\tilde{E}_k}\coth(\frac{\tilde{E}_k}{2T})\,, \tag{B.13}$$

$$= \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{\tilde{E}_k}\left(\frac{1}{2}+\frac{1}{e^{\beta\tilde{E}_k}-1}\right), \tag{B.14}$$

$$= \frac{k^{d+2}S_{d-1}}{d(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{2\tilde{E}_k}\left(1+2n_B\left(\tilde{E}_k\right)\right) \tag{B.15}$$

where the Bose-Einstein distribution,n_B has emerged,

$$n_B(x) = \frac{1}{e^{\beta\, x} - 1} \tag{B.16}$$

and we introduced the renormalized energy $\tilde{E}_k^2(\rho) = Z_k^{-1}\left(U_k'(\rho) + 2\rho U_k''(\rho)\right) + k^2$.

B.2 $O(N)$ models

For LPA', we work with a truncation of the form

$$\Gamma_k[\phi] = \int_{\mathbf{x}} \left\{ U_k(\rho) + \frac{1}{2} Z_k (\partial\phi)^2 \right\} \tag{B.17}$$

where ϕ is an N component field and the wave function renormalization Z_k is allowed to run. Much of the results of the previous section carry through here. From the above ansatz we have

$$\Gamma^{(2)}_{ij,k}[\phi, q, p]\Big|_{\phi=\text{uni}} = \left\{ U_k'(\rho) + \delta_{i0} 2\rho U_k''(\rho) + Z_k\, q^2 \right\} \delta_{ij}\, (2\pi)^d \delta^d(q+p). \tag{B.18}$$

The flow for the effective potential is found to be

$$\begin{aligned} \partial_t U_k(\rho) &= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{\partial_t R_k(q)}{U_k'(\rho) + 2\rho U_k''(\rho) + Z_k q^2 + R_k(q)} \right. \\ &+ \left. (N-1)\, \frac{\partial_t R_k(q)}{U_k'(\rho) + Z_k q^2 + R_k(q)} \right\}. \end{aligned} \tag{B.19}$$

Using the Litum regulator $R_k(q) = Z_k\,(k^2 - q^2)\Theta(k^2 - q^2)$, we have a similar result from the last section:

$$\partial_t U_k(\rho) = \frac{k^{d+2} S_{d-1}}{d(2\pi)^d} \left(1 - \frac{\eta_k}{d+2}\right) \left\{ \frac{Z_k}{m_{\sigma,k}^2 + Z_k k^2} + (N-1)\, \frac{Z_k}{m_{\pi,k}^2 + Z_k k^2} \right\} \tag{B.20}$$

Adding back in the Matsubara sums,

$$\begin{aligned} \partial_t U_k(\rho) &= \frac{k^{d+2} S_{d-1}}{d(2\pi)^d} \left(1 - \frac{\eta_k}{d+2}\right) \left\{ \frac{1}{2\tilde{E}_{\sigma,k}(\rho)} \left(1 + 2 n_B\left(\tilde{E}_{\sigma,k}\right)\right) \right. \\ &+ \left. (N-1)\, \frac{1}{2\tilde{E}_{\pi,k}} \left(1 + 2 n_B\left(\tilde{E}_{\pi,k}\right)\right) \right\}, \end{aligned} \tag{B.21}$$

where again Bose-Einstein distribution, n_B is defined as

$$n_B(x) = \frac{1}{e^{\beta\, x} - 1} \tag{B.22}$$

and we introduced the renormalized energies $\tilde{E}_{i,k}^2 = Z_k^{-1} m_{i,k}^2 + k^2$.

B.3 Expression for anomalous dimension

In what follows, we identify the running anomalous dimensions from the flow of the transverse wavefunction renormalization:

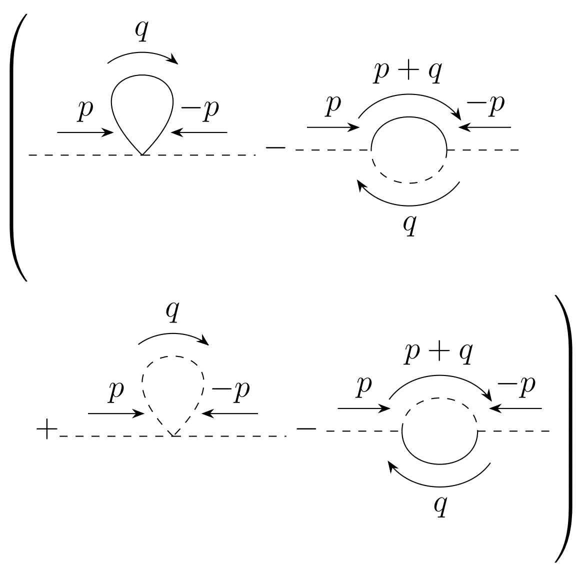
$$\begin{aligned} Z_k &= \lim_{p\to 0} \frac{1}{2}\frac{\partial^2}{\partial p^2}\left\{\frac{\partial^2 \partial_t\, \Gamma_k}{\partial\, \pi_i(p)\, \partial \pi_i(-p)}\right\}\,, && \text{(B.23)} \\ &= \lim_{p\to 0} \frac{1}{2}\frac{\partial^2}{\partial p^2}\left(\partial_t \Gamma_k^{(2),\perp}\right)\,. && \text{(B.24)} \end{aligned}$$

Define the anomalous dimension as $\partial_t Z_k = -\eta_k\, Z_k$. In what follows, we will use

$$\tilde{\partial}_t = \int d^d l\, \partial_t R_k(l^2) \frac{\delta}{\delta R_k(l^2)}\,. \tag{B.25}$$

Given the diagrammatic expression

$$\partial_t \Gamma_k^{(2),\perp}(p) = \frac{1}{2}\int \bar{d}^d q\, \tilde{\partial}_t$$



(B.26)

and the observation that at LPA and LPA', we have $\partial_p \Gamma_k^{(3)} = \partial_p \Gamma_k^{(4)} = 0$, things drastically simplify. At these orders the only momentum dependence lies in the propagators and then the only contribution to the wavefunction renormalization's flow is due to loops with external momenta flowing through the loops. In other words,

$$\begin{aligned}
\partial_t Z_k &= \lim_{p\to 0}\frac{1}{2}\,\tilde{\partial}_t\int_{\mathbf{q}}\frac{\partial}{\partial p^2}\Bigg(-2G_\sigma\Gamma^{(3)}_{\pi\sigma\pi}G_\pi\Gamma^{(3)}_{\pi\pi\sigma}\Bigg)\,, && \text{(B.27)}\\
&= -\frac{1}{2}(\Gamma^{(3)}_{\pi\sigma\pi})^2\,\tilde{\partial}_t\int_{\mathbf{q}}G_\pi\lim_{p\to 0}\Bigg(\partial_{p^2}G_\sigma((\mathbf{p}+\mathbf{q})^2)\Bigg)\,. && \text{(B.28)}
\end{aligned}$$

Following through with the properties we have derived

$$\begin{aligned}
\partial_t Z_k &= -(\Gamma^{(3)}_{\pi\sigma\pi})^2\lim_{p\to 0}\tilde{\partial}_t\frac{\partial}{\partial p^2}\int_{\mathbf{q}}G_\pi\Bigg(G_\sigma(q^2)+(p^2+2\mathbf{p}\cdot\mathbf{q})G'_\sigma(q^2)+2(\mathbf{p}\cdot\mathbf{q})^2\,G''(q^2)\Bigg)\,,\\
&= -(\Gamma^{(3)}_{\pi\sigma\pi})^2\lim_{p\to 0}\tilde{\partial}_t\frac{\partial}{\partial p^2}\Bigg(\int_{\mathbf{q}}G_\pi G_\sigma+p^2\int_{\mathbf{q}}G_\pi G'_\sigma+\frac{2p^2}{d}\int_{\mathbf{q}}q^2\,G''\Bigg)\,, && \text{(B.29)}\\
&= -(\Gamma^{(3)}_{\pi\sigma\pi})^2\,\tilde{\partial}_t\Bigg(\int_{\mathbf{q}}G_\pi G'_\sigma+\frac{2}{d}\int_{\mathbf{q}}q^2\,G_\pi G''_\sigma\Bigg)\,, && \text{(B.30)}\\
&= -Z_k(\sigma U''_k)^2\,4c_d k^{d-6}\tilde{\partial}_t\Bigg(\int_y y^{\frac{d}{2}-1}G_\pi(y)G'_\sigma(y)+\frac{2}{d}\int_y y^{\frac{d}{2}}G_\pi(y)G''_\sigma(y)\Bigg) && \text{(B.31)}
\end{aligned}$$

using $\int\frac{d\Omega_d}{(2\pi)^d}=4c_d$. In the above we moved to a dimensionless variable $y=\frac{q^2}{k^2}$ and defined

$$G_i(y) = \Bigg(y(1+r(y))+\bar{m}_i^2\Bigg)^{-1} \qquad \text{(B.32)}$$

where $R_k = Z_k\,k^2 y r(y)$ for general regulators. Performing integration by parts on the G''_σ term gives (the boundary term vanishes)

$$\begin{aligned}
\partial_t Z_k &= -4c_d Z_k\,\bar{\rho}(\bar{U}''_k)^2\tilde{\partial}_t\Bigg(\int_y y^{\frac{d}{2}-1}G_\pi G'_\sigma-\int_y y^{\frac{d}{2}-1}G_\pi G'_\sigma-\frac{2}{d}\int_y y^{\frac{d}{2}}G'_\pi G'_\sigma\Bigg) && \text{(B.33)}\\
&= \frac{8c_d}{d}Z_k\,\bar{\rho}(\bar{U}''_k\bar{\rho})^2\tilde{\partial}_t\int_y y^{\frac{d}{2}}G'_\pi G'_\sigma\,. && \text{(B.34)}
\end{aligned}$$

We can obtain a more explicit expression given the following relations:

$$G_i(y) = \left(y(1+r(y)) + \bar{m}_i^2\right)^{-1}, \tag{B.35}$$

$$G_i'(y) = -G_i^2(y)\left(1 + r + yr'\right), \tag{B.36}$$

$$\tilde{\partial}_y G(y) = -G^2(y) \times -(\eta_k r + 2yr')y\,, \tag{B.37}$$

$$\tilde{\partial}_y G^n(y) = ny(\eta_k r + 2yr')G^{(n+1)}(y)\,, \tag{B.38}$$

$$\tilde{\partial}_y (r') = (\tilde{\partial}_y r)' = -(\eta r + 2yr')'\,, \tag{B.39}$$

$$\tilde{\partial}_y (1 + r + yr') = -(\eta_k r + y(4+\eta_k)r' + 2y^2 r'')\,. \tag{B.40}$$

Using the above expressions,

$$\begin{aligned}\frac{\partial_t Z_k}{Z_k} &= \frac{8c_d}{d}\bar{\rho}(\bar{U}_k''(\bar{\rho}))^2 \int_y y^{\frac{d}{2}} G_\pi^2 G_\sigma^2 (1+r+yr')\Big((G_\pi + G_\sigma)(1+r+yr')2y(\eta_k r + 2yr') \\ &- 2(\eta_k r + (\eta_k+4)yr' + 2y^2 r'')\Big)\,, \end{aligned}\tag{B.41}$$

$$\begin{aligned}\frac{\partial_t Z_k}{Z_k} &= \frac{16c_d}{d}\bar{\rho}(\bar{U}_k''(\bar{\rho}))^2 \int_y y^{\frac{d}{2}} G_\pi^2 G_\sigma^2 (1+r+yr')\Big((G_\pi + G_\sigma)(1+r+yr')y(\eta_k r + 2yr') \\ &- (\eta_k r + (\eta_k+4)yr' + 2y^2 r'')\Big)\,. \end{aligned}\tag{B.42}$$

Using $\eta_k = -\dfrac{\partial_t Z_k}{Z_k}$,

$$\begin{aligned}\eta_k &= -\frac{16c_d}{d}\bar{\rho}(\bar{U}_k''(\bar{\rho}))^2 \int_y y^{\frac{d}{2}} G_\pi^2 G_\sigma^2 (1+r+yr')\Big((G_\pi + G_\sigma)(1+r+yr')y(\eta_k r + 2yr') \\ &\quad -(\eta_k r + (\eta_k+4)yr' + 2y^2 r'')\Big)\,, \end{aligned}\tag{B.43}$$

$$= \frac{16v_d}{d}\bar{\rho}(\bar{U}_k''(\bar{\rho}))^2\, m_{2,2}^d \tag{B.44}$$

where

$$\begin{aligned}m_{2,2}^d &= -\int_y y^{\frac{d}{2}} G_\pi^2 G_\sigma^2 (1+r+yr')\Big((G_\pi + G_\sigma)(1+r+yr')y(\eta_k r + 2yr') \\ &\quad -(\eta_k r + (\eta_k+4)yr' + 2y^2 r'')\Big)\,. \end{aligned}\tag{B.45}$$

Using the optimized theta regulator $r(y) = (\frac{1}{y} - 1)\Theta(1-y)$, we have

$$\eta_k \quad = \quad \frac{16c_d}{d}\bar{\rho}\left(\bar{U}_k''(\bar{\rho})\right)^2 \frac{1}{\left(1+\bar{U}_k'(\bar{\rho})\right)^2 \left(1+\bar{U}_k'(\bar{\rho})+2\bar{\rho}\,\bar{U}_k''(\bar{\rho})\right)^2}\,, \tag{B.46}$$

or in terms of the masses

$$\eta_k \quad = \quad \frac{16c_d}{d}\bar{\rho}\left(\bar{U}_k''(\bar{\rho})\right)^2 \frac{1}{\left(1+\bar{m}_\pi^2\right)^2 \left(1+\bar{m}_\sigma^2\right)^2}\,. \tag{B.47}$$

To obtain the result for finite temperature, we again add in by hand the Matsubara sum and p_0 frequency to the masses ($\bar{p}_0$ since we are dimensionless):

$$\eta_k \quad = \quad \frac{16c_d}{d}\bar{\rho}\left(\bar{U}_k''(\bar{\rho})\right)^2 \frac{T}{k}\sum_{n=-\infty}^{\infty} \frac{1}{\left(1+\bar{p}_0^2+\bar{m}_\pi^2\right)^2 \left(1+\bar{p}_0^2+\bar{m}_\sigma^2\right)^2}\,. \tag{B.48}$$

APPENDIX

C

DERIVATION OF $\mathcal{O}\left(\partial^2\right)$ FLOWS

C.1 Vertex Functions in Momentum Space

Starting from the $O(\partial^2)$ expansion of the effective action,

$$\Gamma_k^{\partial^2}[\phi] \quad = \quad \int_r \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho)(\partial\phi)^2 + \frac{1}{4} Y_k(\rho)(\partial\rho)^2 \right\}, \tag{C.1}$$

we can begin deriving the expressions for the vertex functions needed to evaluate the diagrammatic expressions:

$$\begin{aligned}
\Gamma^{(1)}_{k,i}[\phi] &= \phi_i\, U'_k(\rho) - \frac{1}{2}\phi_i Z'_k(\rho)\,(\partial\phi)^2 - Z_k(\rho)\partial^2\phi_i \\
&\quad -\frac{1}{4}\phi_i\, Y'_k(\rho)\,(\partial\rho)^2 - \frac{1}{2}Y_k(\rho)\,\partial^\mu\phi_i\partial_\mu\rho - \frac{1}{2}Y_k(\rho)\phi_i\partial^2\rho\,, \qquad \text{(C.2)}\\
\Gamma^{(2)}_{k,ij}[\phi] &= \delta_{ij}U'_k(\rho) + \phi_i\phi_j U''_k(\rho) - \frac{1}{2}\delta_{ij}Z'_k(\rho)\,(\partial\phi)^2 - \frac{1}{2}\phi_i\phi_j Z''_k(\rho)\,(\partial\phi)^2 \\
&\quad -\frac{1}{2}\phi_i Z'_k(\rho)\partial_\mu\phi_j\partial^\mu\delta(x-x') - Z_k(\rho)\partial^2\delta(x-x') \\
&\quad -\frac{1}{4}\delta_{ij}Y'_k(\rho)(\partial\rho)^2 - \frac{1}{4}\phi_i\phi_j Y''_k(\rho)(\partial\rho)^2 - \frac{1}{4}\phi_i\phi_j\partial_\mu\rho\partial^\mu\delta(x-x') \\
&\quad -\frac{1}{2}\phi_j Y'_k(\rho0\partial^\mu\phi_i\partial_\mu\rho - \frac{1}{2}Y_k(\rho)\partial^\mu\delta_{ij}\delta(x-x')\partial_\mu\rho - \frac{1}{2}\phi_j Y_k(\rho)\partial^\mu\phi_i\partial_\mu\delta(x-x') \\
&\quad -\frac{1}{2}\delta_{ij}Y_k(\rho)\partial^2\rho - \frac{1}{2}\phi_i\phi_j Y'_k(\rho)\partial^2\rho - \frac{1}{2}Y_k(\rho)\phi_i\phi_j\partial^2\delta(x-x')\,, \qquad \text{(C.3)}
\end{aligned}$$

$$\Gamma^{(2)}_{k,\perp}[\phi,q,p]\big|_{\phi=uni} = \Big(U'_k(\rho) + Z_k(\rho)\,p^2\Big)(2\pi)^d\,\delta^d(q+p)\,, \qquad \text{(C.4)}$$

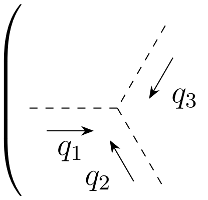
$$\Gamma^{(2)}_{k,||}[\phi,q,p]\big|_{\phi=uni} = \Big(U'_k(\rho) + 2\rho\,U''_k(\rho) + \big(Z_k(\rho) + \rho\,Y_k(\rho)\big)\,p^2\Big)(2\pi)^d\,\delta^d(q+p)\,. \qquad \text{(C.5)}$$

We can drop the delta functions and write the reduced vertex functions. Below we also introduce $Z_{\perp,k} = Z_k$ and $Z_{\|,k} = Z_{\perp,k} + \rho Y_k(\rho)$:

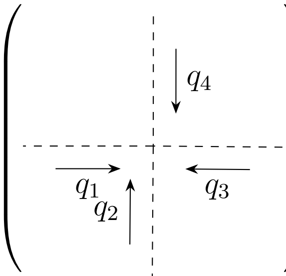
$$\begin{aligned}
\Gamma^{(2)}_{k,\perp}(p) &= \frac{1}{\phi}U'_k(\phi) + Z_{\perp,k}(\phi)\,p^2\,, \qquad \text{(C.6)}\\
\Gamma^{(2)}_{k,||}(p) &= U''_k(\phi) + Z_{\|,k}\,p^2\,. \qquad \text{(C.7)}
\end{aligned}$$

For the three and four point functions, we first define

$$\Gamma^{(3)}_{ijk}(q_1, q_2, q_3) = \left(\quad \right), \tag{C.8}$$



$$\Gamma^{(4)}_{ijkl}(q_1, q_2, q_3, q_4) = \left(\quad \right). \tag{C.9}$$



With the above convention, the general three-point function for a uniform field is found to be

$$\begin{aligned}
\Gamma^{(3)}_{k,ijk}(q_1, q_2, q_3) \quad = \quad & \Bigg\{ \frac{\partial^3 U_k}{\partial\phi_i \partial\phi_j \partial\phi_k} \\
& - Z_k'(\rho) \left(\phi_i \delta_{jk} q_2 \cdot q_3 + \phi_j \delta_{ki} q_1 \cdot q_3 + \phi_k \delta_{ij} q_1 \cdot q_2\right) \\
& - \frac{1}{2} Y_k'(\rho) \phi_i \phi_j \phi_k \left(q_1 \cdot q_2 + q_1 \cdot q_3 + q_2 \cdot q_3\right) \\
& - \frac{1}{2} Y_k(\rho) \left(q_1 \cdot q_2 \left(\phi_i \delta_{jk} + \phi_j \delta_{ik}\right) + q_2 \cdot q_3 \left(\phi_j \delta_{ik} + \phi_k \delta_{ij}\right)\right. \\
& \qquad \left. + q_1 \cdot q_3 \left(\phi_i \delta_{jk} + \phi_k \delta_{ij}\right)\right) \Bigg\} (2\pi)^d \delta(q_1 + q_2 + q_3) \,.
\end{aligned} \tag{C.10}$$

For the reduced three-point function and taking $q_3 = -(q_1 + q_2)$,

$$\begin{aligned}
\Gamma^{(3)}_{k,ijk}(q_1, q_2) \quad = \quad & \Bigg\{ \frac{\partial^3 U_k}{\partial\phi_i \partial\phi_j \partial\phi_k} \\
& + Z_k'(\rho) \left(\phi_i \delta_{jk} \left(q_1 \cdot q_2 + q_2^2\right) + \phi_j \delta_{ik} \left(q_1^2 + q_1 \cdot q_2\right) - \phi_k \delta_{ij} q_1 \cdot q_2\right) \\
& + \frac{1}{2} Y_k'(\rho) \phi_i \phi_j \phi_k \left(q_1^2 + q_2^2 + q_1 \cdot q_2\right) \\
& + \frac{1}{2} Y_k(\rho) \left(\phi_i \delta_{jk} q_1^2 + \phi_j \delta_{ki} q_2^2 + \phi^k \delta_{ij} \left(q_1^2 + q_2^2 + 2 q_1 \cdot q_2\right)\right) \Bigg\}.
\end{aligned} \tag{C.11}$$

The expressions for the four point function are much more cumbersome. We will split the contributions from U_k, Z_k, and Y_k. The U_k is somewhat trivial. For Z_k on a uniform field, we obtain

$$
\begin{aligned}
\Gamma^{(4)}_{ijkl} \sim -\frac{1}{2}\Bigg\{ & 2\left(\delta_{kl} Z'_k(\rho) + \phi_k \phi_l Z''_k(\rho)\right) \delta_{ij} q_1 \cdot q_2 \\
& +2\left(\delta_{jl} Z'_k(\rho) + \phi_j \phi_l Z''_k(\rho)\right) \delta_{ik} q_1 \cdot q_3 \\
& +2\left(\delta_{jk} Z'_k(\rho) + \phi_j \phi_k Z''_k(\rho)\right) \delta_{ik} q_1 \cdot q_4 \\
& +2\left(\delta_{il} Z'_k(\rho) + \phi_i \phi_l Z''_k(\rho)\right) \delta_{jk} q_2 \cdot q_3 \\
& +2\left(\delta_{ik} Z'_k(\rho) + \phi_i \phi_k Z''_k(\rho)\right) \delta_{jl} q_2 \cdot q_4 \\
& +2\left(\delta_{ij} Z'_k(\rho) + \phi_i \phi_j Z''_k(\rho)\right) \delta_{kl} q_3 \cdot q_4 \Bigg\}.
\end{aligned} \tag{C.12}
$$

The contribution from Y_k on a uniform field yields

$$
\begin{aligned}
\Gamma^{(4)}_{ijkl} \sim -\frac{1}{2}\Bigg\{ & Y''_k(\rho)\phi_i\phi_j\phi_k\phi_l \left(q.q_2 + q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& +Y_k(\rho)\delta_{jk}\delta_{il}\left(q_1 \cdot q_2 + q_1 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& +Y_k(\rho)\delta_{ik}\delta_{jl}\left(q_1 \cdot q_2 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4\right) \\
& +Y_k(\rho)\delta_{ij}\delta_{kl}\left(q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4\right) \\
& +Y'_k(\rho)\left(\delta_{ij}\phi_k\phi_l + \delta_{ik}\phi_j\phi_l + \delta_{il}\phi_j\phi_k + \delta_{jk}\phi_i\phi_l + \delta_{jl}\phi_i\phi_k + \delta_{kl}\phi_i\phi_j\right) \\
& \quad \times \left(q_1 \cdot q_2 + q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& \quad - Y'_k(\rho)\left(q_1 \cdot q_2 \delta_{ij}\phi_k\phi_l + q_1 \cdot q_3 \delta_{ik}\phi_j\phi_l + q_1 \cdot q_4 \delta_{il}\phi_j\phi_k\right. \\
& \qquad \left. q_2 \cdot q_3 \delta_{jk}\phi_i\phi_l + q_2 \cdot q_4 \delta_{jl}\phi_i\phi_k + q_3 \cdot q_4 \delta_{kl}\phi_i\phi_j\right) \Bigg\}.
\end{aligned} \tag{C.13}
$$

All together for the four-point function on a uniform field, we obtain

$$
\begin{aligned}
\Gamma^{(4)}_{ijkl} = \Bigg\{ & \frac{\partial^4 U_k}{\partial\phi_i \partial\phi_j \partial\phi_k \partial\phi_l} \\
- \Bigg\{ & \left(\delta_{kl} Z'_k(\rho) + \phi_k \phi_l Z''_k(\rho)\right) \delta_{ij} q_1 \cdot q_2 \\
& + \left(\delta_{jl} Z'_k(\rho) + \phi_j \phi_l Z''_k(\rho)\right) \delta_{ik} q_1 \cdot q_3 \\
& + \left(\delta_{jk} Z'_k(\rho) + \phi_j \phi_k Z''_k(\rho)\right) \delta_{ik} q_1 \cdot q_4 \\
& + \left(\delta_{il} Z'_k(\rho) + \phi_i \phi_l Z''_k(\rho)\right) \delta_{jk} q_2 \cdot q_3 \\
& + \left(\delta_{ik} Z'_k(\rho) + \phi_i \phi_k Z''_k(\rho)\right) \delta_{jl} q_2 \cdot q_4 \\
& + \left(\delta_{ij} Z'_k(\rho) + \phi_i \phi_j Z''_k(\rho)\right) \delta_{kl} q_3 \cdot q_4 \Bigg\} \\
-\frac{1}{2}\Bigg\{ & Y''_k(\rho) \phi_i \phi_j \phi_k \phi_l \left(q . q_2 + q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& + Y_k(\rho) \delta_{jk} \delta_{il} \left(q_1 \cdot q_2 + q_1 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& + Y_k(\rho) \delta_{ik} \delta_{jl} \left(q_1 \cdot q_2 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4\right) \\
& + Y_k(\rho) \delta_{ij} \delta_{kl} \left(q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4\right) \\
& + Y'_k(\rho) \left(\delta_{ij} \phi_k \phi_l + \delta_{ik} \phi_j \phi_l + \delta_{il} \phi_j \phi_k + \delta_{jk} \phi_i \phi_l + \delta_{jl} \phi_i \phi_k + \delta_{kl} \phi_i \phi_j\right) \\
& \quad \times \left(q_1 \cdot q_2 + q_1 \cdot q_3 + q_1 \cdot q_4 + q_2 \cdot q_3 + q_2 \cdot q_4 + q_3 \cdot q_4\right) \\
& \quad - Y'_k(\rho) \left(q_1 \cdot q_2 \delta_{ij} \phi_k \phi_l + q_1 \cdot q_3 \delta_{ik} \phi_j \phi_l + q_1 \cdot q_4 \delta_{il} \phi_j \phi_k \right. \\
& \qquad \left. q_2 \cdot q_3 \delta_{jk} \phi_i \phi_l + q_2 \cdot q_4 \delta_{jl} \phi_i \phi_k + q_3 \cdot q_4 \delta_{kl} \phi_i \phi_j\right) \Bigg\} \Bigg\} (2\pi)^d \delta(q_1 + q_2 + q_3 + q_4) \,.
\end{aligned}
\tag{C.14}
$$

In summary, evaluating the vertices on uniform fields $\phi_i = \phi\delta_{i1}$:

$$\Gamma^{(3)}_{\perp\perp\parallel}(p,q) =(p+q)^2\,\frac{1}{2}\phi Y_k(\phi) - p\cdot q\, Z'_{\perp,k}(\phi) + \left(\frac{1}{\phi}U''_k(\phi) - \frac{1}{\phi^2}U'_k(\phi)\right)\,, \tag{C.15}$$

$$\Gamma^{(3)}_{\parallel\perp\perp}(p,q) =Z'_{\perp,k}(\phi)\left(p\cdot q + q^2\right) + \frac{1}{2}\phi Y_k(\phi)p^2 + \left(\frac{1}{\phi}U''_k(\phi) - \frac{1}{\phi^2}U'_k(\phi)\right)\,, \tag{C.16}$$

$$\Gamma^{(3)}_{\parallel}(p,q) =(p^2+q^2+p\cdot q)\, Z'_{\parallel,k}(\phi) + U^{(3)}_k(\phi)\,, \tag{C.17}$$

$$\Gamma^{(4)}_{\parallel}(p,-p,q) =(p^2+q^2)Z''_{\parallel,k}(\phi) + U^{(4)}_k(\phi)\,, \tag{C.18}$$

$$\Gamma^{(4)}_{\parallel\parallel\perp\perp}(p,-p,q) =q^2\, Z''_{\perp,k}(\phi) + p^2\frac{1}{\phi}\left(Z'_{\perp,k}(\phi) + \frac{1}{2}\phi^2\, Y'_k(\phi)\right) + \left(\frac{1}{\phi}U'_k(\phi)\right)''\,, \tag{C.19}$$

$$\Gamma^{(4),i}_{kki}(p,-p,q) =(p^2+q^2)\left((N-1)\frac{1}{\phi}Z'_k(\phi) + Y_k(\phi)\right) + \frac{1}{\phi}\frac{\partial}{\partial\phi}\left[\frac{1}{\phi}U'_k(\phi)\right] \tag{C.20}$$

where in the last line $k\neq 1$ and we summed over the index $i\neq 1$. Lastly, the propagators are given by

$$G_\perp(q^2) \;=\; \left(\frac{1}{\phi}U'_k(\phi) + Z_{\perp,k}(\phi)\,q^2 + R_k\right)^{-1}\,, \tag{C.21}$$

$$G_\parallel(q^2) \;=\; \left(U''_k(\phi) + Z_{\parallel,k}(\phi)\,q^2 + R_k\right)^{-1}\,. \tag{C.22}$$

C.2 Flow of $Z_{\perp,k}$

The flow of $Z_{\perp,k}$ is given as

$$\begin{aligned}\partial_t Z_{\perp,k} =& \tilde{\partial}_t \lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q \Big\{G_\perp(q^2)\,\Gamma^{(4),i}_{kki}(p,-p,q,-q) + G_\parallel(q^2)\,\Gamma^{(4)}_{\perp\parallel}(p,-p,q,-q)\\ &- 2G_\parallel(q^2)\,\Gamma^{(3)}_{\perp\perp\parallel}\left(p,q,-(q+p)\right)\,G_\perp\left((q+p)^2\right)\,\Gamma^{(3)}_{\perp\parallel\perp}(-p,p+q,-q)\Big\}\end{aligned} \tag{C.23}$$

where $k,i\neq 1$ and we sum over i.

The first two terms are the simplest since the momentum derivative only hits the four-point functions. The result is

$$\begin{aligned}&\tilde{\partial}_t \lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q \Big\{G_\perp(q^2)\,\Gamma^{(4)}_{\perp}(p,-p,q,-q) + G_\parallel(q^2)\,\Gamma^{(4)}_{\perp\parallel}(p,-p,q,-q)\Big\}\\ =\;& \tilde{\partial}_t\int_q\left\{G_\perp\left((N-1)\,\frac{1}{\phi}Z'_{\perp,k}(\phi) + Y_k\right) + G_\parallel Z''_{\parallel,k}(\phi)\right\}\,.\end{aligned} \tag{C.24}$$

For the last term, we use some symmetries of the full three-point function to write

$$\Gamma^{(3)}_{\perp\parallel\perp}\left(-p,(p+q),-q\right) = \Gamma^{(3)}_{\perp\perp\parallel}\left(-p,-q,p+q\right), \tag{C.25}$$

$$= \Gamma^{(3)}_{\perp\perp\parallel}\left(p,q,-(p+q)\right). \tag{C.26}$$

This allows us to rewrite the last term to have a factor of the form

$$G_\perp\left((q+p)^2\right)\left(\Gamma^{(3)}_{\perp\perp\parallel}(p,q)\right)^2. \tag{C.27}$$

Using

$$G((q+p)^2) = G(q^2)+(p^2+2p\cdot q)G'(q^2)+2(p\cdot q)^2G''(q^2)+O(p^3), \tag{C.28}$$

$$\Gamma^{(3)}_{\perp\perp\parallel}(p,q) = \Gamma^{(3)}_0+Z_k'(\rho)\phi(p^2+p\cdot q) \tag{C.29}$$

$$\Gamma^{(3)}_0 = \phi U_k''(\rho)+\frac{1}{2}\phi Y_k(\rho)q^2 \tag{C.30}$$

we can write

$$\begin{aligned} G_\perp\left((q+p)^2\right)\left(\Gamma^{(3)}_{\perp\perp\parallel}(p,q)\right)^2 =& p^2\left(2G_\perp Z_k'(\rho)\phi\Gamma^{(3)}_0+G_\perp'\left(\Gamma^{(3)}_0\right)^2\right) \\ & (q\cdot p)^2\left(2G_\perp Z_k'(\rho)^2\phi^2+4G_\perp' Z_k'(\rho)\phi\,\Gamma^{(3)}_0+2G_\perp''\left(\Gamma^{(3)}_0\right)^2\right) \\ & +\mathcal{O}\left(q\cdot p,\,p^2(q\cdot p),\,p^4\right). \end{aligned} \tag{C.31}$$

In the above, we can ignore terms odd in q as they will vanish under the integral. Using the further relation

$$\int_q (p\cdot q)^2 f(q^2) = \frac{p^2}{d}\int_{\mathbf{q}} q^2 f(q^2), \tag{C.32}$$

we can simplify the expression to

$$\begin{aligned} G_\perp\left((q+p)^2\right)\left(\Gamma^{(3)}_{\perp\perp\parallel}(p,q)\right)^2 =& p^2\left(2G_\perp Z_k'(\rho)\phi\Gamma^{(3)}_0+G_\perp'\left(\Gamma^{(3)}_0\right)^2\right) \\ & \frac{p^2q^2}{d}\left(G_\perp Z_k'(\rho)^2\phi^2+4G_\perp' Z_k'(\rho)\phi\,\Gamma^{(3)}_0+2G_\perp''\left(\Gamma^{(3)}_0\right)^2\right) \\ & +\mathcal{O}\left(q\cdot p,\,p^2(q\cdot p),\,p^4\right). \end{aligned} \tag{C.33}$$

All together we then have

$$\lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q G_\parallel G_\perp\left((q+p)^2\right)\left(\Gamma^{(3)}_{\perp\perp\parallel}(p,q)\right)^2=\int_q G_\parallel\Bigg\{2G_\perp Z'_k(\rho)\phi\Gamma^{(3)}_0+G'_\perp\left(\Gamma^{(3)}_0\right)^2$$
$$\frac{q^2}{d}\left(G_\perp Z'_k(\rho)^2\phi^2+4G'_\perp Z'_k(\rho)\phi\,\Gamma^{(3)}_0+2G''_\perp\left(\Gamma^{(3)}_0\right)^2\right)\Bigg\} \tag{C.34}$$

and the total flow is given by

$$\begin{aligned}\partial_t Z_{\perp,k}=&\tilde{\partial}_t\int_q\Bigg\{G_\perp\left((N-1)\frac{1}{\phi}Z'_{\perp,k}(\phi)+Y_k\right)+G_\parallel Z''_{\parallel,k}(\phi)\\&-2G_\parallel\Bigg\{2G_\perp Z'_k(\rho)\phi\left(\phi U''_k(\rho)+\frac{1}{2}Y_k(\rho)\phi q^2\right)+G'_\perp\left(\phi U''_k(\rho)+\frac{1}{2}Y_k(\rho)\phi q^2\right)^2\\&\frac{q^2}{d}\left(G_\perp Z'_k(\rho)^2\phi^2+4G'_\perp Z'_k(\rho)\phi\left(\phi U''_k(\rho)+\frac{1}{2}Y_k(\rho)\phi q^2\right)\right.\\&\left.+2G''_\perp\left(\phi U''_k(\rho)+\frac{1}{2}Y_k(\rho)\phi q^2\right)^2\right)\Bigg\}\Bigg\}.\end{aligned} \tag{C.35}$$

C.3 Flow of $Z_{\parallel,k}$

The flow of $Z_{\parallel,k}$ is given as

$$\begin{aligned}\partial_t Z_{\parallel,k}=&\tilde{\partial}_t\lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q\Big\{G_\parallel(q^2)\,\Gamma^{(4)}_\parallel(p,-p,q,-q)+(N-1)G_\perp(q^2)\,\Gamma^{(4)}_{\parallel\perp}(p,-p,q,-q)\\&-G_\parallel(q^2)\,\Gamma^{(3)}_\parallel\left(p,q,-(q+p)\right)\,G_\parallel\left((q+p)^2\right)\,\Gamma^{(3)}_\parallel(-p,p+q,-q)\\&-(N-1)G_\perp(q^2)\,\Gamma^{(3)}_{\parallel\perp\perp}\left(p,q,-(q+p)\right)\,G_\perp\left((q+p)^2\right)\,\Gamma^{(3)}_{\parallel\perp\perp}(-p,p+q,-q)\Big\}\end{aligned} \tag{C.36}$$

Again, the tadpole contributions are the simplest, giving

$$\begin{aligned}&\lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q\Big\{G_\parallel(q^2)\,\Gamma^{(4)}_\parallel(p,-p,q,-q)+(N-1)G_\perp(q^2)\,\Gamma^{(4)}_{\parallel\perp}(p,-p,q,-q)\Big\}\\&\quad=\int_q\Bigg\{G_\parallel(q^2)\left(Z'_{\parallel,k}(\rho)+2\rho Z''_{\parallel,k}(\rho)\right)+(N-1)G_\perp(q^2)\left(Z'_k(\rho)+2\rho Y'_k(\rho)\right)\Bigg\}.\end{aligned} \tag{C.37}$$

Now for the third term. Let us use another place holder Γ_0 and Γ_1

$$\Gamma^{(3)}_{k,\|}(p,q) = \Gamma_0 + \Gamma_1 \left(p^2 + p\cdot q\right) \ , \tag{C.38}$$

$$\Gamma_0 = \sqrt{2\rho}\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) + \sqrt{2\rho}\left(Z'_k(\rho) + \rho Y'_k(\rho) + Y_k(\rho)\right) q^2 \,, \tag{C.39}$$

$$= \sqrt{2\rho}\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) + \sqrt{2\rho} Z'_{\|,k}(\rho) q^2 \,, \tag{C.40}$$

$$\Gamma_1 = \sqrt{2\rho}\left(Z'_k(\rho) + \rho Y'_k(\rho) + Y_k(\rho)\right) \ , \tag{C.41}$$

$$= \sqrt{2\rho} Z'_{\|,k}(\rho) \,. \tag{C.42}$$

Again, using symmetries of the three-point function, we can write and expand the factor

$$\begin{aligned} G_\|\left((q+p)^2\right) \quad \left(\Gamma^{(3)}_\|(p,q)\right)^2 &= p^2\left(2G_\|\Gamma_0\Gamma_1 + G'_\|\left(\Gamma_0\right)^2\right) \\ &+ \tfrac{p^2q^2}{d}\left(G_\|\left(\Gamma_1\right)^2 + 4G'_\|\,\Gamma_0\,\Gamma_1 + 2G''_\|\left(\Gamma_0\right)^2\right) \\ &+ \mathcal{O}\left(\, q\cdot p,\ p^2(q\cdot p),\ p^4\right) \ , \end{aligned} \tag{C.43}$$

or so

$$\begin{aligned} &\lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q G_\|(q^2) G_\|\left((q+p)^2\right) \ \left(\Gamma^{(3)}_\|(p,q)\right)^2 \\ &= \int_q G_\| \Bigg\{ 4\rho G_\|\left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\|,k}(\rho) + Z'_{\|,k}(\rho)^2 q^2\right) \\ &+ 2\rho G'_\|\left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right)^2 + 2\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\|,k}(\rho)\, q^2 + Z'^2_{\|,k}(\rho) q^4\right) \\ &+ \frac{q^2}{d}\left(2\rho G_\| Z'^2_{\|,k}(\rho) + 8\rho G'_\| Z'_{\|,k}(\rho)\left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) + Z'_{\|,k}(\rho) q^2\right)\right) \\ &+ \frac{q^2}{d} 4\rho G''_\|\left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right)^2 + 2\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\|,k}(\rho)\, q^2 + Z'^2_{\|,k}(\rho)\, q^4\right) \Bigg\} . \end{aligned} \tag{C.44}$$

Now for the fourth term, we do the same as for the third term. Let us use the place holders γ, $\gamma_{1,1}$, and $\gamma_{1,p}$:

$$\Gamma^{(3)}_{k,\|\perp\perp}(p,q) = \gamma_0 + \gamma_{1,p}p^2 + \gamma_{1,q}\, q\cdot p\,, \tag{C.45}$$

$$\gamma_0 = \sqrt{2\rho}U_k''(\rho) + \sqrt{2\rho}Z_k'(\rho)q^2\,, \tag{C.46}$$

$$\gamma_{1,q} = \sqrt{2\rho}Z_k'(\rho)\,, \tag{C.47}$$

$$\gamma_{1,p} = \frac{1}{2}\sqrt{2\rho}Y_k(\rho)\,. \tag{C.48}$$

We then have the expression for the fourth term

$$\begin{aligned} G_\perp\left((q+p)^2\right) \quad \left(\Gamma^{(3)}_{\|\perp\perp}(-p,q)\right)^2 &= p^2\left(G_\perp 2\gamma_0\gamma_{1,p} + G'_\perp\gamma_0^2\right) \\ &+ \tfrac{p^2q^2}{d}\left(G_\perp\gamma_{1,q}^2 + 4G'_\perp\gamma_0\gamma_{1,q} + 2G''_\perp\gamma_0^2\right) \\ &+ \mathcal{O}\left(\, q\cdot p,\, p^2(q\cdot p),\, p^4\right)\,, \end{aligned} \tag{C.49}$$

or so

$$\begin{aligned} &\lim_{p\to 0}\frac{\partial}{\partial p^2}\int_q G_\perp(q^2)G_\perp\left((q+p)^2\right)\,\left(\Gamma^{(3)}_{\|\perp\perp}(-p,q)\right)^2 \\ &= \int_q G_\perp\Bigg\{2\rho G_\perp\left(U_k''(\rho)+Z_k'(\rho)q^2\right)Y_k(\rho) + 2\rho G'_\perp\left(U_k''(\rho)+Z_k'(\rho)q^2\right)^2 \\ &+ \frac{q^2}{d}\left(2\rho G_\perp Z_k'^2(\rho) + 8\rho G'_\perp Z_k'(\rho)\left(U_k''(\rho)+Z_k'(\rho)q^2\right) + 4\rho G''_\perp\left(U_k''(\rho)+Z_k'(\rho)q^2\right)^2\right)\Bigg\}\,. \end{aligned} \tag{C.50}$$

All together then, the flow of $Z_\parallel$ is given by

$$\begin{aligned}
\partial_t Z_{\parallel,k} = \tilde{\partial}_t \int_q \Bigg\{ & G_\parallel(q^2)\left(Z'_{\parallel,k}(\rho) + 2\rho Z''_{\parallel,k}(\rho)\right) + (N-1)G_\perp(q^2)\left(Z'_k(\rho) + 2\rho Y'_k(\rho)\right) \\
& - G_\parallel \Bigg\{ 4\rho G_\parallel \left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\parallel,k}(\rho) + Z'_{\parallel,k}(\rho)^2 q^2\right) \\
& + 2\rho G'_\parallel \left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right)^2 + 2\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\parallel,k}(\rho)\, q^2 + Z'^2_{\parallel,k}(\rho) q^4\right) \\
& + \frac{q^2}{d}\left(2\rho G_\parallel Z'^2_{\parallel,k}(\rho) + 8\rho G'_\parallel Z'_{\parallel,k}(\rho)\left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) + Z'_{\parallel,k}(\rho) q^2\right)\right) \\
& + \frac{q^2}{d} 4\rho G''_\parallel \left(\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right)^2 + 2\left(3U''_k(\rho) + 2\rho U'''_k(\rho)\right) Z'_{\parallel,k}(\rho)\, q^2 + Z'^2_{\parallel,k}(\rho)\, q^4\right) \Bigg\} \\
& - (N-1)G_\perp \Bigg\{ 2\rho G_\perp \left(U''_k(\rho) + Z'_k(\rho) q^2\right) Y_k(\rho) + 2\rho G'_\perp \left(U''_k(\rho) + Z'_k(\rho) q^2\right)^2 \\
& + \frac{q^2}{d}\left(2\rho G_\perp Z'^2_k(\rho) + 8\rho G'_\perp Z'_k(\rho)\left(U''_k(\rho) + Z'_k(\rho) q^2\right) + 4\rho G''_\perp \left(U''_k(\rho) + Z'_k(\rho) q^2\right)^2\right) \Bigg\} \Bigg\}
\end{aligned} \tag{C.51}$$

APPENDIX

D

ON THE NON-PERTURBATIVE NATURE OF THE ϵ-EXPANSION

For the sake of simplicity, in this section we consider a special case of $N = 1$. Our conclusions trivially extend to an arbitrary N.

Consider the Widom equation of state (see e.g. Ref. [437]):

$$\begin{aligned}
f(x) =& 1 + x \\
&+ \frac{1}{6}\epsilon\left(-x\ln\left(\frac{27}{4}\right) + (x+3)\ln(x+3) - 3\ln(3)\right) \\
&+ \frac{1}{648}\epsilon^2\left(9(x+9)\ln^2(x+3) + 50(x+3)\ln(x+3) + \ldots\right) \\
&+ \epsilon^3\left(\frac{\ln^2(x+3)(675+246x+25x^2)}{1944(x+3)} + \ldots\right)
\end{aligned} \tag{D.1}$$

where in the two and three loop contributions we explicitly displayed only a few principal terms. We also need the derivative

$$f'(x) = 1 + \frac{1}{6}\epsilon\left(\ln\left(\frac{4(x+3)}{27}\right)+1\right) + \frac{1}{648}\epsilon^2\left(\frac{4(17x+78)\ln(x+3)}{x+3}+\ldots\right)$$
$$+\epsilon^3\left(\frac{(25x(x+6)+63)\ln^2(x+3)}{1944(x+3)^2}+\ldots\right). \tag{D.2}$$

For the location of the singularity at one-loop order we then have

$$\beta\delta f(x_c) - x_c f'(x_c) = \frac{x+3}{2} + \frac{\epsilon}{12}\left((x+9)\log(x+3) - x\left(2+\log\frac{27}{4}\right) - 9\log 3\right) = 0 \tag{D.3}$$

with the approximate solution $x_c = -3 + \epsilon\left(\ln\frac{1}{\epsilon}+\ldots\right)$ where the ellipses include a constant term, nested logarithms and their ratios.

Evaluating $f'(x)$ at $x_c = -3+\epsilon\ln\frac{1}{\epsilon}$ reveals the problem. In Eq. (D.2) let us separately consider the two loop

$$\frac{1}{648}\epsilon^2\left(\frac{4(17x+78)\ln(x+3)}{x+3}+\ldots\right)\Bigg|_{x\to -3+\epsilon\ln\frac{1}{\epsilon}} = -\frac{\epsilon}{6} - \frac{\epsilon\log\left(\log\left(\frac{1}{\epsilon}\right)\right)}{6\log(\epsilon)} + \ldots \tag{D.4}$$

and the three-loop terms

$$\epsilon^3\left(\frac{(25x(x+6)+63)\ln^2(x+3)}{1944(x+3)^2}+\ldots\right)\Bigg|_{x\to -3+\epsilon\ln\frac{1}{\epsilon}} = -\frac{\epsilon}{12} - \frac{\epsilon\log\left(\log\left(\frac{1}{\epsilon}\right)\right)}{6\log(\epsilon)} + \ldots \tag{D.5}$$

We see that despite the loop counting, both contributions are of order ϵ (that is the same as the one loop term)! Higher order loop terms also contaminate ϵ^1 order and similar terms involving nested logarithms. This necessitates all loop order resummation and thus brings us to the main conclusion of this appendix that the corrections to the location of YLE singularity are not perturbative in ϵ. Note that this does not prevent us from extracting the leading order correction to z_c. This correction is of order of ϵ; higher order contributions suffer from the non-perturbative contribution.