ABSTRACT

DELLINGER, BRIAN JACOB. Mechanical Equivalents to Hybrid Automata. (Under the direction of Dr. Jon Doyle.)

Hybrid automata have been proposed for modeling systems with interoperating discrete and continuous parts, but the resulting system abstracts away all mechanism in describing the influence of each part on the other. While this distances the system from any specific implementation, it also obscures any mechanical character of the interactions themselves. I show that, given reasonable conditions, a hybrid automaton can be recast as a *hybrid mechanical* system in which the influence of both the discrete and continuous parts of the system on each other are represented as forces between mechanical bodies. Formally, given a hybrid automaton A, I provide a method for constructing a hybrid mechanical system M_A such that the behaviors of M_A are isomorphic to those of A. I illustrate this transformation in terms of selected applications.

Mechanical Equivalents to Hybrid Automata

by
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BIOGRAPHY

Brian Dellinger was born in High Point, NC, the son of Robert and Wanda Dellinger. He completed his undergraduate work at Grove City College in 2007, double majoring in Computer Science and Mathematics, before beginning his graduate studies at North Carolina State University under Dr. Jon Doyle. In 2012, Brian married Dr. Allison Dellinger; a year later, the two returned to Grove City College, where Brian teaches Computer Science as an assistant professor.

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Thanks as well to the members of my committee, Dr. Donald Bitzer, Dr. Ronald Endicott, and Dr. Robert St. Amant, for their patience and help as I slowly moved towards the finish line. I’m particularly indebted to Don, who gave me my first chance to really teach, and who helped me confirm that yes, this is what I want to do with the rest of my life.

When I entered grad school, Allison Kling was my closest friend. Today, my wife, Dr. Allison Dellinger (née Kling), is my constant support, and her love and patience have helped these last few years speed by. I’m thankful as well for my parents: my father Robert, the original Dr. Dellinger, and my mother Wanda. I love you all.

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Introduction

1.1 Hybrid systems

Many modern devices can be divided into two parts: the physical device itself, and the computer or other *controller* that determines the device's actions.A mechanical lift or elevator, for instance, might be programmed to cut on or off as its platform moves; the pulleys, engines, and environment could all be considered physical components of the elevator, while the program that switches it on or off would be the controller. As computers become smaller and more ubiquitous, examples of such *hybrid systems* are likely to be ever more commonplace.

One might describe the physical portion of a hybrid system using a set of numbers: its position, its temperature, and so on. For this portion, time is a continuum. As time passes, these descriptive numbers may change through a continuous range: a car rolls forward, a furnace slowly cools, and so on. By contrast, a computational controller is normally thought of as experiencing a succession of discrete instants, perhaps corresponding to the ticks of a computer's internal processor. In each discrete instant, the controller's state is one of only finitely many possibilities. A light switch, for instance, is either on or off; a computer's memory may be described by any of a great many binary strings, but the options are still ultimately finite. The combined hybrid system, then, is by parts both continuous and discrete, with each portion potentially affecting the behavior of the other.

Hybrid automata [15, 2, 5, 18] are a common means of modeling such systems. A hybrid automaton's states can be divided into a continuous part, representing the physical system, and a discrete part, representing the controller. Each controller state defines a regime of smooth motion for the continuous portion; that is, the controller state determines the possible changes over time in the continuous part. Entering certain regions of the continuous state space may trigger a discontinuous state change in both the controller and continuous portions, with the new controller state defining a new smooth motion for the continuous state.

To expand the example of an automatic elevator, and following the example in [15], a hybrid au-

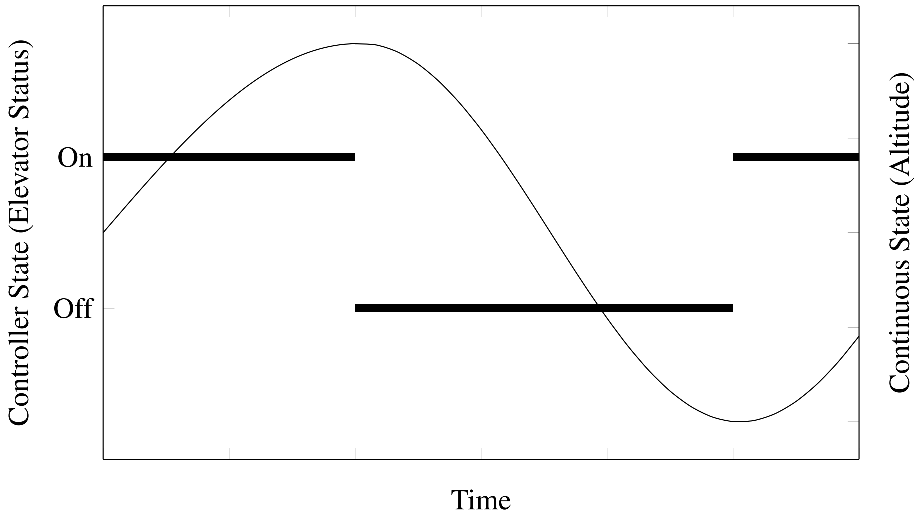


Figure 1.1: Status over time of the hybrid automaton for an elevator. As long as the controller is *On*, the platform rises; once the platform reaches a certain point, the controller transitions to *Off*, and the platform begins to drop again. When the platform drops below some point, the controller transitions back to *On*, and the process repeats.

tomaton might represent the elevator's altitude as a single real-valued variable that changes continuously over time. The discrete part of the automaton, representing the state of the elevator's controller, consists of only two states, *Off* and *On*. The continuous portion can trigger changes in the discrete part in order to keep the attached platform from rising too high or dropping too low. Thus, whenever the platform rises above a certain point, the elevator transitions from *On* to *Off*. As the platform drops, the system eventually switches back. The discrete state, in turn, determines how the platform rises (when *On*) or falls (when *Off*) over time. Figure 1.1 illustrates an example of this process.

1.2 Problem and motivation

A hybrid automaton thus requires two very different sorts of interactions. Its continuous portion can be viewed as a mechanical system, with motion through the continuous state space as the result of a field of forces over the space. The controller, on the other hand, is an automaton; the influence of the continuous portion on its states is one of state transitions rather than mechanical changes. A potential disadvantage of this formalization is that it separates the mechanics of the automaton portion from the bodies it is meant to influence. Thus, while the controller's state affects the forces on the continuous part, the controller itself has no overtly mechanical interaction with the rest of the world. The interaction by which the motion of the continuous portion produces discrete changes in the controller is similarly lacking in detail.

These separate rules for changes in the discrete and continuous portions of the automaton, and the lack of a clear means of interaction between them, obscure any mechanical character of the original system. So, for instance, one might wish to examine mechanical properties of an elevator, applying classic results from physics or related disciplines to make statements about its future behavior without the need to actually perform simulations of that behavior. In a hybrid systems representation, however, it is unclear how such results might be applied, given the separate rules for the discrete and continuous portions. It is this problem which I will address in the following pages. In particular, I will define a single set of mechanical interactions producing all behaviors of an arbitrary hybrid automaton, with this construction forming a foundation for future analysis of automaton behavior.

A secondary objective of this approach is to permit more formal investigation of mind-body interactions. A common modern theory of mind views the mind as a kind of computer; from the perspective of hybrid systems, one might view the mind as a controller (or set of controllers) and the body as the continuous portion acted upon by the controllers. In effect, the mind's state determines the actions of the body; actions alter the observed world, in turn creating changes in the mind, which produce new actions, and so on. As with hybrid automata, however, the languages used for mind and body differ wildly; one can characterize physical actions in mechanical or physics terminology, which is generally disjoint from the terminology of beliefs, desires, and intentions that is sometimes used to describe mental states. By characterizing hybrid automata via a single set of interactions, I hope to provide a common mechanics-based description which might be used to characterize mind-body interactions. It bears emphasizing that this work does not, itself, attempt to characterize minds in this common language; it is purely my objective here to construct a unified description for hybrid automaton, which might permit such mind-body descriptions in the future.

1.3 Approach

In this dissertation, I show that one can describe both components of a hybrid automaton as parts of the same universe, exerting forces on each other and the outside world via mechanical laws. Further, I show that it is possible to describe changes in both the discrete and continuous portions as the results of these forces. Such a model offers a clear mechanical interpretation of two portions' interactions; furthermore, it preserves this mechanical character without any dependence on details beyond those present in the hybrid automaton.

In the longer term, a fully mechanical interpretation might offer additional benefits. Classical mechanics is a well-studied field, with many known results that allow greater insight into a system's behavior. Once hybrid systems are cast in terms of mechanics, it might be possible to translate some of these results into hybrid equivalents, providing similar intuitions for control theory. Several theories of mind can also be cast in hybrid systems terms, suggesting further applications for research in artificial intelligence.

To describe hybrid automata in fully mechanical terms, I turn to *hybrid mechanics* [12, 9]. Where rational mechanics [19, 23] typically describes a single universe of bodies, spaces, and forces, hybrid mechanics describes the interactions of two or more sub-universes, with each one having its own bodies, spaces, and so on. So, for instance, a body in one sub-universe might exert forces on bodies in another sub-universe. Hybrid automata, then, could be re-characterized as systems joining a controller universe and a physical universe, with the respective state spaces becoming simply spaces in which various bodies are located. State changes in a hybrid automaton would then correspond to movement in the mechanical universe, and interactions between the automaton's continuous and discrete portions could be expressed as forces between bodies in both universes.

1.4 Contribution and results

1.4.1 Formalization of conditional force systems

To support mechanical representation of a hybrid automaton, I provide a formal treatment of *conditional force systems*, that is, systems in which the forces for the physical sub-universe depend on the states of the mechanical controller sub-universe. As part of this treatment, I provide a formal definition of mechanical controller states, showing how motions of the mechanical universe correspond to changes in the mechanical controller state over time. I then demonstrate conditions under which one can isolate the effects of each individual mechanical controller, viewing the forces of the entire system as the sum of the contributions of each individual controller.

1.4.2 Formalization of controller-mediated forces

I show that, in many cases, forces that vary based on the state of a controller can be regarded as forces of that controller on other bodies. More importantly, under certain conditions this view of forces does not change the resultant force active on any body. In other words, it is possible in many cases to represent the influence of the controller of a hybrid system as *hybrid forces* of that controller on the physical portion; I say in such cases that the controller *mediates* these hybrid forces. Given an arbitrary conditional force system satisfying the conditions for mediation, I construct a hybrid force system in which all changes in forces are mediated and prove that it preserves all resultant forces of the original system.

1.4.3 Hybrid mechanical transformation of hybrid automata

Using the notion of a conditional force system, I provide a transformation from an arbitrary hybrid automaton to a hybrid mechanical system such that the behaviors of the two systems are isomorphic. To do so, I detail controller and physical bodies, spaces, times, and masses; I then construct state sets for all mechanical controllers and define the force systems conditional on each state. Next, I show that

$$\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "beh"] & M(A) \arrow[d, "beh"] \\
beh(A) \arrow[r, leftrightarrow, "\sim"] & beh(M(A))
\end{tikzcd}$$

Figure 1.2: A graphical illustration of the objective of this research. Given an arbitrary hybrid automaton A, I construct a transformation M such that $M(A)$ is a hybrid mechanical system that is equivalent in behavioral terms to A. In particular, I have $beh(A) \sim beh(M(A))$, that is, there is a one-to-one correspondence between behaviors in the two sets.

certain controllers in the resulting mechanical system satisfy the conditions for force mediation, and so that these controllers can mediate forces on the physical portion.

1.4.4 Definition and proof of hybrid mechanical bisimulation of hybrid automata

I formally define a notion of hybrid mechanical *bisimulation* of a hybrid automaton; that is, I provide conditions sufficient to show that the behaviors of the two systems are isomorphic. More formally, let $\mathcal{A}$ be the set of all hybrid automata, $\mathcal{M}$ be the set of all hybrid mechanical systems, $\mathcal{H}$ be the set of *behaviors* (or possible histories of system states), and $beh : (\mathcal{A} \cup \mathcal{M}) \to 2^{\mathcal{H}}$ be a function mapping any hybrid automaton or hybrid mechanical system to the set of its behaviors. I provide a transformation function $M : \mathcal{A} \to \mathcal{M}$ such that, for any hybrid automaton A in $\mathcal{A}$ satisfying certain natural requirements, $M(A)$ is a hybrid mechanical system that is provably equivalent to A, that is, a system such that there is a bijection from $beh(A)$ to $beh(M(A))$. Figure 1.2 illustrates the resulting correspondence.

Finally, I prove that these conditions hold for the provided transformation, and so that the behaviors of the resulting mechanical system are indeed isomorphic to those of the original automaton.

1.5 Plan of the exposition

In the remainder of this chapter, I summarize existing work in hybrid mechanics, hybrid automata, or bisimulation, followed by a brief summary of basic mathematical concepts needed for the remaining work (Chapter 2). I then present detailed descriptions of hybrid automata (Chapter 3) and rational mechanical systems (Chapters 4 - 5), paying particular attention to the subclass of hybrid mechanical systems that focus on controller-plant interactions (Chapter 6). With the relevant terms defined, I then describe a function transforming an arbitrary hybrid automaton to a hybrid mechanical system (Chapters 7 and 8).

Having presented the mechanical transformation, I present several examples of the transformation in action (Chapter 9), detailing both classic hybrid automata and the resultant mechanical systems. Next, I provide a further conversion to show that controller-dependent forces in this transformation can be assigned to the controllers themselves (Chapter 10). I then formally define a notion of mechanical bisimulation (Chapter 11) and prove the that the transformation presented in Chapter 7 is correct, that is, that there the original hybrid automaton and the corresponding hybrid mechanical system bisimulate each other (Chapter 12). Finally, I expand on applications and extensions of the transformation in computer science and philosophy (Chapter 13).

1.6 Related work

In the following paragraphs, I briefly summarize the basics of hybrid automata and hybrid mechanical systems. A full formal summary of hybrid automata as described in [3, 18] and others would require more space and mathematical exposition than is desirable at this point; thus, I describe these formalizations only briefly here, deferring a full treatment to Chapters 2 - 3. Likewise, I defer a full description of hybrid mechanics to Chapters 4 - 6.

1.6.1 Hybrid automata

Hybrid systems evolve in part out of *Discrete Event Systems* [20, 21]. A Discrete Event System (DES) is a timed automaton with an attached event generator. Unlike hybrid automata, a DES has only a single state space; instead of altering in response to movement in the continuous space, the automaton changes state in response to the generator, which irregularly produces some event. In control theory terms, the timed automaton is the *controller* and the symbol generator is the *plant*. In a *controlled DESs*, certain events can be "turned off" to guarantee that they will not be generated until turned on again; a controlled DES, then, allows for more careful management of which states are entered.

Hybrid automata [3, 15] extend DESs by replacing the event generator with a set of variables whose values change over time in ways dependent on the current controller state. Instead of spontaneously generating events, certain values of the continuous variables can cause state changes in the automaton. These changes may or may not be deterministic, or a particular hybrid automaton may have a mix of deterministic and nondeterministic transitions. Multiple works provide real-world examples suitable for modeling as hybrid systems, with the list of examples including computer disk drives [7], automated cars and traffic-management systems [24], smart power grids [25], and protein regulation in organisms [14].

Timed automata [4] are a major subcategory of hybrid automata in which all discrete transitions are based on the value of one or more *clock variables*, where a clock variable begins at a value of 0 and increases continuously at a rate of 1 unit per second. *Stopwatch automata* relax this requirement slightly, allowing clock variables to "pause" and remain constant for periods of time.

A major question for hybrid automata is that of *reachability*, that is, of which states of the automaton can actually be produced by the possible inputs. Unfortunately, the reachability problem is undecidable for many even comparatively simple automata [2]. Further discussion of various kinds of timed automata, as well as a more detailed summary of the major decidability results, can be found in [22].

Hybrid input/output automata [18], or *HIOAs*, further revise this picture by designating variables as either *inputs* or *outputs*. Inputs variables here are those whose values determine the states of the controller; output variables, by contrast, are those whose values are determined by the controller's state. Given several independent hybrid systems and certain restrictions, the HIOA formalism allows two or more automata to be *composed* into a single, combined automaton. The work that follows derives in part from HIOAs, as the input/output division parallels the structure of the derived hybrid mechanical systems. As well, automata composition is a natural starting place for discussing combinations of controllers, another subject of interest.

Switched systems [6] isolate the continuous portion of a hybrid system. Rather than model the automaton, a switched system considers only some position function $x(t)$ over $\mathbb{R}^n$, where n is a positive integer. At any given instant, $x(t)$ evolves according to one of several functions $f_1, f_2, \ldots f_N$, where each such function maps values of $x(t)$ onto derivatives. A switched system can transition from one such function to another either based on evolution of the continuous state, as in a hybrid automaton, or based on some external command. Discrete transitions are thus more abstract than in a hybrid automaton.

1.6.2 Hybrid mechanics

Much of the mechanical framing of what follows descends from *rational continuum mechanics*, particularly as laid out in [23]. Broadly, rational mechanics provides a mathematical description of time, space, bodies, forces, and other related concepts. Description in rational mechanics typically works from the very general to the specific; so, for instance, one develops theorems and laws that apply to bodies of any sort, and then further develops laws for those bodies possessing a particular quality (rigidity, say). This top-down description makes rational mechanics particularly easy to adapt to sets of assumptions that differ from those typical in real-world physics.

Where rational mechanics describes interactions only within a single universe of bodies, *hybrid mechanics* [11, 12] extends it to cover two or more possibly-overlapping universes. Bodies in such a framing might simultaneously be members of two or more universes, with each universe having its own spatial dimensions, and bodies in one universe might exert forces on bodies in another. A hybrid case of particular note is the two-universe case where one universe consists of *physical* bodies and the other of *mental* or *controller* bodies. Such a mechanical framing might be used to model mind-body interactions as forces of one universe on another. More recent work [9] provides a mathematical basis for extending controller-dependent mechanical forces to hybrid forces acting on the controllers themselves.

Chapter 2

Mathematical Foundations

The following sections outline some of the basic notation used in later portions of this paper. I borrow here heavily from Lynch [18], and those familiar with her notation can largely skip this section.

2.1 Intervals

Let U be a totally ordered set with ordering $\leq$. An *interval* J of U is a convex, nonempty subset of U; that is, an interval must contain at least one element, and for any two elements $j_1, j_2 \in J$, all elements of U between j_1 and j_2 must also be in J. Given $j_1, j_2 \in J$, with $j_1 \leq j_2$, one writes $[j_1, j_2]$ to indicate the interval from j_1 to j_2; that is, $[j_1, j_2] = \{u \in U \mid j_1 \leq u \leq j_2\}$. One writes $min(J)$ and $max(J)$ to indicate the minimum and maximum elements of J, if they exist. An interval is *left-closed* if it has a minimum element; otherwise, it is *left-open*. Similarly, an interval is *right-closed* (*right-open*) if it has (does not have) a maximum element. An interval that is both left-closed and right-closed is a *closed* interval.

Given an interval J and an element $j' \in U$ such that $j + j'$ and $j - j'$ are defined for all elements $j \in J$, one writes $J + j'$ to indicate the interval such that

$$J + j' \stackrel{\text{def}}{=} \{j + j' \mid j \in J\}. \tag{2.1}$$

Likewise, one writes $J - j'$ to indicate the interval such that

$$J - j' \stackrel{\text{def}}{=} \{j - j' \mid j \in J\}. \tag{2.2}$$

In general, intervals can be defined as subsets of many kinds of sets. In what follows, I am primarily concerned with intervals in time; thus, I will most commonly consider intervals of ordered subsets of $\mathbb{R}$, with ordering $\leq$ defined in the usual way for real numbers.

2.2 Functions, limits, and differentiability

Given a function f, I write $dom(f)$ to indicate the domain of f. For any set U, the *restriction* of f to U is the function g such that $dom(g) = dom(f) \cap U$ and, for all x in $dom(g)$, $g(x) = f(x)$. I write the restriction of f to U as $f \lceil U$.

As usual, one says that a function f over an open interval J is *differentiable* at some $j \in J$ iff the value

$$\lim_{x \to j} \frac{f(x) - f(j)}{x - j} \tag{2.3}$$

exists. One says that f is differentiable on J iff f is differentiable at every such $j \in J$; in such cases, one writes $\dot{f}$ for the derivative of f. Thus, for all $j \in J$,

$$\dot{f}(j) \stackrel{\text{def}}{=} \lim_{x \to j} \frac{f(x) - f(j)}{x - j}. \tag{2.4}$$

One says that a function is *twice differentiable* just in case its derivative is also differentiable, writing $\ddot{f}$ for this derivative of the derivative, or *second derivative*.

Less commonly, one may speak of *left-* or *right-differentiable* functions. Let J be an interval, f a function over J, and j a point in J. Recall that one says that the limit of f at j is L,

$$\lim_{x \to j} f(x) = L, \tag{2.5}$$

iff for any $\epsilon > 0$, there exists some $\delta > 0$ such that, whenever

$$0 < |x - j| < \epsilon, \tag{2.6}$$

it must also be the case that

$$|f(x) - f(j)| < \delta. \tag{2.7}$$

In other words, as one chooses points x arbitrarily close to j, the value $f(x)$ must also become arbitrarily close to $f(j)$. One can also speak of *the limit from the left*, or *from the right*, in which one considers only points smaller than j (from the left) or larger than j (from the right) in determining the limit. Thus, the limit from the left of f at j is L,

$$\lim_{x \to j^-} f(x) = L, \tag{2.8}$$

iff for any $\epsilon > 0$, there exists some $\delta > 0$ such that, whenever

$$0 < j - x < \epsilon, \tag{2.9}$$

it must also be the case that

$$|f(x) - f(j)| < \delta. \tag{2.10}$$

Likewise, the limit from the right of f at j is L,

$$\lim_{x \to j^+} f(x) = L, \tag{2.11}$$

iff for any $\epsilon > 0$, there exists some $\delta > 0$ such that, whenever

$$0 < x - j < \epsilon, \tag{2.12}$$

it must also be the case that

$$|f(x) - f(j)| < \delta. \tag{2.13}$$

Note that $\lim_{x \to j} f(x) = L$ exists iff both the limit from the left and the limit from the right at j exist and agree. In general, however, the limits from the left and right need not agree. Consider, for instance, the absolute value function $f(x) = |x|$ defined over the real numbers. The slopes from the left and right as one approaches 0 do not agree, and so

$$\begin{aligned} \lim_{x \to 0^-} \dot{f}(0) &= -1 \\ \lim_{x \to 0^+} \dot{f}(0) &= 1, \end{aligned} \tag{2.14}$$

Then both the limit from the left and from the right exist at 0, but since they do not agree, the limit does not exist.

Conceptually, to restrict attention to left- or right-differentiability is to restrict (2.4) to the limit from the left or the right, respectively. Thus, a function f over an open or left-open, right-closed interval J is left-differentiable at some $j \in J$ iff the function

$$\dot{f}_-(x) \stackrel{\text{def}}{=} \lim_{x \to j^-} \frac{f(x) - f(j)}{x - j} \tag{2.15}$$

is defined at j, and one says that f is left-differentiable on J iff it is left-differentiable at every $j \in J$. Similarly, a function f over some open or right-open, left-closed interval J is right-differentiable at some $j \in J$ iff the function

$$\dot{f}_+(x) \stackrel{\text{def}}{=} \lim_{x \to j^+} \frac{f(x) - f(j)}{x - j} \tag{2.16}$$

is defined at j, and one says that f is right-differentiable on J iff it is right-differentiable at every $j \in J$. If a function is left- or right-differentiable, one says that it is *semidifferentiable*.

Note that the above definition of derivative is given only for open intervals. Closed intervals pose something of a complication, since at either endpoint of an interval, the function may not be defined over values approaching the endpoint from one direction or the other. This problem can be resolved by restricting attention to left- or right-differentiability. Given a function f over a left-closed interval J with infimum j_0, one says that f is differentiable at j_0 iff it is right-differentiable at j_0; in this case,

$$\dot{f}(j_0) \stackrel{\text{def}}{=} \lim_{x \to j_0^+} \frac{f(x) - f(j_0)}{x - j_0}. \tag{2.17}$$

Likewise, given a function f over a right-closed interval J with supremum j_1, one says that f is differentiable at j_1 iff it is left-differentiable at j_1, and

$$\dot{f}(j_1) \stackrel{\text{def}}{=} \lim_{x \to j_1^-} \frac{f(x) - f(j_1)}{x - j_1}. \tag{2.18}$$

No special treatment is needed to handle a left-differentiable function over a right-closed (but left-open) interval; the left-derivative at the right endpoint is defined as normal. Similar statements may be made for a right-differentiable function over a left-closed (but right-open) function. One cannot, however, define a left-derivative at a left endpoint of the interval, since any points "to the left of" that endpoint necessarily lie outside the interval itself. Again, one may make similar statements for a right-derivative at the right-endpoint of an interval.

If a function is differentiable over its entire domain, one says simply that it is differentiable. Likewise, if a function is left-differentiable over its entire domain (except its left endpoint, if it exists), one says that it is left-differentiable; if it is right-differentiable over its entire domain (except its right endpoint, if it exists), one says that it is right-differentiable.

2.3 Manifolds

In general terms, a *manifold* is a topological space that locally resembles $\mathbb{R}^n$. For instance, if one looks at only a very small section of a sphere's surface at a time, the surface appears very nearly flat; in other words, a small section of the sphere's surface resembles the plane $\mathbb{R}^2$. Thus, a sphere is a manifold locally resembling $\mathbb{R}^2$. A *differentiable* manifold is a manifold that is sufficiently like $\mathbb{R}^n$ that calculus can be performed on it. For interested readers, [1] provides a much more comprehensive treatment of the subject.

More formally, a manifold is a topological space $\mathfrak{M}$, with every point in that space associated with a *local coordinate system*, or *chart*. Given a point $x \in \mathfrak{M}$, the chart for x is a local neighborhood U of that point, plus a homeomorphism $h : U \to \mathbb{R}^n$ that maps every element in the local neighborhood to a subset of the vector space $\mathbb{R}^n$. If the charts for two points in $\mathfrak{M}$ overlap, there must be some homeomorphism that translates one coordinate system into the other. In a n-manifold, every chart maps to a subset of a n-dimensional vector space. More complex manifolds may have different charts map to subsets of different-dimensional vector spaces, but such complex spaces are not treated here. A differentiable manifold, then, is a manifold for which every chart's mapping, and the mappings between overlapping charts, are *diffeomorphisms*; that is, these mappings are differentiable bijections whose inverses are also differentiable.

Given a manifold $\mathfrak{M}$ and an interval J, a *curve* in $\mathfrak{M}$ is a continuous function $f : J \to \mathfrak{M}$. For any chart h, then, he composition $h \circ f$ is then a curve through $\mathbb{R}^n$. If this function $h \circ f$ is differentiable, then one writes $\dot{f}$ to indicate its derivative in the chart h.

Let $\mathfrak{M}$ be a differentiable n-manifold and x a point in $\mathfrak{M}$. A *tangent* at x is a pair (x, v), where v is in $\mathbb{R}^n$; intuitively, one can often think of v as the derivative of a curve in $\mathfrak{M}$ as seen in some chart. The *tangent space* at x, written $T_x\mathfrak{M}$, is the set of all such tangents of x; since a chart maps points in the manifold to $\mathbb{R}^n$, then, the tangent space at x is isomorphic to $\mathbb{R}^n$. More formally,

$$T_x\mathfrak{M} = \{x\} \times \mathbb{R}^n. \tag{2.19}$$

The *tangent bundle* of $\mathfrak{M}$, written $T\mathfrak{M}$, is the union of all such tangent spaces for all points in $\mathfrak{M}$. That is,

$$T\mathfrak{M} \stackrel{\text{def}}{=} \bigsqcup_{x \in \mathfrak{M}} T_x\mathfrak{M}. \tag{2.20}$$

In the simple case I treat here, the tangent bundle is all pairs (x, v) such that x is in $\mathfrak{M}$ and v is a vector in $\mathbb{R}^n$.

2.3.1 Vector fields

Given a manifold $\mathfrak{M}$, a *vector field* on $\mathfrak{M}$ is a mapping $\mathbf{v} : \mathfrak{M} \to T\mathfrak{M}$ that associates each point $x \in \mathfrak{M}$ with some tangent in $T_x\mathfrak{M}$. In other words, for any point $x \in \mathfrak{M}$, the tangent $\mathbf{v}(x) = (x, v)$ is in $T_x\mathfrak{M}$. One writes $\mathfrak{X}(\mathfrak{M})$ to denote the set of all vector fields for $\mathfrak{M}$.

Let $\chi : \mathfrak{M} \to \mathfrak{N}$ be a mapping between manifolds. A vector field $\mathbf{v} : \mathfrak{M} \to T\mathfrak{N}$ is said to *cover* χ iff, for any $x \in \mathfrak{M}$,

$$\mathbf{v}(x) \in T_{\chi(x)}\mathfrak{N}. \tag{2.21}$$

In other words, $\mathbf{v}$ maps every point $x \in \mathfrak{M}$ to a tangent in the tangent bundle of $\chi(x)$. If χ is invertible, then the function $\boldsymbol{v} : \mathbf{v} \circ \chi^{-1}$ is a vector field on $\chi(\mathfrak{M})$.

An *integral curve* for a vector field $\mathbf{v}$ is a continuous, differentiable function $f : J \to \mathfrak{M}$ such that, for every $t \in J$ and any choice of chart h,

$$\mathbf{v}(f(t)) = (f(t), \dot{f}(t)). \tag{2.22}$$

Recall that $\dot{f}$ is the derivative of $h \circ f$.

Suppose that f is a mapping from an interval in time; one might wish that the vector field varied with time, so that different instants in the interval would produce different vector fields. If J is a time interval, a *time-varying vector field* is a mapping $\mathbf{v} : \mathfrak{M} \times J \to T\mathfrak{M}$ that associates every point $x \in \mathfrak{M}$ and instant $t \in J$ with a tangent in $T_x\mathfrak{M}$. Again, f is an integral curve for a time-varying vector field $\mathbf{v}$ iff it is a continuous, differentiable function such that, for every $t \in J$,

$$(f(t), \dot{f}(t)) = \mathbf{v}(f(t), t). \tag{2.23}$$

2.3.2 Discrete manifolds

The above discussion focuses on continuous and differentiable manifolds, but discrete spaces can also be manifolds. Indeed, every discrete or countable space with the discrete topology is a 0-manifold, that is, a manifold that associates every element in every local neighborhood with $\mathbb{R}^0 = \{0\}$. Since 0 is the only element of $\mathbb{R}^0$, there is only one possible mapping for any point $x \in \mathfrak{M}$. In particular, $\mathbb{R}^0$ itself is a 0-manifold containing only one point, with only one tangent at that point; thus,

$$T\mathbb{R}^0 = T\{0\} = \{0\} \times \{0\} = \{(0, 0)\}. \tag{2.24}$$

2.3.3 Deformations

Let $\mathfrak{M}$ be a manifold, and let U be an open subset of that manifold. A *deformation* of U is a continuous mapping $\chi : J \to (U \to \mathfrak{M})$ that maps every member of an interval J to a mapping from U into $\mathfrak{M}$. In later chapters, I will be concerned where cases where J is an interval in time, and U is the image of a placement of a body in some space; in such cases, a deformation describes a continuous motion of that body over time.

One can view a deformation instead as a mapping $\chi : U \times J \to \mathfrak{M}$, writing $\chi(x, t)$ to indicate the location in $\mathfrak{M}$ to which $x \in U$ is mapped at $t \in J$. It is common to be interested in the mappings of all values in U at a particular $t \in J$, which can be denoted via the function $\chi_t : U \to \mathfrak{M}$. Thus,

$$\chi_t(x) \stackrel{\text{def}}{=} \chi(x, t). \tag{2.25}$$

Likewise, one may be interested in all mappings of a particular point $x \in U$ across all values in J. Again, one denotes the restriction of χ to such a value x via the function $\chi_x : J \to \mathfrak{M}$. That is,

$$\chi_x(t) \stackrel{\text{def}}{=} \chi(x, t). \tag{2.26}$$

2.4 Trajectories

Let U be a totally ordered set with ordering $\leq$. Given a set V and an interval J of U, a *J-trajectory* for V is a mapping $\tau : J \to V$. That is, τ associates each element of J with an element of V. A J-trajectory is said to be *closed* iff J is closed; likewise, a J-trajectory is left- or right-closed if and only if J is left- or right-closed, respectively. For a fixed choice of U, the set of all J-trajectories for V for all possible intervals J in U is called the *trajectories* for V, and any member of that set can be referred to simply as a *trajectory*. A *point trajectory* is a trajectory over a single point, that is, such that $dom(\tau) = [u, u]$ for some element u in U.

I write $\tau.ftime$ to indicate the infimum of the domain of a trajectory τ; that is,

$$\tau.ftime \stackrel{\text{def}}{=} \inf(dom(\tau)). \tag{2.27}$$

Likewise, I write $\tau.ltime$ to indicate the supremum of the domain of τ; that is,

$$\tau.ltime \stackrel{\text{def}}{=} \sup(dom(\tau)). \tag{2.28}$$

If τ is right-closed and $\tau.ltime = c$, then $\tau(c)$ is called the *last state* of τ, denoted $\tau.lstate$. Likewise, if τ is left-closed and $\tau.ftime = c$, then $\tau(c)$ is called the *first state* of τ, denoted $\tau.fstate$. If a trajectory is differentiable, then one can similarly write the derivatives of the first and last points in the trajectory (if they exist), via a slight abuse of notation. If τ is left-closed, then

$$\dot{\tau}.fstate \stackrel{\text{def}}{=} \dot{\tau}(\tau.ftime), \tag{2.29}$$

and if τ is right-closed, then

$$\dot{\tau}.lstate \stackrel{\text{def}}{=} \dot{\tau}(\tau.ltime). \tag{2.30}$$

Let V be a set, and let τ be a left-closed trajectory for V. A *prefix* of τ is some left-closed trajectory τ' such that $dom(\tau') \subseteq dom(\tau)$, with $\tau.fstate = \tau'.fstate$, and $\tau' = \tau\lceil dom(\tau')$. In other words, a prefix of τ is any trajectory for V formed by restricting τ to some sub-interval of its domain that begins at the left endpoint of $dom(\tau)$. One indicates that τ' is a prefix of τ by writing $\tau' \leq \tau$.

2.4.1 Trajectory shifting

Let U be a totally ordered set with ordering $\leq$, and let U be closed under addition such that, for any $a, b, c \in U$,

$$a \leq b \text{ iff } a + c \leq b + c. \tag{2.31}$$

Given a set V, an interval J of U, and a constant $c \in U$, let J' be the subset of U such that j is in J iff $j + c$ is in J'. Note that, by (2.31), J' is an interval.

Let τ be a J-trajectory for V; then I write $\tau + c$ to indicate the J'-trajectory for V such that, for each $j \in J$, the trajectory $\tau + c$ maps $j + c$ to the same value as τ maps j. In other words, if $\tau' = \tau + c$, then for all elements $j \in J$,

$$\tau(j) = \tau'(j + c). \tag{2.32}$$

In effect, $\tau + c$ is the same trajectory as τ, but with its domain shifted forward by c. Likewise, one writes $\tau - c$ to indicate the trajectory formed by shifting the domain of τ backward by c. That is,

$$\tau - c \stackrel{\text{def}}{=} \tau + (-c). \tag{2.33}$$

Given a trajectory τ and some $j \in dom(\tau)$, let I write $\tau \trianglerighteq j$ be the trajectory such that

$$\tau \trianglerighteq j = (\tau\lceil[j, \infty)) + (\tau.ftime - j)). \tag{2.34}$$

In other words, $\tau \trianglerighteq j$ is formed by omitting everything before some point j in the domain of τ, shifting so that the new trajectory has the same left endpoint as the old one. One calls any such trajectory a *suffix* of τ.

2.4.2 Trajectory sets

Before proceeding much further, it will be useful to introduce some further notation for sets of trajectories. Let U be a totally ordered set with ordering $\leq$, closed under addition as in (2.31), and let V be any set. A (U, V)-set of trajectories is any set of trajectories $\mathcal{T}$ such that, for any trajectory $\tau \in \mathcal{T}$, there is an interval J of U such that τ is an J-trajectory for V. In other words, $\mathcal{T}$ is a (U, V)-set of trajectories iff each trajectory in the set maps an interval of U into V. In what follows, I assume that each set of trajectories discussed is a (U, V)-set for appropriate choices of U and V.

A set of left-closed trajectories $\mathcal{T}$ is *prefix closed* (equivalently, has the property of *prefix closure*) if and only if, for any trajectories τ, τ' such that τ is in $\mathcal{T}$ and $\tau' \leq \tau$, it is the case that τ' is in $\mathcal{T}$. That is, a set of left-closed trajectories is prefix closed iff all prefixes of all trajectories in that set are also in the set.

A set of left-closed trajectories $\mathcal{T}$ is *suffix closed* if and only if, for any $\tau \in \mathcal{T}$, for all $j \in dom(\tau)$ it is the case that $\tau \trianglerighteq j$ is in $\mathcal{T}$. That is, a set of left-closed trajectories is suffix closed iff all suffixes of all trajectories in that set are also in the set.

For a trajectory τ and left-closed trajectory τ', the *concatenation* $\tau \frown \tau'$ is the trajectory formed by shifting the domain of τ' forward so that $\tau'.ftime = \tau.ltime$ and combining the two trajectories. In other words, τ' is shifted so that it begins at the point that τ ends. If τ is right-closed, the two trajectories will have a point in common, namely $\tau.ltime$. In this case, τ determines the value at that point.

More formally, let $\tau'' = \tau' + (\tau.ltime - \tau'.ftime)$; that is, let τ'' be τ' shifted so that its domain begins where the domain of τ ends. Then $\tau \frown \tau'$ is defined to be the trajectory such that

$$dom(\tau \frown \tau') = dom(\tau) \cup dom(\tau''), \tag{2.35}$$

where, for any value $j \in dom(\tau \frown \tau')$,

$$(\tau \frown \tau')(j) = \tau(j) \tag{2.36}$$

if j is in $dom(\tau)$, and

$$(\tau \frown \tau')(j) = \tau''(j) \tag{2.37}$$

otherwise.

In [18], Lynch identifies three natural restrictions on the set of trajectories for a hybrid automaton, which I repeat below. Lynch requires all trajectories to be left-closed over an interval beginning at 0, which is more restrictive than the definition of trajectory I use above; therefore, I begin the list with an additional restriction to this effect. For convenience, I refer to any set satisfying these restrictions as a *legal* set of trajectories.

More formally, given appropriate choices of U and V, I say that $\mathcal{T}$ is a legal (U, V)-set (or, where unambiguous, simply that it is legal) if it satisfies the following properties.

First, all trajectories in $\mathcal{T}$ must be left-closed with a common first instant; that is, for any trajectories $\tau, \tau' \in \mathcal{T}$, the element $\tau.ftime$ is in $dom(\tau)$, the element $\tau'.ftime$ is in $dom(\tau')$, and $\tau.ftime = \tau'.ftime$.

Second, $\mathcal{T}$ must be prefix closed. Recall that a set of left-closed trajectories is prefix closed iff restricting any trajectory in the set to any sub-interval of its domain beginning at its left endpoint produces another trajectory in the set; more formally, for any trajectories τ, τ' such that τ is in $\mathcal{T}$ and $\tau' \leq \tau$, it is the case that τ' is in $\mathcal{T}$.

Third, $\mathcal{T}$ must be suffix closed. Recall that a set is suffix closed iff the latter portion of any trajectory in $\mathcal{T}$ is also a trajectory in $\mathcal{T}$; more formally, for any trajectory $\tau \in \mathcal{T}$ and element $j \in dom(\tau)$, it is the case that $\tau \trianglerighteq j$ is in $\mathcal{T}$.

Finally, $\mathcal{T}$ must be *concatenation closed*. Let $\tau_0, \tau_1, \tau_2, \ldots$ be a sequence of trajectories in $\mathcal{T}$ such that, for any non-final index i, τ_i is closed, and $\tau_i.lstate = \tau_{i+1}.fstate$. In other words, the final state of each trajectory matches the first state of the next trajectory in the sequence. One says that $\mathcal{T}$ is concatenation closed iff, for any such sequence, the trajectory $\tau_0 \frown \tau_1 \frown \tau_2 \frown \ldots$ is in $\mathcal{T}$.

Note that, for any appropriate choice of U and V, the empty set is trivially a legal (U, V)-set of trajectories; thus, there is always at least one legal set of trajectories for any appropriate choice of U and V.

2.4.3 Trajectory projection

Given sets X, Y, let V be a subset of $X \times Y$, and let τ be a trajectory for V. I may be interested in, say, the value of the X-component of τ without being concerned with the Y-component; that is, I may be interested in the *projection* of τ onto X.

If V is a subset of $Y_1 \times Y_2 \times \ldots$ for some finite or infinite set of sets $\{Y_1, Y_2, \ldots\}$, then one defines the ith projection function $\Pi_i : V \to Y_i$ to be the function such that, for any value $y \in V$, with $y = (y_1, y_2, \ldots y_i \ldots)$, one has $\Pi_i(y) = y_i$. In other words, the ith projection function restricts V to its ith component. If the component sets over whose product V is defined are named, as in the $V \subseteq X \times Y$ example above, one instead subscripts the projection function with the names of those sets. Thus, in the above example, Π_X and Π_Y are the functions such that, for any $(x, y) \in V$, one has $\Pi_X(x, y) = x$ and $\Pi_Y(x, y) = y$.

One can extend the definition of the projection function to cover trajectories as well as specific values. If V is a subset of $Y_1 \times Y_2 \times \ldots$ and τ is a trajectory for V, then $\Pi_i(\tau)$ is the trajectory of Y_i such that $dom(\Pi_i(\tau)) = dom(\tau)$ and, for any value j in $dom(\tau)$,

$$\Pi_i(\tau)(j) \stackrel{\text{def}}{=} \Pi_i(\tau(j)). \tag{2.38}$$

In other words, $\Pi_i(\tau)$ has at every point the same value as the ith projection of τ.

Finally, given a set U of objects for which a particular projection function Π is defined, one writes $\Pi(U)$ to indicate the set consisting of the projections of all members of U. That is,

$$\Pi(U) \stackrel{\text{def}}{=} \{x \mid \exists y \in U \, [x = \Pi(y)]\}. \tag{2.39}$$

2.4.4 Differentiably legal trajectories

In later chapters, I will wish to describe mechanical bodies whose motions coincide with the trajectories of a hybrid automaton. Motions of a mechanical body depend on the forces on that body, following the familiar Newton-Euler equation

$$f = ma. \tag{2.40}$$

Thus, to define the motion of a body, it is necessary to define the forces on that body, and in particular to define forces proportionate to the desired acceleration of the body. Since I wish the motion of these bodies to match the automaton trajectories, then, the force on the body must be proportionate to the second derivative of a trajectory, and so this second derivative must exist.

More precisely, second-degree differentiability may be a stronger condition than is actually required. If the second derivative of a function exists, it must be continuous. The force on a body, however, need not be continuous; for instance, a body can experience sudden impulse forces, or instantaneous changes in force. Thus, it seems sufficient to require that a trajectory be differentiable at all points in its domain, with a derivative that is both left- and right-differentiable at all points in its domain (except the left and right endpoints, respectively).

For any such function f, let $\ddot{f}_L$, the *left-favoring second derivative*, be the function as follows: for any point in $dom(\tau)$ except the left endpoint, $\ddot{f}_L$ is identical to the left-derivative of the derivative of f.

At that left endpoint, $\ddot{f}_L$ is identical to the right-derivative of the derivative of f. Let $g = \dot{f}$; then, for any $x \in dom(f)$,

$$\ddot{f}_L(x) = \begin{cases} \dot{g}_{-}x, & x > \inf(dom(f)) \\ \dot{g}_{+}x, & x = \inf(dom(f)). \end{cases} \tag{2.41}$$

Note that any function f whose domain is a single point cannot be differentiable, much less second-degree differentiable; in such cases, where x is the one point in the domain of f, I define $\ddot{f}_L(x) = 0$. I say that any such function over whose domain $\ddot{f}_L$ is defined is *second-degree doubly semidifferentiable*.

In general, I drop the subscripted L from such second derivatives, writing simply $\ddot{f}$ to indicate the left-favoring second derivative. When considering derivatives of a trajectory, I again extend the definition of the projection function so that, for any $j \in dom(\tau)$,

$$\Pi_i(\dot{\tau})(j) \stackrel{\text{def}}{=} \Pi_i(\dot{\tau}(j)) \tag{2.42}$$

and likewise for $\ddot{\tau}$.

Lynch's definition of legality does not precisely align with the above goals. Suppose V is a differentiable manifold, and so trajectories in $\mathcal{T}$ may be differentiable. Under the above definition of concatenation closure, the concatenation of a set of differentiable trajectories may not, itself, be differentiable. In particular, consider the point of "join" between τ_i, for some non-final index i, and τ_{i+1}. For ease of reference, let $j = \tau_i.ltime$, and let $j' = \tau_{i+1}.ftime$. While both $\dot{\tau}_i(j)$ and $\dot{\tau}_{i+1}(j')$ must exist if τ_i and τ_{i+1} are differentiable, there is no guarantee that they have the same value; in other words, that point in the resultant concatenated trajectory may not be differentiable.

As this will be a matter of some significance later on, I introduce a variation on concatenation closure, which I term *differentiable concatenation closure*. Given U, a totally ordered set, and V, a differentiable manifold, let $\mathcal{T}$ be a (U, V)-set of trajectories that are left-closed, second-degree doubly semidifferentiable, and have a common first instant. I say that $\mathcal{T}$ is *differentiably concatenation closed* iff any concatenation of elements of $\mathcal{T}$ that is also second-degree doubly semidifferentiable is an element of $\mathcal{T}$.

As an example, let $\tau_0, \tau_1, \tau_2, \ldots$ be an arbitrary sequence of second-degree doubly semidifferentiable trajectories in $\mathcal{T}$, such that, for any non-final index i, τ_i is closed, $\tau_i.lstate = \tau_{i+1}.fstate$, and $\dot{\tau}_i(\tau_i.ltime) = \dot{\tau}_{i+1}(\tau_{i+1}.ftime)$, if those derivatives exist. In other words, the final state of each trajectory matches the first state of the next trajectory in the sequence in both value and derivative. Then $\mathcal{T}$ is differentiably concatenation closed iff, for any such sequence, the trajectory $\tau = \tau_0 \frown \tau_1 \frown \tau_2 \frown \ldots$ is in $\mathcal{T}$; note that this trajectory must be second-degree doubly differentiable.

At a glance, it might appear that any differentiably concatenation closed set will also be concatenation closed. Indeed, any concatenation closed set whose members are also second-degree doubly semidifferentiable will also be differentiably concatenation closed. However, suppose that there exists a set of second-degree doubly differentiable trajectories, and suppose further that the members of this set can be concatenated as per concatenation closure. Under concatenation closure, that concatenation would belong to the set of trajectories, even though, as noted above, it might not be differentiable. The set might therefore be concatenation closed but not differentiably concatenation closed; in other words, proving that the component trajectories are second-degree doubly semidifferentiable is not sufficient to prove differentiable concatenation closure.

Given these definitions, I say that a (U, V)-set of trajectories is a *differentiably legal* (U, V)-set (or simply that it is differentiably legal) iff: all trajectories are left-closed with a common first instant; all trajectories are second-degree doubly differentiable; the set is prefix closed; the set is suffix closed; and the set is differentiably concatenation closed.

2.5 Summary of notation

In each subsequent chapter in which new notation is presented, I conclude with a brief section summarizing any new symbols, as below. Items in each such list are given in the order in which they appear in the preceding chapter. A full summary of all notation, separated by chapter, is provided in Appendix A.

U	a set
$\leq$	a total order on a set
J	a interval in a total ordering
$[j_1, j_2]$	the interval from j_1 to j_2
$min(J)$	minimal element of an interval (if exists)
$max(J)$	maximum element of an interval (if exists)
$J + j'$	interval translated up by j'
$J - j'$	interval translated down by j'
f	a function
$dom(f)$	domain of function f
$\dot{f}$	derivative of function f
$\ddot{f}$	second derivative of function f
$\dot{f}_-$	left-derivative of function f
$\dot{f}_+$	right-derivative of function f
$\lim x \to j^-$	limit from the left
$\lim x \to j^+$	limit from the right
$f \lceil U$	restriction of function f to set U
$\mathfrak{M}$	a differentiable manifold
$T_x\mathfrak{M}$	tangent space to manifold $\mathfrak{M}$ at point x
$T\mathfrak{M}$	tangent bundle for manifold $\mathfrak{M}$

$\mathbf{v}$	vector field on a manifold
$\mathfrak{X}(\mathfrak{M})$	set of all vector fields on manifold $\mathfrak{M}$
χ	deformation of a subset of a manifold
τ	a trajectory
$\tau.ftime$	the infimum of the domain of trajectory τ
$\tau.ltime$	the supremum of the domain of trajectory τ
$\tau.fstate$	the value of a left-closed trajectory τ at its infimum
$\tau.lstate$	the value of a right-closed trajectory τ at its supremum
$\leq$	prefix relation between two trajectories
$\tau + c$	trajectory translated up by c
$\tau - c$	trajectory translated down by c
$\tau \trianglerighteq j$	suffix of trajectory τ beginning at j
$\mathfrak{T}$	set of trajectories
$\tau \frown \tau'$	concatenation of trajectories τ and τ'
Π_i	projection of a tuple onto its ith component
$\Pi_i(\tau)$	projection of a trajectory τ onto its ith component
$\Pi(U)$	set containing the projection of all members of the set U
$\ddot{f}_L$	left-favoring second derivative of function f

Chapter 3

Hybrid Automata

I can now describe the core mathematics of hybrid automata. As before, I borrow heavily from Lynch [18], particularly in her discussion of trajectories. The following sections present a construction of a hybrid automaton (Section 3.1) and a discussion of the executions of an automaton (Section 3.2). I then compare this construction to those in the works of Lynch, Henzinger, and Alur (Section 3.3) and provide a brief example of a hybrid automaton under several constructions (Section 3.4).

3.1 Construction

For the purposes of this paper, a hybrid automaton is a tuple $(V, X, Q_0, D, \mathcal{T}, \Gamma)$. In the following sections, I provide definitions for each of the terms of this tuple.

3.1.1 Time

For a hybrid automaton, *time* Γ consists of a nonempty, left-closed, fully-ordered set, with some ordering relation $\leq$, that is closed under addition. I refer to any particular value in Γ as an *instant*, denoted t. There is, in general, no requirement that the set of instants forms a continuum; one might, for instance, define time to consist of only the odd nonnegative integers. Likewise, instants need not be real-valued. More complex representations might draw on the idea of *infinitesimal numbers*, allowing an infinite succession of instants after a single real-valued instant and before encountering another, or might represent instants via ordered tuples of any complexity. Later in this dissertation, for example, I describe a set of instants of the form (r, z), where r is a real number greater than or equal to 0 and z is an element of $\{0, 1, 2, 3\}$, with an appropriate ordering.

Regardless, the set of instants must be left-closed, as every automaton is required to have an "initial" state, which it occupies during the first instant in time. A natural simplifying assumption is to consider only those choices of time which are left-closed intervals in $\mathbb{R}$, the real numbers, with the natural ordering. Without further loss of generality, I assume that the left endpoint of any such interval is 0. I here restrict my discussion to automata of that form.

Assumption 1. *Hybrid automaton time Γ is a interval in $\mathbb{R}$, ordered via $<$, that is left-closed at 0.*

I refer to any interval of Γ as a *time interval*.

3.1.2 System states

As noted in Chapter 1, a hybrid automaton can be divided into two portions: the *controller*, representing the computer or other system whose states determine physical behaviors, and the *plant*, the real-world objects or properties affected by the controller. While "plant" is the traditional term in control theory, it may sound peculiar to those who, like this author, come from a computer science background; to avoid this confusion, I typically refer to the real-world portion of the automaton as the *environment*.

Different authors divide controller and environment in various ways. Henzinger and Alur [15, 2], for instance, specify that the controller has a discrete state space, while the environment's state space is a continuum. Thus, in this framing one can speak interchangeably of the "controller" or the "discrete" part of the automaton, as I have done in explanations to this point. I call any automata following Henzinger and Alur in this and in the other differences described below an *ACHH automaton*, taking the name from the authors of [2].

Lynch [18] provides a more complex description, in which the state spaces of both controller and environment can either be discrete or form continua. State spaces may also be the product of discrete sets and continua; for instance, a state might be defined by an ordered pair (x, y) where x is in $\mathbb{R}$ and y is in $\mathbb{Z}_2$, that is, the set $\{0, 1\}$. Lynch also divides variables of both sorts into *input* variables, whose value can change independently of the state of the rest of the controller and/or environment, and *output* variables, whose values depend on the remainder of the controller and/or environment. The primary function of the input/output distinction is to aid in the composition of several automata into one; the outputs of one automaton become the inputs of another, and vice-versa. I term any automaton following Lynch's model an *LSV automaton*, again taking the acronym from the authors of [18].

In this paper, I do not attempt to model composition of automata, and so I omit further treatment of the LSV input/output divide. The division between controller states and environment states is highly relevant, however, as mechanical representation of the controller's influence is the primary aim of this paper. Thus, I follow ACHH automata in saying that every automaton A has an associated *controller state space V* and *environment state space X*.

The set of all possible *system states* of A, then, is $Q \stackrel{\text{def}}{=} V \times X$. Of these, some subset Q_0 are the *initial states* of the automaton. Intuitively, at the first instant in Γ, A is required to be in one of these initial states; this restriction is treated more formally in the discussion of executions, below. The *initial function* $\texttt{Initial} : V \to 2^X$ maps each controller state v to the set of all environment states $\mathbf{x}$ such that $(v, \mathbf{x}) \in Q_0$.

I again extend my notation for projection to a pair of functions, $\Pi_V : Q \to V$ and $\Pi_X : Q \to X$ that map a system state to its component controller and environment states, respectively. In other words, for any system state $q = (v, \mathbf{x})$, I have $\Pi_V(q) = v$ and $\Pi_X(q) = \mathbf{x}$.

ACHH automata assume that the environment state space is $\mathbb{R}^n$ for some integer $n > 0$; in other words, a particular environment state is some finite-dimensional real-valued tuple. A hybrid automaton representing the motion of a single point body in three-space might have $X = \mathbb{R}^3$; more generally, a system with k point bodies might have $X = \mathbb{R}^{3k}$. The dimensions of X need not be positions, however; some or all of them might represent velocities, accelerations, or other derivatives of motion. Still others might represent properties unrelated to motion, such as temperature, magnetic field strength, gravitational attraction, and so on.

Still more generally, the environment state space might be any sort of manifold that could be imagined; one might, for instance, have a set of dimensions representing coordinates on the surface of a sphere. Such constructions are valid hybrid automata, but including them complicates the explanation to follow. In the interests of simplicity, then, in this paper I follow ACHH automata in restricting attention to formulations where $X = \mathbb{R}^n$. I expect, but do not demonstrate, that the results presented here also hold for more general manifolds.

3.1.3 Transitions and trajectories

Hybrid automata change state by one of two methods. First, over a given time interval, an automaton follows a particular trajectory, during which its controller or environment states may change. Second, such intervals are connected by discrete, instantaneous *transitions* from one system state to another. Thus, the behavior of an automaton may be described by its following some trajectory for an interval, making a discrete transition, following another trajectory for some interval, and so on.

All automata, then, include a (Γ, Q)-set of trajectories, denoted $\mathcal{T}$, describing the "interval motion" of the automaton. I restrict my discussion to those automata for whom $\mathcal{T}$ is a differentiably legal (Γ, Q)-set. (The empty set is always differentiably legal, and so there is guaranteed to be at least one differentiably legal set for any choice of Γ and Q.) In other words, if $\mathcal{T}$ is the set of all trajectories of a given hybrid automaton, then all trajectories in $\mathcal{T}$ must be left-closed, second-degree doubly semidifferentiable, and have a common first instant, and $\mathcal{T}$ must be prefix closed, suffix closed, and differentiably concatenation closed. Following the pattern for automaton time, I assume the common first instant of all trajectories is 0.

Assumption 2. *$\mathcal{T}$ is a differentiably legal (Γ, Q)-set of trajectories, that is, a (Γ, Q)-set of trajectories that is prefix closed, suffix closed, and differentiably concatenation closed, and such that all trajectories in the set are left-closed with a common first instant. Furthermore, the common first instant is* 0.

Conceptually, the requirement for differentiable legality means that every trajectory has a defined velocity at every instant; taking the left derivative of the velocity as the acceleration where possible, and the right derivative otherwise, it must also have a defined acceleration at every instant. In physics, the acceleration of a body is proportionate to the total force on it, as in the expression $f = ma$, and so this assumption guarantees that it is possible to define forces proportionate to the trajectory's acceleration at every instant.

The set of all discrete transitions for the automaton is expressed by the *transition relation* $D \subseteq Q \times Q$, where D is the set of all ordered pairs (q, q') such that the system can discretely transition from q to q'. For any such pair $(q, q') \in D$, q' is said to be a *successor* of q.

I extend the earlier notation for projection to cover discrete transitions, writing $\Pi_Y(q, q')$ to indicate $(\Pi_Y(q), \Pi_Y(q'))$ for any system states q, q' and system state component Y. In what follows, I will most often be concerned with Π_V and Π_X.

Given this interest in the component controller and environment states, it will sometimes be desirable to express a discrete transition in terms of the controller and environment states, that is, in a form similar to $((v, \mathbf{x}), (v', \mathbf{x}'))$. Unfortunately, while technically accurate, this description sometimes borders on the unreadable. For this reason, I substitute the notation $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}')$.

3.1.3.1 Constant controller states

In an LSV automaton, any part of the system state can change over the duration of a trajectory. In the ACHH automaton description, however, it is assumed that the controller state changes only via discrete transitions, that is, that for any trajectory $\tau \in \mathcal{T}$, there exists a controller state $v \in V$ such that, for all instants $t, t' \in dom(\tau)$,

$$\Pi_V(\tau(t)) = \Pi_V(\tau(t')) = v. \tag{3.1}$$

I say that any trajectory for which this condition holds has a *constant controller state*, and I refer to v as the constant controller state of τ. In keeping with this framing, I restrict my discussion to automata whose trajectories have constant controller states, as summarized in the following assumption.

Assumption 3. *Let A be a hybrid automaton with a set of trajectories $\mathcal{T}$. All trajectories in τ have constant controller states; that is, for every trajectory $\tau \in \mathcal{T}$ and any instants $t, t' \in dom(\tau)$,*

$$\Pi_V(\tau(t)) = \Pi_V(\tau(t')). \tag{3.2}$$

3.1.3.2 Degenerate transitions

I exclude any degenerate transitions of the form $q \rightarrowtail q$, that is, transitions that do not change any part of the system state. Such transitions introduce awkward notation later in the paper, and as they do not change anything in the automaton, omitting them does not meaningfully restrict the automaton's behavior.

Assumption 4. *There are no transitions of the form (q, q), where q is in Q.*

Similarly, it will be awkward in the construction of executions in a few paragraphs if it is possible to make a transition from which it is impossible to follow any trajectories. Thus, I assume that there is at least one trajectory from each state reachable via a transition. If D does not satisfy this requirement, eliminating the non-compliant transitions has no effect on the behavior of the automaton, since such transitions could not be followed in any event.

Assumption 5. *Given a hybrid automaton $A = \{V, X, Q_0, D, \mathfrak{T}, \Gamma\}$, for any states $q, q' \in Q$ such that (q, q') is in D, there is at least one non-point trajectory $\tau \in \mathfrak{T}$ such that $\tau.\mathit{fstate} = q'$. In other words, no transition is blocking.*

3.1.4 Definition of a hybrid automaton

A particular hybrid automaton, then, is defined by a tuple $(V, X, Q_0, D, \mathfrak{T}, \Gamma)$, where V and X are the controller and environment state spaces, Q_0 is a set of initial states, D is a set of discrete transitions, $\mathfrak{T}$ is a differentiably legal (Γ, Q)-set of trajectories with constant controller states and common first instants of 0, and Γ is a left-closed interval in $\mathbb{R}$ beginning at 0. Where ambiguous, I write the tuple for a hybrid automaton A as $(V_A, X_A, Q_{0A}, D_A, \mathfrak{T}_A, \Gamma_A)$, or, given a set of hybrid automata $A_1, A_2, \ldots$, I write the tuple for A_i as $(V_i, X_i, Q_{0i}, D_i, \mathfrak{T}_i, \Gamma_i)$.

3.2 Executions

Given a hybrid automaton $A = (V, X, Q_0, D, \mathfrak{T}, \Gamma)$, an *execution fragment* α is a finite or countably infinite sequence $\tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$, where the trajectories $\tau_0, \tau_1, \tau_2, \ldots$ are in $\mathfrak{T}$, the transitions $a_1, a_2, \ldots$ are in D, and for each index $i > 0$, $a_i = (\tau_{i-1}.\mathit{lstate}, \tau_i.\mathit{fstate})$. In other words, the discrete transitions connect the last state of each non-final trajectory to the first state of the next trajectory. One refers to the set of all possible execution fragments of A as frags_A, dropping the subscript where unambiguous..

An execution fragment α with initial trajectory τ_0 is said to be *from* the system state $\tau_0.\mathit{fstate}$. If $\tau_0.\mathit{fstate}$ is in Q_0, then α is called an *execution*; an execution, then, is an execution fragment that is from an initial state of the automaton. The set of all possible executions of A is written execs_A, dropping the subscript where unambiguous. One writes $\alpha.\mathit{fstate}$, the *first state* of α, to indicate $\tau_0.\mathit{fstate}$.

One calls an execution fragment *closed* if it consists of a finite number of trajectories and transitions, with a closed final trajectory. By contrast, an execution fragment with no final trajectory, or whose final trajectory is open, is *right-open*. Given a closed execution fragment α with final trajectory τ_k, one writes $\alpha.\mathit{lstate}$, the *last state* of α, to indicate $\tau_k.\mathit{lstate}$. A system state q is *reachable* iff, for some closed execution α, $\alpha.\mathit{lstate} = q$, and one refers to the set of all reachable system states as Q_{reach}.

I again extend my notation for projection to cover executions, such that, for any execution fragment

$\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ and system state component Y,

$$\Pi_Y(\alpha) \stackrel{\text{def}}{=} \Pi_Y(\tau_0)\Pi_Y(a_1)\Pi_Y(\tau_1)\Pi_Y(a_2)\Pi_Y(\tau_2)\ldots \tag{3.3}$$

In other words, $\Pi_Y(\alpha)$ is the sequence formed by Y-projecting each transition and trajectory of α.

I will most often be concerned with the projection of an execution fragment onto its environment states, which I call the *trace* of that fragment and denote $trace(\alpha)$. Thus,

$$trace(\alpha) \stackrel{\text{def}}{=} \Pi_X(\alpha). \tag{3.4}$$

Given a hybrid automaton A, I write $tracefrags_A$ to indicate the set of the traces of all execution fragments of A, or $traces_A$ to indicate the set of the traces of all executions of A.

3.2.1 Executions and time

Intuitively, an execution describes the behavior of an automaton over time; the automaton progresses first through the states of τ_0, makes transition a_1, proceeds through the states in τ_1, and so on. I can describe this behavior by extending the definition of α to a mapping from an interval in time to system states, that is, such that $\alpha : J \rightarrow Q$, where J is some time interval.

Doing so requires some preparatory work, however. Given an interval J and a J-trajectory τ, the *duration* of τ is the distance between its first and last instants, that is, $\tau.ltime - \tau.ftime$. Given my statement in Assumption 2 that all trajectories in $\mathcal{T}$ are left-closed at 0, I can drop the last term of this expression, and so the duration of any such trajectory is $\tau.ltime$. I denote the duration of given trajectory τ via $dur(\tau)$.

Given an instant t and an execution fragment α, I say that t is *i-bounded* iff it would occur during the i-indexed trajectory in α. More formally, t is i-bounded iff it falls between the sum of the durations of the first i trajectories in α and the sum of the durations of the first $i+1$ trajectories in α. That is, if $\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ and contains at least $i+1$ trajectories, I say that t is i-bounded iff

$$\sum_{j=0}^{i} dur(\tau_j) \leq t < \sum_{j=0}^{i+1} dur(\tau_j). \tag{3.5}$$

If an execution fragment has a finite number of trajectories, and i is the index of the final trajectory, I say that t is i-bounded iff

$$\sum_{j=0}^{i} dur(\tau_j) \leq t. \tag{3.6}$$

With these definitions in place, one can relate execution fragments and time more directly. In particular, one defines the first and last instants for an execution fragment as follows. The first instant of an execution, denoted $\alpha.\mathit{ftime}$, is simply $\tau_0.\mathit{ftime}$; for a closed execution fragment, the last instant $\alpha.\mathit{ltime}$ is simply the sum of the durations of the component trajectories. That is, if α contains n trajectories for any countable n,

$$\alpha.\mathit{ltime} \stackrel{\text{def}}{=} \sum_{j=0}^{n} \mathit{dur}(\tau_j) \tag{3.7}$$

Given an execution fragment

$$\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots, \tag{3.8}$$

let J be the closed time interval $[\alpha.\mathit{ftime}, \alpha.\mathit{ltime}]$, if α is closed, or the half-open interval $[\alpha.\mathit{ftime}, \infty)$ otherwise. Then the mapping $\alpha : J \to Q$ is defined as follows as follows. For any instant $t \in J$ such that t is i-bounded for some integer $i \geq 0$,

$$\alpha(t) \stackrel{\text{def}}{=} \begin{cases} \tau_i(t), & i = 0 \\ \tau_i'(t), & i > 0 \end{cases} \tag{3.9}$$

where

$$\tau_i' = \tau_i + \sum_{j=0}^{i-1} \mathit{dur}(\tau_j). \tag{3.10}$$

Intuitively, each trajectory is shifted forward by an amount equal to the sum of the durations of all previous trajectories. Each instant in $\mathit{dom}(\alpha)$, then, is i-bounded by some trajectory index i, and the state in α for that instant is simply the corresponding state in the τ_i trajectory once shifted forward.

If all trajectories composing an execution fragment are differentiable, the fragment itself is said to be differentiable. In such cases, the derivative mapping $\dot{\alpha} : J \to Q$ is defined much as the mapping above. For any instant $t \in J$ such that t is i-bounded for some integer $i \geq 0$,

$$\dot{\alpha}(t) \stackrel{\text{def}}{=} \begin{cases} \dot{\tau}_i(t), & i = 0 \\ \dot{\tau}_i'(t), & i > 0 \end{cases} \tag{3.11}$$

where

$$\tau_i' = \tau_i + \sum_{j=0}^{i-1} \mathit{dur}(\tau_j). \tag{3.12}$$

Let $\alpha = \tau_0 a_1 \alpha_1 \ldots$ be an execution fragment. Then, as noted above,

$$\alpha.\mathit{fstate} \stackrel{\text{def}}{=} \tau_0.\mathit{fstate}. \tag{3.13}$$

Likewise, if α is closed with final trajectory τ_n, then

$$\alpha.\mathit{lstate} \stackrel{\text{def}}{=} \tau_n.\mathit{lstate}. \tag{3.14}$$

Similarly, and by a slight abuse of notation, if α is differentiable, then

$$\dot{\alpha}.\mathit{fstate} \stackrel{\text{def}}{=} \dot{\tau}_0.\mathit{fstate}, \tag{3.15}$$

and, if α is both differentiable and closed, with final trajectory τ_n, then

$$\dot{\alpha}.\mathit{lstate} \stackrel{\text{def}}{=} \dot{\tau}_n.\mathit{lstate}. \tag{3.16}$$

3.2.2 Concatenating executions

It may at times be useful to indicate that an execution fragment is formed by appending two execution fragments end-to-end. Suppose there exists a closed execution fragment $\alpha = \tau_0 a_1 \tau_1 \ldots a_n \tau_n$ and another execution fragment $\alpha' = \tau_{n+1} a_{n+2} \tau_{n+2} \ldots$ such that

$$\begin{aligned} \alpha.\mathit{lstate} &= \alpha'.\mathit{fstate} \\ \dot{\alpha}.\mathit{lstate} &= \dot{\alpha'}.\mathit{fstate}. \end{aligned} \tag{3.17}$$

Equivalently,

$$\begin{aligned} \tau_n.\mathit{lstate} &= \tau_{n+1}.\mathit{fstate} \\ \dot{\tau}_n.\mathit{lstate} &= \dot{\tau}_{n+1}.\mathit{fstate}. \end{aligned} \tag{3.18}$$

Note that in this case, the concatenation $\tau_n \frown \tau_{n+1}$ is in $\mathcal{T}$. Then the concatenation of α and α', denoted $\alpha \frown \alpha'$, is

$$\alpha \frown \alpha' \stackrel{\text{def}}{=} \tau_0 a_1 \tau_1 \ldots a_n (\tau_n \frown \tau_{n+1}) a_{n+2} \tau_{n+2} \ldots . \tag{3.19}$$

It follows that

$$(\alpha \frown \alpha')(t) = \begin{cases} \alpha(t), & t \leq \alpha.\mathit{ltime} \\ \alpha'(t - \alpha.\mathit{ltime}), & t > \alpha.\mathit{ltime}. \end{cases} \tag{3.20}$$

3.2.3 Zeno automata

An unrestricted definition of transition relations and trajectories allows for degenerate conditions. As a simple example, consider an automaton with a transition relation containing both $q \rightarrowtail q'$ and $q' \rightarrowtail q$, for some choice of q, q' in Q. Such an automaton might discretely transition from q to q' and back again an infinite number of times within any finite period of time. Such automata are called *Zeno automata*, while automata that are not capable of such behavior are called *non-Zeno automata* [15]. Other Zeno automata might proceed through an infinite succession of states $q, q', q'' \ldots$, again with each transition immediately enabling another.

Formally, a Zeno automaton is one for which there is some infinite execution $\tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ such that the sum of the durations of all trajectories converges to a finite value, that is, such that

$$\sum_{j=0}^{\infty} dur(\tau_j) = c, \tag{3.21}$$

for some finite value c.

In practical terms, a Zeno automaton might perform an infinite amount of computation within a finite amount of time. Regrettably, such systems are not generally mechanically realizable, and so I restrict consideration to non-Zeno automata.

3.3 Comparison with alternative constructions

As noted earlier, the preceding notation deviates somewhat from existing work in hybrid automata. I pause here briefly to expand on my deviations from the LSV model, on the one hand, and the ACHH formulation, on the other. I then briefly describe a translation between the two approaches. Those familiar with either the LSV or ACHH formulations may find this discussion useful in summarizing the differences in approach.

3.3.1 Differences from Lynch

Those familiar with the LSV hybrid input-output automata will note that I have deviated somewhat from its organization in the above description. As discussed above, LSV automata do not present the controller/environment divide, discussed above, which is common to other hybrid automata descriptions [2, 15]. Instead, LSV automata instead categorize variables as inputs or outputs, a division particularly helpful in combining multiple automata. I focus here on the controller/environment division instead, as it provides a more natural counterpart to the universes of a hybrid mechanical system. Once the core bisimulation has been completed, however, a natural extension would be to develop it so as to respect the input/output divide. Ideally, such an extension would allow appropriate hybrid mechanical systems to be combined in the same way as the hybrid I/O automata they simulate.

I also differ from LSV automata in including time as a member of the tuple summarizing an automaton. As noted earlier, LSV automata takes time to be the real numbers greater than or equal to 0. I allow for some variation in the definition of time in general, codifying the requirement that time be a left-closed interval in $\mathbb{R}$ starting at 0 in Assumption 1.

3.3.2 Differences from Henzinger and Alur

Earlier work in hybrid automata, such as that by Alur in [2] and Henzinger in [15], is framed rather differently from the above description. In particular, one does not describe ACHH automata primarily in terms of trajectories. I have favored a LSV automaton-based formalization instead primarily because comparing trajectories is a straightforward way of showing bisimulation. In what follows, I briefly consider some of the components discussed by ACHH automata and show that they can be derived from the above LSV formalization.

3.3.2.1 Activities and the activity labeling function

Recall that, in an ACHH automaton, V is a discrete state space and X is a continuum, typically a continous subset of $\mathbb{R}^n$ for some positive integer n. Such automata associate each system state $q = (v, \mathbf{x})$ with a set of time-differentials of $\mathbf{x}$, called the *activities* of q. More formally, an activity is a function $a : \Gamma \to X$. Let $\mathfrak{A}$ be the set of all such activities; then the *activity labeling function* $\mathsf{Act} : Q \to 2^{\mathfrak{A}}$ maps each system state to its activities.

Let $q \in Q$ be a system state, and let $a \in \mathsf{Act}(q)$ be any activity. Then, for any instant $t' \in \Gamma$, ACHH automata require that there must exist some function a' such that, for all $t \in \Gamma$,

$$a(t) = a'(t + t') \tag{3.22}$$

and a' is in $\mathsf{Act}(q)$. In other words, given any activity of a system state, there exist equivalent activities shifted forward or back in time by any amount. Henzinger calls this property of the activity labeling function *time-invariance*.

Intuitively, the system state changes over time so that the derivative of the environment state is always one of the activities of its current system state. In LSV automaton terms, for any continuous trajectory τ and instant t in which $\tau(t) = (v, \mathbf{x})$, there exists some activity $a \in \mathsf{Act}(v, \mathbf{x})$ such that

$$\Pi_X(\dot{\tau}(t)) = a(t). \tag{3.23}$$

The controller state, meanwhile, remains constant except for changes during instantaneous transitions. One says that any such trajectory is a *solution* of the activity function

In later chapters, I typically omit explicit discussion of the activity labeling function in favor of discussing the trajectories themselves. Thus, while any trajectory in $\mathcal{T}$ is presumed to be a solution for some activity labeling function, the particular function is left undefined. I here require only that, for any set of trajectories $\mathcal{T}$, there must exist some time-invariant activity labeling function such that every trajectory in $\mathcal{T}$ is a solution of that function. I codify this requirement in the following assumption.

Assumption 6. *Let A be a hybrid automaton, and let $\mathcal{T}$ be the trajectory set for that hybrid automaton.*

Then there exists some time-invariant activity labeling function Act *such that a trajectory* τ *is in* $\mathfrak{T}$ *iff it is a solution of* Act.

3.3.2.2 The invariant set

Each controller state $v \in V$ also has an associated *invariant set*, a subset of X; the *invariant function* $Inv : V \to 2^X$ maps each controller state to its invariant set. If the environment portion of A would move out of its invariant set, the automaton is required instead to take some transition in D such that it is in the invariant of its new discrete state. An automaton that can enter an environment state outside its invariant, with no appropriate transition to a state in the invariant, is said to be *blocking*.

3.3.2.3 Edges, guard sets, and the reset function

Consider any pair of controller states v, v' such that, for some environment states $\mathbf{x} \in Inv(v)$ and $\mathbf{x}' \in Inv(v')$, the pair $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}')$ is in D. For any such case, the ordered pair (v, v') is an *edge* of the automaton. An edge, then, is a pair of controller states such that some transition is possible from the first state to the second. The set of all edges is thus

$$E \stackrel{\text{def}}{=} \{(v, v') \in V \times V | \exists \mathbf{x} \in Inv(v) \exists \mathbf{x}' \in Inv(v')[(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}') \in D]\}. \tag{3.24}$$

The *guard* or *guard set* of an edge (v, v') is the set of all environment states $\mathbf{x} \in Inv(v)$ such that, for some environment state $\mathbf{x}' \in Inv(v')$, the pair $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}')$ is in D. The guard function $Guard : E \to 2^X$ maps each edge to its guard set. The *reset function* $Reset : E \times X \to 2^X$ maps each edge (v, v') and continuous state $\mathbf{x}$ to the set of all environment states $\mathbf{x}'$ such that $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}')$ is in D. That is,

$$Reset((v, v'), \mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{x}' \in Inv(v') \mid (v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}') \in D\}. \tag{3.25}$$

3.3.2.4 Summary

In this perspective, then, a given hybrid automaton is defined by a particular choice of values for the tuple $(V, X, Q_0, \mathsf{Act}, Inv, D, \Gamma)$, where V is the controller space, X is the environment space, Q_0 is the set of initial states, Act is the activity labeling function, Inv is the invariant function, D is the set of discrete transitions, and Γ is time..

3.3.3 Conversions between alternative constructions

3.3.3.1 Conversion from LSV automata to ACHH automata

By way of a brief example, consider some hybrid automaton in the LSV formulation, with tuple $(Q, Q_0, D, \mathfrak{T}, \Gamma)$. This definition deviates somewhat from the LSV automaton standard, in that LSV automata do not define time as part of the standard tuple and, as noted earlier, also divides the system states into the product of input and output variables. As this distinction is irrelevant to ACHH automata, I ignore it here.

To be eligible for conversion into a ACHH automaton formulation, the set of system states, Q, must be decomposable into the product of a discrete state space V and a continuous state space X, where X is a subset of $\mathbb{R}^n$ for some positive integer n. Furthermore, each trajectory $\tau \in \mathcal{T}$ must have a constant system state, and the trajectory as a whole must be continuous and differentiable.

Under these constraints, several components translate directly; V and X are immediately defined, and Q_0, D, and Γ are identical across the two models. It remains only to define Act and Inv.

Given a state $q \in Q$, the activities of q are simply the projections of $\dot{\tau}(t)$ onto X, for all choices of $\tau \in \mathcal{T}$, $t \in \Gamma$ such that τ is in q in instant t. That is,

$$\mathsf{Act}(q) = \{\Pi_X(\dot{\tau}(t)) \mid (\tau \in \mathcal{T}) \wedge (t \in \mathit{dom}(\tau)) \wedge (\tau(t) = q)\}. \tag{3.26}$$

The invariant of a controller state v, likewise, is simply those states $\mathbf{x}$ for which $(v, \mathbf{x})$ is reachable and from which the automaton can either follow another trajectory or make a transition. That is,

$$\begin{aligned} \mathit{Inv}(v) = \{\mathbf{x} \mid (\mathbf{x} \in X) \wedge ((v, \mathbf{x}) \in Q_{\mathit{reach}}) \\ \wedge (\exists \tau \in \mathcal{T}[\Pi_{\mathbf{x}}(\tau.\mathit{fstate}) = \mathbf{x}] \vee \exists \mathbf{x}' \in X, \exists v' \in V[(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}') \in D])\}. \end{aligned} \tag{3.27}$$

Given the above conversions, the edge set, guard set, and reset function can be derived as per their definitions.

3.3.3.2 Conversion from ACHH automata to LSV automata

It is also possible to convert much of a ACHH automaton, represented via a tuple $(V, X, Q_0, \mathsf{Act}, \mathit{Inv}, D, \Gamma)$, to a LSV automaton. Again, I disregard input/output variables here. As before, Q_0, D, and Γ are unaltered in the translation, and Q is defined immediately as $Q = V \times X$. It remains to derive the trajectory set, $\mathcal{T}$.

Intuitively, $\mathcal{T}$ is a (Γ, Q)-set of differentiable trajectories starting at 0. Each trajectory in $\mathcal{T}$ has a constant controller state and has environment states such that, at every instant, the change in the environment state is one of the possible actions of the current system state. Furthermore, the projection of the trajectory onto its environment state must lie entirely within the invariant of its single controller state. More formally, $\mathcal{T}$ is the (Γ, Q)-set of trajectories such that the following conditions hold:

$$\begin{aligned} \mathcal{T} = \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is differentiable}\big) \\ \wedge \forall t \in \mathit{dom}(\tau)\Big[\big(\Pi_X(\dot{\tau}(t)) \in \mathsf{Act}(\tau(t))\big) \wedge \big(\Pi_X(\tau(t)) \in \mathit{Inv}(\Pi_V(\tau(t)))\big) \end{aligned}$$

$$\wedge \forall t' \in dom(\tau)\big(\Pi_V(\tau(t)) = \Pi_V(\tau(t'))\big)\Big]. \quad (3.28)$$

3.3.4 Alternate constructions and differentiable legality

Having provided constructions for LSV and ACHH-style hybrid automata, I turn next to converting from one of these alternatives to the automaton tuple used for this paper. As noted, the tuple used in this paper is already similar to that used by LSV automata, substituting the controller/environment divide for her input/output division. Translating a ACHH automaton to the form used in this paper, then, is effectively the same as the transformation to a LSV automaton tuple, above. However, in either case, Assumption 2 requires restricting consideration to only differentiably legal sets of trajectories.

Thus, to translate from a ACHH automaton to one in this paper's style, I add an additional restriction to (3.28), namely that all trajectories must be second-degree doubly semidifferentiable. In other words, the construction of $\mathcal{T}$ can be extended as follows:

$$\mathcal{T} = \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is second-degree doubly semidifferentiable}\big)$$
$$\wedge \forall t \in dom(\tau)\Big[\big(\Pi_X(\dot\tau(t)) \in \mathsf{Act}(\tau(t))\big) \wedge \big(\Pi_X(\tau(t)) \in \mathit{Inv}(\Pi_V(\tau(t)))\big)$$
$$\wedge \forall t' \in dom(\tau)\big(\Pi_V(\tau(t)) = \Pi_V(\tau(t'))\big)\Big]. \quad (3.29)$$

No such restriction is inherent in the design of a ACHH automaton, for which an activity labeling function can provide right-angle turns or other contortions as desired. Thus, an automaton under this additional restriction may not fully capture all behaviors of the original ACHH automaton. Given this additional restriction, however, the resultant set of trajectories is differentiably legal.

Theorem 1. *Given an ACHH automaton with characteristic tuple $(V, X, Q_0, \mathsf{Act}, \mathit{Inv}, D, \Gamma)$, let $\mathcal{T}$ be the $(\Gamma, (V, X))$-set of trajectories such that, for any trajectory $\tau \in \mathcal{T}$:*

- *τ is left-closed.*
- *$\tau.\mathit{ftime} = 0$.*
- *τ is second-degree doubly semidifferentiable.*
- *For all instants $t \in dom(\tau)$, $\Pi_X(\dot\tau(t))$ is in $\mathsf{Act}(\tau(t))$.*
- *For all instants $t \in dom(\tau)$, $\Pi_X(\tau(t))$ is in $\mathit{Inv}(\Pi_V(\tau(t)))$.*
- *For all instants $t, t' \in dom(\tau)$, $\Pi_V(\tau(t)) = \Pi_V(\tau(t'))$.*

Then $\mathcal{T}$ is a differentiably legal set of trajectories.

Proof. Four things are necessary for a set of trajectories to be differentiably legal: for all trajectories to be left-closed with a common first instant of 0, for the set to be prefix closed, for the set to be suffix closed, and for the set to be differentiably concatenation closed. The first of these holds trivially.

To show that the set is prefix closed, consider some trajectory $\tau \in \mathcal{T}$. By construction, τ is second-degree doubly semidifferentiable and left closed with $\tau.\mathit{ftime} = 0$. Also, for any $t \in \mathit{dom}(\tau)$,

$$\begin{aligned} &\Pi_X(\dot{\tau})(t) \in \mathsf{Act}(\tau(t)) \\ &\Pi_X(\tau(t)) \in \mathit{Inv}(\Pi_V(\tau(t))), \end{aligned} \tag{3.30}$$

and, for any $t' \in \mathit{dom}(\tau)$,

$$\Pi_V(\tau(t)) = \Pi_V(\tau(t')). \tag{3.31}$$

Consider any trajectory τ' such that $\tau' \leq \tau$. By the definition of prefix, τ' is second-degree doubly sem differentiable and left-closed with $\tau'.\mathit{ftime} = 0$. Since τ' is just a restriction of τ to a sub-interval, for any $t \in \mathit{dom}(\tau')$,

$$\begin{aligned} &\Pi_X(\dot{\tau'}(t)) \in \mathsf{Act}(\tau'(t)) \\ &\Pi_X(\tau'(t)) \in \mathit{Inv}(\Pi_V(\tau')(t)), \end{aligned} \tag{3.32}$$

and, for any $t' \in \mathit{dom}(\tau')$,

$$\Pi_V(\tau'(t)) = \Pi_V(\tau'(t')). \tag{3.33}$$

Thus, τ' is in $\mathcal{T}$, and so $\mathcal{T}$ is prefix closed. By an almost identical argument, $\mathcal{T}$ is suffix closed.

I now show that $\mathcal{T}$ is differentiably concatenation closed. Let $\tau_0, \tau_1, \tau_2, \ldots$ be a sequence of trajectories in $\mathcal{T}$ such that, for any non-final index i, τ_i is closed, $\tau_i.\mathit{lstate} = \tau_{i+1}.\mathit{fstate}$, and $\dot{\tau}_i(\tau_i.\mathit{ltime}) = \dot{\tau}_{i+1}(\tau_{i+1}.\mathit{ftime})$, that is, so that each trajectory begins where the previous one ended, with the same derivative. Let $\tau = \tau_0 \frown \tau_1 \frown \tau_2 \ldots$; showing that $\mathcal{T}$ is differentiably concatenation closed is equivalent to showing that τ is in $\mathcal{T}$. It is trivially the case that, for any instant $t \in \mathit{dom}(\tau)$,

$$\Pi_X(\tau(t)) \in \mathit{Inv}(\Pi_V(\tau(t))), \tag{3.34}$$

since the environment states of τ are simply the environment states of its components, all of which are by definition within their respective invariants. Likewise, it is trivial to show that τ is left-closed with first instant 0, since τ_0 is left-closed with first instant 0 by the definition of $\mathcal{T}$.

Note as well that τ must be differentiable. For any instant $t \in \mathit{dom}(\tau)$, it is either the case that t falls “inside” one of the component trajectories or that it falls on the point of connection between two of them. If it falls inside the component trajectory, then $\dot{\tau}(t)$ exists and

$$\Pi_X(\dot{\tau}(t)) \in \mathsf{Act}(\tau(t)), \tag{3.35}$$

since by the definition of $\mathcal{T}$ this property must hold for the (appropriately shifted) component trajectory. Likewise, if it falls on the point of connection between two component trajectories τ_i and τ_{i+1}, then by assumption

$$\dot{\tau}(t) = \dot{\tau}_i(\tau_i.ltime) = \dot{\tau}_{i+1}(0). \tag{3.36}$$

By the definition of $\mathcal{T}$,

$$\Pi_X(\dot{\tau}_{i+1}(0)) \in \mathsf{Act}(\tau_{i+1}(0)), \tag{3.37}$$

and so it follows that

$$\Pi_X(\dot{\tau}(t)) \in \mathsf{Act}(\tau_{i+1}(0)), \tag{3.38}$$

or, equivalently, that

$$\Pi_X(\dot{\tau}(t)) \in \mathsf{Act}(\tau(t)). \tag{3.39}$$

Similarly, $\dot{\tau}$ is left-differentiable at every point except 0. Each component trajectory has a left-differentiable derivative, by the definition of $\mathcal{T}$. Then at any point within $dom(\dot{\tau})$, except the left end-point, the left-derivative must exist. Thus, $\dot{\tau}$ is left-differentiable at all points except 0; by a similar argument, it is right-differentiable at every point except its last instant, if any. Thus, it is second-degree doubly semidifferentiable.

As a final step to prove that $\mathcal{T}$ is differentiably concatenation closed, I show that τ has a constant controller state, that is, that for any instants $t_i, t_j \in dom(\tau)$,

$$\Pi_V(\tau(t_i)) = \Pi_V(\tau(t_j)). \tag{3.40}$$

By the definition of $\mathcal{T}$, it is the case that $\forall t, t' \in dom(\tau_i)$,

$$\Pi_V(\tau_i(t)) = \Pi_V(\tau_i(t')). \tag{3.41}$$

But for each non-final index i, $\tau_i.lstate = \tau_{i+1}.fstate$, and so

$$\Pi_V(\tau_i.lstate) = \Pi_V(\tau_{i+1}.fstate). \tag{3.42}$$

It therefore follows that, for any $t \in dom(\tau_i)$ and $t' \in dom(\tau_{i+1})$,

$$\Pi_V(\tau_i(t)) = \Pi_V(\tau_{i+1}(t')). \tag{3.43}$$

By induction, this may be extended across an arbitrary number of component trajectories. Thus, it must be the case that all of the trajectories in this sequence have the same constant controller state, that is, that for any τ_i, τ_j in the sequence and instants $t_i \in dom(\tau_i)$, $t_j \in dom(\tau_j)$,

$$\Pi_V(\tau_i(t_i)) = \Pi_V(\tau_j(t_j)). \tag{3.44}$$

It follows that, for any instants $t_i, t_j \in dom(\tau)$,

$$\Pi_V(\tau(t_i)) = \Pi_V(\tau(t_j)). \tag{3.45}$$

Thus, τ has a constant controller state, and so $\mathfrak{T}$ is differentiably concatenation closed. It therefore follows that it is differentiably legal. □

Likewise, in converting a LSV automaton to the style of this paper, I restrict attention to only differentiably legal trajectories. In general, differentiable legality does not imply legality, the property of concern to LSV automata, because differential concatenation closure does not imply concatenation closure. A set which is both concatenation closed and differentiably concatenation closed will therefore be both legal and differentiably legal.

3.4 Example: The mechanical elevator

To illustrate the concepts discussed above, I provide a hybrid systems description of a mechanical elevator, followed by a brief representation of the elevator under both the ACHH and the LSV formulations. I also show the transformation from the ACHH to the LSV automaton, which suffices to demonstrate that the resulting set of trajectories is differentiably legal. Much of this discussion and the accompanying illustration are based on a similar example given by Henzinger [15], though in that formulation the automaton represented an automated furnace.

An automatic elevator is either on or off; while on, its height continuously rises to some threshold (say, 75m), at which point it switches off. The platform then drops steadily until it reaches some lower bound (say, 45m), at which point the elevator switches on again. To this description, I add the following. First, the elevator is initially on, with a starting height somewhere between 50m and 60m. Second, while the elevator is on, the platform rises at a rate in the range of 0m to 4m per unit of time; while it is off, the platform drops at a rate in the range of -3m to 0m per unit of time. I restrict consideration here to motions of the platform which are second-degree doubly semidifferentiable, which suffices to make the LSV automaton identical to the construction followed in this paper.

Figure 3.1 summarizes the behavior of the automaton. Discrete states are represented as circles, with each discrete state labeled first with its name and then with the changes in $\mathbf{x}$ for which there exist trajectories in this state. Arrows indicate transitions between discrete states and are labeled with the continuous state requirements for the transition to take place.

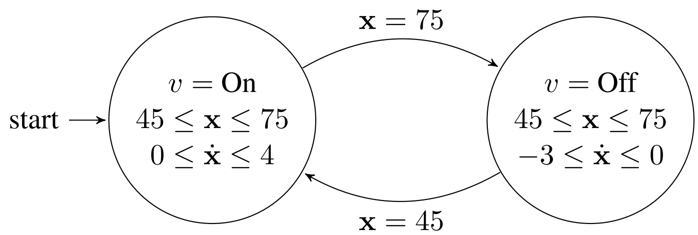


Figure 3.1: A hybrid automaton model for a mechanical elevator. Circles are discrete states of the controller. Each state is labeled with its name followed by acceptable values of and changes in the continuous state. Arrows are transitions between states, with the label on each transition indicating the conditions under which the controller makes that transition

3.4.1 ACHH representation

First, consider the elevator in the ACHH automaton formulation. As before, to represent an ACHH-style hybrid automaton requires defining a tuple $(V, X, Q_0, \mathsf{Act}, \mathit{Inv}, D, \Gamma)$.

As a hybrid automaton, the elevator has two controller states, On and Off, while its environment state space is the interval $[0, \infty)$, that is, from the ground up. The initial states of the system Q_0 are any pairs $(\mathit{On}, \mathbf{x})$ such that $50 \leq \mathbf{x} \leq 60$. The transition relation D consists of only two transitions, the "off switch" $(\mathit{On}, 75) \rightarrowtail (\mathit{Off}, 75)$ and the "on switch" $(\mathit{Off}, 45) \rightarrowtail (\mathit{On}, 45)$. Time is a continuum, with instants corresponding to the nonnegative elements of the real numbers, $\mathbb{R}^{\geq 0}$, and ordered in the natural way.

It remains to define the activity labeling function, Act, and the invariant function, Inv. Of these, the invariant function is simplest. Both controller states have the same invariant region in X, namely the region $[45, 75]$, and so

$$\mathit{Inv}(\mathit{On}) = \mathit{Inv}(\mathit{Off}) = [45, 75]. \tag{3.46}$$

The activity labeling function, on the other hand, maps the two controller states to differing set of actions. While the elevator is on, the platform rises at 0m to 4m per unit of time; thus,

$$\mathsf{Act}(\mathit{On}) = \Big\{ a \in \mathfrak{A} \,\Big|\, \forall t \in \Gamma\big(a(t) \in [0, 4]\big) \Big\}. \tag{3.47}$$

Likewise, while off, the platform drops at a rate in the range of -3m to 0m per unit of time, and so

$$\mathsf{Act}(\mathit{Off}) = \Big\{ a \in \mathfrak{A} \,\Big|\, \forall t \in \Gamma\big(a(t) \in [-3, 0]\big) \Big\}. \tag{3.48}$$

In summary, then, under the ACHH framing, the automaton is defined by

$$V \quad = \quad \{\mathit{On}, \mathit{Off}\} \tag{3.49}$$

$$
\begin{aligned}
X &= [0,\infty) \\
Q_0 &= \{\mathit{On}\} \times [50,60] \\
\mathsf{Act}(v) &= \begin{cases} \{a \in \mathfrak{A} \mid \forall t \in \Gamma\,(a(t) \in [0,4])\}, & v = \mathit{On} \\ \{a \in \mathfrak{A} \mid \forall t \in \Gamma\,(a(t) \in [-3,0])\}, & v = \mathit{Off} \end{cases} \\
\mathit{Inv}(v) &= [45,75], \forall v \in V \\
D &= \{(\mathit{On},75) \rightarrowtail (\mathit{Off},75), (\mathit{Off},45) \rightarrowtail (\mathit{On},45)\} \\
\Gamma &= \mathbb{R}^{\geq 0}.
\end{aligned} \tag{3.50}
$$

3.4.2 LSV representation

I now construct the LSV representation of the same automaton. While it would be possible to derive the LSV representation “from scratch,” I instead construct it from the ACHH representation using the transformation presented earlier. Doing so guarantees that both representations are identical and that the resulting set of trajectories is differentiably legal, as per Theorem 1.

As noted earlier, the two framings share several components, and so I can simply repeat the definitions for the controller states V, environment states X, initial states Q_0, transition relation D, and time Γ. It remains to define the set of trajectories, $\mathcal{T}$. Recall from (3.29) that

$$
\begin{aligned}
\mathcal{T} = \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is second-degree doubly semidifferentiable}\big) \\
\wedge\, \forall t \in \mathit{dom}(\tau)\Big[\big(\Pi_X(\dot\tau(t)) \in \mathsf{Act}(\tau(t))\big) \wedge \big(\Pi_X(\tau(t)) \in \mathit{Inv}(\Pi_V(\tau(t)))\big) \\
\wedge\, \forall t' \in \mathit{dom}(\tau)\big(\Pi_V(\tau(t)) = \Pi_V(\tau(t'))\big)\Big].
\end{aligned} \tag{3.51}
$$

Substituting in the relevant definitions from (3.49), the trajectory set consists of all trajectories beginning at 0 that are second-degree doubly semidifferentiable, and such that, for any trajectory τ and instant $t \in \mathit{dom}(\tau)$, $45 \leq \Pi_X(\tau)(t) \leq 75$ and, for all such instants t, either $\Pi_V(\tau)(t) = \mathit{On}$ and $0 \leq \Pi_X(\dot\tau)(t) \leq 4$, or $\Pi_V(\tau)(t) = \mathit{Off}$ and $-3 \leq \Pi_X(\dot\tau)(t) \leq 0$.

Summarizing, then, I represent the LSV formulation of the elevator automaton via the tuple $(V, X, Q_0, D, \mathcal{T}, \Gamma)$, where

$$
\begin{aligned}
V &= \{\mathit{On}, \mathit{Off}\} \qquad (3.52) \\
X &= [0,\infty) \\
Q_0 &= \{\mathit{On}\} \times [50,60] \\
D &= \{(\mathit{On},75) \rightarrowtail (\mathit{Off},75), (\mathit{Off},45) \rightarrowtail (\mathit{On},45)\} \\
\mathcal{T} &= \Big\{\tau \,\Big|\, \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is second-degree doubly semidifferentiable}\big)
\end{aligned}
$$

$$\wedge\ \forall t \in dom(\tau)\big(45 \leq \Pi_X(\tau)(t) \leq 75\big)$$

$$\wedge\ \Big[\big([\tau_V(t) = On] \wedge [0 \leq \Pi_X(\dot{\tau}(t)) \leq 4]\big) \vee \big([\Pi_V(\tau(t)) = Off] \wedge [-3 \leq \Pi_X(\dot{\tau}(t)) \leq 0]\big)\Big]\Big\}$$

$$\Gamma = \mathbb{R}^{\geq 0}.$$

It follows from Theorem 1 that $\mathfrak{T}$ is a differentiably legal set of trajectories.

3.5 Summary of notation

Γ	time
$\mathbb{R}$	the real numbers
A	a hybrid automaton
V	automaton controller state space
v	an automaton controller state
X	automaton environment state space
$\mathbf{x}$	automaton environment state
Q	automaton system states
q	an automaton system state
Q_0	initial states of a hybrid automaton
D	transition relation of a hybrid automaton
(q, q')	a discrete transition from system state q to q'
$q \rightarrowtail q'$	a discrete transition from system state q to q'
α	an execution fragment
$frags_A$	set of all execution fragments for an automaton A
$execs_A$	set of all executions for an automaton A
Q_{reach}	set of all reachable system states in an automaton
$trace(\alpha)$	trace of an execution fragment α
$tracefrags_A$	set of the traces of all execution fragments of an automaton A
$traces_A$	set of all traces of all executions of an automaton A
$dur(\tau)$	duration of trajectory τ
$\dot{\alpha}$	derivative of a differentiable execution fragment α
$\alpha.fstate$	first state of the first trajectory in execution fragment α
$\alpha.lstate$	last state of a right-closed execution fragment α
$\dot{\alpha}.fstate$	derivative of an execution fragment α at its first instant
$\dot{\alpha}.lstate$	derivative of a right-closed execution fragment α at its last instant
$\alpha \frown \alpha'$	concatenation of execution fragments α and α'

a	an activity
$\mathfrak{A}$	set of all possible activities
Act	activity labeling function
Inv	invariant set
(v, v')	edge of a hybrid automaton
E	set of all edges of a hybrid automaton
$Guard$	guard set
$Reset$	reset function

Chapter 4

Mechanics

Before discussing a conversion from hybrid automata to hybrid mechanics, I first present the mathematical foundations of rational mechanics, as in [19, 23]. This chapter introduces a variety of mechanical concepts from the perspective of traditional rational mechanics, dividing them into the sub-areas of kinematics and dynamics, with a final section describing properties for a particular mechanical system. Chapter 5 then revisits all of the concepts presented here, extending them into hybrid equivalents.

4.1 Kinematics

In mechanics, *kinematics* is the study of bodies, their positions and placements, and their movement across time.

4.1.1 Universes and bodies

In rational mechanics, a *universe* Ω consists of a set of structures called *bodies*; in what follows, I assume that every body consists of a set of *body points*. The set of all bodies in a universe forms a boolean lattice; that is, for any bodies $B_1, B_2 \in \Omega$, their union $B_1 \sqcup B_2$, intersection $B_1 \sqcap B_2$, and complements $\overline{B_1}$ and $\overline{B_2}$ are themselves bodies in Ω. Bodies in this lattice are ordered based on the *part-of* or *subbody* relation $\sqsubseteq$ on $\Omega \times \Omega$; that is, $A \sqsubseteq B$ indicates that A is a part of or subbody of B. Similarly, the *proper part* or *proper subbody* relation, denoted $\sqsubset$, imposes an ordering such that $A \sqsubset B$ iff $A \sqsubseteq B$ and $A \neq B$. Any two bodies are always parts of their union, and the intersection of any two bodies is always a part of both of them.

The union of all bodies in Ω forms its greatest element, the *universal body* $\top$, which has all bodies as subbodies and contains all body points. Likewise, the set contains a least element, the *null body* $\bot$, which is a subbody of all bodies in Ω and contains no body points. Two bodies $A, B \in \Omega$ are *separate* iff $A \sqcap B = \bot$. The complement or *exterior* of a body B, written $\overline{B}$, is the unique body such that $B \sqcap \overline{B} = \bot$ and $B \sqcup \overline{B} = \top$.

A *subuniverse* Ω' of Ω is a subset of Ω such that its elements form a boolean lattice under the $\sqsubseteq$ relation.

4.1.1.1 Body topology

Bodies have a topological structure. In particular, let Θ be the set of all open subsets of $\top$; then $(\top, \Theta)$ is a topology of $\top$. Given such a topology, one writes $\texttt{clo}\ B$ to indicate the closure of B in this topology and $\texttt{int}\ B$ to indicate the interior, largest open set in B. A *regularly open* set of body points is a set B such that $B = \texttt{int}\ \texttt{clo}\ B$, in other words, such that the interior of the set's closure is the original set. In practical terms, a regularly open set has no "missing" points in its interior, such as a single point omitted from an otherwise-complete open interval.

Given these definitions, there exists some topology $(\top, \Theta_\top)$ on $\top$ such that every subbody B is a regularly open set of body points, that is, a set of body points with no missing points in its interior.

4.1.2 Space and time

The following presents a relatively simple treatment of space and time. More complex treatments of the subject from the perspective of rational mechanics are possible, as Doyle does in [12]. However, such treatments introduce complexities that are largely irrelevant to my primary concerns here; for those interested in a fuller treatment, Doyle provides a much more comprehensive discussion of the subject.

A *mechanical space* $\mathcal{S}$ in rational mechanics is a real differentiable manifold consisting of a set of *positions* or *points*. Every mechanical space has an associated distance metric defined over this set of positions. While in general many complex spatial topographies are possible, I simplify things by considering only mechanical spaces that are isomorphic to vector spaces. Thus, for some $n \geq 0$,

$$\mathcal{S} \cong \mathbb{R}^n. \tag{4.1}$$

where $\cong$ indicates an isomorphism.

Time Γ is defined similarly to previous chapters, as a nonempty set with an ordering $\leq$, with any particular value in this set referred to as an *instant*. A particular choice of position and instant, then, defines a unique event in *space-time*, the product $\mathcal{S} \times \Gamma$. Colloquially, an instant t_1 (or an event occurring in t_1) is said to be *before* another instant t_2 (or an event occurring in t_2) precisely when $t_1 < t_2$. One calls two such instants (or events) *simultaneous* precisely when $t_1 = t_2$.

4.1.3 Placement and motion

Given a body $B \in \Omega$, a *placement* of B in $\mathcal{S}$ assigns a position in space to each body point in B in a way that fits with the topologies of both the body and space. More formally, a placement of B is a mapping $\chi : B \to \mathcal{S}$ that is continuous with respect to $(\top, \Theta_\top)$ and $(\mathcal{S}, \Theta_\mathcal{S})$. Note that, since the null body contains no body points, its only placement is

$$\chi(\perp) = \emptyset. \tag{4.2}$$

A body's placement also determines the placement of any subbodies, whose placements must agree with the superbody on all shared body points; thus, the placement $\chi : \top \to \mathcal{S}$ of the universal body $\top$ determines the placement of all other bodies. One can then write $\chi(B)$ to indicate the restriction of χ to the body points some $B \sqsubseteq \top$.

For simplicity, one often restricts consideration to *regular* placements, the placements of a body that do not cause the body to tear, colocate, or puncture. Given these restrictions, a regular placement of a body B is a diffeomorphism between the body points of B and the image $\chi(B)$. I write $\mathcal{C}(\top, \mathcal{S})$, the *placement space* of $\top$, to indicate the set of all regular placements of the universal body; more generally, for any body $B \in \Omega$, one writes $\mathcal{C}(B, \mathcal{S})$ to indicate the set of all regular placements of B in $\mathcal{S}$. Since a regular placement is diffeomorphic, if B is an n-dimensional manifold, its placement is also an n-dimensional manifold. One writes $T\chi(B)$ to indicate the tangent bundle of the image of B. Then, if B is an n-dimensional manifold,

$$T\chi(B) \cong \chi(B) \times \mathbb{R}^n. \tag{4.3}$$

The placement of a body may be different in differing instants, and so one can extend the notion of placements to be a family of functions $\chi_t : \top \to \mathcal{S}$. Each such function gives the placement of the universal body for the instant $t \in \Gamma$, and so also gives the placement of all other bodies for t.

Recall from Section 2.3.3 that a *deformation* of a subset of a manifold is a mapping $\chi : J \to (U \to \mathfrak{M})$, where U is an open set in $\mathfrak{M}$ and J is an J. Letting $\mathfrak{M} = \mathcal{S}$ and $U = B$ for some $B \in \Omega$, the set of homeomorphisms $(U \to \mathfrak{M})$ becomes the set of placements of B, that is, $\mathcal{C}(B, \mathcal{S})$. Then the original deformation is a mapping $\chi : J \to \mathcal{C}(B, \mathcal{S})$, that is, a continuous mapping from time to placements of B. Such deformations are called *motions* of B.

One more commonly considers motions to be mappings $\chi : \top \times \Gamma \to \mathcal{C}(\top, \mathcal{S})$ such that $\chi(b, t)$ associates every body point b and instant t with a point in space. As with deformations in general, one can hold either the body point or the instant constant to provide a mapping over the remaining variable, writing χ_b or χ_t, respectively. Thus,

$$\chi_b(t) = \chi_t(b) = \chi(b, t). \tag{4.4}$$

This relationship is illustrated in Figure 4.1. Under this view, for any point body B, I abuse notation to write $\chi(B, t)$ to indicate both the image of the single body point in B at t, and to refer to the particular point in $\mathcal{S}$ to which that function maps.

$$\begin{tikzcd}
B \arrow[dr, "\chi_t"] & \\
\Gamma \times B \arrow[u, "\pi_2"] \arrow[r, "\chi"] \arrow[d, "\pi_1"'] & \mathcal{S} \\
\Gamma \arrow[ur, "\chi_b"'] &
\end{tikzcd}$$

Figure 4.1: The mappings χ, χ_b, and χ_t. The first allows both the choice of body point and instant to vary freely; the second holds the body point fixed, while the instant varies; the third holds the instant fixed, while the body point varies.

4.1.4 Velocity, configuration, and acceleration

The *material velocity*, or simply *velocity*, of a body point is the rate of change of the position of that point over time. Under the assumption that $\mathcal{S} = \mathbb{R}^n$, then, a body point's velocity is a vector in $\mathbb{R}^n$. Given a motion χ, a body point b, and an instant $t \in \Gamma$, one writes $\dot{\chi}_t(b)$ to indicate the velocity of b at t, or one writes $\dot{\chi}(b, t)$ if both the choice of instant and body point can vary. More generally, $\dot{\chi}_t(B)$ is a mapping from the body points of B to velocity values, and $\dot{\chi}(B, t)$ is a mapping from body points of B and instants to velocity values. As with position, if B consists of a single body point, I abuse notation to use $\dot{\chi}_t(B)$ to indicate both the mapping of that body point and the particular velocity value to which the point is mapped.

For current purposes, it will be useful to consider pairs of the form $(\chi_t, \dot{\chi}_t)$, that is, ordered pairs such that χ_t is the assignment of position to every body point in $\top$ at some instant t, and likewise $\dot{\chi}_t$ is the assignment of velocity to every body point in $\top$ at t. In such cases, given a body point b, the ordered pair $(\chi_t(b), \dot{\chi}_t(b))$ is a tangent at $\chi_t(b)$.

In later chapters, it will often be useful to specify such ordered pairs without reference to a particular motion. One could certainly use $(\chi, \dot{\chi})$ in such cases, but this would force awkward subscripting in proofs where I must represent several separate motions and ordered pairs of this form. Thus, I commonly substitute the notation $(w, \dot{w})$ for such position and velocity mappings. So, for instance, I might say, "Given an ordered pair $(w, \dot{w})$, let χ be a motion such that, at some instant t, it is the case that $\chi_t = w$ and $\dot{\chi}_t = \dot{w}$.'

4.1.4.1 Material velocity fields and configuration spaces

The *material velocity field* at instant t in motion χ, then, is the mapping $\mathbf{v}_t : \top \to T\chi_t(\top)$ that associates every body point b with the position $\chi_t(b)$ and the velocity $\dot{\chi}_t(b)$. In manifold terms, the material velocity field is thus a vector field covering the placement χ_t at t. Following the notation for vector fields, one writes $\mathfrak{X}(B, \mathcal{S})$ for the space of all material velocity fields over a body B. In what follows, I am most often concerned with $\mathfrak{X}(\top, \mathcal{S})$, that is, the space of all material velocity fields over $\top$.

Note that the space of placements of $\top$ is some subset of the set $(\top \to \mathcal{S})$. Abusing notation to write $\mathcal{S}$ for both the mechanical space and the corresponding vector space, then, the set of all assignments of velocity to body points is similarly some subset of $(\top \to \mathcal{S})$. As noted, a material velocity field maps every body point to a position and a velocity, and so $\mathfrak{X}(\top, \mathcal{S})$ is isomorphic to some subset of $(\top \to \mathcal{S}) \times (\top \to \mathcal{S})$.

I more commonly indicate a particular material velocity field by separating these two mappings, writing $(w, \dot{w})$ to indicate the mappings from body points to positions and velocity values, respectively. Written in this fashion, one can think of a material velocity field as one kind of *configuration* of $\top$. In general, mechanics uses the term "configuration" to refer to a range of possible tuples that provide a partial or complete "snapshot" of the behavior of a system. For instance, a configuration in general might include further spatial derivatives, or it might contain non-positional information such as electric charge; on the other hand, a configuration of rigid bodies might not need to specify the position and velocity of all body points. For the purposes of this paper, however, the term "configuration" will always be used to refer to such an ordered pair of mappings $(w, \dot{w})$. One can then refer to $\mathfrak{X}(\top, \mathcal{S})$ as the *configuration space* of $\top$, that is, the set of all configurations of $\top$.

As a simple example, suppose one wished to model a particular example of planetary motion using nine point bodies, each taking positions in $\mathbb{R}^3$. In such a case, the placement space for each body would be $\mathbb{R}^3$, and so the placement space for the universal body would be $(\mathbb{R}^3)^9 = \mathbb{R}^{27}$. The configuration space of such a system is then the product of the space of placements of $\top$ and the space of assignments of velocity vectors to all nine body points, and so

$$\begin{aligned} \mathfrak{X}(\top, \mathcal{S}) &\cong \mathbb{R}^{27} \times \mathbb{R}^{27} \\ &= \mathbb{R}^{54}. \end{aligned} \tag{4.5}$$

4.1.4.2 Acceleration

The *acceleration* of a body is the change in the velocity of its body points over time. If $\mathcal{S}$ is an n-manifold and a motion χ for $\top$ has a differentiable derivative, then its acceleration is the function $\ddot{\chi} : \top \times \Gamma \to \mathbb{R}^n$, where at every instant $\ddot{\chi}$ is the second derivative of χ.

One can define a *material acceleration field* over a body just as one defines a material velocity field; in other words, a material acceleration field associates each body point with a placement and an acceleration value. Such constructions play no part in this paper, however, and so I omit further discussion of them. As with velocity, given a point body b, I write $\ddot{\chi}(b)$ to indicate both the acceleration mapping and the particular acceleration vector to which the single body point is mapped.

4.1.4.3 Discrete velocity and acceleration

The above definitions of velocity and acceleration are insufficient if time and space are discrete, as in such cases the motion of b is not continuous, much less differentiable. To resolve this problem, one can define discrete velocity and acceleration as follows. Let Γ consist of the natural numbers, ordered in the normal way; that is, $\Gamma = \{0, 1, 2, \ldots\}$, and any instant i precedes an instant j iff $i < j$. Given any body point b and instant $i > 0$, one can define the velocity of b at i to be the change in its placement from $i - 1$ to i. That is,

$$\dot{\chi}(b, i) = \chi(b, i) - \chi(b, i - 1). \tag{4.6}$$

Likewise, one can define the acceleration at any non-final $i > 0$ to be the difference between the velocities at $i + 1$ and i. That is,

$$\ddot{\chi}(b, i) = \dot{\chi}(b, i + 1) - \dot{\chi}(b, i). \tag{4.7}$$

or, in terms of placement,

$$\begin{aligned} \ddot{\chi}(b, i) &= [\chi(b, i + 1) - \chi(b, i)] - [\chi(b, i) - \chi(b, i - 1)] \\ &= \chi(b, i + 1) - \chi(b, i) - \chi(b, i) + \chi(b, i - 1). \end{aligned} \tag{4.8}$$

Given this construction, discrete velocity is always *trailing*, based on the placements of preceding instants; discrete acceleration is always *leading*, based on the velocity of future instants.

4.2 Dynamics

Given the basic framework of kinematics, *dynamics* extends it with ideas of mass, momentum, and force.

4.2.1 Mass

A body may have an associated nonnegative real-valued *mass*, and one refers to such bodies as *massy bodies*. One write $\mathcal{M}$ to indicate the set of possible mass values. One typically assumes that all bodies have nonnegative real-valued mass, that is, that $\mathcal{M} = \mathbb{R}^{\geq 0}$, but other constructions are possible; in later chapters, for instance, it will sometimes be useful to construct bodies for which the space of masses is $\mathbb{Z}_2$, that is, where all massy bodies have a mass of either 0 or 1. Following the ordinary assumption that mass is invariant over time, one writes simply $m(B)$ to indicate the mass of massy body B, where $m : \Omega \to \mathcal{M}$.

Masses are additive on separate bodies, so that for any separate bodies $B_1, B_2 \in \Omega$,

$$m(B_1 \sqcup B_2) = m(B_1) + m(B_2), \tag{4.9}$$

or, more generally, for any (possibly overlapping) bodies $B_1, B_2 \in \Omega$,

$$m(B_1 \sqcup B_2) = m(B_1) + m(B_2) - m(B_1 \sqcap B_2). \tag{4.10}$$

For bodies consisting of a discrete set of body points, this means that a body's mass is the sum of the masses of its body points; for continuum bodies, the mass of the whole is found by integrating over the set of body points.

Since $\perp$ is separate from all bodies, it follows from (4.9) that for any body $B \in \Omega$,

$$m(B \sqcup \perp) = m(B) + m(\perp), \tag{4.11}$$

and so, since $B \sqcup \perp = B$,

$$m(B) = m(B) + m(\perp). \tag{4.12}$$

Subtracting $m(B)$ from both sides, it follows that

$$m(\perp) = 0. \tag{4.13}$$

4.2.2 Momentum

Momentum is often thought of as simply the product of mass and velocity, $p = mv$. In general, the space of momenta is the vector space formed by the set of velocity vectors over the mass value scalars; where masses are $\mathbb{R}^{\geq 0}$, then, momenta are values in the vector space $\mathcal{S}$. Formally, one defines momentum p to be a mapping from each body, its mass, its placement, and a particular instant to a vector in this space. For a body point b, its momentum at instant t is

$$p(b, m(b), \chi(b), t) \overset{\text{def}}{=} m(b)\dot{\chi}(b, t). \tag{4.14}$$

This is, in effect, the initial expression $p = mv$. If masses are real-valued, then the set of all momenta values is simply the vector space $\mathcal{S}$.

Momentum is additive; thus, for a body B consisting of finite sets of body points, the momentum of B is the sum of the momenta of its body points,

$$p(B, m(B), \chi(B), t) = \sum_{b \in B} p(b, m(b), \chi(b), t). \tag{4.15}$$

As with mass, the momentum of a continuum body is the integral of the momenta of its set of body points.

4.2.3 Force systems

Mechanics represents a *force* as a vector associated with an ordered pair of bodies. A *force system* associates each pair of separate bodies in Ω with a member of an additive, commutative group $\mathcal{V}$ of force values, where the space of force values is the same as the space of momenta; then as in the preceding section, if masses are real-valued,

$$\mathcal{V} = \mathcal{S}. \tag{4.16}$$

This group must contain some null element $\mathbf{0}$, called the *null force*. That is, a force system is a mapping $f : (\Omega \times \Omega)_0 \to \mathcal{V}$, where $(\Omega \times \Omega)_0$ is the set of all ordered pairs of separate bodies in Ω. The notation $f(A, B)$, read "The force of B on A," indicates the force value associated with the pair (A, B), where A, B are separate bodies in Ω. The *resultant force* on a body B is the force of its exterior on it, that is, $f(B, \overline{B})$.

Noll's axioms of mechanics require that every force system satisfy three properties: that it be additive, null-passive, and pairwise equilibriated. A force system is *additive* iff, given any three separate bodies, the sum of the forces of any two on the third is the same as the force of their union on that body. Likewise, the force of the third body on the union of the other two should be the sum of its forces on the two subbodies. That is, for separate bodies $A, B_1, B_2 \in \Omega$,

$$\begin{aligned} f(A, B_1 \sqcup B_2) &= f(A, B_1) + f(A, B_2) \\ f(B_1 \sqcup B_2, A) &= f(B_1, A) + f(B_2, A). \end{aligned} \tag{4.17}$$

A force system is *null-passive* iff the force of the null body on any other body is the null force $\mathbf{0}$. Likewise, the force of any other body on $\perp$ should be $\mathbf{0}$. That is, for any $B \in \Omega$,

$$f(B, \perp) = f(\perp, B) = \mathbf{0}. \tag{4.18}$$

Finally, a force system is *pairwise equilibriated* if the forces of any two separate bodies on each other are inverses. Intuitively, pairwise equilibriation is the Newtonian claim that for every action, there is an equal and opposite reaction. Formally, for separate bodies $A, B \in \Omega$,

$$f(A, B) = -f(B, A). \tag{4.19}$$

Given that a force system satisfies the above properties, one can extend the original mapping f to cover non-separate bodies; thus, the mapping becomes $f : \Omega \times \Omega \to \mathcal{V}$. The set of all such force systems, for a particular universe Ω and set of force values $\mathcal{V}$, is written $\mathcal{F}(\Omega, \mathcal{V})$.

4.2.4 Force variations

Forces present in a universe may vary with time, and so one can define a function $\hat{f} : \Gamma \to \mathcal{F}(\Omega, \mathcal{V})$ that maps each instant to a force system. One calls such a function $\hat{f}$ a *force variation*. For any instant $t \in \Gamma$, one writes f_t to indicate the force system associated with t; that is, $f_t = \hat{f}(t)$. The expression $f_t(A, B)$ is read as "the force of B on A at time t."

Given two force variations $\hat{f}, \hat{f}'$, the sum $\hat{f} + \hat{f}'$ and difference $\hat{f} - \hat{f}'$ are the force variations that at every instant t are equal to the sum or difference of the force systems f_t and f'_t. In other words, for every instant $t \in \Gamma$,

$$\begin{aligned}(\hat{f} + \hat{f}')(t) &\stackrel{\text{def}}{=} \hat{f}(t) + \hat{f}'(t) \\ (\hat{f} - \hat{f}')(t) &\stackrel{\text{def}}{=} \hat{f}(t) - \hat{f}'(t).\end{aligned} \tag{4.20}$$

One writes $\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$ to indicate the set of all force variations.

4.3 Mechanical systems

4.3.1 System laws

The preceding sections have described the components - bodies, placements, assignments of mass, and so on - that underlie any given mechanical system. Any specific system will thus have a particular set of bodies, with a particular mass function, and so on. Such a system will also need a set of *system laws* that govern the evolution of the system over time.

As a simple example, a classic mechanical system intended to model planetary motion might include the law of universal gravitation,

$$\hat{f}(t)(A, B) = G\frac{m(A)m(B)}{[\chi(A, t) - \chi(B, t)]^2}, \tag{4.21}$$

or, as it is more commonly written,

$$f = G\frac{m_1 m_2}{r^2}, \tag{4.22}$$

where G is the gravitational constant. Together with whatever other laws the planetary modeling system includes (perhaps, in some particularly catastrophic systems, laws for collision!), the law of universal gravitation determines the force on all bodies over time; thus, it determines their acceleration, and so their motion.

More generally, system laws may define the evolution of other properties of the mechanical system besides forces. They might, for instance, govern the dissipation of heat, or the conduction of electrical current. Such laws might be purely functions of the position of various body points, as in the example above, but they might also be functions of velocity, or of time, or other properties of the system. Regardless, in principle, a particular set of system laws and a particular set of starting conditions suffice to determine the possible configurations of the mechanical system at all future instants.

4.3.1.1 Types of system laws

For the purposes of this paper, I restrict attention to systems in which the only laws of interest are those regarding forces between pairs of bodies, ignoring laws regarding heat, conduction, and so on. Further, I restrict attention to systems in which the laws are functions only of configurations of the universal body, or some subbody thereof. Even given such a restriction, several possible constructions for laws suggest themselves. A law might be *autonomous*, with the possible forces the same for every instant with a given position or velocity, or it might be *time-varying*, where identical positions and velocities might produce different possible forces at different times. Similarly, laws may be *deterministic*, such that a particular assignment of position and velocity identifies a singular force system or force variation, or *nondeterministic*, such that a particular choice of position and velocity identifies a set of possible force systems or variations. One can think of a nondeterministic system as a description of a set of system laws that might hold in a given configuration, such that we are ignorant as to which of the laws will actually hold for a given motion of the system.

In the simplest case, every choice of configuration for $\top$ identifies a single force system. In such cases, whatever laws apply to the system can be summarized via an *autonomous deterministic law function* $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to \mathcal{F}(\Omega, \mathcal{V})$. In other words, given a configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, one writes $\mathfrak{L}(w, \dot{w})$ to indicate the force system specified by the system law function. Given a motion χ, an instant t, and any pair of bodies A, B, then, the force of B on A at t is given by

$$\mathfrak{L}(\chi_t, \dot{\chi}_t)(A, B). \tag{4.23}$$

By contrast to this deterministic approach, an *autonomous nondeterministic law function* is a mapping $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\mathcal{F}(\Omega, \mathcal{V})}$, that is, a mapping from any assignment of position and velocity to $\top$ to a set of forces. Such sets are often termed *force inclusions* and will be discussed in greater detail in Chapter 6. One calls a mechanical system in which the law function is autonomous and (non)deterministic an *autonomous (non)deterministic mechanical system.*

One may define a law function to be time-varying, instead of autonomous. For instance, a system law might specify that some forces exist with a particular duration, or at particular intervals; one can imagine a system where full modeling of the moon was beyond the area of interest, but modeling the periodic effects of the moon's gravity on the earth was still desirable. One might not represent the moon as a body with a mass and a position, in such cases; instead, one might simply define a system law in which lunar gravity is represented as a force that waxes and wanes over time. Typically, one represents such behavior via a function of the form

$$\mathfrak{L} : \Gamma \to (\mathfrak{X}(\top, \mathcal{S}) \to \mathcal{F}(\Omega, \mathcal{V})). \tag{4.24}$$

In such cases, each instant is associated with an autonomous system law.

For the purposes of this paper, however, (4.24) will not be the most useful way to think of time-varying system laws. Later sections, including 4.3.1.2, focus on equivalence classes of configurations, where elements of a class are equivalent with respect to the system law function. The notation for such discussion is awkward if system laws are written in the above form, where time is mapped to mappings from configurations. Thus, in the remainder of this paper, I view system laws as mappings from configurations to force variations, where a force variation is a mapping from time to force systems.

More formally, any law of the form in (4.24) can be rewritten as a mapping

$$\mathfrak{L} : \Gamma \times \mathfrak{X}(\top, \mathcal{S}) \to \mathcal{F}(\Omega, \mathcal{V}) \tag{4.25}$$

and so as a mapping

$$\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to (\Gamma \to \mathcal{F}(\Omega, \mathcal{V})). \tag{4.26}$$

The set of all possible mappings $\Gamma \to \mathcal{F}(\Omega, \mathcal{V})$ is simply the set $\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$, that is, the set of all force variations over Γ, as discussed in Section 4.2.4. Following the notation in that section, then, one can rewrite any time-varying vector field as a mapping

$$\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma). \tag{4.27}$$

A *time-varying deterministic law function*, then, is just such a mapping $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$, where for any pair of bodies A, B, the force of B on A at $t \in \Gamma$ is given by

$$\mathfrak{L}(\chi_t, \dot{\chi}_t)(t)(A, B). \tag{4.28}$$

Contrasting with the above deterministic definition, a *time-varying nondeterministic law function* is a mapping $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)}$. A set in $2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)}$, that is, a set of force variations, is called a *force variation inclusion*. For the purposes of this paper, I restrict attention to mechanical systems for which the system laws are represented by a time-varying nondeterministic law function. Where unambiguous, I typically drop the qualifiers and refer simply to a law function.

4.3.1.2 Equivalence classes of system laws

Conceptually, two configurations are equivalent with respect to a given system law function if both configurations map to the same force system or variation, for deterministic law functions, or the same set of force systems or variations for nondeterministic law functions. Practically, one can think of such configurations as having the same effect; the force (or possible forces) on every body, and so the acceleration, is the same for both configurations.

More formally, given a system law function $\mathfrak{L}$ of whatever sort, one can define an equivalence relation $\equiv_{\mathfrak{L}}$ over $\mathfrak{X}(\top, \mathcal{S})$ such that, for any configurations $(w, \dot{w}), (w', \dot{w}') \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_{\mathfrak{L}} (w', \dot{w}') \text{ iff } \mathfrak{L}(w, \dot{w}) = \mathfrak{L}(w', \dot{w}'). \tag{4.29}$$

This equivalence relation imposes a partition $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ in which any equivalence class $[(w, \dot{w})]$ consists of all configurations mapped to the same values as $(w, \dot{w})$ by $\mathfrak{L}$.

One can then extend the previous definition of a system law function to a mapping from equivalence classes to force systems or variations (or sets thereof). So, for instance, given a time-varying nondeterministic law function $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)}$, one can construct an alternative mapping

$$\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}} \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)} \tag{4.30}$$

taking equivalence classes of configurations to sets of force variations. This equivalence class-based construction has the practical benefit that it may allow a simpler description of $\mathfrak{L}$, particularly if $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ is a finite set of equivalence classes. It will also be valuable in later chapters for describing *mechanical controller states*, which will eventually be taken to be equivalence classes of configurations.

4.3.2 Induced motion

As noted above, the resultant force on a body is the force of that body's exterior on it. An *inertial frame* is a frame of reference such that the momentum on a body is constant iff its resultant force is 0. The Newton-Euler laws of motion state that, relative to any inertial frame, the resultant force on a body (that is, the force of its exterior on it) is equal to the change in the body's momentum. More formally, the Newton-Euler Law says that for a body B in Ω and instant t in Γ, in any inertial frame of B,

$$f_t(B, \overline{B}) = \dot{p}(B, m(B), \chi(B), t). \tag{4.31}$$

Taking the derivative of momentum with respect to time and recalling equation (4.14), this expression yields

$$f_t(B, \overline{B}) = \dot{m}(B)\dot{\chi}(B, t) + m(B)\ddot{\chi}(B, t). \tag{4.32}$$

If mass is constant over time, $\dot{m}(B) = 0$, and so the above expression reduces to

$$f_t(B, \overline{B}) = m(B)\ddot{\chi}(B, t), \tag{4.33}$$

or, in its more common form, $f = ma$.

One can think of a system law as something similar to a vector field over configuration space; for every assignment of position and velocity to the body points of the universal body, it specifies the force (and thus, the acceleration) on every body point. A motion obeying the Newton-Euler laws is then one in which every body's acceleration is equal to this force divided by the body's mass. If the system law is taken to be a vector field over configuration space, then such a motion is an integral curve for that vector field. I call such motions *induced motions* for the system law function.

More formally, suppose $\mathfrak{L}$ is a time-varying deterministic system law function. I say that a motion χ of the universal body is an *induced motion* for $\mathfrak{L}$ iff, at every instant, the motion of every body obeys the Newton-Euler Laws for the force variation specified by $\mathfrak{L}$. In other words, χ is an induced motion iff, for every instant $t \in \Gamma$ and body $B \in \Omega$,

$$\mathfrak{L}(\chi_t, \dot{\chi}_t)(t)(B, \overline{B}) = m(B)\ddot{\chi}(B, t). \tag{4.34}$$

In later sections, it will sometimes be useful to discuss a time-varying deterministic system law function for which every configuration maps to a single force variation $\hat{f}$, that is, a function $\mathfrak{L}$ such that, for any configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$,

$$\mathfrak{L}(w, \dot{w}) = \hat{f}. \tag{4.35}$$

Clearly in such cases, the equivalence class $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ contains only a single element, that is, the class mapping all configurations to $\hat{f}$. Thus,

$$\mathfrak{L}(\mathfrak{X}(\top, \mathcal{S})) = \hat{f}. \tag{4.36}$$

By a slight abuse of notation, in such cases I say that a motion is an induced motion for $\hat{f}$ iff it would be an induced motion for $\mathfrak{L}$. This avoids the need to define a trivial system law function and then immediately replace it with the appropriate force variation.

Suppose instead that $\mathfrak{L}$ is a time-varying nondeterministic system law function, and let $\equiv_{\mathfrak{L}}$ be the corresponding equivalence relation. Since $\mathfrak{L}$ is nondeterministic, each configuration maps to a set of force variations. Suppose further that, for some interval of a motion χ, all of the configurations through which χ passes are part of the same equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$. In other words, at every instant in some interval, the configuration of the universal body at that instant maps to the same set of force variations. A natural assumption is that there exists some single force variation in that set that is active at all points in the interval. Informally, one can think of such a system as choosing a force variation whenever the motion enters a new equivalence class, using that force variation as long as the motion remains in the equivalence class, and switching to a new force variation when the equivalence class changes. The nondeterminism, in this case, amounts to uncertainty as to which law will be in effect for the duration of the current equivalence class.

More formally, again let χ be a motion of the universal body. At every instant $t \in \Gamma$, the configuration $(\chi_t, \dot{\chi}_t)$ must be in some equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$. One can define a *configuration-time* function $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L}) : \Gamma \to \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ mapping each instant to the corresponding equivalence class; that is, for any instant $t \in \Gamma$,

$$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(t) \stackrel{\text{def}}{=} [(\chi_t, \dot{\chi}_t)]. \tag{4.37}$$

Such a mapping partitions Γ into a set of intervals $\mathscr{I}_{\chi,\mathfrak{L}}$ such that, given an interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, for any instants $t, t' \in J$,

$$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(t) = \mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})_{\mathfrak{L}}(t'), \tag{4.38}$$

and such that this property does not hold for any larger interval J' containing J. In other words, $\mathscr{I}_{\chi,\mathfrak{L}}$ is the unique set of intervals such that every interval is as large as possible without containing instants that map to different equivalence classes. By a slight extension of notation, one can write $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(J)$ to refer to the equivalence class mapped to by all instants in J.

Then, intuitively, for each interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, it seems reasonable that there should be a single force variation $\hat{f} \in \mathfrak{L}(\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(J))$ such that the motion of every body obeys the Newton-Euler Laws relative to $\hat{f}$ at every instant in J. Formally, χ is an induced motion for $\mathfrak{L}$ iff, for every interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, there exists some force variation $\hat{f} \in \mathfrak{L}(\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(J))$ such that for every instant $t \in J$ and body $B \in \Omega$,

$$f_t(B, \overline{B}) = m(B)\ddot{\chi}(B, t). \tag{4.39}$$

4.3.3 Mechanical elements of a system

I say that the *elements of a mechanical system* are the components discussed to this point: the bodies, space, time, masses, and forces described in the preceding sections. One then defines a particular *mechanical system* by pairing particular choices for each of these elements with a particular system law function. The result is a tuple $(\Omega, \mathcal{S}, \Gamma, m, \mathfrak{L})$, where Ω is the universe of bodies, $\mathcal{S}$ is the mechanical space, Γ is the ordered set of instants, m is the mass function, and $\mathfrak{L}$ is the time-varying nondeterministic system law function.

4.3.4 Initial configurations

One does not typically consider the initial assignment of position, velocity, and other properties of interest to a mechanical system as itself part of the system in question. Instead, one considers various assignments of initial properties to a given system and then traces the change in the system over time due to its particular system laws. However, as noted in Chapter 3, hybrid automata typically specify a set of initial system states. Thus, in constructing a mechanical equivalent to a hybrid automaton, I must specify a corresponding set of initial mechanical configurations. Rather than be concerned purely with a mechanical system, then, later chapters will describe the construction of a tuple containing both the base system and a set of initial configurations. One might formally call such a tuple an *initialized mechanical system*; as I will generally only be concerned with such systems, I typically drop the "initialized" and refer to them simply as mechanical systems.

Thus, for any initialized mechanical system, I provide a subset of $\mathfrak{X}(\top, \mathcal{S})$ as a set of *initial configurations*, denoted $\mathfrak{X}_0(\top, \mathcal{S})$. Given an initial configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, I say that a motion χ *begins* in $(w, \dot{w})$, or that χ is a motion *beginning in* $(w, \dot{w})$, iff

$$\begin{aligned} \chi(\top, 0) &= w \\ \dot{\chi}(\top, 0) &= \dot{w}. \end{aligned} \tag{4.40}$$

4.4 Conclusion

To characterize an initialized mechanical system according to the features of interest in this dissertation, then, requires a tuple $(\Omega, \mathcal{S}, \Gamma, m, \mathfrak{L}, \mathfrak{X}_0(\top, \mathcal{S}))$, where Ω is the set of all bodies, $\mathcal{S}$ is a mechanical space, Γ is time, $\mathfrak{L}$ is the time-varying nondeterministic law function, and $\mathfrak{X}_0(\top, \mathcal{S})$ is the initial configuration set.

4.5 Summary of notation

Ω	the universe of bodies
B	a body
$B \sqcup B'$	union of bodies B and B'
$B \sqcap B'$	intersection of bodies B and B'
$\sqsubseteq$	subbody relation on $\Omega \times \Omega$
$\top$	the universal body
$\bot$	the null body
Θ	set of all open subsets of $\top$
$(\top, \Theta)$	a topology of $\top$
$\overline{B}$	exterior of body B
$\mathtt{clo}\, B$	closure of body B
$\mathtt{int}\, B$	interior of body B
$\mathcal{S}$	a mechanical space
t	an instant in time
Γ	time
χ	placement function

$\mathcal{C}(B, \mathcal{S})$	set of all placements of body B in mechanical space $\mathcal{S}$
$\cong$	isomorphic to
χ_t	placement function for instant t
b	a body point
$\dot{\chi}$	velocity function
$\ddot{\chi}$	acceleration function
$\mathfrak{X}(\top, \mathcal{S})$	configuration space of the universal body
w	one placement of a body
$\dot{w}$	one assignment of velocity to a body
$(w, \dot{w})$	a configuration of the universal body
$\mathcal{M}$	set of mass values
m	mass function
p	momentum function
$\mathcal{V}$	set of force values
$\mathbf{0}$	the null force
f	force system
$f(A, B)$	force of a body B on body A
$f(B, \overline{B})$	resultant force on body B
$\mathcal{F}(\Omega, \mathcal{V})$	set of all force systems for universe Ω
$\hat{f}$	force variation
f_t	force system associated with instant t
$\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$	set of all force variations for universe Ω
$\mathfrak{L}$	system law function
$\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$	equivalence class of configurations mapped to the same force systems by system law function $\mathfrak{L}$
$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$	configuration-time function associating every instant with an equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$
$\mathscr{I}_{\chi, \mathfrak{L}}$	partition of Γ into intervals imposed by $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$
$\mathfrak{X}_0(\top, \mathcal{S})$	initial configuration set for universal body $\top$

Chapter 5

Hybrid Mechanics

I now turn to *hybrid mechanics* [12], which considers universes constructed from two or more *factor* universes while preserving many of the features of their factors. Hybrid mechanics provides a natural parallel to the structure of a hybrid automaton. Different factors of a hybrid universe might correspond to the parts of an automaton: one factor space to model the controller, for instance, and another to model the environment. Interactions between controller and environment would then correspond to interactions between bodies of different factor universes, perhaps in the form of forces or heatings.

In the following pages, then, I provide the terminology of hybrid mechanics, showing how a hybrid universe can be constructed from a set of factor universes. As before, I divide this chapter into discussions of *hybrid kinematics* and *hybrid dynamics*, followed by a discussion of the additional properties which must be given for a *hybrid mechanical system*.

5.1 Hybrid kinematics

5.1.1 Hybrid universes and bodies

Let U be an indexed set of universes $\{\Omega_1, \Omega_2, \ldots\}$. I write $\mathcal{I}$ to indicate the set of indices for U, so that for every index $i \in \mathcal{I}$, the universe Ω_i is in U. Each universe Ω_i has its own associated space $\mathcal{S}_i$, mass function m_i, time Γ_i, and force values $\mathcal{V}_i$. Differing universes may, but need not, share bodies or dimensions. A *hybrid mechanical system* is a combination of these universes, merging their bodies, spaces, time, forces, and other mechanical features into a single system. In particular, a *hybrid universe* Ω is a Boolean lattice formed by summing these factor lattices,

$$\Omega = \bigoplus_{i \in \mathcal{I}} \Omega_i. \tag{5.1}$$

Lattice theory requires that the sum of a set of lattices be a lattice, and so the hybrid universe is, itself, a universe; as before, then, it is a lattice with universal and null bodies and a sub-body relationship. In particular, the hybrid universal body $\top$ is the join of all body points from all factor universes, that is,

$$\top = \bigsqcup_{i \in \mathfrak{I}} \top_i. \tag{5.2}$$

Likewise, the hybrid null body $\bot$ is the intersection of the null bodies of all factor universes, that is,

$$\bot = \bigsqcap_{i \in \mathfrak{I}} \bot_i. \tag{5.3}$$

As well, this lattice sum preserves the subbody relationship in factor universes. Thus, if one body is a subbody of another in some factor universe, it will also be a subbody in the hybrid universe. Each factor universe Ω_i is then a sublattice of the lattice of the hybrid universe.

It is in general possible that a body has non-null components in two or more factor universes. For any such body $B \in \Omega$, I write $\mathcal{I}(B)$ to indicate the set of all factor universes for which B has a non-null component. That is,

$$\mathcal{I}(B) \stackrel{\text{def}}{=} \{i \mid B \sqcap \top_i \neq \bot\}. \tag{5.4}$$

I refer to $\mathcal{I}(B)$ as the *spray* of B.

5.1.2 Hybrid space

The hybrid space $\mathbb{S}$ is the coproduct, or disjoint union, of all of the factor spaces of Ω; that is,

$$\mathbb{S} = \bigoplus_{i \in \mathfrak{I}} \mathbb{S}_i. \tag{5.5}$$

The tangent bundle for $\mathbb{S}$ is likewise the coproduct of if its factor bundles, and so

$$T\mathbb{S} = \bigoplus_{i \in \mathfrak{I}} T\mathbb{S}_i. \tag{5.6}$$

5.1.3 Hybrid time

Much like the standard case, *hybrid time* Γ is a nonempty set with some full order $\leq$ whose degree is at least equal to that of each factor time Γ_i. As a further requirement, for each factor universe Ω_i, there must exist an onto mapping $\hat{t}_i : \Gamma \to \Gamma_i$ so that, for any two instants $t, t' \in \Gamma$ such that $t \leq t'$, it is the case that $\hat{t}_i(t) \leq \hat{t}_i(t')$

Practically speaking, this requirement states that, as one advances from one hybrid instant to a later one, all factor times either remain in the same instant or also advance. The rate at which the

$$
\begin{tikzcd}
B_{\mathrm{p}} \arrow[r, "\chi_{\mathrm{p}}"] & \chi_{\mathrm{p}}(B) \\
B \arrow[u, "\Pi_{\mathrm{p}}"] \arrow[r, "\chi"] \arrow[d, "\Pi_{\mathrm{c}}"'] & \chi(B) \arrow[u, "\Pi_{\mathrm{p}}"'] \arrow[d, "\Pi_{\mathrm{c}}"] \\
B_{\mathrm{c}} \arrow[r, "\chi_{\mathrm{c}}"] & \chi_{\mathrm{c}}(B)
\end{tikzcd}
$$

Figure 5.1: Hybrid placement function for a hybrid mechanical system with two factor systems, labeled p and c respectively. The projection into a factor universe of the placement of body B is identical to the placement within that space of the corresponding projection of B.

various universes do so need not be related, however, even if their definitions of time are identical. Thus, one factor universe with continuous time might advance steadily, while another continuous-time factor universe advances in rapid leaps followed by periods of stagnation. The above requirements guarantee only that time will never "flow backward" in any factor universe and, since the mappings are onto, that there will be at least one hybrid instant mapping to any instant of any given factor universe.

As a hypothetical example of this process, one might envision two neighboring cultures who keep time in their own way: one measuring the days with each set of the sun, the other counting eras with each new child born. As a hybrid time, one might choose the set of seconds, mapping each second onto both the corresponding day in the first measure and the corresponding era in the second.

5.1.4 Hybrid placement and motion

Since universes may be separate or overlapping, any particular body may have non-null components in one, some, or all of the factor systems. This introduces some complexity into the notation for hybrid body placement; conceptually, a body's hybrid placement assigns the part of the body that actually exists in each factor universe to positions in the corresponding factor space. More formally, given a body B, a hybrid placement function $\chi : B \to \mathcal{S}$ is a mapping such that, for any index $i \in \mathcal{I}$, the function

$$\chi_i = \Pi_i \circ \chi \lceil (B \sqcap \top_i) \tag{5.7}$$

is a placement of $B \sqcap \top_i$ in $\mathcal{S}_i$. As usual, the placement of a body defines the placement of its subbodies, and so a placement of the hybrid universal body $\chi(\top)$ defines placements for all bodies in the hybrid universe. Figure 5.1 illustrates a two-universe example, showing the relationship between the placement and projection functions.

The set of all hybrid placements of a hybrid body B is denoted $\mathcal{C}(B, \mathcal{S})$; if only some ith factor universe is of interest, one writes $\mathcal{C}(B, \mathcal{S}_i)$ to indicate the restriction of those placements to the ith factor space and the corresponding subbody of B. In other words,

$$\mathcal{C}(B, \mathcal{S}_i) = \Pi_i\big(\mathcal{C}(B \sqcap \top_i, \mathcal{S})\big). \tag{5.8}$$

In the simplest case, all factor universes are separate, and so a hybrid placement function maps every body point of $\top$ to a position in exactly one factor space. In such cases, the set of all placements of a body B is isomorphic to the set formed by finding the corresponding subbody of B for each universe of its spray, finding the placement sets of those subbodies in their respective factor spaces, and taking the product of the placement sets. That is,

$$\mathcal{C}(B, \mathcal{S}) \stackrel{\text{def}}{=} \prod_{i \in \mathcal{I}(B)} \mathcal{C}(B, \mathcal{S}_i). \tag{5.9}$$

In the more general case where bodies may be shared between universes, specifying the placement space is more complex, since each body point may be mapped to one or more positions in different factor spaces. In the construction presented in this paper, however, I will eventually assume all factor universes are separate, and so I omit treatment of such complex spaces.

If a body has a non-null component in some ith factor universe, then its motion in the ith mechanical system is simply the motion of that non-null component in its corresponding space. In other words, writing χ_i for the motion in the ith factor system,

$$\chi_i = \Pi_i \circ (\chi \lceil \top_i). \tag{5.10}$$

One can technically speak of the motion of a body in some factor system in which it has no non-null component. However, recall that the motion of a body is a mapping from some time interval to its placement space. A null body has no body points, and so there are no mappings from its body points to the corresponding space; thus, its corresponding placement space is empty, and so it has no corresponding motion.

5.1.4.1 Hybrid velocity, configuration, and acceleration

The hybrid velocity of a body is the product of its velocity in each factor system in its spray. Thus, a hybrid velocity field over a body B is a mapping $\mathbf{v} : B \to \mathcal{S}$ such that, for any index $i \in \mathcal{I}$, the function

$$\mathbf{v}_i = \Pi_i \circ \mathbf{v}\lceil(B \sqcap \top_i) \tag{5.11}$$

is a velocity field $B \sqcap \top_i$ in $\mathcal{S}_i$. As in the non-hybrid case, $\mathbf{v}_i$ maps each body point to both a position and a velocity value. Given a placement function χ, if only the velocity value is of interest, one writes

$\dot{\chi}(B, t)$ to indicate the corresponding mapping of body points in B to velocity values at instant t.

A *hybrid configuration* of the universal body is then an ordered pair of mappings $(w, \dot{w})$ such that w is an mapping from every body point in $\top$ to placements in every factor system in its spray, and likewise $\dot{w}$ is a mapping from every body point in $\top$ to velocity values in every factor system in its spray. As with position and velocity, I write $(w_i, \dot{w}_i)$ to indicate a restriction to the ith component; thus,

$$w = \prod_{i \in \mathcal{I}(B)} w_i, \tag{5.12}$$

and likewise for $\dot{w}$.

As with placements, if one assumes that all factor universes are separate, one can specify a configuration by specifying configurations of the sub-bodies for each factor universe, and so

$$\mathfrak{X}(B, \mathcal{S}) \stackrel{\text{def}}{=} \prod_{i \in \mathcal{I}(B)} \mathfrak{X}(B, \mathcal{S}_i). \tag{5.13}$$

Again, I omit treatment of the more complex case in which some bodies may be part of multiple factor systems. Regardless, one writes $\mathfrak{X}(B, \mathcal{S}_i)$ to indicate the restriction of $\mathfrak{X}(B, \mathcal{S})$ to the ith factor universe.

A *hybrid acceleration* of a body B is a product of an acceleration of that body in each factor space in its spray, that is, a function $\ddot{\chi}$ such that, for any index $i \in \mathcal{I}$, the function

$$\ddot{\chi}_i = \Pi_i \circ \ddot{\chi} \lceil (B \sqcap \top_i) \tag{5.14}$$

is an acceleration of $B \sqcap \top_i$ in $\mathcal{S}_i$.

5.2 Hybrid dynamics

5.2.1 Hybrid mass

As noted above, a hybrid body (or part thereof) may be part of more than one factor universe, and each of these universes may have its own measure of the mass of B. The *hybrid mass* of B, denoted m, is an ordered tuple of such that, for any $i \in \mathcal{I}(B)$, the ith component $[m(B)]_i$ is equal to the mass of the component of B in the corresponding factor universe. Thus,

$$[m(B)]_i = m_i(B \sqcap \top_i). \tag{5.15}$$

By arguments similar to those for (4.13), for any index $i \in \mathcal{I}$,

$$[m(\bot)]_i = m_i(\bot_i) = 0. \tag{5.16}$$

The space of hybrid masses is then

$$\mathcal{M} = \prod_{i \in \mathcal{I}} \mathcal{M}_i. \tag{5.17}$$

5.2.2 Hybrid momentum

As noted earlier, in rational mechanics, the space of possible momenta is the same as the vector space of velocities. Hybrid momenta likewise occupy the same space as that of hybrid velocities, with the hybrid momentum of a body defined by the vector products of its momenta in each factor space. Thus, if $[p(B, m(B), \chi(B), t]_i$ is the ith component of a body's momentum and p_i is the momentum function for the ith factor universe, then for any body B in Ω and hybrid instant t in Γ,

$$[p(B, m(B), \chi(B), t]_i \stackrel{\text{def}}{=} p_i(B_i, m_i(B_i), \chi_i(B_i), t_i), \tag{5.18}$$

where $B_i = B \sqcap \top_i$. The space of hybrid momenta for $\top$ is thus the product of the factor spaces of momenta.

5.2.3 Hybrid force systems and variations

The hybrid force value set is the product of the factor sets of force values, that is,

$$\mathcal{V} \stackrel{\text{def}}{=} \prod_{i \in \mathcal{I}} \mathcal{V}_i. \tag{5.19}$$

Since factor sets of force values are identical to factor spaces of momenta, the hybrid force value set is likewise equal to the space of hybrid momenta.

A hybrid force system f, then, is a force system mapping bodies in Ω to hybrid force values, $f : \Omega \times \Omega \to \mathcal{V}$. As a force system, it must still be null-passive, additive, and pairwise equilibriated for all bodies in Ω. As in the non-hybrid case, one writes $\mathcal{F}(\Omega, \mathcal{V})$ for the set of all force systems for a particular hybrid universe Ω and set of force values $\mathcal{V}$.

Likewise, a *hybrid force variation* is a function $\hat{f} : \Gamma \to \mathcal{F}(\Omega, \mathcal{V})$, that is, a mapping from time to hybrid force systems. One writes $\hat{\mathcal{F}}(\Omega, \mathcal{V}, J)$ for the set of all hybrid force variations. One similarly calls a set of hybrid force systems a *hybrid force inclusion*, and one calls a set of hybrid force variations a *hybrid force variation inclusion*.

Note that, by the above constructions, there may be forces on a body along dimensions in which it has no non-null component. Consider, for instance, a hybrid mechanical system composed of two factor mechanical systems M_1, M_2, where $\mathcal{S}_1$ is the real number line and $\mathcal{S}_2$ is the zero-one space $\mathbb{Z}_2$. In the former case, the space of force values $\mathcal{V}_1$ is $\mathbb{R}$ over $\mathbb{R}$; in the latter, $\mathcal{V}_2$ is $\mathbb{Z}_2$ over $\mathbb{Z}_2$. I write a hybrid force value as (v_1, v_2), where v_1 is in $\mathcal{V}_1$ and v_2 is in $\mathcal{V}_2$. Suppose, then, that there is some body B_1 in Ω_1 such that $\mathcal{I}(B_1) = \{1\}$; that is, B_1 has no non-null components in the second factor mechanical system.

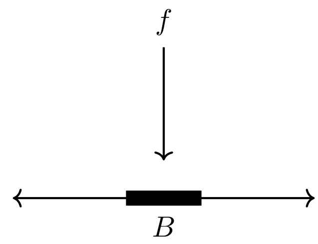


Figure 5.2: Hybrid force f on a body B, with a nonzero component in a dimension in which the body takes no position. One can imagine that the body is only able to slide back and forth along the horizontal axis; a force directly orthogonal to that dimension thus causes no motion.

Thus, the body points of B take placements in $\mathcal{S}_1 = \mathbb{R}$, but take no placements in $\mathcal{S}_2 = \mathbb{Z}_2$. Suppose further a force of $(\langle 0 \rangle, \langle 1 \rangle)$ is exerted on this body. Traditionally, a net force on a body results in a change of the momentum of that body, typically through acceleration in the corresponding dimension(s). What, if anything, should be the effect of a force along the $\mathbb{Z}_2$ dimension on a body that does not exist in that dimension?

Multiple mathematically-sound answers may be defined. Perhaps the simplest, and one of the most natural, is to say that there is no effect. Forces along the $\mathbb{Z}_2$ axis are effectively perpendicular to the range of motion for B, and so they cause no change in its momentum. In traditional mechanics, forces do not cause motion in dimensions to which they are orthogonal; if a force is orthogonal to all of a body's dimensions of motion, then, it should cause no movement at all. I illustrate this concept in Figure 5.2.

While straightforward, this definition does not seem entirely satisfying and requires the abandonment of standard assumptions that result in, for instance, the conservation of momentum. Extending the example above, suppose there exists some body B_2 such that $\mathcal{I}(B_2) = \{2\}$, that is, such that B_2 is a member of the second factor universe, and suppose that in some set of conditions the force of B_2 on B_1 is

$$f(B_1, B_2) = (\langle 0 \rangle, \langle 1 \rangle). \tag{5.20}$$

By pairwise equilibriation, then,

$$f(B_2, B_1) = (\langle 0 \rangle, \langle 1 \rangle). \tag{5.21}$$

Suppose further than no other body exerts nonzero forces on either B_1 or B_2. As noted above, B_1 takes no placement in $\mathcal{S}_2$, and so its momentum is unchanged. B_2, however, does take placements in $\mathcal{S}_2$; then its momentum will change as a result of the above force. Since it is the only body whose momentum changes, the momentum of the hybrid universal body must change, and so momentum is not conserved.

In the following chapters, I attempt to frame solutions that will not depend on the answer to this question. Towards this end, I attempt wherever possible to make sure that the resultant forces on a body along any inapplicable dimensions sum to $\mathbf{0}$.

5.3 Hybrid mechanical systems

5.3.1 Hybrid system laws

Section 4.3.1 described the idea of *system laws*, that is, of functions defining the forces or other behaviors of a mechanical system in terms of the position, velocity, or other properties of that system. One extends system laws to a hybrid system by extending the inputs of such functions to include properties of all factor systems. Thus, a hybrid system law might in principle determine forces based on placements of a hybrid body, on the current hybrid instant, on temperatures of bodies in several factor universes, and so on. As before, I restrict myself to system laws that are derived from the configuration of the universal body; under that restriction, an *autonomous deterministic hybrid system law function* is a function $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to \mathcal{F}(\Omega, \mathcal{V})$, that is, a mapping from the configurations of the hybrid universal body to the set of hybrid force systems. Given an autonomous deterministic hybrid system law function $\mathfrak{L}$, then, for any particular pair of hybrid bodies $A, B \in \Omega$, the hybrid force of B on A at $t \in \Gamma$ is given by

$$\mathfrak{L}(\chi_t, \dot{\chi}_t)(A, B). \tag{5.22}$$

As in the non-hybrid case, I similarly define an *autonomous nondeterministic hybrid system law*, denoted $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\mathcal{F}(\Omega,\mathcal{V})}$, that maps configurations to hybrid force inclusions. I likewise define *time-varying (non)deterministic hybrid system laws*, where a time-varying hybrid system law is a mapping to hybrid force variations (if deterministic) or hybrid force variation inclusions (if nondeterministic). In what follows, I consider only cases where the system laws are represented by a time-varying nondeterministic hybrid system law, that is, a function $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma)}$. For simplicity, I typically drop the qualifiers and refer simply to hybrid system laws.

One may be concerned only with the forces specified by a hybrid system law function (of whatever sort) in a particular factor mechanical system. In such cases, one writes $\mathfrak{L}_i$ to indicate the projection onto that system; that is,

$$\mathfrak{L}_i \stackrel{\text{def}}{=} \Pi_i \circ \mathfrak{L}. \tag{5.23}$$

As in the case of non-hybrid system laws, given a system law function (of whatever sort) $\mathfrak{L}$, one can define an equivalence relation $\equiv_{\mathfrak{L}}$ over $\mathfrak{X}(\top, \mathcal{S})$ such that, for any configurations $(w, \dot{w}), (w', \dot{w}') \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_{\mathfrak{L}} (w', \dot{w}') \text{ iff } \mathfrak{L}(w, \dot{w}) = \mathfrak{L}(w', \dot{w}'). \tag{5.24}$$

This equivalence relation imposes a partition $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ in which any equivalence class $[(w, \dot{w})]$ consists of all hybrid configurations mapped to the same values as $(w, \dot{w})$ by $\mathfrak{L}$.

One can then extend the preceding definition of hybrid system law functions to be mappings from equivalence classes in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ to force systems or variations (or sets thereof). So, for instance, a time-varying nondeterministic hybrid system law function is a function

$$\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}} \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)} \tag{5.25}$$

such that, for any $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$,

$$\mathfrak{L}[(w, \dot{w})] \stackrel{\text{def}}{=} \mathfrak{L}(w, \dot{w}). \tag{5.26}$$

5.3.2 Hybrid induced motion

Recall from Section 4.3.2 that an induced motion is one obeying the Newton-Euler Laws, that is, a motion whose second derivative exists at all points, and such that the acceleration of every body is proportionate to the force of that body's exterior on it at every instant. One extends this notion to hybrid bodies in the natural way, save for the restriction noted in Section 5.2.3; that is, a body need not obey the Newton-Euler laws for any factor mechanical system in which it has no non-null components.

Formally, a motion χ of the hybrid universal body is a *hybrid induced motion* for a time-varying deterministic hybrid system law function $\mathfrak{L}$ iff, at every instant, the motion of every body in every factor space in its spray obeys the Newton-Euler Laws for the corresponding component of the force variation specified by $\mathfrak{L}$. In other words, χ is a hybrid induced motion iff, for every instant $t \in \Gamma$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\mathfrak{L}_i(\chi_t, \dot{\chi}_t)(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{5.27}$$

Likewise, suppose $\mathfrak{L}$ is a time-varying nondeterministic hybrid system law function. Let χ be a motion of $\top$, and let $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L}) : \Gamma \to \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$ be the function mapping each instant to the corresponding equivalence class; that is, for any instant $t \in \Gamma$,

$$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(t) = [(\chi_t, \dot{\chi}_t)]. \tag{5.28}$$

Let $\mathscr{I}_{\chi,\mathfrak{L}}$ be the partition of Γ such that, for any interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, for any instants $t, t' \in J$,

$$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(t) = \mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(t'), \tag{5.29}$$

and such that this property does not hold for any larger interval J' containing J. By a slight extension of notation, one can write $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(J)$ to refer to the equivalence class mapped to by all instants in J.

Then χ is a hybrid induced motion for $\mathfrak{L}$ iff for each such interval there exists a force variation specified by $\mathfrak{L}$ such that the motion of every body in every factor space in its spray obeys the Newton-

Euler Laws for the corresponding component of that force variation. In other words, χ is a hybrid induced motion iff, for each interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, there is some force variation $\hat{f} \in \mathfrak{L}(J)$ such that for every instant $t \in \Gamma$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{5.30}$$

Similar constructions exist for autonomous hybrid system law functions; I omit detailed construction of such constructions, as they will play no further part in this paper.

5.3.3 Elements of a hybrid mechanical system

The *elements* of a hybrid mechanical system are precisely those components described in the previous sections, and one defines a particular *hybrid mechanical system* by defining a tuple $(\Omega, \mathcal{S}, \Gamma, m, \mathfrak{L})$, where Ω is the hybrid universe of bodies, $\mathcal{S}$ is the hybrid mechanical space, Γ is the ordered set of hybrid instants, m is the hybrid mass function, and $\mathfrak{L}$ is the time-varying nondeterministic hybrid system law function.

5.3.4 Hybrid initial configurations

As with non-hybrid mechanical systems, I typically provide not only the elements of a hybrid mechanical system, but also some set of initial configurations of the hybrid universal body. I call a system including such a set an *initialized hybrid mechanical system*, dropping the "initialized" where unambiguous.

I call any assignment of position and velocity to all body points at the first instant in time a *initial hybrid tangent* of the hybrid universal body. Following the pattern from Section 4.3.4, I write $\mathfrak{X}_0(\top, \mathcal{S})$ to indicate the *initial hybrid configuration set*, the set of all acceptable initial hybrid configurations.

As with non-hybrid configurations, given a hybrid initial configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, I say that a motion χ *begins* in $(w, \dot{w})$, or that χ is a motion *beginning in* $(w, \dot{w})$, iff

$$\begin{aligned} \chi(\top, 0) &= w \\ \dot{\chi}(\top, 0) &= \dot{w}. \end{aligned} \tag{5.31}$$

5.4 Conclusion

To characterize an initialized hybrid mechanical system according to the features of interest in this dissertation, then, requires a tuple $(\Omega, \mathcal{S}, \Gamma, m, \mathfrak{L}, \mathfrak{X}_0(\top, \mathcal{S}))$, where Ω is the set of all hybrid bodies, $\mathcal{S}$ is a hybrid mechanical space, Γ is hybrid time, $\mathfrak{L}$ is a time-varying nondeterministic hybrid system law function, and $\mathfrak{X}_0(\top, \mathcal{S})$ is an initial hybrid configuration set.

5.5 Summary of notation

$\mathcal{J}$	set of indices for a universe
Ω_i	universe for ith factor mechanical system
Ω	hybrid universe
$\top_i$	universal body for ith factor mechanical system
$\top$	hybrid universal body
$\mathcal{I}(B)$	spray of hybrid body B
$\mathcal{S}_i$	mechanical space for ith factor mechanical system
$\mathcal{S}$	hybrid mechanical space
Γ_i	time for ith factor mechanical system
Γ	hybrid time
$\hat{t}_i$	mapping from hybrid time to ith factor time
χ_i	placement function for ith factor mechanical system
$\chi_{\mathcal{I}(B)}$	spray placement for body B
$\mathcal{C}(B, \mathcal{S}_i)$	placement space of body B in ith factor mechanical system
$\mathcal{C}(B, \mathcal{S})$	hybrid placement space of body B
$\mathbf{v}_i$	material velocity field for ith factor mechanical system
$\mathbf{v}$	hybrid material velocity field
$\mathfrak{X}(B, \mathcal{S})$	hybrid configuration space of body B
$\mathfrak{X}(\top, \mathcal{S})$	hybrid configuration space of the hybrid universal body
$\mathfrak{X}(B, \mathcal{S}_i)$	set of configurations of body B in ith factor mechanical system
m_i	mass function for ith factor mechanical system
m	hybrid mass function
p_i	momentum function for ith factor mechanical system
p	hybrid momentum function
$\mathcal{V}_i$	force values for ith factor mechanical system
$\mathcal{V}$	hybrid force values
f	hybrid force system
$\mathcal{F}(\Omega, \mathcal{V})$	set of all hybrid force systems for universe Ω
$\hat{f}$	hybrid force variation
f_t	hybrid force system associated with hybrid instant t
$\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$	set of all hybrid force variations for hybrid universe Ω
$\mathfrak{L}$	hybrid system law function
$\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$	equivalence class of configurations mapped to the same force systems by hybrid system law function $\mathfrak{L}$

$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$	configuration-time function associating every hybrid instant with an equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$
$\mathscr{I}_{\chi,\mathfrak{L}}$	partition of Γ into intervals imposed by $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$
w	one hybrid placement of a body
$\dot{w}$	one assignment of hybrid velocity to a body
$(w, \dot{w})$	hybrid configuration for a body
$\mathfrak{X}_0(\top, \mathcal{S})$	initial hybrid configuration set for hybrid universal body $\top$

Chapter 6

Mechanical Controllers and Conditional Force Systems

Having defined a hybrid mechanical system, I now adapt the notion of a controller to such a system. Much of this chapter is a reproduction of my work in [9]. Throughout, I reproduce verbatim theorems and illustrations from that paper.

6.1 Mechanical controllers

Previous chapters have described controllers as components of a hybrid automaton. Before proceeding to discuss a mechanical representation of this portion of the automaton, I first present concepts and language for a mechanical controller. To do so, I draw heavily on my previous work in [9].

In mechanical terms, a controller is a body C with a set of *states*, denoted $\Psi(C)$. States may be left abstract or associated with some property of the controller or other bodies. In later chapters, I most often associate a controller's state with subsets of the configuration space of the universal body. So, for instance, some configurations will correspond to state ψ_1, while others correspond to ψ_2; the full set $\Psi(C)$ is then a partition of $\mathfrak{X}(\top, \mathcal{S})$.

In any particular instant, a controller occupies exactly one state, though it may transition from one state to another over time. Each state, in turn, corresponds to a set of possible force systems over the universe; thus, the force that a particular body exerts on another depends on the current state of the controller. More generally, in a system with multiple controllers, the force of one body on another may depend on the states of all controllers.

For the duration of this paper, I restrict myself to discussing mechanical systems in which there are at least two significant subuniverses, the *physical universe* Ω_p and the *controller universe* Ω_c, characterized by the following discussion. To refer to components of the unified system, I omit the subscripts, so that Ω to refer to the full universe, and so on. Since each subuniverse is a boolean lattice in its own right, each contains a universal body. I refer to these as the *universal physical body* and *universal controller body* (or simply *universal controller*), denoted $\top_{\mathrm{p}}$ and $\top_{\mathrm{c}}$, respectively. Likewise, I refer to the two universes' respective null bodies as the *physical null body* $\bot_{\mathrm{p}}$ and the *controller null body* $\bot_{\mathrm{c}}$. For any body $B \in \Omega_{\mathrm{p}}$, I write $\overline{B}^{\mathrm{p}}$ to indicate the unique *physical exterior* of B, the body in Ω_{p} such that $B \sqcap \overline{B}^{\mathrm{p}} = \bot_{\mathrm{p}}$ and $B \sqcup \overline{B}^{\mathrm{p}} = \top_{\mathrm{p}}$. Likewise, for a body $C \in \Omega_{\mathrm{c}}$, the *controller exterior* $\overline{C}^{\mathrm{c}}$ is the body in Ω_{c} such that $C \sqcap \overline{C}^{\mathrm{c}} = \bot_{\mathrm{c}}$ and $C \sqcup \overline{C}^{\mathrm{c}} = \top_{\mathrm{c}}$. If $\top_{\mathrm{p}} \neq \top$, one refers to any subbodies of $\top \sqcap \overline{\top_{\mathrm{p}}}$ as *nonphysical bodies*; similarly, if $\top_{\mathrm{c}} \neq \top$, one refers to any subbodies of $\top \sqcap \overline{\top_{\mathrm{c}}}$ as *noncontroller bodies*.

The bodies in Ω_{c} are *controllers*, where a controller C is a body with an associated set of *states* such that forces between some bodies in Ω_{p} depend on the state of C. That is, for different values of these controller properties, various physical bodies exert different forces on each other. I represent this dependence by assigning to each controller C a nonempty set of states $\Psi(C)$. The *conditional force system* $f[\cdot] : \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega_{\mathrm{p}}, \mathcal{V}_{\mathrm{p}})$ associates each state of the universal controller with a force system over the physical universe. For any given *universal state* $\psi \in \Psi(\top_{\mathrm{c}})$, then, $f[\psi]$ is a force system, and $f[\psi](A, B)$ is the *conditional force* of B on A in state ψ.

All controllers are required to *state-project* onto their sub-controllers. That is, given any two controllers $C, C' \in \Omega_{\mathrm{c}}$ such that $C' \sqsubseteq C$, each state $\psi \in \Psi(C)$ is associated with exactly one state $\psi' \in \Psi(C')$ such that, whenever C is in ψ, C' is in ψ'. In this case, I say that ψ projects onto ψ'. I indicate this fact by writing $C'(\psi) = \psi'$, read "The projection of ψ onto C' is ψ'." Practically, this means that the states of a controller determine the states of its subcontrollers; the state of the universal controller thus determines the state of all other controllers in Ω_{c}.

Each state of a controller is required to have a unique state projection relative to the other states of that controller. Thus, given a controller C that is partitioned into two subcontrollers C_1, C_2, for any states $\psi_1 \in \Psi(C_1), \psi_2 \in \Psi(C_2)$, there is at most one state $\psi \in \Psi(C)$ such that $C_1(\psi) = \psi_1$ and $C_2(\psi) = \psi_2$. I can therefore extend the above notation for conditional force systems. Given some partition $C_1, C_2, \ldots$ of the universal controller and a state $\psi \in \Psi(\top_{\mathrm{c}})$ with $\psi_1 = C_1(\psi), \psi_2 = C_2(\psi), \ldots$, I can write the force system $f[\psi]$ instead as $f[\psi_1, \psi_2, \ldots]$, with the order of the states irrelevant.

6.1.1 Conditional force inclusions

I briefly note one extension to the idea of a conditional force system. Recall from earlier discussions that, where a differential equation associates exactly one time derivative with each point in some space, a differential inclusion instead associates a set of time derivatives with each point in the space.

Similarly, while a conditional force system associates a force system with each universal controller state, a *conditional force inclusion* associates a set of force systems with each controller state. More formally, a conditional force inclusion is a mapping $\mathfrak{f}[\cdot] : \Psi(\top_{\mathrm{c}}) \to 2^{\mathcal{F}(\Omega_{\mathrm{p}}, \mathcal{V}_{\mathrm{p}})}$. Thus, given a particular universal state ψ, $\mathfrak{f}[\psi]$ is a set of force systems.

6.1.2 Independence

In general, given a controller and one state for each of two subcontrollers, there may be no state of the supercontroller that projects onto the states of both subcontrollers. I say that controllers C_1 and C_2 are *mutually state independent* iff, for any states $\psi_1 \in \Psi(C_1), \psi_2 \in \Psi(C_2)$, there is exactly one state $\psi \in \Psi(C_1 \sqcup C_2)$ such that $C_1(\psi) = \psi_1$ and $C_2(\psi) = \psi_2$. If a controller C and its controller complement $\overline{C}^{\mathrm{c}}$ are mutually state independent, I say simply that C is *state independent*. In effect, if all controllers are state independent, the states of the universal controller are just the products of the states of any partition of that controller.

I restrict my discussion here to systems of controllers that are *force independent*. A controller C is force independent iff, for any states $\psi, \psi' \in \Psi(C)$ and $v, v' \in \Psi(\overline{C}^{\mathrm{c}})$ such that there are universal states projecting onto $[\psi, v]$, $[\psi, v']$, $[\psi', v]$, and $[\psi', v']$, I have

$$f[\psi, v] - f[\psi', v] = f[\psi, v'] - f[\psi', v']. \tag{6.1}$$

Intuitively, a controller is force independent iff for any of its states ψ, ψ', changing its state from ψ to ψ' always changes the conditional force system by the same amount, regardless of the state of its controller complement. If a controller is both state independent and force independent, I say simply that it is *independent*.

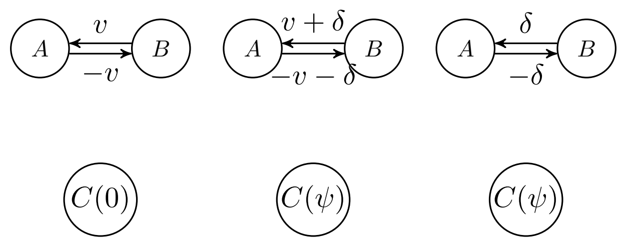
6.1.3 Differential Controllers

Many systems can reasonably be thought off as having a passive or "off" state in which controllers are not responsible for any of the forces between bodies. A motorized elevator, for instance, might have an On setting in which the elevator is actively raised, but in its Off setting the elevator is passively acted on by gravity and other forces. This suggests that a natural concept for controllers is to identify some set of baseline forces in the passive state, while describing other states in terms of the forces they add to this baseline.

In practice, it may be impossible to identify an indisputably passive state of a system. Perhaps the elevator actually controls two systems at once, and allowing one of them to switch to Off necessarily turns the other to On. The underlying concept of a passive or *null state* of the controller, however, is not dependent on these particulars; I can effectively designate any universal state as the null state, viewing all other states as having controller-dependent forces modifying these base conditions.

Thus, I identify an arbitrary state in $\Psi(\top_{\mathrm{c}})$ as the *universal null state*, denoted 0. For any subcontroller C, the projection of the universal null state onto C is the *null state* of C. I formally denote this projection as $C(0)$ or 0_C, though in practice I will typically write simply 0 for the null state of any controller, relying on context to indicate which projection is being discussed.

I can then describe any other state $\psi \in \Psi(\top_{\mathrm{c}})$ in terms of the difference between the forces con-



(a) (b) (c)

Figure 6.1: Three different forces between two bodies A and B. In (a), the null force $f[0]$ is shown. In (b), the force given some alternate universal state $(C(\psi), 0)$ is shown; note that this force can be viewed as the "base" forces of $f[0]$ plus some additional force δ. In (c), only this differential force $f^*[C, \psi]$ is present.

ditional on it and those conditional on the null state, $f[\psi] - f[0]$. I call this difference the *conditional difference*, denoted as the partial function $f^*[\cdot, \cdot] : \Omega_c \times \Psi(\top_c) \to \mathcal{F}(\Omega_p, \mathcal{V}_p)$. For any state ψ in $\Psi(\top_c)$, the conditional difference $f^*[\top_c, \psi]$ is

$$f^*[\top_c, \psi] = f[\psi, 0] - f[0]. \tag{6.2}$$

I can generalize this definition to provide the conditional difference for other state independent controllers. For any state ψ in $\Psi(\top_c)$ and state independent controller C, then, the conditional difference $f^*[C, \psi]$ is

$$f^*[C, \psi] = f[C(\psi), 0] - f[0]. \tag{6.3}$$

Figure 6.1 illustrates an example of $f[0]$, $f[C(\psi), 0]$, and $f^*[C, \psi]$. Note that it is not necessary, for instance, that the forces in $f[0]$ all be $\mathbf{0}$; the universal null state need not be indicative of universally null forces.

6.1.3.1 Null complementation

In the above case, I require that the controller C be state independent. Recall that state independence means that any choice of states for C and $\overline{C}^c$ defines a unique state of the universal controller; for controllers in general, there need not be any such compatible universal state. State independence thus guarantees the existence of a universal state $(C(\psi), 0)$ for arbitrary choice of ψ. I do not actually require full state independence for this condition to hold, however. It is sufficient that, given a choice of the universal null state 0, any controller projection of 0 together with any state of that controller's complement defines a unique universal state.

In other words, let C be an arbitrary controller, and let $0, \psi_1, \psi_2$ be states in $\Psi(\top_c)$, where $\psi_1 \neq 0 \neq \psi_2$ and at least one of $C(\psi_1) \neq C(0)$ or $\overline{C}^c(\psi_2) \neq \overline{C}^c(0)$. The definition of conditional difference does not require that the pair $(C(\psi_1), \overline{C}^{\mathrm{c}}(\psi_2))$ define any universal state, since this pair can never appear in the definition of f^*; thus, full state independence is unnecessary. The definition does, however, require that $(C(\psi_1), \overline{C}^{\mathrm{c}}(0))$ and $(C(\psi_2), \overline{C}^{\mathrm{c}}(0))$ identify universal states, and likewise for any other choices of state and controller.

Suppose that, for some choice of controller C and universal state ψ, there exists a universal state of the form $(C(\psi), 0)$, that is, a universal state ψ' such that

$$C(\psi') = C(\psi) \tag{6.4}$$

$$\overline{C}^{\mathrm{c}}(\psi') = \overline{C}^{\mathrm{c}}(0). \tag{6.5}$$

In such cases, I say that the state $C(\psi)$ is *null complemented with respect to* 0. Given such a null complemented state, I can then define

$$f^*[C, \psi] \stackrel{\text{def}}{=} f[C(\psi), 0] - f[0]. \tag{6.6}$$

If f^* is defined for any state ψ and some set of controllers $\{C_1, C_2, \ldots\}$, I say that it is *complete* for that set. If it is complete for any controller in Ω_{c}, I say simply that it is *complete*. I can drop the "with respect to 0" in this phrasing, as any definition of f^* requires me to first fix a choice of 0. Since $f^*[C, \psi]$ is the difference of two force systems when defined, it is itself a force system; if f^* is complete, then, it is a conditional force system.

I say that a controller C is *null complemented with respect to* 0, or simply *null complemented* where unambiguous, iff each state in $\Psi(C)$ is null complemented with respect to 0. In such a case, $f^*[C, \psi]$ is defined for any choice of ψ. Further, a controller can only be null complemented if $\perp_{\mathrm{c}}$ has a single state, the projection of 0 onto $\perp_{\mathrm{c}}$.

Theorem 2. *Some controller is null complemented with respect to* 0 *if and only if* $\perp_{\mathrm{c}}(0)$ *is the only state of* $\perp_{\mathrm{c}}$*, that is, if and only if*

$$\Psi(\perp_{\mathrm{c}}) = \{\perp_{\mathrm{c}}(0)\}. \tag{6.7}$$

Proof. First suppose $C \in \Omega_{\mathrm{c}}$ is null complemented with respect to 0. By definition, for each $\psi \in \Psi(\top_{\mathrm{c}})$ there exists some $\psi' \in \Psi(\top_{\mathrm{c}})$ such that

$$\begin{aligned} C(\psi') &= C(\psi) & (6.8)\\ \overline{C}^{\mathrm{c}}(\psi') &= \overline{C}^{\mathrm{c}}(0). & (6.9) \end{aligned}$$

Because $\perp_{\mathrm{c}} \sqsubseteq C$, (6.8) implies

$$\perp_{\mathrm{c}}(\psi') = \perp_{\mathrm{c}}(\psi), \tag{6.10}$$

and because $\bot_{\mathrm{c}} \sqsubseteq \overline{C}^{\mathrm{c}}$, (6.9) implies

$$\bot_{\mathrm{c}}(\psi') = \bot_{\mathrm{c}}(0). \tag{6.11}$$

From these last two equalities I obtain

$$\bot_{\mathrm{c}}(\psi) = \bot_{\mathrm{c}}(0). \tag{6.12}$$

By hypothesis, this must hold for each $\psi \in \Psi(\top_{\mathrm{c}})$, and because state projection is onto, $\bot_{\mathrm{c}}$ has the single state $\bot_{\mathrm{c}}(0)$.

Now suppose instead that $\bot_{\mathrm{c}}$ has only a single state, and choose $0 \in \Psi(\top_{\mathrm{c}})$. The sole state $\bot_{\mathrm{c}}(0)$ of $\bot_{\mathrm{c}}$ is null complemented with respect to 0 because 0 itself is such that $\bot_{\mathrm{c}}(0) = \bot_{\mathrm{c}}(0)$ and $\top_{\mathrm{c}}(0) = \top_{\mathrm{c}}(0)$. Thus $\bot_{\mathrm{c}}$ is null complemented with respect to 0. □

The null complementation of a particular controller need not imply the null complementation of its complement, or of other controllers in general. I say that two controllers C_1, C_2 are *mutually null-state independent with respect to* 0 iff C_1, C_2, and $C_1 \sqcup C_2$ are all null complemented with respect to 0. Likewise, I say that a controller C is *null-state independent with respect to* 0 iff C and $\overline{C}^{\mathrm{c}}$ are mutually null-state independent; equivalently, C is null-state independent iff C and $\overline{C}^{\mathrm{c}}$ are null complemented.

Theorem 3. *For every choice of a null state, if some controller is null-state independent, then the null-state independent controllers form a subuniverse of* Ω_{c} *that includes* $\top_{\mathrm{c}}$ *and* $\bot_{\mathrm{c}}$.

Proof. Let $0 \in \Psi(\Omega_{\mathrm{c}})$. The definition of null-state independence is symmetric with respect to controller exteriors, so I can demonstrate Boolean closure of the null-state independent controllers by showing that the meet $C_1 \sqcap C_2$ of null-state independent controllers $C_1, C_2 \in \Omega_{\mathrm{c}}$ is itself null-state independent.

Choose $\psi_{12} \in \Psi(C_1 \sqcap C_2)$. I need to show that there exists a global state of the form $(\psi_{12}, 0)$. To do this, first extend ψ_{12} to any $\psi_1 \in \Psi(C_1)$ such that

$$(C_1 \sqcap C_2)(\psi_1) = \psi_{12}. \tag{6.13}$$

Then there exists $\psi_1' \in \Psi(\top_{\mathrm{c}})$ with

$$\psi_1' = (C_1(\psi_1), 0), \tag{6.14}$$

so let

$$\psi_2 = C_2(\psi_1'). \tag{6.15}$$

Then there exists $\psi_2' \in \Psi(\top_{\mathrm{c}})$ with

$$\psi_2' = (C_2(\psi_1'), 0). \tag{6.16}$$

I can decompose $C_2(\psi_1')$ into

$$(C_2 \sqcap C_1)(\psi_1') \quad = \quad \psi_{12} \tag{6.17}$$

$$(C_2 \sqcap \overline{C_1}^{\mathrm{c}})(\psi_1') \;=\; \psi_{2-1},$$

where by construction I have

$$\begin{aligned}(C_2 \sqcap \overline{C_1}^{\mathrm{c}})(\psi_1') &= (C_2 \sqcap \overline{C_1}^{\mathrm{c}})(C_1(\psi_1), 0) \\ &= \overline{C_1}^{\mathrm{c}}(0) \\ &= 0.\end{aligned} \tag{6.18}$$

This demonstrates that $C_1 \sqcap C_2$ is null complemented.

Now choose $v_{12} \in \Psi(\overline{C_1 \sqcap C_2}^{\mathrm{c}})$. I need to show that there exists a global state of the form $(v_{12}, 0)$. Because $\overline{C_1 \sqcap C_2}^{\mathrm{c}} = \overline{C_1}^{\mathrm{c}} \sqcup \overline{C_2}^{\mathrm{c}}$, first restrict v_{12} to $\overline{C_1}^{\mathrm{c}}$ to obtain

$$\overline{C_1}^{\mathrm{c}}(v_{12}) = v_1. \tag{6.19}$$

Because C_1 is null-state independent, this means there exists a global state $\psi_1' \in \Psi(\top_{\mathrm{c}})$ such that

$$\psi_1' = (0, v_1). \tag{6.20}$$

But now by projection I have

$$\begin{aligned}(C_1 \sqcap C_2)(\psi_1') &= (C_1 \sqcap C_2)(0, v_1) \\ &= C_1(0) \\ &= 0.\end{aligned} \tag{6.21}$$

This shows that $\overline{C_1 \sqcap C_2}^{\mathrm{c}}$ is null complemented as well, so together these results show that the meet of null-state independent controllers is null-state independent. This means that Boolean combinations of null-state independent controllers are null-state independent as well. If in addition some controller is null-state independent, then $\top_{\mathrm{c}}$ and $\bot_{\mathrm{c}}$, as the join and meet of the controller and its controller complement, must be null-state independent as well. $\square$

While the original choice of a universal null state is arbitrary, there are obvious advantages in choosing a universal null state that allows many or all controllers to be null-state independent. I say that a universal state 0 is *complement complete* iff every controller is null complemented with respect to 0; clearly, in this case all controllers will also be null-state independent. It follows that f^* is complete iff 0 is complement complete.

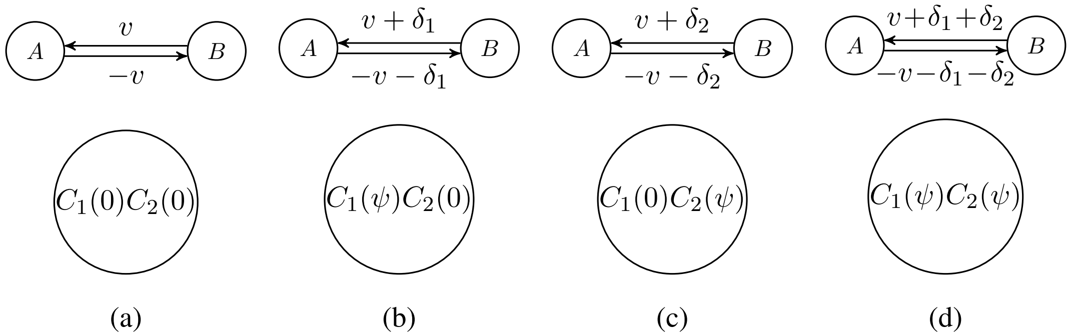


Figure 6.2: Example of mutual controller-additivity. In (a), the global state is the null state 0, and $f[0](A, B) = v$. In (b), the state of C_1 has changed, while that of C_2 remains 0, producing an increment of δ_1. In (c,) the state of C_2 has changed, while that of C_1 remains 0, producing an increment of δ_2. In (d), with both controllers changing states, the force changes by the sum of the increments.

6.1.4 Additive controllers

With the conditional difference defined, I can begin to consider the combined effects of the conditional differences of multiple controllers. I say that two controllers C_1, C_2 are *mutually controller-additive* with respect to a differential force system f^* iff, for any state ψ in $\Psi(\top_c)$ such that $(C_1 \sqcup C_2)(\psi)$, $C_1(\psi)$, $C_2(\psi)$, and $(C_1 \sqcap C_2)(\psi)$ are null complemented, I have

$$f^*[C_1 \sqcup C_2, \psi] = f^*[C_1, \psi] + f^*[C_2, \psi] - f^*[C_1 \sqcap C_2, \psi]. \tag{6.22}$$

In other words, C_1 and C_2 are mutually controller-additive if the above equation holds whenever all of the conditional differences exist. I say that a controller C is *controller-additive* with respect to f^* iff C and $\overline{C}^c$ are mutually controller-additive with respect to f^*, and I say that f^* is *controller-additive* iff every controller in Ω_c is controller-additive with respect to f^*. Figure 6.2 illustrates a pair of controller-additive controllers.

In general, controllers need not be controller-additive, though the null controller is mutually controller-additive with any other controller.

Corollary 4. *For every differential force system f^*, the null controller $\bot_c$ is mutually controller-additive with every controller.*

Proof. Let $C \in \Omega_c$. The null controller $\bot_c$ is mutually controller-additive with C with respect to f^* just in case for each $\psi \in \Psi(\top_c)$

$$f^*[C \sqcup \bot_c, \psi] = f^*[C, \psi] + f^*[\bot_c, \psi] - f^*[C \sqcap \bot_c, \psi] \tag{6.23}$$

holds whenever all these conditional forces exist. Simplifying the lattice operations yields

$$f^*[C,\psi] = f^*[C,\psi] + f^*[\perp_{\mathrm{c}},\psi] - f^*[\perp_{\mathrm{c}},\psi], \tag{6.24}$$

which is always true when all these forces exist. □

Given a complete force system f^*, any force independent controller is also guaranteed to be mutually controller-additive with any other controller.

Theorem 5. *Controllers C_1 and C_2 are mutually controller-additive with respect to a differential force system f^* of a conditional force system f whenever C_1 is force independent with respect to f and f^* is complete for C_1 and C_2.*

Proof. Suppose f^* is complete for $\{C_1, C_2\}$ and that C_1 is force independent with respect to f. Since f^* is complete for C_1 and C_2, by Theorem 3, f^* is complete for a subuniverse containing C_1 and C_2; denote this universe by $\Omega_{\mathrm{c}}[C_1, C_2]$.

Let ψ be in $\Psi(\top_{\mathrm{c}})$, and decompose it into the substates

$$\begin{aligned}
\psi_1 &= C_1(\psi) \\
\psi_2 &= C_2(\psi) \\
\psi_{12} &= (C_1 \sqcap C_2)(\psi) \\
\psi_{1-2} &= (C_1 \sqcap \overline{C_2}^{\mathrm{c}})(\psi) \\
\psi_{2-1} &= (\overline{C_1}^{\mathrm{c}} \sqcap C_2)(\psi) \\
\psi_{-1-2} &= (\overline{(C_1 \sqcup C_2)}^{\mathrm{c}})(\psi).
\end{aligned} \tag{6.25}$$

Similarly, decompose 0 into the components 0_1, 0_2, 0_{1-2}, 0_{2-1}, 0_{12}, 0_{-1-2}.

Because f^* is complete with respect to $\Omega_{\mathrm{c}}[C_1, C_2]$, the universal states $[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}]$, $[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}]$, $[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}]$, and $[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}]$ exist. Because C_1 is force independent, I have

$$\begin{aligned}
&f[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}] - f[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}] \\
&\qquad = f[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}] - f[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}].
\end{aligned} \tag{6.26}$$

Rearranging terms yields

$$\begin{aligned}
&f[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}] \\
&\quad = f[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}] + f[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}] - f[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}].
\end{aligned} \tag{6.27}$$

Subtracting $f[0]$ from both sides, adding $f[0] - f[0]$ to the right-hand side, and associating terms produces

$$\begin{aligned}(f[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}] - f[0])\\ &= (f[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}] - f[0])\\ &\quad + (f[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}] - f[0])\\ &\quad - (f[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}] - f[0]). \quad (6.28)\end{aligned}$$

Each of the associated terms in (6.28) constitutes the conditional differences for ψ corresponding to $C_1 \sqcup C_2$, C_1, C_2, and $C_1 \sqcap C_2$. I thus obtain

$$f^*[C_1 \sqcup C_2, \psi] = f^*[C_1, \psi] + f^*[C_2, \psi] - f^*[C_1 \sqcap C_2, \psi], \tag{6.29}$$

and conclude that C_1 and C_2 are mutually controller-additive with respect to f^*. $\square$

It follows that any controller that is force independent and null-state independent will also be controller-additive.

Theorem 6. *If a differential force system f^* is complete, then each controller is controller-additive with respect to f^* if and only if it is force independent with respect to a conditional force system f.*

Proof. Suppose that f^* is complete. As a controller is force independent if and only if its complement is, it follows immediately from Theorem 5 that if C is force independent with respect to f, then it is controller additive with respect to f^*. To prove the other direction of the biimplication, assume that C is controller-additive with respect to f^*. Choose $\psi, \psi' \in \Psi(C)$ and $v, v' \in \Psi(\overline{C}^{c})$. To show that C is force independent, I need to show that

$$f[\psi, v] - f[\psi', v] = f[\psi, v'] - f[\psi', v']. \tag{6.30}$$

By definition of f^* I have

$$\begin{aligned} f[\psi, v] &= f[0] + f^*[\top_c, (\psi, v)] \quad (6.31)\\ f[\psi', v] &= f[0] + f^*[\top_c, (\psi', v)]\\ f[\psi, v'] &= f[0] + f^*[\top_c, (\psi, v')]\\ f[\psi', v'] &= f[0] + f^*[\top_c, (\psi', v')], \end{aligned}$$

so by subtraction I have

$$f[\psi, v] - f[\psi', v] = f^*[\top_c, (\psi, v)] - f^*[\top_c, (\psi', v)] \tag{6.32}$$

$$f[\psi, v'] - f[\psi', v'] \quad = \quad f^*[\top_c, (\psi, v')] - f^*[\top_c, (\psi', v')].$$

By the controller additivity of C, I have

$$\begin{aligned} f[\psi, v] - f[\psi', v] &= f^*[C, (\psi, v)] + f^*[\overline{C}^c, (\psi, v)] \\ &\quad -f^*[C, (\psi', v)] - f^*[\overline{C}^c, (\psi', v)] \\ f[\psi, v'] - f[\psi', v'] &= f^*[C, (\psi, v')] + f^*[\overline{C}^c, (\psi, v')] \\ &\quad -f^*[C, (\psi', v')] - f^*[\overline{C}^c, (\psi', v')]. \end{aligned} \tag{6.33}$$

Appealing to the definition of f^* again, I note that

$$\begin{aligned} f^*[C, (\psi, v)] &= f^*[C, (\psi, 0))] \\ f^*[\overline{C}^c, (\psi, v)] &= f^*[\overline{C}^c, (0, v))], \end{aligned} \tag{6.34}$$

so by similar considerations I may reduce the previous equations to

$$\begin{aligned} f[\psi, v] - f[\psi', v] &= f^*[C, (\psi, 0)] + f^*[\overline{C}^c, (0, v)] \\ &\quad -f^*[C, (\psi', 0)] - f^*[\overline{C}^c, (0, v)] \\ &= f^*[C, (\psi, 0)] - f^*[C, (\psi', 0)] \\ f[\psi, v'] - f[\psi', v'] &= f^*[C, (\psi, 0)] + f^*[\overline{C}^c, (0, v')] \\ &\quad -f^*[C, (\psi', 0)] - f^*[\overline{C}^c, (0, v')] \\ &= f^*[C, (\psi, 0)] - f^*[C, (\psi', 0)]. \end{aligned} \tag{6.35}$$

Combining (6.36) and (6.36) I obtain (6.30), demonstrating that C is force independent with respect to f. $\square$

6.2 Extending controllers

The preceding results can easily be modified to allow controller states to vary over time. Recall from Section 4.2.4 that one writes $\hat{f} : \Gamma \to \mathcal{F}(\Omega, \mathcal{V})$ to indicate a force variation, that is, a mapping from instants in time to force systems. One often writes f_t for $\hat{f}(t)$, that is,

$$f_t = \hat{f}(t). \tag{6.36}$$

Relying on this notation, I provide extensions of conditional force systems and conditional force inclusions to allow these systems to vary over time.

6.2.1 The state-time function

As I consider conditional force variations, it will be useful to make statements about which state a controller occupies in different instants, that is, to have a formal expression for conditions of the form, "For any instant in which the state of C is $\psi \ldots$" Toward that end, I call a function associating each instant with a universal controller state a *state-time function*, denoted $\texttt{state} : \Gamma \to \Psi(\top_\mathrm{c})$. Thus, I write $\texttt{state}(t) = \psi$ to indicate that the universal state is ψ during instant t. I can define a similar function $\texttt{state}_C$ for each controller $C \in \Omega_\mathrm{c}$; thus, $\texttt{state}_C : \Gamma \to \Psi(C)$ is the mapping that associates each instant with a state of C. Each such mapping must respect the projection of controllers onto their substates, and so, given a state-time function $\texttt{state}$, for any controller $C \in \Omega_\mathrm{c}$ and instant $t \in \Gamma$,

$$\texttt{state}_C(t) \stackrel{\text{def}}{=} C(\texttt{state}(t)). \tag{6.37}$$

To reduce subscripting, I commonly write $\texttt{state}_C(t)$ as $\texttt{state}(C, t)$.

Let J be an interval in time, and let $\texttt{state} : \Gamma \to \Psi(\top_\mathrm{c})$ be a state-time function. For any controller $C \in \Omega_\mathrm{c}$ with state $\psi \in \Psi(C)$, I say that J is an *interval of C for ψ given* $\texttt{state}$ iff, for any instant $t \in J$,

$$\texttt{state}(C, t) = \psi. \tag{6.38}$$

Where the choice of $\texttt{state}$ is unambiguous, I refer to J as simply an interval of C for ψ. I extend my earlier notation for $\texttt{state}$, writing $\texttt{state}(C, J) = \psi$ to indicate that J is an interval of C for ψ. If $C = \top_\mathrm{c}$, I omit the first argument as normal; in such cases, I say simply that J is an interval for ψ.

For any controller C and state $\psi \in \Psi(C)$, let $J \subseteq \Gamma$ be an interval of C for ψ given $\texttt{state}$. I say that J is a *maximal* interval of C for ψ (given $\texttt{state}$) iff there is no larger interval of C for ψ containing J, that is, if there is no interval J' such that $J \subset J'$ and $\texttt{state}(C, J') = \psi$. The *maximal intervals of* C, denoted $\mathscr{I}_C$, are the set of all such non-empty maximal intervals of C for any states in $\Psi(C)$. The state-time function $\texttt{state}_C$ associates every instant in Γ with a state in $\Psi(C)$, and so every instant is part of some maximal interval for some state in $\Psi(C)$. Thus, the elements of $\mathscr{I}_C$ form a partition of time; that is,

$$\bigcup_{J \in \mathscr{I}_C} J = \Gamma, \tag{6.39}$$

and, for any intervals $J, J' \in \mathscr{I}_C$, $J \cap J' = \emptyset$.

Theorem 7. *Given any controller $C \in \Omega_\mathrm{c}$ and state-time function* $\texttt{state}$*, let $\mathscr{I}_C$ be the maximal intervals for C. Then the elements of $\mathscr{I}_C$ form a partition of Γ.*

Proof. By the definition of $\texttt{state}$, for any instant $t \in \Gamma$, there exists some state $\psi \in \Psi(C)$ such that $\texttt{state}(C, t) = \psi$. Then $\{t\}$ is an interval for ψ. Either $\{t\}$ is a maximal interval for ψ, or it is a proper sub-interval of a maximal interval for ψ; in either case, it is a sub-interval of some maximal interval for

ψ, and so of some element of $\mathscr{I}_C$. Since t is an arbitrary instant, every instant is part of some element of $\mathscr{I}_C$, and so

$$\bigcup_{J \in \mathscr{I}_C} J = \Gamma. \tag{6.40}$$

Also, since `state` maps every instant to exactly one state of C, there do not exist any instants in two or more elements of $\mathscr{I}_C$, and so, for any intervals $J, J' \in \mathscr{I}_C$ such that $J \neq J'$,

$$J \cap J' = \emptyset. \tag{6.41}$$

Thus, the elements of $\mathscr{I}_C$ form a partition of Γ. $\square$

In what follows, I am most often concerned with $\mathscr{I}_{\top_c}$, the maximal intervals for $\top_c$.

6.2.2 Conditional force systems over time

I have previously referred to conditional force inclusions; more generally, a *force inclusion* $\mathfrak{f}$ is a set of force systems; that is, $\mathfrak{f} \subseteq 2^{\mathcal{F}(\Omega,\mathcal{V})}$. A *force variation inclusion* is a set $\hat{\mathfrak{f}}$ whose members are force variations. In other words, a force variation inclusion is a set $\hat{\mathfrak{f}} \subseteq 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma)}$, and every member of $\hat{\mathfrak{f}}$ is a function $\hat{f} : \Gamma \to \mathcal{F}(\Omega, \mathcal{V})$. I write the set of all force variation inclusions as $\hat{\mathfrak{F}}(\Omega, \mathcal{V}, \Gamma)$.

In what has been said previously, conditional force systems are *autonomous*; that is, the force system mapped to by a particular state does not vary with time. Where time is of interest, I refer instead to *conditional force variations*, mappings from states to force variations. Thus, a conditional force variation is a mapping $\hat{f}[\cdot] : \Psi(\top_c) \to \hat{\mathcal{F}}(\Omega_p, \mathcal{V}_p, \Gamma)$. Given a conditional force variation $\hat{f}[\cdot]$, for any instant $t \in \Gamma$ and universal state $\psi \in \Psi(\top_c)$, I write either $\hat{f}[\psi](t)$ or $f_t[\psi]$ to indicate the force system at time t. Thus, $f_t[\psi](A, B)$ is the conditional force of B on A in state ψ at time t. I extend the notation for expressing conditional differences in the same way.

Similarly, I extend the notion of a conditional force inclusion to a *conditional force variation inclusion*, that is, a mapping from controller states to sets of force variations. Formally, a conditional force variation inclusion is a mapping $\hat{\mathfrak{f}}[\cdot] : \Psi(\top_c) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma)}$ that associates every universal controller state with a force variation inclusion.

6.2.3 Extension of definitions and results

The definitions and results in this chapter generalize straightforwardly to force variations, force inclusions, and force variation inclusions. In some cases, the results need no extension; state independence, for instance, is defined purely in terms of which combinations of states are possible, and so it needs no extension. Similarly, null complementation is a property of the possible states, and so requires no extension. Force independence, by contrast, depends on the particular forces possible in various states, and so it requires an extended definition. In the following sections, then, I extend the definitions for force independence and conditional difference

6.2.3.1 Extension of force independence

I extend force independence as follows. Given a conditional force variation $\hat{f}[\cdot]$, a controller C is *force independent* (with respect to $\hat{f}[\cdot]$) iff, for any states $\psi, \psi' \in \Psi(C)$ and $\overline{\psi}, \overline{\psi'} \in \Psi(\overline{C}^{\mathrm{c}})$ such that there are universal states projecting onto $[\psi, \overline{\psi}]$, $[\psi, \overline{\psi'}]$, $[\psi', \overline{\psi}]$, and $[\psi', \overline{\psi'}]$, it is the case that, for all instants $t \in \Gamma$,

$$f_t[\psi, \overline{\psi}] - f_t[\psi', \overline{\psi}] = f_t[\psi, \overline{\psi'}] - f_t[\psi', \overline{\psi'}]. \tag{6.42}$$

In other words, a controller is force independent with respect to $\hat{f}[\cdot]$ iff at every instant $t \in \Gamma$, the controller is force independent with respect to the conditional force system $f_t[\cdot]$.

Defining force independence for a conditional force inclusion $\mathfrak{f}[\cdot]$ is somewhat more complex. Intuitively, a controller is force independent if, whenever there exist appropriate other states, the difference in the forces conditional on those states could produce the forces conditional on the actual state of the controller. For a force inclusion, this amounts to saying that, for every force system f in the inclusion for the current state, there must exist force systems in each of the other appropriate inclusions that could produce f. Thus, for each choice of the first force system, any choice of the other three force systems is permitted.

Some more formal construction is needed here. Given a conditional force inclusion $\mathfrak{f}[\cdot]$, a controller C is *force independent* (with respect to $\mathfrak{f}[\cdot]$) iff, for any states $\psi, \psi' \in \Psi(C)$ and $\overline{\psi}, \overline{\psi'} \in \Psi(\overline{C}^{\mathrm{c}})$ such that there are universal states projecting onto $[\psi, \overline{\psi}]$, $[\psi, \overline{\psi'}]$, $[\psi', \overline{\psi}]$, and $[\psi', \overline{\psi'}]$, it is the case that, for any force system $f_{\psi,\overline{\psi}} \in \mathfrak{f}[\psi, \overline{\psi}]$, there exist force systems $f_{\psi',\overline{\psi}} \in \mathfrak{f}[\psi', \overline{\psi}]$, $f_{\psi,\overline{\psi'}} \in \mathfrak{f}[\psi, \overline{\psi'}]$, and $f_{\psi',\overline{\psi'}} \in \mathfrak{f}[\psi', \overline{\psi'}]$ such that

$$f_{\psi,\overline{\psi}} - f_{\psi',\overline{\psi}} = f_{\psi,\overline{\psi'}} - f_{\psi',\overline{\psi'}}. \tag{6.43}$$

The controller complement of such a controller is also force independent. I show this result in the following theorem.

Theorem 8. *Given a conditional force inclusion $\mathfrak{f}[\cdot]$, if a controller C is force independent with respect to $\mathfrak{f}[\cdot]$, then its controller complement $\overline{C}^{\mathrm{c}}$ is force independent with respect to $\mathfrak{f}[\cdot]$.*

Proof. Suppose that some controller C is force independent with respect to $\mathfrak{f}[\cdot]$. Let $\psi, \psi' \in \Psi(C)$ and $v, v' \in \Psi(\overline{C}^{\mathrm{c}})$ be any states such that $[\psi, v]$, $[\psi, v']$, $[\psi', v]$, and $[\psi', v']$ all exist, if any such choice of states are possible; if no such choices exist, the proof holds trivially. Since C is force independent, for any force system $f_{\psi,v} \in \mathfrak{f}[\psi, v, v]$, there exist force systems $f_{\psi',v} \in \mathfrak{f}[\psi', v]$, $f_{\psi,v'} \in \mathfrak{f}[\psi, v']$, and $f_{\psi',v'} \in \mathfrak{f}[\psi', v']$ such that

$$f_{\psi,v}(t) - f_{\psi',v}(t) = f_{\psi,v'}(t) - f_{\psi',v'}(t). \tag{6.44}$$

Switching the order in which substates are listed, for any force system $f_{v,\psi} \in \mathfrak{f}[v, \psi]$, there exist force systems $f_{v,\psi'} \in \mathfrak{f}[v, \psi']$, $f_{v',\psi} \in \mathfrak{f}[v', \psi]$, and $f_{v',\psi'} \in \mathfrak{f}[v', \psi']$ such that

$$f_{v,\psi}(t) - f_{v,\psi'}(t) = f_{v',\psi}(t) - f_{v',\psi'}(t). \tag{6.45}$$

Moving the middle two terms to the other side yields

$$f_{v,\psi}(t) - f_{v',\psi}(t) = f_{v,\psi'}(t) - f_{v',\psi'}(t). \tag{6.46}$$

But this is precisely the requirement for $\overline{C}^{\mathrm{c}}$ to be force independent with respect to $\mathfrak{f}[\cdot]$. □

Conditional force variation inclusions are handled in much the same way as conditional force inclusions, save for the extension from force systems to force variations. Thus, given a conditional force variation inclusion $\hat{\mathfrak{f}}[\cdot]$, a controller C is *force independent* (with respect to $\hat{\mathfrak{f}}[\cdot]$) iff, for any states $\psi, \psi' \in \Psi(C)$ and $\overline{\psi}, \overline{\psi'} \in \Psi(\overline{C}^{\mathrm{c}})$ such that there are universal states projecting onto $[\psi, \overline{\psi}]$, $[\psi, \overline{\psi'}]$, $[\psi', \overline{\psi}]$, and $[\psi', \overline{\psi'}]$, it is the case that, for any force variation $\hat{f}_{\psi,\overline{\psi}} \in \hat{\mathfrak{f}}[\psi, \overline{\psi}]$, there exist force systems $\hat{f}_{\psi',\overline{\psi}} \in \hat{\mathfrak{f}}[\psi', \overline{\psi}]$, $\hat{f}_{\psi,\overline{\psi'}} \in \hat{\mathfrak{f}}[\psi, \overline{\psi'}]$, and $\hat{f}_{\psi',\overline{\psi'}} \in \hat{\mathfrak{f}}[\psi', \overline{\psi'}]$ such that

$$\hat{f}_{\psi,\overline{\psi}}(t) - \hat{f}_{\psi',\overline{\psi}}(t) = \hat{f}_{\psi,\overline{\psi'}}(t) - \hat{f}_{\psi',\overline{\psi'}}(t). \tag{6.47}$$

The controller complement of a controller force independent with respect to a conditional force variation inclusion is also force independent with respect to that inclusion, as stated in the following corollary.

Corollary 9. *Given a conditional force variation inclusion $\hat{\mathfrak{f}}[\cdot]$, if a controller C is force independent with respect to $\hat{\mathfrak{f}}[\cdot]$, then its controller complement $\overline{C}^{\mathrm{c}}$ is force independent with respect to $\hat{\mathfrak{f}}[\cdot]$.*

Proof. This proof is functionally identical to the proof of Theorem 8, replacing force systems with force variations. □

6.2.3.2 Extension of conditional difference

I now extend conditional difference to force variations, writing $\hat{f}^*$ for such variations. Given a force variation $\hat{f}$, for any universal state $\psi \in \Psi(\top_{\mathrm{c}})$, the *conditional difference variation* $\hat{f}^*[\top_{\mathrm{c}}, \psi]$ for $\hat{f}$ is

$$\hat{f}^*[\top_{\mathrm{c}}, \psi] \stackrel{\text{def}}{=} \hat{f}[\psi] - \hat{f}[0]. \tag{6.48}$$

For any state $\psi \in \Psi(\top_{\mathrm{c}})$ and null complemented controller C, the conditional difference variation $f^*[C, \psi]$ over J is

$$\hat{f}^*[C, \psi] \stackrel{\text{def}}{=} \hat{f}[C(\psi), 0] - \hat{f}[0]. \tag{6.49}$$

For any choice of C and ψ, I write $f^*_t[C, \psi]$ for $\hat{f}^*[C, \psi](t)$ where possible.

Earlier definitions for differential force systems translate directly to differential force variations. Two controllers C_1, C_2 are mutually controller-additive with respect to a differential force variation $\hat{f}^*$ iff, for any state $\psi \in \Psi(\top_c)$ such that $(C_1 \sqcup C_2)(\psi)$, $C_1(\psi)$, $C_2(\psi)$, and $(C_1 \sqcap C_2)(\psi)$ are all null complemented,

$$\hat{f}^*[C_1 \sqcup C_2, \psi] = \hat{f}^*[C_1, \psi] + \hat{f}^*[C_2, \psi] - \hat{f}^*[C_1 \sqcap C_2, \psi]. \tag{6.50}$$

A controller C is controller-additive with respect to $\hat{f}^*$ iff C and $\overline{C}^{\mathrm{c}}$ are mutually controller-additive with respect to $\hat{f}^*$. $\hat{f}^*$ is controller-additive iff every controller in Ω_c is controller-additive with respect to $\hat{f}^*$. If $\hat{f}^*$ is defined for any state ψ and some set of controllers $\{C_1, C_2, \ldots\}$, I say it is complete for that set.

The conditional difference theorems in this chapter extend from force systems to force variations in a natural way; given a result shown for force systems, showing that the same result holds for force variation versions is equivalent to stating that the original proofs hold for all instants in the domain of a force variation. Thus, I provide corresponding corollaries.

Corollary 10. *For every differential force variation $\hat{f}^*$, the null controller $\perp_c$ is mutually controller-additive with every controller.*

Proof. Since Corollary 4 holds for all instants in Γ, this corollary holds. □

Corollary 11. *Controllers C_1 and C_2 are mutually controller-additive with respect to a differential force variation $\hat{f}^*$ of a conditional force variation $\hat{f}$ whenever C_1 is force independent with respect to $\hat{f}$ and $\hat{f}^*$ is complete for C_1 and C_2.*

Proof. Since Corollary 5 holds for all instants in Γ, this corollary holds. □

Corollary 12. *If a differential force variation $\hat{f}^*$ is complete, then each controller is controller-additive with respect to $\hat{f}^*$ if and only if it is force independent with respect to a conditional force variation $\hat{f}$.*

Proof. Since Theorem 6 holds for all instants in Γ, this corollary holds. □

I also extend the notion of controller-additivity to conditional force inclusions. I begin by restricting attention to inclusions and choices of 0 for which $|\mathfrak{f}[0]| = 1$, that is, such that $\mathfrak{f}[0] = \{f_0\}$ for some force system f_0. Such a requirement makes intuitive sense if the null state is viewed as an "off" position for the mechanical system; while forces in f may not literally be zero, the singular force system nonetheless represents a singular set of base forces, with other states providing one or more alternatives. Indeed, this restriction may assist in the selection of a meaningful choice of null state.

In general, this requirement may prove to be unnecessary. All else aside, however, it simplifies the treatment of force inclusions in the following pages, and it will be sufficient for my needs in this paper. If any more general treatment of the null state in inclusions is possible, I leave it to future work.

In particular, then, let $\mathfrak{f}[0] = \{f\}$ for some force system f. In such a case, for any state $\psi \in \Psi(\top_c)$, the *conditional differential force inclusion* $\mathfrak{f}^*[\top_c, \psi]$ is

$$\mathfrak{f}^*[\top_c, \psi] \stackrel{\text{def}}{=} \{f' - f \mid f' \in \mathfrak{f}[\psi]\}. \tag{6.51}$$

Generalizing, for any state $\psi \in \Psi(\top_c)$ and state independent controller C, the conditional difference $\mathfrak{f}^*[C, \psi]$ is

$$\mathfrak{f}^*[C, \psi] \stackrel{\text{def}}{=} \{f' - f \mid f' \in \mathfrak{f}[C(\psi), 0]\}. \tag{6.52}$$

Let $f_{\mathbf{0}}$, the *passive force system*, be the force system such that only null forces exist between any two bodies; that is, for any $A, B \in \Omega$,

$$f_{\mathbf{0}}(A, B) \stackrel{\text{def}}{=} \mathbf{0}. \tag{6.53}$$

Then $\mathfrak{f}^*[\bot_c, \psi] = \{f_{\mathbf{0}}\}$.

Lemma 1. *If $\mathfrak{f}^*$ is a differential force inclusion such that $|\mathfrak{f}[0]| = 1$, then for any state $\psi \in \Psi(\top_c)$,*

$$\mathfrak{f}^*[\bot_c, \psi] = \{f_{\mathbf{0}}\}. \tag{6.54}$$

Proof. By definition, if $\mathfrak{f}[0] = \{f\}$ for some force system f, then

$$\mathfrak{f}^*[\bot_c, \psi] = \{f' - f \mid f' \in \mathfrak{f}[\bot_c(\psi), 0]\}. \tag{6.55}$$

But $\mathfrak{f}[\bot_c(\psi), 0] = \mathfrak{f}[0]$, and so $f' = f$. Then

$$\begin{aligned}\mathfrak{f}^*[\bot_c, \psi] &= \{f - f\} \\ &= \{f_{\mathbf{0}}\}.\end{aligned} \tag{6.56}$$

$\square$

In defining controller-additivity for force inclusions, I follow the pattern of force independence, above, following similar reasoning. Thus, given a conditional differential force inclusion $\mathfrak{f}^*$, I say that two controllers C_1, C_2 are mutually controller-additive with respect to $\mathfrak{f}^*$ iff, for any state $\psi \in \Psi(\top_c)$ such that state projections $(C_1 \sqcup C_2)(\psi)$, $C_1(\psi)$, $C_2(\psi)$, and $(C_1 \sqcap C_2)(\psi)$ are all null complemented, for any force system $f_1^* \in \mathfrak{f}^*[C_1 \sqcup C_2, \psi]$, there exists some force systems $f_2^* \in \mathfrak{f}^*[C_1, \psi]$, $f_3^* \in \mathfrak{f}^*[C_2, \psi]$, and $f_4^* \in \mathfrak{f}^*[C_1 \sqcap C_2, \psi]$ such that

$$f_1^* = f_2^* + f_3^* - f_4^*. \tag{6.57}$$

A controller C is controller-additive with respect to $\mathfrak{f}^*$ iff C and $\overline{C}^c$ are mutually controller-additive with respect to $\mathfrak{f}^*$. $\mathfrak{f}^*$ is controller-additive iff every controller is controller-additive with respect to $\mathfrak{f}^*$. Again, the conditional theorems in this chapter extend to force inclusions with no structural changes, and so I provide corresponding corollaries.

Corollary 13. *For every differential force inclusion* $\mathfrak{f}^*$ *such that* $|\mathfrak{f}[0]| = 1$*, the null controller* $\bot_{\mathrm{c}}$ *is mutually controller-additive with every controller.*

Proof. The truth of this corollary follows immediately from Corollary 4 and (6.57). Consider the definition of force independence in (6.57); letting $C_1 = \bot_{\mathrm{c}}$, one has $\bot_{\mathrm{c}} \sqcap C_2 = \bot_{\mathrm{c}}$ and $\bot_{\mathrm{c}} \sqcup C_2 = C_2$. The definition of force independence then becomes the claim that, for any $\psi \in \Psi(\top_{\mathrm{c}})$ such that $\bot_{\mathrm{c}}(\psi)$ and $C_2(\psi)$ are all null complemented, for any force system $f_1 \in \mathfrak{f}[C_2(\psi), 0]$, there exists some force systems $f_2, f_4 \in \mathfrak{f}[\bot_{\mathrm{c}}(\psi), 0]$ and $f_3 \in \mathfrak{f}[C_2(\psi), 0]$ such that

$$f_1^* = f_2^* + f_3^* - f_4^*, \tag{6.58}$$

Note that f_1 and f_3 are in the same inclusion, as are f_2 and f_4. Letting $f_3 = f_1$ and $f_2 = f_4$, this equation is always true. □

Corollary 14. *Let* $\mathfrak{f}$ *be any differential force inclusion for which* $|\mathfrak{f}[0]| = 1$*. Controllers* C_1 *and* C_2 *are mutually controller-additive with respect to* $\mathfrak{f}^*$ *whenever* C_1 *is force independent with respect to* $\mathfrak{f}$ *and* $\mathfrak{f}^*$ *is complete for* C_1 *and* C_2*.*

Proof. Let f be the force system such that $\mathfrak{f}[0] = \{f_0\}$.

Suppose f^* is complete for $\{C_1, C_2\}$ and that C_1 is force independent with respect to f. Since f^* is complete for C_1 and C_2, by Theorem 3, f^* is complete for a subuniverse containing C_1 and C_2; denote this universe by $\Omega_{\mathrm{c}}[C_1, C_2]$.

Let ψ be in $\Psi(\top_{\mathrm{c}})$, and decompose it into the substates

$$\begin{aligned}
\psi_1 &= C_1(\psi) \\
\psi_2 &= C_2(\psi) \\
\psi_{12} &= (C_1 \sqcap C_2)(\psi) \\
\psi_{1-2} &= (C_1 \sqcap \overline{C_2}^{\mathrm{c}})(\psi) \\
\psi_{2-1} &= (\overline{C_1}^{\mathrm{c}} \sqcap C_2)(\psi) \\
\psi_{-1-2} &= (\overline{(C_1 \sqcup C_2)}^{\mathrm{c}})(\psi).
\end{aligned}$$

Similarly, decompose 0 into the components $0_1, 0_2, 0_{1-2}, 0_{2-1}, 0_{12}, 0_{-1-2}$.

Because f^* is complete with respect to $\Omega_{\mathrm{c}}[C_1, C_2]$, there must exist universal states that can be decomposed as $[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}]$, $[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}]$, $[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}]$, and $[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}]$. Then since controller C_1 is force independent, for any force system $f_w \in \mathfrak{f}[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}]$, there must exist force systems $f_x \in \mathfrak{f}[\psi_{12}, 0_{1-2}, \psi_{2-1}, 0_{-1-2}]$,

$f_y \in \mathfrak{f}[\psi_{12}, \psi_{1-2}, 0_{2-1}, 0_{-1-2}]$, and $f_z \in \mathfrak{f}[\psi_{12}, 0_{1-2}, 0_{2-1}, 0_{-1-2}]$ such that

$$f_w - f_x = f_y - f_z. \tag{6.59}$$

Rearranging terms yields

$$f_w = f_x + f_y - f_z. \tag{6.60}$$

Subtracting f from both sides, adding $f - f$ to the right-hand side, and associating terms produces

$$(f_w - f) = (f_x - f) + (f_y - f) - (f_z - f)\,. \tag{6.61}$$

Each of the associated terms in (6.61) constitutes a force system in the conditional differences for ψ corresponding to $C_1 \sqcup C_2$, C_1, C_2, and $C_1 \sqcap C_2$. In other words, f_w is in $\mathfrak{f}[\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2}]$ iff the differential force system $f^*_w = f_w - f$ is in $\mathfrak{f}^*[C_1 \sqcup C_2, (\psi_{12}, \psi_{1-2}, \psi_{2-1}, 0_{-1-2})]$. Equivalently, f^*_w is in $\mathfrak{f}^*[C_1 \sqcup C_2, \psi]$. Similar constructions exist for the other three force systems.

Then for any choice of $f^*_w \in \mathfrak{f}^*[C_1 \sqcup C_2, \psi]$,

$$f^*_w = f^*_x + f^*_y - f^*_z, \tag{6.62}$$

where

$$\begin{aligned} f^*_x &= f_x - f \in \mathfrak{f}^*[C_1, \psi] \\ f^*_y &= f_y - f \in \mathfrak{f}^*[C_2, \psi] \\ f^*_z &= f_z - f \in \mathfrak{f}^*[C_1 \sqcap C_2, \psi]. \end{aligned} \tag{6.63}$$

Then C_1 and C_2 are mutually controller additive with respect to $\mathfrak{f}^*$. □

Corollary 15. *Let $\mathfrak{f}$ be any differential force inclusion for which $|\mathfrak{f}[0]| = 1$. If $\mathfrak{f}^*$ is complete, then each controller is controller-additive with respect to $\mathfrak{f}^*$ if it is force independent with respect to $\mathfrak{f}$.*

Proof. By assumption, $\mathfrak{f}[0] = \{f\}$ for some force system f. Suppose that $\mathfrak{f}^*$ is complete. It follows immediately from Corollary 14 that if C is force independent with respect to $\mathfrak{f}$, then it is controller additive with respect to $\mathfrak{f}^*$. □

Conditional force variation inclusions are handled in much the same way as conditional force inclusions, save for the extension from force systems to force variations. Thus, let $\hat{f}$ be the force variation such that $\hat{\mathfrak{f}}[0] = \{\hat{f}\}$. If a conditional differential force variation inclusion $\hat{\mathfrak{f}}^*$ exists, I say that two controllers C_1, C_2 are mutually controller-additive with respect to $\hat{\mathfrak{f}}^*$ iff, for any state $\psi \in \Psi(\top_\mathrm{c})$ such that $(C_1 \sqcup C_2)(\psi)$, $C_1(\psi)$, $C_2(\psi)$, and $(C_1 \sqcap C_2)(\psi)$ are all null complemented, for any force variation $\hat{f}^*_1 \in \hat{\mathfrak{f}}^*[C_1 \sqcup C_2, \psi]$, there exists some force variations $\hat{f}^*_2 \in \hat{\mathfrak{f}}^*[C_1, \psi]$, $\hat{f}^*_3 \in \hat{\mathfrak{f}}^*[C_2, \psi]$, and

$\hat{f}_4^* \in \hat{\mathfrak{f}}^*[C_1 \sqcap C_2, \psi]$ such that

$$\hat{f}_1^* = \hat{f}_2^* + \hat{f}_3^* - \hat{f}_4^*. \tag{6.64}$$

Let $\hat{f}_{\mathbf{0}}$ be the force variation such that, for any instant $t \in \Gamma$,

$$\hat{f}_{\mathbf{0}}(t) = f_{\mathbf{0}}. \tag{6.65}$$

I call this force variation the *passive force variation*. Then for any state $\psi \in \Psi(\top_{\mathrm{c}})$, it follows that $\hat{\mathfrak{f}}^*[\perp_{\mathrm{c}}, \psi] = \{\hat{f}_{\mathbf{0}}\}$.

Lemma 2. *If $\hat{\mathfrak{f}}^*$ is a differential force variation inclusion such that $|\hat{\mathfrak{f}}[0]| = 1$, then for any state $\psi \in \Psi(\top_{\mathrm{c}})$,*

$$\hat{\mathfrak{f}}^*[\perp_{\mathrm{c}}, \psi] = \{\hat{f}_{\mathbf{0}}\}. \tag{6.66}$$

Proof. Since Lemma 1 holds at every instant, this lemma holds. □

A controller C is controller-additive with respect to $\hat{\mathfrak{f}}^*$ iff C and $\overline{C}^{\mathrm{c}}$ are mutually controller-additive with respect to $\hat{\mathfrak{f}}^*$. $\hat{\mathfrak{f}}^*$ is controller-additive iff every controller is controller-additive with respect to $\hat{\mathfrak{f}}^*$. Again, functionally equivalent versions of the above theorems hold for conditional force variation inclusions.

Corollary 16. *For every differential force variation inclusion $\hat{\mathfrak{f}}^*$ such that $|\hat{\mathfrak{f}}[0]| = 1$, the null controller $\perp_{\mathrm{c}}$ is mutually controller-additive with every controller.*

Proof. This proof is functionally identical to the proof of Corollary 13, with force systems replaced by force variations. □

Corollary 17. *Let $\hat{\mathfrak{f}}$ be any differential force variation inclusion for which $|\hat{\mathfrak{f}}[0]| = 1$. Controllers C_1 and C_2 are mutually controller-additive with respect to $\hat{\mathfrak{f}}^*$ whenever C_1 is force independent with respect to $\hat{\mathfrak{f}}$ and $\hat{\mathfrak{f}}^*$ is complete for C_1 and C_2.*

Proof. This proof is functionally identical to the proof of Corollary 14, with force systems replaced by force variations. □

Corollary 18. *Let $\hat{\mathfrak{f}}$ be any differential force variation inclusion for which $|\hat{\mathfrak{f}}[0]| = 1$. If $\hat{\mathfrak{f}}^*$ is complete, then each controller is controller-additive with respect to $\hat{\mathfrak{f}}^*$ if it is force independent with respect to $\hat{\mathfrak{f}}$.*

Proof. This proof is functionally identical to the proof of Corollary 15, with force systems replaced by force variations. □

6.3 Mechanical definition of controller states

To this point, the exact nature of "mechanical controller states" has been left undefined; in what follows, I will provide a formal definition. Recall from Section 5.3.4 that the *hybrid configuration space* of a hybrid body B for a hybrid space $\mathcal{S}$, denoted $\mathfrak{X}(B, \mathcal{S})$, is the set of all pairs $(w, \dot{w})$, where w is a hybrid placement of B in $\mathcal{S}$ and $\dot{w}$ is a hybrid vector field over the body points of B. Given a hybrid mechanical system M, I will most often be concerned with $\mathfrak{X}(\top, \mathcal{S})$, that is, with configurations of the universal body $\top$ in hybrid space $\mathcal{S}$. In particular, for the purposes of this dissertation, a mechanical controller state is a nonempty subset of $\mathfrak{X}(\top, \mathcal{S})$, with the set of all states of any particular controller forming a partition of $\mathfrak{X}(\top, \mathcal{S})$. In other words, to say that a controller C is in a particular state is equivalent to stating that the body points of $\top$ have one of a particular set of assignments of placement and velocity. Then for any state ψ and configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ in ψ, I can write ψ as the equivalence class $[(w, \dot{w})]$.

Given an assignment of position and velocity to all body points of $\top$ at all instants, then, the state of $\top_{\mathrm{c}}$ is determined at every instant. In other words, every motion χ of $\top$ determines a state-time function $\texttt{state}_\chi$. More formally, let χ be any motion of $\top$, and let $\Psi(\top_{\mathrm{c}})$ be a partition of $\mathfrak{X}(\top, \mathcal{S})$. Then at every instant $t \in \Gamma$,

$$\texttt{state}_\chi(t) = [(\chi_t, \dot{\chi}_t)]. \tag{6.67}$$

I call $\texttt{state}_\chi$ the state-time function for χ; where the choice of motion is unambiguous, I drop the subscript.

6.3.1 Mechanical state projection

As described in Section 6.1, the states of non-universal controllers are formally defined as projections of the states of the universal controller. In practice, it is often simpler to first define states for all controllers in some partition of the universal controller, and then to define universal controller states that will produce the desired state projections. Suppose that there exist two separate controllers $C_1, C_2 \in \Omega_{\mathrm{c}}$, and suppose further that $\Psi(C_1)$ and $\Psi(C_2)$, the state sets for these bodies, have been defined. Let $C = C_1 \sqcup C_2$; then the states of C are the non-empty intersections of the states of C_1 and C_2. In other words, for any $\psi_1 \in \Psi(C_1)$ and $\psi_2 \in \Psi(C_2)$, let the compound state notation (ψ_1, ψ_2) correspond to the equivalence class

$$(\psi_1, \psi_2) \stackrel{\text{def}}{=} \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid (w, \dot{w}) \in \psi_1 \cap \psi_2\}. \tag{6.68}$$

Clearly in this case, $(\psi_1, \psi_2) = \psi_1 \cap \psi_2$. Then the states of C are all such nonempty equivalence classes,

$$\Psi(C) \stackrel{\text{def}}{=} \{(\psi_1, \psi_2) \mid \big(\psi_1 \in \Psi(C_1)\big) \wedge \big(\psi_2 \in \Psi(C_2)\big) \wedge \big((\psi_1, \psi_2) \neq \emptyset\big)\}. \tag{6.69}$$

In this case, each state in $\Psi(C)$ maps onto its component controllers in the obvious way, so that

$$\begin{aligned} C_1(\psi_1, \psi_2) &= \psi_1 \\ C_2(\psi_1, \psi_2) &= \psi_2. \end{aligned} \tag{6.70}$$

This definition can be easily extended to allow for controllers which are the join of any number of separate components. Thus, let U be any set of separate controllers such that, for every controller $C \in U$, the set of states $\Psi(C)$ is defined. Let C' be the join of all such controllers, that is,

$$C' = \bigsqcup_{C \in U} C. \tag{6.71}$$

Let $\equiv_{C'}$ be the equivalence relation over $\mathfrak{X}(\top, \mathcal{S})$ such that two configurations are related iff they are in the same states of every controller in U. In other words,

$$(w, \dot{w}) \equiv_{C'} (w', \dot{w}') \tag{6.72}$$

iff

$$\forall C \in U \Big(\exists \psi_C \in \Psi(C) \big[(w, \dot{w}) \in \psi_C \wedge (w', \dot{w}') \in \psi_C \big] \Big). \tag{6.73}$$

This relation over configuration space imposes a partition $\mathfrak{X}(\top, \mathcal{S})/\!\equiv_{C'}$ in which all members of an equivalence class $[(w, \dot{w})] \in \mathfrak{X}(\top, \mathcal{S})/\!\equiv_{C'}$ are members of the same states of the component controllers of C'. The elements of this partition are then the states of C', that is,

$$\Psi(C') \stackrel{\text{def}}{=} \mathfrak{X}(\top, \mathcal{S})/\!\equiv_{C'}. \tag{6.74}$$

Given the above definition, every state of the supercontroller is a subset of exactly one state of each component subcontroller. Formally, for any state $\psi' \in \Psi(C')$ and component controller $C \in U$, there exists a unique state $\psi \in \Psi(C)$ such that $\psi' \subseteq \psi$. This state is the projection of ψ' onto C, that is,

$$C(\psi') = \psi. \tag{6.75}$$

The mechanical controller states of any other joins of controllers in U are defined similarly, with the states of any such join being the equivalence classes induced by intersections of the states in its component controllers. Let C, C', C'' be any controllers such that $C \sqsubseteq C' \sqsubseteq C''$, and such that each such controller is either an element of U or the join of elements of U. By the above construction, the states of C'' are subsets of the states of C', which are themselves subsets of the states of C. In other words, for any state $\psi'' \in \Psi(C'')$, there must exist unique states $\psi' \in \Psi(C')$ and $\psi \in \Psi(C)$ such that

$\psi'' \subseteq \psi' \subseteq \psi$. In this case,

$$C'(\psi'') = \psi' \tag{6.76}$$

and

$$C(\psi'') = C(\psi') = \psi. \tag{6.77}$$

To specify all mechanical controller states of a hybrid mechanical system, then, it is sufficient to provide mechanical controller states for a set U of mechanical controllers such that every element of Ω_c is either in U or is a join of elements in U.

6.3.2 Conditional forces and hybrid mechanics

As discussed in Section 5.4, an (initialized) hybrid mechanical system is a tuple $(\Omega, \mathcal{S}, \Gamma, m, \mathfrak{L}, \mathfrak{X}_0(\top, \mathcal{S}))$, where Ω is the set of all hybrid bodies, $\mathcal{S}$ is a hybrid mechanical space, Γ is hybrid time, $\mathfrak{L}$ is a time-varying nondeterministic hybrid system law function, and $\mathfrak{X}_0(\top, \mathcal{S})$ is an initial hybrid configuration set. Recall that $\mathfrak{L}$ is a function $\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S}) \to 2^{\hat{\mathfrak{F}}(\Omega, \mathcal{V}, \Gamma)}$ mapping configurations of the hybrid universal body to hybrid force variation inclusions. Alternatively, given a motion χ, one can define an equivalence relation $\equiv_{\mathfrak{L}}$ such that, for any equivalence class $[(w, \dot{w})] \in \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$, the system law function maps all elements of $[(w, \dot{w})]$ to the same set of force variations. Thus, one can rewrite $\mathfrak{L}$ as a mapping

$$\mathfrak{L} : \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}} \to 2^{\hat{\mathfrak{F}}(\Omega, \mathcal{V}, \Gamma)}, \tag{6.78}$$

that is, a mapping from a partition of $\mathfrak{X}(\top, \mathcal{S})$ to force variation inclusions.

Recall as well from Section 6.2.2 that a conditional force variation inclusion is a function $\hat{\mathfrak{f}}[\cdot] : \Psi(\top_c) \to 2^{\hat{\mathfrak{F}}(\Omega, \mathcal{V}, \Gamma)}$, that is, a function mapping from universal controller states to force variation inclusions. But by Section 6.3, universal controller states are simply equivalence classes over the configuration space of the hybrid universal body. It follows that a conditional force variation inclusion is a time-varying nondeterministic hybrid system law function; in particular, it is a system law function in which the states of the universal controller are precisely the equivalence classes of $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$, that is,

$$\Psi(\top_c) = \mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}} . \tag{6.79}$$

Given a motion χ, the state-time function for χ is the configuration-time function for $\mathfrak{L}$ and χ; that is,

$$\mathtt{state}_\chi = \mathfrak{X}\text{-time}(\chi, \mathfrak{L}). \tag{6.80}$$

Thus, one can substitute a particular mechanical state function Ψ and conditional force variation inclusion $\hat{\mathfrak{f}}[\cdot]$ for any given system law function $\mathfrak{L}$. I follow this pattern in subsequent chapters, writing a hybrid mechanical system via a tuple $(\Omega, \mathcal{S}, \Gamma, m, \Psi, \hat{\mathfrak{f}}[\cdot], \mathfrak{X}_0(\top, \mathcal{S}))$, where Ω is the set of hybrid mechanical bodies, $\mathcal{S}$ is hybrid space, Γ is hybrid time, Ψ is the function mapping mechanical controller bodies to sets of mechanical controller states, $\hat{\mathfrak{f}}[\cdot]$ is a conditional force variation inclusion, and $\mathfrak{X}_0(\top, \mathcal{S})$ is a hybrid initial configuration set.

6.3.3 Mechanical states and induced motion

By Section 5.3.2, given a time-varying deterministic hybrid system law function $\mathfrak{L}$ and a configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, an induced motion for $\mathfrak{L}$ is a motion $\chi : \top \times \Gamma \to \mathcal{C}(\top, \mathcal{S})$ such that, for every instant $t \in \Gamma$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\mathfrak{L}_i(\chi_t, \dot{\chi}_t)(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{6.81}$$

If $\mathfrak{L}$ is instead nondeterministic, then an induced motion for $\mathfrak{L}$ is a motion χ such that, for every interval $J \in \mathscr{I}_{\chi,\mathfrak{L}}$, there exists some $\hat{f} \in \mathfrak{L}(\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})(J))$ such that for every instant $t \in \Gamma$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{6.82}$$

By the discussion above, there exists an equivalent notion of induced motion for any conditional force variation inclusion $\hat{\mathfrak{f}}[\cdot]$.

I proceed towards this equivalence in several steps. First, I introduce the idea of motions that are induced only over some interval in time; conceptually, such motions are guaranteed to follow the Newton-Euler Laws during some interval $J \subseteq \Gamma$, but their behavior is otherwise unconstrained. More formally, let $\mathfrak{L}$ be a hybrid system law function of whatever sort. A motion χ is an induced motion for $\mathfrak{L}$ *during* interval $J \subseteq \Gamma$ iff it satisfies the requirements for being an induced motion if restricted to J; it may or may not satisfy these requirements at instants not in J. Thus, if $\mathfrak{L}$ is deterministic and time-varying, χ is an induced motion for $\mathfrak{L}$ during J iff for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\mathfrak{L}_i(\chi_t, \dot{\chi}_t)(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{6.83}$$

Constructions for other forms of system law function are similar.

Next, let $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ be a configuration, $\hat{\mathfrak{f}}[\cdot]$ a conditional force variation inclusion, and χ a second-degree doubly semidifferentiable motion of $\top$. Let `state` be a state-time function for χ, and let $\mathscr{I}_{\top_c}$ the the maximal interval set for `state`. Finally, let $J \in \mathscr{I}_{\top_c}$ be a maximal interval such that, for some state $\psi \in \Psi(\top_c)$, $\mathtt{state}(J) = \psi$. I say that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ during J iff there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ over J such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{6.84}$$

Then χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ iff it is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ during all intervals in $\mathscr{I}_{\top_c}$.

6.4 Summary of notation

C	a mechanical controller
$\Psi(C)$	states of controller C
Ω_{p}	physical universe
Ω_{c}	controller universe
$\top_{\mathrm{p}}$	universal physical body
$\top_{\mathrm{c}}$	universal controller body
$\perp_{\mathrm{p}}$	physical null body
$\perp_{\mathrm{c}}$	controller null body
$\overline{B}^{\mathrm{p}}$	physical exterior of body B
$\overline{B}^{\mathrm{c}}$	controller exterior of body B
$f[\cdot]$	conditional force system
$f[\psi]$	force system conditional on mechanical controller state ψ
$C(\psi)$	projection of mechanical controller state ψ onto mechanical controller C
$\mathfrak{f}[\cdot]$	conditional force inclusion
$\mathfrak{f}[\psi]$	force inclusion conditional on mechanical controller state ψ
$f^*[\cdot,\cdot]$	conditional difference
$f^*[C,\psi]$	conditional difference given mechanical controller C in state ψ
$\mathtt{state}$	state-time function
$\mathtt{state}(t)$	state of universal mechanical controller in instant t
$\mathtt{state}_C$	state-time function for mechanical controller C
$\mathtt{state}_C(t)$	state of mechanical controller C in instant t
$\mathtt{state}(C,t)$	state of mechanical controller C in instant t
$\mathtt{state}(J)$	state of universal mechanical controller during interval J (if it exists)
$\mathtt{state}(C,J)$	state of mechanical controller C during interval J (if it exists)
$\mathscr{I}_C$	maximal intervals for mechanical controller C
$\hat{f}[\cdot]$	conditional force variation
$\hat{f}_t[\psi], \hat{f}[\psi](t)$	force system conditional on mechanical controller state ψ at instant t
$\hat{f}^*[\cdot,\cdot]$	conditional differential force variation
$\hat{f}^*[C,\psi]$	conditional differential force variation given mechanical controller C in state ψ
$f_{\mathbf{0}}$	passive force system

$\hat{f}_\mathbf{0}$	passive force variation
$\mathfrak{f}$	force inclusion
$\mathfrak{f}[\cdot]$	conditional force inclusion
$\mathfrak{f}^*[\cdot,\cdot]$	conditional differential force inclusion
$\mathfrak{f}^*[C,\psi]$	conditional differential force inclusion given mechanical controller C in state ψ
$\hat{\mathfrak{f}}$	force variation inclusion
$\hat{\mathfrak{f}}[\cdot]$	conditional force variation inclusion
$\hat{\mathfrak{f}}^*[\cdot,\cdot]$	conditional differential force variation inclusion
$\hat{\mathfrak{f}}^*[C,\psi]$	conditional differential force variation inclusion given mechanical controller C in state ψ
$\equiv_C$	equivalence class over $\mathfrak{X}(\top,\mathcal{S})$ based on membership in the same state of controller C
$\texttt{state}_\chi$	state-time function for motion χ

Chapter 7

Transformation from Automata to Mechanics

To this point, I have primarily only described existing mathematical formalisms for hybrid automata and hybrid mechanics. In what follows, I define a transformation from an arbitrary hybrid automaton

$$A = (V_A, X_A, Q_{0A}, D_A, \mathcal{T}_A, \Gamma_A), \tag{7.1}$$

as defined in Section 3.3.2.4, to a hybrid mechanical system

$$M = (\top_M, \mathcal{S}_M, \Gamma_M, m_M, \Psi_M, \mathfrak{f}[\cdot]_M, \mathfrak{X}_0(\top, \mathcal{S})_M), \tag{7.2}$$

as defined in Section 6.3.2. I generally omit the subscript from all components of both tuples except for Γ_A and Γ_M, retaining the subscript there in order to avoid ambiguity as to which time is being referenced.

The resultant transformation will produce a *physical* and a *controller* mechanical system, each with a set of bodies $\Omega_\mathrm{p}, \Omega_\mathrm{c}$, space $\mathcal{S}_\mathrm{p}, \mathcal{S}_\mathrm{c}$, time $\Gamma_\mathrm{p}, \Gamma_\mathrm{c}$, and mass $m_\mathrm{p}, m_\mathrm{c}$. I will also define a set of mechanical controller states for the universal controller body $\top_\mathrm{c}$. I then provide a construction of a single hybrid mechanical system, with bodies Ω, space $\mathcal{S}$, time Γ_M, mass m, mapping from controller bodies to states Ψ, and set of initial configurations $\mathfrak{X}_0(\top, \mathcal{S})$. In Chapter 8, I add to this construction, defining a set of force variation inclusions over Ω conditional on the state of $\top_\mathrm{c}$. The final result, then, is a hybrid mechanical system M; in Chapter 12, I will prove that M bisimulates the hybrid automaton A.

As discussed in Section 3.1.2, I restrict attention in this paper to automata such that the environment state space X is $\mathbb{R}^n$ for some integer $n > 0$. I expect, but do not demonstrate, that the results presented here also hold for more general manifolds.

7.1 Mechanical representation of the automaton environment

In the general case, the environment portion of a hybrid automaton may represent many different properties of the system to be modeled. Some dimensions of the environment space may represent the placement of one or more bodies; other dimensions may represent such bodies' velocity, acceleration, or further derivatives of motion. Still other dimensions may represent non-positional features of interest, such as temperature or the strength of a magnetic field on a body. If the details of the original system to be modeled are known, it may be possible to reconstruct a mechanical universe including these additional mechanical features. In such cases, it might also be possible to decompose the automaton's environment state, recognizing one dimension of the environment state as a body's position, another as a separate body's velocity, and so on.

None of this information is evident, however, given only the original automaton tuple. The environment state itself does not distinguish dimensions of position from those of velocity, acceleration, or temperature. By that same token, it does not inherently indicate any decomposition into separate bodies. Lacking such information, it is difficult to see how one might reliably reproduce the number of bodies or particular properties originally modeled. Since clarifying information cannot be assumed, I will provide a mechanical construction for the physical environment which does not depend on any knowledge other than the tuple itself. My discussion of the mechanical system for the remainder of the chapter will assume the construction I provide, but at the end of this section I will briefly discuss how these results might be adjusted for alternate constructions based on greater initial information.

7.1.1 Physical time and universe

Time, at least, seems straightforward to translate from an automaton to a mechanical system, as it serves a similar role in both systems; in each case, the rate at which the system's other properties change is defined relative to time. Thus, in my construction, physical time is identical to automaton time,

$$\Gamma_{\mathrm{p}} \stackrel{\mathrm{def}}{=} \Gamma_A. \tag{7.3}$$

By Assumption 1, then, physical time is a left-closed interval of $\mathbb{R}^{\geq 0}$ with first instant 0. Hybrid time may or may not be identical to Γ_A; as normal for a hybrid mechanical system, however, there must be some onto order-preserving mapping from hybrid time to physical time. Since automaton and physical time are identical, in what follows, I speak of a particular automaton instant t_A as an instant in either automaton or physical time.

Construction of the physical universe is slightly more complex. As argued above, it is not possible in general to determine from an automaton what bodies were present in the original system of interest. Thus, the simplest approach would seem to be to declare that the physical universal body consists of a single body point; the physical universe would then consist only of the physical universal and null bodies,

$$\Omega_{\mathrm{p}} = \{\top_{\mathrm{p}}, \bot_{\mathrm{p}}\}. \tag{7.4}$$

Changes in the automaton's environment state then correspond to motions of the physical universal body, and it remains only to define physical forces that would induce appropriate motions of this body.

While such an approach is appealingly simple, this last requirement displays its fundamental limitation. Recall from Axiom 4.33 that, for any induced motion, the acceleration of a body is proportionate to the force of its exterior on it. In a hybrid system where the controller and physical sub-universes do not exert forces on each other, this means that a physical body's acceleration is proportionate to the force of its physical exterior on it. In the construction above, however, the only physical bodies are $\top_{\mathrm{p}}$ and $\bot_{\mathrm{p}}$, the physical universal and null bodies. Since force systems are null passive, there can be no nonzero resultant forces on $\top_{\mathrm{p}}$, and so its acceleration must likewise be zero. In other words, such a system can model only a trivial and unnatural class of automata for which the environment state changes at a constant rate. To allow for more complex motions, the physical universe must contain at least two non-null bodies.

I thus provide the following construction for the physical universe. I define the physical universal body $\top_{\mathrm{p}}$ to be the join of a pair of separate bodies G and N. That is,

$$\begin{aligned} G \sqcap N &= \bot_{\mathrm{p}} \\ G \sqcup N &= \top_{\mathrm{p}}. \end{aligned} \tag{7.5}$$

The body G, which I call the *physical correspondent body*, consists of only a single body point. As will be discussed further below, I make no assumptions regarding the construction or internal structure of N, the *physical exterior of* G. Thus, at a minimum, the physical universe consists of the bodies

$$\Omega_{\mathrm{p}} = \{\top_{\mathrm{p}}, G, N, \bot_{\mathrm{p}}\}. \tag{7.6}$$

Any other bodies in Ω_{p} must be sub-bodies of N or the joins of such sub-bodies with G. As I do not treat the structure of N, however, I rely on the above definition; for all purposes in this paper, it suffices to treat N as though it were a point body.

Conceptually, N serves only as an external physical body which can exert forces on G so as to produce any necessary motion. Such constructions are not uncommon in mechanics, with some body or bodies of interest identified as a *great system* for which forces and other interactions are described in detail, while any remaining bodies are left abstracted. Here, G serves as the great system for the physical universe, and so I omit any discussion of construction, motion, masses, and so on for N.

7.1.2 Physical space, placement, and mass

I now turn to the construction of physical space. Again, rather than attempt to determine which components of the automaton's environment state are positional, I define physical space to be identical to the environment state space, and so

$$\mathcal{S}_{\mathrm{p}} \stackrel{\mathrm{def}}{=} X. \tag{7.7}$$

Effectively, then, since I cannot distinguish positional dimensions from non-positional dimensions, I simply treat all dimensions of X as positional. As noted above, I restrict attention to automata such that $X = \mathbb{R}^n$ for some $n > 0$, and so $\mathcal{S}_{\mathrm{p}} = \mathbb{R}^n$.

Since G consists of a single body point, the placement space of G in $\mathcal{S}_{\mathrm{p}}$ is

$$\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) = \mathbb{R}^n \tag{7.8}$$

Treating N as a single point, one likewise obtains for the physical universal body

$$\begin{aligned} \mathcal{C}(\top_{\mathrm{p}}, \mathcal{S}_{\mathrm{p}}) &= (\mathbb{R}^n)^{|\top_{\mathrm{p}}|} \\ &\cong (\mathbb{R}^n)^2 \\ &\cong \mathbb{R}^{2n}. \end{aligned} \tag{7.9}$$

Note that both physical space $\mathcal{S}_{\mathrm{p}}$ and the placement space of G are isomorphic to $\mathbb{R}^n$ (trivially, in the case of $\mathcal{S}_{\mathrm{p}}$). Thus, these spaces are isomorphic, and so

$$\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) \cong \mathcal{S}_{\mathrm{p}}. \tag{7.10}$$

Then by (7.7), the placement space for G is isomorphic to the automaton environment space X. Again, I identify the two spaces via the natural isomorphism, and so

$$\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) \cong X. \tag{7.11}$$

Given this isomorphism, I commonly write X as a shorthand for $\mathcal{C}(G, \mathcal{S}_{\mathrm{p}})$.

The placement of G thus provides a natural correspondent to automaton environment states, with changes in the automaton environment state corresponding to motions of G in $\mathcal{S}_{\mathrm{p}}$. Thus, I will refer to a single value $\mathbf{x}$ variously as an environment state of the automaton, as a point in physical space, and as a placement of the universal physical body. Intuitively, given an execution α of A, there should be a motion of G in $\mathcal{S}_{\mathrm{p}}$ such that at every automaton instant, the environment state of A is isomorphic (under the natural isomorphism) to G's placement in $\mathcal{S}_{\mathrm{p}}$ in the corresponding physical instant.

It remains only to define physical mass, which I take to be the non-negative reals,

$$\mathcal{M}_{\mathrm{p}} \stackrel{\mathrm{def}}{=} \mathbb{R}^{\geq 0}. \tag{7.12}$$

I refer to physical masses via a subscripted "p," as in 1_{p} or 0_{p}, to distinguish them from non-physical masses. In particular, I assign unit mass to the physical correspondent body,

$$m_{\mathrm{p}}(G) \stackrel{\mathrm{def}}{=} 1_{\mathrm{p}}. \tag{7.13}$$

The physical momenta and force values are determined by the definition of $\mathcal{S}$ and $\mathcal{M}$, and so

$$\mathcal{V}_{\mathrm{p}} = X = \mathbb{R}^n. \tag{7.14}$$

7.1.3 Alternative constructions

As noted above, when additional information about the original hybrid automaton is available, the above construction may not be ideal. In general, any construction should agree with the above definition of physical time, but it may disagree as to the construction of physical space and the physical universal body. Very little may be said definitively about the requirements for such constructions; one cannot, for instance, require that the placement space of any body be identical to the environment state of the automaton, as some components of the environment state may be represented as velocities, temperatures, or other properties that are not directly dimensions of position. Ultimately, I require only that there exists some bijection between automaton environment states and some set of mechanical properties for some physical body; the physical forces I describe in later sections should be replaced by whatever forces or other interactions will ensure that, at every instant, the environment state corresponds to the mechanical configuration of this body.

7.2 Mechanical representation of the automaton controller

I now begin construction of the controller universe, Ω_{c}. Throughout this section, I distinguish the *automaton controller* from the *mechanical controller* to be constructed, referring to simple "controllers" only where context makes the distinction unambiguous. Similarly, I speak of "automaton controller states" and "mechanical controller states" wherever the meaning of the word "state" might be ambiguous.

7.2.1 Mechanical representation of controller states

Recall from Section 6.3 that, given a hybrid mechanical system, a mechanical controller state is a subset of $\mathcal{X}(\top, \mathcal{S})$, the hybrid configuration space for $\top$ in $\mathcal{S}$. Given a mechanical controller C, the state set $\Psi(C)$ is a partition of $\mathfrak{X}(\top, \mathbb{S})$. Further, given a state set for all controllers in a set U of mechanical controllers, the mechanical controller states of all joins of bodies in U are simply the non-empty intersections of the mechanical controller states of the component bodies, and the state of such a super-controller state-projects onto the mechanical controller states intersected to produce it.

The construction in Section 6.3 is independent of any particular choice of $\top$ and $\mathbb{S}$. In the following sections, I will provide several possible constructions of $\top$ and $\mathbb{S}$, with the definition of $\mathfrak{X}(\top, \mathbb{S})$ following from each such construction.

As noted above, $\top_{\mathrm{p}}$ consists of two separate bodies: a physical correspondent body G whose positions I will describe and a physical exterior of G whose properties I will ignore. It will therefore be impossible to precisely define $\mathfrak{X}(\top, \mathbb{S})$, since such a definition must include the placement space of N. Instead, I will more generally define $\mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathbb{S})$, where

$$\mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathbb{S}) = \mathfrak{X}(G, \mathbb{S}_{\mathrm{p}}) \times \mathfrak{X}(\top_{\mathrm{c}}, \mathbb{S}_{\mathrm{c}}). \tag{7.15}$$

Given a particular construction of N, the full configuration space $\mathfrak{X}(\top, \mathbb{S})$ could trivially be reconstructed from such a definition; in particular,

$$\mathfrak{X}(\top, \mathbb{S}) = \mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathbb{S}) \times \mathfrak{X}(N, \mathbb{S}_{\mathrm{p}}). \tag{7.16}$$

Thus, despite only providing a definition for $\mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathbb{S})$, I can still speak of mechanical controller states as subsets of $\mathfrak{X}(\top, \mathbb{S})$, with the understanding that N plays no role in differentiating mechanical controller states.

7.2.2 Plan of the explanation

A number of models for a mechanical controller suggest themselves; unfortunately, several of the most seemingly-straightforward approaches fail to emulate even some simple hybrid automata. To clarify this situation, I present several simple approaches, each preceded by the question, "Is such-and-such an approach sufficient?" I then examine the limitations of that approach and, where applicable, define a limited class of hybrid automata for which the approach suffices. Finally, I present a solution which is sufficient for automata of the sort discussed to this point.

In each of these unsuccessful approaches, I assume that mechanical controller time is identical to physical time, and likewise that hybrid time is identical to physical time; the mapping from hybrid time to each factor time is then simply the identity. While different approaches may require different constructions of the universal controller body $\top_{\mathrm{c}}$, I assume throughout that, given whatever construction applies to a particular approach, the hybrid universal body is the join of the universal physical body and the universal controller body, that is,

$$\top = \top_{\mathrm{p}} \sqcup \top_{\mathrm{c}}. \tag{7.17}$$

Likewise, for whatever controller space $\mathcal{S}_\mathrm{c}$ is constructed in an attempt, hybrid space is the product of physical space and controller space, that is,

$$\mathcal{S} = \mathcal{S}_\mathrm{p} \times \mathcal{S}_\mathrm{c}. \tag{7.18}$$

More specifically, in each of the non-final approaches that follows, I will maintain the constructions of $\top_\mathrm{p}$ and $\mathcal{S}_\mathrm{p}$ described above, and so

$$\mathcal{C}(G, \mathcal{S}_\mathrm{p}) = \mathbb{R}^n. \tag{7.19}$$

Each such example will assume a single controller with a controller space $\mathcal{S}_\mathrm{c} = \mathbb{Z}_1$, and so

$$\mathcal{C}(\top_\mathrm{c}, \mathcal{S}_\mathrm{c}) = \mathbb{Z}_1 \tag{7.20}$$

and

$$\mathcal{S} = \mathbb{R}^n \times \mathbb{Z}_1. \tag{7.21}$$

The placement space of $G \sqcup \top_\mathrm{c}$ is the placement space of G times that of $\top_\mathrm{c}$; thus,

$$\mathcal{C}(G \sqcup \top_\mathrm{c}, \mathcal{S}) = \mathbb{R}^n \times \mathbb{Z}_1. \tag{7.22}$$

The configuration space $\mathfrak{X}(G \sqcup \top_\mathrm{c}, \mathcal{S})$ is then the product of this space with itself,

$$\mathfrak{X}(G \sqcup \top_\mathrm{c}, \mathcal{S}) = (\mathbb{R}^n \times \mathbb{Z}_1) \times (\mathbb{R}^n \times \mathbb{Z}_1). \tag{7.23}$$

In my final, successful construction, the above assumptions do not apply, and I provide instead a full description of all factor and hybrid components.

7.2.3 Is a simple atomic controller over system states sufficient?

Perhaps the simplest model for a mechanical equivalent to the automaton is to define a single atomic mechanical controller $\top_\mathrm{c}$ consisting of a single body point, and to define the states of that controller in terms of the motion of the physical correspondent body G. Since motion of the controller plays no part in representing its state in this construction, the definition of controller space is largely arbitrary; for simplicity, it can be taken to be a single point, $\mathbb{Z}_1$.

Mechanical controller states of such a mechanical controller seem naturally to correspond to automaton controller states of the automaton controller; thus, for each automaton controller state $v \in V$, I will define a mechanical controller state $\psi_v \in \Psi(\top_\mathrm{c})$. Intuitively, each such state ψ_v is the set of all points in $\mathfrak{X}(\top, \mathcal{S})$ such that v and the physical placement of G define an automaton system state.

Formally, for any $v \in V$,

$$\psi_v = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, \big(v, w_{\mathrm{p}}(G)\big) \in Q\right\}. \tag{7.24}$$

The states of $\top_{\mathrm{c}}$ are then the set of all such states, that is,

$$\Psi(\top_{\mathrm{c}}) = \{\psi_v \,|\, v \in V\}. \tag{7.25}$$

While appealingly simple, the above description does not in general suffice to describe a hybrid automaton. One of the foundational assumptions of mechanical controllers is that, in a particular instant, a given mechanical controller occupies exactly one mechanical controller state. If mechanical controller states are a partition of $\mathfrak{X}(\top, \mathcal{S})$ based solely on the placement of G, then any particular placement of G can be in at most one mechanical controller state. Equivalently, for any environment state $\mathbf{x}$ in X, there exists at most one controller state v in V such that $(v, \mathbf{x})$ is in Q.

Unfortunately, this condition fails to hold for any automaton containing at least two automaton controller states. Recall that the set of system states Q is defined to be the product of the automaton controller and environment state spaces, that is, that $Q = V \times X$. It follows that for any distinct controller states $v, v' \in V$ and environment state $\mathbf{x} \in X$, both $(v, \mathbf{x})$ and $(v', \mathbf{x})$ are in Q. Therefore, for any point $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that $w_{\mathrm{p}}(G) = \mathbf{x}$, it must be the case that $(w, \dot{w})$ is in both ψ_v and $\psi_{v'}$. Since mechanical controller states are equivalence classes, this implies that $\psi_v = \psi_{v'}$, which is inconsistent with distinct mechanical controller states being separate equivalence classes.

7.2.4 Is a simple atomic controller over reachable states sufficient?

A slight modification to the first model would be to define mechanical controller states based on a partition only of reachable system states, that is, of system states that can be reached by some execution of the automaton. Again, let the universal mechanical controller be an atomic body $\top_{\mathrm{c}}$ consisting of a single body point with placements in $\mathbb{Z}_1$. For each automaton controller state $v \in V$, let ψ_v be the mechanical controller state such that

$$\psi_v = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, (v, w_{\mathrm{p}}(G)) \in Q_{reach}\right\}. \tag{7.26}$$

and

$$\Psi(\top_{\mathrm{c}}) = \{\psi_v \,|\, v \in V\}. \tag{7.27}$$

This construction requires only a weaker version of the earlier assumption: for any configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ and environment state $\mathbf{x} \in X$ such that $w_{\mathrm{p}}(G) = \mathbf{x}$, there exists at most one controller state $v \in V$ such that $(v, \mathbf{x})$ is in Q_{reach}. I refer to automata for which this condition holds as *environment state partitionable* automata. Again, however, there is no guarantee in general that automata

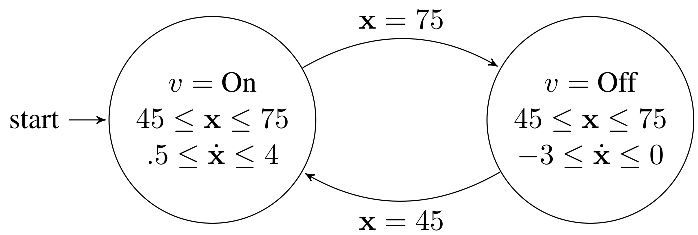


Figure 7.1: The illustration for the hybrid automaton model of a mechanical elevator, reproduced. Note that the system will pass through state $(On, 50)$ while the platform is rising and, once switched off, will pass through $(\mathit{Off}, 50)$ on the way back down. Thus, both $(On, 50)$ and $(\mathit{Off}, 50)$ are in Q_{reach}, and so it is impossible to partition Q_{reach} based on controller states.

will satisfy this requirement.

As a simple example, recall again the mechanical elevator. In the (reachable) middle altitude, the automaton controller could be either in state Off, with the platform gradually dropping, or in On, with the platform slowly rising. So, for instance, both $(\mathit{Off}, 50)$ and $(On, 50)$ are in Q_{reach}, and so this automaton is not environment state partitionable. Figure 7.1 illustrates this flaw.

7.2.5 Is a simple atomic controller over trajectories sufficient?

One might propose to eliminate the conflict displayed above by shifting focus to automaton trajectories, rather than simply states. Notably, by Assumption 3, every trajectory has exactly one automaton controller state, and by Assumption 2, all trajectories of A must be differentiable. Thus, one might define a particular mechanical controller state in terms of the traces and trace derivatives of all trajectories with the corresponding automaton controller state. Formally, again letting $\top_{\mathrm{c}}$ be an atomic body consisting of a single body point with placements in $\mathbb{Z}_1$, let each mechanical controller state ψ_v be the equivalence class so that

$$\psi_v = \left\{ (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \middle| \exists \tau \in \mathcal{T} \left(\exists t \in \Gamma_A \left[\left(w_{\mathrm{p}}(G) = \Pi_X(\tau(t)) \right) \wedge \left(\dot{w}_{\mathrm{p}}(G) = \Pi_X(\dot{\tau}(t)) \right) \wedge \left(v = \Pi_V(\tau(t)) \right) \right] \right) \right\} \tag{7.28}$$

In other words, ψ_v is the set of all configurations of $\top$ such that, for some trajectory τ with a constant controller state of v, there is a point in the trace of τ where the value of the trace is w and the derivative of the trace is $\dot{w}$.

Unfortunately, this approach reaches the same problem as those above; a single tangent to a trace, and so a single point in $\mathfrak{X}(\top, \mathcal{S})$, might be common to two or more trajectories with different automaton controller states. Thus, the assumption of separate equivalence classes for distinct states is again

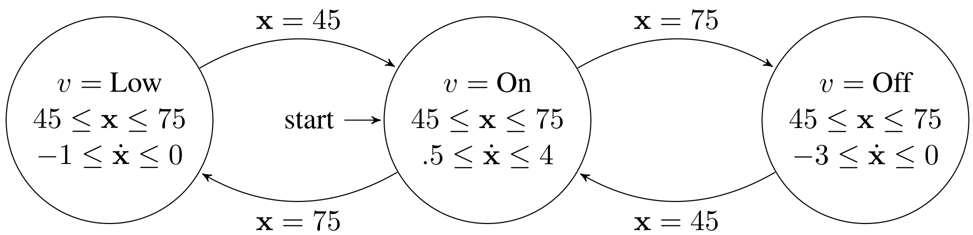


Figure 7.2: A modified elevator, adding a new state, *Low*. On reaching $\mathbf{x} = 75$, the automaton can transition from On to either *Off* or *Low*; in either case, it returns to On at $\mathbf{x} = 45$. The state *Low* functions identically to *Off* except that the range of possible values for $\dot{\mathbf{x}}$ is $[-1, 0]$ instead of $[-3, 0]$. The controller states of this automaton cannot be uniquely associated with tangents to the traces of trajectories with that controller state.

violated.

As an example, consider the hybrid automaton in Figure 7.2. Here, the elevator is extended to have three settings: On, *Off*, and a new state *Low*. The behavior of the first two states is the same as in previous elevators; this time, however, on reaching a high altitude the system can move from On to either of *Off* and *Low*. In *Low*, the elevator exerts just enough force to slow the descent. Thus, the platform drops more gradually than it may in *Off*; notably, however, the range of values for $\dot{\mathbf{x}}$ in both *Off* and *Low* includes $[-1, 0]$. Therefore, a point at which $\mathbf{x} = 50$ and $\dot{\mathbf{x}} = .5$ is reachable when the controller state is either of *Off* or *Low*. Thus, the traces of trajectories cannot in general be uniquely associated with controller states. I refer to the category of hybrid automata that do admit such a partition as *trajectory trace partitionable*.

7.2.6 Is a refined atomic controller sufficient?

The recurring issue in all approaches to this point has been that a single configuration in $\mathfrak{X}(\top, \mathcal{S})$ might be associated with two or more automaton controller states, which is inconsistent with an isomorphism between automaton controller states and mechanical controller states. This suggests a different direction for refinement of the atomic controller. Under this revision, instead of identifying mechanical controller states with automaton controller states, one identifies mechanical controller states with sets of automaton controller states. If a particular point and derivative in X (and so, a particular point in $\mathfrak{X}(\top, \mathcal{S})$) could lie along trajectories with different automaton controller states, then one simply defines a single mechanical controller state consisting of all such points in $\mathfrak{X}(\top, \mathcal{S})$ for that set of automaton controller states.

More formally, let the *reachability image* $L : V \to 2^X$ be the mapping from each automaton controller state $v \in V$ to the set of all environment states $\mathbf{x} \in X$ for which $(v, \mathbf{x})$ is a reachable state of the automaton. That is,

$$L(v) \stackrel{\text{def}}{=} \{\mathbf{x} \,|\, (v, \mathbf{x}) \in Q_{reach}\}\,. \tag{7.29}$$

Likewise, let the *reachability preimage* $L^{-1} : X \to 2^V$ be the mapping from each environment state $\mathbf{x}$ to the set of all automaton controller states v for which $(v, \mathbf{x})$ is a reachable state of the automaton. That is,

$$L^{-1}(\mathbf{x}) \stackrel{\text{def}}{=} \{v \,|\, (v, \mathbf{x}) \in Q_{reach}\} \,. \tag{7.30}$$

I can then define an equivalence relation $\equiv_V$ over X so that two environment states $\mathbf{x}, \mathbf{x}'$ are equivalent with respect to $\equiv_V$ if and only if they have the same reachability preimage. That is,

$$\mathbf{x} \equiv_V \mathbf{x}' \;\text{ iff }\; L^{-1}(\mathbf{x}) = L^{-1}(\mathbf{x}'). \tag{7.31}$$

The equivalence relation thus imposes a partition $X/\equiv_V$ in which all elements of an equivalence class $[\mathbf{x}]$ have the same reachability preimage as $\mathbf{x}$. In other words, every element of $X/\equiv_V$ can be reached while in any of the same set of automaton controller states. Each such partition element is then a state in the hybrid mechanical system, so that, given a point body controller $\top_\mathrm{c}$ with placement space $\mathbb{Z}_1$,

$$\Psi(\top_\mathrm{c}) = \left\{\psi_{[\mathbf{x}]} \,\middle|\, [\mathbf{x}] \in X/\equiv_V\right\} \tag{7.32}$$

where, for each $\psi_{[\mathbf{x}]}$ in $\Psi(\top_\mathrm{c})$,

$$\psi_{[\mathbf{x}]} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, w_\mathrm{p}(G) \equiv_V \mathbf{x}\right\}. \tag{7.33}$$

Thus, in the example of the two-state elevator from Section 3.4, the resultant mechanical controller would have two states. That is, $\Psi(\top_\mathrm{c}) = \{\psi_{[\![50]\!]}, \psi_{[\![0]\!]}\}$, where

$$\psi_{[\![50]\!]} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, 45 \leq w_\mathrm{p}(G) \leq 75\right\} \tag{7.34}$$

and

$$\psi_{[\![0]\!]} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, \big(w_\mathrm{p}(G) < 45\big) \vee \big(w_\mathrm{p}(G) > 75\big)\right\}. \tag{7.35}$$

The meaning of these states may be somewhat clearer if they are relabeled according to the reachability preimage of their members. Under this labeling, I have $\psi_{\{On, Off\}}$, $\psi_\emptyset$, where

$$\psi_{\{On, Off\}} = \psi_{[\![50]\!]} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, 45 \leq w_\mathrm{p}(G) \leq 75\right\} \tag{7.36}$$

and

$$\psi_\emptyset = \psi_{[\![0]\!]} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, \big(w_\mathrm{p}(G) < 45\big) \vee \big(w_\mathrm{p}(G) > 75\big)\right\}. \tag{7.37}$$

Unfortunately, this approach again falls short. To illustrate the problem, I provide an example automaton for which it is insufficient.

$$
\begin{aligned}
V &= \{v_1, v_2, v_3, v_4, v_5, v_6\} \\
X &= \mathbb{R} \\
Q_0 &= \{(v_1, 1), (v_2, -1)\} \\
D &= \{(v_1, 2) \rightarrowtail (v_3, 10), \\
&\quad (v_2, -2) \rightarrowtail (v_4, 10), \\
&\quad (v_3, 11) \rightarrowtail (v_5, 100), \\
&\quad (v_4, 11) \rightarrowtail (v_6, -100)\} \\
\mathcal{T} &= \{\tau \mid \forall t \in dom(\tau)[\lim_{t' \to t} \tau(t') = \tau(t)] \\
&\quad \wedge \Big(\big[(\Pi_V(\tau(t)) = v_1) \wedge (1 \le \Pi_X(\tau(t)) \le 2) \wedge (\Pi_X(\dot{\tau}(t)) = 1)\big] \\
&\quad \vee \big[(\Pi_V(\tau(t)) = v_2) \wedge (-2 \le \Pi_X(\tau(t)) \le -1) \wedge (\Pi_X(\dot{\tau}(t)) = -1)\big] \\
&\quad \vee \big[(\Pi_V(\tau(t)) = v_3) \wedge (10 \le \Pi_X(\tau(t)) \le 11) \wedge (\Pi_X(\dot{\tau}(t)) = 1)\big] \\
&\quad \vee \big[(\Pi_V(\tau(t)) = v_4) \wedge (10 \le \Pi_X(\tau(t)) \le 11) \wedge (\Pi_X(\dot{\tau}(t)) = 1)\big] \\
&\quad \vee \big[(\Pi_V(\tau(t)) = v_5) \wedge (100 \le \Pi_X(\tau(t))) \wedge (\Pi_X(\dot{\tau}(t)) = 1)\big] \\
&\quad \vee \big[(\Pi_V(\tau(t)) = v_6) \wedge (\Pi_X(\tau(t)) \le -100) \wedge (\Pi_X(\dot{\tau}(t)) = -1)\big] \Big) \} \\
\Gamma &= \mathbb{R}^{\ge 0}.
\end{aligned} \tag{7.38}
$$

Figure 7.3 illustrates this automaton. To summarize: the hybrid automaton begins in either state $(v_1, 1)$ or $(v_2, -1)$. In the former case, its environment state's value increases over time to $(v_1, 2)$, at which point the automaton transitions to $(v_3, 10)$. The environment state then continues increasing to $(v_3, 11)$, at which point the automaton transitions to $(v_5, 100)$, and the environment state continues increasing. In the latter case, on the other hand, the environment state's value drops from -1 to -2, at which point the automaton transitions to $(v_4, 10)$. The environment state then continues increasing to $(v_4, 11)$, at which point the automaton transitions to $(v_6, -100)$, and the environment state continues decreasing. Note that under no circumstance does the automaton begin in v_1 and end in v_6 or begin in v_2 and end in v_5.

Under the given transformation to a hybrid mechanical system, six mechanical controller states are generated. I list them here, writing $\{i\}$ for the equivalence class containing v_i:

$$
\begin{aligned}
\psi_{\{1\}} &= \big\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\big|\, w_{\mathrm{p}}(G) \in [1, 2]\big\} \\
\psi_{\{2\}} &= \big\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\big|\, w_{\mathrm{p}}(G) \in [-2, -1]\big\} \\
\psi_{\{3,4\}} &= \big\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\big|\, w_{\mathrm{p}}(G) \in [10, 11]\big\}
\end{aligned} \tag{7.39}
$$

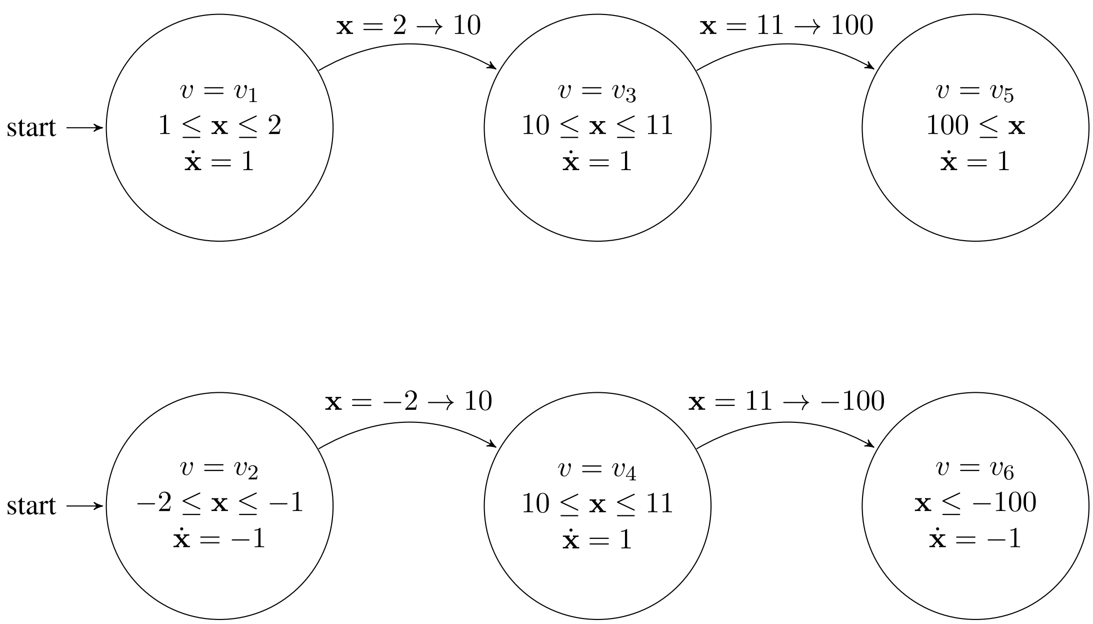


Figure 7.3: A six-state automaton. Note that there is no possible path leading from v_1 to v_6 (and likewise, no path from v_2 to v_5). The trajectories of v_3 and v_4 are identical, however, and so an approach that divides mechanical controller states based on the placement and velocity of points in $\top_\mathrm{p}$ does not preserve this requirement.

$$
\begin{aligned}
\psi_{\{5\}} &= \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, w_\mathrm{p}(G) \in [100, \infty)\right\} \\
\psi_{\{6\}} &= \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, w_\mathrm{p}(G) \in (-\infty, -100]\right\} \\
\psi_{\{\}} &= \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, w_\mathrm{p}(G) \in (-100, -2) \cup (-1, 1) \cup (2, 10) \cup (11, 100)\right\}.
\end{aligned}
$$

Note that $\psi_{\{\}}$ contains all physical placements of G corresponding to unreachable environment states. Figure 7.4 illustrates this division of states.

Most crucially, note that automaton controller states v_3 and v_4 are combined into a single mechanical controller state $\psi_{\{3,4\}}$, since the set of traces for all trajectories of the automaton while in either of those

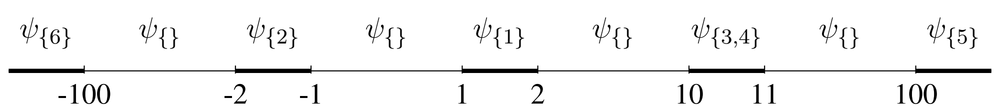


Figure 7.4: Mechanical controller states for the automaton in Figure 7.3. The number line is the placement space for G. States of the mechanical controller are regions in this placement space, with each such region labeled according to the automaton controller states in the reachability preimage of its members. Most crucially, note that automaton controller states v_3 and v_4 are combined into a single state, $\psi_{\{3,4\}}$, and so the resultant mechanical system cannot distinguish between the two.

states is identical to that in the other state. Once the hybrid mechanical system enters $\psi_{\{3,4\}}$, it thus becomes impossible to determine whether it was previously in $\psi_{\{1\}}$ or $\psi_{\{2\}}$, and, likewise, it becomes impossible to distinguish whether to next enter $\psi_{\{5\}}$ or $\psi_{\{6\}}$. Thus, there would seem to be no way to enforce the original automaton's requirement that the system never begin in v_1 and later enter v_6 (equivalently, begin in $\psi_{\{1\}}$ and later enter $\psi_{\{6\}}$). Consideration of trace derivatives (or, equivalently, of G's velocity) does not resolve this issue, as the traces for v_3 and v_4 are identical in both position and velocity.

In other words, the automaton may have two or more states whose trajectory traces are identical, even though the execution fragment traces involving the two states may look markedly different. (Recall that, under Assumption 3, a trajectory has only a single automaton controller state, while an execution fragment may involve several such trajectories linked by discrete transitions.) Similar constructions exist even under the more restrictive assumptions that will be adopted in the remainder of this paper, though such constructions are more difficult to intuitively follow.

In practical terms, a hybrid automaton lacks any way to "remember" the states it occupied previously. Thus, any information not provided in the automaton's environment state cannot be preserved across transitions, and the automaton's environment state does not, in the general case, provide sufficient information to resolve these ambiguities.

7.2.7 Sufficient approach: sensor/effector decomposition of controllers

Fundamentally, the underlying issue in all these inadequate "solutions" is that they attempt to reduce the states of the automaton controller to nothing more than regions of the configuration space of $\top$. Such an approach does not capture any properties of an automaton controller state that cannot be reduced to motion in the environment state space - such as the transitions into and out of that automaton controller state! - and so it is insufficient.

To accurately describe the transitions of an arbitrary hybrid automaton, then, requires three effectively independent pieces of information: the environment state, the controller state, and the set of acceptable transitions associated with each combined system state. Of these, the environment state is represented already in the physical portion of the hybrid mechanical system. I represent the remaining pieces using two kinds of mechanical controller: *sensor controllers*, or simply *sensors* where unambiguous, and *effector controllers*, or *effectors* where unambiguous. Broadly, I will define a set of conditional forces so that the state of the effectors determines forces on on the physical universe, based on the motion of G, while the sensors determine forces on the rest of the controller system, based on the current placements of both the effectors and the physical universe.

7.2.7.1 Controller space

In what follows, no controller body point will require more than two possible positions, and so I define the controller space to be a pair of isolated points with the discrete topology, both homeomorphic to $\mathbb{R}^0$. In particular, then,

$$\mathcal{S}_{\mathrm{c}} \stackrel{\mathrm{def}}{=} \mathbb{Z}_2. \tag{7.40}$$

The potential velocities, accelerations, and other derivatives of motion for this space likewise have values in $\{0, 1\}$. A body point with a velocity of 1 will alternate between positions 0 and 1 on sequential discrete instants, and similarly, a body point with an acceleration of 1 will alternate between velocities 0 and 1 on sequential discrete instants. A more detailed description of discrete motion over time may be found in Section 4.1.4.

7.2.7.2 Controller time

Controller time presents something of a riddle; before presenting a formal definition, I here provide some description of the intuitions driving that definition.

From the perspective of a hybrid automaton, a change in automaton controller state is instantaneous, as opposed to the gradual change in the environment state. From the perspective of a mechanical controller, however, forces must act on a controller at rest to set it into motion, it must be allowed time to move, and then forces must act again to bring it to rest. To allow this entire process to occur "instantly," multiple discrete steps must be possible within a single instant of continuous time.

Fortunately, the number of steps required for a change in the position of a body in $\mathcal{S}_{\mathrm{c}}$ is finite. Informally, it suffices to have an initial instant of stillness, one instant in which the mechanical controller bodies are set in motion, one in which they complete their motion, and one in which they return to stillness. Thus, it is sufficient to have four mechanical controller instants for every physical instant; from the perspective of the hybrid mechanical system, the controller undergoes a sequence of discrete changes, while from the perspective of the physical mechanical system, the change in state occurs in a single instant. In (7.3), I defined physical time Γ_{p} to be a left-closed interval in $\mathbb{R}^{\geq 0}$ beginning at 0. I therefore define mechanical controller time Γ_{c} to be a left-closed interval of $\mathbb{R}^{\geq 0} \times \mathbb{Z}_4$, with first instant $(0, 0)$. In other words, every controller instant that is a pair (r, z), where r is a nonnegative real number and z is in $\{0, 1, 2, 3\}$. I define an ordering over the controller instants so that for any choice of $r, r' \in \mathbb{R}^{\geq 0}$ and $z, z' \in \mathbb{Z}_4$,

$$(r, z) < (r', z') \text{ iff either } r < r', \text{ or } r = r' \text{ and } z < z'. \tag{7.41}$$

In other words, the z-portion of an instant is only relevant in determining its ordering when comparing it to another instant with the same real-valued portion. In such cases, the lower z-value is ordered first; otherwise, the numbers are simply ordered by their r values as normal.

7.2.7.3 Effector controller

I assume that $\top_c$ contains a set of separate body points isomorphic to V. I refer to each such body point as an *effector body point*. Let $e : V \to \top_c$ be the function mapping each automaton controller state to its corresponding effector body point; in what follows, I commonly write $e(v)$ using the shorthand notation e_v, that is,

$$e_v \stackrel{\text{def}}{=} e(v). \tag{7.42}$$

For each effector body point, there is a point body in Ω_c consisting only of that body point; by some abuse of notation, I use e_v interchangeably for both the body point and the *effector point body* consisting only of that body point.

The set of all effector body points is the *effector controller*, E. That is,

$$E \stackrel{\text{def}}{=} \{e_v \,|\, v \in V\}\,. \tag{7.43}$$

It follows that

$$|E| = |V|. \tag{7.44}$$

7.2.7.4 Sensor controller

The four insufficient approaches discussed above all fail to consistently answer a single question: Given the controller and environment states of the original automaton, what transitions are possible? The necessary task, then, is to directly encode this information into the mechanical system. A straightforward way of doing so would seem to be to construct *sensor* controllers whose states indicate which transitions are possible; that is, a sensor should be in one state when one set of transitions are possible, a different state for another set, and so on, with a different force system conditional on each state. Since each state of the automaton controller can define a different range of possible transitions, a reasonable construction is to provide one sensor for each automaton controller state.

Thus, I assume that $\top_c$ contains a set of separate body points isomorphic to V and separate from the effector. I refer to such body points as *sensor body points*. Let $s : V \to \top_c$ be the function mapping each automaton controller state to its corresponding sensor body point; again, I commonly write $s(v)$ as s_v, and so

$$s_v \stackrel{\text{def}}{=} s(v). \tag{7.45}$$

For each sensor body point, there is a point body in Ω_c consisting only of that body point; I call any such body a *sensor point body*. By some abuse of notation, I use s_v interchangeably for both the body point and the point body consisting only of that body point. As the sensor body points are separate from

$$
\begin{tikzcd}
& \top_{\mathrm{c}} & \\
S \arrow[ur, "\sqsubset"] & & E \arrow[ul, "\sqsubset"'] \\
& \bot_{\mathrm{c}} \arrow[ul, "\sqsubset"] \arrow[ur, "\sqsubset"'] &
\end{tikzcd}
$$

Figure 7.5: Relationship between the sensor, effector, and the universal and null controller bodies. The sensor and effector are separate and together comprise the universal body.

the effector body points, for any $v, v' \in V$,

$$s_v \not\sqsubseteq E \tag{7.46}$$

The set of all sensor body points is the *sensor controller*, S. That is,

$$S \stackrel{\text{def}}{=} \{s_v \,|\, v \in V\}\,. \tag{7.47}$$

It follows that

$$|S| = |V|. \tag{7.48}$$

The universal controller body $\top_{\mathrm{c}}$ is the join of the preceding controller bodies,

$$\top_{\mathrm{c}} \stackrel{\text{def}}{=} S \sqcup E. \tag{7.49}$$

Since the sensor and effector are separate and each contain $|V|$ body points, it follows that

$$|\top_{\mathrm{c}}| = 2|V|. \tag{7.50}$$

Every set of body points in $\top_{\mathrm{c}}$ defines a separate controller body, and so the controller universe is

$$\Omega_{\mathrm{c}} \stackrel{\text{def}}{=} \{C_E \sqcap C_S \,|\, (C_E \subseteq E) \wedge (C_S \subseteq S)\}\,. \tag{7.51}$$

I illustrate the relationship between the various controller bodies in Figure 7.5.

7.2.7.5 Controller placement

Following the normal rules for factor bodies, controllers take positions in controller space; that is, the placement of a controller C is a continuous mapping $\chi_{\mathrm{c}} : C \to \mathcal{S}_{\mathrm{c}}$, and the set of all placements of C

is written $\mathcal{C}(C, \mathcal{S}_\mathrm{c})$. I do not require that separate controllers have separate placements, however, and so two controllers can occupy the same position in controller space. Indeed, since controller space contains only two positions, it may be unavoidable that some controllers colocate. The placement space for the universal controller body is then

$$\begin{aligned}\mathcal{C}(\top_\mathrm{c}, \mathcal{S}_\mathrm{c}) &= \mathbb{Z}_2^{S \cup E} \\ &= \mathbb{Z}_2^{2|V|}.\end{aligned} \tag{7.52}$$

Likewise,

$$\begin{aligned}\mathfrak{X}(\top_\mathrm{c}, \mathcal{S}_\mathrm{c}) &= \mathbb{Z}_2^{2|V|} \times \mathbb{Z}_2^{2|V|} \\ &= \mathbb{Z}_2^{4|V|}.\end{aligned}$$

7.2.7.6 Controller mass

As with controller space, I define the set of controller masses to be the set $\{0, 1\}$,

$$\mathcal{M}_\mathrm{c} \stackrel{\mathrm{def}}{=} \mathbb{Z}_2. \tag{7.53}$$

Following the pattern for physical masses, I attach a subscripted "c" to any controller masses. I assign a unit controller mass to each controller point body, and so for all $v \in V$,

$$m_\mathrm{c}(e_v) \stackrel{\mathrm{def}}{=} 1_\mathrm{c} \tag{7.54}$$

and

$$m_\mathrm{c}(s_v) \stackrel{\mathrm{def}}{=} 1_\mathrm{c}. \tag{7.55}$$

Then the space of controller momenta, and so of controller force values, is likewise the vector space $\mathbb{Z}_2$,

$$\mathcal{V}_\mathrm{c} = \mathbb{Z}_2. \tag{7.56}$$

7.3 The hybrid mechanical system

I can now define the hybrid mechanical system for the hybrid automaton. In doing so, I rely on the definitions of the physical and controller systems from the preceding section, defining the components of the hybrid system in terms of these factor systems. Toward that end, I briefly recap the definitions to this point.As defined so far, given a hybrid automaton $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$ such that

$$X = \mathbb{R}^n, \tag{7.57}$$

there exists a physical mechanical system such that

$$\begin{aligned}
\Omega_{\mathrm{p}} &= \{\top_{\mathrm{p}}, G, N, \bot_{\mathrm{p}}\} \\
\mathcal{S}_{\mathrm{p}} &= X = \mathbb{R}^n \\
\Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\
\mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\
m_{\mathrm{p}}(G) &= 1_{\mathrm{p}} \\
\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) &\cong X = \mathbb{R}^n \\
\mathfrak{X}(G, \mathcal{S}_{\mathrm{p}}) &\cong X \times X = \mathbb{R}^{2n} \\
\mathcal{V}_{\mathrm{p}} &= X = \mathbb{R}^n
\end{aligned} \tag{7.58}$$

and a controller mechanical system such that

$$\begin{aligned}
E &= \{e_v \mid v \in V\} && (7.59) \\
\forall v, v' \in V[(v \neq v') &\rightarrow (e_v \neq e_{v'})] \\
S &= \{s_v \mid v \in V\} && (7.60) \\
\forall v, v' \in V[(v \neq v') &\rightarrow (s_v \neq s_{v'})] \\
E \sqcap S &= \bot_{\mathrm{c}} \\
\top_{\mathrm{c}} &= E \sqcup S \\
\Omega_{\mathrm{c}} &= 2^{E \sqcup S} \\
\mathcal{S}_{\mathrm{c}} &= \mathbb{Z}_2 \\
\Gamma_{\mathrm{c}} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \\
\mathcal{M}_{\mathrm{c}} &= \mathbb{Z}_2 \\
\forall v \in V, m_{\mathrm{c}}(e_v) &= 1_{\mathrm{c}} \\
\forall v \in V, m_{\mathrm{c}}(s_v) &= 1_{\mathrm{c}} \\
\mathcal{C}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{2|V|} \\
\mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{4|V|} \\
\mathcal{V}_{\mathrm{c}} &= \mathbb{Z}_2.
\end{aligned}$$

7.3.1 Hybrid universal body and universe

The hybrid universe is the lattice sum of the physical and controller universes, that is,

$$\Omega = \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}}, \tag{7.61}$$

with the subbody relationships implied by that lattice sum. The universal body for the hybrid system is the join of its factor universal bodies, $\top = \top_{\mathrm{p}} \sqcup \top_{\mathrm{c}}$. In the discussion of the default construction of Ω_{p}, I allowed for the possibility of alternate constructions for the physical universe; as the definition of $\top$ is provided in terms of $\top_{\mathrm{p}}$, however, this definition should hold regardless of the chosen construction. Similar statements can be made regarding the construction of hybrid space, below.

7.3.2 Assumed separation of physical and controller universes

A reasonable question to this point might be, "Is there any overlap between the physical and controller universes - that is, does $\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} = \bot$?" At this point, I define the two universes to be separate, and so, indeed, $\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} = \bot$. I do so primarily for simplicity of notation, as this definition allows me to make statements such as "There are no nonzero forces on any physical body" without considering, for instance, whether the effector might also be a physical body in some way unrelated to its role as effector. Likewise, it simplifies discussion of self-forces to be assured that, say, a force of $\top_{\mathrm{p}}$ on $\top_{\mathrm{c}}$ is not a self-force.

Assumption 7. *The physical and controller universes are separate; that is,*

$$\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} = \bot. \tag{7.62}$$

With that said, there is no inherent reason that the two universes must be entirely separate, save for simplicity of notation. A reasonable extension of the work I present here would be to allow overlap between the two universes.

7.3.3 Hybrid space and time

Hybrid space $\mathbb{S}$ is the coproduct $\mathbb{S}_{\mathrm{p}} \oplus \mathbb{S}_{\mathrm{c}}$; thus,

$$\mathbb{S} = \mathbb{R}^n \oplus \mathbb{Z}_2. \tag{7.63}$$

As with controller time, I define hybrid time to be identical to controller time, with the same ordering. Thus, hybrid time Γ_M is a left-closed interval of $\mathbb{R}^{\geq 0} \times \mathbb{Z}_4$ with first instant $(0, 0)$, and for any choice of $r, r' \in \mathbb{R}^{\geq 0}$ and $z, z' \in \mathbb{Z}_4$,

$$(r, z) < (r', z') \text{ iff either } r < r' \text{ or } r = r' \text{ and } z < z'. \tag{7.64}$$

Given this definition, I can define a pair of mappings $\hat{t}_{\mathrm{p}} : \Gamma_M \to \Gamma_{\mathrm{p}}$ and $\hat{t}_{\mathrm{c}} : \Gamma_M \to \Gamma_{\mathrm{c}}$ taking hybrid instants to physical and controller instants, respectively. In particular, for any instant $(r, z) \in \Gamma$, where

r is an element of $\mathbb{R}^{\geq 0}$ and z is an element of $\mathbb{Z}_4$, the mapping $\hat{t}_{\mathrm{p}}$ simply drops the z-value, so that

$$\hat{t}_{\mathrm{p}}(r, z) \stackrel{\text{def}}{=} r, \tag{7.65}$$

and the mapping $\hat{t}_{\mathrm{c}}$ is the identity, so that

$$\hat{t}_{\mathrm{c}}(r, z) \stackrel{\text{def}}{=} (r, z). \tag{7.66}$$

Let $(r, z), (r', z') \in \Gamma_M$ be any hybrid mechanical instants such that $(r, z) \leq (r', z')$. Then $r \leq r'$, and so

$$\hat{t}_{\mathrm{p}}(r, z) \leq \hat{t}_{\mathrm{p}}(r', z'). \tag{7.67}$$

Since $\hat{t}_{\mathrm{c}}$ is the identity, clearly

$$\hat{t}_{\mathrm{c}}(r, z) \leq \hat{t}_{\mathrm{c}}(r', z'). \tag{7.68}$$

In other words, the mappings from hybrid mechanical time to each factor universe preserve the ordering of all instants in Γ_M.

I also provide a mapping from automaton time to hybrid time. Per Assumption 1, automaton time is a left-closed interval of $\mathbb{R}$ beginning at 0. Thus, I define a function $\hat{t}_M : \Gamma_A \to \Gamma_M$ as follows. For any instant $t \in \Gamma_A$,

$$\hat{t}_M(t) \stackrel{\text{def}}{=} (t, 0). \tag{7.69}$$

In other words, each real-valued instant of automaton time maps to the first hybrid instant with the same real value. Note that this mapping preserves my earlier intuition that every automaton instant maps to the same real-valued physical instant, as

$$\hat{t}_{\mathrm{p}}(\hat{t}_M(t)) = \hat{t}_{\mathrm{p}}(t, 0) = t. \tag{7.70}$$

For any instants $(r, z), (r', 0) \in \Gamma_M$, where r, r' are in $\mathbb{R}^{\geq 0}$ and z is in $\mathbb{Z}_4$, the sum $(r, z) + (r', 0)$ is defined; in particular,

$$(r, z) + (r', 0) \stackrel{\text{def}}{=} (r + r', z). \tag{7.71}$$

If $r \geq r'$, the difference $(r, z) - (r', 0)$ is also defined; in particular,

$$(r, z) - (r', 0) \stackrel{\text{def}}{=} (r - r', z). \tag{7.72}$$

The difference is not defined if $r < r'$, since hybrid instants cannot have negative terms. Note that, given this definition, for any instants $t, t', (r, 0) \in \Gamma_M$, it is the case that

$$t + (r, 0) = t' \tag{7.73}$$

iff

$$t' - (r, 0) = t. \tag{7.74}$$

Note as well that, for any instants $r, r' \in \Gamma_A$,

$$\begin{aligned}\hat{t}_M(r) + \hat{t}_M(r') &= (r, 0) + (r', 0) \\ &= (r + r', 0) \\ &= \hat{t}_M(r + r').\end{aligned} \tag{7.75}$$

Likewise, if $r \geq r'$,

$$\begin{aligned}\hat{t}_M(r) - \hat{t}_M(r') &= (r, 0) - (r', 0) \\ &= (r - r', 0) \\ &= \hat{t}_M(r - r').\end{aligned} \tag{7.76}$$

7.3.4 Hybrid mass

Following Section 5.2.1, a body has mass values for all factor mechanical systems, but has 0 mass for any factor system in which it has no non-null component. The set of hybrid mass values is thus the product of the sets of mass values for the physical and controller mechanical systems, that is,

$$\begin{aligned}\mathcal{M} &\stackrel{\text{def}}{=} \mathcal{M}_{\text{p}} \times \mathcal{M}_{\text{c}} \\ &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_2.\end{aligned} \tag{7.77}$$

As described in Section 5.2.1, the physical and controller masses of any particular body are then the physical and controller masses of that body's physical and controller components, respectively. That is, for $i \in \{p, c\}$,

$$[m(B)]_i = m_i(B \sqcap \top_i), \tag{7.78}$$

where B_i is the component of some body B in the ith universe.

I have previously assigned each effector body point and sensor body point a fixed unit controller mass. Thus, for each $v \in V$,

$$\begin{aligned}[m(e_v)]_{\text{c}} &= m_{\text{c}}(e_v) = 1_{\text{c}} \\ [m(s_v)]_{\text{c}} &= m_{\text{c}}(s_v) = 1_{\text{c}},\end{aligned} \tag{7.79}$$

By Assumption 7, physical and controller universes are separate. Then by the above argument, for each

$v \in V$,

$$[m(e_v)]_{\mathrm{p}} = m_{\mathrm{p}}(\perp) = 0_{\mathrm{p}} \tag{7.80}$$
$$[m(s_v)]_{\mathrm{p}} = m_{\mathrm{p}}(\perp) = 0_{\mathrm{p}}.$$

Likewise, I have assumed that the physical correspondent body has a fixed unit physical mass; I have not specified the mass of its physical exterior, as I am unconcerned throughout with the details of that body. Thus,

$$[m(G)]_{\mathrm{p}} = m_{\mathrm{p}}(G) = 1_{\mathrm{p}}. \tag{7.81}$$

As with the physical masses of controllers,

$$[m(G)]_{\mathrm{c}} = m_{\mathrm{c}}(\perp) = 0_{\mathrm{c}}. \tag{7.82}$$

In general, I will only be concerned with the physical or controller mass of any particular body; where I wish to express both components of mass, I write an ordered pair beginning with the physical mass. Thus, for example,

$$m(G) = (1_{\mathrm{p}}, 0_{\mathrm{c}}). \tag{7.83}$$

By some abuse of notation, in what follows I will sometimes write $m_{\mathrm{p}}(B)$ for $[m(B)]_{\mathrm{p}}$, and likewise $m_{\mathrm{c}}(B)$ for $[m(B)]_{\mathrm{c}}$. In such cases, even if B has nonphysical (or noncontroller) components, I refer only to the mass of the physical or controller components of that body, respectively. In other words,

$$m_{\mathrm{p}}(B) = m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}) \tag{7.84}$$
$$m_{\mathrm{c}}(B) = m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}).$$

7.3.5 Hybrid placement and configuration spaces

Since the controller and physical universal bodies are separate, the placement space of $\top$ is the coproduct of their placement spaces. By (7.9) and (7.52), then,

$$\begin{aligned} \mathcal{C}(\top, \mathcal{S}) &= \mathcal{C}(\top_{\mathrm{p}}, \mathcal{S}_{\mathrm{p}}) \times \mathcal{C}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) \\ &\cong \mathbb{R}^{2n} \times \mathbb{Z}_2^{2|V|}. \end{aligned} \tag{7.85}$$

Separation of the physical and controller universes also implies that a physical body has no controller motion, and likewise that controller bodies have no physical motion. Thus, a motion of G is a curve in $\mathcal{S}_{\mathrm{p}}$, that is, $\mathbb{R}^n$; if N is taken to be a point body, then a motion of $\top_{\mathrm{p}}$ is a pair of such curves. Since $\mathcal{S}_{\mathrm{c}}$ is discrete, any motion of a controller body is locally constant; in other words, a motion of a controller body divides hybrid time into a series of intervals, with the position of the controller body alternating between 0 and 1 across subsequent intervals.

The set of all configurations of the corresponding bodies may be constructed similarly. Recall from Section 5.3.4 that, when all factor universes are separate, the configuration space of a hybrid body is the product of the configuration spaces of its components in each factor universe. Thus, the configuration space of $G \sqcup \top_{\mathrm{c}}$ is

$$\begin{aligned} \mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &= \mathfrak{X}(G, \mathcal{S}_{\mathrm{p}}) \times \mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) \\ &\cong \mathbb{R}^{2n} \times \mathbb{Z}_2^{4|V|}. \end{aligned} \tag{7.86}$$

Likewise, given a construction of N,

$$\mathfrak{X}(\top, \mathcal{S}) \cong \mathbb{R}^{2n} \times \mathbb{Z}_2^{4|V|} \times \mathfrak{X}(N, \mathcal{S}_{\mathrm{p}}). \tag{7.87}$$

7.3.6 States of the mechanical controllers

Having defined a hybrid universe, I can define states of the controllers in terms of configurations of $\top$, the universal body. Again, I provide a brief restatement of the results to this point. As defined so far, given a hybrid automaton $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$ such that

$$X = \mathbb{R}^n, \tag{7.88}$$

there exists a physical mechanical system such that

$$\begin{aligned} \Omega_{\mathrm{p}} &= \{\top_{\mathrm{p}}, G, N, \perp_{\mathrm{p}}\} \\ \mathcal{S}_{\mathrm{p}} &= X = \mathbb{R}^n \\ \Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\ \mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\ m_{\mathrm{p}}(G) &= 1_{\mathrm{p}} \\ \mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) &\cong X = \mathbb{R}^n \\ \mathfrak{X}(G, \mathcal{S}_{\mathrm{p}}) &\cong X \times X = \mathbb{R}^{2n} \\ \mathcal{V}_{\mathrm{p}} &= X = \mathbb{R}^n \end{aligned} \tag{7.89}$$

and a controller mechanical system such that

$$E = \{e_v \mid v \in V\} \tag{7.90}$$

$$\forall v, v' \in V[(v \neq v') \to (e_v \neq e_{v'})]$$

$$S = \{s_v \mid v \in V\} \tag{7.91}$$

$$
\begin{aligned}
\forall v, v' \in V[(v \neq v') \to (s_v \neq s_{v'})]\\
E \sqcap S = \bot_{\mathrm{c}}\\
\top_{\mathrm{c}} = E \sqcup S\\
\Omega_{\mathrm{c}} = 2^{E \sqcup S}\\
\mathcal{S}_{\mathrm{c}} = \mathbb{Z}_2\\
\Gamma_{\mathrm{c}} = \mathbb{R}^{\geq 0} \times \mathbb{Z}_4\\
\mathcal{M}_{\mathrm{c}} = \mathbb{Z}_2\\
\forall v \in V, m_{\mathrm{c}}(e_v) = 1_{\mathrm{c}}\\
\forall v \in V, m_{\mathrm{c}}(s_v) = 1_{\mathrm{c}}\\
\mathcal{C}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) \cong \mathbb{Z}_2^{2|V|}\\
\mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) \cong \mathbb{Z}_2^{4|V|}\\
\mathcal{V}_{\mathrm{c}} = \mathbb{Z}_2.
\end{aligned}
$$

This provides a hybrid mechanical system such that

$$
\begin{aligned}
\Omega &= \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}} \qquad (7.92)\\
\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} &= \bot\\
\mathcal{S} &= \mathcal{S}_{\mathrm{p}} \oplus \mathcal{S}_{\mathrm{c}}\\
\Gamma_M &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4\\
\hat{t}_{\mathrm{p}}(r, z) &= r\\
\hat{t}_{\mathrm{c}}(r, z) &= (r, z)\\
\mathcal{M} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_2\\
m(G) &= (1_{\mathrm{p}}, 0_{\mathrm{c}})\\
\forall v \in V, m(e_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}})\\
\forall v \in V, m(s_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}})\\
\mathcal{C}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X \times \mathbb{Z}_2^{2|V|} = \mathbb{R}^n \times \mathbb{Z}_2^{|V|}\\
\mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X^2 \times \mathbb{Z}_2^{4|V|} = \mathbb{R}^{2n} \times \mathbb{Z}_2^{4|V|}.
\end{aligned}
$$

7.3.6.1 States of the effector

I now define the states of the effector. Following the pattern described in Section 6.3, I begin by defining states for the point subbodies of the effector, defining the states of the effector in terms of the nonempty intersections of those sets.

The construction in the following pages may appear somewhat mysterious; hopefully, some commentary on the rationale may make it more plain. Loosely speaking, my goal is that at every mechanical instant, the state of the effector should indicate the full system state of the automaton at a corresponding automaton instant. Since the automaton may in general transition from any state to any other state, the effector be able to perform similarly arbitrary changes from a mechanical controller state associated with a given automaton controller state to one labeled with a different automaton controller state.

Some preliminary work is necessary before producing the effector point body state sets. For each automaton controller state $v \in V$, let ψ_v be the set of all configurations $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ so that e_v either has a position of 0 and a velocity of 1, or a position of 1 and a velocity of 0, while all other effector point bodies have either both a position and a velocity of 0 or both a position and a velocity of 1. In other words, the position and velocity of e_v are different values in $\mathbb{Z}_2$, while position and velocity of all other effector point bodies are the same values in $\mathbb{Z}_2$. Formally,

$$\psi_v \stackrel{\text{def}}{=} \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, (w(e_v) \neq \dot{w}(e_v)) \wedge \left(\forall v' \neq v \in V \left[w(e_{v'}) = \dot{w}(e_{v'})\right]\right)\right\}. \tag{7.93}$$

It is easy to show that ψ_v is nonempty; for instance, it contains the configuration such that all physical body points have positions and velocities of 0^n, and all controller bodies have positions and velocities of 0 except for e_v, which has a position of 1.

Let $\overline{\psi}_v$ be the set of all configurations $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ not in ψ_v; that is,

$$\overline{\psi}_v \stackrel{\text{def}}{=} \mathfrak{X}(\top, \mathcal{S}) \setminus \psi_v. \tag{7.94}$$

Again, it is easy to show that $\overline{\psi}_v$ is nonempty; for instance, it contains the configuration such that all physical body points have positions and velocities of 0^n, and all controller bodies have positions and velocities of 0.

The mutual complements $\overline{\psi}_v$ and ψ_v constitute a partition of $\mathfrak{X}(\top, \mathcal{S})$. For any automaton controller states $v, v' \in V$ such that $v \neq v'$, the mechanical controller states ψ_v and $\psi_{v'}$ are disjoint, as shown in the following lemma.

Lemma 3. *For any automaton controller states $v, v' \in V$ such that $v \neq v'$, the mechanical controller states ψ_v and $\psi_{v'}$ are disjoint, that is,*

$$\psi_v \cap \psi_{v'} = \emptyset. \tag{7.95}$$

Proof. Suppose it were not so; that is, suppose that there exists some configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that $(w, \dot{w})$ is in $\psi_v \cap \psi_{v'}$. By the construction of ψ_v, the position and velocity of any effector point other than e_v must equal each other. In particular, considering $e_{v'}$,

$$w(e_{v'}) = \dot{w}(e_{v'}). \tag{7.96}$$

By the construction of $\psi_{v'}$, however, the position and velocity of $e_{v'}$ must not equal each other. That is,

$$w(e_{v'}) \neq \dot{w}(e_{v'}). \tag{7.97}$$

This is a contradiction, and so no such configurations exist. Thus,

$$\psi_v \cap \psi_{v'} = \emptyset. \tag{7.98}$$

□

Let the set ψ_V be the union of all sets ψ_v, for all $v \in V$. That is,

$$\psi_V \stackrel{\text{def}}{=} \bigcup_{v \in V} \psi_v. \tag{7.99}$$

I write $\overline{\psi}_V$ for the complement of ψ_V, that is,

$$\overline{\psi}_V \stackrel{\text{def}}{=} \mathfrak{X}(\top, \mathcal{S}) \setminus \psi_V. \tag{7.100}$$

By DeMorgan's Laws, it follows that

$$\overline{\psi}_V = \bigcap_{v \in V} \overline{\psi}_v. \tag{7.101}$$

In other words, $\overline{\psi}_V$ is the intersection of all states $\overline{\psi}_v$; equivalently, it is the mutual exterior of all states ψ_v. Again, $\overline{\psi}_V$ is nonempty, as the example configuration given for $\overline{\psi}_v$ is also an element of $\overline{\psi}_V$.

Clearly ψ_V and $\overline{\psi}_V$ partition $\mathfrak{X}(\top, \mathcal{S})$. By the construction of ψ_V, it follows that the set

$$\{\psi_v \mid v \in V\} \cup \left\{\overline{\psi}_V\right\} \tag{7.102}$$

is a partition of $\mathfrak{X}(\top, \mathcal{S})$.

Let $\equiv_G$ be the equivalence relation over $\mathfrak{X}(\top, \mathcal{S})$ such that any two configurations $(w, \dot{w})$ and $(w', \dot{w}')$ are related iff they assign the same position and velocity to G. In other words,

$$(w, \dot{w}) \equiv_G (w', \dot{w}') \text{ iff } \left[\left(w_{\text{p}}(G) = w'_{\text{p}}(G)\right) \wedge \left(\dot{w}_{\text{p}}(G) = \dot{w}'_{\text{p}}(G)\right)\right]. \tag{7.103}$$

Let $\mathfrak{X}(\top, \mathcal{S}) / \equiv_G$ be the partition imposed by this relation. Since the physical placement space of G is isomorphic to X, for any such configurations $(w, \dot{w})$ and $(w', \dot{w}')$, let $\mathbf{x} \in X$ be the automaton environment state such that

$$\mathbf{x} \cong w_{\text{p}}(G) = w'_{\text{p}}(G). \tag{7.104}$$

Likewise, let $\dot{\mathbf{x}} \in X$ be the environment state derivative such that

$$\dot{\mathbf{x}} \cong \dot{w}_{\mathrm{p}}(G) = \dot{w}'_{\mathrm{p}}(G). \tag{7.105}$$

Then I write $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ to indicate the equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$ consisting of all configurations agreeing on the placement and velocity of G. That is,

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} \stackrel{\text{def}}{=} \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w_{\mathrm{p}}(G) = \mathbf{x}\big) \wedge \big(\dot{w}_{\mathrm{p}}(G) = \dot{\mathbf{x}}\big)\}. \tag{7.106}$$

In general, the equivalence class $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ may contain many configurations. As the equivalence relation constrains the assignment of position and velocity only for the body points of G, two configurations might be in the same equivalence class and yet assign wildly different positions or velocities to the body points of N or $\top_{\mathrm{c}}$. Minimally, for any choice of $\mathbf{x}, \dot{\mathbf{x}} \in X$, the equivalence class $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ exists and is nonempty; for instance, it contains the configuration such that the position of G is isomorphic to $\mathbf{x}$, the vector field over G is isomorphic to $\dot{\mathbf{x}}$, all remaining physical bodies have positions and velocities of 0^n, and all controller bodies have positions and velocities of 0.

For any automaton controller state $v \in V$, let $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ be the intersection of ψ_v and $\psi_{\mathbf{x},\dot{\mathbf{x}}}$, that is,

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \stackrel{\text{def}}{=} \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}. \tag{7.107}$$

Then for any $v \in V$, the set

$$\{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_v\} \tag{7.108}$$

is a partition of $\mathfrak{X}(\top, \mathcal{S})$, as I show in the following lemma. In particular, I define this partition to be the set of mechanical controller states for effector point body e_v. In other words, for each $v \in V$,

$$\Psi(e_v) \stackrel{\text{def}}{=} \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_v\}. \tag{7.109}$$

Note that in this case, $\Psi(e_v)$ is isomorphic to $(\mathbb{R}^n \times \mathbb{R}^n) + 1$.

Lemma 4. *For any $v \in V$, the set $\Psi(e_v)$ is a partition of $\mathfrak{X}(\top, \mathcal{S})$.*

Proof. The nonemptiness of $\overline{\psi}_v$ has already been shown; I show now that all remaining elements of $\Psi(e_v)$ are nonempty. For any $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, consider a conifugration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that $w_{\mathrm{c}}(e_v) = 1$ and $\dot{w}_{\mathrm{c}}(e_v) = 0$; $w_{\mathrm{p}}(G) = \mathbf{x}$ and $\dot{w}_{\mathrm{p}}(G) = \dot{\mathbf{x}}$; all other controller body points have positions and velocities of 0; and all other physical body points have positions and velocities of 0^n. By (7.93), this configuration is in ψ_v. By (7.106), it is in $\psi_{\mathbf{x},\dot{\mathbf{x}}}$. Then by (7.107), it is in the set $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, and so every such set is nonempty.

I now show that all elements of $\Psi(e_v)$ are pairwise disjoint. Consider any set $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$; by definition,

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} = \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}. \tag{7.110}$$

Consider any other set $\psi_{v,\mathbf{x}',\dot{\mathbf{x}}'} \neq \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$; again, by definition,

$$\psi_{v,\mathbf{x}',\dot{\mathbf{x}}'} = \psi_v \cap \psi_{\mathbf{x}',\dot{\mathbf{x}}'}. \tag{7.111}$$

It follows that $\psi_{\mathbf{x},\dot{\mathbf{x}}} \neq \psi_{\mathbf{x}',\dot{\mathbf{x}}'}$. Then since $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ and $\psi_{\mathbf{x}',\dot{\mathbf{x}}'}$ are different equivalence classes from the same partition, they must be disjoint. It follows that $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ and $\psi_{v,\mathbf{x}',\dot{\mathbf{x}}'}$ are subsets of disjoint sets, and so they are disjoint as well. Also, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v, and so by definition it is disjoint from $\overline{\psi}_v$. Thus, all elements of $\Psi(e_v)$ are pairwise disjoint.

Finally, the elements of $\Psi(e_v)$ collectively comprise $\mathfrak{X}(\top, \mathcal{S})$. Consider any arbitrary configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$. If $(w, \dot{w})$ is in $\overline{\psi}_v$, then it is in an element of $\Psi(e_v)$, and so the proof is satisfied. Otherwise, $(w, \dot{w})$ must be in ψ_v. Since $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$ is a partition of $\mathfrak{X}(\top, \mathcal{S})$, there must be some $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$ such that $(w, \dot{w})$ is an element of $\psi_{\mathbf{x},\dot{\mathbf{x}}}$. Then by definition, $(w, \dot{w})$ is an element of state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(e_v)$, since

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} = \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}. \tag{7.112}$$

Then in any event, $(w, \dot{w})$ is an element of some member of $\Psi(e_v)$, and so all elements of $\mathfrak{X}(\top, \mathcal{S})$ are members of some element of $\Psi(e_v)$. Thus, $\Psi(e_v)$ is a partition of $\mathfrak{X}(\top, \mathcal{S})$. □

Let C be any mechanical controller formed by joining some collection of effector point bodies; then the mechanical controller states of C are defined by the construction in Section 6.3. That is, $\Psi(C)$ is the set of equivalence classes such that, for any $(w, \dot{w}), (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_C (w', \dot{w}') \tag{7.113}$$

iff

$$\forall e_v \sqsubseteq C \left(\exists \psi \in \Psi(e_v) \left[(w, \dot{w}) \in \psi \wedge (w', \dot{w}') \in \psi\right]\right). \tag{7.114}$$

That is, two elements of $\mathfrak{X}(\top, \mathcal{S})$ are in the same equivalence class, and thus the same state of C, if they share a controller state for each effector point body that is a subbody of C.

In particular, then, the states of the effector are all states $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, where v is in V and $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ is an equivalence class imposed by $\equiv_G$, together with $\overline{\psi}_V$. That is,

$$\Psi(E) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \left\{\overline{\psi}_V\right\}. \tag{7.115}$$

I show that this result holds in the following theorem; note that, in this case, $\Psi(E)$ is isomorphic to $(V \times \mathbb{R}^n \times \mathbb{R}^n) + 1$.

Theorem 19. *The states of E are*

$$\Psi(E) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \left\{\overline{\psi}_V\right\}. \tag{7.116}$$

Proof. By (7.43), E is the body composed of the set of all effector body points,

$$E = \{e_v \,|\, v \in V\}. \tag{7.117}$$

Equivalently, it is the join of all effector point bodies,

$$E = \bigsqcup_{v \in V} e_v. \tag{7.118}$$

By Section 6.3, $\Psi(E)$ consists of the nonempty intersections of one mechanical controller state from each such body. That is, $\Psi(E)$ is the set of equivalence classes such that, for any $(w, \dot{w}), (w', \dot{w}') \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_E (w', \dot{w}') \tag{7.119}$$

iff

$$\forall v \in V \left(\exists \psi \in \Psi(e_v) \left[(w, \dot{w}) \in \psi \wedge (w', \dot{w}') \in \psi\right]\right). \tag{7.120}$$

Consider any set ψ formed by intersecting one mechanical controller state from the state set of each effector point body. By (7.109), for each $v \in V$,

$$\Psi(e_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \{\overline{\psi}_v\}. \tag{7.121}$$

Thus, either all of the states intersected to produce ψ are of the form $\overline{\psi}_v$; or all but one of them are of this form, with the remaining state of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$; or two or more of them are of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Suppose that all of the mechanical controller states intersected to produce ψ are of the form $\overline{\psi}_v$. That is,

$$\psi = \bigcap_{v \in V} \overline{\psi}_v. \tag{7.122}$$

But then by (7.100), $\psi = \overline{\psi}_V$. Since $\overline{\psi}_V$ is nonempty, it must be a state of E.

Suppose instead that exactly one mechanical controller state intersected to produce ψ is of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, for some arbitrary $v \in V$ and $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}$. That is,

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \left(\bigcap_{v' \neq v \in V} \overline{\psi}_{v'}\right). \tag{7.123}$$

By Lemma 3, for any $v' \in V$ such that $v \neq v'$, the mechanical controller states ψ_v and $\psi_{v'}$ are disjoint.

By (7.94), $\psi_{v'}$ and $\overline{\psi}_{v'}$ constitute a partition of $\mathfrak{X}(\top, \mathcal{S})$, and so it follows that ψ_v is a subset of $\overline{\psi}_{v'}$. Since $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is the intersection of ψ_v and $\psi_{\mathbf{x},\dot{\mathbf{x}}}$, it is a subset of ψ_v, and so it is a subset of $\overline{\psi}_{v'}$. But then

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_{v'} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}, \tag{7.124}$$

and so

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \left(\bigcap_{v' \neq v \in V} \overline{\psi}_{v'} \right) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{7.125}$$

Then $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$. Since $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a state of e_v, it is nonempty, and so it is a state of E. Since the choice of $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ was arbitrary, every such effector point body state is also a mechanical controller state of the effector.

Suppose instead that two or more of the states intersected to produce ψ are of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}, \psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$, for some $v, v' \in V$ such that $v \neq v'$, and some $\psi_{\mathbf{x},\dot{\mathbf{x}}}, \psi_{\mathbf{x}',\dot{\mathbf{x}}'} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$. It follows that

$$\psi \subseteq \left(\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}\right). \tag{7.126}$$

By definition, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v, and likewise $\psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$ is a subset of $\psi_{v'}$. But by Lemma 3,

$$\psi_v \cap \psi_{v'} = \emptyset. \tag{7.127}$$

Then $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ and $\psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$ must likewise be disjoint, and so their only common subset is the empty set. Thus, $\psi = \emptyset$, and so it is not a state of E.

Then the nonempty intersections are $\overline{\psi}_V$ and every $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, and so

$$\Psi(E) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \left\{\overline{\psi}_V\right\}. \tag{7.128}$$

□

7.3.6.2 Examples of effector states

To illustrate the different mechanical controller states of the effector, suppose that a hybrid automaton A has three automaton controller states, denoted v_1, v_2, v_3. The effector would then be a set of three body points, $\{e_{v_1}, e_{v_2}, e_{v_3}\}$. For simplicity, I refer to these points simply as e_1, e_2, e_3, and I write any state $\psi_{v_i,\mathbf{x},\dot{\mathbf{x}}}$ as $\psi_{i,\mathbf{x},\dot{\mathbf{x}}}$.

Suppose that, for some configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, all effector body points have a velocity of 0, with $w(e_1) = 1$ and, for all $i \neq 1$, $w(e_i) = 0$. Suppose further that the assignments of position and velocity to the body points of G are isomorphic to $\mathbf{x}$ and $\dot{\mathbf{x}}$, respectively, for some $\mathbf{x}, \dot{\mathbf{x}} \in X$. It follows that $(w, \dot{w})$ is in effector state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$. This case is demonstrated in Figure 7.6.

Alternatively, suppose that all effector body points except e_1 and one other are at rest; without loss

	e_1	e_2	e_3	G
Position of bodies	1	0	0	$\mathbf{x}$
Velocity of bodies	0	0	0	$\dot{\mathbf{x}}$

Figure 7.6: Effector in $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$, with all subbodies of the effector at rest.

	e_1	e_2	e_3	G
Position of bodies	0	0	1	$\mathbf{x}$
Velocity of bodies	1	0	1	$\dot{\mathbf{x}}$

Figure 7.7: Effector in $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$, with two effector body points in motion.

	e_1	e_2	e_3	G
Position of bodies	0	0	1	$\mathbf{x}$
Velocity of bodies	0	0	0	$\dot{\mathbf{x}}$

Figure 7.8: Effector in $\psi_{3,\mathbf{x},\dot{\mathbf{x}}}$, with all subbodies of the effector at rest.

of generality, I choose e_3 as the second effector body point. Here, all effector body points except for the second chosen body point are in position 0, as illustrated in Figure 7.7, while the assignment of position and velocity to body points of G are as before. Again, any configuration in $\mathfrak{X}(\top, \mathcal{S})$ satisfying these conditions must be a member of an effector state that is a subset of ψ_1, because the only effector body point whose position and velocity differ is e_1.

If the preceding configuration is altered so that all effector body points have a velocity of 0, the effector will instead be in the mechanical controller state corresponding to the second effector body point, in this case e_3. Thus, the effector is in $\psi_{3,\mathbf{x},\dot{\mathbf{x}}}$ in Figure 7.8.

Note that, as originally desired for the states of effector, there is an unambiguous correspondence between states of the effector and those of the automaton. Thus, it is easy to tell whether the states of the effector and automaton "match" in a given instant.

Further, just as the automaton may in general transition from any state to any other state, the effector can perform similarly arbitrary changes from a mechanical controller state labeled with a given automaton state to one labeled with a different automaton state. One can send the effector from a state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ to another state $\psi_{v',\mathbf{x},\dot{\mathbf{x}}}$ by moving e_v from 1 to 0 and moving $e_{v'}$ from 0 to 1, while leaving all separate bodies unmoved, and then bringing all bodies back to a velocity of 0. This progression exactly mirrors the assignments of position and velocity shown in Figures 7.6 - 7.8; in other words, these figures show the transition from a configuration in ψ_1 to a configuration in ψ_3 across successive instants..

7.3.6.3 States of the sensor point bodies

I will now define the states of each sensor point body s_v, with each mechanical controller state of a particular sensor point body corresponding to a different set of automaton controller states to which an automaton might transition.

In general, hybrid automata permit discrete transitions of the form $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x}')$, where v, v' are in V and $\mathbf{x}, \mathbf{x}'$ are in X. Transitions in which $\mathbf{x} \neq \mathbf{x}'$ pose two problems for the framing presented here. First, they suggest that each sensor point body must indicate not only an automaton controller state to which transitions are possible, but also the point or points in environment space to which the transition may shift. In principle, every possible set of environment states might define a separate mechanical controller state for a sensor point body. This seems undesirable.

Perhaps more persuasively, there is a basic mechanical concern with allowing such transitions. Fundamentally, they suggest an instantaneous change in the automaton environment, and hence in some physical body. The physical body effectively "teleports," moving from one placement to another arbitrary placement in the space of a single physical instant. In mechanical terms, such motions are discontinuities and unrealizable through finite forces. In the original automaton, perhaps such changes represent shifts in acceleration or other quantities that need not be continuous; again, however, in the general case, it is not possible to identify which quantities correspond to properties other than placement.

For these reasons, I here restrict consideration to automata whose transitions are all of the form $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x})$, that is, transitions in which only the automaton controller state changes. Thus, knowing the set of possible controller automaton state changes associated with a particular placement is sufficient to know all possible transitions associated with that placement.

Assumption 8. *All transitions in D are of the form $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x})$ for some $v, v' \in V$ and some $\mathbf{x} \in X$. In other words, transitions preserve environment states.*

A similar assumption is required for velocity, to prevent instantaneous changes in the momentum of an object.

Assumption 9. *Given a hybrid automaton A and an execution fragment $\alpha = \tau_0 a_1 \tau_1 a_2 \ldots$ of that automaton, for any component trajectories τ_i, τ_{i+1} in the execution fragment,*

$$\Pi_X(\dot{\tau}_i.lstate) = \Pi_X(\dot{\tau}_{i+1}.fstate). \tag{7.129}$$

In other words, transitions preserve environment state derivatives.

Given these requirements, I define the *transition image* $\mathcal{D} : V \times X \to V$ be the set of all automaton controller states to which it is possible to transition from a particular system state. I commonly write $\mathcal{D}_v(\mathbf{x})$ to indicate $\mathcal{D}(v, \mathbf{x})$; thus

$$\mathcal{D}_v(\mathbf{x}) = \{v' \in V \mid ((v, \mathbf{x}), (v', \mathbf{x})) \in D\}. \tag{7.130}$$

I can then define an equivalence relation $\equiv_{\mathcal{D},v}$ over X so that two environment states $\mathbf{x}, \mathbf{x}'$ are equivalent with respect to $\equiv_{\mathcal{D},v}$ if and only if they have the same transition image. That is,

$$\mathbf{x} \equiv_{\mathcal{D},v} \mathbf{x}' \text{ iff } \mathcal{D}_v(\mathbf{x}) = \mathcal{D}_v(\mathbf{x}'). \tag{7.131}$$

The equivalence relation thus imposes a partition $X/\equiv_{\mathcal{D},v}$ in which all elements of an equivalence class $[\![\mathbf{x}]\!]$ have the same transition image as $\mathbf{x}$. Loosely speaking, every environment state in some $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$ can transition from v to the same set of other automaton controller states.

For each such element, I define the set $\psi_{[\![\mathbf{x}]\!]}$ to be the set of all configurations $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that $w(G)$ uniquely identifies an element of $[\![\mathbf{x}]\!]$. In other words,

$$\psi_{[\![\mathbf{x}]\!]} \stackrel{\text{def}}{=} \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\right\}. \tag{7.132}$$

The set of all such sets is a partition of $\mathfrak{X}(\top, \mathcal{S})$ based on the placement of G, since $X/\equiv_{\mathcal{D},v}$ is a partition and the placement space of G is isomorphic to X.

I write $\psi_{v,[\![\mathbf{x}]\!]}$ for the intersection of ψ_v and $\psi_{[\![\mathbf{x}]\!]}$,

$$\psi_{v,[\![\mathbf{x}]\!]} \stackrel{\text{def}}{=} \psi_v \cap \psi_{[\![\mathbf{x}]\!]}. \tag{7.133}$$

Note that ψ_v constrains only the placement of E, while $\psi_{[\![\mathbf{x}]\!]}$ constrains only the placement of G; thus, $\psi_{v,[\![\mathbf{x}]\!]}$ is nonempty. For instance, it contains the configuration such that G has a placement isomorphic to $\mathbf{x}$, e_v has a position of 1 and a velocity of 0, and in all other cases, physical body points have positions and velocities of 0^n and controller body points have positions and velocities of 0.

For any $v \in V$, the set

$$\{\psi_{v,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_v\} \tag{7.134}$$

is a partition of $\mathfrak{X}(\top, \mathcal{S})$, as I show in the following lemma. In particular, I define this partition to be the set of mechanical controller states for the sensor point body s_v. In other words, for each $v \in V$,

$$\Psi(s_v) \stackrel{\text{def}}{=} \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right\} \cup \left\{\overline{\psi}_v\right\} \tag{7.135}$$

Lemma 5. *For any automaton controller state $v \in V$, the set $\Psi(s_v)$ is a partition of $\mathfrak{X}(\top, \mathcal{S})$.*

Proof. Previous remarks have shown that all elements of $\Psi(s_v)$ are nonempty; it remains only to show that they are pairwise disjoint and collectively comprise $\mathfrak{X}(\top, \mathcal{S})$.

Consider any set $\psi_{v,[\![\mathbf{x}]\!]}$; by definition,

$$\psi_{v,[\![\mathbf{x}]\!]} = \psi_v \cap \psi_{[\![\mathbf{x}]\!]} \tag{7.136}$$

Consider any other set $\psi_{v,[\![\mathbf{x}']\!]} \neq \psi_{v,[\![\mathbf{x}]\!]}$; again, by definition,

$$\psi_{v,[\![\mathbf{x}']\!]} = \psi_v \cap \psi_{[\![\mathbf{x}']\!]} \tag{7.137}$$

It follows that $\psi_{[\![\mathbf{x}]\!]} \neq \psi_{[\![\mathbf{x}']\!]}$, and so that $[\![\mathbf{x}]\!] \neq [\![\mathbf{x}']\!]$. Then since $[\![\mathbf{x}]\!]$ and $[\![\mathbf{x}']\!]$ are different equivalence classes from the same partition, they must be disjoint. It follows that $\psi_{v,[\![\mathbf{x}]\!]}$ and $\psi_{v,[\![\mathbf{x}']\!]}$ are subsets of disjoint sets, and so they are disjoint as well. Also, $\psi_{v,[\![\mathbf{x}]\!]}$ is a subset of ψ_v, and so by definition it is disjoint from $\overline{\psi}_v$. Thus, all elements of $\Psi(s_v)$ are pairwise disjoint.

Consider some arbitrary configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$. Suppose $(w, \dot{w})$ is in $\overline{\psi}_v$; then it is an element of some member of $\Psi(s_v)$. Suppose instead that it is not in $\overline{\psi}_v$; then by (7.100), it must be in ψ_v. Since $X/\equiv_{\mathcal{D},v}$ is a partition, there must also be some $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$ such that $(w, \dot{w})$ is in $\psi_{[\![\mathbf{x}]\!]}$. Then $(w, \dot{w})$ is in $\psi_v \cap \psi_{[\![\mathbf{x}]\!]}$, and so by 7.133 it is in $\psi_{v,[\![\mathbf{x}]\!]}$. Then in any event, it is a member of some element of $\Psi(s_v)$, and so $\Psi(s_v)$ is a partition of $\mathfrak{X}(\top, \mathcal{S})$. □

Let C be any mechanical controller formed by joining some collection of sensor point bodies; then the mechanical controller states of C are defined by the construction in Section 6.3. That is, $\Psi(C)$ is the set of equivalence classes such that, for any $(w, \dot{w}), (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_C (w', \dot{w}') \tag{7.138}$$

iff

$$\forall s_v \sqsubseteq C \left(\exists \psi \in \Psi(s_v) \left[(w, \dot{w}) \in \psi \wedge (w', \dot{w}') \in \psi\right]\right). \tag{7.139}$$

In particular, the states of the sensor are all states $\psi_{v,[\![\mathbf{x}]\!]}$, where v is in V and $[\![\mathbf{x}]\!]$ is an equivalence class imposed by $\equiv_{\mathcal{D},v}$, together with $\overline{\psi}_V$. That is,

$$\Psi(S) = \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, (v \in V) \wedge \left([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right)\right\} \cup \left\{\overline{\psi}_V\right\}. \tag{7.140}$$

I show this result in the following theorem. I show that this result holds in the following theorem; note that, in this case, $\Psi(S)$ is isomorphic to $(V \times X/\equiv_{\mathcal{D},v}) + 1$.

Theorem 20. *The states of the sensor are*

$$\Psi(S) = \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, (v \in V) \wedge \left([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right)\right\} \cup \left\{\overline{\psi}_V\right\}. \tag{7.141}$$

Proof. By (7.47), S is the body composed of the set of all sensor body points,

$$S = \{s_v \mid v \in V\}. \tag{7.142}$$

Equivalently, S is the join of all sensor point bodies,

$$S = \bigsqcup_{v \in v} s_v. \tag{7.143}$$

By Section 6.3, $\Psi(S)$ consists of the nonempty intersections of one mechanical controller state from each sensor point body. That is, $\Psi(S)$ is the set of equivalence classes such that, for any $(w, \dot{w}), (w', \dot{w}') \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_S (w', \dot{w}') \tag{7.144}$$

iff

$$\forall v \in V \left(\exists \psi \in \Psi(s_v) \left[(w, \dot{w}) \in \psi \wedge (w', \dot{w}') \in \psi \right] \right). \tag{7.145}$$

Consider any set ψ formed by intersecting one mechanical controller state from the state set of each sensor point body. By (7.135), for each $v \in V$,

$$\Psi(s_v) = \left\{ \psi_{v, [\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D}, v} \right\} \cup \left\{ \overline{\psi}_v \right\}. \tag{7.146}$$

Then it must be the case either that all of the states intersected to produce ψ are of the form $\overline{\psi}_v$; or that all but one of them are of this form, with the remaining state of the form $\psi_{v, [\![\mathbf{x}]\!]}$; or that two or more of them are of the form $\psi_{v, [\![\mathbf{x}]\!]}$.

Suppose that all of the mechanical controller states intersected to produce ψ are of the form $\overline{\psi}_v$. That is, suppose

$$\psi = \bigcap_{v \in V} \overline{\psi}_v. \tag{7.147}$$

But then by (7.100), $\psi = \overline{\psi}_V$. Since $\overline{\psi}_V$ is nonempty, it is a state of S.

Suppose instead that exactly one mechanical controller state intersected to produce ψ is of the form $\psi_{v, [\![\mathbf{x}]\!]}$, for some arbitrary $v \in V$ and $\mathbf{x} \in X$. That is, suppose

$$\psi = \psi_{v, [\![\mathbf{x}]\!]} \cap \left(\bigcap_{v' \neq v \in V} \overline{\psi}_{v'} \right). \tag{7.148}$$

By Lemma 3, for any $v' \in V$ such that $v \neq v'$, the mechanical controller states ψ_v and $\psi_{v'}$ are disjoint. Since $\psi_{v'}$ and $\overline{\psi}_{v'}$ constitute a partition of $\mathfrak{X}(\top, \mathcal{S})$, it follows that ψ_v is a subset of $\overline{\psi}_{v'}$. Since $\psi_{v, [\![\mathbf{x}]\!]}$ is a subset of ψ_v, it is also a subset of $\overline{\psi}_{v'}$. But then

$$\psi_{v, [\![\mathbf{x}]\!]} \cap \overline{\psi}_{v'} = \psi_{v, [\![\mathbf{x}]\!]}, \tag{7.149}$$

and so

$$\psi_{v,[\![\mathbf{x}]\!]} \cap \left(\bigcap_{v' \neq v \in V} \overline{\psi}_{v'} \right) = \psi_{v,[\![\mathbf{x}]\!]}. \tag{7.150}$$

Then $\psi = \psi_{v,[\![\mathbf{x}]\!]}$. Since $\psi_{v,[\![\mathbf{x}]\!]}$ is nonempty, it is a state of S. Since the original choice of $\psi_{v,[\![\mathbf{x}]\!]}$ was arbitrary, all such sets are states of the sensor.

Suppose instead that two or more of the states intersected to produce ψ are of the form $\psi_{v,[\![\mathbf{x}]\!]}, \psi_{v',[\![\mathbf{x}']\!]}$, for some $v, v' \in V$ such that $v \neq v'$, and some $\mathbf{x}, \mathbf{x}' \in X$. It follows that

$$\psi \subseteq \left(\psi_{v,[\![\mathbf{x}]\!]} \cap \psi_{v',[\![\mathbf{x}']\!]} \right). \tag{7.151}$$

By construction, $\psi_{v,[\![\mathbf{x}]\!]}$ is a subset of ψ_v; similarly, $\psi_{v',[\![\mathbf{x}']\!]}$ is a subset of $\psi_{v'}$. Thus,

$$\psi \subseteq (\psi_v \cap \psi_{v'}). \tag{7.152}$$

But by Lemma 3,

$$\psi_v \cap \psi_{v'} = \emptyset. \tag{7.153}$$

Then $\psi = \emptyset$, and so it is not a state of S.

Then the nonempty intersections are $\overline{\psi}_V$ and every $\psi_{v,[\![\mathbf{x}]\!]}$, and so

$$\Psi(S) = \left\{ \psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, (v \in V) \wedge \left([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v} \right) \right\} \cup \left\{ \overline{\psi}_V \right\}. \tag{7.154}$$

□

7.3.6.4 States of the universal controller

As defined in (7.49), the universal controller body $\top_\mathrm{c}$ is the join of the effector and the sensor. Let C be any mechanical controller formed by joining some collection of effector and sensor point bodies; then as per Section 6.3, the mechanical controller states of C. That is, $\Psi(C)$ is the set of equivalence classes such that, for any $(w, \dot{w}), (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$,

$$(w, \dot{w}) \equiv_C (w', \dot{w}') \tag{7.155}$$

iff, for all body points $b \sqsubseteq C$,

$$\exists \psi \in \Psi(b) \left[(w, \dot{w}) \in \psi \wedge (w', \dot{w}') \in \psi \right]. \tag{7.156}$$

In particular, the states of the universal controller body are

$$\Psi(\top_{\mathrm{c}}) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\ \big)\} \cup \{\overline{\psi}_V\}. \tag{7.157}$$

I prove this result in the following pages. First, however, I note that the set of universal controller states is identical to $\Psi(E)$. In other words, the state of the effector determines the state of the universal controller; it follows that it also determines the state of the sensor. It might be argued that, in such cases, it is unnecessary to track the states of the effector and sensor separately, and that, in fact, it suffices to have $\top_{\mathrm{c}} = E$. Indeed, such alternate constructions appear plausible, with the states of the sensor reduced to groups of states of the effector. Such a reduction would complicate construction of conditional forces in later sections, however. While an effector-only construction might well be possible, the division of force systems into "influence of the effector" and "influence of the sensor" allows for a more natural expression of the behavior of the overall system.

Lemma 6. *For any $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$,*

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_{[\![\mathbf{x}]\!]}. \tag{7.158}$$

Proof. Consider any arbitrary configuration $(w, \dot{w}) \in \psi_{\mathbf{x},\dot{\mathbf{x}}}$. By (7.106),

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w_{\mathrm{p}}(G) = \mathbf{x}\big) \wedge \big(\dot{w}_{\mathrm{p}}(G) = \dot{\mathbf{x}}\big)\}, \tag{7.159}$$

and so $w_{\mathrm{p}}(G) = \mathbf{x}$ and $\dot{w}_{\mathrm{p}}(G) = \dot{\mathbf{x}}$. By (7.132),

$$\psi_{[\![\mathbf{x}]\!]} = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\}. \tag{7.160}$$

Clearly $\mathbf{x}$ is in $[\![\mathbf{x}]\!]$, and so $w(G)$ uniquely identifies an environment state in $[\![\mathbf{x}]\!]$. Then $(w, \dot{w})$ is in $\psi_{[\![\mathbf{x}]\!]}$. Since $(w, \dot{w})$ is an arbitrary element of $\psi_{\mathbf{x},\dot{\mathbf{x}}}$, it follows that

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_{[\![\mathbf{x}]\!]}. \tag{7.161}$$

□

Lemma 7. *For any $v \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$, if $\mathbf{x}$ is in $[\![\mathbf{x}']\!]$ then*

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{7.162}$$

If $\mathbf{x}$ is not in $[\![\mathbf{x}']\!]$, then

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \emptyset. \tag{7.163}$$

Proof. Suppose that $\mathbf{x}$ is in $[\![\mathbf{x}']\!]$. Then $\psi_{[\![\mathbf{x}']\!]} = \psi_{[\![\mathbf{x}]\!]}$, and by Lemma 6,

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_{[\![\mathbf{x}]\!]} = \psi_{[\![\mathbf{x}']\!]}. \tag{7.164}$$

Intersecting both sides of this expression with ψ_v gives

$$(\psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}) \subseteq \left(\psi_v \cap \psi_{[\![\mathbf{x}']\!]}\right). \tag{7.165}$$

But $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} = \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}$ and $\psi_{v,[\![\mathbf{x}']\!]} = \psi_v \cap \psi_{[\![\mathbf{x}']\!]}$. Substituting these into (7.165) gives

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_{v,[\![\mathbf{x}']\!]}, \tag{7.166}$$

and so

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{7.167}$$

Suppose instead that $\mathbf{x}$ is not a member of $\psi_{[\![\mathbf{x}']\!]}$. Then $\psi_{[\![\mathbf{x}']\!]} \neq \psi_{[\![\mathbf{x}]\!]}$; since the sets of the form $\psi_{[\![\mathbf{x}]\!]}$ collectively form a partition of $\mathfrak{X}(\top, \mathcal{S})$, it follows that

$$\psi_{[\![\mathbf{x}]\!]} \cap \psi_{[\![\mathbf{x}']\!]} = \emptyset. \tag{7.168}$$

Again, by Lemma 6, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of $\psi_{[\![\mathbf{x}]\!]}$, and so

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{[\![\mathbf{x}']\!]} = \emptyset. \tag{7.169}$$

Since $\psi_{v',[\![\mathbf{x}']\!]}$ is a subset of $\psi_{[\![\mathbf{x}']\!]}$, it follows that

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \emptyset. \tag{7.170}$$

□

Theorem 21. *The states of the universal controller body are*

$$\Psi(\top_{\mathrm{c}}) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\ \big)\} \cup \{\overline{\psi}_V\}. \tag{7.171}$$

Proof. By construction, $\top_{\mathrm{c}}$ is the join of the effector and the sensor,

$$\top_{\mathrm{c}} = E \sqcup S, \tag{7.172}$$

where the effector and the sensor are disjoint bodies,

$$E \sqcap S = \bot. \tag{7.173}$$

By the definition of supercontroller states in Section 6.3, $\Psi(\top_c)$ consists of the nonempty intersections of one mechanical controller state from each such body. That is, $\Psi(\top_c)$ is the nonempty elements of the set

$$\{\psi \cap \psi' \mid \big(\psi \in \Psi(E)\big) \wedge \big(\psi' \in \Psi(S)\big)\}. \tag{7.174}$$

By Theorem 19,

$$\Psi(E) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\}. \tag{7.175}$$

By Theorem 20,

$$\Psi(S) = \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\big)\} \cup \{\overline{\psi}_V\}. \tag{7.176}$$

Consider any set ψ formed by intersecting one mechanical controller state from the state set of each of the effector and the sensor. By the above state set definitions, there are four categories of possibilities for the construction of ψ:

1. For some $v, v' \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$,

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v',[\![\mathbf{x}']\!]}. \tag{7.177}$$

2. For some $v \in V$ and $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$,

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_V. \tag{7.178}$$

3. For some $v' \in V$ and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$,

$$\psi = \overline{\psi}_V \cap \psi_{v',[\![\mathbf{x}']\!]}. \tag{7.179}$$

4. ψ is the intersection

$$\psi = \overline{\psi}_V \cap \overline{\psi}_V. \tag{7.180}$$

The proof proceeds by cases.

Case 1: Suppose that, for some $v, v' \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$,

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v',[\![\mathbf{x}']\!]}. \tag{7.181}$$

Note that $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v, and $\psi_{v',[\![\mathbf{x}']\!]}$ is a subset of $\psi_{v'}$. By Lemma 3, ψ_v and $\psi_{v'}$ are disjoint unless $v = v'$. It follows that ψ is empty unless $v = v'$.

Any such empty intersections cannot be states of $\top_c$. Discarding them, suppose further that $v = v'$,

and so

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]}. \tag{7.182}$$

Either $\mathbf{x}$ is a member of $\psi_{[\![\mathbf{x}']\!]}$ or it is not. In the former case, by Lemma 7,

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{7.183}$$

In the latter case, again by Lemma 7,

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \emptyset. \tag{7.184}$$

Thus, the only potentially nonempty intersections are every choice of $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(E)$. Since these sets are mechanical controller states of the effector, they must be nonempty. Thus, every such set is a state of $\top_{\mathrm{c}}$.

Case 2: Suppose that, for some $v \in V$ and $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$,

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_V. \tag{7.185}$$

By definition, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v, while $\overline{\psi}_V$ is a subset of $\overline{\psi}_v$. Since ψ_v and $\overline{\psi}_v$ are disjoint, it follows that $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ and $\overline{\psi}_V$ are disjoint. Then their intersection is empty, and so

$$\begin{aligned}\psi &= \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_V \\ &= \emptyset.\end{aligned} \tag{7.186}$$

Then ψ is not a mechanical controller state.

Case 3: Suppose that, for some $v' \in V$ and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$,

$$\psi = \overline{\psi}_V \cap \psi_{v',[\![\mathbf{x}']\!]}. \tag{7.187}$$

By definition, $\overline{\psi}_V$ is a subset of $\overline{\psi}_{v'}$. By contrast, $\psi_{v',[\![\mathbf{x}']\!]}$ is a subset of $\psi_{v'}$. Since $\overline{\psi}_{v'}$ and $\psi_{v'}$ are disjoint, it follows that $\overline{\psi}_V$ and $\psi_{v',[\![\mathbf{x}']\!]}$ are disjoint. Then their intersection is empty, and so

$$\begin{aligned}\psi &= \overline{\psi}_V \cap \psi_{v',[\![\mathbf{x}']\!]} \\ &= \emptyset.\end{aligned} \tag{7.188}$$

Then ψ is not a mechanical controller state.

Case 4: Suppose that, for some $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$,

$$\psi = \overline{\psi}_V \cap \overline{\psi}_V. \tag{7.189}$$

Trivially, in this case $\psi = \overline{\psi}_V$. Since $\overline{\psi}_V$ is a mechanical controller state, namely for the mechanical controller E, it is nonempty; thus, it is a state of $\top_\mathrm{c}$.

Conclusion: The nonempty intersections of the states of E and S produced by Case 1 are precisely

$$\{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\big)\}. \tag{7.190}$$

Cases 2 and 3 do not produce any nonempty intersections, and Case 4 produces only the nonempty intersection $\overline{\psi}_V$. Then the nonempty intersections of E and S, and so the states of $\top_\mathrm{c}$, are precisely those originally given in this theorem. □

Note that, given this construction, $\Psi(\top_\mathrm{c})$ is isomorphic to $(V \times X \times X) + 1$, since every choice of system state, environment state, and environment state derivative defines a separate mechanical controller state; equivalently,

$$\Psi(\top_\mathrm{c}) \cong (Q \times X) + 1. \tag{7.191}$$

The primary impetus for this very large state set is the state set for the effector, which must determine the forces on G based on all three of the above factors.

The state set for the sensor may be somewhat smaller, depending on the number of equivalence classes in each $X/\equiv_{\mathcal{D},v}$. For each choice of $v \in V$,

$$1 \leq |X/\equiv_{\mathcal{D},v}| \leq |X|, \tag{7.192}$$

and so, since $Q = V \times X$,

$$|V| + 1 \leq |\Psi(S)| \leq |Q| + 1. \tag{7.193}$$

Further constraining the size of the sensor state set requires restricting the set of automaton transitions.

7.3.7 Hybrid initial configuration set

In keeping with my definition of a initialized hybrid mechanical system in Section 5.3.4, I provide an initial configuration set, that is, a set of positions and velocities for $\top$ at instant $(0, 0)$. Intuitively, for any trajectory beginning in an initial automaton system state $(v, \mathbf{x})$, there should be a corresponding mechanical configuration such that the effector is in state ψ_v the position of G is isomorphic to $\mathbf{x}$, and the velocity of G is the initial derivative of that trajectory's trace. I define this condition more formally by defining a function $\mathtt{Mech} : \mathcal{T} \to \mathfrak{X}(\top, \mathcal{S})$, as follows.

Consider any automaton trajectory $\tau \in \mathcal{T}$ automaton instant $t \in \mathit{dom}(\tau)$. Let $\tau(t) = (v, \mathbf{x})$. Then $\mathtt{Mech}(\tau) = (w, \dot{w})$, where $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ is the hybrid configuration such that the following conditions hold. First, the placement and velocity of the physical correspondent body in $\mathcal{S}_\mathrm{p}$ are the same as the projection of τ's value and first derivative onto the environment state, that is,

$$\begin{aligned} w_{\mathtt{p}}(G) &= \Pi_X(\tau(t)) \\ \dot{w}_{\mathtt{p}}(G) &= \Pi_X(\dot{\tau}(t)). \end{aligned} \tag{7.194}$$

Second, the effector is in state ψ_v with all body points at velocity 0, that is,

$$\begin{aligned} w_{\mathtt{c}}(e_v) &= 1 \\ \forall v' \neq v \in V\big[w_{\mathtt{c}}(e_{v'}) &= 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathtt{c}}(e_{v'}) &= 0\big]. \end{aligned} \tag{7.195}$$

Likewise, all sensor body points have positions and velocities of 0 in $\mathcal{S}_{\mathtt{c}}$, that is

$$\begin{aligned} \forall v' \in V\big[w_{\mathtt{c}}(s_{v'}) = 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathtt{c}}(s_{v'}) = 0\big]. \end{aligned} \tag{7.196}$$

Note that this is a complete specification of position and velocity to all body points, except for those in N; as usual, I make no specifications regarding what assignment of position and velocity `Mech` makes to that body.

By (7.194), $\mathtt{Mech}(\tau, t)$ is in the set $\psi_{\mathbf{x},\dot{\mathbf{x}}}$; by (7.195), $\mathtt{Mech}(\tau, t)$ is in state ψ_v. Then it is in the intersection of those sets; that is, $\mathtt{Mech}(\tau, t)$ is in the universal mechanical controller state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, where $\dot{\mathbf{x}} = \Pi_X(\dot{\tau}.\mathit{fstate})$. The initial configuration set, then, is the set of all such configurations such that $\tau(t)$ is in Q_0, that is,

$$\mathfrak{X}_0(\top, \mathcal{S}) \stackrel{\text{def}}{=} \{\mathtt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in \mathit{dom}(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{7.197}$$

Note that this definition excludes any automaton states in Q_0 from which there are no trajectories. Such states are effectively irrelevant to the behavior of the automaton, in any event; in Henzinger/Alur terms, they are immediately blocking, and so removing them from the automaton would not restrict its behavior at any automaton instant after the first. It therefore seems reasonable to restrict consideration to automata for which there is at least one trajectory from every initial automaton state, as I do in the following assumption.

Assumption 10. *Given a hybrid automaton $A = \{V, X, Q_0, D, \mathcal{T}, \Gamma\}$, for any state $q \in Q_0$, there is at least one non-point trajectory $\tau \in \mathcal{T}$ such that $\tau.\mathit{fstate} = q$. In other words, no initial state is blocking.*

7.3.8 Summary of construction without forces

To conclude this chapter, it seems prudent to again summarize the results to this point; thus, this section simply reproduces and consolidates existing definitions. In the next chapter, I will extend this construction to include conditional forces.

Given a hybrid automaton $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$ such that

$$X = \mathbb{R}^n, \tag{7.198}$$

there exists a physical mechanical system such that

$$\begin{aligned}
\Omega_{\mathrm{p}} &= \{\top_{\mathrm{p}}, G, N, \bot_{\mathrm{p}}\} && (7.199)\\
\mathcal{S}_{\mathrm{p}} &= X = \mathbb{R}^n\\
\Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0}\\
\mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0}\\
m_{\mathrm{p}}(G) &= 1_{\mathrm{p}}\\
\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) &\cong X = \mathbb{R}^n\\
\mathfrak{X}(G, \mathcal{S}_{\mathrm{p}}) &\cong X \times X = \mathbb{R}^{2n}\\
\mathcal{V}_{\mathrm{p}} &= X = \mathbb{R}^n
\end{aligned}$$

and a controller mechanical system such that

$$\begin{aligned}
E &= \{e_v \mid v \in V\} && (7.200)\\
\forall v, v' \in V[(v \neq v') &\to (e_v \neq e_{v'})]\\
S &= \{s_v \mid v \in V\} && (7.201)\\
\forall v, v' \in V[(v \neq v') &\to (s_v \neq s_{v'})]\\
E \sqcap S &= \bot_{\mathrm{c}}\\
\top_{\mathrm{c}} &= E \sqcup S\\
\Omega_{\mathrm{c}} &= 2^{E \sqcup S}\\
\mathcal{S}_{\mathrm{c}} &= \mathbb{Z}_2\\
\Gamma_{\mathrm{c}} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4\\
\mathcal{M}_{\mathrm{c}} &= \mathbb{Z}_2\\
\forall v \in V, m_{\mathrm{c}}(e_v) &= 1_{\mathrm{c}}\\
\forall v \in V, m_{\mathrm{c}}(s_v) &= 1_{\mathrm{c}}\\
\mathcal{C}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{2|V|}
\end{aligned}$$

$$\begin{aligned}\mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{4|V|}\\ \mathcal{V}_{\mathrm{c}} &= \mathbb{Z}_2.\end{aligned}$$

This provides a hybrid mechanical system such that

$$\begin{aligned}\Omega &= \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}} \\ \top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} &= \bot \\ \mathcal{S} &= \mathcal{S}_{\mathrm{p}} \oplus \mathcal{S}_{\mathrm{c}} \\ \Gamma_M &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \\ \hat{t}_{\mathrm{p}}(r, z) &= r \\ \hat{t}_{\mathrm{c}}(r, z) &= (r, z) \\ \mathcal{M} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_2 \\ m(G) &= (1_{\mathrm{p}}, 0_{\mathrm{c}}) \\ \forall v \in V, m(e_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}}) \\ \forall v \in V, m(s_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}}) \\ \mathcal{C}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X \times \mathbb{Z}_2^{2|V|} = \mathbb{R}^n \times \mathbb{Z}_2^{|V|} \\ \mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X^2 \times \mathbb{Z}_2^{4|V|} = \mathbb{R}^{2n} \times \mathbb{Z}_2^{4|V|}.\end{aligned} \tag{7.202}$$

The initial configuration set $\mathfrak{X}_0(\top, \mathcal{S})$ is the set

$$\mathfrak{X}_0(\top, \mathcal{S}) = \{\mathtt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in dom(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{7.203}$$

where $\mathtt{Mech}(\tau, t)$, for any $\tau \in \mathcal{T}$ and $t \in dom(\tau)$, is the configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that

$$\begin{aligned}w_{\mathrm{p}}(G) &= \Pi_X(\tau(t)) \\ \dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau})(t)) \\ w_{\mathrm{c}}(e_v) &= 1 \\ \forall v' \neq v \in V&\big[w_{\mathrm{c}}(e_{v'}) = 0\big] \\ \forall v' \in V&\big[\dot{w}_{\mathrm{c}}(e_{v'}) = 0\big] \\ \forall v' \in V&\big[w_{\mathrm{c}}(s_{v'}) = 0\big] \\ \forall v' \in V&\big[\dot{w}_{\mathrm{c}}(s_{v'}) = 0\big].\end{aligned} \tag{7.204}$$

The equivalence class $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$ is defined by the relation

$$(w, \dot{w}) \equiv_G (w', \dot{w}') \text{ iff } \big[(w_{\mathrm{p}}(G) = w'_{\mathrm{p}}(G)) \wedge (\dot{w}_{\mathrm{p}}(G) = \dot{w}'_{\mathrm{p}}(G))\big], \tag{7.205}$$

and the equivalence class $X/\equiv_{\mathcal{D},v}$ is defined by the relation

$$\mathbf{x} \equiv_{\mathcal{D},v} \mathbf{x}' \text{ iff } \mathcal{D}_v(\mathbf{x}) = \mathcal{D}_v(\mathbf{x}'). \tag{7.206}$$

Then for each $v \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$, I have defined subsets of $\mathfrak{X}(\top, \mathcal{S})$ as follows:

$$\begin{aligned}
\psi_v &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_v) \neq \dot{w}(e_v)\big) \wedge \big(\forall v' \neq v \in V[w(e_{v'}) = \dot{w}(e_{v'})]\big)\} \\
\overline{\psi}_v &= \mathfrak{X}(\top, \mathcal{S}) - \psi_v \\
\overline{\psi}_V &= \bigcap_{v \in V} \overline{\psi}_v \\
\psi_{v,\mathbf{x},\dot{\mathbf{x}}} &= \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}} \\
\psi_{[\![\mathbf{x}]\!]} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\} \\
\psi_{v,[\![\mathbf{x}]\!]} &= \psi_v \cap \psi_{[\![\mathbf{x}]\!]}.
\end{aligned} \tag{7.207}$$

Then the states of the various controllers are:

$$\begin{aligned}
\forall v \in V\big[\Psi(e_v) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}\big] \\
\Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\} \\
\forall v \in V\big[\Psi(s_v) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_v\}\big] \\
\Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\big)\} \cup \{\overline{\psi}_V\} \\
\Psi(\top_{\mathrm{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\big)\} \cup \{\overline{\psi}_V\}.
\end{aligned} \tag{7.208}$$

7.4 Summary of notation

M	a hybrid mechanical system
Γ_M	hybrid mechanical time
Γ_A	automaton time
Γ_{p}	physical time
$\mathcal{S}_{\mathrm{p}}$	physical space
G	the physical correspondent body
N	physical exterior of G
$\mathcal{C}(B, \mathcal{S}_{\mathrm{p}})$	physical placement space of physical body B
$\mathfrak{X}(B, \mathcal{S}_{\mathrm{p}})$	physical configurations of physical body B
$\cong$	isomorphic

$\mathbf{x}$	automaton environment state, or the isomorphic placement of G
L	reachability image
L^{-1}	reachability preimage
$\equiv_V$	reachability equivalence relation over X
$X/\equiv_V$	partition imposed by $\equiv_V$
$[\mathbf{x}]$	element of the partition $X/\equiv_V$
$v_{[\mathbf{x}]}$	mechanical controller state defined in terms of $[\mathbf{x}]$
$\mathcal{S}_\mathrm{c}$	mechanical controller space
Γ_c	mechanical controller time
e	an effector body point, or an effector point body
E	the effector controller
s	a sensor body point, or a sensor point body
S	the sensor controller
$\mathcal{C}(C, \mathcal{S}_\mathrm{c})$	controller placement space of controller body C
$\mathfrak{X}(C, \mathcal{S}_\mathrm{c})$	controller configurations of controller body C
$\hat{t}_\mathrm{p}$	mapping from Γ_M to Γ_p
$\hat{t}_\mathrm{c}$	mapping from Γ_M to Γ_c
$\hat{t}_M$	mapping from Γ_A to Γ_M
$\mathcal{M}_\mathrm{p}$	set of physical mass values
$\mathcal{M}_\mathrm{c}$	set of controller mass values
m_p	physical mass function
m_c	controller mass function
$0_\mathrm{c}, 1_\mathrm{c}$	controller mass values
$0_\mathrm{p}, 1_\mathrm{p}$	physical mass values
ψ_v	mechanical controller state defined in terms of placement of e_v
$\overline{\psi}_v$	complement of ψ_v with respect to $\mathfrak{X}(\top, \mathcal{S})$
ψ_V	union of all mechanical controller states ψ_v
$\overline{\psi}_V$	complement of ψ_V with respect to $\mathfrak{X}(\top, \mathcal{S})$
$\equiv_G$	equivalence relation over $\mathfrak{X}(\top, \mathcal{S})$ defined in terms of placement and velocity of G
$\mathfrak{X}(\top, \mathcal{S})/\equiv_G$	partition imposed by $\equiv_G$
$\psi_{\mathbf{x},\dot{\mathbf{x}}}$	element of the partition $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$
$\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$	mechanical controller state, intersection of ψ_v and $\psi_{\mathbf{x},\dot{\mathbf{x}}}$
$\mathcal{D}$	transition image of an automaton system state

$\mathcal{D}_v$	transition image of an automaton environment state, given automaton controller state v
$\equiv_{\mathcal{D},v}$	equivalence relation over X defined in terms of $\mathcal{D}_v$
$X/\equiv_{\mathcal{D},v}$	partition of X imposed by $\equiv_{\mathcal{D},v}$
$[\![\mathbf{x}]\!]$	element of the partition $X/\equiv_{\mathcal{D},v}$ containing automaton environment state $\mathbf{x}$
$\psi_{[\![\mathbf{x}]\!]}$	mechanical controller state defined in terms of $[\![\mathbf{x}]\!]$
$\psi_{v,[\![\mathbf{x}]\!]}$	mechanical controller state, intersection of ψ_v and $\psi_{[\![\mathbf{x}]\!]}$
`Mech`	mapping from trajectories to configurations

Chapter 8

Mechanical Forces for the Transformation

Chapter 7 provided the construction of a hybrid mechanical system without forces, that is, a tuple

$$(\top_M, \mathcal{S}_M, \Gamma_M, m_M, \Psi_M, \mathfrak{X}_0(\top, \mathcal{S})_M). \tag{8.1}$$

Construction of mechanical forces from the original automaton is complex and lengthy, and so I present it here as a separate chapter. In other words, this chapter provides the construction of $\hat{\mathfrak{f}}[\cdot]_M$, the remaining element of the tuple for an initialized hybrid mechanical system M.

In Chapter 6, I discussed the idea of a conditional force variation inclusion over the physical universe, that is, a function of the form $\hat{\mathfrak{f}}[\cdot] : \Psi(\top_\mathrm{c}) \to 2^{\hat{\mathcal{F}}(\Omega_\mathrm{p}, \mathcal{V}_\mathrm{p}, \Gamma_M)}$. Such a construction is sufficient if one is concerned only with motions of the physical universe; in this chapter, however, I will consider motions of controller bodies as well. Thus, I define the conditional force variation inclusion $\hat{\mathfrak{f}}[\cdot]$ to be a mapping to sets of hybrid force variations over the hybrid universe, that is, a function $\hat{\mathfrak{f}}[\cdot] : \Psi(\top_\mathrm{c}) \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M)}$. Aside from this change, I retain all definitions from Chapter 6, and all results from that chapter hold for this alternative construction. In Chapter 10, I will show which of the controllers I describe below have the properties of state independence, force independence, and so on.

As described in Section 5.2.3, the hybrid force values $\mathcal{V}$ are

$$\mathcal{V} = \prod_{i \in \mathcal{I}} \mathcal{V}_i, \tag{8.2}$$

and so, for this system,

$$\mathcal{V} \stackrel{\text{def}}{=} \mathbb{R}^n \times \mathbb{Z}_2. \tag{8.3}$$

Following my notation from Section 6.2.2, a conditional force variation inclusion is a function $\hat{\mathfrak{f}}[\cdot] : \Psi(\top_\mathrm{c}) \to 2^{\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M)}$, that is, a function that associates each state in $\Psi(\top_\mathrm{c})$ with a set of force variations. In what follows, I assume that there are no forces other than those I describe or those derived from my description. In particular, for any choice of universal state $\psi \in \Psi(\top_\mathrm{c})$, $\hat{\mathfrak{f}}[\psi]$ is set of force variations, and so each of its elements must be null-passive, additive, and pairwise equilibriated at every instant. I do not attempt to describe the forces for every possible pair of bodies, relying instead on these three properties to imply the forces between, for instance, superbodies of a body pair I describe.

My approach here is to define a set of conditional force variation inclusions, beginning simply and constructing more complex inclusions from these first steps. First, for each effector point body e_v, I define a conditional force variation inclusion $\hat{\mathfrak{f}}_{e,v}[\cdot] : \Psi(e_v) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma_M)}$. I then do the same for each sensor point body s_v, defining a conditional force variation inclusion $\hat{\mathfrak{f}}_{s,v}[\cdot] : \Psi(s_v) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma_M)}$. The conditional force variation inclusion for the effector, denoted $\hat{\mathfrak{f}}_E[\cdot] : \Psi(E) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma_M)}$, is formed by taking each state of the effector, state-projecting it onto its effector point bodies, and summing one force variation from each resulting inclusion. The conditional force variation for the sensor, denoted $\hat{\mathfrak{f}}_S[\cdot] : \Psi(S) \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma_M)}$, is produced similarly. Thus, for any state $\psi \in \Psi(E)$,

$$\hat{\mathfrak{f}}_E[\psi] = \left\{ \sum_{v\in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{e,v}[e_v(\psi)] \right\}, \tag{8.4}$$

and, for any state $\psi \in \Psi(S)$,

$$\hat{\mathfrak{f}}_S[\psi] = \left\{ \sum_{v\in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{s,v}[s_v(\psi)] \right\}. \tag{8.5}$$

The conditional force variation inclusion for the universal controller $\top_{\mathrm{c}}$ is likewise formed from the conditional force variation inclusions for the effector and the sensor. Thus, for any state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{\mathfrak{f}}[\psi] = \left\{ \hat{f}_E + \hat{f}_S \,\middle|\, \left(\hat{f}_E \in \hat{\mathfrak{f}}_E[E(\psi)]\right) \wedge \left(\hat{f}_S \in \hat{\mathfrak{f}}_S[S(\psi)]\right) \right\}. \tag{8.6}$$

Note that all forces in each of the above inclusions are of the form $(\mathbf{u}, \mathbf{v})$, where $\mathbf{u}$ is a vector over $\mathcal{S}_{\mathrm{p}}$, that is, $\mathbb{R}^n$, and $\mathbf{v}$ is a vector over $\mathcal{S}_{\mathrm{c}}$, that is, $\mathbb{Z}_2$. As the product of two vector spaces is not in general a vector space, $(\mathbf{u}, \mathbf{v})$ is not a vector. Thus, I write the vectors $\mathbf{u}$ and $\mathbf{v}$ separately in the definitions that follow. In particular, I write $\mathbf{0}_{\mathrm{p}}$ to represent the vector $\langle 0, \ldots 0\rangle$ over $\mathbb{R}^n$. Likewise, I write $\mathbf{0}_{\mathrm{c}}$ for the vector $\langle 0\rangle$ in $\mathbb{Z}_2$, and I write $\mathbf{0}$ for their product $(\mathbf{0}_{\mathrm{p}}, \mathbf{0}_{\mathrm{c}})$.

Recall from Section 6.2.3.2 that the *passive force variation*, denoted $\hat{f}_{\mathbf{0}}$, is the unique force variation in which no body exerts nonzero force on another in any instant. Formally, for any instant $t \in \Gamma_M$ and bodies $A, B \in \Omega$,

$$\hat{f}_{\mathbf{0}}(t)(A, B) = \mathbf{0}. \tag{8.7}$$

In subsequent sections, I will declare some force variations to be the passive force variation, or will state that a force variation agrees with the passive force variation except where otherwise specified.

8.1 Hybrid forces of effector bodies

8.1.1 Hybrid forces of the effector point bodies

In a hybrid automaton, the state changes of the environment depend upon the state of the controller. Under the mechanical transformation provided here, environment state changes correspond to motions of the physical correspondent body, and automaton controller states correspond roughly to states of E. Thus, motions of $\top_{\mathrm{p}}$ should depend on the state of E. Toward this end, I will define $\hat{\mathfrak{f}}_{e,v}[\cdot]$ for each $e \in E$, and finally construct $\hat{\mathfrak{f}}_E[\cdot]$ in terms of these component systems as described above.

The force on a body over time should explain its motion over time; my construction of the forces conditional on the effector, then, is determined by the desired motions of the physical body. As discussed in Section 7.1, physical space is constructed so that the physical placement space of G is isomorphic to the automaton environment space, that is,

$$\mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) \cong X. \tag{8.8}$$

My intent is to provide a construction of forces so that, for any execution α of A, there should be an induced motion χ of $\top$ such that, for every $t \in \Gamma_A$,

$$\chi_{\mathrm{p}}(G, t) \cong \Pi_X(\alpha(t)). \tag{8.9}$$

(Recall that physical and automaton time are identical, and so I can interpret t as an instant in either set.) In other words, at every instant in physical or automaton time, the placement of the physical universal body should be isomorphic to the automaton environment state.

I can now cast this description in terms of the automaton controller states. Recall from (7.208) that

$$\Psi(e_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \left\{\overline{\psi}_v\right\}. \tag{8.10}$$

Let $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ be some state of e_v, for some $v \in V$ and $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$. Let $\hat{f} \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M)$ be the force variation such that, for some trajectory $\tau \in \mathcal{T}$, the following conditions hold:

- The domain of $\hat{f}$ is all instants in hybrid mechanical time; that is,

$$dom(\hat{f}) = \Gamma_M, \tag{8.11}$$

- τ has a constant automaton controller state of v; that is,

$$\forall t \in dom(\tau)\, [\Pi_V(\tau(t)) = v]\,, \tag{8.12}$$

- At automaton instant 0, the environment state component of τ is $\mathbf{x}$, while the derivative of the

environment state component of τ is $\dot{\mathbf{x}}$; that is,

$$(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}) \wedge (\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}}) , \tag{8.13}$$

- For all instants $t \in \mathit{dom}(\hat{f})$, the force of N on G is isomorphic to the second derivative of the environment state component of τ at 0 times $\mathbf{0}_\mathrm{c}$; that is,

$$\forall t \in \mathit{dom}(\hat{f}) \left[\hat{f}(t)(G, N) = (\Pi_X(\ddot{\tau}.\mathit{fstate}), \mathbf{0}_\mathrm{c}) \right] , \tag{8.14}$$

  and

- Except as required by the above, $\hat{f} = \hat{f}_\mathbf{0}$.

In other words, $\hat{f}$ is a force variation over all hybrid instants; in every instant, it defines a force on G isomorphic to the second derivative of the trajectory's environment state at the trajectory's first instant, when the trajectory's automaton controller state, environment state, and environment state derivative are v, $\mathbf{x}$, and $\dot{\mathbf{x}}$, respectively. Except as implied by this requirement (via additivity and pairwise equilibriation), the force variation exerts no nonzero forces. I call the force variation satisfying the above requirements for some trajectory τ the *force variation for* τ, relabeling it from $\hat{f}$ to the more specific $\hat{f}^\tau$. Note that any such force variation is constant; that is, it exerts the same force on all bodies in all instants.

Some discussion of this definition may be helpful. In Assumption 2, I required that all hybrid automata be *differentiably legal*; among other things, this assumption ensures that all trajectories are second-degree doubly semidifferentiable. It therefore follows that $\ddot{\tau}$ must exist at all automaton instants, and so the above force definition is defined for all trajectories at all mechanical instants. Given an induced motion χ for $\hat{f}^\tau$, at every physical instant $t \in \Gamma_\mathrm{p}$, the physical acceleration of G must be isomorphic to $\Pi_X(\ddot{\tau}.\mathit{fstate})$, that is,

$$\forall t \in \Gamma_\mathrm{p} \left[\ddot{\chi}_\mathrm{p}(G, t) = \Pi_X(\ddot{\tau}.\mathit{fstate}) \right] . \tag{8.15}$$

I show this result in the following theorem.

Theorem 22. *Given a trajectory $\tau \in \mathcal{T}$, let χ be any induced motion for $\hat{f}^\tau$. Then at every hybrid mechanical instant $t \in \Gamma_\mathrm{p}$, the physical acceleration of G must be isomorphic to $\Pi_X(\ddot{\tau}.\mathit{fstate})$, that is,*

$$\forall t \in \Gamma_\mathrm{p} \left[\ddot{\chi}_\mathrm{p}(G, t) = \Pi_X(\ddot{\tau}.\mathit{fstate}) \right] . \tag{8.16}$$

Proof. By Assumption 7, G is a physical and non-controller body. Then by the definition of hybrid induced motion in Section 5.3.2, for any instant $(r, z) \in \Gamma_M$,

$$\hat{f}^{\tau}_{\mathrm{p}}(r,z)(G \sqcap \top_{\mathrm{p}}, \overline{G \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{p}}(G \sqcap \top_{\mathrm{p}})\ddot{\chi}_{\mathrm{p}}\left(G \sqcap \top_{\mathrm{p}}, r\right). \tag{8.17}$$

Since G is a physical body, $G \sqcap \top_{\mathrm{p}}$ is simply G, and so

$$\hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \overline{G}) = m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G, r). \tag{8.18}$$

By (7.199) and (7.202), all physical bodies have unit physical mass. Thus, this expression becomes

$$\begin{aligned} \hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \overline{G}) &= 1_{\mathrm{p}}\ddot{\chi}_{\mathrm{p}}(G, r) \\ &= \ddot{\chi}_{\mathrm{p}}(G, r). \end{aligned} \tag{8.19}$$

By the construction of $\top_{\mathrm{p}}$ and $\top$, the body $\overline{G}$ may be divided into two separate parts, namely $\top_{\mathrm{c}}$ and N. Since force systems are additive,

$$\hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \overline{G}) = \hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \top_{\mathrm{c}}) + \hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, N). \tag{8.20}$$

But by the definition of $\hat{f}$,

$$\begin{aligned} \hat{f}^{\tau}(r,z)(G, N) &= (\Pi_X(\ddot{\tau}.\mathit{fstate}), \mathbf{0}_{\mathrm{c}}) \\ \hat{f}^{\tau}(r,z)(G, \top_{\mathrm{c}}) &= \mathbf{0}, \end{aligned} \tag{8.21}$$

and so

$$\begin{aligned} \hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \overline{G}) &= \Pi_X(\ddot{\tau}.\mathit{fstate}) + \mathbf{0}_{\mathrm{p}} \\ &= \Pi_X(\ddot{\tau}.\mathit{fstate}). \end{aligned} \tag{8.22}$$

The right-hand sides of (8.19) and (8.22) are both equal to $\hat{f}^{\tau}_{\mathrm{p}}(r,z)(G, \overline{G})$. Setting them equal to each other,

$$\ddot{\chi}_{\mathrm{p}}(G, r) = \Pi_X(\ddot{\tau}.\mathit{fstate}). \tag{8.23}$$

□

Intuitively, then, when G is in position $\Pi_X(\tau.\mathit{fstate})$ with velocity $\Pi_X(\dot{\tau}.\mathit{fstate})$, the force system $\hat{f}^{\tau}$ ensures that it will have an acceleration of $\Pi_X(\ddot{\tau}.\mathit{fstate})$. When G changes position, the state of e_v changes, and so G experiences some new force, and so some new acceleration. Broadly speaking, the overall effect is that at every point in the motion of G, its position, velocity, and acceleration match the value, derivative, and second derivative of the trace of some automaton trajectory.

It may be objected that, if the intent of the above is to in some sense replicate the behavior of a trajectory at points where it "looks like" the mechanical controller state, it is overly reductive to consider only those trajectories that look like the mechanical controller state at automaton instant 0. However, as noted above, by Assumption 2, all trajectories are assumed to be differentiably legal. As such, all trajectories are suffix-closed. Thus, for any trajectory in $\mathcal{T}$ that resembles the mechanical controller state at an automaton instant after 0, there is a suffix of that trajectory in $\mathcal{T}$ that resembles the mechanical controller state precisely at instant 0.

For each $v \in V$, let $\mathcal{T}_v$ be the set of all trajectories in $\mathcal{T}$ with a constant automaton controller state of v; that is,

$$\mathcal{T}_v \stackrel{\text{def}}{=} \left\{\tau \in \mathcal{T} \,\middle|\, \forall t \in dom(\tau)\left[\Pi_V(\tau(t)) = v\right]\right\}. \tag{8.24}$$

By Assumption 3, all trajectories have a constant automaton controller state, and so the set of all such sets $\mathcal{T}_v$ is a partition of τ. I further refine the elements of this partition as follows. For any $v \in V$, $\mathbf{x} \in X$, and $\dot{\mathbf{x}} \in X$, let $\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$ be the set of all trajectories in $\mathcal{T}_v$ with an initial environment state projection of $\mathbf{x}$ and environment state projection derivative of $\dot{\mathbf{x}}$. That is,

$$\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \stackrel{\text{def}}{=} \left\{\tau \in \mathcal{T}_v \,\middle|\, \big(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}\big) \wedge \big(\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}}\big)\right\}. \tag{8.25}$$

Finally, I define the force variation inclusion $\hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]$ to be the set of all force variations $\hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M)$, for all trajectories $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$. That is,

$$\hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] \stackrel{\text{def}}{=} \left\{\hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}. \tag{8.26}$$

The above defines forces for all mechanical controller states of the effector point body e_v except for $\overline{\psi}_v$. In the remaining state $\overline{\psi}_v \in \Psi(e_v)$, the effector should be passive, and so the inclusion contains only $\hat{f}_{\mathbf{0}}$. Thus,

$$\hat{\mathfrak{f}}_{e,v}[\overline{\psi}_v] \stackrel{\text{def}}{=} \{\hat{f}_{\mathbf{0}}\}. \tag{8.27}$$

Summarizing the above definitions, the conditional force variation inclusion for each effector point body is

$$\hat{\mathfrak{f}}_{e,v}[\psi] = \begin{cases} \left\{\hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_v. \end{cases} \tag{8.28}$$

8.1.2 Hybrid forces of the effector

As noted in (8.4), for any mechanical controller state $\psi \in \Psi(E)$, the force variation inclusion conditional on ψ for the effector is the set of all sums of one force variation from each effector point body's inclusion, conditional on the state-projection of ψ onto that body. In other words, for any state

$\psi \in \Psi(E)$,

$$\hat{\mathfrak{f}}_E[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{e,v}[e_v(\psi)] \right\}. \tag{8.29}$$

This sum produces the force variation inclusion

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{ \hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{8.30}$$

as shown in the following theorem.

Theorem 23. *For any mechanical controller state $\psi \in \Psi(E)$, the force variation inclusion for the effector is*

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{ \hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{8.31}$$

Proof. Recall from (7.208) that

$$\Psi(E) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \left\{\overline{\psi}_V\right\}. \tag{8.32}$$

For any choice of $v \in V$ and $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, the mechanical controller state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(E)$ is also a state of e_v. Clearly this mechanical controller state is subset of itself, and so its projection onto e_v is the identity,

$$e_v(\psi_{v,\mathbf{x},\dot{\mathbf{x}}}) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{8.33}$$

By definition, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v. But by Lemma 3, for any $v, v' \in V$ such that $v \neq v'$, the sets ψ_v and $\psi_{v'}$ are disjoint. It follows that, for any effector point body $e_{v'}$ such that $v' \neq v$, the mechanical controller state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is disjoint from any $\psi_{v',\mathbf{x}',\dot{\mathbf{x}}'} \in \Psi(e_{v'})$. Then $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of the only remaining mechanical controller state of $e_{v'}$, that is, $\overline{\psi}_{v'}$. Then

$$e_{v'}(\psi_{v,\mathbf{x},\dot{\mathbf{x}}}) = \overline{\psi}_{v'}. \tag{8.34}$$

By (8.4),

$$\hat{\mathfrak{f}}_E[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{e,v}[e_v(\psi)] \right\}. \tag{8.35}$$

Then for any mechanical controller state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(E)$,

$$\hat{\mathfrak{f}}_E[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] = \left\{ \hat{f} + \sum_{v' \neq v \in V} \hat{f}_{v'} \,\middle|\, \left(\hat{f} \in \hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]\right) \wedge \forall v' \neq v \in V \left(\hat{f}_{v'} \in \hat{\mathfrak{f}}_{e,v'}[\overline{\psi}_{v'}]\right) \right\}. \tag{8.36}$$

By (8.28), $\hat{\mathfrak{f}}_{e,v'}[\overline{\psi}_{v'}] = \{\hat{f}_{\mathbf{0}}\}$ for all $v' \in V$. Thus, the above expression reduces to

$$\hat{\mathfrak{f}}_E[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] = \left\{ \hat{f} + \sum_{v' \neq v \in V} \hat{f}_{\mathbf{0}} \;\middle|\; \hat{f} \in \hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] \right\}. \tag{8.37}$$

Since $\hat{f}_{\mathbf{0}}$ contains no nonzero terms, for any $\hat{f} \in \hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]$,

$$\hat{f} + \sum_{v' \neq v \in V} \hat{f}_{\mathbf{0}} = \hat{f}. \tag{8.38}$$

Thus, (8.37) reduces to

$$\begin{aligned} \hat{\mathfrak{f}}_E[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] &= \left\{ \hat{f} \;\middle|\; \hat{f} \in \hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] \right\} \\ &= \hat{\mathfrak{f}}_{e,v}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]. \end{aligned} \tag{8.39}$$

Then by the definition of $\hat{\mathfrak{f}}_{e,v}[\cdot]$ in (8.28),

$$\hat{\mathfrak{f}}_E[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] = \left\{ \hat{f}^{\tau} \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \;\middle|\; \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}. \tag{8.40}$$

The above accounts for all mechanical controller states in $\Psi(E)$ except for $\overline{\psi}_V$. Consider $\overline{\psi}_V$; by (7.207),

$$\overline{\psi}_V = \bigcap_{v \in V} \overline{\psi}_v. \tag{8.41}$$

It follows that, for any effector point body e_v,

$$e_v(\overline{\psi}_V) = \overline{\psi}_v. \tag{8.42}$$

Again, by (8.4),

$$\hat{\mathfrak{f}}_E[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \;\middle|\; \hat{f}_v \in \hat{\mathfrak{f}}_{e,v}[e_v(\psi)] \right\}, \tag{8.43}$$

and so, letting $\psi = \overline{\psi}_V$,

$$\hat{\mathfrak{f}}_E[\overline{\psi}_V] = \left\{ \sum_{v \in V} \hat{f}_v \;\middle|\; \hat{f}_v \in \hat{\mathfrak{f}}_{e,v}[\overline{\psi}_v] \right\}. \tag{8.44}$$

By (8.28), $\hat{\mathfrak{f}}_{e,v}[\overline{\psi}_v] = \{\hat{f}_{\mathbf{0}}\}$ for all $v \in V$. Thus, the above expression reduces to

$$\hat{\mathfrak{f}}_E[\overline{\psi}_V] = \left\{ \sum_{v \in V} \hat{f}_{\mathbf{0}} \right\} \tag{8.45}$$

$$= \{\hat{f}_{\mathbf{0}}\}.$$

Then for any mechanical controller state $\psi \in \Psi(E)$,

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{ \hat{f}^\tau \in \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \left\{ \hat{f}_{\mathbf{0}} \right\}, & \psi = \overline{\psi}_V, \end{cases} \tag{8.46}$$

□

8.2 Hybrid forces of sensor bodies

8.2.1 Hybrid forces of the sensor point bodies

By (7.208), for each sensor point body s_v,

$$\Psi(s_v) = \left\{ \psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v} \right\} \cup \left\{ \overline{\psi}_v \right\}. \tag{8.47}$$

In the following paragraphs, I will define a conditional force variation inclusion $\hat{\mathfrak{f}}_{s,v}[\cdot]$ over the states of each such sensor point body s_v.

Conceptually, for any mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]} \in \Psi(s_v)$, the motions induced by force variations in $\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]$ correspond to automaton controller state changes. That is, these force variations should in some sense correspond to the set of automaton system states $(v', \mathbf{x})$ to which the automaton controller can transition from $(v, \mathbf{x})$, where $v \neq v'$. In other words, the force variations conditional on $\psi_{v,[\![\mathbf{x}]\!]}$ should induce motions of the effector causing it to change from some state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ to some other state $\psi_{v',\mathbf{x},\dot{\mathbf{x}}}$.

Recall from (7.131) that each equivalence class $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$ specifies a set of automaton controller states $\mathcal{D}_v(\mathbf{x})$ such that, for each $v' \in \mathcal{D}_v(\mathbf{x})$, the transition $(v, \mathbf{x}) \rightarrowtail (v', \mathbf{x})$ is in D. I now begin construction of an appropriate force system to alter the position of E so that it moves from any state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ to state $\psi_{v',\mathbf{x},\dot{\mathbf{x}}}$.

For some $r \in \mathbb{R}^{\geq 0}$ and automaton controller states $v, v' \in V$, with $v \neq v'$, let $\hat{f}$ be the force variation such that

$$\begin{aligned} f_{(r,0)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,0)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,1)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,1)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,2)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \end{aligned} \tag{8.48}$$

	e_1	e_2	e_3	G
Position of subbodies	1	0	0	$\mathbf{x}$
Velocity of subbodies	0	0	0	$\dot{\mathbf{x}}$
Acceleration of subbodies	0	0	0	0

Figure 8.1: Effector at time $(r, 0)$, with state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$.

$$\begin{aligned}
f_{(r,2)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\
f_{(r,3)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\
f_{(r,3)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}),
\end{aligned}$$

and, except where required by the above via additivity,

$$\hat{f} = f_\mathbf{0}. \tag{8.49}$$

Note that, since $\mathbf{1}_\mathrm{c}$ is its own additive inverse in $\mathbb{Z}_2$, the above forces are pairwise equilibriated. I call any force variation satisfying these requirements a *v–v' transition force variation active at r*, written $\hat{f}_{v\text{–}v',r}$, and refer to the set of all such systems for all choices of r as the v–v' transition force variations. Clearly in this case, $\hat{f}_{v\text{–}v',r} = \hat{f}_{v'\text{–}v,r}$

The effects of such a force variation may be somewhat opaque; to clarify them, consider a system like that previously discussed for Figure 7.6. That is, suppose that A has three controller states, denoted v_1, v_2, v_3. As in that example, I write the effector point bodies as e_1, e_2, e_3 and any effector point body state $\psi_{v_i,\mathbf{x},\dot{\mathbf{x}}}$ as $\psi_{i,\mathbf{x},\dot{\mathbf{x}}}$. Without loss of generality, let $e_v = e_1$ and $e_{v'} = e_3$. Then for some $r \in \mathbb{R}^{\geq 0}$, let χ be a induced motion for $\hat{f}_{1\text{–}3,r}$ such that, for some $\mathbf{x}, \dot{\mathbf{x}} \in X$,

$$\begin{aligned}
\chi_\mathrm{p}(G, r) &= \mathbf{x} \\
\dot{\chi}_\mathrm{p}(G, r) &= \dot{\mathbf{x}} \\
\chi_\mathrm{c}(e_1, (r, 0)) &= 1 \\
\forall v' \neq v_1 \in V, \chi_\mathrm{c}(e_{v'}, (r, 0)) &= 0 \\
\forall v \in V, \dot{\chi}_\mathrm{c}(e_v, (r, 0)) &= 0,
\end{aligned} \tag{8.50}$$

with any assignment of position and velocity to all remaining body points. I call any such motion, for any effector point e_v, a *$(v, \mathbf{x}, \dot{\mathbf{x}})$-motion active at r*. By the definition of effector states in (7.208), the configuration $(\chi_{(r,0)}, \dot{\chi}_{(r,0)})$ is in state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(E)$. Figure 8.1 illustrates such a system.

	e_1	e_2	e_3	G
Position of subbodies	1	0	0	$\mathbf{x}$
Velocity of subbodies	0	0	0	$\dot{\mathbf{x}}$
Acceleration of subbodies	1	0	1	0

Figure 8.2: Effector at time $(r, 1)$, with state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$.

Given the force variation described in (8.48), at instant $(r, 1)$,

$$\begin{aligned} f_{(r,1)}(e_1, e_3) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,1)}(e_1, \overline{e_1 \sqcup e_3}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,1)}(e_3, e_1) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,1)}(e_3, \overline{e_1 \sqcup e_3}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}). \end{aligned} \tag{8.51}$$

Summing the first two and last two lines, at $(r, 1)$, the forces of the exteriors of e_1, e_3 on these bodies is

$$\begin{aligned} f_{(r,1)}(e_1, \overline{e_1}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,1)}(e_3, \overline{e_3}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}). \end{aligned} \tag{8.52}$$

Since effector body points have unit controller mass, this implies that

$$\ddot{\chi}_\mathrm{c}(e_1, (r, 1)) = \ddot{\chi}_\mathrm{c}(e_3, (r, 1)) = 1. \tag{8.53}$$

Recall from Section 4.1.4 that discrete acceleration is leading, while discrete velocity is trailing. Thus, at $(r, 1)$, the two effector body points have a acceleration of 1 in $\mathcal{S}_\mathrm{c}$, but both still have a velocity of 0 in $\mathcal{S}_\mathrm{c}$ and have unchanged positions. The result is illustrated in Figure 8.2. Note that, per the definition of $\Psi(E)$ in (7.208), it is still true that $(\chi_{(r,1)}, \dot{\chi}_{(r,1)})$ is in state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$ of E, since only e_1 has unequal position and velocity.

In the subsequent instant, $(r, 2)$, several things change. First, as a result of their acceleration in the previous instant, both e_1 and e_3 have a velocity of 1 in $\mathcal{S}_\mathrm{c}$. Second, as a result of their new velocities, the two bodies change positions, with e_1 moving from 1 to 0 in $\mathcal{S}_\mathrm{c}$ and e_3 moving from 0 to 1 in $\mathcal{S}_\mathrm{c}$. Finally, both bodies again experience unit-magnitude forces in $\mathbb{Z}_2$, and so both bodies again have an acceleration of 1 in $\mathcal{S}_\mathrm{c}$. The result is summarized in Figure 8.3. Again, per the definition of $\Psi(E)$ in (7.208), it is still true that $(\chi_{(r,2)}, \dot{\chi}_{(r,2)})$ is in state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$ of E, since only e_1 has an unequal position and velocity.

Finally, in instant $(r, 3)$, there are no nonzero forces on the effector bodies. All accelerations are

	e_1	e_2	e_3	G
Position of subbodies	0	0	1	$\mathbf{x}$
Velocity of subbodies	1	0	1	$\dot{\mathbf{x}}$
Acceleration of subbodies	1	0	1	0

Figure 8.3: Effector at time $(r, 2)$, with state $\psi_{1,\mathbf{x},\dot{\mathbf{x}}}$.

	e_1	e_2	e_3	G
Position of subbodies	0	0	1	$\mathbf{x}$
Velocity of subbodies	0	0	0	$\dot{\mathbf{x}}$
Acceleration of subbodies	0	0	0	0

Figure 8.4: Effector at time $(r, 3)$, with state $\psi_{3,\mathbf{x},\dot{\mathbf{x}}}$.

therefore 0; as a result of the acceleration in the previous instant, all velocities also return to 0. Thus, all positions remain as they were in the previous instant, with $\chi_c(e_1, (r, 3)) = 0$ and $\chi_c(e_3, (r, 3)) = 1$. By the definition of $\Psi(E)$ in (7.208), then, $(\chi_{(r,3)}, \dot{\chi}_{(r,3)})$ is in state $\psi_{3,\mathbf{x},\dot{\mathbf{x}}}$ of E, since only e_3 has unequal position and velocity. This result is summarized in Figure 8.4.

More generally, then, the effect of the force variation described in (8.48) is that $(\chi_{(r,z)}, \dot{\chi}_{(r,z)})$ is in ψ_v for $z \in \{0, 1, 2\}$, while $(\chi_{(r,3)}, \dot{\chi}_{(r,3)})$ is in $\psi_{v'}$. In other words, $\hat{f}$ is a force variation that causes E to change from a substate of ψ_v at $(r, 0)$ to a substate of $\psi_{v'}$ at $(r, 3)$. I show this result more formally in the following theorem.

Theorem 24. *Let $\mathbf{x} \in X$ be an automaton environment state, $\dot{\mathbf{x}} \in X$ an environment state derivative, and let $v, v' \in V$ be a pair of automaton controller states. For some $r \in \mathbb{R}^{\geq 0}$, let $\hat{f}_{v\text{–}v',r}$ be the v–v' transition force variation active at r, let χ be a $(v, \mathbf{x}, \dot{\mathbf{x}})$-motion active at r, and let* `state` *be the state-time function for χ. Then*

$$\texttt{state}(E, (r, i)) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \tag{8.54}$$

for $i = \{0, 1, 2\}$, and

$$\texttt{state}(E, (r, 3)) = \psi_{v',\mathbf{x},\dot{\mathbf{x}}}. \tag{8.55}$$

Proof. By (8.50), since χ is a $(v, \mathbf{x}, \dot{\mathbf{x}})$-motion active at r,

$$\begin{aligned} \chi_p(G, r) &= \mathbf{x} \qquad (8.56) \\ \dot{\chi}_p(G, r) &= \dot{\mathbf{x}} \end{aligned}$$

$$\chi_{\mathrm{c}}(e_v, (r, 0)) = 1$$
$$\forall v' \neq v \in V, \chi_{\mathrm{c}}(e_{v'}, (r, 0)) = 0$$
$$\forall v' \in V, \dot{\chi}_{\mathrm{c}}(e_{v'}, (r, 0)) = 0.$$

By the definition of the states of the effector, $\mathtt{state}(E, (r, 0)) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Instant $(r, 1)$**:** By the definition of discrete velocity and acceleration in Section 4.1.4, for any $b \in \Omega$ and instant $(r, z) \in \Gamma_M$ with $z \geq 1$,

$$\dot{\chi}(b, (r, z)) = \chi(b, (r, z)) - \chi(b, (r, z - 1)). \tag{8.57}$$

Likewise, if $z \leq 2$,

$$\ddot{\chi}(b, (r, z)) = \dot{\chi}(b, (r, z + 1)) - \dot{\chi}(b, (r, z)). \tag{8.58}$$

Restricting consideration to the placement of e_v in controller space, this gives

$$\dot{\chi}_{\mathrm{c}}(e_v, (r, 1)) = \chi_{\mathrm{c}}(e_v, (r, 1)) - \chi_{\mathrm{c}}(e_v, (r, 0)) \tag{8.59}$$

$$\ddot{\chi}_{\mathrm{c}}(e_v, (r, 0)) = \dot{\chi}_{\mathrm{c}}(e_v, (r, 1)) - \dot{\chi}_{\mathrm{c}}(e_v, (r, 0)). \tag{8.60}$$

As noted in (8.56),

$$\chi_{\mathrm{c}}(e_v, (r, 0)) = 1 \tag{8.61}$$
$$\dot{\chi}_{\mathrm{c}}(e_v, (r, 0)) = 0.$$

By the construction of $\hat{f}$,

$$f_{(r,0)}(e_v, \overline{e_v}) = (\mathbf{0}_{\mathrm{p}}, \mathbf{0}_{\mathrm{c}}), \tag{8.62}$$

and so the controller-space acceleration of e_v is likewise 0,

$$\ddot{\chi}_{\mathrm{c}}(e_v, (r, 0)) = 0. \tag{8.63}$$

Substituting into (8.60) to solve for $\dot{\chi}_{\mathrm{c}}(e_v, (r, 1))$ gives

$$\ddot{\chi}_{\mathrm{c}}(e_v, (r, 0)) = \dot{\chi}_{\mathrm{c}}(e_v, (r, 1)) - \dot{\chi}_{\mathrm{c}}(e_v, (r, 0)) \tag{8.64}$$
$$0 = \dot{\chi}_{\mathrm{c}}(e_v, (r, 1)) - 0$$
$$0 = \dot{\chi}_{\mathrm{c}}(e_v, (r, 1)).$$

Now substituting into (8.59) to solve for $\chi_{\mathrm{c}}(e_v, (r, 1))$ gives

$$\dot{\chi}_{\mathrm{c}}(e_v, (r, 1)) = \chi_{\mathrm{c}}(e_v, (r, 1)) - \chi_{\mathrm{c}}(e_v, (r, 0)) \tag{8.65}$$

$$
\begin{aligned}
0 &= \chi_{\mathrm{c}}(e_v, (r,1)) - 1 \\
1 &= \chi_{\mathrm{c}}(e_v, (r,1)).
\end{aligned}
$$

By a similar argument,

$$\dot{\chi}_{\mathrm{c}}(e_{v'}, (r,1)) = 0 \tag{8.66}$$

and

$$\chi_{\mathrm{c}}(e_{v'}, (r,1)) = 0. \tag{8.67}$$

Summarizing,

$$
\begin{aligned}
\chi_{\mathrm{c}}(e_v, (r,1)) &= 1 \\
\dot{\chi}_{\mathrm{c}}(e_v, (r,1)) &= 0 \\
\ddot{\chi}_{\mathrm{c}}(e_v, (r,1)) &= 1 \\
\chi_{\mathrm{c}}(e_{v'}, (r,1)) &= 0 \\
\dot{\chi}_{\mathrm{c}}(e_{v'}, (r,1)) &= 0 \\
\ddot{\chi}_{\mathrm{c}}(e_{v'}, (r,1)) &= 1.
\end{aligned} \tag{8.68}
$$

In other words, at instant $(r,1)$, both e_v and $e_{v'}$ are still in their original positions, with velocity 0 in $\mathcal{S}_{\mathrm{c}}$. Thus, by the definition of effector states in (7.208), $\mathtt{state}(E, (r,1)) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Instant $(r,2)$**:** Since χ is an induced motion, at every instant $t_i \in \Gamma_M$, for any controller body point c,

$$\hat{f}_{\mathrm{c}}(t_i)(c, \overline{c}) = m_{\mathrm{c}}(c)\ddot{\chi}_{\mathrm{c}}(c, t_i). \tag{8.69}$$

Letting $c = e_v$, at instant $(r,1)$,

$$\hat{f}_{\mathrm{c}}(r,1)(e_v, \overline{e_v}) = \hat{f}_{\mathrm{c}}(r,1)(e_v, e_{v'}) = \mathbf{1}_{\mathrm{c}}, \tag{8.70}$$

which implies

$$m_{\mathrm{c}}(e_v)\ddot{\chi}_{\mathrm{c}}(e_v, (r,1)) = \mathbf{1}_{\mathrm{c}}. \tag{8.71}$$

By (7.200), $m_{\mathrm{c}}(e_\psi) = 1_{\mathrm{c}}$, and so the mass term can be dropped, giving

$$\ddot{\chi}_{\mathrm{c}}(e_v, (r,1)) = 1. \tag{8.72}$$

By the definition of discrete acceleration above,

$$\ddot{\chi}_{\mathrm{c}}(e_v, (r,1)) = \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) - \dot{\chi}_{\mathrm{c}}(e_v, (r,1)). \tag{8.73}$$

Substituting 1 for $\ddot{\chi}_{\mathrm{c}}(e_v, (r,1))$ in this expression gives

$$1 = \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) - \dot{\chi}_{\mathrm{c}}(e_v, (r,1)). \tag{8.74}$$

Substituting for $\dot{\chi}_{\mathrm{c}}(e_v, (r,1))$ via (8.64) gives

$$\begin{aligned} \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) - \dot{\chi}_{\mathrm{c}}(e_v, (r,1)) &= 1 \\ \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) - 0 &= 1 \\ \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) &= 1. \end{aligned} \tag{8.75}$$

Substituting (8.67) and (8.75) into the definition of discrete velocity gives

$$\begin{aligned} \chi_{\mathrm{c}}(e_v, (r,2)) - \chi_{\mathrm{c}}(e_v, (r,1)) &= \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) \\ \chi_{\mathrm{c}}(e_v, (r,2)) - 1 &= 1 \\ \chi_{\mathrm{c}}(e_v, (r,2)) &= 0. \end{aligned} \tag{8.76}$$

By a similar argument,

$$\dot{\chi}_{\mathrm{c}}(e_{v''}, (r,2)) = 1. \tag{8.77}$$

and

$$\chi_{\mathrm{c}}(e_{v''}, (r,2)) = 1. \tag{8.78}$$

Summarizing,

$$\begin{aligned} \chi_{\mathrm{c}}(e_v, (r,2)) &= 0 \\ \dot{\chi}_{\mathrm{c}}(e_v, (r,2)) &= 1 \\ \ddot{\chi}_{\mathrm{c}}(e_v, (r,2)) &= 1 \\ \chi_{\mathrm{c}}(e_{v'}, (r,2)) &= 1 \\ \dot{\chi}_{\mathrm{c}}(e_{v'}, (r,2)) &= 1 \\ \ddot{\chi}_{\mathrm{c}}(e_{v'}, (r,2)) &= 1. \end{aligned} \tag{8.79}$$

In other words, at instant $(r,2)$, e_v and $e_{v'}$ have changed places relative to their positions at $(r,1)$. Both now have a velocity of 1 in $\mathcal{S}_{\mathrm{c}}$. By the definition of effector states in (7.208), $\texttt{state}(E, (r,2)) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Instant $(r,3)$**:** A construction functionally identical to the above shows that the additional forces at $(r,2)$ serve to modify both subbodies so that

$$\dot{\chi}_{\mathrm{c}}(e_v, (r,3)) = \dot{\chi}_{\mathrm{c}}(e_{v''}, (r,3)) = 0, \tag{8.80}$$

while

$$\begin{aligned}\chi_{\mathrm{c}}(e_v,(r,3)) &= 0 \\ \chi_{\mathrm{c}}(e_{v'},(r,3)) &= 1.\end{aligned} \tag{8.81}$$

Thus, by the definition of effector states, $\mathtt{state}(E,(r,3)) = \psi_{v',\mathbf{x},\dot{\mathbf{x}}}$. Since all bodies are once more at rest and there are no nonzero forces at any later instants, $\mathtt{state}(E,(r,3)) = \psi_{v''}$. $\square$

I define the conditional force variation inclusion $\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]$ to contain all v–v' transition force variations active at any $r \in \mathbb{R}^{\geq 0}$ such that the automaton transition relation contains $(v,\mathbf{x}) \rightarrowtail (v',\mathbf{x})$. Such force variations allow the effector to shift from a substate of ψ_v to a substate of $\psi_{v'}$ during any physical instant. As the original automaton may also perform none of its possible transitions, remaining instead in its original automaton controller state, $\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]$ also contains the passive force variation. Formally, for each automaton controller state $v \in V$ and equivalence class $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$,

$$\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}] \stackrel{\text{def}}{=} \left\{ \hat{f}_{v\text{–}v',r} \;\middle|\; \left[v' \in \mathcal{D}_v(\mathbf{x})\right] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}_{\mathbf{0}} \right\}. \tag{8.82}$$

The above covers all mechanical controller states of s_v except for $\overline{\psi}_v$. Note that this mechanical controller state corresponds to no environment state. In practical terms, s_v is in $\overline{\psi}_v$ whenever some other sensor point body $s_{v'}$ is in some mechanical controller state $\psi_{v',[\![\mathbf{x}']\!]}$. In such cases, the other sensor point body should determine the forces on E; thus, the forces conditional on s_v should be passive. Formally, for each automaton controller state $v \in V$,

$$\hat{\mathfrak{f}}_{s,v}[\overline{\psi}_v] \stackrel{\text{def}}{=} \{\hat{f}_{\mathbf{0}}\}. \tag{8.83}$$

Summarizing the preceding results, then, for any automaton controller state $v \in V$ and mechanical controller state $\psi \in \Psi(s_v)$,

$$\hat{\mathfrak{f}}_{s,v}[\psi] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \;\middle|\; [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_v. \end{cases} \tag{8.84}$$

8.2.2 Hybrid forces of the sensor

As noted in (8.5), for any mechanical controller state $\psi \in \Psi(S)$, the force variation inclusion for the sensor conditional on ψ is the set of all sums of one force variations from the inclusion for each effector point body, conditional on the state-projection of ψ onto that body. In other words, for any state $\psi \in \Psi(S)$,

$$\hat{\mathfrak{f}}_S[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \;\middle|\; \hat{f}_v \in \hat{\mathfrak{f}}_{s,v}[s_v(\psi)] \right\}. \tag{8.85}$$

This sum produces the force variation inclusion

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{8.86}$$

as shown in the following theorem.

Theorem 25. *For any mechanical controller state $\psi \in \Psi(S)$, the force variation inclusion for the effector is*

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{8.87}$$

Proof. Recall from (7.208) that

$$\Psi(S) = \left\{ \psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, (v \in V) \wedge \left([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right) \right\} \cup \{\overline{\psi}_V\}. \tag{8.88}$$

For any choice of $v \in V$ and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$, the mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]} \in \Psi(S)$ is also a state of s_v. Clearly this mechanical controller state is subset of itself, and so its projection onto s_v is the identity,

$$s_v(\psi_{v,[\![\mathbf{x}]\!]}) = \psi_{v,[\![\mathbf{x}]\!]}. \tag{8.89}$$

By definition, $\psi_{v,[\![\mathbf{x}]\!]}$ is a subset of ψ_v. But by Lemma 3, for any $v, v' \in V$ such that $v \neq v'$, the sets ψ_v and $\psi_{v'}$ are disjoint. It follows that, for any sensor point body $s_{v'}$ such that $v' \neq v$, the mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]}$ is disjoint from any $\psi_{v',[\![\mathbf{x}']\!]} \in \Psi(s_{v'})$. Then $\psi_{v,[\![\mathbf{x}]\!]}$ is a subset of the only remaining mechanical controller state of $s_{v'}$, that is, $\overline{\psi}_{v'}$. Then

$$s_{v'}(\psi_{v,[\![\mathbf{x}]\!]}) = \overline{\psi}_{v'}. \tag{8.90}$$

By (8.4),

$$\hat{\mathfrak{f}}_S[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{s,v}[s_v(\psi)] \right\}. \tag{8.91}$$

Then for any mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]} \in \Psi(S)$,

$$\hat{\mathfrak{f}}_E[\psi_{v,[\![\mathbf{x}]\!]}] = \left\{ \hat{f} + \sum_{v' \neq v \in V} \hat{f}_{v'} \,\middle|\, \left(\hat{f} \in \hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]\right) \wedge \left(\hat{f}_{v'} \in \hat{\mathfrak{f}}_{s,v'}[\overline{\psi}_{v'}]\right) \right\}. \tag{8.92}$$

By (8.28), $\hat{\mathfrak{f}}_{s,v'}[\overline{\psi}_{v'}] = \{\hat{f}_{\mathbf{0}}\}$ for all $v' \in V$. Thus, the above expression reduces to

$$\hat{\mathfrak{f}}_S[\psi_{v,[\![\mathbf{x}]\!]}] = \left\{ \hat{f} + \sum_{v' \neq v \in V} \hat{f}_{\mathbf{0}} \,\middle|\, \hat{f} \in \hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}] \right\}. \tag{8.93}$$

Since $\hat{f}_{\mathbf{0}}$ contains no nonzero forces, for any $\hat{f} \in \hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]$,

$$\hat{f} + \sum_{v' \neq v \in V} \hat{f}_{\mathbf{0}} = \hat{f}. \tag{8.94}$$

It follows that

$$\begin{aligned} \hat{\mathfrak{f}}_S[\psi_{v,[\![\mathbf{x}]\!]}] &= \left\{ \hat{f} \,\middle|\, \hat{f} \in \hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}] \right\} \\ &= \hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}]. \end{aligned} \tag{8.95}$$

Then by the definition of $\hat{\mathfrak{f}}_{s,v}[\cdot]$ in (8.84),

$$\hat{\mathfrak{f}}_S[\psi_{v,[\![\mathbf{x}]\!]}] = \left\{ \hat{f}_{v \to v',r} \,\middle|\, \left[v' \in \mathcal{D}_v(\mathbf{x})\right] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}. \tag{8.96}$$

The above accounts for all mechanical controller states in $\Psi(S)$ except for $\overline{\psi}_V$. Consider $\overline{\psi}_V$; by (7.207),

$$\overline{\psi}_V = \bigcap_{v \in V} \overline{\psi}_v. \tag{8.97}$$

It follows that, for any sensor point body s_v,

$$s_v(\overline{\psi}_V) = \overline{\psi}_v. \tag{8.98}$$

Again, by (8.5),

$$\hat{\mathfrak{f}}_S[\psi] = \left\{ \sum_{v \in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{s,v}[s_v(\psi)] \right\}, \tag{8.99}$$

and so, letting $\psi = \overline{\psi}_V$,

$$\hat{\mathfrak{f}}_S[\overline{\psi}_V] = \left\{ \sum_{v \in V} \hat{f}_v \,\middle|\, \hat{f}_v \in \hat{\mathfrak{f}}_{s,v}[\overline{\psi}_v] \right\}. \tag{8.100}$$

By (8.84), $\hat{\mathfrak{f}}_{s,v}[\overline{\psi}_v] = \{\hat{f}_{\mathbf{0}}\}$ for all $v \in V$. Thus, the above expression reduces to

$$\hat{\mathfrak{f}}_S[\overline{\psi}_V] = \left\{ \sum_{v \in V} \hat{f}_{\mathbf{0}} \right\} \tag{8.101}$$

$$= \{\hat{f}_{\mathbf{0}}\}.$$

Then for any mechanical controller state $\psi \in \Psi(S)$,

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{ \hat{f}_{v \multimap v',r} \,\middle|\, \left[v' \in \mathcal{D}_v(\mathbf{x})\right] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{8.102}$$

□

8.3 Combined hybrid forces

The combined hybrid force variation inclusion is then the set of all sums of one force variation from appropriate inclusions of the effector and the sensor. As stated in (8.6), then, for any mechanical controller state $\psi \in \Psi(\top_c)$,

$$\hat{\mathfrak{f}}[\psi] = \left\{ \hat{f}_E + \hat{f}_S \,\middle|\, \left(\hat{f}_E \in \hat{\mathfrak{f}}_E[E(\psi)]\right) \wedge \left(\hat{f}_S \in \hat{\mathfrak{f}}_S[S(\psi)]\right) \right\}. \tag{8.103}$$

It follows that

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^\tau + \hat{f}_{v \multimap v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{8.104}$$

as shown in the following theorem.

Theorem 26. *For any state $\psi \in \Psi(\top_c)$, the force variation inclusion for the universal controller body is*

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^\tau + \hat{f}_{v \multimap v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{8.105}$$

Proof. Recall from (7.208) that

$$\begin{aligned} \Psi(\top_c) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\!\equiv_G)\} \cup \{\overline{\psi}_V\} \\ \Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\!\equiv_G)\} \cup \{\overline{\psi}_V\} \\ \Psi(S) &= \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, (v \in V) \wedge \left([\![\mathbf{x}]\!] \in X/\!\equiv_{\mathcal{D},v}\right)\right\} \cup \{\overline{\psi}_V\}. \end{aligned} \tag{8.106}$$

Note that any mechanical controller state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(\top_c)$ is also in $\Psi(E)$; it follows that its state-projection onto the effector is the identity,

$$E(\psi_{v,\mathbf{x},\dot{\mathbf{x}}}) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{8.107}$$

By Lemma 6, $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ is a subset of $\psi_{[\![\mathbf{x}]\!]}$; it follows that

$$\psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_v \cap \psi_{[\![\mathbf{x}]\!]}, \tag{8.108}$$

and so

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \subseteq \psi_{v,[\![\mathbf{x}]\!]}, \tag{8.109}$$

where $\psi_{v,[\![\mathbf{x}]\!]}$ is in $\Psi(S)$. Then

$$S(\psi_{v,\mathbf{x},\dot{\mathbf{x}}}) = \psi_{v,[\![\mathbf{x}]\!]}. \tag{8.110}$$

Then for any $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(\top_{\mathrm{c}})$,

$$\hat{\mathfrak{f}}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] = \left\{ \hat{f}_E + \hat{f}_S \,\middle|\, \left(\hat{f}_E \in \hat{\mathfrak{f}}_E[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]\right) \wedge \left(\hat{f}_S \in \hat{\mathfrak{f}}_S[\psi_{v,[\![\mathbf{x}]\!]}]\right) \right\}. \tag{8.111}$$

By Theorem 23, $\hat{f}_E$ is some $\hat{f}^\tau$, for some $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$. By Theorem 25, either $\hat{f}_S$ is some $\hat{f}_{v-v',r}$ or it is $\hat{f}_{\mathbf{0}}$. Then, substituting

$$\hat{\mathfrak{f}}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}] = \left\{ \hat{f}^\tau + \hat{f}_{v-v',r} \,\middle|\, \left[\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right] \wedge \left[v' \in \mathcal{D}_v(\mathbf{x})\right] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}. \tag{8.112}$$

The above covers all mechanical controller states in $\Psi(\top_{\mathrm{c}})$ except for $\overline{\psi}_V$. Note that $\overline{\psi}_V$ is also a member of both $\Psi(E)$ and $\Psi(S)$, and so

$$\begin{aligned} E(\overline{\psi}_V) &= \overline{\psi}_V \\ S(\overline{\psi}_V) &= \overline{\psi}_V. \end{aligned} \tag{8.113}$$

It follows that

$$\hat{\mathfrak{f}}[\overline{\psi}_V] = \left\{ \hat{f}_E + \hat{f}_S \,\middle|\, \left(\hat{f}_E \in \hat{\mathfrak{f}}_E[\overline{\psi}_V]\right) \wedge \left(\hat{f}_S \in \hat{\mathfrak{f}}_S[\overline{\psi}_V]\right) \right\}. \tag{8.114}$$

But $\hat{\mathfrak{f}}_E[\overline{\psi}_V] = \hat{\mathfrak{f}}_S[\overline{\psi}_V] = \{\hat{f}_{\mathbf{0}}\}$, and so

$$\begin{aligned} \hat{\mathfrak{f}}[\overline{\psi}_V] &= \{\hat{f}_{\mathbf{0}} + \hat{f}_{\mathbf{0}}\} \\ &= \{\hat{f}_{\mathbf{0}}\}. \end{aligned} \tag{8.115}$$

Then for any state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} + \hat{f}_{v \to v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{8.116}$$

□

8.4 Mechanical system summary

I can finally summarize all components of the hybrid mechanical system. Given a hybrid automaton $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$ such that

$$X = \mathbb{R}^n, \tag{8.117}$$

there exists a physical mechanical system such that

$$\begin{aligned} \Omega_{\mathrm{p}} &= \{\top_{\mathrm{p}}, G, N, \bot_{\mathrm{p}}\} && (8.118) \\ \mathcal{S}_{\mathrm{p}} &= X = \mathbb{R}^n \\ \Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\ \mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\ m_{\mathrm{p}}(G) &= 1_{\mathrm{p}} \\ \mathcal{C}(G, \mathcal{S}_{\mathrm{p}}) &\cong X = \mathbb{R}^n \\ \mathfrak{X}(G, \mathcal{S}_{\mathrm{p}}) &\cong X \times X = \mathbb{R}^{2n} \\ \mathcal{V}_{\mathrm{p}} &= X = \mathbb{R}^n && (8.119) \end{aligned}$$

and a controller mechanical system such that

$$\begin{aligned} E &= \{e_v \mid v \in V\} && (8.120) \\ \forall v, v' \in V[(v \neq v') &\to (e_v \neq e_{v'})] \\ S &= \{s_v \mid v \in V\} && (8.121) \\ \forall v, v' \in V[(v \neq v') &\to (s_v \neq s_{v'})] \\ E \sqcap S &= \bot_{\mathrm{c}} \\ \top_{\mathrm{c}} &= E \sqcup S \\ \Omega_{\mathrm{c}} &= 2^{E \sqcup S} \\ \mathcal{S}_{\mathrm{c}} &= \mathbb{Z}_2 \\ \Gamma_{\mathrm{c}} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_{\mathrm{c}} &= \mathbb{Z}_2 \\
\forall v \in V, m_{\mathrm{c}}(e_v) &= 1_{\mathrm{c}} \\
\forall v \in V, m_{\mathrm{c}}(s_v) &= 1_{\mathrm{c}} \\
\mathcal{C}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{2|V|} \\
\mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &\cong \mathbb{Z}_2^{4|V|} \\
\mathcal{V}_{\mathrm{c}} &= \mathbb{Z}_2.
\end{aligned}
$$

This provides a hybrid mechanical system such that

$$
\begin{aligned}
\Omega &= \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}} \\
\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} &= \bot \\
\mathcal{S} &= \mathcal{S}_{\mathrm{p}} \oplus \mathcal{S}_{\mathrm{c}} \\
\Gamma_M &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \\
\hat{t}_{\mathrm{p}}(r, z) &= r \\
\hat{t}_{\mathrm{c}}(r, z) &= (r, z) \\
\mathcal{M} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_2 \\
m(G) &= (1_{\mathrm{p}}, 0_{\mathrm{c}}) \\
\forall v \in V, m(e_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}}) \\
\forall v \in V, m(s_v) &= (0_{\mathrm{p}}, 1_{\mathrm{c}}) \\
\mathcal{C}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X \times \mathbb{Z}_2^{2|V|} = \mathbb{R}^n \times \mathbb{Z}_2^{|V|} \\
\mathfrak{X}(G \sqcup \top_{\mathrm{c}}, \mathcal{S}) &\cong X^2 \times \mathbb{Z}_2^{4|V|} = \mathbb{R}^{2n} \times \mathbb{Z}_2^{4|V|} \\
\mathcal{V} &= X \times \mathbb{Z}_2^{2|V|} = \mathbb{R}^n \times \mathbb{Z}_2^{2|V|}.
\end{aligned}
\tag{8.122}
$$

The initial configuration set $\mathfrak{X}_0(\top, \mathcal{S})$ is the set

$$
\mathfrak{X}_0(\top, \mathcal{S}) = \{\texttt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in dom(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{8.123}
$$

where $\texttt{Mech}(\tau, t)$, for any $\tau \in \mathcal{T}$ and $t \in dom(\tau)$, is the configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that

$$
\begin{aligned}
w_{\mathrm{p}}(G) &= \Pi_X(\tau(t)) \\
\dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}(t)) \\
w_{\mathrm{c}}(e_v) &= 1 \\
\forall v' \neq v \in V \big[w_{\mathrm{c}}(e_{v'}) &= 0 \big] \\
\forall v' \in V \big[\dot{w}_{\mathrm{c}}(e_{v'}) &= 0 \big]
\end{aligned}
\tag{8.124}
$$

$$\forall v' \in V\big[w_{\mathrm{c}}(s_{v'}) = 0\big]$$
$$\forall v' \in V\big[\dot{w}_{\mathrm{c}}(s_{v'}) = 0\big].$$

The equivalence class $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$ is defined by the relation

$$(w, \dot{w}) \equiv_G (w', \dot{w}') \text{ iff } \big[(w_{\mathrm{p}}(G) = w'_{\mathrm{p}}(G)) \wedge (\dot{w}_{\mathrm{p}}(G) = \dot{w}'_{\mathrm{p}}(G))\big], \tag{8.125}$$

and the equivalence class $X/\equiv_{\mathcal{D},v}$ is defined by the relation

$$\mathbf{x} \equiv_{\mathcal{D},v} \mathbf{x}' \text{ iff } \mathcal{D}_v(\mathbf{x}) = \mathcal{D}_v(\mathbf{x}'). \tag{8.126}$$

Then for each $v \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$, I have defined subsets of $\mathfrak{X}(\top, \mathcal{S})$ as follows:

$$\begin{aligned}
\psi_v &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_v) \neq \dot{w}(e_v)\big) \wedge \big(\forall v' {\neq} v {\in} V[w(e_{v'}) = \dot{w}(e_{v'})]\big)\} \qquad (8.127)\\
\overline{\psi}_v &= \mathfrak{X}(\top, \mathcal{S}) - \psi_v\\
\overline{\psi}_V &= \bigcap_{v \in V} \overline{\psi}_v\\
\psi_{v,\mathbf{x},\dot{\mathbf{x}}} &= \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}}\\
\psi_{[\![\mathbf{x}]\!]} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\}\\
\psi_{v,[\![\mathbf{x}]\!]} &= \psi_v \cap \psi_{[\![\mathbf{x}]\!]}.
\end{aligned}$$

Then the states of the various controllers are:

$$\begin{aligned}
\forall v \in V\big[\Psi(e_v) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}\big] \qquad (8.128)\\
\Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\}\\
\forall v \in V\big[\Psi(s_v) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_v\}\big]\\
\Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\ \big)\} \cup \{\overline{\psi}_V\}\\
\Psi(\top_{\mathrm{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\ \big)\} \cup \{\overline{\psi}_V\}.
\end{aligned}$$

I write $\hat{f}^{\tau}$ to indicate a force variation such that, for some trajectory $\tau \in \mathcal{T}$, the following conditions hold:

- The domain of $\hat{f}$ is all instants in hybrid mechanical time; that is,

$$dom(\hat{f}) = \Gamma_M, \tag{8.129}$$

- τ has a constant automaton controller state of v; that is,

$$\forall t \in dom(\tau)\left[\Pi_V(\tau(t)) = v\right], \tag{8.130}$$

- At automaton instant 0, the environment state component of τ is $\mathbf{x}$, while the derivative of the environment state component of τ is $\dot{\mathbf{x}}$; that is,

$$(\Pi_X(\tau.fstate) = \mathbf{x}) \wedge (\Pi_X(\dot{\tau}.fstate) = \dot{\mathbf{x}}), \tag{8.131}$$

- For all instants $t \in dom(\hat{f})$, the force of N on G is isomorphic to the second derivative of the environment state component of τ at 0 times $\mathbf{0}_\mathrm{c}$; that is,

$$\forall t \in dom(\hat{f})\left[\hat{f}(t)(G, N) = (\Pi_X(\ddot{\tau}.fstate), \mathbf{0}_\mathrm{c})\right], \tag{8.132}$$

  and

- Except as required by the above, $\hat{f} = \hat{f}_\mathbf{0}$.

For any $v, v' \in V$ and $r \in \mathbb{R}^{\geq 0}$, I write $\hat{f}_{v\text{--}v',r}$ for the force variation such that

$$\begin{aligned} f_{(r,0)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,0)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,1)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,1)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,2)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,2)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\ f_{(r,3)}(e_v, e_{v'}) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\ f_{(r,3)}(e_{v'}, e_v) &= (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}), \end{aligned} \tag{8.133}$$

and except as required by the above,

$$\hat{f}_{v\text{--}v',r} = \hat{f}_\mathbf{0}. \tag{8.134}$$

The conditional force variation inclusions for the various controller bodies are then

$$\forall v \in V\left[\hat{\mathfrak{f}}_{e,v}[\psi] = \begin{cases} \left\{\hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_v \end{cases}\right] \tag{8.135}$$

$$\forall v \in V\left[\hat{\mathfrak{f}}_{s,v}[\psi] = \begin{cases} \left\{\hat{f}_{v\text{--}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right]\right\} \cup \{\hat{f}_\mathbf{0}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_v \end{cases}\right]$$

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{\hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_V \end{cases}$$

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{\hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right]\right\} \cup \{\hat{f}_\mathbf{0}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_V \end{cases}$$

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{\hat{f}^\tau + \hat{f}_{v\text{–}v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right]\right\} \cup \left\{\hat{f}^\tau \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_V. \end{cases}$$

Combining the discussions of the previous pages, then, the hybrid mechanical system can be defined by the tuple $M = (\Omega, \mathcal{S}, \Gamma_M, m, \Psi, \hat{\mathfrak{f}}[\cdot], \mathfrak{X}_0(\top, \mathcal{S}))$, where Ω is the hybrid universe, Γ_M is hybrid time, $\mathfrak{X}_0(\top, \mathcal{S})$ is the set of initial hybrid configurations, m is the hybrid mass function, Ψ is an assignment of mechanical controller states to every mechanical controller, and $\mathfrak{f}[\cdot]$ is the conditional force variation inclusion over $\Psi(\top_\mathrm{c})$. I refer to the hybrid mechanical system constructed from a hybrid automaton A according to the above rules as the *hybrid mechanical counterpart* to A.

8.5 Summary of notation

$\hat{\mathfrak{f}}_{e,v}[\cdot]$	conditional force variation inclusion, conditional on state of effector point body e_v
$\hat{\mathfrak{f}}_{s,v}[\cdot]$	conditional force variation inclusion, conditional on state of sensor point body s_v
$\hat{\mathfrak{f}}_E[\cdot]$	conditional force variation inclusion, conditional on state of effector
$\hat{\mathfrak{f}}_S[\cdot]$	conditional force variation inclusion, conditional on state of sensor
$\hat{f}_\mathbf{0}$	passive force variation
$\hat{f}^\tau$	the force variation for trajectory τ
$\mathcal{T}_v$	set of all trajectories in $\mathcal{T}$ with constant automaton controller state v
$\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$	set of all trajectories in $\mathcal{T}$ with constant controller state v, environment state projection $\mathbf{x}$, and environment state projection derivative $\dot{\mathbf{x}}$
$f_{v\text{–}v',r}$	v–v' transition force variation active at r

Chapter 9

Transformation Examples

To this point, the constructions provided have been largely abstract and detached from any practical examples. In this chapter, I provide concrete examples of the translation from hybrid automata to hybrid mechanical systems. I do so via two examples: the mechanical elevator discussed in earlier chapters, and a classic hybrid automaton example, the inverted pendulum.

9.1 The mechanical elevator

Consider first the mechanical elevator presented in Section 3.4. For convenience, I repeat the description of that problem:

The elevator is either on or off; while on, its height continuously rises to some threshold (say, 75m), at which point it switches off. The platform then drops steadily until it reaches some lower bound (say, 45m), at which point the elevator switches on again. The elevator is initially on, with a starting height somewhere between 50m and 60m. While the elevator is on, the platform rises at a rate in the range of 0m to 4m per unit of time; while it is off, the platform drops at a rate in the range of -3m to 0m per unit of time.

9.1.1 Hybrid automaton representation

Rather than rework the full description presented in the earlier discussion of the elevator, I here summarize the results of the Lynch-style construction in Section . In particular, I represent the Lynch formulation of the automaton via the tuple $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$, where

$$\begin{aligned} V &= \{\mathit{On}, \mathit{Off}\} \\ X &= [0, \infty) \\ Q_0 &= \{\mathit{On}\} \times [50, 60] \end{aligned} \tag{9.1}$$

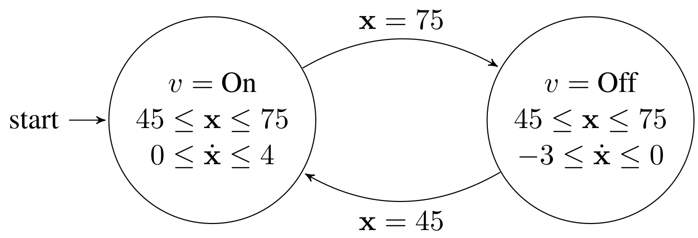


Figure 9.1: A hybrid automaton model for a mechanical elevator.

$$
\begin{aligned}
&D = \{(On, 75) \rightarrowtail (Off, 75), (Off, 45) \rightarrowtail (On, 45)\} \\
&\mathcal{T} = \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is second-degree doubly semidifferentiable}\big) \\
&\quad \wedge\ \forall t \in dom(\tau)\big(45 \leq \Pi_X(\tau)(t) \leq 75\big) \\
&\quad \wedge\ \Big[\big([\tau_V(t) = On] \wedge [0 \leq \Pi_X(\dot{\tau}(t)) \leq 4]\big) \vee \big([\Pi_V(\tau(t)) = Off] \wedge [-3 \leq \Pi_X(\dot{\tau}(t)) \leq 0]\big)\Big]\} \\
&\Gamma = \mathbb{R}^{\geq 0}.
\end{aligned}
$$

It follows from Theorem 1 that $\mathcal{T}$ is a differentiably legal set of trajectories.

Figure 9.1 repeats the illustration of this system original presented in Figure 3.1.

9.1.2 Hybrid mechanical representation

I now construct a hybrid mechanical system $M = (\Omega, \mathcal{S}, \Gamma_M, \mathfrak{X}_0(\top, \mathcal{S}), m, \Psi, \hat{\mathfrak{f}}[\cdot])$ from the automaton A, following the rules presented in Chapter 7.

9.1.2.1 Bodies and universes

As usual, the hybrid universe Ω decomposes into separate physical and controller universes. The physical universe, as always, consists of the correspondent point body G and an undetailed remainder body N, together with the physical universal and null bodies. Thus, at a minimum,

$$\Omega_\text{p} = \{\top_\text{p}, G, N, \bot_\text{p}\}, \tag{9.2}$$

plus whatever other bodies are implied by any particular construction of N.

The controller universe contains one effector point body and one sensor point body for each automaton controller state, with all sensor and effector point bodies separate from each other. As usual, the join of the effector point bodies is the effector body, and likewise the join of the sensor point bodies is the sensor body. There also exist universal and null controller bodies, where the universal controller body is the join of the sensor and effector, and the null controller body is their intersection. Thus,

$$\begin{aligned}
E &= \{e_{\mathit{On}}, e_{\mathit{Off}}\} \\
e_{\mathit{On}} &\neq e_{\mathit{Off}} \\
S &= \{s_{\mathit{On}}, s_{\mathit{Off}}\} \\
s_{\mathit{On}} &\neq s_{\mathit{Off}} \\
E \sqcap S &= \bot_{\mathrm{c}} \\
E \sqcup S &= \top_{\mathrm{c}} \\
\Omega_{\mathrm{c}} &= \{C_E \sqcup C_S \mid (C_E \sqsubseteq E) \wedge (C_S \sqsubseteq S)\}.
\end{aligned} \tag{9.3}$$

This provides a hybrid universe such that

$$\Omega = \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}}. \tag{9.4}$$

9.1.2.2 Space, time, and mass

Physical space is identical to automaton environment space. The mechanical elevator has only a single environmental dimension, and so

$$\mathcal{S}_{\mathrm{p}} = [0, \infty). \tag{9.5}$$

Controller space is always the set $\{0, 1\}$, and so

$$\mathcal{S}_{\mathrm{c}} = \mathbb{Z}_2. \tag{9.6}$$

Hybrid space is the coproduct of the two factor spaces; thus,

$$\mathcal{S} = [0, \infty) \oplus \mathbb{Z}_2. \tag{9.7}$$

Physical, controller, and hybrid time are defined independently of the particular automaton, and so

$$\begin{aligned}
\Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\
\Gamma_{\mathrm{c}} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \\
\Gamma_M &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4.
\end{aligned} \tag{9.8}$$

Hybrid time maps to physical and controller time in the usual way, and so for any $(r, z) \in \Gamma_M$,

$$\begin{aligned}
\hat{t}_{\mathrm{p}}(r, z) &= r \\
\hat{t}_{\mathrm{c}}(r, z) &= (r, z).
\end{aligned} \tag{9.9}$$

Likewise, the set of physical masses is always the nonnegative real numbers, and the set of controller masses is always $\mathbb{Z}_2$, and so

$$\begin{aligned} \mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\ \mathcal{M}_{\mathrm{c}} &= \mathbb{Z}_2. \end{aligned} \tag{9.10}$$

The set of hybrid masses is the product of these two factor sets,

$$\mathcal{M} = \mathbb{R}^{\geq 0} \times \mathbb{Z}_2. \tag{9.11}$$

All physical point bodies have unit physical mass and zero controller mass, and so

$$\begin{aligned} m_{\mathrm{p}}(G) &= 1_{\mathrm{p}} \\ m_{\mathrm{c}}(G) &= 0_{\mathrm{c}}. \end{aligned} \tag{9.12}$$

Likewise, all controller point bodies have unit controller mass and zero physical mass, and so

$$\begin{aligned} m_{\mathrm{p}}(e_{On}) = m_{\mathrm{p}}(e_{Off}) = m_{\mathrm{p}}(s_{On}) = m_{\mathrm{p}}(s_{Off}) = 0_{\mathrm{p}} \\ m_{\mathrm{c}}(e_{On}) = m_{\mathrm{c}}(e_{Off}) = m_{\mathrm{c}}(s_{On}) = m_{\mathrm{c}}(s_{Off}) = 1_{\mathrm{c}}. \end{aligned} \tag{9.13}$$

9.1.2.3 Initial configurations

The hybrid configuration space is the product of the physical and controller configuration spaces; taking N to be a point body, one has

$$\begin{aligned} \mathfrak{X}(\top_{\mathrm{p}}, \mathcal{S}_{\mathrm{p}}) &= [0, \infty) \times \mathbb{R} \\ \mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &= \mathbb{Z}_2^{4|V|} = \mathbb{Z}_2^8 \\ \mathfrak{X}(\top, \mathcal{S}) &= [0, \infty) \times \mathbb{R} \times \mathbb{Z}_2^8. \end{aligned} \tag{9.14}$$

Recall from (7.197) that the initial configurations are the set

$$\mathfrak{X}_0(\top, \mathcal{S}) = \{\mathtt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in dom(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{9.15}$$

where, for any trajectory $\tau \in \mathcal{T}$ and automaton instant $t \in dom(\tau)$ such that $\tau(t) = (v, \mathbf{x}) \in Q_0$, the value $\mathtt{Mech}(\tau, t)$ is defined to be the configuration $(w, \dot{w})$, where

$$\begin{aligned} w_{\mathrm{p}}(G) &= \Pi_X(\tau(t)) \\ \dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}(t)) \end{aligned} \tag{9.16}$$

$$
\begin{aligned}
w_{\mathrm{c}}(e_v) &= 1\\
\forall v' \neq v \in V\big[w_{\mathrm{c}}(e_{v'}) &= 0\big]\\
\forall v' \in V\big[\dot{w}_{\mathrm{c}}(e_{v'}) &= 0\big]\\
\forall v' \in V\big[w_{\mathrm{c}}(s_{v'}) &= 0\big]\\
\forall v' \in V\big[\dot{w}_{\mathrm{c}}(s_{v'}) &= 0\big].
\end{aligned}
$$

In this particular case, then, the initial configurations are precisely those such that

$$
\begin{aligned}
w_{\mathrm{p}}(G) &\in [50, 60] \qquad (9.17)\\
\dot{w}_{\mathrm{p}}(G) &\in [0, 4]\\
w_{\mathrm{c}}(e_{\mathit{On}}) &= 1\\
w_{\mathrm{c}}(e_{\mathit{Off}}) &= 0\\
\dot{w}_{\mathrm{c}}(e_{\mathit{On}}) = \dot{w}_{\mathrm{c}}(e_{\mathit{Off}}) &= 0\\
w_{\mathrm{c}}(s_{\mathit{On}}) = w_{\mathrm{c}}(s_{\mathit{Off}}) &= 0\\
\dot{w}_{\mathrm{c}}(s_{\mathit{On}}) = \dot{w}_{\mathrm{c}}(s_{\mathit{Off}}) &= 0.
\end{aligned}
$$

9.1.2.4 Mechanical controller states

Mechanical controller states are described slightly more abstractly, since the trajectories themselves are somewhat abstract in the original construction. Recall from (7.207) that the mechanical controller state ψ_v is defined to be

$$
\psi_v = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_v) \neq \dot{w}(e_v)\big) \wedge \big(\forall v' \neq v \in V[w(e_{v'}) = \dot{w}(e_{v'})]\big)\}. \qquad (9.18)
$$

Then

$$
\begin{aligned}
\psi_{\mathit{On}} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{\mathit{On}}) \neq \dot{w}(e_{\mathit{On}})\big) \wedge \big(w(e_{\mathit{Off}}) = \dot{w}(e_{\mathit{Off}})\big)\} \qquad (9.19)\\
\psi_{\mathit{Off}} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{\mathit{Off}}) \neq \dot{w}(e_{\mathit{Off}})\big) \wedge \big(w(e_{\mathit{On}}) = \dot{w}(e_{\mathit{On}})\big)\}.
\end{aligned}
$$

The complements of these mechanical states are then

$$
\begin{aligned}
\overline{\psi}_{\mathit{On}} &= \mathfrak{X}(\top, \mathcal{S}) - \psi_{\mathit{On}} \qquad (9.20)\\
&= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{\mathit{On}}) = \dot{w}(e_{\mathit{On}})\big) \vee \big(w(e_{\mathit{Off}}) \neq \dot{w}(e_{\mathit{Off}})\big)\}\\
\overline{\psi}_{\mathit{Off}} &= \mathfrak{X}(\top, \mathcal{S}) - \psi_{\mathit{Off}}\\
&= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{\mathit{Off}}) = \dot{w}(e_{\mathit{Off}})\big) \vee \big(w(e_{\mathit{On}}) \neq \dot{w}(e_{\mathit{On}})\big)\}.
\end{aligned}
$$

Finally,

$$\begin{aligned}\overline{\psi}_V &= \bigcap_{v\in V} \overline{\psi}_v \\ &= \overline{\psi}_{On} \cap \overline{\psi}_{Off} \\ &= \Big\{ (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\Big|\, \Big[\big(w(e_{Off}) = \dot{w}(e_{Off})\big) \wedge \big(w(e_{On}) = \dot{w}(e_{On})\big)\Big] \\ &\qquad \vee \Big[\big(w(e_{Off}) \neq \dot{w}(e_{Off})\big) \wedge \big(w(e_{On}) \neq \dot{w}(e_{On})\big)\Big]\Big\}.\end{aligned} \tag{9.21}$$

Recall from (7.103) that the equivalence class $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$ is defined by the relation

$$(w, \dot{w}) \equiv_G (w', \dot{w}') \text{ iff } \big[(w_{\mathrm{p}}(G) = w'_{\mathrm{p}}(G)) \wedge (\dot{w}_{\mathrm{p}}(G) = \dot{w}'_{\mathrm{p}}(G))\big]. \tag{9.22}$$

Similarly, in (7.131), the equivalence class $X/\equiv_{\mathcal{D},v}$ is defined by the relation

$$\mathbf{x} \equiv_{\mathcal{D},v} \mathbf{x}' \text{ iff } \mathcal{D}_v(\mathbf{x}) = \mathcal{D}_v(\mathbf{x}'). \tag{9.23}$$

Since the only transitions possible in the original automaton are at $(On, 75) \rightarrowtail (Off, 75)$ and $(Off, 45) \rightarrowtail (On, 45)$, it follows that

$$\begin{aligned}\mathcal{D}_{On}(75) &= \{Off\} \\ \mathcal{D}_{Off}(45) &= \{On\},\end{aligned} \tag{9.24}$$

and, in all other pairs $(v, \mathbf{x})$ not covered by the above,

$$\mathcal{D}_v(\mathbf{x}) = \emptyset. \tag{9.25}$$

Thus,

$$\begin{aligned}X/\equiv_{\mathcal{D},On} &= \{\{75\}, \mathbb{R} - \{75\}\} \\ X/\equiv_{\mathcal{D},Off} &= \{\{45\}, \mathbb{R} - \{45\}\}.\end{aligned} \tag{9.26}$$

Again in (7.207), for each $v \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$, I defined subsets of $\mathfrak{X}(\top, \mathcal{S})$ as follows:

$$\begin{aligned}\psi_{v,\mathbf{x},\dot{\mathbf{x}}} &= \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}} \\ \psi_{[\![\mathbf{x}]\!]} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\} \\ \psi_{v,[\![\mathbf{x}]\!]} &= \psi_v \cap \psi_{[\![\mathbf{x}]\!]}.\end{aligned} \tag{9.27}$$

By (7.208),

$$\begin{aligned}
\forall v \in V\big[\Psi(e_v) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}\big] \\
\Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\} \\
\forall v \in V\big[\Psi(s_v) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_v\}\big] \\
\Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\ \big)\} \cup \{\overline{\psi}_V\} \\
\Psi(\top_{\mathbf{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G\ \big)\} \cup \{\overline{\psi}_V\}.
\end{aligned} \tag{9.28}$$

Then, in particular, the states of the various controllers for the mechanical elevator are:

$$\begin{aligned}
\Psi(e_{On}) &= \{\psi_{On,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_{On}\} \\
\Psi(e_{Off}) &= \{\psi_{Off,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_{Off}\} \\
\Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\} \\
\Psi(s_{On}) &= \{\psi_{On,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_{On}\} \\
\Psi(s_{Off}) &= \{\psi_{Off,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_{Off}\} \\
\Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\ \big)\} \cup \{\overline{\psi}_V\} \\
\Psi(\top_{\mathbf{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top,\mathcal{S})/\equiv_G\ \big)\} \cup \{\overline{\psi}_V\}.
\end{aligned} \tag{9.29}$$

9.1.2.5 Sensor forces

Recall from (8.82) that for any sensor point body s_v and mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]} \in \Psi(s_v)$,

$$\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}] = \left\{\hat{f}_{v\text{–}v',r} \,\middle|\, \big[v' \in \mathcal{D}_v(\mathbf{x})\big] \wedge \big[r \in \mathbb{R}^{\geq 0}\big]\right\} \cup \{f_{\mathbf{0}}\}\,. \tag{9.30}$$

As noted above,

$$\mathcal{D}_{On}(75) = \{Off\} \tag{9.31}$$

and, for any $\mathbf{x} \neq 75$,

$$\mathcal{D}_{On}(\mathbf{x}) = \emptyset. \tag{9.32}$$

Then letting $v = On$,

$$\hat{\mathfrak{f}}_{s,On}[\psi_{On,[\![75]\!]}] = \left\{\hat{f}_{On\text{–}Off,r} \,\middle|\, r \in \mathbb{R}^{\geq 0}\right\} \cup \{f_{\mathbf{0}}\}\,, \tag{9.33}$$

and, for any $\mathbf{x} \neq 75$,

$$\hat{\mathfrak{f}}_{s,On}[\psi_{On,[\![\mathbf{x}]\!]}] = \{f_{\mathbf{0}}\}\,. \tag{9.34}$$

By a similar argument, letting $v = \mathit{Off}$,

$$\hat{\mathfrak{f}}_{s,\mathit{Off}}[\psi_{\mathit{Off},[\![45]\!]}] = \left\{ \hat{f}_{\mathit{Off}\text{–}\mathit{On},r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\}\,, \tag{9.35}$$

and, for any $\mathbf{x} \neq 45$,

$$\hat{\mathfrak{f}}_{s,\mathit{On}}[\psi_{\mathit{Off},[\![\mathbf{x}]\!]}] = \{f_{\mathbf{0}}\}\,. \tag{9.36}$$

Via (8.86), the sensor force variation is

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{f_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{9.37}$$

and so, substituting via the above states,

$$\hat{\mathfrak{f}}_S[\psi] = \begin{cases} \left\{ \hat{f}_{\mathit{On}\text{–}\mathit{Off},r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\}\,, & \psi = \psi_{\mathit{On},[\![75]\!]} \\ \left\{ \hat{f}_{\mathit{Off}\text{–}\mathit{On},r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\}\,, & \psi = \psi_{\mathit{Off},[\![45]\!]} \\ \{f_{\mathbf{0}}\}, & \text{otherwise.} \end{cases} \tag{9.38}$$

9.1.2.6 Effector forces

Recall from (8.28) that the for any effector point body e_v and state $\psi \in \Psi(e_v)$,

$$\hat{\mathfrak{f}}_{e,v}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_v, \end{cases} \tag{9.39}$$

where

$$\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} = \left\{ \tau \in \mathcal{T}_v \,\middle|\, \left(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}\right) \wedge \left(\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}}\right) \right\} \tag{9.40}$$

and $\hat{f}^{\tau}$ is the force variation such that, for some trajectory $\tau \in \mathcal{T}$, the following conditions hold:

- The domain of $\hat{f}$ is all instants in hybrid mechanical time; that is,

$$\mathit{dom}(\hat{f}) = \Gamma_M, \tag{9.41}$$

- τ has a constant automaton controller state of v; that is,

$$\forall t \in \mathit{dom}(\tau)\, [\Pi_V(\tau(t)) = v]\,, \tag{9.42}$$

- At automaton instant 0, the environment state component of τ is $\mathbf{x}$, while the derivative of the

environment state component of τ is $\dot{\mathbf{x}}$; that is,

$$(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}) \wedge (\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}})\,, \tag{9.43}$$

- For all instants $t \in \mathit{dom}(\hat{f})$, the force of N on G is naturally isomorphic to the second derivative of the environment state component of τ at 0 times $\mathbf{0}_\mathrm{c}$; that is,

$$\forall t \in \mathit{dom}(\hat{f}) \left[\hat{f}(t)(G, N) = (\Pi_X(\ddot{\tau}.\mathit{fstate}), \mathbf{0}_\mathrm{c})\right], \tag{9.44}$$

  and

- Except as required by the above by pairwise equilibriation and additivity, $\hat{f} = f_{\mathbf{0}}$.

As noted in (9.1),

$$\begin{aligned}\mathcal{T} = \{\tau \mid &\big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \big(\tau \text{ is second-degree doubly semidifferentiable}\big)\\ &\wedge\ \forall t \in \mathit{dom}(\tau)\big(45 \leq \Pi_X(\tau)(t) \leq 75\big)\\ &\wedge\ \Big[\big([\tau_V(t) = \mathit{On}] \wedge [0 \leq \Pi_X(\dot{\tau}(t)) \leq 4]\big) \vee \big([\Pi_V(\tau(t)) = \mathit{Off}] \wedge [-3 \leq \Pi_X(\dot{\tau}(t)) \leq 0]\big)\Big]\}.\end{aligned}$$

Consider any set $\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$, and any trajectory $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$. It follows that

$$\begin{aligned}\Pi_V(\tau.\mathit{fstate}) &= v\\ \Pi_X(\tau.\mathit{fstate}) &= \mathbf{x}\\ \Pi_X(\dot{\tau}.\mathit{fstate}) &= \dot{\mathbf{x}}.\end{aligned} \tag{9.45}$$

The definition of $\mathcal{T}$ does not directly constrain the second derivative of any trajectory in $\mathcal{T}$, save that it must stay within the derivative bounds defined above. Practically, this requires the following constraints:

- if $\Pi_V(\tau.\mathit{fstate}) = \mathit{On}$ and $\Pi_X(\dot{\tau}.\mathit{fstate}) = 0$, then $\Pi_X(\ddot{\tau}.\mathit{fstate}) \geq 0$,
- if $\Pi_V(\tau.\mathit{fstate}) = \mathit{Off}$ and $\Pi_X(\dot{\tau}.\mathit{fstate}) = 0$, then $\Pi_X(\ddot{\tau}.\mathit{fstate}) \leq 0$,
- if $\Pi_V(\tau.\mathit{fstate}) = \mathit{On}$ and $\Pi_X(\dot{\tau}.\mathit{fstate}) = 4$, then $\Pi_X(\ddot{\tau}.\mathit{fstate}) \leq 0$, and
- if $\Pi_V(\tau.\mathit{fstate}) = \mathit{Off}$ and $\Pi_X(\dot{\tau}.\mathit{fstate}) = -3$, then $\Pi_X(\ddot{\tau}.\mathit{fstate}) \geq 0$.

For any non-boundary velocity, the value of $\Pi_X(\ddot{\tau}.\mathit{fstate})$ is unconstrained.

Then let $f^{\ddot{\mathbf{x}}}$ be any force variation such that, for all instants $t \in \Gamma_M$,

$$f^{\ddot{\mathbf{x}}}(t)(G, N) = (\ddot{\mathbf{x}}, 0_\mathrm{c}) \tag{9.46}$$

and, except as required by the above and the definition of a force system, for any bodies $A, B \in \Omega$,

$$f^{\ddot{\mathbf{x}}}(t)(A,B) = (0_{\mathrm{p}}, 0_{\mathrm{c}}). \tag{9.47}$$

Then for any effector body point e_{On} and mechanical controller state $\psi \in \Psi(e_{On})$,

$$\hat{\mathfrak{f}}_{e,On}[\psi] = \begin{cases} \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [0 < \dot{\mathbf{x}} < 4] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\leq 0} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 4] \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_{On} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.48}$$

By a similar argument, for any effector body point e_{Off} and mechanical controller state $\psi \in \Psi(e_{Off})$,

$$\hat{\mathfrak{f}}_{e,Off}[\psi] = \begin{cases} \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R} \right\}, & [\psi = \psi_{Off,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [-3 < \dot{\mathbf{x}} < 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\leq 0} \right\}, & [\psi = \psi_{Off,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{Off,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = -3] \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_{Off} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.49}$$

Via (8.30), for any mechanical controller state $\psi \in \Psi(E)$, the force variation inclusion for the effector is

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{9.50}$$

Then, combining the above conditional force variation inclusions,

$$\hat{\mathfrak{f}}_{e,On}[\psi] = \begin{cases} \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [0 < \dot{\mathbf{x}} < 4] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\leq 0} \right\}, & [\psi = \psi_{On,\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 4] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R} \right\}, & [\psi = \psi_{\mathit{Off},\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [-3 < \dot{\mathbf{x}} < 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\leq 0} \right\}, & [\psi = \psi_{\mathit{Off},\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = 0] \\ \left\{ \hat{f}^{\ddot{\mathbf{x}}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \ddot{\mathbf{x}} \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{\mathit{Off},\mathbf{x},\dot{\mathbf{x}}}] \wedge [45 \leq \mathbf{x} \leq 75] \wedge [\dot{\mathbf{x}} = -3] \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.51}$$

Note the peculiarity here that the "otherwise" condition produces an empty set of force variations. Considering the states accounted for by the previous conditions, "otherwise" is all states $\psi_{On,\mathbf{x},\dot{\mathbf{x}}}$ such that $\mathbf{x} < 45$, $\mathbf{x} > 75$, $\dot{\mathbf{x}} < 0$, or $\dot{\mathbf{x}} > 4$, as well as all states $\psi_{\mathit{Off},\mathbf{x},\dot{\mathbf{x}}}$ such that $\mathbf{x} < 45$, $\mathbf{x} > 75$, $\dot{\mathbf{x}} < -3$, or $\dot{\mathbf{x}} > 0$. In other words, it consists of all states corresponding to unreachable areas of the original automaton, such as cases where the elevator has gotten too high or too low, or is rising or falling too quickly. Such cases are undefined in the original automaton; indeed, the automaton *blocks*, or halts, if it would enter such a state. Thus, the automaton has no trajectories continuing from such states. If the corresponding mechanical system is to truly bisimulate the automaton, then, it cannot define any motion conditional on such states, and so the corresponding sets of force variations are empty.

9.1.2.7 Combined hybrid forces

Recall from (8.104) that, for any mechanical controller state $\psi \in \Psi(\top_{\mathbf{c}})$,

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} + \hat{f}_{v \to v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{9.52}$$

Then, combining the conditional force variation inclusion definitions for the sensor and effector,

$$\hat{\mathfrak{f}}_{e,On}[\psi] = \begin{cases} \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[0<\dot{\mathbf{x}}<4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[-3<\dot{\mathbf{x}}<0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\wedge r\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=-3] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[0<\dot{\mathbf{x}}<4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{On\text{--}Off,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\right\}, & [\psi=\psi_{On,75,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[-3<\dot{\mathbf{x}}<0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}}, \hat{f}^{\ddot{\mathbf{x}}}+\hat{f}_{Off\text{--}On,r} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{Off,45,\dot{\mathbf{x}}}]\wedge[\dot{\mathbf{x}}=-3] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\right\}, & [\psi=\psi_{On,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45\leq\mathbf{x}<75]\wedge[0<\dot{\mathbf{x}}<4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{On,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45\leq\mathbf{x}<75]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\right\}, & [\psi=\psi_{On,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45\leq\mathbf{x}<75]\wedge[\dot{\mathbf{x}}=4] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}\right\}, & [\psi=\psi_{Off,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45<\mathbf{x}\leq 75]\wedge[-3<\dot{\mathbf{x}}<0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\leq 0}\right\}, & [\psi=\psi_{Off,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45<\mathbf{x}\leq 75]\wedge[\dot{\mathbf{x}}=0] \\ \left\{\hat{f}^{\ddot{\mathbf{x}}} \,\middle|\, \ddot{\mathbf{x}}\in\mathbb{R}^{\geq 0}\right\}, & [\psi=\psi_{Off,\mathbf{x},\dot{\mathbf{x}}}]\wedge[45<\mathbf{x}\leq 75]\wedge[\dot{\mathbf{x}}=-3] \\ \{f_{\mathbf{0}}\}, & \psi=\overline{\psi}_V \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.53}$$

9.2 The inverted pendulum

The inverted pendulum problem is one of the classic control theory examples; while numerous implementations exist, I will focus on a variant of the version presented by Johansson in [17].

Consider a wheeled cart sitting on a flat surface. Mounted on the top of the cart is an inverted pendulum; that is, the bob of the pendulum is at the top, at the end of a rigid pole. As the pendulum rocks back and forth, the cart naturally also moves in a direction opposite that of the pendulum. A computer-controlled motor connects pendulum to cart; by varying when the motor is running, and in

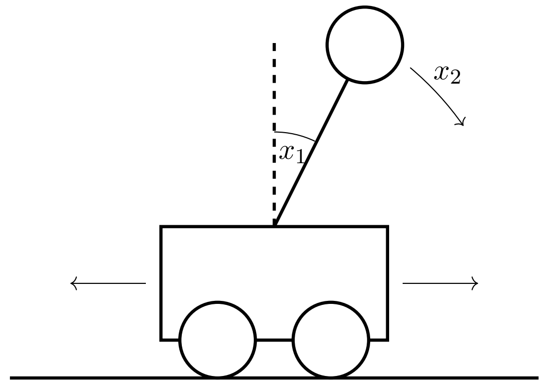


Figure 9.2: Inverted pendulum on cart.

which direction, the motion of the pendulum can be altered. The overall objective of the problem is to bring the pendulum to rest in a straight vertical line, at which point the cart will also be at rest. This problem is illustrated in Figure 9.2.

The activity of the inverted pendulum can be described using only a handful of parameters: x_1, the angle between the pendulum and a vertical line; x_2, the angular velocity of the bob; and u, the force applied by the motor. The force u is bounded to some range $[u_{\text{min}}, u_{\text{max}}]$. In addition, the pendulum bob has a fixed mass, which is acted on by a gravitational acceleration g. Note that the motion of the cart does not need to be expressed here, as the cart's position is a function of the bob's position and their relative masses.

The change in the dynamics of the pendulum can be described via

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g \sin x_1 - u \cos x_1. \end{aligned} \tag{9.54}$$

Johansson proposes a three-state hybrid automaton to control the pendulum's behavior, calling the three automaton controller states *pos-max*, *neg-max*, and *loc-stab*. In general terms, *pos-max* runs the motor at its maximum force in the positive direction, that is, $u = u_{\text{max}}$. The *neg-max* state runs the motor in the opposite direction, that is, $u = u_{\text{min}}$. The automaton switches between these states based on the value of the function

$$\beta(x_1, x_2) = [x_2^2/2 + g(\cos x_1 - 1)]x_2 \cos x_1. \tag{9.55}$$

When $\beta(x_1, x_2)$ is positive, the system changes state to *pos-max*; when it is negative, the system changes state to *neg-max*. Johansson also introduces a third state, *loc-stab*, that handles automaton behavior when x_1 is within some value θ of 0, that is, when the pendulum is nearly vertical and might rapidly change back and forth between the above states. I omit this third state in the interests of simplicity. The resulting scheme is illustrated in Figure 9.3.

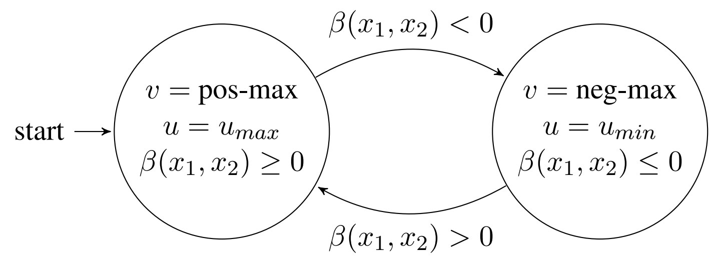


Figure 9.3: A hybrid automaton model for an inverted pendulum on a cart. Here x_1 is the angle from the pendulum to a vertical line, x_2 is the angular momentum of the pendulum, u is the acceleration due to the motor, and $\beta(x_1, x_2) = [x_2^2/2 + g(\cos x_1 - 1)]x_2 \cos x_1$.

To the above, I add the starting requirement that the automaton begins in state *pos-max*, with the pendulum initially vertical, with a unit forward angular velocity.

9.2.1 Hybrid automaton representation

In considering the above automaton, one problem immediately suggests itself: in previous constructions, I have required that all parameters are second-degree doubly semidifferentiable. This requirement is satisfied for x_1, that is, the angle by which the bob is out of vertical alignment, which makes intuitive sense; in a real-world implementation of the inverted pendulum, the force on the bob (and so its acceleration) must exist at every point.

The same cannot be said for x_2, the angular velocity of the bob. Again, this seems intuitively reasonable; while velocity must be differentiable, for the reasons given above, it need not be twice differentiable. The third parameter, u, also presents a challenge; the motion of u is not differentiable at all, as the motor is presented as being able to instantly switch from *pos-max* to *pos-min*.

These problems can be easily resolved, as we are given the meaning of each of the variables in advance. So, for instance, x_2 is simply the derivative of x_1. On translating from the automaton to the hybrid mechanical system, then, I can disregard all references to x_2, instead substituting $\dot{x}_1$.

Handling u is even simpler. Note that u plays no role in determining the transitions of the automaton; that is, it is not an argument to the function β. Its purpose is solely to communicate the current behavior of the automaton. We can thus remove u from the automaton entirely without changing its behavior; instead, in each state, we explicitly specify the changes in position and velocity for u_{max} or u_{min} as appropriate. The result is given in Figure 9.4.

Note that these amendments are possible only because we know the original description of the automaton, and so are able to account for the specific properties of each component of X. In the more general case, this information is not available; as noted in Chapter 7, one may be given a tuple with no word of explanation. This poses a significant limitation, as automata like this one, where derivatives or

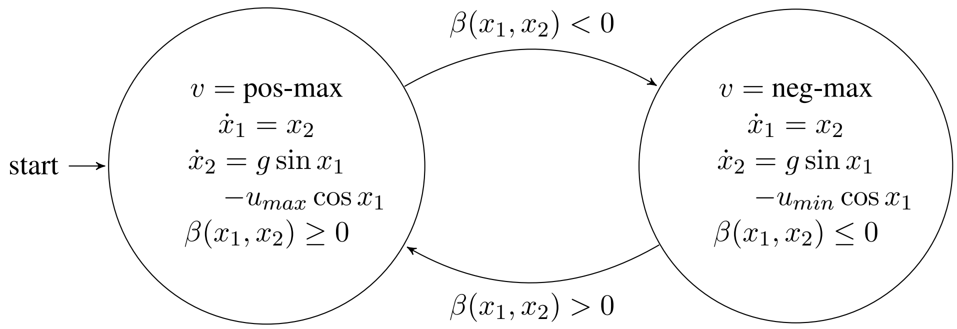


Figure 9.4: A revised version of the hybrid automaton for the inverted pendulum. The variable x_2 is the derivative of x_1, and so it will be eliminated on translating to a hybrid mechanical system. The variable u can likewise be replaced with the constants u_{max} and u_{min}.

other properties of the automaton are encoded into the environment state, are very common. Exploring a more general-case policy for handling such automata is thus a major area for future work.

In any event, the revised description has two controller states, pos and neg. The environment state is the product of $\mathbf{x}_1$ and $\mathbf{x}_2$. Assuming without loss of generality that $\mathbf{x}_1$ is measured in radians, and that the pendulum cannot swing below the horizon in either direction, the range of values for $\mathbf{x}_1$ is $[-\pi/2, \pi/2]$, and the range of values for its derivative is in principle $[-\infty, \infty]$. Initially, the automaton is in state pos; the pendulum is vertical, with $\mathbf{x}_1 = 0$, and has a unit forward velocity, with $\mathbf{x}_2 = 1$. The pendulum can switch states in either direction whenever $\beta(\mathbf{x}_1, \mathbf{x}_2)$ is 0, and so it has discrete transitions from all such system states. Finally, the trajectories of the automaton are all those motions such that $\mathbf{x}_2$ is the derivative of $\mathbf{x}_1$, and such that $\beta(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ and the derivative of $\mathbf{x}_2$ is $g \sin \mathbf{x}_1 - u_{max} \cos \mathbf{x}_1$ in controller state pos, or such that $\beta(\mathbf{x}_1, \mathbf{x}_2) \leq 0$ and the derivative of $\mathbf{x}_2$ is $g \sin \mathbf{x}_1 - u \cos \mathbf{x}_1$ in controller state neg. More formally, the revised formulation of the inverted pendulum can be represented via the tuple $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$, where

$$
\begin{aligned}
V &= \{pos, neg\} \qquad (9.56)\\
X &= [-\pi, \pi] \times [-\infty, \infty]\\
Q_0 &= \{(pos, (0, 1))\}\\
D &= \{(pos, \mathbf{x}) \rightarrowtail (neg, \mathbf{x}), (neg, \mathbf{x}) \rightarrowtail (pos, \mathbf{x}) \mid \beta(\mathbf{x}) = 0\}\\
\mathcal{T} &= \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \forall t \in dom(\tau)\big([\Pi_{X_1}(\dot{\tau}(t)) = \Pi_{X_2}(\tau(t))]\big)\\
&\qquad \wedge \Big[\big([\Pi_V(\tau(t) = pos)] \wedge [\Pi_{X_2}(\dot{\tau}(t)) = g \sin \Pi_{X_1}(\tau(t)) - u_{max} \cos \Pi_{X_1}(\tau(t))]\\
&\qquad\qquad \wedge [\beta(\Pi_X(\tau(t))) \geq 0]\big)\\
&\qquad\quad \vee \big([\Pi_V(\tau(t) = neg)] \wedge [\Pi_{X_2}(\dot{\tau}(t)) = g \sin \Pi_{X_1}(\tau(t)) - u_{min} \cos \Pi_{X_1}(\tau(t))]\\
&\qquad\qquad \wedge [\beta(\Pi_X(\tau(t))) \leq 0]\big)\Big]
\end{aligned}
$$

$\Gamma = \mathbb{R}^{\geq 0}$.

9.2.2 Hybrid mechanical representation

I now construct a hybrid mechanical system $M = (\Omega, \mathcal{S}, \Gamma_M, \mathfrak{X}_0(\top, \mathcal{S}), m, \Psi, \hat{\mathfrak{f}}[\cdot])$ from the automaton A, following the rules presented in Chapter 7.

9.2.2.1 Bodies and universes

As usual, the hybrid universe Ω decomposes into separate physical and controller universes. The physical universe, as always, consists of the correspondent point body G and an undetailed remainder body N, together with the physical universal and null bodies. Thus, at a minimum,

$$\Omega_{\mathrm{p}} = \{\top_{\mathrm{p}}, G, N, \bot_{\mathrm{p}}\}, \tag{9.57}$$

plus whatever other bodies are implied by any particular construction of N.

The controller universe contains one effector point body and one sensor point body for each automaton controller state, with all sensor and effector point bodies separate from each other. As usual, the join of the effector point bodies is the effector body, and likewise the join of the sensor point bodies is the sensor body. There also exist universal and null controller bodies, where the universal controller body is the join of the sensor and effector, and the null controller body is their intersection. Thus,

$$\begin{aligned} E &= \{e_{pos}, e_{neg}\} \\ e_{pos} &\neq e_{neg} \\ S &= \{s_{pos}, s_{neg}\} \\ s_{pos} &\neq s_{neg} \\ E \sqcap S &= \bot_{\mathrm{c}} \\ E \sqcup S &= \top_{\mathrm{c}} \\ \Omega_{\mathrm{c}} &= \{C_E \sqcup C_S \mid (C_E \sqsubseteq E) \wedge (C_S \sqsubseteq S)\}. \end{aligned} \tag{9.58}$$

This provides a hybrid universe such that

$$\Omega = \Omega_{\mathrm{p}} \oplus \Omega_{\mathrm{c}}. \tag{9.59}$$

9.2.2.2 Space, time, and mass

Physical space is in general identical to automaton environment space; in this case, as discussed above, that claim comes with a slight caveat. As previously noted, in the mechanical system I discard $\mathbf{x}_2$, replacing any references to it with $\dot{\mathbf{x}}_1$. Thus, I treat X as if it consisted only of the values of $\mathbf{x}_1$, and so

$$\mathcal{S}_{\mathrm{p}} = [-\pi/2, \pi/2]. \tag{9.60}$$

Controller space is always the set $\{0, 1\}$, and so

$$\mathcal{S}_{\mathrm{c}} = \mathbb{Z}_2. \tag{9.61}$$

Hybrid space is the coproduct of the two factor spaces; thus,

$$\mathcal{S} = [-\pi/2, \pi/2] \oplus \mathbb{Z}_2. \tag{9.62}$$

Physical, controller, and hybrid time are defined independently of the particular automaton, and so

$$\begin{aligned}
\Gamma_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\
\Gamma_{\mathrm{c}} &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4 \\
\Gamma_M &= \mathbb{R}^{\geq 0} \times \mathbb{Z}_4.
\end{aligned} \tag{9.63}$$

Hybrid time maps to physical and controller time in the usual way, and so for any $(r, z) \in \Gamma_M$,

$$\begin{aligned}
\hat{t}_{\mathrm{p}}(r, z) &= r \\
\hat{t}_{\mathrm{c}}(r, z) &= (r, z).
\end{aligned} \tag{9.64}$$

Likewise, the set of physical masses is always the nonnegative real numbers, and the set of controller masses is always $\mathbb{Z}_2$, and so

$$\begin{aligned}
\mathcal{M}_{\mathrm{p}} &= \mathbb{R}^{\geq 0} \\
\mathcal{M}_{\mathrm{c}} &= \mathbb{Z}_2.
\end{aligned} \tag{9.65}$$

The space of hybrid masses is then the product of these two sets,

$$\mathcal{M} = \mathbb{R}^{\geq 0} \times \mathbb{Z}_2. \tag{9.66}$$

All physical point bodies have unit physical mass and zero controller mass, and so

$$\begin{aligned}
m_{\mathrm{p}}(G) &= 1_{\mathrm{p}} \\
m_{\mathrm{c}}(G) &= 0_{\mathrm{c}}.
\end{aligned} \tag{9.67}$$

Likewise, all controller point bodies have unit controller mass and zero physical mass, and so

$$\begin{aligned} m_{\mathrm{p}}(e_{pos}) = m_{\mathrm{p}}(e_{neg}) = m_{\mathrm{p}}(s_{pos}) = m_{\mathrm{p}}(s_{neg}) = 0_{\mathrm{p}} \\ m_{\mathrm{c}}(e_{pos}) = m_{\mathrm{c}}(e_{neg}) = m_{\mathrm{c}}(s_{pos}) = m_{\mathrm{c}}(s_{neg}) = 1_{\mathrm{c}}. \end{aligned} \tag{9.68}$$

9.2.2.3 Initial configurations

The hybrid configuration space is the product of the physical and controller configuration spaces; taking N to be a point body, one has

$$\begin{aligned} \mathfrak{X}(\top_{\mathrm{p}}, \mathcal{S}_{\mathrm{p}}) &= \mathbb{R}^2 \\ \mathfrak{X}(\top_{\mathrm{c}}, \mathcal{S}_{\mathrm{c}}) &= \mathbb{Z}_2^{4|V|} = \mathbb{Z}_2^8 \\ \mathfrak{X}(\top, \mathcal{S}) &= \mathbb{R}^2 \times \mathbb{Z}_2^8. \end{aligned} \tag{9.69}$$

Recall from (7.197) that the initial configurations are the set

$$\mathfrak{X}_0(\top, \mathcal{S}) = \{\mathtt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in dom(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{9.70}$$

where, for any trajectory $\tau \in \mathcal{T}$ and automaton instant $t \in dom(\tau)$ such that $\tau(t) = (v, \mathbf{x}) \in Q_0$, the value $\mathtt{Mech}(\tau, t)$ is defined to be the configuration $(w, \dot{w})$, where

$$\begin{aligned} w_{\mathrm{p}}(G) &= \Pi_X(\tau(t)) \\ \dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}(t)) \\ w_{\mathrm{c}}(e_v) &= 1 \\ \forall v' \neq v \in V\big[w_{\mathrm{c}}(e_{v'}) &= 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathrm{c}}(e_{v'}) &= 0\big] \\ \forall v' \in V\big[w_{\mathrm{c}}(s_{v'}) &= 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathrm{c}}(s_{v'}) &= 0\big]. \end{aligned} \tag{9.71}$$

In this particular case, then, the initial configurations are precisely those such that

$$\begin{aligned} w_{\mathrm{p}}(G) &= 0 \\ \dot{w}_{\mathrm{p}}(G) &= 1 \\ w_{\mathrm{c}}(e_{pos}) &= 1 \\ w_{\mathrm{c}}(e_{neg}) &= 0 \\ \dot{w}_{\mathrm{c}}(e_{pos}) = \dot{w}_{\mathrm{c}}(e_{neg}) &= 0 \end{aligned} \tag{9.72}$$

$$w_{\mathsf{c}}(s_{pos}) = w_{\mathsf{c}}(s_{neg}) = 0$$
$$\dot{w}_{\mathsf{c}}(s_{pos}) = \dot{w}_{\mathsf{c}}(s_{neg}) = 0.$$

9.2.2.4 Mechanical controller states

I now show a construction of the mechanical controller states, producing a set of universal mechanical controller states $\Psi(\top_{\mathsf{c}})$ such that there is an isomorphism from $\Psi(\top_{\mathsf{c}})$ to $(2 \times [-\pi/2, \pi/2] \times \mathbb{R}^2) + 1$.

Recall from (7.207) that the mechanical controller state ψ_v is defined to be

$$\psi_v = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_v) \neq \dot{w}(e_v)\big) \wedge \big(\forall v' \neq v \in V[w(e_{v'}) = \dot{w}(e_{v'})]\big)\}. \tag{9.73}$$

Then

$$\begin{aligned} \psi_{pos} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{pos}) \neq \dot{w}(e_{pos})\big) \wedge \big(w(e_{neg}) = \dot{w}(e_{neg})\big)\} \\ \psi_{neg} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{neg}) \neq \dot{w}(e_{neg})\big) \wedge \big(w(e_{pos}) = \dot{w}(e_{pos})\big)\}. \end{aligned} \tag{9.74}$$

The complements of these mechanical states are then

$$\begin{aligned} \overline{\psi}_{pos} &= \mathfrak{X}(\top, \mathcal{S}) - \psi_{pos} \\ &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{pos}) = \dot{w}(e_{pos})\big) \vee \big(w(e_{neg}) \neq \dot{w}(e_{neg})\big)\} \\ \overline{\psi}_{neg} &= \mathfrak{X}(\top, \mathcal{S}) - \psi_{neg} \\ &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_{neg}) = \dot{w}(e_{neg})\big) \vee \big(w(e_{pos}) \neq \dot{w}(e_{pos})\big)\}. \end{aligned} \tag{9.75}$$

Finally,

$$\begin{aligned} \overline{\psi}_V &= \bigcap_{v \in V} \overline{\psi}_v \\ &= \overline{\psi}_{pos} \cap \overline{\psi}_{neg} \\ &= \left\{ (w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, \Big[\big(w(e_{neg}) = \dot{w}(e_{neg})\big) \wedge \big(w(e_{pos}) = \dot{w}(e_{pos})\big)\Big] \right. \\ &\qquad \left. \vee \Big[\big(w(e_{neg}) \neq \dot{w}(e_{neg})\big) \wedge \big(w(e_{pos}) \neq \dot{w}(e_{pos})\big)\Big] \right\}. \end{aligned} \tag{9.76}$$

Recall from (7.103) that the equivalence class $\mathfrak{X}(\top, \mathcal{S})/{\equiv_G}$ is defined by the relation

$$(w, \dot{w}) \equiv_G (w', \dot{w}') \text{ iff } \big[(w_{\mathsf{p}}(G) = w'_{\mathsf{p}}(G)) \wedge (\dot{w}_{\mathsf{p}}(G) = \dot{w}'_{\mathsf{p}}(G))\big]. \tag{9.77}$$

Similarly, in (7.131), the equivalence class $X/\equiv_{\mathcal{D},v}$ is defined by the relation

$$\mathbf{x} \equiv_{\mathcal{D},v} \mathbf{x}' \text{ iff } \mathcal{D}_v(\mathbf{x}) = \mathcal{D}_v(\mathbf{x}'). \tag{9.78}$$

Let U be the set of all environment states $\mathbf{x}$ such that $\beta(\mathbf{x}_1, \mathbf{x}_2) = 0$, that is,

$$U = \{\mathbf{x} \in X \mid \beta(\mathbf{x}_1, \mathbf{x}_2) = 0\}. \tag{9.79}$$

Then for any $\mathbf{x} \in U$,

$$\begin{aligned} \mathcal{D}_{pos}(\mathbf{x}) &= \{neg\} \\ \mathcal{D}_{neg}(\mathbf{x}) &= \{pos\}, \end{aligned} \tag{9.80}$$

and, in all other pairs $(v, \mathbf{x})$ where $\mathbf{x} \notin U$,

$$\mathcal{D}_v(\mathbf{x}) = \emptyset. \tag{9.81}$$

Thus,

$$\begin{aligned} X/\equiv_{\mathcal{D},pos} &= \{U, \overline{U}\} \\ X/\equiv_{\mathcal{D},neg} &= \{U, \overline{U}\}. \end{aligned} \tag{9.82}$$

Again in (7.207), for each $v \in V$, $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$, I defined subsets of $\mathfrak{X}(\top, \mathcal{S})$ as follows:

$$\begin{aligned} \psi_{v,\mathbf{x},\dot{\mathbf{x}}} &= \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}} \\ \psi_{[\![\mathbf{x}]\!]} &= \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\} \\ \psi_{v,[\![\mathbf{x}]\!]} &= \psi_v \cap \psi_{[\![\mathbf{x}]\!]}. \end{aligned} \tag{9.83}$$

By (7.208),

$$\begin{aligned} \forall v \in V\big[\Psi(e_v) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}\big] \\ \Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\} \\ \forall v \in V\big[\Psi(s_v) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\} \cup \{\overline{\psi}_v\}\big] \\ \Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\ \big)\} \cup \{\overline{\psi}_V\} \\ \Psi(\top_{\mathrm{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\ \big)\} \cup \{\overline{\psi}_V\}. \end{aligned} \tag{9.84}$$

These definitions require some adjusting for this particular problem, since, again, the hybrid mechanical system drops the environment state variable $\mathbf{x}_2$ in favor of $\dot{\mathbf{x}}_1$. In particular, rather than defining

$$\psi_{[\![\mathbf{x}]\!]} = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid w_{\mathrm{p}}(G) \in [\![\mathbf{x}]\!]\}, \tag{9.85}$$

I have

$$\psi_{[\![\mathbf{x}]\!]} = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid (w_{\mathrm{p}}(G), \dot{w}_{\mathrm{p}}(G)) \in [\![\mathbf{x}]\!]\}. \tag{9.86}$$

Likewise, I define $\psi_{\mathbf{x},\dot{\mathbf{x}}}$ to be

$$\psi_{\mathbf{x},\dot{\mathbf{x}}} = \left\{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \,\middle|\, \big((w(G), \dot{w}(G)) = \mathbf{x}\big) \wedge \big((\dot{w}(G), \ddot{w}(G)) = \dot{\mathbf{x}}\big)\right\}. \tag{9.87}$$

Subject to these revised definitions, the states of the various controllers for the mechanical elevator are:

$$\begin{aligned}
\Psi(e_{pos}) &= \{\psi_{pos,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_{pos}\} \\
\Psi(e_{neg}) &= \{\psi_{neg,\mathbf{x},\dot{\mathbf{x}}} \mid \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_{neg}\} \\
\Psi(E) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \{\overline{\psi}_V\} \\
\Psi(s_{pos}) &= \{\psi_{pos,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/{\equiv_{\mathcal{D},v}}\} \cup \{\overline{\psi}_{pos}\} \\
\Psi(s_{neg}) &= \{\psi_{neg,[\![\mathbf{x}]\!]} \mid [\![\mathbf{x}]\!] \in X/{\equiv_{\mathcal{D},v}}\} \cup \{\overline{\psi}_{neg}\} \\
\Psi(S) &= \{\psi_{v,[\![\mathbf{x}]\!]} \mid \big(v \in V\big) \wedge \big([\![\mathbf{x}]\!] \in X/{\equiv_{\mathcal{D},v}}\ \big)\} \cup \{\overline{\psi}_V\} \\
\Psi(\top_{\mathrm{c}}) &= \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \mid \big(v \in V\big) \wedge \big(\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\ \big)\} \cup \{\overline{\psi}_V\}.
\end{aligned} \tag{9.88}$$

9.2.2.5 Sensor forces

Recall from (8.82) that for any sensor point body s_v and mechanical controller state $\psi_{v,[\![\mathbf{x}]\!]} \in \Psi(s_v)$,

$$\hat{\mathfrak{f}}_{s,v}[\psi_{v,[\![\mathbf{x}]\!]}] = \left\{\hat{f}_{v\text{–}v',r} \,\middle|\, \big[v' \in \mathcal{D}_v(\mathbf{x})\big] \wedge \big[r \in \mathbb{R}^{\geq 0}\big]\right\} \cup \{f_{\mathbf{0}}\}\,. \tag{9.89}$$

As noted above, for any $\mathbf{x} \in U$,

$$\begin{aligned}
\mathcal{D}_{pos}(\mathbf{x}) &= \{neg\} \\
\mathcal{D}_{neg}(\mathbf{x}) &= \{pos\}
\end{aligned} \tag{9.90}$$

and, for any $(v, \mathbf{x})$ such that $\mathbf{x} \notin U$,

$$\mathcal{D}_v(\mathbf{x}) = \emptyset. \tag{9.91}$$

Then letting $v = pos$,

$$\begin{aligned}
\hat{\mathfrak{f}}_{s,pos}[\psi_{pos,U}] &= \left\{\hat{f}_{pos\text{–}neg,r} \,\middle|\, r \in \mathbb{R}^{\geq 0}\right\} \cup \{f_{\mathbf{0}}\} \\
\hat{\mathfrak{f}}_{s,pos}[\psi_{pos,\overline{U}}] &= \{f_{\mathbf{0}}\}\,.
\end{aligned} \tag{9.92}$$

By a similar argument, letting $v = neg$,

$$\hat{\mathfrak{f}}_{s,neg}[\psi_{neg,U}] = \left\{ \hat{f}_{neg\text{–}pos,r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\} \tag{9.93}$$
$$\hat{\mathfrak{f}}_{s,pos}[\psi_{neg,\overline{U}}] = \{f_{\mathbf{0}}\}\,.$$

Via (8.86), the sensor force variation is

$$\hat{\mathfrak{f}}_{S}[\psi] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \{f_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{9.94}$$

and so, substituting via the above states,

$$\hat{\mathfrak{f}}_{S}[\psi] = \begin{cases} \left\{ \hat{f}_{pos\text{–}neg,r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\}\,, & \psi = \psi_{pos,U} \\ \left\{ \hat{f}_{neg\text{–}pos,r} \,\middle|\, r \in \mathbb{R}^{\geq 0} \right\} \cup \{f_{\mathbf{0}}\}\,, & \psi = \psi_{neg,U} \\ \{f_{\mathbf{0}}\}, & \text{otherwise.} \end{cases} \tag{9.95}$$

9.2.2.6 Effector forces

Recall from (8.28) that the for any effector point body e_v and state $\psi \in \Psi(e_v)$,

$$\hat{\mathfrak{f}}_{e,v}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_v, \end{cases} \tag{9.96}$$

where

$$\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} = \left\{ \tau \in \mathcal{T}_v \,\middle|\, \left(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}\right) \wedge \left(\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}}\right) \right\} \tag{9.97}$$

and $\hat{f}^{\tau}$ is the force variation such that, for some trajectory $\tau \in \mathcal{T}$, the following conditions hold:

- The domain of $\hat{f}$ is all instants in hybrid mechanical time; that is,

$$\mathit{dom}(\hat{f}) = \Gamma_M, \tag{9.98}$$

- τ has a constant automaton controller state of v; that is,

$$\forall t \in \mathit{dom}(\tau)\, [\Pi_V(\tau(t)) = v]\,, \tag{9.99}$$

- At automaton instant 0, the environment state component of τ is $\mathbf{x}$, while the derivative of the

environment state component of τ is $\dot{\mathbf{x}}$; that is,

$$(\Pi_X(\tau.\mathit{fstate}) = \mathbf{x}) \wedge (\Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}}) \,, \tag{9.100}$$

- For all instants $t \in \mathit{dom}(\hat{f})$, the force of N on G is naturally isomorphic to the second derivative of the environment state component of τ at 0 times $\mathbf{0}_\mathrm{c}$; that is,

$$\forall t \in \mathit{dom}(\hat{f}) \left[\hat{f}(t)(G,N) = (\Pi_X(\ddot{\tau}.\mathit{fstate}), \mathbf{0}_\mathrm{c})\right], \tag{9.101}$$

  and

- Except as required by the above via pairwise equilibriation and additivity, $\hat{f} = f_{\mathbf{0}}$.

As noted in (9.56),

$$\begin{aligned}
\mathcal{T} = \{\tau \mid \big(\tau \text{ is left-closed}\big) \wedge \big(\tau.\mathit{ftime} = 0\big) \wedge \forall t \in \mathit{dom}(\tau)\big([\Pi_{X_1}(\dot{\tau}(t)) = \Pi_{X_2}(\tau(t))]\big) \\
\wedge \Big[\big([\Pi_V(\tau(t) = \mathit{pos})] \wedge [\Pi_{X_2}(\dot{\tau}(t)) = g \sin \Pi_{X_1}(\tau(t)) - u_{max} \cos \Pi_{X_1}(\tau(t))] \\
\wedge [\beta(\Pi_X(\tau(t))) \geq 0]\big) \\
\vee \big([\Pi_V(\tau(t) = \mathit{neg})] \wedge [\Pi_{X_2}(\dot{\tau}(t)) = g \sin \Pi_{X_1}(\tau(t)) - u_{min} \cos \Pi_{X_1}(\tau(t))] \\
\wedge [\beta(\Pi_X(\tau(t))) \leq 0]\big)\Big].
\end{aligned}$$

Consider any set $\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$, and any trajectory $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$. It follows that

$$\begin{aligned}
\Pi_V(\tau.\mathit{fstate}) &= v \\
\Pi_X(\tau.\mathit{fstate}) &= \mathbf{x} \\
\Pi_X(\dot{\tau}.\mathit{fstate}) &= \dot{\mathbf{x}}.
\end{aligned} \tag{9.102}$$

Then let $f^{+,\mathbf{x}}$ be the force variation such that, for all instants $t \in \Gamma_M$,

$$f^{+,\mathbf{x}}(t)(G,N) = (g \sin \mathbf{x}_1 - u_{max} \cos \mathbf{x}_1, 0_\mathrm{c}) \tag{9.103}$$

and, except as required by the above and the definition of a force system, for any bodies $A, B \in \Omega$,

$$f^{+,\mathbf{x}}(t)(A,B) = (0_\mathrm{p}, 0_\mathrm{c}). \tag{9.104}$$

Then for any effector body point e_{pos} and mechanical controller state $\psi \in \Psi(e_{pos})$,

$$\hat{\mathfrak{f}}_{e,pos}[\psi] = \begin{cases} \left\{ \hat{f}^{+,\mathbf{x}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \right\}, & [\psi = \psi_{pos,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \geq 0] \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.105}$$

Similarly, let $f^{-,\mathbf{x}}$ be the force variation such that, for all instants $t \in \Gamma_M$,

$$f^{-,\mathbf{x}}(t)(G, N) = (g \sin \mathbf{x}_1 - u_{min} \cos \mathbf{x}_1, 0_{\mathrm{c}}) \tag{9.106}$$

and, except as required by the above and the definition of a force system, for any bodies $A, B \in \Omega$,

$$f^{-,\mathbf{x}}(t)(A, B) = (0_{\mathrm{p}}, 0_{\mathrm{c}}). \tag{9.107}$$

Then for any effector body point e_{neg} and mechanical controller state $\psi \in \Psi(e_{neg})$,

$$\hat{\mathfrak{f}}_{e,neg}[\psi] = \begin{cases} \left\{ \hat{f}^{-,\mathbf{x}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \right\}, & [\psi = \psi_{neg,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \leq 0] \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.108}$$

Via (8.30), for any mechanical controller state $\psi \in \Psi(E)$, the force variation inclusion for the effector is

$$\hat{\mathfrak{f}}_E[\psi] = \begin{cases} \left\{ \hat{f}^{+,\mathbf{x}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \right\}, & [\psi = \psi_{pos,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \geq 0] \\ \left\{ \hat{f}^{-,\mathbf{x}} \in \mathcal{F}(\top, \mathcal{V}, \Gamma_M) \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \right\}, & [\psi = \psi_{neg,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \leq 0] \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.109}$$

Again, note that the final empty inclusion corresponds to blocking states of the automaton, such as states $(v, \mathbf{x})$ where $\beta(\mathbf{x}) > 0$ and yet $v = neg$. The behavior of the automaton is not defined in such states, and so it is inappropriate to specify any possible motions of the mechanical system, and so any forces.

9.2.2.7 Combined hybrid forces

Recall from (8.104) that, for any mechanical controller state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} + \hat{f}_{v \to v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{9.110}$$

Then, combining the conditional force variation inclusion definitions for the sensor and effector,

$$\hat{\mathfrak{f}}_{e,pos}[\psi] = \begin{cases} \left\{ \hat{f}^{+,\mathbf{x}}, \hat{f}^{+,\mathbf{x}} + \hat{f}_{pos\text{–}neg,r} \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \wedge r \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{pos,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \geq 0] \\ \left\{ \hat{f}^{-,\mathbf{x}}, \hat{f}^{-,\mathbf{x}} + \hat{f}_{neg\text{–}pos,r} \,\middle|\, \mathbf{x} \in [-\pi/2, \pi/2] \times \mathbb{R} \wedge r \in \mathbb{R}^{\geq 0} \right\}, & [\psi = \psi_{neg,\mathbf{x},\dot{\mathbf{x}}}] \wedge [\beta(\mathbf{x}) \leq 0] \\ \{f_{\mathbf{0}}\}, & \psi = \overline{\psi}_V \\ \emptyset, & \text{otherwise.} \end{cases} \tag{9.111}$$

Note that this expression is much simpler than that for the elevator, despite the added complexity of resolving $\mathbf{x}_1$ and $\mathbf{x}_2$ into a single variable. Primarily, this complexity reduction comes from the fact that the inverted pendulum specifies a single velocity for $\mathbf{x}_2$ in each automaton controller state; equivalently, it specifies a single acceleration for $\mathbf{x}_1$, and so a single force system.

Chapter 10

Interpreting Controller-Dependent Forces

In previous chapters, I have discussed the idea of forces dependent on the state of a universal controller. In those sections, however, the forces might not necessarily be forces of the controller itself; in other words, a controller might determine the forces between two bodies, both separate from the controller itself.

One might prefer instead to view forces dependent on the state of a controller as forces of that controller itself. Thus, rather than determining forces between two separate bodies, the controller might itself exert forces on both of those bodies, producing the same external force on all bodies as in the original description. In many cases, one can choose freely between these two representations, as the resultant behavior of the system is identical. Indeed, under some reasonable assumptions, one may even be able to assign some forces to particular subcontrollers, making those controllers directly responsible for changes in forces dependent on their states.

This chapter builds on and reproduces results from [9].

10.1 Controller interaction forces

To this point, I have considered only conditional forces between physical bodies. My ultimate aim, however, has been to consider such forces as instead being forces of the controllers directly on the physical universe, that is, hybrid forces on bodies in both factor universes. My objective here is to redistribute forces across this hybrid universe so that forces dependent on the state of a controller are clearly interacting with that controller, while preserving the resultant forces on each body in the hybrid universe.

More formally, given a conditional force system

$$f[\cdot] : \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega_{\mathrm{p}}, \mathcal{V}), \tag{10.1}$$

I construct a conditional hybrid force system of the form

$$h[\cdot] : \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega, \mathcal{V}) \tag{10.2}$$

such that, for every body $B \in \Omega$ and state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$h[\psi](B, \overline{B}) = f[\psi](B \sqcap \top_{\mathrm{p}}, \overline{B}^{\mathrm{p}}). \tag{10.3}$$

In particular, if B is in Ω_{p}, then

$$h[\psi](B, \overline{B}) = f[\psi](B, \overline{B}^{\mathrm{p}}). \tag{10.4}$$

By preserving the resultant forces on all physical bodies, I preserve mechanical equivalence under h and f; thus, for instance, bodies under either conditional force system will experience the same changes in inertia.

In the following sections, I reverse the order of my construction in the previous section. Where previously I first defined f and then derived from it $f[0]$ and f^*, here I first define base and differential forces $h[0]$ and h^* such that

$$h[0] \in \mathcal{F}(\Omega, \mathcal{V}) \tag{10.5}$$

and

$$h^*[\cdot, \cdot] : \Omega_{\mathrm{c}} \times \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega, \mathcal{V}). \tag{10.6}$$

I then construct h so that, mirroring the case for f,

$$h^*[C, \psi] = h[C(\psi), 0] - h[0], \tag{10.7}$$

or, equivalently,

$$h[C(\psi), 0] = h^*[C, \psi] + h[0]. \tag{10.8}$$

By first constructing $h[0]$ and h^*, I can preserve additional features of the original force system. In particular, I construct $h[0]$ so as to preserve the base forces under $f[0]$, and I construct h^* so as to reassign the conditional forces dependent on a particular universal state. More formally, for any null-state independent controller C and body $A \in \Omega_{\mathrm{p}}$, I construct h^* so that

$$h^*[C, \psi](A, \overline{A}) = f^*[C, \psi](A, \overline{A}^{\mathrm{p}}). \tag{10.9}$$

More generally, for any $A \in \Omega$, I preserve the resultant forces on the physical components of A, so that

$$h^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A \sqcap \top_{\mathrm{p}}}) = f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}). \tag{10.10}$$

10.1.1 Base interaction forces

I begin by defining the *null hybrid forces* $h[0]$. To do so, suppose first that $\Psi(\top_{\mathrm{c}})$ is singular; that is, suppose that there is only a single universal state. Since no alternatives exist, this state must be chosen as the universal null state, and so I can represent it as 0. In such a case, forces never vary with the state of controllers, because the states of the controllers themselves never vary. Thus, there are no nonzero conditional forces, and so no nonzero physical forces on nonphysical bodies. I can then construct $h[0]$ so that it preserves forces on physical bodies while setting all physical forces on nonphysical bodies to zero. To this end, I define $h[0]$ to be

$$h[0](A, B) \overset{\text{def}}{=} f[0](A \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \tag{10.11}$$

for any $A, B \in \Omega$.

I require $h[0]$ to be a force system, which is entailed by the following lemma.

Lemma 8. *If Ω' is a subuniverse of Ω and $g' : \Omega' \times \Omega' \to \mathcal{V}$ is a force system defined over Ω', then g can be extended to a force system g over Ω that agrees with g' on Ω' and is* $\mathbf{0}$ *on bodies wholly outside the universal body $\top'$ of Ω' by defining*

$$g(A, B) \overset{\text{def}}{=} g'(A \sqcap \top', B \sqcap \top') \tag{10.12}$$

for all $A, B \in \Omega'$.

Proof. For g to be a force system, it must be additive, null passive, and pairwise equilibriated. Consider these properties in order.

Since g' is a force system, it is additive. Then for any $D \in \Omega$ and separate bodies $A, B \in \Omega$,

$$\begin{aligned} g(A, D) + g(B, D) &= g'(A \sqcap \top', D \sqcap \top') + g'(B \sqcap \top', D \sqcap \top') \\ &= g'((A \sqcap \top') \sqcup (B \sqcap \top'), D \sqcap \top'). \end{aligned} \tag{10.13}$$

Meet distributes over join, and so, factoring out a $\top'$ from the first term,

$$\begin{aligned} g(A, D) + g(B, D) &= g'((A \sqcup B) \sqcap \top', D \sqcap \top') \\ &= g(A \sqcup B, D). \end{aligned} \tag{10.14}$$

By a similar argument,

$$g(D, A) + g(D, B) = g(D, A \sqcup B). \tag{10.15}$$

Then g is additive.

Consider next null passivity. The meet of any body with the null body is the null body, and so, for any $A \in \Omega$,

$$g(A, \bot) = g'(A \sqcap \top', \bot \sqcap \top') \qquad (10.16)$$
$$= g'(A \sqcap \top', \bot).$$

Since g' is a force system, it is null passive, and so

$$g'(A \sqcap \top', \bot) = \mathbf{0}. \qquad (10.17)$$

Then

$$g(A, \bot) = \mathbf{0}. \qquad (10.18)$$

By a similar argument,

$$g(\bot, A) = \mathbf{0}. \qquad (10.19)$$

Then g is null passive.

Consider finally pairwise equilibriation. For any $A, B \in \Omega$,

$$g(A, B) = g'(A \sqcap \top', B \sqcap \top'). \qquad (10.20)$$

Since g' is a force system, it is pairwise equilibriated, and so

$$g'(A \sqcap \top', B \sqcap \top') = -g'(B \sqcap \top', A \sqcap \top') \qquad (10.21)$$
$$= -g(B, A).$$

Then

$$g(A, B) = -g(B, A), \qquad (10.22)$$

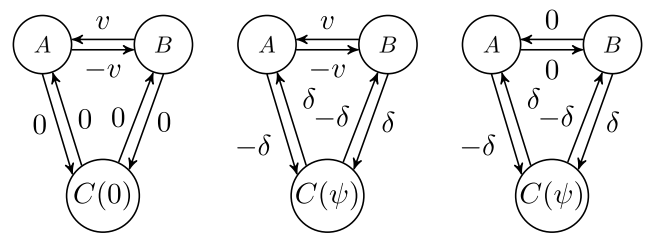
and so g is pairwise equilibriated. Then g is a force system. □

The construction of $h[0]$ is simply a more specific case of this lemma, letting $\Omega' = \Omega_{\mathrm{p}}$, $g' = f[0]$, and $g = h[0]$. Therefore, $h[0]$ is a force system.

Theorem 27. *The mapping $h[0]$ is a force system over Ω that preserves resultant forces on physical bodies with respect to $f[0]$.*

Proof. That $h[0]$ is a force system follows immediately from Lemma 8, letting $\Omega' = \Omega_{\mathrm{p}}$, $g' = f[0]$, and $g = h[0]$. It remains to be shown that it preserves resultant forces on physical bodies. Consider any $A \in \Omega_{\mathrm{p}}$; then

$$h[0](A, \overline{A}) = f[0](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) \qquad (10.23)$$



(a) (b) (c)

Figure 10.1: Variation in hybrid forces between A and B as the state of a nonphysical controller C varies, corresponding to the three cases displayed in Figure 6.1. In (a), controllers are in the null state 0, and $h[0](A, B) = v$. In (b), controllers are in a state ψ in which $h[\psi](A, B)$ remains v but C adds the force δ to the force on A via $h[\psi](A, C)$ and subtracts the force δ from the force on B via $h[\psi](B, C)$. (c) depicts the same state as (b), but here the force values denote the differential hybrid forces $h^*[C, \psi] = h[\psi] - h[0]$.

$$= f[0](A, \overline{A}^{\mathrm{p}}),$$

Then by the definition of preserving resultant force given in (10.3), $h[0]$ preserves resultant force on physical bodies with respect to $f[0]$. □

10.1.2 Differential interaction forces

I now turn to construct the *differential hybrid force systems* $h^*[C, \psi]$. To provide an intuitive sense of the justification for this construction, I first imagine that all controllers and physical bodies are disjoint, that is, that $\top_{\mathrm{p}} \sqcap \top_{\mathrm{c}} = \bot$. I can then remap forces from physical bodies to controllers with no concern that these controllers may themselves be physical bodies as well. This initial framing provides me with three key intuitions; ultimately, these intuitions lead me to a definition of h^* that will also cover cases in which controllers may also be physical bodies. Figure 10.1 provides an illustration of this original intuition.

The first motivating idea is to say that, for any C, separate body A, and state ψ, the system $h^*[C, \psi]$ should reassign differential forces on A that depend on the state of C to C itself. That is, for any $A \in \Omega$ and null-state independent controller $C \in \Omega_{\mathrm{c}}$ such that $A \sqcap C = \bot$, I require

$$h^*[C, \psi](A, C) = f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}). \tag{10.24}$$

The second motivating idea is to say that, for any two bodies A, B separate from a controller C, the system $h^*[C, \psi]$ should assign zero forces between A and B for any state ψ. This dovetails neatly with the previous idea; as the differential forces dependent on C have been assigned to C, they cannot also be assigned to any pair of bodies separate from C. More formally, for any $A, B \in \Omega$ and null-state independent controller $C \in \Omega_{\mathrm{c}}$ such that $A \sqcap C = B \sqcap C = \perp$, I require

$$h^*[C, \psi](A, B) = \mathbf{0}. \tag{10.25}$$

The third motivating idea is that the differential force of a controller on itself should always be zero; that is, the force of a controller on itself does not alter as the controller changes state. More formally, for any null-state independent controller $C \in \Omega_{\mathrm{c}}$, I require

$$h^*[C, \psi](C, C) = \mathbf{0}. \tag{10.26}$$

I say that a differential force system of the form described for h^* is *force adjoint* to a differential force system f^* iff it satisfies (10.24), (10.25), and (10.26) for all bodies $A, B \in \Omega$, null-state independent controllers in Ω_{c}, and states $\psi \in \Psi(\top_{\mathrm{c}})$. I can now derive a formula for h^* under the assumption that it is a differential force system force adjoint to f^* and that all controllers are controller-additive with respect to h^*. I can then prove that the resultant formula for h^* satisfies these properties as long as all controllers are controller-additive with respect to f^*.

Theorem 28. *If a controller-additive differential force system h^* is force adjoint to a complete differential force system f^*, then*

$$h^*[C, \psi](A, B) = f^*[(B \sqcap C), \psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + f^*[(A \sqcap C), \psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \tag{10.27}$$

for each controller $C \in \Omega_{\mathrm{c}}$, state $\psi \in \Psi(\top_{\mathrm{c}})$, and bodies $A, B \in \Omega$.

Proof. Consider any controller $C \in \Omega_{\mathrm{c}}$, state $\psi \in \Psi(\top_{\mathrm{c}})$, and bodies $A, B \in \Omega$. One can divide C into four parts, based on its overlap with the other bodies; thus,

$$C = (A \sqcap B \sqcap C) \sqcup (A \sqcap \overline{B} \sqcap C) \sqcup (\overline{A} \sqcap B \sqcap C) \sqcup (\overline{A} \sqcap \overline{B} \sqcap C). \tag{10.28}$$

Since h^* is controller-additive, then,

$$\begin{aligned} h^*[C, \psi] = h^*[A \sqcap B \sqcap C, \psi] + h^*[A \sqcap \overline{B} \sqcap C, \psi] \\ + h^*[\overline{A} \sqcap B \sqcap C, \psi] + h^*[\overline{A} \sqcap \overline{B} \sqcap C, \psi]. \end{aligned} \tag{10.29}$$

By (10.25),

$$h^*[\overline{A} \sqcap \overline{B} \sqcap C, \psi](A, B) = \mathbf{0}, \tag{10.30}$$

and so $h^*[C, \psi](A, B)$ reduces to the first three terms, that is,

$$h^*[C, \psi](A, B) = h^*[A \sqcap B \sqcap C, \psi](A, B) + h^*[A \sqcap \overline{B} \sqcap C, \psi](A, B) \tag{10.31}$$

$$+ h^*[\overline{A} \sqcap B \sqcap C, \psi](A, B).$$

Consider the last term of this expression, that is, $h^*[\overline{A} \sqcap B \sqcap C, \psi](A, B)$. Clearly A is separate from the controller $\overline{A} \sqcap B \sqcap C$. By (10.25), the force of the part of B separate from this controller on A is $\mathbf{0}$. Then restricting attention to the remaining part of B,

$$h^*[\overline{A} \sqcap B \sqcap C, \psi](A, B) = h^*[\overline{A} \sqcap B \sqcap C, \psi](A, \overline{A} \sqcap B \sqcap C). \tag{10.32}$$

Consider the second term of (10.31), that is, $h^*[A \sqcap \overline{B} \sqcap C, \psi](A, B)$. By a similar argument,

$$h^*[A \sqcap \overline{B} \sqcap C, \psi](A, B) = h^*[A \sqcap \overline{B} \sqcap C, \psi](A \sqcap \overline{B} \sqcap C, B). \tag{10.33}$$

Consider the first term of (10.31), that is, $h^*[A \sqcap B \sqcap C, \psi](A, B)$. The controller $A \sqcap B \sqcap C$ is separate from all parts of A and B except itself, and so by a similar argument,

$$\begin{aligned} h^*[A \sqcap B \sqcap C, \psi](A, B) = {} & h^*[A \sqcap B \sqcap C, \psi](A, A \sqcap B \sqcap C) \\ & + h^*[A \sqcap B \sqcap C, \psi](A \sqcap B \sqcap C, B) \\ & - h^*[A \sqcap B \sqcap C, \psi](A \sqcap B \sqcap C, A \sqcap B \sqcap C). \end{aligned} \tag{10.34}$$

By (10.26), the last of these terms is $\mathbf{0}$, and so

$$\begin{aligned} h^*[A \sqcap B \sqcap C, \psi](A, B) = {} & h^*[A \sqcap B \sqcap C, \psi](A, A \sqcap B \sqcap C) \\ & + h^*[A \sqcap B \sqcap C, \psi](A \sqcap B \sqcap C, B). \end{aligned} \tag{10.35}$$

Then substituting (10.32), (10.33), and (10.35) into (10.31),

$$\begin{aligned} h^*[C, \psi](A, B) = {} & h^*[A \sqcap B \sqcap C, \psi](A, A \sqcap B \sqcap C) \\ & + h^*[A \sqcap B \sqcap C, \psi](A \sqcap B \sqcap C, B) \\ & + h^*[A \sqcap \overline{B} \sqcap C, \psi](A \sqcap \overline{B} \sqcap C, B) \\ & + h^*[\overline{A} \sqcap B \sqcap C, \psi](A, \overline{A} \sqcap B \sqcap C). \end{aligned} \tag{10.36}$$

By pairwise equilibriation, the second and third terms can have their arguments reordered to produce

$$\begin{aligned} h^*[C, \psi](A, B) = {} & h^*[A \sqcap B \sqcap C, \psi](A, A \sqcap B \sqcap C) \\ & - h^*[A \sqcap B \sqcap C, \psi](B, A \sqcap B \sqcap C) \\ & - h^*[A \sqcap \overline{B} \sqcap C, \psi](B, A \sqcap \overline{B} \sqcap C) \\ & + h^*[\overline{A} \sqcap B \sqcap C, \psi](A, \overline{A} \sqcap B \sqcap C). \end{aligned} \tag{10.37}$$

Each of these terms is a force of the form in (10.24), and so, by that requirement,

$$\begin{aligned} h^*[C,\psi](A,B) = f^*[A \sqcap B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) \\ - f^*[A \sqcap B \sqcap C,\psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ - f^*[A \sqcap \overline{B} \sqcap C,\psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ + f^*[\overline{A} \sqcap B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.38}$$

Since f^* is controller-additive, one can combine the first and fourth terms, and the second and third terms, to produce

$$h^*[C,\psi](A,B) = f^*[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) - f^*[A \sqcap C,\psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.39}$$

Since f^* is pairwise equilibriated, one can reverse the body arguments in the second term to produce

$$h^*[C,\psi](A,B) = f^*[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + f^*[A \sqcap C,\psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}), \tag{10.40}$$

as initially required. □

I therefore define $h^*[C,\psi]$ to match this expression for each $A, B \in \Omega$ and $C \in \Omega_{\mathrm{c}}$. That is, I define h^* to be

$$h^*[C,\psi](A,B) \stackrel{\text{def}}{=} f^*[(B \sqcap C),\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + f^*[(A \sqcap C),\psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \tag{10.41}$$

It follows that $h^*[C,\psi]$ is a force system if f^* is a complete, controller-additive differential force system.

Lemma 9. *If f^* is a complete differential force system and h^* is defined by (10.41), then $h^*[C,\psi]$ is null-passive and pairwise equilibriated for each controller $C \in \Omega_{\mathrm{c}}$ and state $\psi \in \Psi(\top_{\mathrm{c}})$.*

Proof. Consider first null-passivity. By (10.41), for any body $A \in \Omega$, controller $C \in \Omega_{\mathrm{c}}$, and state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\begin{aligned} h^*[C,\psi](A,\perp) &= f^*[(\perp \sqcap C),\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + f^*[(A \sqcap C),\psi](\overline{\perp} \sqcap \top_{\mathrm{p}}, \perp \sqcap \top_{\mathrm{p}}) \\ h^*[C,\psi](\perp,B) &= f^*[(B \sqcap C),\psi](\perp \sqcap \top_{\mathrm{p}}, \overline{\perp} \sqcap \top_{\mathrm{p}}) + f^*[(\perp \sqcap C),\psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.42}$$

The meet of $\perp$ and any body is $\perp$, the exterior of $\perp$ is $\top$, and the meet of $\top$ and any body is that body. Thus, the above reduces to

$$\begin{aligned} h^*[C,\psi](A,\perp) &= f^*[\perp,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[(A \sqcap C),\psi](\top_{\mathrm{p}}, \perp) \\ h^*[C,\psi](\perp,B) &= f^*[(B \sqcap C),\psi](\perp, \top_{\mathrm{p}}) + f^*[\perp,\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.43}$$

Since f^* is a force system, it is null-passive. Thus,

$$\begin{aligned} f^*[(A \sqcap C), \psi](\top_{\mathrm{p}}, \bot) &= \mathbf{0} \\ f^*[(B \sqcap C), \psi](\bot, \top_{\mathrm{p}}) &= \mathbf{0}, \end{aligned} \tag{10.44}$$

Then (10.43) reduces to

$$\begin{aligned} h^*[C, \psi](A, \bot) &= f^*[\bot, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ h^*[C, \psi](\bot, B) &= f^*[\bot, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.45}$$

By (6.3),

$$\begin{aligned} f^*[\bot, \psi] &= f[\bot(\psi), 0] - f[0] \\ &= f[0] - f[0] \\ &= f_{\mathbf{0}}, \end{aligned} \tag{10.46}$$

where $f_{\mathbf{0}}$ is the passive force system, as per Section 6.2.3.2. Then

$$\begin{aligned} h^*[C, \psi](A, \bot) &= f_{\mathbf{0}}(A, \bot) \\ &= \mathbf{0} \\ h^*[C, \psi](\bot, B) &= f_{\mathbf{0}}(\bot, B) \\ &= \mathbf{0}. \end{aligned} \tag{10.47}$$

Thus, $h^*[C, \psi]$ is null-passive.

Next, consider pairwise equilibriation. By (10.41), for any bodies $A, B \in \Omega$, controller $C \in \Omega_{\mathrm{c}}$, and state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$h^*[C, \psi](A, B) = f^*[(B \sqcap C), \psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + f^*[(A \sqcap C), \psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \tag{10.48}$$

Since f^* is a force system, it is pairwise equilibriated. Then reversing the body arguments in the above terms,

$$h^*[C, \psi](A, B) = -f^*[(B \sqcap C), \psi](\overline{A} \sqcap \top_{\mathrm{p}}, A \sqcap \top_{\mathrm{p}}) - f^*[(A \sqcap C), \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.49}$$

Reversing the order of the terms themselves,

$$\begin{aligned} h^*[C, \psi](A, B) = &-f^*[(A \sqcap C), \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ &- f^*[(B \sqcap C), \psi](\overline{A} \sqcap \top_{\mathrm{p}}, A \sqcap \top_{\mathrm{p}}) \end{aligned} \tag{10.50}$$

$$
\begin{aligned}
= -\big[&f^*[(A \sqcap C), \psi](B \sqcap \top_\mathrm{p}, \overline{B} \sqcap \top_\mathrm{p}) \\
&+ f^*[(B \sqcap C), \psi](\overline{A} \sqcap \top_\mathrm{p}, A \sqcap \top_\mathrm{p})\big].
\end{aligned}
$$

But by (10.41), the term inside the brackets is precisely $h^*[C, \psi](B, A)$. Then

$$
h^*[C, \psi](A, B) = -h^*[C, \psi](B, A) \tag{10.51}
$$

and so $h^*[C, \psi]$ is pairwise equilibriated. □

Theorem 29. *If f^* is a complete controller-additive differential force system and h^* is defined by (10.41), then $h^*[C, \psi]$ is a force system for each controller $C \in \Omega_\mathrm{c}$ and state $\psi \in \Psi(\top_\mathrm{c})$.*

Proof. Consider any controller $C \in \Omega_\mathrm{c}$ and state $\psi \in \Psi(\top_\mathrm{c})$. By Lemma 9, $h^*[C, \psi]$ is null-passive and pairwise equilibriated. It remains to be shown that it is additive.

Let $A_1, A_2, B \in \Omega$ be bodies separate from each other and from C. I begin by showing that

$$
h^*[C, \psi](A_1 \sqcap A_2, B) = h^*[C, \psi](A_1, B) + h^*[C, \psi](A_2, B). \tag{10.52}
$$

By (10.41),

$$
\begin{aligned}
h^*[C, \psi](A_1, B) &= f^*[(B \sqcap C), \psi](A_1 \sqcap \top_\mathrm{p}, \overline{A_1}^\mathrm{p}) + f^*[(A_1 \sqcap C), \psi](\overline{B}^\mathrm{p}, B \sqcap \top_\mathrm{p}) \qquad (10.53)\\
h^*[C, \psi](A_2, B) &= f^*[(B \sqcap C), \psi](A_2 \sqcap \top_\mathrm{p}, \overline{A_2}^\mathrm{p}) + f^*[(A_2 \sqcap C), \psi](\overline{B}^\mathrm{p}, B \sqcap \top_\mathrm{p}).
\end{aligned}
$$

Summing these two terms,

$$
\begin{aligned}
h^*[C, \psi]&(A_1, B) + h^*[C, \psi](A_2, B) \qquad (10.54)\\
&= f^*[(B \sqcap C), \psi](A_1 \sqcap \top_\mathrm{p}, \overline{A_1}^\mathrm{p}) + f^*[(A_1 \sqcap C), \psi](\overline{B}^\mathrm{p}, B \sqcap \top_\mathrm{p}) \\
&\quad + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_\mathrm{p}, \overline{A_2}^\mathrm{p}) + f^*[(A_2 \sqcap C), \psi](\overline{B}^\mathrm{p}, B \sqcap \top_\mathrm{p}).
\end{aligned}
$$

Since A_1 and A_2 are separate, so are $A_1 \sqcap C$ and $A_2 \sqcap C$. Then since f^* is controller-additive, one can sum the second and fourth terms above, giving

$$
\begin{aligned}
h^*[C, \psi]&(A_1, B) + h^*[C, \psi](A_2, B) \qquad (10.55)\\
&= f^*[(B \sqcap C), \psi](A_1 \sqcap \top_\mathrm{p}, \overline{A_1}^\mathrm{p}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_\mathrm{p}, \overline{A_2}^\mathrm{p}) \\
&\quad + f^*[(A_2 \sqcup A_2) \sqcap C, \psi](\overline{B}^\mathrm{p}, B \sqcap \top_\mathrm{p}).
\end{aligned}
$$

Again, A_1 and A_2 are separate, and so one can partition their exteriors into

$$
\overline{A_1}^\mathrm{p} = \overline{A_1 \sqcup A_2}^\mathrm{p} \sqcup (A_2 \sqcap \top_\mathrm{p}) \tag{10.56}
$$

$$\overline{A_2}^{\mathrm{p}} = \overline{A_1 \sqcup A_2}^{\mathrm{p}} \sqcup (A_1 \sqcap \top_{\mathrm{p}}).$$

Since f^* is a force system, it is additive. Then one can divide the first two terms of (10.55) by this partition, so that

$$\begin{aligned} f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1}^{\mathrm{p}}) &= f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) \\ &\quad + f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, A_2 \sqcap \top_{\mathrm{p}}) \\ f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_2}^{\mathrm{p}}) &= f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) \\ &\quad + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, A_1 \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.57}$$

Adding these two expressions,

$$\begin{aligned} &f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_2}^{\mathrm{p}}) \\ &\quad = f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, A_2 \sqcap \top_{\mathrm{p}}) \\ &\quad\quad + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, A_1 \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.58}$$

Since f^* is a force system, it is pairwise equilibriated. Then the second and fourth terms on the right cancel, and so

$$\begin{aligned} &f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_2}^{\mathrm{p}}) \\ &\quad = f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}). \end{aligned} \tag{10.59}$$

Since f^* is additive, the remaining two right hand terms combine to produce

$$\begin{aligned} &f^*[(B \sqcap C), \psi](A_1 \sqcap \top_{\mathrm{p}}, \overline{A_1}^{\mathrm{p}}) + f^*[(B \sqcap C), \psi](A_2 \sqcap \top_{\mathrm{p}}, \overline{A_2}^{\mathrm{p}}) \\ &\quad = f^*[(B \sqcap C), \psi]((A_1 \sqcup A_2) \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}). \end{aligned} \tag{10.60}$$

Substituting this back into (10.55),

$$\begin{aligned} h^*[C, \psi](A_1, B) + h^*[C, \psi](A_2, B) &= f^*[(B \sqcap C), \psi]((A_1 \sqcup A_2) \sqcap \top_{\mathrm{p}}, \overline{A_1 \sqcup A_2}^{\mathrm{p}}) \\ &\quad + f^*[(A_2 \sqcup A_2) \sqcap C, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.61}$$

But by (10.41), this is precisely $h^*[C, \psi](A_1 \sqcup A_2, B)$, and so

$$h^*[C, \psi](A_1 \sqcap A_2, B) = h^*[C, \psi](A_1, B) + h^*[C, \psi](A_2, B). \tag{10.62}$$

By an identical argument,

$$h^*[C,\psi](B, A_1 \sqcap A_2) = h^*[C,\psi](B, A_1) + h^*[C,\psi](B, A_2). \tag{10.63}$$

Then $h^*[C,\psi]$ is additive. Since it has already been shown to be null-passive and pairwise equilibriated, it is a force system. □

Theorem 30. *If f^* is a complete differential force system, then each controller that is controller-additive with respect to f^* is controller-additive with respect to the differential force system h^* defined in (10.41).*

Proof. Suppose some controller $C \in \Omega_{\mathrm{c}}$ is controller-additive with respect to f^*. By the definition of h^*, for any state $\psi \in \Psi(\top_{\mathrm{c}})$ and bodies $A, B \in \Omega$,

$$\begin{aligned} h^*[C,\psi](A,B) + h^*[\overline{C}^{\mathrm{c}},\psi](A,B) = f^*&[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ &+ f^*[A \sqcap C,\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ &+ f^*[B \sqcap \overline{C}^{\mathrm{c}},\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ &+ f^*[A \sqcap \overline{C}^{\mathrm{c}},\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \end{aligned} \tag{10.64}$$

and

$$h^*[\top_{\mathrm{c}},\psi](A,B) = f^*[B \sqcap \top_{\mathrm{c}},\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[A \sqcap \top_{\mathrm{c}},\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \tag{10.65}$$

Since C is controller-additive with respect to f^*,

$$\begin{aligned} f^*[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[B \sqcap \overline{C}^{\mathrm{c}},\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) &= f^*[B \sqcap \top_{\mathrm{c}},\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ f^*[A \sqcap C,\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) + f^*[A \sqcap \overline{C}^{\mathrm{c}},\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) &= f^*[A \sqcap \top_{\mathrm{c}},\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.66}$$

Then, substituting for the first and third terms, and the second and fourth terms, of (10.64),

$$\begin{aligned} h^*[C,\psi](A,B) + h^*[\overline{C}^{\mathrm{c}},\psi](A,B) = f^*&[B \sqcap \top_{\mathrm{c}},\psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ &+ f^*[A \sqcap \top_{\mathrm{c}},\psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}), \end{aligned} \tag{10.67}$$

and so, substituting via (10.65),

$$h^*[C,\psi](A,B) + h^*[\overline{C}^{\mathrm{c}},\psi](A,B) = h^*[\top_{\mathrm{c}},\psi](A,B). \tag{10.68}$$

Since this holds for any choice of ψ and A, B, it follows that C is controller-additive with respect to h^*. □

Theorem 31. *If f^* is a complete controller-additive differential force system, then h^* as defined by (10.41) is a controller-additive differential force system force adjoint to f^*.*

Proof. Suppose f^* is a complete controller-additive differential force system; then since h^* is the sum of values of f^*, it follows that h^* is also complete. By Theorem 29, h^* is a differential force system. By Theorem 30, h^* is controller-additive. It remains to be shown that h^* is force adjoint to f^*, that is, that the three intuitions in (10.24), (10.25), and (10.26) hold.

First, I show that (10.24) holds. In other words, for any $A \in \Omega$ and null-state independent controller $C \in \Omega_{\mathrm{c}}$ such that $A \sqcap C = \bot$, I wish to show that

$$h^*[C, \psi](A, C) = f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}). \tag{10.69}$$

By the definition of h^*,

$$\begin{aligned} h^*[C, \psi](A, C) &= f^*[C \sqcap C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[A \sqcap C, \psi](\overline{C}^{\mathrm{p}}, C \sqcap \top_{\mathrm{p}}) \\ &= f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[\bot, \psi](\overline{C}^{\mathrm{p}}, C \sqcap \top_{\mathrm{p}}) \\ &= f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathbf{0} \\ &= f^*[C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}). \end{aligned} \tag{10.70}$$

Then (10.24) holds.

Next, I show that (10.25) holds. In other words, for any $A, B \in \Omega$ and null-state independent controller $C \in \Omega_{\mathrm{c}}$ such that $A \sqcap C = B \sqcap C = \bot$, I wish to show that

$$h^*[C, \psi](A, B) = \mathbf{0}. \tag{10.71}$$

Again, by the definition of h^*,

$$\begin{aligned} h^*[C, \psi](A, B) &= f^*[B \sqcap C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[A \sqcap C, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ &= f^*[\bot, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + f^*[\bot, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0}. \end{aligned} \tag{10.72}$$

Then (10.25) holds.

Finally, I show that (10.26) holds. In other words, for any null-state independent controller $C \in \Omega_{\mathrm{c}}$, I wish to show that

$$h^*[C, \psi](C, C) = \mathbf{0}. \tag{10.73}$$

Again, by the definition of h^*,

$$h^*[C,\psi](C,C) = f^*[C \sqcap C,\psi](C \sqcap \top_{\rm p}, \overline{C}^{\rm p}) + f^*[C \sqcap C,\psi](\overline{C}^{\rm p}, C \sqcap \top_{\rm p}) \\ = f^*[C,\psi](C \sqcap \top_{\rm p}, \overline{C}^{\rm p}) + f^*[C,\psi](\overline{C}^{\rm p}, C \sqcap \top_{\rm p}). \tag{10.74}$$

Since f^* is a force system, it is pairwise equilibriated; then, reversing the order of the bodies in the last term,

$$h^*[C,\psi](C,C) = f^*[C,\psi](C \sqcap \top_{\rm p}, \overline{C}^{\rm p}) - f^*[C,\psi](C \sqcap \top_{\rm p}, \overline{C}^{\rm p}) \\ = \mathbf{0}. \tag{10.75}$$

Then (10.26) holds, and so h^* is force adjoint to f^*. □

10.1.3 Total interaction forces

Given the above definitions for $h[0]$ and h^*, I define h for any universal state $\psi \in \Psi(\top_{\rm c})$ to be their sum. More formally,

$$h[\psi] \stackrel{\text{def}}{=} h^*[\top_{\rm c},\psi] + h[0] \tag{10.76}$$

or, more explicitly,

$$h[\psi](A,B) = f[0](A \sqcap \top_{\rm p}, B \sqcap \top_{\rm p}) \\ + f^*[(B \sqcap \top_{\rm c}),\psi](A \sqcap \top_{\rm p}, \overline{A} \sqcap \top_{\rm p}) + f^*[(A \sqcap \top_{\rm c}),\psi](\overline{B} \sqcap \top_{\rm p}, B \sqcap \top_{\rm p}). \tag{10.77}$$

It is straightforward to show that this definition fits neatly with the original formulation for a differential force system, that is, that

$$h^*[C,\psi] = h[C(\psi),0] - h[0] \tag{10.78}$$

if f^* is complete.

Theorem 32. *If f^* is a complete controller-additive differential force system, then h as defined by (10.76) is a conditional force system.*

Proof. By Lemma 27, $h[0]$ is a force system. By Theorem 29, for any choice of $C \in \Omega_{\rm c}$ and $\psi \in \Psi(\top_{\rm c})$, it follows that $h^*[C,\psi]$ is a differential force system. The sum of force systems is a force system, and so $h[C(\psi),0]$ is a force system for any choice of C and ψ. Then h is a conditional force system. □

As yet, I have not shown that this derivation satisfies my original requirement: to preserve the resultant forces on all bodies. I do so in the following theorem.

Theorem 33. *If all controllers are force independent with respect to a conditional force system f such that h^* is complete, then the conditional force system h defined by (10.76) preserves resultant forces in the corresponding conditional force system f, that is, for every body $B \in \Omega_{\mathrm{p}}$ and state $\psi \in \Psi(\top_{\mathrm{c}})$,*

$$h[\psi](B, \overline{B}) = f[\psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.79}$$

Proof. Consider any choice of body $B \in \Omega$, state $\psi \in \Psi(\top_{\mathrm{c}})$, and controller $C \in \Omega_{\mathrm{c}}$. By (10.41),

$$\begin{aligned} h^*[C, \psi](B, \overline{B}) &= f^*[\overline{B} \sqcap C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ &\qquad + f^*[B \sqcap C, \psi](\overline{\overline{B}} \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ &= f^*[\overline{B} \sqcap C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \\ &\qquad + f^*[B \sqcap C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}), \end{aligned} \tag{10.80}$$

noting that $\overline{\overline{B}} = B$. By Theorem 30, h^* is controller-additive. Then since

$$(\overline{B} \sqcap C) \sqcup (B \sqcap C) = C, \tag{10.81}$$

the above reduces to

$$h^*[C, \psi](B, \overline{B}) = f^*[C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \tag{10.82}$$

By (10.11),

$$h[0](B, \overline{B}) = f[0](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.83}$$

By (10.76),

$$h[C(\psi), 0](B, \overline{B}) = h^*[C, \psi](B, \overline{B}) + h[0](B, \overline{B}). \tag{10.84}$$

Substituting via (10.82) and (10.83),

$$h[C(\psi), 0](B, \overline{B}) = f^*[C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) + f[0](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.85}$$

But by (6.3),

$$f^*[C, \psi] = f[C(\psi), 0] - f[0], \tag{10.86}$$

and so

$$f^*[C, \psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) + f[0](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) = f[C(\psi), 0](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.87}$$

Substituting into (10.85),

$$h[C(\psi), 0](B, \overline{B}) = f[C(\psi), 0](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}) \tag{10.88}$$

and so, by (10.3), h preserves resultant forces in f. □

10.2 Extension to force variations and inclusions

As noted at the start of the chapter, the results so far have been previously developed in [9]. That paper, however, concerned itself simply with force systems; this thesis, by contrast, focuses primarily on force variations and inclusions. Thus, I now extend the results presented so far to include such systems.

I begin by extending the idea of a hybrid force system. Summarizing the previous few pages, given a conditional force system $f : \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega_{\mathrm{p}}, \mathcal{V})$, I have defined a conditional hybrid force system

$$h[\cdot] : \Psi(\top_{\mathrm{c}}) \to \mathcal{F}(\Omega, \mathcal{V}) \tag{10.89}$$

such that, for every body $B \in \Omega$ and state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$h[\psi](B, \overline{B}) = f[\psi](B \sqcap \top_{\mathrm{p}}, \overline{B}^{\mathrm{p}}). \tag{10.90}$$

Similarly, given a conditional force variation $\hat{f} : \Psi(\top_{\mathrm{c}}) \to \hat{\mathcal{F}}(\Omega_{\mathrm{p}}, \mathcal{V}, \Gamma_{\mathrm{p}})$, one can define a conditional hybrid force variation

$$\hat{h}[\cdot] : \Psi(\top_{\mathrm{c}}) \to \hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma) \tag{10.91}$$

such that, for every body $B \in \Omega$, instant $t \in \Gamma$ and state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{h}[\psi](t)(B, \overline{B}) = \hat{f}[\psi](\hat{t}_{\mathrm{p}}(t))(B \sqcap \top_{\mathrm{p}}, \overline{B}^{\mathrm{p}}). \tag{10.92}$$

I say that any construction of $\hat{h}[\cdot]$ satisfying this equality preserves resultant forces on physical bodies with respect to $\hat{f}[\cdot]$.

Again, given a conditional force inclusion $\mathfrak{f} : \Psi(\top_{\mathrm{c}}) \to 2^{\mathcal{F}(\Omega_{\mathrm{p}}, \mathcal{V})}$, one can define a conditional hybrid force inclusion

$$\mathfrak{h}[\cdot] : \Psi(\top_{\mathrm{c}}) \to 2^{\mathcal{F}(\Omega, \mathcal{V})} \tag{10.93}$$

such that, for every state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\mathfrak{h}[\psi] = \left\{ h \,\middle|\, \exists f \in \mathfrak{f}[\psi] \left[\forall B \in \Omega \left(h(B, \overline{B}) = f(B \sqcap \top_{\mathrm{p}}, \overline{B}^{\mathrm{p}}) \right) \right] \right\}. \tag{10.94}$$

I say that any construction of $\mathfrak{h}[\cdot]$ satisfying this equality preserves resultant forces on physical bodies with respect to $\mathfrak{f}[\cdot]$.

Finally, given a conditional force variation inclusion $\hat{\mathfrak{f}} : \Psi(\top_{\mathrm{c}}) \to 2^{\hat{\mathcal{F}}(\Omega_{\mathrm{p}}, \mathcal{V}, \Gamma_{\mathrm{p}})}$, one can define a conditional hybrid force variation inclusion

$$\hat{\mathfrak{h}}[\cdot] : \Psi(\top_\mathrm{c}) \to 2^{\hat{\mathfrak{F}}(\Omega,\mathcal{V},\Gamma)} \tag{10.95}$$

such that, for every state $\psi \in \Psi(\top_\mathrm{c})$,

$$\hat{\mathfrak{h}}[\psi] = \left\{ \hat{h} \,\middle|\, \exists f \in \mathfrak{f}[\psi] \left[\forall t \in \Gamma \left[\forall B \in \Omega \left(h(t)(B, \overline{B}) = f(\hat{t}_\mathrm{p}(t))(B \sqcap \top_\mathrm{p}, \overline{B}^\mathrm{p}) \right) \right] \right] \right\}. \tag{10.96}$$

I say that any construction of $\hat{\mathfrak{h}}[\cdot]$ satisfying this equality preserves resultant forces on physical bodies with respect to $\hat{\mathfrak{f}}[\cdot]$.

10.2.1 Extension of base interaction forces

For each of the above constructions, I define a null hybrid force variation or system, as appropriate.

10.2.1.1 Force variations

Given a construction of a hybrid force variation $\hat{h}$ from a force variation $\hat{f}$, I define $\hat{h}[0]$ to be the force variation such that, for any $A, B \in \Omega$ and $t \in \Gamma$,

$$\hat{h}[0](t)(A, B) \stackrel{\mathrm{def}}{=} \hat{f}[0](\hat{t}_\mathrm{p}(t))(A \sqcap \top_\mathrm{p}, B \sqcap \top_\mathrm{p}). \tag{10.97}$$

One can straightforwardly extend Lemma 8 and Theorem 27 to show that $\hat{h}[0]$ is a force variation that preserves resultant forces.

Lemma 10. *If Ω' is a subuniverse of Ω and $\hat{g}' : \Gamma' \to \mathcal{F}(\Omega', \mathcal{V})$ is a force variation defined over Ω', then $\hat{g}'$ can be extended to a force variation $\hat{g} : \Gamma \to \mathcal{F}(\Omega, \mathcal{V})$ over Ω that at any instant $t \in \Gamma$ agrees with $\hat{g}'$ on Ω' at $\hat{t}'(t)$, and is always $\mathbf{0}$ on bodies wholly outside the universal body $\top'$ of Ω'.*

Proof. Since Lemma 8 holds at every instant, this lemma holds. □

Theorem 34. *The mapping $\hat{h}[0]$ is a force variation over Ω that preserves resultant forces on physical bodies with respect to $\hat{f}[0]$.*

Proof. This theorem follows immediately from Lemma 10, letting $\Omega' = \Omega_\mathrm{p}$, $\Gamma' = \Gamma_\mathrm{p}$, $\hat{g}' = \hat{f}[0]$, and $\hat{g} = \hat{h}[0]$. □

10.2.1.2 Force inclusions

Similar extensions are possible for a hybrid force inclusion $\mathfrak{h}$ constructed from a force inclusion $\mathfrak{f}$. As discussed in Section 6.2.3.2, when defining force independence and force additivity for force inclusions, I consider only those cases in which $|\mathfrak{f}(0)| = 1$, that is, such that 0 maps to a single force system. I repeat that restriction here. Letting f be the force system such that $\mathfrak{f}[0] = \{f\}$, then, $\mathfrak{h}[0]$ is the force inclusion such that

$$\mathfrak{h}[0] \stackrel{\text{def}}{=} \left\{ h \in \mathcal{F}(\Omega, \mathcal{V}) \,\middle|\, \forall A, B \in \Omega \left[h(A,B) = f(A \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \right] \right\}. \tag{10.98}$$

Note that in this case, $\mathfrak{h}[0]$ likewise contains a single element. Again, equivalent theorems show that $\mathfrak{h}[0]$ is a force inclusion.

Lemma 11. *If Ω' is a subuniverse of Ω and $\mathfrak{g}' : \Omega' \times \Omega' \to 2^{\mathcal{F}(\Omega,\mathcal{V})}$ is a force inclusion defined over Ω' such that $|\mathfrak{g}'| = 1$, then $\mathfrak{g}'$ can be extended to a force inclusion $\mathfrak{g}$ over Ω whose sole member agrees with the sole member of $\mathfrak{g}'$ on Ω' and is $\mathbf{0}$ on bodies wholly outside the universal body $\top'$ of Ω'.*

Proof. Since Lemma 8 holds for the one force system in $\mathfrak{g}'$, this lemma holds. ☐

Theorem 35. *If $|\mathfrak{f}[0]| = 1$, then the mapping $\mathfrak{h}[0]$ is a force inclusion over Ω whose sole member preserves resultant forces on physical bodies relative to $\mathfrak{f}[0]$.*

Proof. This theorem follows immediately from Lemma 11, letting $\Omega' = \Omega_{\mathrm{p}}$, $\mathfrak{g}' = \mathfrak{f}[0]$, and $\mathfrak{g} = \mathfrak{h}[0]$. ☐

10.2.1.3 Force variation inclusions

Finally, one can provide a similar extension of the results for a hybrid force variation inclusion $\hat{\mathfrak{h}}$ constructed from a force variation inclusion $\hat{\mathfrak{f}}$. Again considering only those cases for which $|\hat{\mathfrak{f}}(0)| = 1$, and letting $\hat{f}$ be the force variation such that $\hat{\mathfrak{f}}[0] = \{\hat{f}\}$, $\hat{\mathfrak{h}}[0]$ is the force variation inclusion such that

$$\hat{\mathfrak{h}}[0] \stackrel{\text{def}}{=} \left\{ \hat{h} \in \mathcal{F}(\Omega, \mathcal{V}, \Gamma) \,\middle|\, \forall A, B \in \Omega \left[\forall t \in \Gamma \left[h_t(A,B) = f_{\hat{t}_{\mathrm{p}}(t)}(A \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \right] \right] \right\}. \tag{10.99}$$

Again, equivalent theorems show that $\hat{\mathfrak{h}}$ is a force variation inclusion.

Lemma 12. *If Ω' is a subuniverse of Ω and $\hat{\mathfrak{g}}' : \Gamma' \to 2^{\hat{\mathcal{F}}(\Omega',\mathcal{V},\Gamma')}$ is a force variation inclusion defined over Ω' such that $|\hat{\mathfrak{g}}'| = 1$, then $\hat{\mathfrak{g}}'$ can be extended to a force variation inclusion $\hat{\mathfrak{g}} : \Gamma \to 2^{\hat{\mathcal{F}}(\Omega,\mathcal{V},\Gamma)}$ over Ω whose sole member agrees at every instant $t \in \Gamma$ with the sole member of $\hat{\mathfrak{g}}'$ on Ω' at $\hat{t}'(t)$, and is always $\mathbf{0}$ on bodies wholly outside the universal body $\top'$ of Ω'.*

Proof. Since Lemma 10 holds for the one force variation in $\hat{\mathfrak{g}}'$, this lemma holds. ☐

Theorem 36. *If $|\hat{\mathfrak{f}}[0] = 1$, then the mapping $\hat{\mathfrak{h}}[0]$ is a force variation inclusion over Ω that preserves resultant forces on physical bodies relative to $\hat{\mathfrak{f}}[0]$.*

Proof. This theorem follows immediately from Lemma 12 by letting $\Omega' = \Omega_{\mathrm{p}}$, $\Gamma' = \Gamma_{\mathrm{p}}$, $\hat{\mathfrak{g}}' = \hat{\mathfrak{f}}[0]$, and $\hat{\mathfrak{g}} = \hat{\mathfrak{h}}[0]$. ☐

10.2.2 Extension of differential interaction forces

The intuitions for extending differential interaction forces to variations and inclusions are functionally identical to those in Section 10.1.2, only extended over time, sets of force systems, or both, as appropriate.

10.2.2.1 Force variations

A differential force variation $\hat{h}^*$ is *force adjoint* to a differential force variation $\hat{f}^*$ iff the following three conditions hold:

- For every instant $t \in \Gamma$, body $A \in \Omega$, null-state independent controller $C \in \Omega_{\mathrm{c}}$, and state $\psi \in \Psi(\top_{\mathrm{c}})$,
$$h^*_t[C,\psi](A,C) = f^*_{\hat{t}_{\mathrm{p}}(t)}[C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}). \tag{10.100}$$
- For any instant $t \in \Gamma$, bodies $A, B \in \Omega$, null-state independent controller $C \in \Omega_{\mathrm{c}}$ separate from A and B, and state $\psi \in \Psi(\top_{\mathrm{c}})$,
$$h^*_t[C,\psi](A,B) = \mathbf{0}. \tag{10.101}$$
- For any instant $t \in \Gamma$, controller $C \in \Omega_{\mathrm{c}}$, and state $\psi \in \Psi(\top_{\mathrm{c}})$,
$$h^*_t[C,\psi](C,C) = \mathbf{0}. \tag{10.102}$$

Given such conditions, I define $\hat{h}^*[C,\psi]$ to be the function such that

$$\begin{aligned}\hat{h}^*[C,\psi](t)(A,B) \stackrel{\text{def}}{=} &\hat{f}^*[(B \sqcap C),\psi](\hat{t}_{\mathrm{p}}(t))(A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) \\ &+ \hat{f}^*[(A \sqcap C),\psi](\hat{t}_{\mathrm{p}}(t))(\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}})\end{aligned} \tag{10.103}$$

for any instant $t \in \Gamma$, bodies $A, B \in \Omega$, controller $C \in \Omega_{\mathrm{c}}$, and $\psi \in \Psi(\top_{\mathrm{c}})$. Again, one can show that this force variation is the unique force variation adjoint to $\hat{f}^*$, and that it is a force variation if $\hat{f}$ is a complete controller additive differential force variation.

Theorem 37. *If a controller-additive differential force variation $\hat{h}^*$ is force adjoint to a complete differential force system $\hat{f}^*$, then*

$$\begin{aligned}\hat{h}^*[C,\psi](t)(A,B) = &\hat{f}^*[(B \sqcap C),\psi](\hat{t}_{\mathrm{p}}(t))(A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) \\ &+ \hat{f}^*[(A \sqcap C),\psi](\hat{t}_{\mathrm{p}}(t))(\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}})\end{aligned} \tag{10.104}$$

for each instant $t \in \Gamma$, controller $C \in \Omega_{\mathrm{c}}$, state $\psi \in \Psi(\top_{\mathrm{c}})$, and bodies $A, B \in \Omega$.

Proof. Since Theorem 28 holds for $\hat{h}^*$ at every instant, this theorem holds. □

Theorem 38. *If $\hat{f}^*$ is a complete controller-additive differential force variation, then $\hat{h}^*$ as defined by (10.103) is a controller-additive differential force variation force adjoint to $\hat{f}^*$.*

Proof. Since Theorems 29, 30, and 31 hold for $\hat{h}^*$ at every instant, this theorem holds. □

10.2.2.2 Force inclusions

Intuitively, one wishes $\mathfrak{h}^*$ to be a mapping from universal controller states to sets of force systems. Some care is required to make sure that the elements of each such set are, indeed, force systems. To illustrate the problem, consider the definition of h^* in (10.41),

$$\begin{aligned} h^*[C,\psi](A,B) =& f^*[(B \sqcap C),\psi](A \sqcap \top_\mathrm{p}, \overline{A} \sqcap \top_\mathrm{p}) \\ &+ f^*[(A \sqcap C),\psi](\overline{B} \sqcap \top_\mathrm{p}, B \sqcap \top_\mathrm{p}). \end{aligned} \tag{10.105}$$

Note that h^* is composed of forces from two separate force systems, and that the particular force systems depend on the choice of bodies A and B. Thus, one cannot simply say that $h^*[C,\psi]$ is the sum of force systems; it is not, as different force systems are summed to produce forces on different bodies. A similar problem exists for force inclusions. Here, though, the situation is somewhat more complex, as each choice of A and B now specifies sets of force systems. I resolve this difficulty via the following construction.

Given a conditional force inclusion $\mathfrak{f}^*$, a *conditional force system assignment* for $\mathfrak{f}^*$ is a mapping $\mathfrak{a} : \Omega_\mathrm{c} \times \Psi(\top_\mathrm{c}) \to \mathcal{F}(\Omega_\mathrm{p}, \mathcal{V})$ that associates each controller $C \in \Omega_\mathrm{c}$ and state $\psi \in \Psi(\top_\mathrm{c})$ with a force system in $\mathfrak{f}^*[C,\psi]$. In other words, for any choice of C and ψ,

$$\mathfrak{a}[C,\psi] \in \mathfrak{f}^*[C,\psi]. \tag{10.106}$$

Since every element of $\mathfrak{f}^*[C,\psi]$ is a force system, $\mathfrak{a}[C,\psi]$ is a force system, and so $\mathfrak{a}$ is a conditional force system. I write $\mathfrak{A}(\mathfrak{f}^*)$ for the set of all conditional force system assignments for $\mathfrak{f}^*$, and I say that a conditional force system assignment *includes* a force system f^* iff, for some controller C and state ψ,

$$\mathfrak{a}[C,\psi] = f^*. \tag{10.107}$$

Let ψ be a state; if $\mathfrak{f}^*$ is a complete controller-additive inclusion such that $|\mathfrak{f}[0]| = 1$, then for any force system $f^* \in \mathfrak{f}^*[\top_\mathrm{c},\psi]$, there exists a controller-additive conditional force system assignment $\mathfrak{a}$ such that $\mathfrak{a}[\top_\mathrm{c},\psi] = f^*$.

Theorem 39. *If $\mathfrak{f}^*$ is a complete controller-additive differential force inclusion such that $|\mathfrak{f}[0]| = 1$, then for any choice of state $\psi \in \Psi(\top_\mathrm{c})$ and force system $f^* \in \mathfrak{f}^*[\top_\mathrm{c},\psi]$, there exists a controller-additive* *conditional force system assignment* $\mathfrak{a} \in \mathfrak{A}(\mathfrak{f}^*)$ *such that*

$$\mathfrak{a}[\top_{\mathrm{c}}, \psi] = f^*. \tag{10.108}$$

Proof. Let $\mathfrak{f}^*$ be a complete controller-additive differential force inclusion. For any state $\psi \in \Psi(\top_{\mathrm{c}})$, let f^* be some force system in $\mathfrak{f}^*[\top_{\mathrm{c}}, \psi]$. Since $\mathfrak{f}^*$ is controller-additive, by (6.57), for any controller $C \in \Omega_{\mathrm{c}}$, there exists some force systems $f^*_C \in \mathfrak{f}^*[C, \psi]$, $f^*_{\overline{C}^{\mathrm{c}}} \in \mathfrak{f}^*[\overline{C}^{\mathrm{c}}, \psi]$, and $f^*_{\perp_{\mathrm{c}}} \in \mathfrak{f}^*[\perp_{\mathrm{c}}, \psi]$ such that

$$f^* = f^*_C + f^*_{\overline{C}^{\mathrm{c}}} - f^*_{\perp_{\mathrm{c}}}. \tag{10.109}$$

By Lemma 1,

$$f^*_{\perp_{\mathrm{c}}} = f_{\mathbf{0}} \tag{10.110}$$

and so contributes no nonzero forces. Then, dropping $f^*_{\perp_{\mathrm{c}}}$ from the above argument, there exist choices of $f^*_C \in \mathfrak{f}^*[C, \psi]$ and $f^*_{\overline{C}^{\mathrm{c}}} \in \mathfrak{f}^*[\overline{C}^{\mathrm{c}}, \psi]$ such that

$$f^* = f^*_C + f^*_{\overline{C}^{\mathrm{c}}}. \tag{10.111}$$

Let $\mathfrak{a}$ be the conditional force system assignment such that, for each controller $C \neq \top_{\mathrm{c}}$,

$$\mathfrak{a}[C, \psi] = f^*_C, \tag{10.112}$$

letting $\mathfrak{a}[\top_{\mathrm{c}}, \psi] = f^*$. Then for any choice of $C \in \Omega_{\mathrm{c}}$,

$$\begin{aligned} \mathfrak{a}[C, \psi] + \mathfrak{a}[\overline{C}^{\mathrm{c}}, \psi] &= f^*_C + f^*_{\overline{C}^{\mathrm{c}}} \\ &= f^* \\ &= \mathfrak{a}[\top_{\mathrm{c}}, \psi]. \end{aligned}$$

Then $\mathfrak{a}$ is controller-additive, and

$$\mathfrak{a}[\top_{\mathrm{c}}, \psi] = f^*. \tag{10.113}$$

□

Given a controller-additive conditional force inclusion $\mathfrak{f}^*$ such that $|\mathfrak{f}[0]| = 1$, I write $\mathfrak{A}_{ca}(\mathfrak{f}^*)$ for the set of all controller-additive conditional force system assignments for $\mathfrak{f}^*$. Then a differential force inclusion $\mathfrak{h}^*$ is *force adjoint* to a differential force inclusion $\mathfrak{f}^*$ such that $|\mathfrak{f}[0]| = 1$ iff the following three conditions hold for every force system $h^* \in \mathfrak{h}^*[C, \psi]$, for every choice of null-state independent $C \in \Omega_{\mathrm{c}}$ and $\psi \in \Psi(\top_{\mathrm{c}})$:

- There exists some conditional force system assignment $\mathfrak{a} \in \mathfrak{A}_{ca}(\mathfrak{f}^*)$ so that, for every body $A \in \Omega$

such that $A \sqcap C = \bot$,

$$h^*(A,C) = \mathfrak{a}[C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}). \tag{10.114}$$

- For any bodies $A, B \in \Omega$ separate from C,

$$h^*(A,B) = \mathbf{0}. \tag{10.115}$$

- For any such choice of C,

$$h^*(C,C) = \mathbf{0}. \tag{10.116}$$

Then I define $\mathfrak{h}^*[C,\psi]$ to be the set such that

$$\begin{aligned}\mathfrak{h}^*[C,\psi] \overset{\text{def}}{=} \Big\{h^* \in \mathcal{F}(\Omega,\mathcal{V}) \,\Big|\, &\exists \mathfrak{a} \in \mathfrak{A}_{ca}(\mathfrak{f}^*) \Big[\forall A,B \in \Omega \\ &\big[h^*(A,B) = \mathfrak{a}[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + \mathfrak{a}[A \sqcap C,\psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}})\big]\Big]\Big\}.\end{aligned} \tag{10.117}$$

Again, one can show that $\mathfrak{h}^*$ is the unique force inclusion that is force adjoint to $\mathfrak{f}^*$, and that it is a force variation if $\mathfrak{f}$ is a complete controller-additive differential force variation.

Theorem 40. *If a controller-additive differential force inclusion $\mathfrak{h}^*$ is force adjoint to a complete differential force inclusion $\mathfrak{f}^*$, then*

$$\begin{aligned}\mathfrak{h}^*[C,\psi] = \Big\{h^* \in \mathcal{F}(\Omega,\mathcal{V}) \,\Big|\, &\exists \mathfrak{a} \in \mathfrak{A}_{ca}(\mathfrak{f}^*) \Big[\forall A,B \in \Omega \\ &\big[h^*(A,B) = \mathfrak{a}[B \sqcap C,\psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + \mathfrak{a}[A \sqcap C,\psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}})\big]\Big]\Big\}.\end{aligned} \tag{10.118}$$

for each controller $C \in \Omega_{\mathrm{c}}$, state $\psi \in \Psi(\top_{\mathrm{c}})$, and bodies $A, B \in \Omega$.

Proof. This proof is structurally identical to Theorem 28, showing that each conditional force system $h^* \in \mathfrak{h}^*[C,\psi]$ must fit the pattern in (10.118) for some conditional force system assignment in $\mathfrak{A}_{ca}(\mathfrak{f}^*)$. □

Theorem 41. *If $\mathfrak{f}^*$ is a complete controller-additive differential force inclusion such that $|\mathfrak{f}[0]| = 1$, and $\mathfrak{h}^*$ is defined by (10.117), then $h^*[C,\psi]$ is a force inclusion for each controller $C \in \Omega_{\mathrm{c}}$ and state $\psi \in \Psi(\top_{\mathrm{c}})$.*

Proof. This proof is structurally identical to Theorem 29, showing that each conditional force system $h^* \in \mathfrak{h}^*[C,\psi]$ must fit the pattern in (10.117) for some conditional force system assignment in $\mathfrak{A}_{ca}(\mathfrak{f}^*)$. □

Theorem 42. *If $\mathfrak{f}^*$ is a complete differential force inclusion such that $|\mathfrak{f}[0]| = 1$, then each controller that is controller-additive with respect to $\mathfrak{f}^*$ is controller-additive with respect to the differential force inclusion $\mathfrak{h}^*$ defined in (10.117).*

Proof. Following the argument of Theorem 39, to show that C is controller-additive with respect to $\mathfrak{h}^*$ requires showing that, for any state $\psi \in \Psi(\top_{\mathrm{c}})$ and force system $h_1^* \in \mathfrak{h}^*[\top_{\mathrm{c}}, \psi]$, there exists some force systems $h_2^* \in \mathfrak{h}^*[C, \psi]$ and $h_3^* \in \mathfrak{h}^*[\overline{C}^{\mathrm{c}}, \psi]$ such that

$$h_1^* = h_2^* + h_3^*. \tag{10.119}$$

Given a controller C that is controller-additive with respect to $\mathfrak{f}^*$, consider any state $\psi \in \Psi(\top_{\mathrm{c}})$ and force system $h_1^* \in \mathfrak{h}^*[\top_{\mathrm{c}}, \psi]$. By (10.117), there exists some conditional force system assignment $\mathfrak{a}_1 \in \mathfrak{A}_{ca}(\mathfrak{f}^*)$ such that, for any bodies $A, B \in \Omega$,

$$h_1^*(A, B) = \mathfrak{a}_1[B \sqcap \top_{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap \top_{\mathrm{c}}, \psi](\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \tag{10.120}$$

Since $\mathfrak{a}_1$ is in $\mathfrak{A}_{ca}(\mathfrak{f}^*)$, there exist force systems $h_2^* \in \mathfrak{h}^*[C, \psi]$ and $h_3^* \in \mathfrak{h}^*[\overline{C}^{\mathrm{c}}, \psi]$ such that, for any $A, B \in \Omega$,

$$\begin{aligned} h_2^*(A, B) &= \mathfrak{a}_1[B \sqcap C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap C, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ h_3^*(A, B) &= \mathfrak{a}_1[B \sqcap \overline{C}^{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap \overline{C}^{\mathrm{c}}, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \end{aligned} \tag{10.121}$$

and so

$$\begin{aligned} h_2^*(A, B) + h_3^*(A, B) =& \mathfrak{a}_1[B \sqcap C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap C, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ &+ \mathfrak{a}_1[B \sqcap \overline{C}^{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap \overline{C}^{\mathrm{c}}, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.122}$$

But since $\mathfrak{a}_1$ is in $\mathfrak{A}_{ca}(\mathfrak{f}^*)$, it is controller-additive, and so

$$\begin{aligned} \mathfrak{a}_1[B \sqcap \top_{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) =& \mathfrak{a}_1[B \sqcap C, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ &+ \mathfrak{a}_1[B \sqcap \overline{C}^{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) \\ \mathfrak{a}_1[A \sqcap \top_{\mathrm{c}}, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) =& \mathfrak{a}_1[A \sqcap C, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ &+ \mathfrak{a}_1[A \sqcap \overline{C}^{\mathrm{c}}, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \end{aligned} \tag{10.123}$$

Then, substituting into (10.122),

$$\begin{aligned} h_2^*(A, B) + h_3^*(A, B) =& \mathfrak{a}_1[B \sqcap \top_{\mathrm{c}}, \psi](A \sqcap \top_{\mathrm{p}}, \overline{A}^{\mathrm{p}}) + \mathfrak{a}_1[A \sqcap \top_{\mathrm{c}}, \psi](\overline{B}^{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \\ =& h_1^*(A, B). \end{aligned} \tag{10.124}$$

Then there exist force systems $h_2^* \in \mathfrak{h}^*[C, \psi]$ and $h_3^* \in \mathfrak{h}^*[\overline{C}^{\mathrm{c}}, \psi]$ such that

$$h_1^* = h_2^* + h_3^*, \tag{10.125}$$

and so C is controller-additive with respect to $\mathfrak{h}^*$. $\square$

Theorem 43. *If $\mathfrak{f}^*$ is a complete controller-additive differential force inclusion, then $\mathfrak{h}^*$ as defined by (10.117) is a controller-additive differential force inclusion force adjoint to $\mathfrak{f}^*$.*

Proof. This proof is structurally identical to Theorem 31, showing that, for any choice of $C \in \Omega_\mathrm{c}$ and $\psi \in \Psi(\top_\mathrm{c})$, each force system $h^* \in \mathfrak{h}^*[C, \psi]$ satisfies (10.114), (10.115), and (10.116). $\square$

10.2.2.3 Force variation inclusions

Given a differential force variation inclusion $\hat{\mathfrak{f}}^*$, a *conditional force variation assignment* for $\hat{\mathfrak{f}}^*$ is a mapping $\hat{\mathfrak{a}} : \Omega_\mathrm{c} \times \Psi(\top_\mathrm{c}) \to \hat{\mathfrak{F}}(\Omega_\mathrm{p}, \mathcal{V}, \Gamma_\mathrm{p})$ such that, for any choice of controller C and state ψ,

$$\hat{\mathfrak{a}}[C, \psi] \in \hat{\mathfrak{f}}^*[C, \psi]. \tag{10.126}$$

Note that any choice of $\hat{\mathfrak{a}}$ is itself a conditional force variation. I write $\hat{\mathfrak{A}}(\hat{\mathfrak{f}}^*)$ for the set of all conditional force variation assignments for $\hat{\mathfrak{f}}^*$ and $\hat{\mathfrak{A}}_{ca}(\hat{\mathfrak{f}}^*)$ for the set of all controller-additive controller force variation assignments. As in the case of inclusions, there is a controller-additive conditional force variation assignment including any force variation $\hat{f} \in \hat{\mathfrak{f}}[C, \psi]$, for any choice of C and ψ.

Theorem 44. *If $\hat{\mathfrak{f}}^*$ is a complete controller-additive differential force variation inclusion such that $|\hat{\mathfrak{f}}[0]| = 1$, then for any choice of state $\psi \in \Psi(\top_\mathrm{c})$ and force variation $\hat{f}^* \in \hat{\mathfrak{f}}^*[\top_\mathrm{c}, \psi]$, there exists a controller-additive conditional force variation assignment $\hat{\mathfrak{a}} \in \hat{\mathfrak{A}}(\hat{\mathfrak{f}}^*)$ such that*

$$\mathfrak{a}[\top_\mathrm{c}, \psi] = f^*. \tag{10.127}$$

Proof. This proof is structurally identical to the proof of Theorem 39. $\square$

Then a differential force variation inclusion $\hat{\mathfrak{h}}^*$ is force adjoint to $\hat{\mathfrak{f}}^*$ iff the following three conditions hold for every force variation $\hat{h}^* \in \hat{\mathfrak{h}}^*[C, \psi]$, for every choice of null-state independent $C \in \Omega_\mathrm{c}$ and $\psi \in \Psi(\top_\mathrm{c})$:

- There is some $\hat{\mathfrak{a}} \in \hat{\mathfrak{A}}_{ca}(\hat{\mathfrak{f}}^*)$ such that, for every instant $t \in \Gamma$ and body $A \in \Omega$ such that $A \sqcap C = \perp$,
$$h_t^*(A, C) = \hat{\mathfrak{a}}_{\hat{t}_\mathrm{p}(t)}(A \sqcap \top_\mathrm{p}, \overline{A} \sqcap \top_\mathrm{p}). \tag{10.128}$$

- For any instant $t \in \Gamma$, bodies $A, B \in \Omega$ separate from C, and state $\psi \in \Psi(\top_\mathrm{c})$,
$$h_t^*(A, B) = \mathbf{0}. \tag{10.129}$$

- For any instant $t \in \Gamma$,
$$h_t^*(C, C) = \mathbf{0}. \tag{10.130}$$

I define $\hat{\mathfrak{h}}^*[C,\psi]$ to be the set such that

$$\hat{\mathfrak{h}}^*[C,\psi] \stackrel{\text{def}}{=} \Big\{ \hat{h}^* \in \mathcal{F}(\Omega, \mathcal{V}, \Gamma) \,\Big|\, \exists \hat{\mathfrak{a}} \in \hat{\mathfrak{A}}_{ca}(\hat{\mathfrak{f}}^*) \Big[\forall A,B \in \Omega \Big[\forall t \in \Gamma \\ \big[h_t^*(A,B) = \hat{\mathfrak{a}}_{\hat{t}_{\mathrm{p}}(t)}(A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + \hat{\mathfrak{a}}_{\hat{t}_{\mathrm{p}}(t)}(\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \big] \Big] \Big] \Big\}. \quad (10.131)$$

Again, one can show that this force variation inclusion is the unique force variation inclusion adjoint to $\hat{\mathfrak{f}}^*$, and that it is a force variation inclusion if $\hat{\mathfrak{f}}$ is a complete controller additive differential force variation inclusion.

Theorem 45. *If a controller-additive differential force variation inclusion $\hat{\mathfrak{h}}^*$ is force adjoint to a complete differential force variation inclusion $\mathfrak{f}^*$, then*

$$\hat{\mathfrak{h}}^*[C,\psi] = \Big\{ \hat{h}^* \in \mathcal{F}(\Omega, \mathcal{V}, \Gamma) \,\Big|\, \exists \hat{\mathfrak{a}} \in \hat{\mathfrak{A}}_{ca}(\hat{\mathfrak{f}}^*) \Big[\forall A,B \in \Omega \Big[\forall t \in \Gamma \\ \big[h_t^*(A,B) = \hat{\mathfrak{a}}_{\hat{t}_{\mathrm{p}}(t)}(A \sqcap \top_{\mathrm{p}}, \overline{A} \sqcap \top_{\mathrm{p}}) + \hat{\mathfrak{a}}_{\hat{t}_{\mathrm{p}}(t)}(\overline{B} \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}) \big] \Big] \Big] \Big\}. \quad (10.132)$$

Proof. This proof is identical to the proof of Theorem 40, replacing force systems with force variations. □

Theorem 46. *If $\hat{\mathfrak{f}}^*$ is a complete controller-additive differential force variation inclusion such that $|\hat{\mathfrak{f}}[0]| = 1$, and $\hat{\mathfrak{h}}^*$ is defined by (10.131), then $\hat{h}^*[C,\psi]$ is a force variation inclusion for each controller $C \in \Omega_{\mathrm{c}}$ and state $\psi \in \Psi(\top_{\mathrm{c}})$.*

Proof. This proof is identical to the proof of Theorem 41, replacing force systems with force variations. □

Theorem 47. *If $\hat{\mathfrak{f}}^*$ is a complete differential force variation inclusion such that $|\hat{\mathfrak{f}}[0]| = 1$, then each controller that is controller-additive with respect to $\hat{\mathfrak{f}}^*$ is controller-additive with respect to the differential force variation inclusion $\hat{\mathfrak{h}}^*$ defined in (10.131).*

Proof. This proof is identical to the proof of Theorem 42, replacing force systems with force variations. □

Theorem 48. *If $\hat{\mathfrak{f}}^*$ is a complete controller-additive differential force variation inclusion, then $\hat{\mathfrak{h}}^*$ as defined by (10.131) is a controller-additive differential force variation inclusion force adjoint to $\hat{\mathfrak{f}}^*$.*

Proof. This proof is identical to the proof of Theorem 43, replacing force systems with force variations. □

10.2.3 Extension of total interaction forces

Finally, one can define $\hat{h}$ (or $\mathfrak{h}$, or $\hat{h}$) by summing the relevant terms in the previous sections. Thus, in the case of a force variation, for each $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{h}[\psi] \stackrel{\text{def}}{=} \hat{h}^*[\top_{\mathrm{c}}, \psi] + \hat{h}[0]. \tag{10.133}$$

In the case of a force inclusion, letting $\mathfrak{h}[0] = \{h\}$,

$$\mathfrak{h}[\psi] \stackrel{\text{def}}{=} \{h + h' \mid h' \in \mathfrak{h}^*[\top_{\mathrm{c}}, \psi]\}. \tag{10.134}$$

In the case of a force variation inclusion, letting $\hat{\mathfrak{h}}[0] = \{\hat{h}\}$,

$$\hat{\mathfrak{h}}[\psi] \stackrel{\text{def}}{=} \{\hat{h} + \hat{h}' \mid \hat{h}' \in \hat{\mathfrak{h}}^*[\top_{\mathrm{c}}, \psi]\}. \tag{10.135}$$

In each case, if the original f-based system is controller-additive, then the resulting h-based system is a conditional force system.

Theorem 49. *If $\hat{f}^*$ is a complete controller-additive differential force variation, then $\hat{h}$ as defined by (10.133) is a conditional force variation.*

Proof. This proof is identical to the proof of Theorem 32, replacing force systems with force variations. □

Theorem 50. *If $\mathfrak{f}^*$ is a complete controller-additive differential force inclusion, then $\mathfrak{h}$ as defined by (10.134) is a conditional force inclusion.*

Proof. Since Theorem 32 holds for every element of $\mathfrak{h}[C, \psi]$, for any choice of $C \in \Omega_{\mathrm{c}}$ and $\psi \in \Psi(\top_{\mathrm{c}})$, this theorem holds. □

Theorem 51. *If $\hat{\mathfrak{f}}^*$ is a complete controller-additive differential force variation inclusion, then $\hat{\mathfrak{h}}$ as defined by (10.135) is a conditional force variation inclusion.*

Proof. This proof is identical to the proof of Theorem 50, replacing force systems with force variations. □

Finally, each of the above constructions preserves resultant forces.

Theorem 52. *If all controllers are force independent with respect to a conditional force variation $\hat{f}$ such that $\hat{h}^*$ is complete, then the conditional force variation $\hat{h}$ defined by (10.133) preserves resultant forces relative to the corresponding conditional force variation $\hat{f}$, that is, for every instant $t \in \Gamma$, body $B \in \Omega$ and state $\psi \in \Psi(\top_{\mathrm{c}})$,*

$$h_t[\psi](B, \overline{B}) = f_{\hat{t}_{\mathrm{p}}(t)}[\psi](B \sqcap \top_{\mathrm{p}}, \overline{B} \sqcap \top_{\mathrm{p}}). \tag{10.136}$$

Proof. Since Theorem 33 holds at every instant, this theorem holds. □

Theorem 53. *If all controllers are force independent with respect to a conditional force inclusion* $\mathfrak{f}$ *such that* $\mathfrak{h}^*$ *is complete, then the conditional force inclusion* $\mathfrak{h}$ *defined by (10.134) preserves resultant forces relative to the corresponding conditional force inclusion* $\mathfrak{f}$*, that is, for any state* $\psi \in \Psi(\top_c)$*, a force system* h *is in* $\mathfrak{h}[\psi]$ *iff there exists a force system* $f \in \mathfrak{f}[\psi]$ *such that for every body* $B \in \Omega$*,*

$$h(B, \overline{B}) = f(B \sqcap \top_p, \overline{B} \sqcap \top_p). \tag{10.137}$$

Proof. This proof is structurally identical to Theorem 33 applied to each force system in $\mathfrak{h}[\psi]$, for any choice of $\psi \in \Psi(\top_c)$. □

Theorem 54. *If all controllers are force independent with respect to a conditional force variation inclusion* $\hat{\mathfrak{f}}$ *such that* $\hat{\mathfrak{h}}^*$ *is complete, then the conditional force variation inclusion* $\hat{\mathfrak{h}}$ *defined by (10.135) preserves resultant forces relative to the corresponding conditional force variation inclusion* $\hat{\mathfrak{f}}$*, that is, for any state* $\psi \in \Psi(\top_c)$*, a force variation* $\hat{h}$ *is in* $\hat{\mathfrak{h}}[\psi]$ *iff there exists a force variation* $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ *such that for every instant* $t \in \Gamma$ *and body* $B \in \Omega$ *,*

$$h_t(B, \overline{B}) = f_t(B \sqcap \top_p, \overline{B} \sqcap \top_p). \tag{10.138}$$

Proof. This proof is identical to the proof of Theorem 53, replacing force systems with force variations. □

10.3 Implications for systems corresponding to hybrid automata

Chapter 7 defined a particular transformation from a hybrid automaton A to a hybrid mechanical system M. In the subsequent proofs, I show which of the above properties hold for the various bodies defined in that transformation.

Note that the results defined previously in this chapter assumed that f is a force system only over physical bodies, whereas the forces I construct in Chapter 7 are over both physical and controller bodies. For the purposes of this section, then, one can disregard all non-physical forces. Indeed, the only nonzero non-physical forces in M are the forces of the sensors on the effector, and those forces are both dependent on the sensor state and already assigned to the sensor bodies; in other words, the controller forces are already mediated by controllers. It remains only to consider the nonzero forces on Ω_p, that is, the forces dependent on the state of the effector.

10.3.1 No separate controllers are mutually state independent

Recall from Section 6.1.2 that two controllers C_1 and C_2 are *mutually state independent* iff, for any states $\psi_1 \in \Psi(C_1)$, $\psi_2 \in \Psi(C_2)$, there is exactly one state $\psi \in \Psi(C_1 \sqcup C_2)$ such that $C_1(\psi) = \psi_1$ and $C_2(\psi) = \psi_2$. Unfortunately, the first result that follows from this definition shows that no separate non-null controllers in Ω_c are mutually state independent.

Theorem 55. *Let A be a hybrid automaton, and let M be the corresponding hybrid mechanical system. Let $C_1, C_2 \in \Omega_\mathrm{c}$ be any pair of separate non-null controllers; then C_1 and C_2 are not mutually state independent.*

Proof. By (7.200),

$$\Omega_\mathrm{c} = \{C_E \sqcup C_S \,|\, (C_E \subseteq E) \wedge (C_S \subseteq S)\}\,. \tag{10.139}$$

It follows that C_1 must be a subbody of E, or it must be a subbody of S, or it must be the join of two such subbodies. The same is true of C_2. Again by (7.200),

$$\begin{aligned} E &= \{e_v \mid v \in V\} \\ S &= \{s_v \mid v \in V\}. \end{aligned} \tag{10.140}$$

It follows that at least one of the following must be true:

- There exist body points e_{v_1}, e_{v_2}, where $v_1 \neq v_2$, such that e_{v_1} is in C_1 and e_{v_2} is in C_2.
- There exist body points e_{v_1}, s_{v_2} such that e_{v_1} is in C_1 and s_{v_2} is in C_2.
- There exist body points s_{v_1}, e_{v_2} such that s_{v_1} is in C_1 and e_{v_2} is in C_2.
- There exist body points s_{v_1}, s_{v_2}, where $v_1 \neq v_2$, such that s_{v_1} is in C_1 and s_{v_2} is in C_2.

The two middle cases are interchangeable, since the labeling of C_1 and C_2 is arbitrary. Thus, there remain three basic cases to prove.

First, I show that C_1 and C_2 are not mutually state independent if there exist body points e_{v_1}, e_{v_2}, where $v_1 \neq v_2$, such that e_{v_1} is in C_1 and e_{v_2} is in C_2. By (7.109), for any e_v,

$$\Psi(e_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}. \tag{10.141}$$

By (7.207),

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} = \psi_v \cap \psi_{\mathbf{x},\dot{\mathbf{x}}} \tag{10.142}$$

and

$$\psi_v = \{(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S}) \mid \big(w(e_v) \neq \dot{w}(e_v)\big) \wedge \big(\forall v' \neq v \in V[w(e_{v'}) = \dot{w}(e_{v'})]\big)\}. \tag{10.143}$$

By the mechanical definition of controller states in Section 6.3, a non-point body controller's mechanical states are subsets of the states of its component bodies. It follows that the states of C_1 are subsets of the states of e_{v_1}, and likewise C_2 for e_{v_2}. Then let ψ_1 be any state of C_1 such that ψ_1 is a subset of $\psi_{v_1,\mathbf{x},\dot{\mathbf{x}}}$, for some $\mathbf{x}, \dot{\mathbf{x}} \in X$. (If no such states exist, then no state of any superbody of C_1 is a subset of $\psi_{v_1,\mathbf{x},\dot{\mathbf{x}}}$. But $\top_\mathrm{c}$ is a superbody of C_1, and by (7.208) its state set does contain such states.) It follows that ψ_1 is a subset of ψ_{v_1}, that is,

$$\psi_1 \subseteq \psi_{v_1}. \tag{10.144}$$

Likewise, let ψ_2 be any state of C_2 such that ψ_2 is a subset of $\psi_{v_2,\mathbf{x}',\dot{\mathbf{x}}'}$, for some $\mathbf{x}', \dot{\mathbf{x}}' \in X$. It follows that ψ_2 is a subset of ψ_{v_2}, that is,

$$\psi_2 \subseteq \psi_{v_2}. \tag{10.145}$$

By Lemma 3, ψ_{v_1} and ψ_{v_2} are disjoint, and so ψ_1 and ψ_2 are disjoint. Then consider $C = C_1 \sqcup C_2$; if C_1 and C_2 were mutually state independent, by definition there would be some state $\psi \in \Psi(C)$ such that $C_1(\psi) = \psi_1$ and $C_2(\psi) = \psi_2$. Suppose there is such a state ψ; then by the mechanical definition of controller states, ψ is a subset of both ψ_1 and ψ_2. But ψ_1 and ψ_2 are disjoint, and so their only subset is the empty set. But by the mechanical definition of controller states, controller states are non-empty; then ψ is not a controller state, and the assumption is violated. Thus, no such state ψ exists, and so C_1 and C_2 are not mutually state independent.

Second, I show that C_1 and C_2 are not mutually state independent if there exist body points s_{v_1}, s_{v_2}, where $v_1 \neq v_2$, such that s_{v_1} is in C_1 and s_{v_2} is in C_2. By (7.135), for any s_v,

$$\Psi(s_v) = \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right\} \cup \left\{\overline{\psi}_v\right\} \tag{10.146}$$

By (7.207),

$$\psi_{v,[\![\mathbf{x}]\!]} = \psi_v \cap \psi_{[\![\mathbf{x}]\!]}. \tag{10.147}$$

Then by an argument identical to the above, one can let ψ_1 be a subset of any $\psi_{v_1,[\![\mathbf{x}]\!]}$, and so a subset of ψ_{v_1}. Likewise, one can let ψ_2 be a subset of any $\psi_{v_2,[\![\mathbf{x}']\!]}$, and so a subset of ψ_{v_2}. It follows that ψ_1 and ψ_2 are disjoint, and so that there is no state $\psi \in \Psi(C_1 \sqcup C_2)$ such that $C_1(\psi) = \psi_1$ and $C_2(\psi) = \psi_2$. Thus, C_1 and C_2 are not mutually state-additive.

Finally, I show that C_1 and C_2 are not mutually state independent if there exist body points e_{v_1}, s_{v_2} such that e_{v_1} is in C_1 and s_{v_2} is in C_2, or if an identical condition holds with the identities of C_1 and C_2 exchanged. As before,

$$\Psi(e_{v_1}) = \{\psi_{v_1,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_{v_1}\} \tag{10.148}$$

and

$$\Psi(s_{v_2}) = \left\{\psi_{v_2,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right\} \cup \left\{\overline{\psi}_{v_2}\right\} \tag{10.149}$$

If $v_1 \neq v_2$, then by arguments similar to the above, one can show that the intersection of any subset of any $\psi_{v_1,\mathbf{x},\dot{\mathbf{x}}}$ and any $\psi_{v_2,[\![\mathbf{x}']\!]}$ must be empty, and so that C_1 and C_2 are not mutually state independent. Suppose instead that $v_1 = v_2$, that is, that

$$\begin{aligned}\Psi(s_{v_2}) &= \Psi(s_{v_1}) &(10.150)\\ &= \left\{\psi_{v_1,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right\} \cup \left\{\overline{\psi}_{v_1}\right\}. &(10.151)\end{aligned}$$

Then let ψ_1 be any state of C_1 that is a subset of some $\psi_{v_1,\mathbf{x},\dot{\mathbf{x}}}$, and so a subset of ψ_{v_1}. Let ψ_2 be any state of C_2 that is a subset of $\overline{\psi}_{v_1}$. (Again, if no such state exists, then no superbody of C_2 can have states that are subsets of $\overline{\psi}_{v_1}$. But $\top_{\mathrm{c}}$ is a superbody of C_2, and by (7.208) it does have such a state, namely $\overline{\psi}_V$.) By definition, ψ_{v_1} and $\overline{\psi}_{v_1}$ are disjoint, and so ψ_1 and ψ_2 are disjoint; then by arguments identical to the preceding cases, C_1 and C_2 are not mutually state independent.

Then any separate non-null choices of C_1 and C_2 are not mutually state independent. $\square$

10.3.2 Sensor and effector point bodies are not null-complemented

Recall from 7.208 that

$$\Psi(\top_{\mathrm{c}}) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in V) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G)\} \cup \{\overline{\psi}_V\}. \tag{10.152}$$

I choose $\overline{\psi}_V$ as the universal null state. The null hybrid force function is then

$$\hat{\mathfrak{h}}[\overline{\psi}_V](A, B) = \hat{\mathfrak{f}}[\overline{\psi}_V](A \sqcap \top_{\mathrm{p}}, B \sqcap \top_{\mathrm{p}}). \tag{10.153}$$

By 8.104, for any $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{\hat{f}^{\tau} + \hat{f}_{v\text{–}v',r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right]\right\} \cup \left\{\hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{10.154}$$

It follows that

$$\hat{\mathfrak{h}}[\overline{\psi}_V] = \{\hat{f}_{\mathbf{0}}\}. \tag{10.155}$$

Recall from Section 6.1.3.1 that a controller C is null-complemented with respect to $\overline{\psi}_V$ iff all of its mechanical controller states are null-complemented with respect to $\overline{\psi}_V$. In other words, for any state $\psi \in \Psi(C)$, there must be some state $\psi' \in \Psi(\top_{\mathrm{c}})$ such that $C(\psi') = \psi$ and $\overline{C}^{\mathrm{c}}(\psi') = \overline{C}^{\mathrm{c}}(\overline{\psi}_V)$. I show first that individual sensor and effector point bodies are not null-complemented.

Theorem 56. *Let A be a hybrid automaton, and let M be the corresponding hybrid mechanical controller. No effector point body $e \in \Omega_{\mathrm{c}}$ is null-complemented with respect to $\overline{\psi}_V$.*

Proof. Suppose some effector point body e was null-complemented with respect to $\overline{\psi}_V$. It follows that, for any state $\psi \in \Psi(e)$, there exists some state $\psi' \in \Psi(\top_{\mathrm{c}})$ such that

$$\begin{aligned} e_v(\psi') &= \psi \\ \overline{e_v}^{\mathrm{c}}(\psi') &= \overline{e_v}^{\mathrm{c}}(\overline{\psi}_V). \end{aligned} \tag{10.156}$$

For ease of reference, let $\psi_{\overline{v}} = \overline{e_v}^{\mathrm{c}}(\overline{\psi}_V)$.

By (7.109),

$$\Psi(e_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_v\}. \tag{10.157}$$

Then consider any state $\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \in \Psi(e_v)$. Let ψ' be the state in $\Psi(\top_{\mathrm{c}})$ such that

$$\begin{aligned} e_v(\psi') &= \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \overline{e_v}^{\mathrm{c}}(\psi') &= \psi_{\overline{v}}. \end{aligned} \tag{10.158}$$

By the definition of $\top_{\mathrm{c}}$, the sensor point body s_v is a subbody of $\overline{e_v}^{\mathrm{c}}$. By the state-projection of mechanical controller states,

$$s_v(\overline{\psi}_V) = s_v(\overline{e_v}^{\mathrm{c}}(\overline{\psi}_V)) = s_v(\psi_{\overline{v}}). \tag{10.159}$$

By the definition of $\overline{\psi}_V$, the state $\overline{\psi}_v$ is a superset of $\overline{\psi}_V$. But by (7.135), $\overline{\psi}_v$ is a state of s_v, and so by the mechanical definition of controller states,

$$s_v(\overline{\psi}_V) = \overline{\psi}_v. \tag{10.160}$$

Then

$$s_v(\psi_{\overline{v}}) = \overline{\psi}_v, \tag{10.161}$$

and so $\psi_{\overline{v}}$ is a subset of $\overline{\psi}_v$. By definition, $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v; but by definition, ψ_v and $\overline{\psi}_v$ are disjoint. Then $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ and $\psi_{\overline{v}}$ are disjoint, and so

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{\overline{v}} = \emptyset. \tag{10.162}$$

Equivalently,

$$e_v(\psi') \cap \overline{e_v}^{\mathrm{c}}(\psi') = \emptyset. \tag{10.163}$$

But $e_v(\psi') \cap \overline{e_v}^{\mathrm{c}}(\psi')$ is exactly ψ', and so $\psi' = \emptyset$. Mechanical controller states must be nonempty, and so this is a contradiction; then e_v is not null-complemented with respect to $\overline{\psi}_V$. □

Corollary 57. *Let A be a hybrid automaton, and let M be the corresponding hybrid mechanical controller. No sensor point body $s \in \Omega_{\mathrm{c}}$ is null-complemented with respect to $\overline{\psi}_V$.*

Proof. This proof is functionally identical to the proof of the preceding theorem. $\square$

10.3.3 Effector-sensor pairs are force independent and null-complemented

For any choice of $v \in V$, I write es_v for the *effector-sensor pair body* $e_v \sqcup s_v$. While the individual effector and sensor point bodies are not null-complemented, for any $v \in V$, the effector-sensor pair body es_v is null-complemented, as is any join of effector-sensor pair bodies. I show this result in the following theorems; first, I show that for any $v \in V$,

$$\Psi(es_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}. \tag{10.164}$$

Lemma 13. *For any $v \in V$, let $es_v = e_v \sqcup s_v$. Then*

$$\Psi(es_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}. \tag{10.165}$$

Proof. By (7.109),

$$\Psi(e_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G\} \cup \{\overline{\psi}_v\}. \tag{10.166}$$

By (7.135),

$$\Psi(s_v) = \left\{\psi_{v,[\![\mathbf{x}]\!]} \,\middle|\, [\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}\right\} \cup \left\{\overline{\psi}_v\right\}. \tag{10.167}$$

Clearly

$$\overline{\psi}_v \cap \overline{\psi}_v = \overline{\psi}_v. \tag{10.168}$$

Since any $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of ψ_v, and ψ_v and $\overline{\psi}_v$ are disjoint, it follows that

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_v = \emptyset. \tag{10.169}$$

By a similar argument, for any $\psi_{v,[\![\mathbf{x}]\!]}$,

$$\psi_{v,[\![\mathbf{x}]\!]} \cap \overline{\psi}_v = \emptyset. \tag{10.170}$$

By Lemma 7, for any $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\equiv_G$, and $[\![\mathbf{x}']\!] \in X/\equiv_{\mathcal{D},v}$, if $\mathbf{x}$ is in $[\![\mathbf{x}']\!]$ then

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{10.171}$$

If $\mathbf{x}$ is not in $[\![\mathbf{x}']\!]$, then

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v,[\![\mathbf{x}']\!]} = \emptyset. \tag{10.172}$$

Then the non-empty intersections of one state from $\Psi(e_v)$ and one state from $\Psi(s_v)$ are $\overline{\psi}_v$ and every choice of $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$. By the mechanical definition of controller states, these are precisely the states of es_v,

and so

$$\Psi(es_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_v\}. \tag{10.173}$$

□

For any $U \subseteq V$, let

$$es_U = \bigsqcup_{v \in U} es_v. \tag{10.174}$$

Then if U is nonempty,

$$\Psi(es_U) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in U) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \{\bigcap_{v \in U} \overline{\psi}_v\}. \tag{10.175}$$

Lemma 14. *Given a hybrid automaton A and corresponding hybrid mechancial system M, let U be any nonempty subset of V, and let*

$$es_U = \bigsqcup_{v \in U} es_v. \tag{10.176}$$

Then

$$\Psi(es_U) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in U) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \{\bigcap_{v \in U} \overline{\psi}_v\}. \tag{10.177}$$

Proof. If $|U| = 1$, then this proof reduces to Lemma 13. Otherwise, by that lemma, for any $v \in V$,

$$\Psi(es_v) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, \psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}\} \cup \{\overline{\psi}_v\}. \tag{10.178}$$

As usual, the mechanical controller states of es_U are the non-empty intersections of one mechanical controller state of each es_v, for each $v \in U$.

The remainder of this theorem is structurally identical to the proof of Theorem 19. For any state ψ of es_U formed by taking such an intersection, either all of the states intersected to produce ψ are of the form $\overline{\psi}_v$; or all but one of them are of this form, with the remaining state of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ for some $v \in U$; or two or more of them are of the form $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

As argued in that theorem, the first case produces $\bigcap_{v \in U} \overline{\psi}_v$; since $\bigcap_{v \in U} \overline{\psi}_v$ contains $\overline{\psi}_V$, which is nonempty, it is also nonempty. Any instance of the second case reduces to $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, with $v \in U$; since this is a state of es_v, it is nonempty. Any instances of the third case are empty. Thus,

$$\Psi(es_U) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in U) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \{\bigcap_{v \in U} \overline{\psi}_v\}. \tag{10.179}$$

□

Let $\Omega_{\mathrm{c},es}$ be the subuniverse of Ω_{c} containing all es_v bodies, together with their joins and intersections. That is,

$$\Omega_{\mathrm{c},es} \stackrel{\mathrm{def}}{=} \left\{ \bigsqcup_{v \in U} es_v \,\middle|\, U \subseteq 2^V \right\}. \tag{10.180}$$

The elements of $\Omega_{\mathrm{c},es}$ are force independent with respect to $\hat{\mathfrak{f}}[\cdot]$ and null-complemented with respect to $\overline{\psi}_V$, as shown in the following proofs.

Theorem 58. *Let A be a hybrid automaton, let M be the corresponding hybrid mechanical controller, and let $\Omega_{\mathrm{c},es}$ be the set defined in (10.180). Then the elements of $\Omega_{\mathrm{c},es}$ are force independent with respect to $\hat{\mathfrak{f}}[\cdot]$.*

Proof. By definition, the elements of $\Omega_{\mathrm{c},es}$ are force independent with respect to $\hat{\mathfrak{f}}[\cdot]$ iff, for any $U \subseteq V$, the body

$$es_U = \bigsqcup_{v \in U} es_v. \tag{10.181}$$

is force independent with respect to $\hat{\mathfrak{f}}[\cdot]$. In other words, the es_v bodies with their joins and intersections form a subuniverse of Ω_{c} of force independent bodies.

If $U = \emptyset$, then $es_U = \bot_{\mathrm{c}}$, and the proof is trivial. Otherwise, if $U = V$, then $es_U = \top_{\mathrm{c}}$, and again the proof is trivial.

Otherwise, consider any states $\psi, \psi' \in \Psi(es_U)$ and $\overline{\psi}, \overline{\psi'} \in \Psi(\overline{es_U}^{\mathrm{c}})$ such that the universal states $[\psi, \overline{\psi}]$, $[\psi, \overline{\psi'}]$, $[\psi', \overline{\psi}]$, and $[\psi', \overline{\psi'}]$ are all defined. (If no such states exist, the proof holds trivially.) By the definition of mechanical states, then, each of the intersections $\psi \cap \overline{\psi}$, $\psi \cap \overline{\psi'}$, $\psi' \cap \overline{\psi}$, and $\psi' \cap \overline{\psi'}$ are nonempty. Recall from (6.47) that es_U is force independent iff, for any such states, for any force system $f_{\psi,\overline{\psi}} \in \mathfrak{f}[\psi, \overline{\psi}]$, there exist force systems $f_{\psi',\overline{\psi}} \in \mathfrak{f}[\psi', \overline{\psi}]$, $f_{\psi,\overline{\psi'}} \in \mathfrak{f}[\psi, \overline{\psi'}]$, and $f_{\psi',\overline{\psi'}} \in \mathfrak{f}[\psi', \overline{\psi'}]$ such that

$$f_{\psi,\overline{\psi}}(t) - f_{\psi',\overline{\psi}}(t) = f_{\psi,\overline{\psi'}}(t) - f_{\psi',\overline{\psi'}}(t). \tag{10.182}$$

By Lemma 14,

$$\Psi(es_U) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in U) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\!\equiv_G)\} \cup \{\bigcap_{v \in U} \overline{\psi}_v\}. \tag{10.183}$$

For simplicity, I write $\overline{\psi}_U$ for $\bigcap_{v \in U} \overline{\psi}_v$. But note $es_V = \top_{\mathrm{c}}$, and so $\overline{es_U}^{\mathrm{c}} = es_{V-U}$. Then

$$\Psi(\overline{es_U}^{\mathrm{c}}) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in (V-U)) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/\!\equiv_G)\} \cup \{\bigcap_{v \in (V-U)} \overline{\psi}_v\}. \tag{10.184}$$

Again, I write $\overline{\psi}_{V-U}$ for $\bigcap_{v \in (V-U)} \overline{\psi}_v$.

Then each of ψ, ψ' must either be some $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, for some $v \in U$, or must be $\overline{\psi}_U$. Likewise, each of $\overline{\psi}, \overline{\psi'}$ must either be some $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, for some $v \in (V-U)$, or must be $\overline{\psi}_{V-U}$. Suppose $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, for $v \in U$, and $\overline{\psi} = \psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$, for $v' \in (V-U)$. Then ψ is a subset of ψ_v and $\overline{\psi}$ is a subset is a subset of $\psi_{v'}$, where $v \neq v'$. Then by Lemma 3, ψ and $\overline{\psi}$ are disjoint, and so their union is empty. This contradicts the above requirement that the four intersections be nonempty.

Then it cannot be the case both that either of ψ, ψ' is some $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ and that either of $\overline{\psi}, \overline{\psi'}$ is some $\psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$. Without loss of generality, suppose that neither $\overline{\psi}$ nor $\overline{\psi'}$ are any $\psi_{v',\mathbf{x}',\dot{\mathbf{x}}'}$. It follows from (10.184) that

$$\overline{\psi} = \overline{\psi'} = \overline{\psi}_{V-U}. \tag{10.185}$$

Then (10.182) reduces to the claim that, for any force system $f_{\psi,\overline{\psi}} \in \mathfrak{f}[\psi, \overline{\psi}]$, there exist force systems $f_{\psi',\overline{\psi}}, f_{\psi',\overline{\psi'}} \in \mathfrak{f}[\psi', \overline{\psi}]$ and $f_{\psi,\overline{\psi'}} \in \mathfrak{f}[\psi, \overline{\psi}]$ such that

$$f_{\psi,\overline{\psi}}(t) - f_{\psi',\overline{\psi}}(t) = f_{\psi,\overline{\psi'}}(t) - f_{\psi',\overline{\psi'}}(t). \tag{10.186}$$

Letting $f_{\psi',\overline{\psi}} = f_{\psi',\overline{\psi'}}$ and $f_{\psi,\overline{\psi'}} = f_{\psi,\overline{\psi}}$, this reduces to

$$f_{\psi,\overline{\psi}}(t) - f_{\psi',\overline{\psi}}(t) = f_{\psi,\overline{\psi}}(t) - f_{\psi',\overline{\psi}}(t), \tag{10.187}$$

which is simply the identity. A similar reduction occurs if one instead assumes that neither of ψ, ψ' is some $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Then for any appropriate choice of $\psi, \psi', \overline{\psi}, \overline{\psi}'$, the equation in (10.182) is satisfied. Thus, any es_U, for any $U \subseteq V$, is force independent. $\square$

Theorem 59. *Let A be a hybrid automaton, let M be the corresponding hybrid mechanical controller, and let $\Omega_{\mathrm{c},es}$ be the set defined in (10.180). The elements of $\Omega_{\mathrm{c},es}$ are null-complemented with respect to $\overline{\psi}_V$.*

Proof. Let U be any subset of V, and let

$$es_U = \bigsqcup_{v \in U} es_v. \tag{10.188}$$

Recall that es_U is null-complemented with respect to $\overline{\psi}_V$ iff, for any state $\psi \in \Psi(es_U)$, there must be some state $\psi' \in \Psi(\top_\mathrm{c})$ such that $es_U(\psi') = \psi$ and $\overline{es_U}^{\mathrm{c}}(\psi') = \overline{es_U}^{\mathrm{c}}(\overline{\psi}_V)$.

If $U = \emptyset$, then $es_U = \bot_\mathrm{c}$; if $U = V$, then $es_U = \top_\mathrm{c}$. In either case, the proof is trivial. Otherwise, by Lemma 14,

$$\Psi(es_U) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in U) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \left\{ \bigcap_{v \in U} \overline{\psi}_v \right\}. \tag{10.189}$$

For simplicity, I write $\overline{\psi}_U$ for $\bigcap_{v \in U} \overline{\psi}_v$. But note $es_V = \top_{\mathrm{c}}$, and so $\overline{es_U}^{\mathrm{c}} = es_{V-U}$. Then

$$\Psi(\overline{es_U}^{\mathrm{c}}) = \{\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \,|\, (v \in (V-U)) \wedge (\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G})\} \cup \left\{ \bigcap_{v \in (V-U)} \overline{\psi}_v \right\}. \tag{10.190}$$

Again, I write $\overline{\psi}_{V-U}$ for $\bigcap_{v \in (V-U)} \overline{\psi}_v$.

By the mechanical definition of controller states,

$$\overline{es_v}^{\mathrm{c}}(\overline{\psi}_V) = \overline{\psi}_{V-U}. \tag{10.191}$$

Let ψ be any state of es_U; by the mechanical definition of controller states, if $\psi \cap \overline{\psi}_{V-U}$ is nonempty, then it is a state of $\top_{\mathrm{c}}$. By (10.189), either $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$, for some $v \in U$ and $\mathbf{x}, \dot{\mathbf{x}} \in X$, or $\psi = \overline{\psi}_U$.

Suppose it is the former case, that is, $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$. By Lemma 3, ψ_v and any $\psi_{v'}$ are disjoint if $v \neq v'$. It follows that ψ is a subset of $\overline{\psi}_{v'}$, and so that $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a subset of $\overline{\psi}_{v'}$. Then

$$\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_{v'} = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \tag{10.192}$$

It follows that

$$\begin{aligned} \psi \cap \overline{\psi}_{V-U} &= \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \left(\bigcap_{v' \in (V-U)} \overline{\psi}_{v'} \right) \\ &= \bigcap_{v' \in (V-U)} \left(\psi_{v,\mathbf{x},\dot{\mathbf{x}}} \cap \overline{\psi}_{v'} \right) \\ &= \bigcap_{v' \in (V-U)} \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ &= \psi_{v,\mathbf{x},\dot{\mathbf{x}}}. \end{aligned} \tag{10.193}$$

By (7.208), $\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ is a state in $\Psi(\top_{\mathrm{c}})$.

Suppose instead that the latter case holds, that is, that $\psi = \overline{\psi}_U$. Then

$$\begin{aligned} \psi \cap \overline{\psi}_{V-U} &= \left(\bigcap_{v \in U} \overline{\psi}_v \right) \cap \left(\bigcap_{v \in (V-U)} \overline{\psi}_v \right) \\ &= \bigcap_{v \in V} \overline{\psi}_v \\ &= \overline{\psi}_V. \end{aligned} \tag{10.194}$$

By (7.208), $\overline{\psi}_V$ is a state in $\Psi(\top_{\mathrm{c}})$.

Then for any $\psi \in \Psi(es_v)$, the set $[\psi, \overline{es_v}^{\mathrm{c}}(\overline{\psi}_V)]$ is a state of $\top_{\mathrm{c}}$, and so es_U is null-complemented with respect to $\overline{\psi}_V$. $\square$

As discussed in Section 6.2.3, given a choice of null state such that $\left|\hat{\mathfrak{f}}[0]\right| = 1$, let $\hat{f}$ be the one force variation in $\hat{\mathfrak{f}}[0]$. Then for any $C \in \Omega_{\mathrm{c}}$, the conditional force variation inclusion $\hat{\mathfrak{f}}^*[C, \psi]$ is

$$\hat{\mathfrak{f}}^*[C, \psi] = \left\{ \hat{f}' - \hat{f} \,\middle|\, \hat{f}' \in \hat{\mathfrak{f}}[C(\psi), 0] \right\}. \tag{10.195}$$

By (7.208),

$$\hat{\mathfrak{f}}[\overline{\psi}_V] = \{\hat{f}_{\mathbf{0}}\}, \tag{10.196}$$

and so $\left|\hat{\mathfrak{f}}[\overline{\psi}_V]\right| = 1$. Then for any controller $C \in \Omega_{\mathrm{c}}$ and mechanical controller state $\psi \in \Psi(\top_{\mathrm{c}})$,

$$\begin{aligned} \hat{\mathfrak{f}}^*[C, \psi] &= \{\hat{f}' - \hat{f}_{\mathbf{0}} \mid \hat{f}' \in \hat{\mathfrak{f}}[C(\psi), 0]\} \\ &= \{\hat{f}' \mid \hat{f}' \in \hat{\mathfrak{f}}[C(\psi), 0]\} \\ &= \hat{\mathfrak{f}}[C(\psi), 0]. \end{aligned} \tag{10.197}$$

Since the above is defined for any choice of controller C and mechanical controller state $\psi \in \Psi(\top_{\mathrm{c}})$, by definition $\hat{\mathfrak{f}}^*$ is complete for Ω_{c}. By Corollary 18, then, the elements of $\Omega_{\mathrm{c},es}$ are controller-additive.

Then let $\hat{\mathfrak{h}}$ be the function defined in (10.135) for any controller in $\Omega_{\mathrm{c},es}$. By Theorem 51, $\hat{\mathfrak{h}}$ is a conditional force variation inclusion. By Theorem 54, it preserves resultant forces on every body in Ω.

10.4 Summary of notation

$h[\cdot]$	conditional hybrid force system
$h[0]$	null hybrid force system
$h^*[\cdot,\cdot]$	conditional differential hybrid force system
$\hat{h}[\cdot]$	conditional hybrid force variation
$\hat{h}[0]$	null hybrid force variation
$\hat{h}^*[\cdot,\cdot]$	conditional differential hybrid force variation
$\mathfrak{h}[\cdot]$	conditional hybrid force inclusion
$\mathfrak{h}[0]$	null hybrid force inclusion
$\mathfrak{h}^*[\cdot,\cdot]$	conditional differential hybrid force inclusion
$\hat{\mathfrak{h}}[\cdot]$	conditional hybrid force variation inclusion
$\hat{\mathfrak{h}}[0]$	null hybrid variation inclusion
$\hat{\mathfrak{h}}^*[\cdot,\cdot]$	conditional differential hybrid force variation inclusion
es_v	effector-sensor pair body for automaton controller state v
es_U	join of all effector-sensor pair bodies es_v, with $v \in U$
$\Omega_{\mathrm{c},es}$	subuniverse of Ω_{c} containing all es_U bodies

Chapter 11

Bisimulation

The objective of the preceding work has been to create a system of rules for constructing a hybrid mechanical system M from an arbitrary hybrid automaton A, such that M can be said to *bisimulate* A. Conceptually, a system S_1 can be said to *simulate* another system S_2 iff, for any behavior of S_2, S_1 is capable of reproducing the effects that behavior. S_1 may also be capable of additional behaviors; it might, for instance, be a more detailed model of the same phenomenon as S_2, representing both S_2's actions and more subtle variations on them. If S_2 also simulates S_1, however, then their behaviors must be exactly matched; in such cases, S_1 and S_2 are said to bisimulate each other.

11.1 Automaton-to-automaton bisimulation

Formal definitions of simulation vary with context. Here, I rely on a modified form of the definition of simulation given by Lynch [18]. Two hybrid automata are said to be *comparable* if they have identical environment state spaces. Let $A_1 = (V_1, X, Q_{01}, D_1, \mathcal{T}_1, \Gamma_1)$ and $A_2 = (V_2, X, Q_{02}, D_2, \mathcal{T}_2, \Gamma_2)$ be two comparable hybrid automata, with system states $Q_1 = V_1 \times X$ and $Q_2 = V_2 \times X$, respectively. Let R be a relation on $Q_1 \times Q_2$; then R is a *simulation* iff it satisfies the following three conditions.

First, every initial state of A_1 must be related to an initial state of A_2. That is, for any state $q_1 \in Q_{01}$, there must be some state $q_2 \in Q_{02}$ such that (q_1, q_2) is an element of R.

Second, let (q_1, q_2) be an element of R. For any discrete transition of A_1 beginning in q_1, there must be a corresponding discrete transition of A_2 beginning in q_2 such that the two transitions exhibit identical traces and the two ending states are also related. Formally, consider any pair of system states $(q_1, q_2) \in R$ and closed execution fragment $\alpha_1 = \tau_1 a_1 \tau_1'$ of A_1, where τ_1, τ_1' are point trajectories in $\mathcal{T}_1$, a_1 is an element of D_1, and $\alpha_1.\mathit{fstate} = q_1$. If R is a simulation, then there exists some closed execution fragment $\alpha_2 = \tau_2 a_2 \tau_2'$ of A_2 such that: τ_2, τ_2' are point trajectories in $\mathcal{T}_2$; a_2 is an element of D_2; $\alpha_2.\mathit{fstate} = q_2$; $\mathit{trace}(\alpha_1) = \mathit{trace}(\alpha_2)$; and $(\alpha_1.\mathit{lstate}, \alpha_2.\mathit{lstate})$ is in R.

Third, again let (q_1, q_2) be an element of R. For any closed trajectory beginning in q_1, there must be a closed trajectory beginning in q_2 with an identical trace such that the two ending states are also related in R. Formally, consider any pair of system states $(q_1, q_2) \in R$ and closed execution fragment $\alpha_1 = \tau_1$, where τ_1 is a closed trajectory in $\mathcal{T}_1$ and $\alpha_1.\mathit{fstate} = q_1$. If R is a simulation, then there exists some closed execution fragment $\alpha_2 = \tau_2$ such that: τ_2 is a closed trajectory in $\mathcal{T}_2$; $\alpha_2.\mathit{fstate} = q_2$; $\mathit{trace}(\alpha_1) = \mathit{trace}(\alpha_2)$; and $(\alpha_1.\mathit{lstate}, \alpha_2.\mathit{lstate})$ is in R.

Intuitively, these requirements collectively guarantee that, for any execution of A_1, there is a corresponding execution of A_2 whose trace is identical. In other words, $\mathit{traces}_{A_1} \subseteq \mathit{traces}_{A_2}$. I restate this result from Lynch below, though I omit her proof.

Theorem 60. *Let $A_1 = (V_1, X, Q_{01}, D_1, \mathcal{T}_1, \Gamma_1)$ and $A_2 = (V_2, X, Q_{02}, D_2, \mathcal{T}_2, \Gamma_2)$ be two comparable hybrid automata, with system states $Q_1 = V_1 \times X$ and $Q_2 = V_2 \times X$, such that there exists some simulation R from A_1 to A_2. In other words, R is a relation on $Q_1 \times Q_2$ such that the following properties hold:*

1. *For any initial state $q_1 \in Q_{01}$ of A_1, there exists an initial state $q_2 \in Q_{02}$ of A_2 such that (q_1, q_2) is in R.*

2. *For any pair $(q_1, q_2) \in R$ and closed execution fragment $\alpha_1 = \tau_1 a_1 \tau_1'$ of A_1, where τ_1, τ_1' are point trajectories in $\mathcal{T}_1$, a_1 is an element of D_1, and $\alpha_1.\mathit{fstate} = q_1$, there exists some closed execution fragment $\alpha_2 = \tau_2 a_2 \tau_2'$ of A_2 such that: τ_2, τ_2' are point trajectories in $\mathcal{T}_2$; a_2 is an element of D_2; $\alpha_2.\mathit{fstate} = q_2$; $\mathit{trace}(\alpha_1) = \mathit{trace}(\alpha_2)$; and $(\alpha_1.\mathit{lstate}, \alpha_2.\mathit{lstate})$ is in R.*

3. *For any pair $(q_1, q_2) \in R$ and execution fragment $\alpha_1 = \tau_1$, where τ_1 is a closed trajectory in $\mathcal{T}_1$ and $\alpha_1.\mathit{fstate} = q_1$, there exists some execution fragment $\alpha_2 = \tau_2$ such that: τ_2 is a closed trajectory in $\mathcal{T}_2$; $\alpha_2.\mathit{fstate} = q_2$; $\mathit{trace}(\alpha_1) = \mathit{trace}(\alpha_2)$; and $(\alpha_1.\mathit{lstate}, \alpha_2.\mathit{lstate})$ is in R.*

Then $\mathit{traces}_{A_1} \subseteq \mathit{traces}_{A_2}$.

A simulation from A_1 to A_2 is said to be a *bisimulation* if its inverse R^{-1} is a simulation from A_2 to A_1. In this case, $\mathit{traces}_{A_2} \subseteq \mathit{traces}_{A_1}$, and so $\mathit{traces}_{A_1} = \mathit{traces}_{A_2}$. If such a relation exists, I say that A_1 *bisimulates* A_2.

11.2 Automaton-to-mechanics bisimulation

To extend the above notation to hybrid mechanics is relatively straightforward, consisting primarily of providing mechanical notions of trajectories, executions, and traces. It also requires some further development on the idea of induced motion.

Throughout the following section, I wish to make arguments regarding a variety of hybrid mechanical systems, without restricting discussion to the particular system constructed in the preceding chapters. Thus, I refer only to a generic hybrid mechanical system $M = (\Omega, \mathcal{S}, \Gamma_M, m, \Psi, \hat{\mathfrak{f}}[\cdot], \mathfrak{X}_0(\top, \mathcal{S}))$. In the general case I refer to the first instant in Γ_M via t_{M0}, using $(0,0)$ only when discussing the first instant of the specific construction in Chapter 7.

11.2.1 Preconditions for automaton-to-mechanics bisimulation

11.2.1.1 Comparable factor mechanical system and body

Consider any hybrid automaton A and hybrid mechanical system M. Suppose that there exists a factor mechanical system M_p of M with some body $G \in \Omega_\mathrm{p}$ such that there is a bijection from X to $\mathcal{C}(G, \mathcal{S}_\mathrm{p})$, the set of placements of G in $\mathcal{S}_\mathrm{p}$. I refer to such a factor mechanical system M_p as a *comparable factor mechanical system* of M for A, and I say that G is a *comparable factor body* of M for A.

11.2.1.2 Motion-concatenation

Consider any hybrid mechanical system M. Let χ be any differentiable motion of the universal body $\top$, with state-time function `state` and maximal interval set $\mathscr{I}_{\top_\mathrm{c}}$. Let $w = \chi_{t_{M0}}$, and let $\dot{w} = \dot{\chi}_{t_{M0}}$. For some instant $t_1 \in \Gamma_M$, let $J \in \mathscr{I}_{\top_\mathrm{c}}$ be the maximal interval containing t_1. Then let

$$\begin{aligned} J' &= \{t \in J \mid t \leq t_1\} \\ J'' &= \{t \in J \mid t > t_1\}. \end{aligned} \tag{11.1}$$

Suppose that χ is an induced motion of $\top$ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during both J' and J''. I say that M *motion-concatenates* at t_1, or that M is a motion-concatenating hybrid mechanical system at t_1, iff, for any such motion, χ is also an induced motion of $\top$ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J. If U is a set of instants and M motion-concatenates at all instants in U, then I say that M motion-concatenates over U, or that it is motion-concatenating over U.

Given a hybrid automaton A, a hybrid mechanical system M, and a mapping from automaton time to hybrid time $\hat{t}_M : \Gamma_A \to \Gamma_M$, let $\hat{t}_M(\Gamma_A)$ be the image of Γ_A under $\hat{t}_M$. That is,

$$\hat{t}_M(\Gamma_A) = \{\hat{t}_M(t_A) \mid t_A \in \Gamma_A\}. \tag{11.2}$$

In what follows, I will most often be interested in showing that a hybrid mechanical system M motion-concatenates over $\hat{t}_M(\Gamma_A)$ for some such automaton and mapping.

11.2.1.3 Force-shifting

Consider any hybrid mechanical system M. Suppose that there exists some $t_1 \in \Gamma_M$ such that, for all $t \in \Gamma_M$, $t + t_1$ is defined. I say that M *force-shifts* at t_1 iff, for any mechanical controller state $\psi \in \Psi(\top_\mathrm{c})$ and any force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$, there exists some other force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ such that, for all instants $t \in dom(\hat{f})$,

$$f_t = f'_{t+t_1}. \tag{11.3}$$

In other words, for any force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$, there must be another force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ whose forces are identical to those in $\hat{f}$ shifted forward by t_1. Note that this does not constrain the forces present in $\hat{f}'$ prior to t_1, as no instant of $\hat{f}$ would map to such instants.

If U is a set of hybrid mechanical instants and M force-shifts at all instants in U, I say that M force-shifts over U. As before, I will most often be concerned with hybrid mechanical systems that force-shift over $\hat{t}_M(\Gamma_A)$.

11.2.1.4 Time additivity-preserving

Consider any hybrid automaton A and hybrid mechanical system M; let $\hat{t}_M : \Gamma_A \to \Gamma_M$ be any mapping from automaton instants to mechanical instants. I say that $\hat{t}_M$ is a *time additivity-preserving* mapping iff the following three properties hold. First, $\hat{t}_M$ must preserve the order of all instants; that is, given two instants $t, t' \in \Gamma_A$,

$$t < t' \text{ iff } \hat{t}_M(t) < \hat{t}_M(t'). \tag{11.4}$$

Second, for any instants $t \in \Gamma_M$ and $t' \in \hat{t}_M(\Gamma_A)$, the sum $t + t'$ is defined; likewise, if $t \geq t'$, the difference $t - t'$ is defined.

Third, for any instants $t, t' \in \Gamma_A$, the sum of the mapping of the two instants must be equal to the mapping of their sums. That is,

$$\hat{t}_M(t + t') = \hat{t}_M(t) + \hat{t}_M(t'). \tag{11.5}$$

Likewise, if $t \geq t'$, the difference of the mappings of the two instants must be equal to the mapping of their differences. That is,

$$\hat{t}_M(t - t') = \hat{t}_M(t) - \hat{t}_M(t'). \tag{11.6}$$

Suppose that there exists some comparable factor mechanical system M_{p} of M, with some comparable factor body $G \in \Omega_{\text{p}}$. By Section 5.1.3, M must define a mapping $\hat{t}_{\text{p}} : \Gamma_M \to \Gamma_{\text{p}}$. I say that a time additivity-preserving mapping $\hat{t}_M$ is *comparably time additivity-preserving* (for M_{p} and G) iff the composition of $\hat{t}_M$ with $\hat{t}_{\text{p}}$ is a bijection. I write this bijection as $\hat{t}_{\text{p}M} = \hat{t}_{\text{p}} \circ \hat{t}_M$.

11.2.1.5 Mechanical comparability

Suppose that there exists some hybrid automaton A and a hybrid mechanical system M such that the following conditions hold. First, there is a comparable factor mechanical system M_{p} of M with some comparable factor body $G \in \Omega_{\text{p}}$. Second, there is some comparably time additivity-preserving mapping $\hat{t}_M : \Gamma_A \to \Gamma_M$ for M_{p} and G. Third, M motion-concatenates over $\hat{t}_M(\Gamma_A)$. Fourth, and finally, M force-shifts over $\hat{t}_M(\Gamma_A)$.

In such cases, I say that A and M are *comparable* for M_p and G. Where the choice of comparable factor mechanical system and body are unambiguous, I omit the last clause, saying only that A and M are comparable, or that M is comparable to A.

11.2.2 Comparison of automaton traces to mechanical traces

Let M be a hybrid mechanical system comparable to some hybrid automaton A for some comparable factor mechanical system M_p and comparable factor body G. Then the *mechanical traces of* M (for M_p and G), denoted $\mathit{traces}_M(M_\mathrm{p}, G)$, are the G components of all induced motions for $\hat{\mathfrak{f}}[\cdot]$ given any initial configuration. Where the choice of M_p and G is unambiguous, I omit the parentheses, referencing only traces_M.

Formally, given a motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$, the *mechanical trace* of χ, denoted $\mathit{trace}(\chi)$, is its restriction to the placements of G in $\mathcal{S}_\mathrm{p}$ in all instants of Γ_p. In other words, $\mathit{trace}(\chi) : \Gamma_\mathrm{p} \to \mathcal{C}(G, \mathcal{S}_\mathrm{p})$ is the function so that, at every instant $t \in \Gamma_\mathrm{p}$,

$$\mathit{trace}(\chi)(t) \stackrel{\mathrm{def}}{=} \chi_\mathrm{p}(G, t). \tag{11.7}$$

I omit the "mechanical" in "mechanical trace" when it is unambiguous to which sort of trace I am referring.

Let $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ be some motion of $\top$, let $\mathtt{state} : \Gamma \to \Psi(\top_\mathrm{c})$ be the state-time function for χ, and let $\mathscr{I}_{\top_\mathrm{c}}$ be the maximal intervals for $\top_\mathrm{c}$. Recall from Sections 5.3.4 and 6.3.3 that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ iff the following conditions hold:

- χ is second-degree doubly semidifferentiable.
- $\chi(\top, t_{M0}) = w$.
- $\dot{\chi}(\top, t_{M0}) = \dot{w}$.
- Given any interval $J \in \mathscr{I}_{\top_\mathrm{c}}$, let $\psi \in \Psi(\top_\mathrm{c})$ be the state such that $\mathtt{state}(J) = \psi$. Then there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ such that, for all instants $t \in J$ and bodies $B \in \Omega$,

$$f_t(B, \overline{B}) = m(B)\ddot{\chi}(B, t). \tag{11.8}$$

I say that a trace of a motion χ is a *mechanical trace of* M iff χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in some configuration in $\mathfrak{X}_0(\top, \mathcal{S})$. Then the mechanical traces of M are the elements of the set containing every such mechanical trace of M. Again, I refer simply to "a trace of M" or "the traces of M" where meaning is unambiguous.

Given an execution fragment α of the automaton A, I say that $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$ iff, for all instants $t_A \in \mathit{dom}(\alpha)$,

$$\Pi_X(\alpha(t_A)) = \chi_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t_A)). \tag{11.9}$$

In other words, an automaton trace and mechanical trace are equal iff the automaton's environment component at every instant t_A is isomorphic to the placement of the comparable factor body at every corresponding instant $\hat{t}_{\mathrm{p}M}(t_A)$.

One larger execution fragment may be formed by concatenating several smaller fragments; in such cases, motions whose mechanical traces are equal to the automaton traces of the smaller fragments may be combined to form a motion whose mechanical trace is equal to the automaton trace of the larger execution fragment, as shown in the following proof.

Lemma 15. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor mechanical system M_{p} and comparable factor body G. Suppose that there exist execution fragments $\alpha, \alpha', \alpha''$ for A such that α' is closed and $\alpha = \alpha' \frown \alpha''$ and, for some motions χ', χ'',*

$$\begin{aligned} \mathit{trace}(\alpha') &= \mathit{trace}(\chi') \\ \mathit{trace}(\alpha'') &= \mathit{trace}(\chi''). \end{aligned} \tag{11.10}$$

Let χ be any motion such that, for every automaton instant $t \in \mathit{dom}(\alpha)$,

$$\chi_{\hat{t}_M(t)} = \begin{cases} \chi'_{\hat{t}_M(t)}, & t \le \hat{t}_M(\alpha'.\mathit{ltime}) \\ \chi''_{\hat{t}_M(t-\alpha'.\mathit{ltime})}, & t > \hat{t}_M(\alpha'.\mathit{ltime}). \end{cases} \tag{11.11}$$

Then

$$\mathit{trace}(\alpha) = \mathit{trace}(\chi). \tag{11.12}$$

Proof. By assumption, $\mathit{trace}(\alpha') = \mathit{trace}(\chi')$; equivalently, for any instant $t \in \mathit{dom}(\alpha')$,

$$\Pi_X(\alpha'(t)) = \chi'_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)). \tag{11.13}$$

Likewise, by assumption $\mathit{trace}(\alpha'') = \mathit{trace}(\chi'')$, and so for any instant $t \in \mathit{dom}(\alpha'')$,

$$\Pi_X(\alpha''(t)) = \chi''_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)). \tag{11.14}$$

By the definition of execution fragment concatenation in Section 3.2.2, for every instant $t \in \mathit{dom}(\alpha)$,

$$\alpha(t) = \begin{cases} \alpha'(t), & t \le \alpha'.\mathit{ltime} \\ \alpha''(t - \alpha'.\mathit{ltime}), & t > \alpha'.\mathit{ltime}. \end{cases} \tag{11.15}$$

Restricting consideration to the environment projection of these fragments gives

$$\Pi_X(\alpha(t)) = \begin{cases} \Pi_X(\alpha'(t)), & t \leq \alpha'.\mathit{ltime} \\ \Pi_X(\alpha''(t - \alpha'.\mathit{ltime})), & t > \alpha'.\mathit{ltime}. \end{cases} \tag{11.16}$$

Substituting via (11.13) and (11.14) gives

$$\Pi_X(\alpha(t)) = \begin{cases} \chi'_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)), & t \leq \alpha'.\mathit{ltime} \\ \chi''_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t - \alpha'.\mathit{ltime})), & t > \alpha'.\mathit{ltime}. \end{cases} \tag{11.17}$$

But by the construction of χ, for every instant $t \in \mathit{dom}(\alpha)$,

$$\chi(\top, \hat{t}_M(t)) = \begin{cases} \chi'(\top, \hat{t}_M(t)), & t \leq \hat{t}_M(\alpha'.\mathit{ltime}) \\ \chi''(\top, \hat{t}_M(t - \alpha'.\mathit{ltime})), & t > \hat{t}_M(\alpha'.\mathit{ltime}). \end{cases} \tag{11.18}$$

Restricting consideration to physical components gives

$$\chi_{\mathrm{p}}(\top_{\mathrm{p}}, \hat{t}_{\mathrm{p}M}(t)) = \begin{cases} \chi'_{\mathrm{p}}(\top_{\mathrm{p}}, \hat{t}_{\mathrm{p}M}(t)), & t \leq \hat{t}_M(\alpha'.\mathit{ltime}) \\ \chi''_{\mathrm{p}}(\top_{\mathrm{p}}, \hat{t}_{\mathrm{p}M}(t - \alpha'.\mathit{ltime})), & t > \hat{t}_M(\alpha'.\mathit{ltime}). \end{cases} \tag{11.19}$$

Restricting consideration further to G gives

$$\chi_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)) = \begin{cases} \chi'_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)), & t \leq \hat{t}_M(\alpha'.\mathit{ltime}) \\ \chi''_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t - \alpha'.\mathit{ltime})), & t > \hat{t}_M(\alpha'.\mathit{ltime}). \end{cases} \tag{11.20}$$

But the right-hand side of this expression is identical to the right-hand side of (11.17) over the same domain; thus, the left-hand sides are equal, and for every instant $t \in \mathit{dom}(\alpha)$,

$$\Pi_X(\alpha(t)) = \chi_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)). \tag{11.21}$$

Then by the definition of mechanical trace,

$$\mathit{trace}(\alpha) = \mathit{trace}(\chi). \tag{11.22}$$

□

11.2.3 Comparison of automaton executions to mechanical motions

Before formally defining simulation in either direction, I provide a few last pieces of supporting terminology. Let A be a hybrid automaton, and let M be a comparable hybrid mechanical system for comparable factor mechanical system M_{p} and comparable factor body G. Since the two systems are comparable, there must be some bijection between automaton environment states and mechanical placements of G; as in previous chapters, I abuse notation somewhat by referring to elements related via this bijection as though they were identical. Let R be a relation on $Q \times \mathfrak{X}(\top, \mathcal{S})$. I define the following properties of R; I will then define simulation and bisimulation in terms of these properties.

11.2.3.1 Related motions and executions

I say that a *related motion* of M for α is any motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- if α is closed, then $(\alpha.\mathit{lstate}, (\chi_{\hat{t}_M(\alpha.\mathit{ltime})}, \dot{\chi}_{\hat{t}_M(\alpha.\mathit{ltime})}))$ is in R.

In other words, a related motion is an induced motion for which the motion of G in $\mathcal{S}_{\mathrm{p}}$ exactly matches the environment component of the execution fragment; if the fragment has a final state, then that state is related to the corresponding placement of $\top$ in χ. If some related motion χ exists for any choice of $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R, then I say that M has related motions for α. Likewise, I say that M has related motions for a set of execution fragments iff there exist related motions of M for all execution fragments in the set. Finally, I say that M has related motions for a trajectory τ precisely when M has related motions for the execution fragment $\alpha = \tau$.

Similarly, for any pair $(q, (w, \dot{w})) \in R$, let χ be an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$. Then a *related execution fragment* of A for χ is any execution fragment $\alpha = \tau_0 a_1 \tau_1 \ldots$ such that

- for all $i \geq 0$, τ_i is in $\mathcal{T}$,
- for all $i > 0$, a_i is in D,
- $\alpha.\mathit{fstate} = q$, and
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$.

In other words, a related execution fragment is any execution fragment whose components are part of A and whose environment component exactly matches the physical component of the motion. If some such execution fragment exists for any choice of q such that $(q, (w, \dot{w}))$ is in R, then I say that M has a related execution fragments for χ.

11.2.3.2 Related initial states

I say that M has *related initial states* for A (in R) iff every initial state of A is related to an initial configuration of M. That is, for any state $q \in Q_0$, there must be a corresponding configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R.

Similarly, I say that A has *related initial states* for M iff every initial configuration of M is related to some initial state of A. That is, for any configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$, there must be a corresponding automaton system state $q \in Q_0$ such that $(q, (w, \dot{w}))$ is in R.

11.2.3.3 Related motions for closed trajectories

Consider any pair $(q, (w, \dot{w})) \in R$, with $q = (v, \mathbf{x}) \in Q$ and $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$. I say that M has *related motions for the closed trajectories* of A (in R) iff, for any closed trajectory beginning in any such system state, M has related motions for that trajectory.

Formally, given any choice of $(q, (w, \dot{w})) \in R$, let $\alpha = \tau$, where τ is a closed trajectory in $\mathcal{T}$ with $\tau.\mathit{fstate} = q$. Then M has related motions for the closed trajectories of A iff, for any such choices, there exists some motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- $(\alpha.\mathit{lstate}, (\chi_{\hat{t}_M(\alpha.\mathit{ltime})}, \dot{\chi}_{\hat{t}_M(\alpha.\mathit{ltime})}))$ is in R.

11.2.3.4 Related motions for discrete transitions

Dealing with discrete transitions is complicated by Assumption 9, that is, by the assumption that transitions preserve the $\mathbf{x}$-component derivatives of the trajectories involved. To aid in the definition, I add one additional piece of terminology. Let α be any closed execution fragment of A with a final trajectory $\tau \in \mathcal{T}$. Let α' be any execution fragment of A such that

$$\alpha' = \tau' a \tau'', \tag{11.23}$$

where

- $\tau' \in \mathcal{T}$ is the point trajectory formed by restricting τ to its final instant, that is, $\tau' = \tau \lceil \tau.\mathit{ltime}$;
- $\tau'' \in \mathcal{T}$ is a trajectory whose initial X-value and derivative agree with the final X-value and derivative of τ, that is,

$$\begin{aligned} \Pi_X(\tau.\mathit{lstate}) &= \Pi_X(\tau''.\mathit{fstate}) \\ \Pi_X(\dot{\tau}.\mathit{lstate}) &= \Pi_X(\dot{\tau}''.\mathit{fstate}); \end{aligned} \tag{11.24}$$

- and $a \in D$ is a transition such that

$$a = (\tau.lstate, \tau''.fstate). \tag{11.25}$$

I call any such execution fragment α' a *point-initiated extension of* α; if α is the single trajectory τ, I also say that α' is a point-initiated extension of τ. It is straightforward to show that, for any such point-initiated extension α', the concatenation $\alpha \frown \alpha'$ is an execution fragment of A.

Lemma 16. *Let A be a hybrid automaton. Let α be any closed execution fragment of A, and let α' be any point-initiated extension of α. Then $\alpha \frown \alpha'$ is an execution fragment of A.*

Proof. Since α is closed, and by Section 3.2.3 it is non-Zeno, it consists of some finite sequence of trajectories and transitions, that is,

$$\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots \tau_k \tag{11.26}$$

for some finite integral k. By the definition of an execution fragment, every trajectory of this execution fragment is in $\mathcal{T}$; every transition is in D; and by Assumptions 8 and 9, every transition preserves environment states and derivatives.

Since α' is a point-initiated extension of α, it must be the case that

$$\alpha' = \tau'_k a_{k+1} \tau_{k+1}, \tag{11.27}$$

where $\tau'_k = \tau_k \lceil \tau_k.ltime$, and $\tau_{k+1} \in \mathcal{T}$ is a trajectory such that

$$\begin{aligned} \Pi_X(\tau_k.lstate) &= \Pi_X(\tau_{k+1}.fstate) \\ \Pi_X(\dot{\tau}_k.lstate) &= \Pi_X(\dot{\tau}_{k+1}.fstate), \end{aligned} \tag{11.28}$$

and $a_{k+1} \in D$ is the transition such that

$$a_{k+1} = (\tau_k.lstate, \tau_{k+1}.fstate). \tag{11.29}$$

Then clearly

$$\begin{aligned} \alpha \frown \alpha' = \alpha &= \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots (\tau_k \frown \tau'_k) a_{k+1} \tau_{k+1} \\ &= \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots \tau_k a_{k+1} \tau_{k+1}. \end{aligned} \tag{11.30}$$

Every trajectory of this execution fragment is in $\mathcal{T}$; every transition is in D; and every transition preserves environment states and derivatives. Thus, $\alpha \frown \alpha'$ is an execution fragment of A. $\square$

Consider any pair $(q, (w, \dot{w})) \in R$, with $q = (v, \mathbf{x}) \in Q$ and $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, and any trajectory $\tau \in \mathcal{T}$ such that $\tau.lstate = q$. I say that M has *related motions for the discrete transitions* of A (in R) iff, for any such system state and trajectory, M has related motions for any closed point-initiated extension of τ.

Formally, given any choice of $(q, (w, \dot{w})) \in R$ and trajectory $\tau \in \mathcal{T}$ such that $\tau.\mathit{lstate} = q$, let, let α be any closed point-initiated extension of τ. Thus, α is an execution fragment such that $\alpha = \tau' a \tau''$, where

- $\tau' \in \mathcal{T}$ is the point trajectory formed by restricting τ to its final instant, that is, $\tau' = \tau\lceil\tau.\mathit{ltime}$;
- $\tau'' \in \mathcal{T}$ is a closed trajectory whose initial X-value and derivative agree with the final X-value and derivative of τ, that is,

$$\begin{aligned}\Pi_X(\tau.\mathit{lstate}) &= \Pi_X(\tau''.\mathit{fstate}) \\ \Pi_X(\dot{\tau}.\mathit{lstate}) &= \Pi_X(\dot{\tau}''.\mathit{fstate});\end{aligned} \tag{11.31}$$

- and $a \in D$ is a transition such that

$$a = (\tau.\mathit{lstate}, \tau''.\mathit{fstate}). \tag{11.32}$$

Then M has related motions for the discrete transitions of A iff, for any such execution fragment, there exists some motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- $(\alpha.\mathit{lstate}, (\chi_{\hat{t}_M(\alpha.\mathit{ltime})}, \dot{\chi}_{\hat{t}_M(\alpha.\mathit{ltime})}))$ is in R.

11.2.3.5 Related executions for trajectories

Consider any pair $(q, (w, \dot{w})) \in R$, with $q \in Q$ and $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$. Suppose there exists some induced motion χ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$. I say that A has *related execution fragments for the motions* of M iff, for any such configuration and motion, A has related execution fragments for χ.

11.2.4 Simulation of a hybrid automaton by a mechanical system

Given a hybrid automaton A, a comparable hybrid mechanical system M, and a relation R as defined in the previous section, R is a *simulation* (or equivalently, M simulates A) iff:

- M has related initial states for A,
- M has related motions for the closed trajectories of A, and

- M has related motions for the discrete transitions of A.

Some statement should be made as to exactly what this complex definition guarantees. When considering one hybrid automaton which simulates another, simulation is a guarantee that the traces of the latter are a subset of those of the former; in other words, if A_2 simulates A_1, then

$$traces_{A_1} \subseteq traces_{A_2}. \tag{11.33}$$

Similarly, if some hybrid mechanical system M simulates a hybrid automaton A,

$$traces_A \subseteq traces_M. \tag{11.34}$$

I show this result in the following lemmas and theorems, which are based on the work in [18]. Notationally, recall from Section 2.1 that, given an interval J and instant t' such that $t + t'$ is defined for all $t \in J$, one writes $J + t'$ to indicate the interval such that

$$J + t' \stackrel{\text{def}}{=} \{t + t' \mid t \in J\}. \tag{11.35}$$

Likewise, if $t - t'$ is defined for all $t \in J$, one writes $J - t'$ to indicate the interval such that

$$J - t' \stackrel{\text{def}}{=} \{t - t' \mid t \in J\}. \tag{11.36}$$

Suppose that χ, χ', and χ'' are motions of a hybrid mechanical system such that χ agrees with χ' prior to some instant and χ'' after that instant. I begin by showing that, under reasonable conditions, if χ' and χ'' are induced motions, χ is also an induced motion.

Lemma 17. *Given a hybrid mechanical system M that motion-concatenates and force-shifts over some set $U \subseteq \Gamma_M$, let $(w_0, \dot{w}_0)$ be any configuration for $\top$. Let $t_1 \in U$ be any instant such that $t - t_1$ is defined for all instants $t \in \Gamma_M$. Suppose that there exist two motions χ, χ' such that*

- *χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$,*
- *$\chi(\top, t_1) = \chi'(\top, t_{M0})$,*
- *$\dot{\chi}(\top, t_1) = \dot{\chi}'(\top, t_{M0})$, and*
- *χ' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$.*

Let χ'' be the motion such that, for every instant $t \in \Gamma_M$,

$$\chi''(\top, t) = \begin{cases} \chi(\top, t), & t \leq t_1 \\ \chi'(\top, t - t_1), & t > t_1. \end{cases} \tag{11.37}$$

Then χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$.

Proof. Let $\mathtt{state}''$ be the state-time function for χ'', and let $\mathscr{I}''_{\top_c}$ be the maximal interval set for $\mathtt{state}''$. For ease of reference, let

$$\begin{aligned} w_1 &= \chi(\top, t_1) = \chi'(\top, t_{M0}) \\ \dot{w}_1 &= \dot{\chi}(\top, t_1) = \dot{\chi}'(\top, t_{M0}). \end{aligned} \tag{11.38}$$

Consider any interval $J'' \in \mathscr{I}''_{\top_c}$. There are three possibilities: either J'' is entirely prior to t_1; or J'' lies entirely in the range subsequent to t_1; or J'' contains t_1.

Case 1 - J'' prior to t_1: Suppose that J'' lies entirely in $[t_{M0}, t_1)$. Then for any instant $t \in J''$,

$$\chi''(\top, t) = \chi(\top, t). \tag{11.39}$$

Then for any instant $t \in J''$,

$$\mathtt{state}''(t) = \mathtt{state}(t). \tag{11.40}$$

Since J'' is in $\mathscr{I}''_{\top_c}$, there exists some state $\psi \in \Psi(\top_c)$ so that

$$\mathtt{state}''(J'') = \psi, \tag{11.41}$$

and so, for any instant $t \in J''$,

$$\mathtt{state}''(t) = \psi. \tag{11.42}$$

It follows that, for any instant $t \in J''$,

$$\begin{aligned} \mathtt{state}(t) &= \mathtt{state}''(t) \\ &= \psi. \end{aligned} \tag{11.43}$$

Then $\mathtt{state}(J'')$ is defined, and in particular

$$\mathtt{state}(J'') = \psi. \tag{11.44}$$

By assumption, χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$. Then there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ such that, for every body $B \in \Omega$ and instant $t \in J''$,

$$\begin{aligned} \hat{f}_{\mathrm{p}}(t)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{p}}}) &= m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t)) \\ \hat{f}_{\mathrm{c}}(t)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{p}}}) &= m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t)). \end{aligned} \tag{11.45}$$

For any instant $t \in J''$, $\chi(t) = \chi''(t)$, and so for any such instant,

$$\begin{aligned}\hat{f}_{\mathrm{p}}(t)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{c}}}) &= m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}''_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t)) \\ \hat{f}_{\mathrm{c}}(t)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{c}}}) &= m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}''_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t)).\end{aligned} \tag{11.46}$$

Then by definition, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J''.

Case 2 - J'' subsequent to t_1: Suppose instead that J'' lies entirely subsequent to t_1, that is, that for every $t \in J''$, it is the case that $t > t_1$.

Then for any instant $t \in J''$,

$$\chi''(\top, t) = \chi'(\top, t - t_1). \tag{11.47}$$

Then for any instant $t \in J''$,

$$\mathtt{state}''(t) = \mathtt{state}'(t - t_1). \tag{11.48}$$

Since J'' is in $\mathscr{I}''_{\top_{\mathrm{c}}}$, there exists some state $\psi \in \Psi(\top_{\mathrm{c}})$ so that

$$\mathtt{state}''(J'') = \psi \tag{11.49}$$

and so, for any instant $t \in J''$,

$$\mathtt{state}''(t) = \psi. \tag{11.50}$$

It follows that, for any instant $t \in J''$,

$$\begin{aligned}\mathtt{state}'(t - t_1) &= \mathtt{state}''(t) \\ &= \psi.\end{aligned} \tag{11.51}$$

Then $\mathtt{state}(J'' - t_1)$ is defined, and in particular

$$\mathtt{state}(J'' - t_1) = \psi. \tag{11.52}$$

By assumption, χ' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_1, \dot{w}_1)$. Then there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ such that, for every body $B \in \Omega$ and instant $t \in (J'' - t_1)$,

$$\begin{aligned}\hat{f}_{\mathrm{p}}(t)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{p}}}) &= m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}'_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t)) \\ \hat{f}_{\mathrm{c}}(t)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{p}}}) &= m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}'_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t)).\end{aligned} \tag{11.53}$$

By (11.47), for any instant $t \in J''$, $\chi'(t - t_1) = \chi''(t)$. Equivalently, for any instant $t \in (J'' - t_1)$,

$$\chi'(t) = \chi''(t + t_1). \tag{11.54}$$

Substituting into (11.53), for every instant $t \in (J'' - t_1)$,

$$\hat{f}_{\mathrm{p}}(t)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}''_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t + t_1)) \tag{11.55}$$
$$\hat{f}_{\mathrm{c}}(t)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}''_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t + t_1)).$$

Since M force-shifts over U, there exists some force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ so that, for every instant $t \in J''$,

$$f_t = f'_{t+t_1}. \tag{11.56}$$

It follows that, for every instant $t \in (J'' - t_1)$,

$$\hat{f}'_{\mathrm{p}}(t + t_1)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}''_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t + t_1)) \tag{11.57}$$
$$\hat{f}'_{\mathrm{c}}(t + t_1)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}''_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t + t_1)).$$

Equivalently, for every instant $t \in J''$,

$$\hat{f}'_{\mathrm{p}}(t)(B \sqcap \top_{\mathrm{p}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}})\ddot{\chi}''_{\mathrm{p}}(B \sqcap \top_{\mathrm{p}}, \hat{t}_{\mathrm{p}}(t)) \tag{11.58}$$
$$\hat{f}'_{\mathrm{c}}(t)(B \sqcap \top_{\mathrm{c}}, \overline{B \sqcap \top_{\mathrm{p}}}) = m_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}})\ddot{\chi}''_{\mathrm{c}}(B \sqcap \top_{\mathrm{c}}, \hat{t}_{\mathrm{c}}(t)).$$

Then by definition, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J''.

Case 3 - J'' contains t_1: In the final case, J'' contains t_1. In this case, divide J'' into two portions, one running from $\inf(J'')$ to t_1 and the other running from just after t_1 to $\sup(J'')$. In other words, let

$$J_2 = \{t \in J'' \mid t \leq t_1\} \tag{11.59}$$
$$J_3 = \{t \in J'' \mid t > t_1\}.$$

Again, since J'' is in $\mathscr{I}''_{\top_{\mathrm{c}}}$, there exists some state $\psi \in \Psi(\top_{\mathrm{c}})$ such that

$$\texttt{state}''(J'') = \psi. \tag{11.60}$$

It follows that

$$\texttt{state}''(J_2) = \texttt{state}''(J_3) = \psi. \tag{11.61}$$

Consider J_2, which lies inside $[t_{M0}, t_1]$; then for any instant $t \in J_2$,

$$\chi''(\top, t) = \chi(\top, t). \tag{11.62}$$

By an argument similar to that in Case 1, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J_2.

Consider instead J_3 ,which lies inside (t_1, ∞); then for any instant $t \in J_3$,

$$\chi''(\top, t) = \chi'(\top, t - t_1). \tag{11.63}$$

By an argument similar to that in Case 2, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J_3.

Then χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during both J_2 and J_3. Since M motion-concatenates over U, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J''.

Conclusion: Then for any interval $J'' \in \mathscr{I}''_{\top_c}$, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ during J''. Then χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$. $\square$

Broadly, the above proof shows that, under certain conditions, two induced motions can be concatenated to produce an induced motion. Then consider two execution fragments

$$\begin{aligned} \alpha &= \tau_0 a_1 \tau_1 \ldots \tau_n \\ \alpha' &= \tau_{n+1} a_{n+2} \tau_{n+2} \ldots \end{aligned} \tag{11.64}$$

such that $\tau_n \frown \tau_{n+1}$ exists and is in $\mathcal{T}$. Then $\alpha \frown \alpha'$ is also an execution fragment, namely

$$\alpha \frown \alpha' = \tau_0 a_1 \tau_1 \ldots (\tau_n \frown \tau_{n+1}) a_{n+2} \tau_{n+2} \ldots \tag{11.65}$$

Suppose that M has related motions for both α and α'. Intuitively, it follows that M has motions whose traces match the automaton traces from $\tau_0.\mathit{fstate}$ to $\tau_n.\mathit{lstate}$, and from $\tau_{n+1}.\mathit{fstate}$ to the end of the execution fragment. By the above lemma, M also has an induced motion whose trace matches the automaton trace of $\tau_n \frown \tau_{n+1}$. Then M has a motion whose trace matches the automaton trace of $\alpha \frown \alpha'$, and so M has related motions for $\alpha \frown \alpha'$.

In other words, M has related motions for $\alpha \frown \alpha'$ if it has related motions for α and α', given reasonable constraints. I show this result formally in the following theorem.

Theorem 61. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system for comparable factor mechanical system M_{p} and comparable factor body G, such that there exists a simulation R from A to M. Let $\hat{t}_M$ be a comparable time additivity-preserving mapping from Γ_A to Γ_M.*

Suppose that there exist two execution fragments $\alpha, \alpha' \in \mathit{frags}_A$ such that α is closed,

$$\begin{aligned} \alpha.\mathit{lstate} &= \alpha'.\mathit{fstate} \\ \Pi_X(\dot{\alpha}.\mathit{lstate}) &= \Pi_X(\dot{\alpha}'.\mathit{fstate}), \end{aligned} \tag{11.66}$$

and M has related motions for both α and α'. Then M has related motions for $\alpha \frown \alpha'$.

Proof. Let $\alpha'' = \alpha \frown \alpha'$.

By assumption, M has related motions for α. Suppose that there are no pairs $(w_0, \dot{w}_0) \in \mathfrak{X}(\top, \mathcal{S})$ such that $(\alpha.\mathit{fstate}, (w_0, \dot{w}_0))$ is in R; then, since $\alpha''.\mathit{fstate} = \alpha.\mathit{fstate}$, it is trivially the case that M has related motions for α''.

Otherwise, consider any pair $(w_0, \dot{w}_0) \in \mathfrak{X}(\top, \mathcal{S})$ such that $(\alpha.\mathit{fstate}, (w_0, \dot{w}_0))$ is in R. Then by the definition of related motion, there must be some induced motion χ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ such that

$$\mathit{trace}(\alpha) = \mathit{trace}(\chi) \tag{11.67}$$

and $(\alpha.\mathit{lstate}, (\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime}))))$ is in R. For ease of notation, let

$$\begin{aligned} w_1 &= \chi(\top, \hat{t}_M(\alpha.\mathit{ltime})) \\ \dot{w}_1 &= \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime})). \end{aligned} \tag{11.68}$$

Then $(\alpha.\mathit{lstate}, (w_1, \dot{w}_1))$ is in R.

By assumption, M has related motions for α'. Since $\alpha'.\mathit{fstate} = \alpha.\mathit{lstate}$ and $(\alpha.\mathit{lstate}, (w_1, \dot{w}_1))$ is in R, the pair $(\alpha'.\mathit{fstate}, (w_1, \dot{w}_1))$ is in R. Then by the definition of related motion, there exists some induced motion χ' for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_1, \dot{w}_1)$ such that

$$\mathit{trace}(\alpha') = \mathit{trace}(\chi') \tag{11.69}$$

and $(\alpha'.\mathit{lstate}, (\chi'(\top, \hat{t}_M(\alpha'.\mathit{ltime})), \dot{\chi}'(\top, \hat{t}_M(\alpha'.\mathit{ltime}))))$ is in R. Again, for ease of notation, let

$$\begin{aligned} w_2 &= \chi'(\top, \hat{t}_M(\alpha'.\mathit{ltime})) \\ \dot{w}_2 &= \dot{\chi}'(\top, \hat{t}_M(\alpha'.\mathit{ltime})). \end{aligned} \tag{11.70}$$

Then $(\alpha'.\mathit{lstate}, (w_2, \dot{w}_2))$ is in R.

Since $\hat{t}_M$ is a comparable time additivity-preserving mapping, for any instant $t \in \Gamma_M$ such that $t \geq \hat{t}_M(\alpha.\mathit{ltime})$, the instant $t - \hat{t}_M(\alpha.\mathit{ltime})$ is defined. Let χ'' be the motion such that, for every $t \in \Gamma_M$,

$$\chi''(\top, t) = \begin{cases} \chi(\top, t), & t \leq \hat{t}_M(\alpha.\mathit{ltime}) \\ \chi'(\top, t - \hat{t}_M(\alpha.\mathit{ltime})), & t > \hat{t}_M(\alpha.\mathit{ltime}). \end{cases} \tag{11.71}$$

Then

$$\begin{aligned} \chi''(\top, t_{M0}) &= \chi(\top, t_{M0}) = w_0 \\ \dot{\chi}''(\top, t_{M0}) &= \dot{\chi}(\top, t_{M0}) = \dot{w}_0. \end{aligned} \tag{11.72}$$

Since M is a comparable hybrid mechanical system, it motion-concatenates and force-shifts over

$\hat{t}_M(\Gamma_A)$. Then since

$$\begin{aligned}\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})) &= w_1 = \chi'(\top, t_{M0}) \\ \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime})) &= \dot{w}_1 = \dot{\chi}'(\top, t_{M0}),\end{aligned} \tag{11.73}$$

by Lemma 17, χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$. Since $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$ and $\mathit{trace}(\alpha') = \mathit{trace}(\chi')$, by Lemma 15,

$$\mathit{trace}(\alpha'') = \mathit{trace}(\chi''). \tag{11.74}$$

Since α'' is the concatenation of α with another execution fragment, plainly either $\alpha''.\mathit{ltime} = \alpha.\mathit{ltime}$ or $\alpha''.\mathit{ltime} > \alpha.\mathit{ltime}$. Suppose it is the former; then $\alpha'' = \alpha$, and so M has related motions for α'' because it has related motions for α. Suppose instead that $\alpha''.\mathit{ltime} > \alpha.\mathit{ltime}$; since $\hat{t}_M$ is order-preserving, it follows that $\hat{t}_M(\alpha''.\mathit{ltime}) > \hat{t}_M(\alpha.\mathit{ltime})$. Then by (11.71),

$$\chi''(\top, \hat{t}_M(\alpha''.\mathit{ltime})) = \chi'(\top, \hat{t}_M(\alpha''.\mathit{ltime}) - \hat{t}_M(\alpha.\mathit{ltime})). \tag{11.75}$$

Since $\hat{t}_M$ is a comparable time additivity-preserving mapping, I can combine the last two terms, and so

$$\chi''(\top, \hat{t}_M(\alpha''.\mathit{ltime})) = \chi'(\top, \hat{t}_M(\alpha''.\mathit{ltime} - \alpha.\mathit{ltime})). \tag{11.76}$$

Equivalently, since $\alpha.\mathit{ltime} + \alpha'.\mathit{ltime} = \alpha''.\mathit{ltime}$,

$$\begin{aligned}\chi''(\top, \hat{t}_M(\alpha''.\mathit{ltime})) &= \chi'(\top, \hat{t}_M(\alpha'.\mathit{ltime})) \\ &= w_2.\end{aligned} \tag{11.77}$$

By a similar argument

$$\dot{\chi}''(\top, \hat{t}_M(\alpha''.\mathit{ltime})) = \dot{w}_2. \tag{11.78}$$

Then since $\alpha''.\mathit{lstate} = \alpha'.\mathit{lstate}$,

$$(\alpha''.\mathit{lstate}, (\chi''(\top, \hat{t}_M(\alpha''.\mathit{ltime})), \dot{\chi}''(\top, \hat{t}_M(\alpha''.\mathit{ltime})))) = (\alpha'.\mathit{lstate}, (w_2, \dot{w}_2)), \tag{11.79}$$

which is in R, as noted above.

Then χ'' is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ such that

$$\mathit{trace}(\alpha'') = \mathit{trace}(\chi'') \tag{11.80}$$

and $(\alpha''.\mathit{lstate}, (\chi''(\top, \hat{t}_M(\alpha''.\mathit{ltime})), \dot{\chi}''(\top, \hat{t}_M(\alpha''.\mathit{ltime}))))$ is in R. In other words, it is a related motion for α''. Such a related motion could be constructed for any choice of $(w_0, \dot{w}_0)$ such that $(\alpha.\mathit{fstate}, (w_0, \dot{w}_0))$ is in R, and so M has related motions for α''. $\square$

I can now begin to demonstrate that M has related motions for any possible execution fragment of A. First, I show that M has related motions for any execution fragment consisting of a single right-open trajectory.

Lemma 18. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor system M_{p} and comparable factor body G, such that there exists a simulation R from A to M. Let $(q, (w, \dot{w}))$ be an element of R, with $q \in Q$ and $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$.*

For any right-open trajectory $\tau \in \mathcal{T}$ such that $\tau.\mathit{fstate} = q$, let α be the execution of A such that $\alpha = \tau$. Then M has related motions for α.

Proof. Consider any countably infinite set of closed trajectories $\{\tau_0, \tau_1, \tau_2, \ldots\}$ such that $\tau = \tau_0 \frown \tau_1 \frown \ldots$ and, for any $i \geq 0$,

$$\begin{aligned} \tau_i.\mathit{lstate} &= \tau_{i+1}.\mathit{fstate} \\ \dot{\tau}_i.\mathit{lstate} &= \dot{\tau}_{i+1}.\mathit{fstate}. \end{aligned} \tag{11.81}$$

Each such trajectory τ_i is thus some suffix of some prefix of τ. By Assumption 2, $\mathcal{T}$ is differentiably legal, and so it is prefix closed and suffix closed, and so each τ_i is a closed trajectory in $\mathcal{T}$. For each such trajectory, let $\alpha_i = \tau_i$; then α_i is the execution fragment consisting of a single closed trajectory in $\mathcal{T}$. Since R is a simulation, M has related motions for the closed trajectories of A, and so there is a related motion for each α_i.

The proof proceeds by induction. For any integer $n \geq 0$, let $\alpha'_n = \tau_0 \frown \tau_1 \frown \ldots \frown \tau_n$. Clearly $\alpha'_0 = \alpha_0$, and so M has related motions for α'_0. Suppose that M has related motions for α'_i for all indices $i \in \{0, 1, 2, \ldots k\}$ for some integer $k \geq 0$. It can be shown that M has related motions for α'_{k+1}, as follows. Note that

$$\begin{aligned} \alpha'_k.\mathit{lstate} &= \tau_k.\mathit{lstate} \\ \dot{\alpha}'_k.\mathit{lstate} &= \dot{\tau}_k.\mathit{lstate}. \end{aligned} \tag{11.82}$$

Then substituting via (11.81),

$$\begin{aligned} \alpha'_k.\mathit{lstate} &= \tau_{k+1}.\mathit{fstate} \\ \dot{\alpha}'_k.\mathit{lstate} &= \dot{\tau}_{k+1}.\mathit{fstate}. \end{aligned} \tag{11.83}$$

Equivalently, since $\alpha_{k+1} = \tau_{k+1}$,

$$\alpha'_k.lstate = \alpha_{k+1}.fstate \tag{11.84}$$
$$\dot{\alpha}'_k.lstate = \dot{\alpha}_{k+1}.fstate.$$

By the definition of α'_{k+1},

$$\alpha'_{k+1} = \alpha'_k \frown \alpha_{k+1}. \tag{11.85}$$

By assumption, M has related motions for α'_k. As noted above, α has related motions for α_{k+1}. Then by Theorem 61, M has related motions for α'_{k+1}. Then by induction, M has related motions for every such concatenation $\alpha'_{k+1} = \tau_0 \frown \tau_1 \frown \ldots \frown \tau_{k+1}$, and so M has related motions for α. □

Conceptually, the previous lemma serves as an inductive base case, showing that M has related motions for an execution fragment consisting of a single trajectory. I now provide inductive steps to show that one can extend an execution fragment while retaining related motions.

Suppose that M has related motions for some execution fragment α. One can extend an execution fragment in two ways: either by appending a discrete transition followed by a trajectory to the execution fragment, or by concatenating a trajectory to the end of the execution fragment. In other words, if M has related motions for

$$\alpha = \tau_0 a_1 \tau_1 \ldots a_n \tau_n, \tag{11.86}$$

then there are two possible inductive extensions to α: either

$$\alpha' = \tau_0 a_1 \tau_1 \ldots a_n \tau_n a_{n+1} \tau_{n+1}, \tag{11.87}$$

for some $a_{n+1} \in D$ and $\tau_{n+1} \in \mathcal{T}$, or

$$\alpha'' = \tau_0 a_1 \tau_1 \ldots a(\tau_n \frown \tau_{n+1}), \tag{11.88}$$

again for some $\tau_{n+1} \in \mathcal{T}$. In the following lemmas, I show that both of these cases produce an execution fragment for which M has related motions.

Lemma 19. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system for comparable factor mechanical system M_{p} and comparable factor body G such that there exists a simulation R from A to M. Let $(q_0, (w_0, \dot{w}_0))$ be an element of R, with $q_0 \in Q$ and $(w_0, \dot{w}_0) \in \mathfrak{X}(\top, \mathcal{S})$. Let α be any closed execution fragment of A such that $\alpha.lstate = q_0$, and let α' be any closed point-initiated extension of α. Then if A has a related motion for α, it must also have a related motion for $\alpha \frown \alpha'$.*

Proof. Let τ be the final trajectory in α. Note first that, since α' is a closed point-initiated extension of α,

$$\alpha' = \tau' a \tau'', \tag{11.89}$$

where

- $\tau' \in \mathfrak{T}$ is the point trajectory formed by restricting τ to its final instant, that is, $\tau' = \tau \lceil \tau.ltime$;
- $\tau'' \in \mathfrak{T}$ is a closed trajectory whose initial X-value and derivative agree with the final X-value and derivative of τ, that is,

$$\begin{aligned} \Pi_X(\tau.lstate) &= \Pi_X(\tau''.fstate) \\ \Pi_X(\dot{\tau}.lstate) &= \Pi_X(\dot{\tau}''.fstate); \end{aligned} \tag{11.90}$$

- and $a \in D$ is a transition such that

$$a = (\tau.lstate, \tau''.fstate). \tag{11.91}$$

If no such construction of α' exists, the proof holds trivially.

Otherwise, suppose that M has related motions for α. Then there exists an induced motion χ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ such that

$$trace(\alpha) = trace(\chi) \tag{11.92}$$

and $(q_1, (\chi_{t_1}, \dot{\chi}_{t_1}))$ is in R. For ease of notation, let

$$\begin{aligned} w_1 &= \chi_{t_1} \\ \dot{w}_1 &= \dot{\chi}_{t_1}. \end{aligned} \tag{11.93}$$

Then $(q_1, (w_1, \dot{w}_1))$ is in R.

By the definition of τ',

$$\tau'.fstate = \tau.lstate = q_1, \tag{11.94}$$

and so

$$\alpha'.fstate = \alpha.lstate = q_1. \tag{11.95}$$

Since R is a simulation, M has related motions for the discrete transitions of A. Then since $(q_1, (w_1, \dot{w}_1))$ is in R and α' is a closed point-initiated extension of α, M has related motions for α'.

Then M has related motions for α and α'. Then by (11.95), (11.90), and Theorem 61, M has related motions for $\alpha \frown \alpha'$. □

Lemma 20. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor mechanical system M_{p} and comparable factor body G, such that there exists a simulation R from A to M. Let $(q_0, (w_0, \dot{w}_0))$ be an element of R, with $q_0 \in Q$ and $(w_0, \dot{w}_0) \in \mathfrak{X}(\top, \mathcal{S})$.*

Let α be any closed execution fragment of A such that $\alpha.\mathit{fstate} = q_0$.

Consider any trajectory $\tau' \in \mathcal{T}$ such that $\tau'.\mathit{fstate} = \alpha.\mathit{lstate}$ and $\Pi_X(\dot{\tau}'.\mathit{fstate}) = \Pi_X(\dot{\alpha}.\mathit{lstate})$. Let $\alpha' = \tau'$. If M has related motions for α, then M has related motions for $\alpha \frown \alpha'$.

Proof. Since M is a comparable hybrid mechanical system, there exists a comparable time additivity-preserving mapping $\hat{t}_M : \Gamma_A \to \Gamma_M$. Since this mapping is comparable time additivity-preserving, for any instants $t, t' \in \Gamma_A$,

$$\left(t < t'\right) \text{ iff } \left(\hat{t}_M(t) < \hat{t}_M(t')\right); \tag{11.96}$$

for any instants $t, t', t'' \in \Gamma_A$ such that $t + t' = t''$,

$$\hat{t}_M(t) + \hat{t}_M(t') = \hat{t}_M(t''); \tag{11.97}$$

and the composition $\hat{t}_{\mathrm{p}M}$ is a bijection.

Suppose M has related motions for α; then there is some induced motion χ for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w_0, \dot{w}_0)$ such that

$$\mathit{trace}(\alpha) = \mathit{trace}(\chi) \tag{11.98}$$

and $(\alpha.\mathit{lstate}, \chi(\hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\hat{t}_M(\alpha.\mathit{ltime})))$ is in R. For ease of reference, let

$$\begin{aligned} q_1 &= \alpha.\mathit{lstate} \\ w_1 &= \chi(\hat{t}_M(\alpha.\mathit{ltime})) \\ \dot{w}_1 &= \dot{\chi}(\hat{t}_M(\alpha.\mathit{ltime})). \end{aligned} \tag{11.99}$$

Then $(q_1, (w_1, \dot{w}_1))$ is in R.

Note that $\alpha'.\mathit{fstate} = \alpha.\mathit{lstate} = q_1$, and α' is either closed or right-open. Suppose α' is closed. Since R is a simulation, M has related motions for the closed trajectories of A. Since α' consists of a single closed trajectory and $(\alpha'.\mathit{fstate}, (w_1, \dot{w}_1))$ is in R, M has related motions for α'. Suppose instead that α' is right-open. Since α' consists of a single right-open trajectory and $(\alpha'.\mathit{fstate}, (w_1, \dot{w}_1))$ is in R, by Lemma 18, M has related motions for α'.

Then in any event, M has related motions for α and α', and by assumption

$$\begin{aligned} \tau.\mathit{fstate} &= \alpha.\mathit{lstate} \\ \Pi_X(\dot{\tau}.\mathit{fstate}) &= \Pi_X(\dot{\alpha}.\mathit{lstate}). \end{aligned} \tag{11.100}$$

Then by Theorem 61, M has related motions for $\alpha \frown \alpha'$. ☐

The preceding lemmas form the skeleton of an inductive proof. Informally, if α is an execution fragment consisting of a single trajectory, then M has related motions for α. Given an execution fragment for which M has related motions, any extension of that fragment by one discrete transition or one trajectory produces an execution fragment for which M also has related motions. Since every non-Zeno execution fragment consists of some countable sequence of discrete transitions and trajectories, M has related motions for all execution fragments. I show this result more formally in the following theorem.

Theorem 62. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor system M_p and comparable factor body G, such that there exists a simulation R from A to M. Let $(q, (w, \dot{w}))$ be an element of R, with $q \in Q$ and $(w, \dot{w}) \in \mathcal{C}(\top, \mathcal{S})$. Then for any execution fragment $\alpha \in \mathit{frags}_A$ such that $\alpha.\mathit{fstate} = q$, M has related motions for α.*

Proof. By definition, α is a finite or infinite sequence of the form $\tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ such that: for every index $i \geq 0$, τ_i is in $\mathcal{T}$; for every index $i > 0$, a_i is in D; for every index $i > 0$, $a_i = (\tau_{i-1}.\mathit{lstate}, \tau_i.\mathit{fstate})$; all non-final trajectories are closed; and the final trajectory is either right-open or closed.

Single trajectory: Suppose α consists of only a single trajectory, that is, $\alpha = \tau_0$. Then τ_0 is either right-open or it is closed. If τ_0 is right-open, then by Lemma 18, M has related motions for α.

If τ_0 is closed, then since M simulates A, M has related motions for the closed trajectories of A, and so M has related motions for τ_0.

Then in either case, if $\alpha = \tau_0$, M has related motions for α.

Multiple trajectories: Suppose instead that α contains at least two trajectories, separated by at least one discrete transition. For every nonfinal index $i \geq 0$, let $\alpha_i = \tau_i$; then α_i consists of a single closed trajectory in $\mathcal{T}$. Since R is a relation, M has related motions for the closed trajectories of A, and so M has related motions for α_i. If there is a final index k for α, then by the "Single trajectory" argument, M has related motions for τ_k.

The proof proceeds by induction. For any integer $n \geq 0$, let $\alpha'_n = \tau_0 a_1 \tau_1 a_2 \ldots \tau_n$. Clearly $\alpha'_0 = \alpha_0$, and so M has related motions for α'_0. Suppose that M has related motions for α'_i for all indices $i \in \{0, 1, 2, \ldots k\}$ for some nonfinal index $k \geq 0$. It can be shown that M has related motions for α'_{k+1}, as follows.

Let τ be the point trajectory consisting of the restriction of τ_k to its final instant, that is,

$$\tau = \tau_k \lceil (\tau_k.\mathit{ltime}). \tag{11.101}$$

Note that in this case,

$$\tau.\mathit{fstate} = \tau.\mathit{lstate} = \tau_k.\mathit{lstate} = \alpha_k.\mathit{lstate}. \tag{11.102}$$

Let τ' be any closed prefix of τ_{k+1}, and let τ'' be the remaining portion of τ_{k+1}. Clearly in this case,

$$\begin{aligned} \tau'.\mathit{lstate} &= \tau''.\mathit{fstate} \\ \Pi_X(\dot{\tau}'.\mathit{lstate}) &= \Pi_X(\dot{\tau}''.\mathit{fstate}). \end{aligned} \tag{11.103}$$

By Assumption 2, $\mathcal{T}$ is differentiably legal, and so it is prefix closed and suffix closed. Since τ is a suffix of τ_k and τ_k is in $\mathcal{T}$, τ must be in $\mathcal{T}$. Likewise, since τ' is a prefix of τ_{k+1} and τ_{k+1} is in $\mathcal{T}$, τ' must be in $\mathcal{T}$. Again, since τ'' is a suffix of τ_{k+1}, τ'' must be in $\mathcal{T}$. Finally, since τ' is a prefix of τ_{k+1}, and τ_k and τ_{k+1} are subsequent trajectories in α, by Assumptions 8 and 9,

$$\begin{aligned} \tau_k.\mathit{lstate} &= \tau_{k+1}.\mathit{fstate} \\ &= \tau'.\mathit{fstate} \\ \Pi_X(\dot{\tau}_k.\mathit{lstate}) &= \Pi_X(\dot{\tau}_{k+1}.\mathit{fstate}) \\ &= \Pi_X(\dot{\tau}'.\mathit{fstate}) \end{aligned} \tag{11.104}$$

Let $\alpha''_{k+1} = \tau a_{k+1} \tau'$; then α''_{k+1} is closed point-initiated extension of α'_k. Since R is a simulation, M has related motions for the discrete transitions of A, and so M has related motions for α''_{k+1}. Again, since τ'' is in $\mathcal{T}$, M has related motions for τ''.

By Theorem 16, $\alpha'_k \frown \alpha''_{k+1}$ exists; then in particular, $\alpha'_{k+1} = \alpha'_k \frown \alpha''_{k+1} \frown \tau''$. By Theorem 61 and (11.104), M has related motions for $\alpha'_k \frown \alpha''_{k+1}$, and so by the same theorem and (11.103), it has related motions for $(\alpha'_k \frown \alpha''_{k+1}) \frown \tau''$. Then M has related motions for α'_{k+1}.

Then by induction, M has related motions for any α'_n, for any $n \geq 0$, and so it has related motions for α. $\square$

The preceding theorem suffices to show that, for any execution fragment α of A, there exists some motion χ for M such that

$$\mathit{trace}(\chi) = \mathit{trace}(\alpha). \tag{11.105}$$

It follows that

$$\mathit{traces}_A \subseteq \mathit{traces}_M. \tag{11.106}$$

I show this result in the following theorem.

Theorem 63. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor universe Ω_{p} and comparable factor body G, such that there exists a simulation R from A to M. Then*

$$\mathit{traces}_A \subseteq \mathit{traces}_M. \tag{11.107}$$

Proof. Let $q \in Q_0$ be any initial state of A. Since R is a simulation, M has related initial states for A, and so there exists some configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R. By Theorem 62, for any execution $\alpha \in \mathit{frags}_A$ such that $\alpha.\mathit{fstate} = q$, M has related motions for α. Then there exists some motion χ such that $\mathit{trace}(\chi)$ is a trace of M and $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$. The set of traces of all such executions is, by definition, the set of traces of A, and so

$$\mathit{traces}_A \subseteq \mathit{traces}_M. \tag{11.108}$$

□

11.2.5 Simulation of a mechanical system by a hybrid automaton

Given a hybrid automaton A, a comparable hybrid mechanical system M, and a relation R as defined in the previous section, R^{-1} is a simulation (or equivalently, A simulates M) iff A has related initial states for M and A has related execution fragments for the closed motions of M. Again, in this case

$$traces_M \subseteq traces_A. \tag{11.109}$$

I show this result in the following theorem.

Theorem 64. *Let A be a hybrid automaton. Let M be a comparable hybrid mechanical system, with comparable factor system M_p and comparable factor body G, such that there exists a relation R from A to M, where R^{-1} is a simulation. Then*

$$traces_M \subseteq traces_A. \tag{11.110}$$

Proof. Let $(w, \dot{w})$ be any initial configuration of M. Since R^{-1} is a simulation, A has related initial states for M, and so there exists some system state $q \in Q_0$ such that $(q, (w, \dot{w}))$ is in R. Let χ be any induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$; again, since R^{-1} is a simulation, A has related execution fragments for χ. In other words, there exists some execution fragment $\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ such that: for all $i \geq 0$, τ_i is in $\mathcal{T}$; for all $i > 0$, a_i is in D; $\alpha.\mathit{fstate} = q$; and $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$. By the definition of the traces for A, $\mathit{trace}(\alpha)$ is in $traces_A$.

By definition, the traces for M are simply the set of all such motions, for all such choices of configuration in $\mathfrak{X}_0(\top, \mathcal{S})$. By the above argument, for any such motion χ, there exists an execution fragment α such that $\mathit{trace}(\alpha)$ is in $traces_A$ and $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$. Then

$$traces_M \subseteq traces_A. \tag{11.111}$$

□

11.2.6 Bisimulation of hybrid automata and mechanical systems

Given a hybrid automaton A, a comparable hybrid mechanical system M, and a relation R as defined in the previous section, R is a *bisimulation* iff both R and R^{-1} are simulations. In this case,

$$traces_A = traces_M, \tag{11.112}$$

as shown in the following theorem.

Theorem 65. *Let A be a hybrid automaton, M a comparable hybrid mechanical system, and R a relation such that R and R^{-1} are both simulations. Then the traces of A and M are equal, that is,*

$$\mathit{traces}_A = \mathit{traces}_M. \tag{11.113}$$

Proof. By Theorem 63,

$$\mathit{traces}_A \subseteq \mathit{traces}_M. \tag{11.114}$$

By Theorem 64,

$$\mathit{traces}_M \subseteq \mathit{traces}_A. \tag{11.115}$$

Then

$$\mathit{traces}_A = \mathit{traces}_M. \tag{11.116}$$

□

11.3 Summary of notation

$\hat{t}_M(\Gamma_A)$	image of Γ_A under $\hat{t}_M$
$\hat{t}_{\mathrm{p}M}$	mapping from Γ_A to Γ_p
traces_M	mechanical traces of hybrid mechanical system M
trace	mechanical trace

Chapter 12

Proof of Bisimulation

Having clearly defined a hybrid mechanical counterpart to a hybrid automaton, and defined a notion of bisimulation, I at last prove that the hybrid mechanical counterpart bisimulates the hybrid automaton.

12.1 Definition of the relation

Given a hybrid automaton $A = (V, X, Q_0, D, \mathcal{T}, \Gamma_A)$ with system states $Q = V \times X$ and its hybrid mechanical counterpart $M = (\Omega, \mathcal{S}, \Gamma_M, m, \Psi, \mathfrak{f}[\cdot], \mathfrak{X}_0(\top, \mathcal{S}))$, I define the relation R on $Q \times \mathfrak{X}(\top, \mathcal{S})$ as follows. Given any system state $q \in Q$ and configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, the relation R contains $(q, (w, \dot{w}))$ iff there is some trajectory $\tau \in \mathcal{T}$ and automaton instant $t \in dom(\tau)$ such that $\tau(t) = q$ and $(w, \dot{w}) = \texttt{Mech}(\tau, t)$. In other words,

$$R \stackrel{\text{def}}{=} \Big\{ \big(q, (w, \dot{w})\big) \,\Big|\, \big(q \in Q\big) \wedge \big((w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})\big) \\ \wedge \big(\exists \tau \in \mathcal{T}, t \in dom(\tau)[(\tau(t) = q) \wedge ((w, \dot{w}) = \texttt{Mech}(\tau, t))]\big) \Big\}. \quad (12.1)$$

Then consider any system state $q = (v, \mathbf{x})$, where $v \in V$, $\mathbf{x} \in X$, and trajectory $\tau \in \mathcal{T}$ such that at some automaton instant $t \in dom(\tau)$ it is the case that $\tau(t) = q$. Let $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ be the configuration such that the following conditions hold:

$$\begin{aligned} w_{\text{p}}(G) &= \mathbf{x} && (12.2) \\ \dot{w}_{\text{p}}(G) &= \Pi_X(\dot{\tau}(t)) \\ \forall v' \in V\ [w_{\text{c}}(s_{v'}) &= 0] \\ \forall v' \in V\ [\dot{w}_{\text{c}}(s_{v'}) &= 0] \\ w_{\text{c}}(e_v) &= 1 \\ \forall v' \neq v \in V\ [w_{\text{c}}(e_{v'}) &= 0] \end{aligned}$$

$$\forall v' \in V\ [\dot{w}_{\mathrm{c}}(e_{v'}) = 0].$$

Then (12.1) amounts to saying that $(q, (w, \dot{w}))$ is in R; more properly, for any system state $(v, \mathbf{x}) \in Q$ and configuration $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, R contains $((v, \mathbf{x}), (w, \dot{w}))$ if and only if there exists a trajectory such that the conditions in (12.1) hold.

Note that (12.1) specifies position and velocity for all body points not in N, since

$$G \sqcup N = \top_{\mathrm{p}} \tag{12.3}$$

and

$$\left(\bigsqcup_{v' \in V} s_{v'} \right) \sqcup \left(\bigsqcup_{v' \in V} e_{v'} \right) = \top_{\mathrm{c}}. \tag{12.4}$$

In other words, ignoring N as usual, for any choice of system state q and trajectory τ there exists exactly one configuration $(w, \dot{w})$ satisfying (12.1).

12.2 Proof

The primary result of this chapter is a proof demonstrating that R as defined above is a bisimulation. Towards this end, I offer several theorems proving that the various requirements for simulation hold for both R and R^{-1}, with a final theorem that links the results to immediately conclude that R is a bisimulation.

12.2.1 Comparability of M and A

Recall from Section 11.2.1.5 that A and M are comparable if the following conditions hold: first, M_{p} is a comparable factor mechanical system of M with comparable factor body G. Second, $\hat{t}_M$ is a comparably time additivity-preserving mapping for M_{p} and G. Third, M motion-concatenates over $\hat{t}_M(\Gamma_A)$. Fourth, and finally, M force-shifts over $\hat{t}_M(\Gamma_A)$.

I will show that all of these conditions hold. I begin by showing that M_{p} is a comparable factor mechanical system of M with comparable factor body G.

Lemma 21. *Let A be a hybrid automaton with hybrid mechanical counterpart M; then M_{p} is a comparable factor mechanical system for A, with comparable factor body $G \in \Omega_{\mathrm{p}}$.*

Proof. By the construction of G in (7.10),

$$\chi_{\mathrm{p}}(G) \cong X. \tag{12.5}$$

Then by the definition of comparable factor bodies, M_{p} is a comparable factor mechanical system for A, with comparable factor body G. □

I then show that the function $\hat{t}_M$, initially defined in (7.69), is comparably time additivity-preserving.

Lemma 22. *Let A be a hybrid automaton with hybrid mechanical counterpart M. Let $\hat{t}_M : \Gamma_A \to \Gamma_M$ be the function defined in (7.69), that is, for any instant $t \in \Gamma_A$,*

$$\hat{t}_M(t) = (t, 0). \tag{12.6}$$

Then for any instant $t \in \Gamma_M$ and $t' \in \hat{t}_M(\Gamma_A)$, the sum $t + t'$ is defined; likewise, if $t \geq t'$, the instant $t - t'$ is defined.

Proof. Let $t = (r, z)$ be any instant in Γ_M, for some $r \in \mathbb{R}^{\geq 0}$ and some $z \in \mathbb{Z}_4$. Let t' be any instant in $\hat{t}_M(\Gamma_A)$. It follows from the definition of $\hat{t}_M$ in (7.69) that

$$t' = (r', 0) \tag{12.7}$$

for some $r' \in \mathbb{R}^{\geq 0}$ and some $z \in \mathbb{Z}_4$. By Section 7.3.3, the sum

$$t + t' = (r, z) + (r', 0) \tag{12.8}$$

is defined; in particular,

$$t + t' = (r, z) + (r', 0) = (r + r', z). \tag{12.9}$$

As described in that same section, if $r \geq r'$, the difference

$$t - t' = (r, z) - (r', 0) \tag{12.10}$$

is also defined; in particular,

$$t - t' = (r, z) - (r', 0) = (r - r', z). \tag{12.11}$$

□

Theorem 66. *Let A be a hybrid automaton with hybrid mechanical counterpart M and comparable factor mechanical system M_p and comparable factor body G. Let $\hat{t}_M : \Gamma_A \to \Gamma_M$ be the function defined in (7.69), that is, for any instant $t \in \Gamma_A$,*

$$\hat{t}_M(t) = (t, 0). \tag{12.12}$$

Then $\hat{t}_M$ is comparably time additivity-preserving for M_p and G.

Proof. For $\hat{t}_M$ to be comparably time additivity-preserving, it must first be time additivity-preserving. Recall from Section 11.2.1.4 that $\hat{t}_M$ is time additivity-preserving iff the following three properties hold. First, $\hat{t}_M$ must preserve the order of all automaton instants; that is, given two automaton instants $t, t' \in \Gamma_A$,

$$t < t' \text{ iff } \hat{t}_M(t) < \hat{t}_M(t'). \tag{12.13}$$

Second, for any hybrid mechanical instants $t \in \Gamma_M$ and $t' \in \hat{t}_M(\Gamma_A)$, the sum $t + t'$ is defined; likewise, if $t \geq t'$, the difference $t - t'$ is defined.

Third, for any automaton instants $t, t' \in \Gamma_A$, the sum of the mechanical mappings of the two instants must be equal to the mapping of their sums. That is,

$$\hat{t}_M(t + t') = \hat{t}_M(t) + \hat{t}_M(t'). \tag{12.14}$$

Likewise, if $t \geq t'$, the difference of the mappings of the two instants must be equal to the mapping of their differences. That is,

$$\hat{t}_M(t - t') = \hat{t}_M(t) - \hat{t}_M(t'). \tag{12.15}$$

Consider these one at a time. Suppose t, t' are any instants int Γ_A; then

$$\begin{aligned} \hat{t}_M(t) &= (t, 0) \\ \hat{t}_M(t') &= (t', 0). \end{aligned} \tag{12.16}$$

By the time ordering in Section 7.3.3, $(t, 0) < (t', 0)$ iff $t < t'$. Equivalently, $\hat{t}_M(t) < \hat{t}_M(t')$ iff $(t < t')$. This satisfies the first requirement.

Consider the second requirement; this is precisely the result from Lemma 22, and so this result holds.

Consider the third requirement. Let t, t' be any instants in Γ_A. By the definition of $\hat{t}_M$, it follows that

$$\begin{aligned} \hat{t}_M(t) &= (t, 0) \\ \hat{t}_M(t') &= (t', 0). \end{aligned} \tag{12.17}$$

Again, by Lemma 22, the sum $\hat{t}_M(t) + \hat{t}_M(t')$ is defined. By Section 7.3.3, in particular

$$\begin{aligned} \hat{t}_M(t) + \hat{t}_M(t') &= (t, 0) + (t', 0) \\ &= (t + t', 0) \\ &= \hat{t}_M(t + t'). \end{aligned} \tag{12.18}$$

Likewise, if $t \geq t'$, the difference $\hat{t}_M(t) - \hat{t}_M(t')$ is defined, and in particular,

$$\hat{t}_M(t) - \hat{t}_M(t') = (t, 0) - (t', 0) \tag{12.19}$$

$$
\begin{aligned}
&= (t - t', 0) \\
&= \hat{t}_M(t - t').
\end{aligned}
$$

This satisfies the third requirement.

Then $\hat{t}_M$ is a time additivity-preserving mapping. It is comparably time additivity-preserving iff the composition $\hat{t}_{pM} = \hat{t}_p \circ \hat{t}_M$ is a bijection. Again, by Section 7.3.3, for any instant $t \in \Gamma_A$,

$$
\begin{aligned}
\hat{t}_{pM}(t) &= \hat{t}_p(\hat{t}_M(t)) \\
&= \hat{t}_p(t, 0) \\
&= t.
\end{aligned}
\tag{12.20}
$$

That is, the mapping $\hat{t}_{pM}$ is the identity, and so it is a bijection. Then $\hat{t}_M$ is comparably time additivity-preserving. $\square$

Next, I show that M motion concatenates over $\hat{t}_M(\Gamma_A)$.

Theorem 67. *Let A be a hybrid automaton with hybrid mechanical counterpart M; then M motion concatenates over $\hat{t}_M(\Gamma_A)$.*

Proof. By the definition of motion-concatenation in Section 11.2.1.2, M motion-concatenates over $\hat{t}_M(\Gamma_A)$ given the following conditions. Suppose that there exists some differentiable motion χ with state-time function $\texttt{state}$ and maximal interval set $\mathscr{I}_{\top_c}$. Let $w = \chi(\top, (0, 0))$, and let $\dot{w} = \dot{\chi}(\top, (0, 0))$. For any hybrid mechanical instant $t_1 \in \hat{t}_M(\Gamma_A)$, let $J \in \mathscr{I}_{\top_c}$ be the maximal interval containing t_1, and let

$$
\begin{aligned}
J' &= \{t \in J \mid t \leq t_1\} \\
J'' &= \{t \in J \mid t > t_1\}.
\end{aligned}
\tag{12.21}
$$

Then M motion-concatenates iff, for any such arrangement, χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J whenever χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during both J' and J''.

If no such induced motion χ exists, then the proof is trivially satisfied. Suppose instead that some such motion exists, that is, that there is some choice of motion χ, instant t_1, maximal interval J, and corresponding intervals J', J'' such that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during both J' and J''. Again, if J'' is empty, the proof is trivially satisfied; suppose instead that it is not empty.

Consider first the instant t_1. Note that $\hat{t}_M(\Gamma_A)$ consists of all instants $(r, 0) \in \Gamma_M$, where r is in $\mathbb{R}^{\geq 0}$. Thus, there exists some value $r_1 \in \mathbb{R}^{\geq 0}$ such that $t_1 = (r_1, 0)$. Since J is in $\mathscr{I}_{\top_c}$, there exists some state $\psi \in \Psi(\top_c)$ such that $\texttt{state}(J) = \psi$, and so

$$
\texttt{state}(J') = \texttt{state}(J'') = \psi. \tag{12.22}
$$

Then since χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J', there exists some force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ over J' such that, for every instant $t' \in J'$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}'_i(t')(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t')). \tag{12.23}$$

Likewise, since χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J'', there exists some force variation $\hat{f}'' \in \hat{\mathfrak{f}}[\psi]$ over J'' such that, for every instant $t'' \in J''$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}'_i(t'')(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t'')). \tag{12.24}$$

By (8.6),

$$\hat{\mathfrak{f}}[\psi] = \left\{ \hat{f}_E + \hat{f}_S \,\middle|\, \left(\hat{f}_E \in \hat{\mathfrak{f}}_E[E(\psi)] \right) \wedge \left(\hat{f}_S \in \hat{\mathfrak{f}}_S[S(\psi)] \right) \right\}. \tag{12.25}$$

In other words, there exists force variations $\hat{f}'_S, \hat{f}''_S \in \hat{\mathfrak{f}}_S[S(\psi)]$ and $\hat{f}'_E, \hat{f}''_E \in \hat{\mathfrak{f}}_E[E(\psi)]$ such that

$$\begin{aligned} \hat{f}' &= \hat{f}'_S + \hat{f}'_E \\ \hat{f}'' &= \hat{f}''_S + \hat{f}''_E, \end{aligned} \tag{12.26}$$

and so, for any instants $t' \in J'$, $t'' \in J''$ and body $B \in \Omega$,

$$\begin{aligned} \hat{f}'(t')(B, \overline{B}) &= \hat{f}'_S(t')(B, \overline{B}) + \hat{f}'_E(t')(B, \overline{B}) \\ \hat{f}''(t'')(B, \overline{B}) &= \hat{f}''_S(t'')(B, \overline{B}) + \hat{f}''_E(t'')(B, \overline{B}). \end{aligned} \tag{12.27}$$

Sensor forces: By (8.86),

$$\hat{\mathfrak{f}}_S[S(\psi)] = \begin{cases} \left\{ \hat{f}_{v\text{–}v',r} \,\middle|\, [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0} \right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}, & \psi = \psi_{v,[\![\mathbf{x}]\!]} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{12.28}$$

Suppose that $S(\psi) = \overline{\psi}_V$. Then, by (12.28),

$$\begin{aligned} \hat{\mathfrak{f}}_S[S(\psi)] &= \hat{\mathfrak{f}}_S[\overline{\psi}_V] \\ &= \{\hat{f}_{\mathbf{0}}\}. \end{aligned} \tag{12.29}$$

Since $\hat{f}'_S$ is in $\hat{\mathfrak{f}}_S[S(\psi)]$, it follows that $\hat{f}'_S = \hat{f}_{\mathbf{0}}$. Likewise, $\hat{f}''_S = \hat{f}_{\mathbf{0}}$.

Then let $\hat{f}_S = \hat{f}_{\mathbf{0}}$; then clearly $\hat{f}_S$ is a force variation in $\hat{\mathfrak{f}}_S[S(\psi)]$ such that

$$\hat{f}_S(t) = \begin{cases} \hat{f}_{\mathbf{0}}(t), & t \leq t_1 \\ \hat{f}_{\mathbf{0}}(t), & t > t_1, \end{cases} \tag{12.30}$$

and so, for every $t \in J$,

$$\hat{f}_S(t) = \begin{cases} \hat{f}'_S(t), & t \le t_1 \\ \hat{f}''_S(t), & t > t_1. \end{cases} \tag{12.31}$$

Suppose instead that $S(\psi) = \psi_{v,[\![\mathbf{x}]\!]}$ for some $v \in V$ and $[\![\mathbf{x}]\!] \in X/\equiv_{\mathcal{D},v}$. Then

$$\hat{\mathfrak{f}}_S[S(\psi)] = \left\{ \hat{f}_{v\text{–}v',r} \;\middle|\; \left[v' \in \mathcal{D}_v(\mathbf{x})\right] \wedge \left[r \in \mathbb{R}^{\ge 0}\right] \right\} \cup \{\hat{f}_{\mathbf{0}}\}. \tag{12.32}$$

Since $\hat{f}'_S$ is in $\hat{\mathfrak{f}}_S[\psi]$, it follows that either $\hat{f}'_S = \hat{f}'_{v\text{–}v',r'}$, for some $v' \ne v \in \mathcal{D}_v(\mathbf{x})$ and some $r' \in \mathbb{R}^{\ge 0}$, or $\hat{f}'_S = \hat{f}_{\mathbf{0}}$. Likewise, either $\hat{f}''_S = \hat{f}_{v\text{–}v'',r''}$, for some $v'' \ne v \in \mathcal{D}_v(\mathbf{x})$ and some $r'' \in \mathbb{R}^{\ge 0}$, or $\hat{f}''_S = \hat{f}_{\mathbf{0}}$.

Before continuing, note by Theorem 24 that, given a force system $\hat{f}_{v\text{–}v',r}$, for any $r \in \mathbb{R}$,

$$\texttt{state}(r,0) = \texttt{state}(r,1) = \texttt{state}(r,2) \ne \texttt{state}(r,3). \tag{12.33}$$

In other words, either all of $(r,0)$, $(r,1)$, and $(r,2)$ are in J, or none of them are; if any of them are, then $(r,3)$ is not. Also note that the only nonzero forces in $\hat{f}_{v\text{–}v',r}$ are at instants $(r,1)$ and $(r,2)$.

Suppose that $\hat{f}' = \hat{f}'_{v\text{–}v',r'}$ for some $v' \ne v \in \mathcal{D}_v(\mathbf{x})$, and some $r' \in \mathbb{R}^{\ge 0}$ such that $(r',1)$ or $(r',2)$ are in J'. Since $t_1 = (r_1,0)$, and $(r',1) < (r',2) < t_1$, it must also be the case that $(r',3) < t_1$; thus $(r',3)$ is in J'. But by (12.33), if $(r'1,)$ or $(r',2)$ are in J, then $(r',3)$ is not, and so $(r',3)$ is not in J'. This is a contradiction; thus, there is no instant $(r',1)$ or $(r',2)$ in J' such that $\hat{f}' = \hat{f}'_{v\text{–}v',r'}$. That is, over the interval J', there are no nonzero forces in $\hat{f}'$. In other words, for any instant $t \in J'$,

$$\hat{f}'_S(t) = \hat{f}_{\mathbf{0}}(t). \tag{12.34}$$

Likewise, suppose that $\hat{f}'' = \hat{f}''_{v\text{–}v'',r''}$ for some $v'' \ne v \in \mathcal{D}_v(\mathbf{x})$, and some $r'' \in \mathbb{R}^{\ge 0}$. If $(r'',1)$ and $(r'',2)$ are not in J'', then again at every instant $t \in J''$,

$$\hat{f}''_S(t) = \hat{f}_{\mathbf{0}}(t). \tag{12.35}$$

In this case, let $\hat{f}_S = \hat{f}_{\mathbf{0}}$; then clearly $\hat{f}_S$ is a force variation in $\hat{\mathfrak{f}}_S[S(\psi)]$ such that

$$\hat{f}_S(t) = \begin{cases} \hat{f}_{\mathbf{0}}(t), & t \le t_1 \\ \hat{f}_{\mathbf{0}}(t), & t > t_1, \end{cases} \tag{12.36}$$

and so, by (12.34) and (12.35), for every $t \in J$,

$$\hat{f}_S(t) = \begin{cases} \hat{f}'_S(t), & t \le t_1 \\ \hat{f}''_S(t), & t > t_1. \end{cases} \tag{12.37}$$

Otherwise, if $(r'', 1)$ is in J'', then by (12.33) so is $(r'', 2)$. Then let $\hat{f}_S = \hat{f}''_{v\text{–}v'', r''}$; then clearly $\hat{f}_S$ is a force variation in $\hat{\mathfrak{f}}_S[S(\psi)]$ such that

$$\hat{f}_S(t) = \begin{cases} \hat{f}_{\mathbf{0}}(t), & t \leq t_1 \\ \hat{\mathfrak{f}}_S[S(\psi)], & t > t_1, \end{cases} \tag{12.38}$$

and so, by (12.34), for every $t \in J$,

$$\hat{f}_S(t) = \begin{cases} \hat{f}'_S(t), & t \leq t_1 \\ \hat{f}''_S(t), & t > t_1. \end{cases} \tag{12.39}$$

Then for any choice of $\psi \in \Psi(S)$ and any $\hat{f}', \hat{f}'' \in \hat{\mathfrak{f}}_S[S(\psi)]$, there exists some force variation $\hat{f}_S \in \hat{\mathfrak{f}}_S[S(\psi)]$ such that, for every instant $t \in J$,

$$\hat{f}_S(t) = \begin{cases} \hat{f}'_S(t), & t \leq t_1 \\ \hat{f}''_S(t), & t > t_1. \end{cases} \tag{12.40}$$

Effector forces: By (8.30),

$$\hat{\mathfrak{f}}_E[E(\psi)] = \begin{cases} \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & E(\psi) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & E(\psi) = \overline{\psi}_V. \end{cases} \tag{12.41}$$

Suppose that $E(\psi) = \overline{\psi}_V$. By an argument identical to the sensor case, if one lets $\hat{f}_E = \hat{f}_{\mathbf{0}}$, then $\hat{f}_E$ is a force variation in $\hat{\mathfrak{f}}_E[E(\psi)]$ such that, for every $t \in J$,

$$\hat{f}_E(t) = \begin{cases} \hat{f}'_E(t), & t \leq t_1 \\ \hat{f}''_E(t), & t > t_1. \end{cases} \tag{12.42}$$

Suppose instead that $E(\psi) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ for some $v \in V$ and $\mathbf{x}, \dot{\mathbf{x}} \in X$. Then

$$\hat{\mathfrak{f}}_E[E(\psi)] = \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\} \tag{12.43}$$

Since $\hat{f}'_E$ is in $\hat{\mathfrak{f}}_E[\psi]$, it follows that $\hat{f}'_E = \hat{f}^{\tau'}$ for some $\tau' \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$. Likewise, $\hat{f}''_E = \hat{f}^{\tau''}$ for some $\tau'' \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$.

Note that $t_1 = (r_1, 0)$ is the final instant in J'; since J'' is not empty, it must contain $(r_1, 1)$. By (12.23), restricting attention to G and the physical universe, for every instant $t' \in J'$,

$$\hat{f}'_{\mathrm{p}}(t')(G, \overline{G}) = m_{\mathrm{p}}(G) \ddot{\chi}_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}}(t')). \tag{12.44}$$

Since only the effector can have nonzero forces on physical bodies, one can restrict $\hat{f}'_{\mathrm{p}}$ to $\hat{f}'_{\mathrm{p},E}$, and so

$$\hat{f}'_{\mathrm{p},E}(t')(G,\overline{G}) = m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,\hat{t}_{\mathrm{p}}(t')). \tag{12.45}$$

Since $\hat{f}'_E = \hat{f}^{\tau'}$, one can substitute the latter for the former, giving

$$\hat{f}^{\tau'}_{\mathrm{p}}(t')(G,\overline{G}) = m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,\hat{t}_{\mathrm{p}}(t')). \tag{12.46}$$

Since $\hat{f}^{\tau'}$ is autonomous, that is, its value does not vary over time, one can replace t' in the last term with any instant in J'. Choosing $t_1 = (r_1, 0)$,

$$\begin{aligned}\hat{f}^{\tau'}_{\mathrm{p}}(t')(G,\overline{G}) &= m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,\hat{t}_{\mathrm{p}}(r_1,0)) \\ &= m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,r_1).\end{aligned} \tag{12.47}$$

By an identical argument, for any $t'' \in J''$,

$$\hat{f}^{\tau''}_{\mathrm{p}}(t'')(G,\overline{G}) = m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,\hat{t}_{\mathrm{p}}(r_1,0)) \tag{12.48}$$

$$= m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G,r_1). \tag{12.49}$$

But the right-hand sides of the preceding equations are equal, and so their left-hand sides are, too. Then for any instants $t' \in J'$, $t'' \in J''$,

$$\hat{f}^{\tau'}_{\mathrm{p}}(t')(G,\overline{G}) = \hat{f}^{\tau''}_{\mathrm{p}}(t'')(G,\overline{G}). \tag{12.50}$$

Equivalently, since $\hat{f}^{\tau'}$ and $\hat{f}^{\tau''}$ have no nonzero controller forces,

$$\hat{f}^{\tau'}(t')(G,\overline{G}) = \hat{f}^{\tau''}(t'')(G,\overline{G}), \tag{12.51}$$

and so

$$\hat{f}'_E = \hat{f}''_E. \tag{12.52}$$

Then let $\hat{f}_E = \hat{f}'_E = \hat{f}''_E$. It follows that, for any choice of $\psi \in \Psi(\top_{\mathrm{c}})$ and any $\hat{f}', \hat{f}'' \in \hat{\mathfrak{f}}_E[E(\psi)]$, there exists some force variation $\hat{f}_E \in \hat{\mathfrak{f}}_E[E(\psi)]$ such that, for every instant $t \in J$,

$$\hat{f}_E(t) = \begin{cases}\hat{f}'_E(t), & t \leq t_1 \\ \hat{f}''_E(t), & t > t_1.\end{cases} \tag{12.53}$$

Combined forces: Then let $\hat{f}_S$ be a force variation in $\hat{\mathfrak{f}}_S[S(\psi)]$ such that, for every instant $t \in J$,

$$\hat{f}_S(t) = \begin{cases} \hat{f}'_S(t), & t \leq t_1 \\ \hat{f}''_S(t), & t > t_1. \end{cases} \tag{12.54}$$

Likewise, let $\hat{f}_E$ be a force variation in $\hat{\mathfrak{f}}_E[E(\psi)]$ such that, for every instant $t \in J$,

$$\hat{f}_E(t) = \begin{cases} \hat{f}'_E(t), & t \leq t_1 \\ \hat{f}''_E(t), & t > t_1. \end{cases} \tag{12.55}$$

By the preceding results, such force variations must exist. Let $\hat{f} = \hat{f}_S + \hat{f}_E$; by the definition of universal forces in (8.6), it follows that $\hat{f}$ is in $\hat{\mathfrak{f}}[\psi]$. By (12.26) and (12.27), for every instant $t \in J$,

$$\hat{f}(t) = \begin{cases} \hat{f}'(t), & t \leq t_1 \\ \hat{f}''(t), & t > t_1. \end{cases} \tag{12.56}$$

By (12.23) and (12.24), then, for every instant $t \in J$ and body $B \in \Omega$,

$$\hat{f}(t)(B, \overline{B}) = \begin{cases} m(B)\ddot{\chi}(B,t), & t \leq t_1 \\ m(B)\ddot{\chi}(B,t), & t > t_1, \end{cases} \tag{12.57}$$

or, more simply,

$$\hat{f}(t)(B, \overline{B}) = m(B)\ddot{\chi}(B,t). \tag{12.58}$$

Then there is a force variation in $\hat{\mathfrak{f}}[\psi]$, namely $\hat{f}$, such that at every instant $t \in J$,

$$\hat{f}(B, \overline{B}, t) = m(B)\ddot{\chi}(B,t). \tag{12.59}$$

Then χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$. Since the above construction holds for any motions χ', χ'' and instant $t \in \hat{t}_M(\Gamma_A)$, M motion concatenates over $\hat{t}_M(\Gamma_A)$. $\square$

Finally, M force-shifts over $\hat{t}_M(\Gamma_A)$.

Theorem 68. *Let A be a hybrid automaton with hybrid mechanical counterpart M; then M force-shifts over $\hat{t}_M(\Gamma_A)$.*

Proof. By the definition of force-shifting, M force-shifts over $\hat{t}_M(\Gamma_A)$ if, for every instant $t_1 \in \hat{t}_M(\Gamma_A)$, the sum $t + t_1$ is defined for all $t \in \Gamma_M$ and, for any mechanical controller state $\psi \in \Psi(\top_c)$ and any force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$, there exists some other force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ such that, for all instants

$t \in dom(\hat{f})$,

$$f_t = f'_{t+t_1}. \tag{12.60}$$

Let t_1 be any instant in $\hat{t}_M(\Gamma_A)$. By Lemma 22, for any instant $t \in \Gamma_M$, the sum $t + t_1$ is defined.

Consider any mechanical controller state $\psi \in \Psi(\top_{\mathrm{c}})$ and any force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$. Let t be any arbitrary instant in $dom(\hat{f})$. By Section 7.3.3,

$$t = (r, z) \tag{12.61}$$

for some $r \in \mathbb{R}^{\geq 0}$ and some $z \in \mathbb{Z}_4$,

$$t_1 = (r_1, 0) \tag{12.62}$$

for some $r_1 \in \mathbb{R}^{\geq 0}$, and

$$t + t_1 = (r + r_1, z). \tag{12.63}$$

By (8.128), either $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ for some $v \in V$ and some $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}$, or $\psi = \overline{\psi}_V$. Suppose it is the former; that is, suppose

$$\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \tag{12.64}$$

for some $v \in V$ and some $\psi_{\mathbf{x},\dot{\mathbf{x}}} \in \mathfrak{X}(\top, \mathcal{S})/{\equiv_G}$. Then by (8.104), either

$$\hat{f} = \hat{f}^{\tau} + \hat{f}_{v\text{–}v',r} \tag{12.65}$$

or

$$\hat{f} = \hat{f}^{\tau}, \tag{12.66}$$

for some $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$, some $v' \in \mathcal{D}_v(\mathbf{x})$, and some $r \in \mathbb{R}^{\geq 0}$. By the definition of $\hat{f}^{\tau}$ in Section 8.1.1,

$$f^{\tau}_t = (\Pi_X(\ddot{\tau}.\mathit{fstate}), \mathbf{0}_{\mathrm{c}}) = f^{\tau}_{t+t_1}. \tag{12.67}$$

As noted above, either $\hat{f} = \hat{f}^{\tau} + \hat{f}_{v\text{–}v',r}$ or $\hat{f} = \hat{f}^{\tau}$. Suppose it is the latter; then let $\hat{f}' = \hat{f}^{\tau}$. It follows that

$$\begin{aligned} \hat{f}'(t + t_1) &= \hat{f}^{\tau}(t + t_1) \\ &= \hat{f}^{\tau}(t) \\ &= \hat{f}(t). \end{aligned} \tag{12.68}$$

Clearly in this case $\hat{f}'$ is in $\hat{\mathfrak{f}}[v]$.

Suppose instead that $\hat{f} = \hat{f}^{\tau} + \hat{f}_{v\text{–}v',r}$. Consider the force variation $\hat{f}_{v\text{–}v',r+r_1}$. By the definition of

$\hat{f}_{v-v',r+r_1}$ in Section 8.2.1,

$$\hat{f}_{v-v',r}(t) = \hat{f}_{v-v',r+r_1}(t + t_1). \tag{12.69}$$

Let $\hat{f}' = \hat{f}^{\tau} + \hat{f}_{v-v',r+r_1}$. By (12.67) and (12.69),

$$\begin{aligned} \hat{f}'(t + t_1) &= \hat{f}^{\tau}(t + t_1) + \hat{f}_{v-v',r+r_1}(t + t_1) \\ &= \hat{f}^{\tau}(t) + \hat{f}_{v-v',r}(t) \\ &= \hat{f}(t). \end{aligned} \tag{12.70}$$

By the definition of $\hat{\mathfrak{f}}[\cdot]$ in (8.104), $\hat{f}'$ is in $\hat{\mathfrak{f}}[\psi]$.

The above handles the case where $\psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$; there remains the case where $\psi = \overline{\psi}_V$. But in this case, $\hat{\mathfrak{f}}[\psi] = \{\hat{f}_{\mathbf{0}}\}$, and so $\hat{f} = \hat{f}_{\mathbf{0}}$. Let $\hat{f}' = \hat{f}_{\mathbf{0}}$; then clearly, for any $t \in \Gamma$,

$$\hat{f}'(t + t_1) = f_{\mathbf{0}} = \hat{f}(t). \tag{12.71}$$

Then in any case, there exists some force variation $\hat{f}' \in \hat{\mathfrak{f}}[\psi]$ such that, for any instant $t \in dom(\hat{f})$,

$$\hat{f}_t = \hat{f}'_{t+t_1}. \tag{12.72}$$

The above argument holds for any choice of $t_1 \in \hat{t}_M(\Gamma_A)$, and so M force shifts over $\hat{t}_M(\Gamma_A)$. $\square$

The preceding theorems have shown that all conditions for comparability are satisfied. Thus, M and A are comparable.

Theorem 69. *Let A be a hybrid automaton with hybrid mechanical counterpart M. M and A are comparable, with comparable factor universe Ω_{p} and comparable factor body G.*

Proof. The truth of this theorem follows immediately from Lemma 21 and Theorems 66, 67, and 68, and the definition of comparability. $\square$

12.2.2 Relation of initial automaton states to initial mechanical controller states

I next prove that, for each initial system state $q \in Q$, there is a corresponding configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R.

Theorem 70. *Let A be a hybrid automaton with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then for any automaton system state $q \in Q_0$, there is a pair $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R. In other words, M has related initial states for A.*

Proof. Consider any $q \in Q_0$; then $q = (v, \mathbf{x})$ for some $v \in V$, $\mathbf{x} \in X$. By Assumption 10, there exists some non-point trajectory $\tau \in \mathcal{T}$ such that $\tau.fstate = q$; by Assumption 2, τ is second-degree doubly semidifferentiable. By the definition of $\mathfrak{X}_0(\top, \mathcal{S})$ in (7.194), the configuration $\texttt{Mech}(\tau, 0)$ is in $\mathfrak{X}_0(\top, \mathcal{S})$. By the definition of R in (12.1), the pair $(\tau.\mathit{fstate}, \texttt{Mech}(\tau))$ is in R. Then for any such system state $q \in Q_0$, there exists a configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$ such that $(q, (w, \dot{w}))$ is in R, and so M has related initial states for A. $\square$

By a similar argument, for any configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$, there is an automaton system state $q \in Q_0$ such that $(q, (w, \dot{w}))$ is in R. In other words, A has related initial states for M.

Corollary 71. *Let A be a hybrid automaton with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then for any pair $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$, there exists a system state $q \in Q_0$ such that $(q, (w, \dot{w}))$ is in R. In other words, A has related initial states for M.*

Proof. Consider any pair $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$. By the definition of $\mathfrak{X}_0(\top, \mathcal{S})$ in (7.194),

$$\mathfrak{X}_0(\top, \mathcal{S}) = \{\texttt{Mech}(\tau, t) \mid [\tau \in \mathcal{T}] \wedge [t \in \mathit{dom}(\tau)] \wedge [\tau(t) \in Q_0]\}. \tag{12.73}$$

In other words, there exists some trajectory $\tau \in \mathcal{T}$ and automaton instant $t \in \mathit{dom}(\tau)$ such that $\texttt{Mech}(\tau, t) = (w, \dot{w})$ and $\tau(t)$ is in Q_0. By the definition of R in (12.1), the pair $(\tau(t), \texttt{Mech}(\tau, t))$ is in R. Then for any configuration $(w, \dot{w}) \in \mathfrak{X}_0(\top, \mathcal{S})$, there exists a system state $q \in Q_0$ such that $(q, (w, \dot{w}))$ is in R, and so A has related initial states for M. $\square$

12.2.3 Relation of single automaton trajectories to single mechanical force systems

I next prove that M has related motions for the closed trajectories of A. Towards that end, I show certain restrictions hold for the forces possible in M. First, I show that, given any motion of $\top$, the acceleration of G is constant during any period in which the universal mechanical controller state does not change.

Lemma 23. *Let A be a hybrid automaton with hybrid mechanical counterpart M. Let $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ be any motion, let $\texttt{state}$ be the state-time function for χ, and let $\mathscr{I}_{\top_{\mathrm{c}}}$ be the maximal interval set given $\texttt{state}$. Then for any interval $J \in \mathscr{I}_{\top_{\mathrm{c}}}$, the acceleration of G is constant across all instants in J.*

Proof. Let $\psi \in \Psi(\top_{\mathrm{c}})$ be the mechanical controller state such that $\texttt{state}(J) = \psi$. By the definition of induced motion in Section 6.3.3, there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ such that, at every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.74}$$

Note that G has no controller component, and so to show that its acceleration is constant, it is sufficient to consider its physical acceleration. In other words, one needs to show only that

$$\hat{f}_{\mathrm{p}}(t)(G, \overline{G}) = m_{\mathrm{p}}(G)\ddot{\chi}_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}}(t)). \tag{12.75}$$

By (8.135), the only possible nonzero physical forces contributed by any $\hat{f} \in \hat{\mathfrak{f}}[\psi]$ are those contributed by some $\hat{f}^\tau$, for some $\tau \in \mathcal{T}$. But by definition, $\hat{f}^\tau$ is constant across all instants; then the acceleration of G is constant across all instants. $\square$

Given the above constraints on the motion of G, I can show that M has related motions for the closed trajectories of A.

Theorem 72. *Let A be a hybrid automaton with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then M has related motions for the closed trajectories of A.*

Proof. Consider any choice of system state $q \in Q$ and closed trajectory $\tau \in \mathcal{T}$ such that $\tau.\mathit{fstate} = q$; if no such system state and trajectory exist, then the proof is trivially satisfied. By the definition of system states, there exist automaton controller and environment states $v \in V$ and $\mathbf{x} \in X$ such that $q = (v, \mathbf{x})$. Let $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ be the configuration such that $\mathtt{Mech}(\tau, 0) = (w, \dot{w})$; that is, by the definition of `Mech` in Section 7.3.7,

$$\begin{aligned} w_{\mathrm{p}}(G) &= \Pi_X(\tau(0)) \\ \dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}(0)) \\ w_{\mathrm{c}}(e_v) &= 1 \\ \forall v' \neq v \in V &\big[w_{\mathrm{c}}(e_{v'}) = 0\big] \\ \forall v' \in V &\big[\dot{w}_{\mathrm{c}}(e_{v'}) = 0\big] \\ \forall v' \in V &\big[w_{\mathrm{c}}(s_{v'}) = 0\big] \\ \forall v' \in V &\big[\dot{w}_{\mathrm{c}}(s_{v'}) = 0\big]. \end{aligned} \tag{12.76}$$

By the definition of R in (12.1), $(q, (w, \dot{w}))$ is in R. Let α be the execution fragment such that $\alpha = \tau$. By definition, M has related motions for the closed trajectories of A iff, for any such construction of $(q, (w, \dot{w}))$ and execution fragment α, there exists some motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- $(\alpha.\mathit{lstate}, (\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime}))))$ is in R.

Let J_1 be the hybrid mechanical time interval $[(0,0), (\alpha.\mathit{ltime}, 0)]$.

Construction of χ: Let χ be any motion such that, at every instant $(r, z) \in J_1$,

$$\begin{aligned} \chi_{\mathrm{p}}(G, r) &= \Pi_X(\alpha(r)) \\ \forall v' \in V &\left[\chi_{\mathrm{c}}(s_{v'}, (r, z)) = 0\right] \end{aligned} \tag{12.77}$$

$$\chi_{\mathrm{c}}(e_v,(r,z)) = 1$$
$$\forall v' \neq v \in V\ \big[\chi_{\mathrm{c}}(e_{v'},(r,z)) = 0\big].$$

Note that $G \sqcup N = \top_{\mathrm{p}}$ and $(\bigsqcup_{v' \in V} e_{v'}) \sqcup (\bigsqcup_{v' \in v} s_{v'}) = \top_{\mathrm{c}}$, and so this is a complete specification of placement for all body points not in N during J_1. By Assumption 2, α is differentiably legal, and so it is second-degree doubly semidifferentiable. It follows that $\dot{\chi}$ and $\ddot{\chi}$ are defined, and in particular that at every such instant $(r,z) \in J_1$,

$$\begin{aligned} \dot{\chi}_{\mathrm{p}}(G,r) &= \Pi_X(\dot{\alpha}(r)) \\ \forall v' \in V\ \big[\dot{\chi}_{\mathrm{c}}(s_{v'},(r,z)) &= 0\big] \\ \forall v' \in V\ \big[\dot{\chi}_{\mathrm{c}}(e_{v'},(r,z)) &= 0\big]. \end{aligned} \tag{12.78}$$

and

$$\begin{aligned} \ddot{\chi}_{\mathrm{p}}(G,r) &= \Pi_X(\ddot{\alpha}(r)) \\ \forall v' \in V\ \big[\ddot{\chi}_{\mathrm{c}}(s_{v'},(r,z)) &= 0\big] \\ \forall v' \in V\ \big[\ddot{\chi}_{\mathrm{c}}(e_{v'},(r,z)) &= 0\big]. \end{aligned} \tag{12.79}$$

Let `state` be the state-time function for χ, and let $\mathscr{I}_{\top_{\mathrm{c}}}$ be the set of maximal state intervals given `state`.

Prove induced motion: Recall from the definition of induced motion in Section 5.3.2 and the discussion of initial configurations in Section 5.3.4 that χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$ iff

$$\chi(\top,(0,0)) = w, \tag{12.80}$$

$$\dot{\chi}(\top,(0,0)) = \dot{w}, \tag{12.81}$$

and, for any interval $J \in \mathscr{I}_{\top_{\mathrm{c}}}$, there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\mathtt{state}(J)]$ over J such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.82}$$

Consider first the requirement in (12.80) that $\chi(G,(0,0)) = w$. By (12.77), letting $(r,z) = (0,0)$,

$$\begin{aligned} \chi_{\mathrm{p}}(G,0) &= \Pi_X(\alpha(0)) \\ \forall v' \in V\ \big[\chi_{\mathrm{c}}(s_{v'},(0,0)) &= 0\big] \\ \chi_{\mathrm{c}}(e_v,(0,0)) &= 1 \\ \forall v' \neq v \in V\ \big[\chi_{\mathrm{c}}(e_{v'},(0,0)) &= 0\big]. \end{aligned} \tag{12.83}$$

But this is precisely the definition of w given in (12.76), and so $\chi(G,(0,0)) = w$.

Similarly, consider the requirement in (12.81) that $\dot{\chi}(\top,(0,0)) = \dot{w}$. By (12.78), letting $(r,z) = (0,0)$,

$$\begin{aligned} \dot{\chi}_{\mathrm{p}}(G,0) &= \Pi_X(\dot{\alpha}(0)) \\ \forall v' \in V \left[\dot{\chi}_{\mathrm{c}}(s_{v'},(0,0)) = 0\right] \\ \forall v' \in V \left[\dot{\chi}_{\mathrm{c}}(e_{v'},(0,0)) = 0\right]. \end{aligned} \tag{12.84}$$

Again, this is precisely the definition of $\dot{w}$ given in (12.76), and so $\dot{\chi}(G,(0,0)) = \dot{w}$.

To show (12.82), consider any arbitrary interval $J \in \mathscr{I}_{\top_{\mathrm{c}}}$; let $\psi \in \Psi(\top_{\mathrm{c}})$ be the mechanical controller state so that $\mathtt{state}(J) = \psi$. By (12.77) and (12.78), for any instant $(r,z) \in J$,

$$\chi_{\mathrm{c}}(e_v,(r,z)) = 1 \neq 0 = \dot{\chi}_{\mathrm{c}}(e_v,(r,z)) \tag{12.85}$$

and, for any $v' \neq v \in V$,

$$\chi_{\mathrm{c}}(e_{v'},(r,z)) = 0 = \dot{\chi}_{\mathrm{c}}(e_{v'},(r,z)). \tag{12.86}$$

Then by (8.127), ψ must be a subset of ψ_v. Let (r',z') be an arbitrary instant in J, and let

$$\begin{aligned} \mathbf{x}' &= \Pi_X(\alpha(r')) = \Pi_X(\tau(r')) \\ \dot{\mathbf{x}}' &= \Pi_X(\dot{\alpha}(r')) = \Pi_X(\dot{\tau}(r')) \\ \ddot{\mathbf{x}}' &= \Pi_X(\ddot{\alpha}(r')) = \Pi_X(\ddot{\tau}(r')). \end{aligned} \tag{12.87}$$

Then since by (12.77), (12.78), and (12.79),

$$\begin{aligned} \chi_{\mathrm{p}}(G,r') &= \Pi_X(\alpha(r')) = \mathbf{x}' \\ \dot{\chi}_{\mathrm{p}}(G,r') &= \Pi_X(\dot{\alpha}(r')) = \dot{\mathbf{x}}' \\ \ddot{\chi}_{\mathrm{p}}(G,r') &= \Pi_X(\ddot{\alpha}(r')) = \ddot{\mathbf{x}}', \end{aligned} \tag{12.88}$$

again by (8.127) it must be the case that ψ is a subset of $\psi_{\mathbf{x}',\dot{\mathbf{x}}'}$. Then

$$\begin{aligned} \psi &\subseteq \psi_v \cap \psi_{\mathbf{x}',\dot{\mathbf{x}}'} \\ &= \psi_{v,\mathbf{x}',\dot{\mathbf{x}}'}. \end{aligned} \tag{12.89}$$

By (8.128) $\psi_{v,\mathbf{x}',\dot{\mathbf{x}}'}$ is a mechanical controller state in $\Psi(\top_{\mathrm{c}})$; since universal states are disjoint, it follows that

$$\psi = \psi_{v,\mathbf{x}',\dot{\mathbf{x}}'}. \tag{12.90}$$

By (8.135),

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{\hat{f}^{\tau} + \hat{f}_{v \multimap v', r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right]\right\} \cup \left\{\hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}\right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V. \end{cases} \tag{12.91}$$

Let τ' be the suffix of τ starting in automaton instant r'; it follows from (12.77) that

$$\begin{aligned} \tau'.\mathit{fstate} = \tau(r') &= (v, \mathbf{x}') \\ \Pi_X(\dot{\tau}'.\mathit{fstate}) = \Pi_X(\dot{\tau}(r')) &= \dot{\mathbf{x}}' \\ \Pi_X(\ddot{\tau}'.\mathit{fstate}) = \Pi_X(\ddot{\tau}(r')) &= \ddot{\mathbf{x}}'. \end{aligned} \tag{12.92}$$

Then τ' is in $\mathcal{T}_{v,\mathbf{x}',\dot{\mathbf{x}}'}$. Let $\hat{f} = \hat{f}^{\tau'}$; by (12.87), it follows that $\hat{f}$ is in $\hat{\mathfrak{f}}[\psi]$. By the definition of $\hat{f}^{\tau'}$ in 8.1.1, $\hat{f}$ is the force variation such that:

- The domain of $\hat{f}$ is all instants in hybrid mechanical time; that is,

$$\mathit{dom}(\hat{f}) = \Gamma_M, \tag{12.93}$$

- For all instants $t \in \mathit{dom}(\hat{f})$, the force of N on G is isomorphic to the second derivative of the environment state component of τ' at 0 times $\mathbf{0}_{\mathrm{c}}$; that is,

$$\forall t \in \mathbb{R}^{\geq 0} \left[f_t(G, N) = \left(\Pi_X(\ddot{\tau}'.\mathit{fstate}), \mathbf{0}_{\mathrm{c}}\right)\right], \tag{12.94}$$

  and

- Except as required by the above via pairwise equilibriation and additivity, $\hat{f} = \hat{f}_{\mathbf{0}}$.

By Lemma 23, the acceleration of G is constant for all instants in J; that is, for any instant $(r, z) \in J$,

$$\ddot{\chi}_{\mathrm{p}}(G, r) = \ddot{\chi}_{\mathrm{p}}(G, r'). \tag{12.95}$$

Substituting via (12.88),

$$\ddot{\chi}_{\mathrm{p}}(G, r) = \ddot{\mathbf{x}}'. \tag{12.96}$$

By (12.88), $\ddot{\mathbf{x}}' = \Pi_X(\ddot{\tau}'.\mathit{fstate})$, and so

$$\ddot{\chi}_{\mathrm{p}}(G, r) = \Pi_X(\ddot{\tau}'.\mathit{fstate}). \tag{12.97}$$

But by (12.94),

$$\hat{f}_{\mathrm{p}}(t)(G, N) = \Pi_X(\ddot{\tau}'.\mathit{fstate}) \tag{12.98}$$

and so

$$\hat{f}_{\mathrm{p}}(r)(G, N) = \ddot{\chi}(G, (r, z)). \tag{12.99}$$

By (8.118), $m_{\mathrm{p}}(G) = 1_{\mathrm{p}}$, and so, equivalently,

$$\hat{f}_{\mathrm{p}}(r)(G, N) = m_{\mathrm{p}}(G)\ddot{\chi}(G, (r, z)). \tag{12.100}$$

By the definition of $\hat{f}$, there are no nonzero forces of any other body on G, and so

$$\begin{aligned} \hat{f}_{\mathrm{p}}(r)(G, \overline{G}) &= \hat{f}_{\mathrm{p}}(r)(G, N) \\ &= m_{\mathrm{p}}(G)\ddot{\chi}(G, (r, z)). \end{aligned} \tag{12.101}$$

Likewise, any nonphysical body has an acceleration of $\mathbf{0}_{\mathrm{c}}$ during all instants of J. By the definition of $\hat{f}$, the force on any such body is also $\mathbf{0}_{\mathrm{c}}$ during all instants of J; that is, for any $C \in \Omega_{\mathrm{c}}$,

$$\begin{aligned} \hat{f}_{\mathrm{c}}(r, z)(C, \overline{C}) &= \mathbf{0} \\ &= m_{\mathrm{c}}(C)\ddot{\chi}_{\mathrm{c}}(C, (r, z)). \end{aligned} \tag{12.102}$$

Then for every instant (r, z) in J, where $\mathtt{state}(J) = \psi$, there exists some force variation in $\hat{\mathfrak{f}}(\psi)$, namely $\hat{f} = \hat{f}^{\tau'}$, such that for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.103}$$

In other words, χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J. Since the choice of interval J was arbitrary, the above holds for all intervals in $\mathscr{I}_{\top_{\mathrm{c}}}$. In other words, χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$.

Prove equal traces: The proof of equal traces follows directly from the construction of χ given in (12.77). In particular, for any instant $t \in \Gamma_A$,

$$\chi_{\mathrm{p}}(G, t) = \chi_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)) = \Pi_X(\alpha(t)), \tag{12.104}$$

and so, by definition,

$$\mathit{trace}(\chi) = \mathit{trace}(\alpha). \tag{12.105}$$

Prove endpoints in relation: Finally, I show that $(\alpha.\mathit{lstate}, (\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime}))))$ is in R. For ease of notation, let

$$\begin{aligned} q' &= (v, \mathbf{x}') = \alpha.\mathit{lstate} \\ w' &= \chi(\top, \hat{t}_M(\alpha.\mathit{ltime})) \end{aligned} \tag{12.106}$$

$$\dot{w}' = \dot{\chi}(\top, \hat{t}_M(\alpha.ltime)).$$

Clearly there exists some trajectory and instant for which the automaton system state of the trajectory is q', since in particular $\tau.lstate = q'$. By the definition of χ given in (12.77),

$$\begin{aligned} w'_{\mathrm{p}}(G) &= \Pi_X(\alpha.lstate) = \mathbf{x}' \\ \forall v' \in V\ \big[w'_{\mathrm{c}}(s_{v'}) &= 0\big] \\ w'_{\mathrm{c}}(e_v) &= 1 \\ \forall v' \neq v \in V\ \big[w'_{\mathrm{c}}(e_{v'}) &= 0\big], \end{aligned} \tag{12.107}$$

and, by (12.78),

$$\begin{aligned} \dot{w}'_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}.lstate) \\ \forall v' \in V\ [\dot{w}'_{\mathrm{c}}(s_{v'}) &= 0] \\ \forall v' \in V\ [\dot{w}'_{\mathrm{c}}(e_{v'}) &= 0]. \end{aligned} \tag{12.108}$$

But these are precisely the requirements in the definition of R given in (12.2). Thus, $(q', (w', \dot{w}'))$ is in R.

Conclusion: Then χ is a related motion of M for α. Since the original choice of $(q, (w, \dot{w})) \in R$ and closed trajectory τ such that $\tau.fstate = q$ were otherwise unconstrained, there exists such a related motion for every such pair in R and corresponding trajectory. Thus, M has related motions for the closed trajectories of A. □

12.2.4 Relation of discrete transitions to mechanical controller transitions

I next prove that M has related motions for the discrete transitions of A.

Theorem 73. *Let A be a hybrid automaton, with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then M has related motions for the discrete transitions of A.*

Proof. Consider any pair $(q, (w, \dot{w})) \in R$, with $q = (v, \mathbf{x}) \in Q$ and $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, and any trajectory $\tau \in \mathcal{T}$ such that $\tau.lstate = q$. If no such pair and trajectory exist, then the proof is trivially satisfied.

Otherwise, let α be any closed point-initiated extension of τ, that is, an execution fragment such that $\alpha = \tau_1 a \tau_2$, where

- $\tau_1 \in \mathcal{T}$ is the point trajectory formed by restricting τ to its final instant, that is, $\tau_1 = \tau \lceil \tau.ltime$;
- $\tau_2 \in \mathcal{T}$ is a trajectory whose initial X-value and derivative agree with the final X-value and

derivative of τ, that is,

$$\Pi_X(\tau.\mathit{lstate}) = \Pi_X(\tau_2.\mathit{fstate}) \qquad (12.109)$$
$$\Pi_X(\dot{\tau}.\mathit{lstate}) = \Pi_X(\dot{\tau}_2.\mathit{fstate});$$

- and $a \in D$ is a transition such that

$$a = (\tau.\mathit{lstate}, \tau_2.\mathit{fstate}). \qquad (12.110)$$

Again, if no such execution fragment α exists, the proof is trivially satisfied.

Otherwise, M has related motions for the discrete transitions of A iff, for any such execution fragment, there exists some motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- $(\alpha.\mathit{lstate}, (\chi_{\hat{t}_M(\alpha.\mathit{ltime})}, \dot{\chi}_{\hat{t}_M(\alpha.\mathit{ltime})}))$ is in R.

Let $q_1 = \tau.\mathit{lstate} = \tau_1.\mathit{lstate}$, and let $q_2 = \tau_2.\mathit{fstate}$. By (12.109), q_1 and q_2 have matching environment state components, and so there exist automaton controller and environment states $v_1, v_2 \in V$ and $\mathbf{x}_1 \in X$ such that $q_1 = (v_1, \mathbf{x}_1)$ and $q_2 = (v_2, \mathbf{x}_1)$. Let $\dot{\mathbf{x}}_1$ be the value such that

$$\dot{\mathbf{x}}_1 = \Pi_X(\dot{\tau}.\mathit{lstate}) = \Pi_X(\dot{\tau}_2.\mathit{fstate}) = \Pi_X(\dot{\alpha}.\mathit{fstate}). \qquad (12.111)$$

Let $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ be the configuration such that $\texttt{Mech}(\tau, \tau.\mathit{ltime}) = (w, \dot{w})$; that is, by the definition of `Mech` in Section 7.3.7,

$$\begin{aligned} w_{\mathrm{p}}(G) &= \Pi_X(\tau.\mathit{lstate}) = \mathbf{x}_1 \qquad (12.112)\\ \dot{w}_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}.\mathit{lstate}) = \dot{\mathbf{x}}_1 \\ w_{\mathrm{c}}(e_{v_1}) &= 1 \\ \forall v' \neq v \in V\big[w_{\mathrm{c}}(e_{v_2}) &= 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathrm{c}}(e_{v_2}) &= 0\big] \\ \forall v' \in V\big[w_{\mathrm{c}}(s_{v_2}) &= 0\big] \\ \forall v' \in V\big[\dot{w}_{\mathrm{c}}(s_{v_2}) &= 0\big]. \end{aligned}$$

By the definition of R in (12.1), $(q, (w, \dot{w}))$ is in R.

By definition, M has related motions for the discrete transitions of A iff, for any such construction, there exists some motion $\chi : \top \times \Gamma_M \to \mathcal{C}(\top, \mathcal{S})$ such that

- χ is an induced motion for $\mathfrak{f}[\cdot]$ beginning in $(w, \dot{w})$,
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$, and
- $(\alpha.\mathit{lstate}, (\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime}))))$ is in R.

Construction of χ: Let χ be any motion such that the following conditions hold: at physical instant 0,

$$\begin{aligned}
\chi_{\mathrm{p}}(G, 0) = \Pi_X(\alpha(0)) = \mathbf{x}_1 \\
\chi_{\mathrm{c}}(e_{v_1}, (0,0)) = 1 \\
\chi_{\mathrm{c}}(e_{v_2}, (0,0)) = 0 \\
\forall v \neq v_1, v_2 \in V, \chi_{\mathrm{c}}(e_v, (0,0)) = 0 \\
\forall v \in V, \chi_{\mathrm{c}}(s_v, (0,0)) = 0.
\end{aligned} \tag{12.113}$$

At instant $(0,1)$,

$$\begin{aligned}
\chi_{\mathrm{c}}(e_{v_1}, (0,1)) = 1 \\
\chi_{\mathrm{c}}(e_{v_2}, (0,1)) = 0 \\
\forall v \neq v_1, v_2 \in V, \chi_{\mathrm{c}}(e_v, (0,1)) = 0 \\
\forall v \in V, \chi_{\mathrm{c}}(s_v, (0,1)) = 0.
\end{aligned} \tag{12.114}$$

At instant $(0,2)$,

$$\begin{aligned}
\chi_{\mathrm{c}}(e_{v_1}, (0,2)) = 0 \\
\chi_{\mathrm{c}}(e_{v_2}, (0,2)) = 1 \\
\forall v \neq v_1, v_2 \in V, \chi_{\mathrm{c}}(e_v, (0,2)) = 0 \\
\forall v \in V, \chi_{\mathrm{c}}(s_v, (0,2)) = 0.
\end{aligned} \tag{12.115}$$

At instant $(0,3)$,

$$\begin{aligned}
\chi_{\mathrm{c}}(e_{v_1}, (0,3)) = 0 \\
\chi_{\mathrm{c}}(e_{v_2}, (0,3)) = 1 \\
\forall v \neq v_1, v_2 \in V, \chi_{\mathrm{c}}(e_v, (0,3)) = 0 \\
\forall v \in V, \chi_{\mathrm{c}}(s_v, (0,3)) = 0.
\end{aligned} \tag{12.116}$$

At every subsequent instant (r, z) in J,

$$\begin{aligned} \chi_{\mathrm{p}}(G, r) &= \Pi_X(\alpha(r)) \\ \chi_{\mathrm{c}}(e_{v_1}, (r, z)) &= 0 \\ \chi_{\mathrm{c}}(e_{v_2}, (r, z)) &= 1 \\ \forall v \neq v_1, v_2 \in V, \chi_{\mathrm{c}}(e_v, (r, z)) &= 0 \\ \forall v \in V, \chi_{\mathrm{c}}(s_v, (r, z)) &= 0. \end{aligned} \tag{12.117}$$

By Assumption 2, the components of α are differentiably legal, and so they are second-degree doubly semidifferentiable. It follows that $\dot{\alpha}$ and $\ddot{\alpha}$ are defined, and so that $\dot{\chi}$ and $\ddot{\chi}$ are defined. Given the preceding positions, it is straightforward to derive the corresponding velocities: at instant $(0, 0)$,

$$\begin{aligned} \dot{\chi}_{\mathrm{p}}(G, 0) &= \Pi_X(\dot{\alpha}(0)) = \dot{\mathbf{x}}_1 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(e_v, (0, 0)) &= 0 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(s_v, (0, 0)) &= 0. \end{aligned} \tag{12.118}$$

At instant $(0, 1)$,

$$\begin{aligned} \forall v \in V, \dot{\chi}_{\mathrm{c}}(e_v, (0, 1)) &= 0 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(s_v, (0, 1)) &= 0. \end{aligned} \tag{12.119}$$

At instant $(0, 2)$,

$$\begin{aligned} \dot{\chi}_{\mathrm{c}}(e_{v_1}, (0, 2)) &= 1 \\ \dot{\chi}_{\mathrm{c}}(e_{v_2}, (0, 2)) &= 1 \\ \forall v \neq v_1, v_2 \in V, \dot{\chi}_{\mathrm{c}}(e_v, (0, 2)) &= 0 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(s_v, (0, 2)) &= 0. \end{aligned} \tag{12.120}$$

At instant $(0, 3)$,

$$\begin{aligned} \dot{\chi}_{\mathrm{c}}(e_{v_1}, (0, 3)) &= 0 \\ \dot{\chi}_{\mathrm{c}}(e_{v_2}, (0, 3)) &= 0 \\ \forall v \neq v_1, v_2 \in V, \dot{\chi}_{\mathrm{c}}(e_v, (0, 3)) &= 0 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(s_v, (0, 3)) &= 0. \end{aligned} \tag{12.121}$$

At every subsequent instant (r, z) in J,

$$\begin{aligned} \dot{\chi}_{\mathrm{p}}(G, r) &= \Pi_X(\dot{\alpha}(r)) \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(e_v, (r, z)) &= 0 \\ \forall v \in V, \dot{\chi}_{\mathrm{c}}(s_v, (r, z)) &= 0. \end{aligned} \tag{12.122}$$

Likewise, given the preceding velocities, it is straightforward to derive the corresponding accelerations: at instant $(0, 0)$,

$$\begin{aligned} \ddot{\chi}_{\mathrm{p}}(G, 0) &= \Pi_X(\ddot{\alpha}(0)) \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(e_v, (0, 0)) &= 0 \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(s_v, (0, 0)) &= 0. \end{aligned} \tag{12.123}$$

At instant $(0, 1)$,

$$\begin{aligned} \ddot{\chi}_{\mathrm{c}}(e_{v_1}, (0, 1)) &= 1 \\ \ddot{\chi}_{\mathrm{c}}(e_{v_2}, (0, 1)) &= 1 \\ \forall v \neq v_1, v_2 \in V, \ddot{\chi}_{\mathrm{c}}(e_v, (0, 1)) &= 0 \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(s_v, (0, 1)) &= 0. \end{aligned}$$

At instant $(0, 2)$,

$$\begin{aligned} \ddot{\chi}_{\mathrm{c}}(e_{v_1}, (0, 2)) &= 1 \\ \ddot{\chi}_{\mathrm{c}}(e_{v_2}, (0, 2)) &= 1 \\ \forall v \neq v_1, v_2 \in V, \ddot{\chi}_{\mathrm{c}}(e_v, (0, 2)) &= 0 \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(s_v, (0, 2)) &= 0. \end{aligned}$$

At instant $(0, 3)$,

$$\begin{aligned} \forall v \in V, \ddot{\chi}_{\mathrm{c}}(e_v, (0, 3)) &= 0 \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(s_v, (0, 3)) &= 0. \end{aligned}$$

At every subsequent instant (r, z) in J,

$$\begin{aligned} \ddot{\chi}_{\mathrm{p}}(G, r) &= \Pi_X(\ddot{\alpha}(r)) \\ \forall v \in V, \ddot{\chi}_{\mathrm{c}}(e_v, (r, z)) &= 0 \end{aligned} \tag{12.124}$$

$$\forall v \in V, \ddot{\chi}_{\mathrm{c}}(s_v, (r, z)) = 0.$$

Let `state` be the state-time function for χ, and let $\mathscr{I}_{\top_{\mathrm{c}}}$ be the set of maximal state intervals given `state`.

Prove induced motion: Recall from the definition of induced motion in Section 6.3.3 and the discussion of initial configurations in Section 5.3.4 that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ iff

$$\chi(\top, (0,0)) = w, \tag{12.125}$$

$$\dot{\chi}(\top, (0,0)) = \dot{w}, \tag{12.126}$$

and, for any interval $J \in \mathscr{I}_{\top_{\mathrm{c}}}$, there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\mathtt{state}(J)]$ over J such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.127}$$

Consider first the requirement in (12.125) that $\chi(G, (0,0)) = w$. The statement of $\chi_{(0,0)}$ in (12.113) is is precisely the definition of w given in (12.76), and so $\chi(G, (0,0)) = w$.

Similarly, consider the requirement in (12.126) that $\dot{\chi}(\top, (0,0)) = \dot{w}$. The statement of $\dot{\chi}_{(0,0)}$ in (12.118) is is precisely the definition of $\dot{w}$ given in (12.76), and so $\dot{\chi}(G, (0,0)) = \dot{w}$.

To show (12.127), consider the first interval $J_1 \in \mathscr{I}_{\top_{\mathrm{c}}}$, that is, the interval containing $(0,0)$. Let $\psi_1 \in \Psi(\top_{\mathrm{c}})$ be the mechanical controller state so that $\mathtt{state}(J_1) = \psi_1$. Note that, by (12.113) and (12.118),

$$\chi_{\mathrm{c}}(e_{v_1}, (0,0)) = 1 \neq 0 = \dot{\chi}_{\mathrm{c}}(e_{v_1}, (0,0)) \tag{12.128}$$

and, for any $v \neq v_1 \in V$,

$$\chi_{\mathrm{c}}(e_v, (0,0)) = 0 = \dot{\chi}_{\mathrm{c}}(e_v, (0,0)). \tag{12.129}$$

By (8.127), then, ψ_1 must be a subset of ψ_{v_1}. Again, by (12.113) and (12.118),

$$\begin{aligned} \chi_{\mathrm{p}}(G, 0) &= \mathbf{x}_1 \\ \dot{\chi}_{\mathrm{p}}(G, 0) &= \dot{\mathbf{x}}_1. \end{aligned} \tag{12.130}$$

Again by (8.127), then, ψ_1 is a subset of $\psi_{\mathbf{x}_1, \dot{\mathbf{x}}_1}$. Then

$$\begin{aligned} \psi_1 &\subseteq \psi_{v_1} \cap \psi_{\mathbf{x}_1, \dot{\mathbf{x}}_1} \\ &= \psi_{v_1, \mathbf{x}_1, \dot{\mathbf{x}}_1}. \end{aligned} \tag{12.131}$$

By (8.128) $\psi_{v_1, \mathbf{x}_1, \dot{\mathbf{x}}_1}$ is a mechanical controller state in $\Psi(\top_{\mathrm{c}})$; since universal states are disjoint, it follows that

$$\psi_1 = \psi_{v_1,\mathbf{x}_1,\dot{\mathbf{x}}_1}. \tag{12.132}$$

By similar arguments,

$$\mathtt{state}(0,1) = \mathtt{state}(0,2) = \psi_{v_1,\mathbf{x}_1,\dot{\mathbf{x}}_1}. \tag{12.133}$$

By another similar argument,

$$\mathtt{state}(0,3) = \psi_{v_2,\mathbf{x}_1,\dot{\mathbf{x}}_1}, \tag{12.134}$$

and so

$$J_1 = [(0,0),(0,2)]. \tag{12.135}$$

By (8.104),

$$\hat{\mathfrak{f}}[\psi_1] = \left\{ \hat{f}^{\tau} + \hat{f}_{v_1 \to v',r} \,\middle|\, [\tau \in \mathcal{T}_{v_1,\mathbf{x}_1,\dot{\mathbf{x}}_1}] \wedge \left[v' \in \mathcal{D}_{v_1}(\mathbf{x}_1)\right] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v_1,\mathbf{x}_1,\dot{\mathbf{x}}_1} \right\}. \tag{12.136}$$

Plainly τ_1 is in $\mathcal{T}_{v_1,\mathbf{x}_1,\dot{\mathbf{x}}_1}$. By the definition of $X/\equiv_{\mathcal{D},v}$, since there is a transition in D from $(v_1,\mathbf{x}_1)$ to $(v_2,\mathbf{x}_1)$, it must be the case that v_2 is in $\mathcal{D}_v(\mathbf{x}_1)$. Then let $\hat{f}_1 = \hat{f}^{\tau_1} + \hat{f}_{v_1 \to v_2,0}$; by (12.136), $\hat{f}_1$ is in $\hat{\mathfrak{f}}[\psi_1]$. The force variations $\hat{f}^{\tau_1}$ and $\hat{f}_{v_1 \to v_2,0}$ are defined in (8.129) and (8.133), respectively; summing those forces, $\hat{f}_1$ is the force variation such that

$$\begin{aligned}
&\forall t \in \Gamma_M \left[f_t(G,N) = (\Pi_X(\ddot{\tau}_1(0), \mathbf{0}_\mathrm{c}) = (\Pi_X(\ddot{\alpha}(0), \mathbf{0}_\mathrm{c}) \right] \\
&\quad \hat{f}_1(0,0)(e_{v_1}, e_{v_2}) = (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\
&\quad \hat{f}_1(0,0)(e_{v_2}, e_{v_1}) = (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\
&\quad \hat{f}_1(0,1)(e_{v_1}, e_{v2}) = (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\
&\quad \hat{f}_1(0,1)(e_{v_2}, e_{v_1}) = (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\
&\quad \hat{f}_1(0,2)(e_{v_1}, e_{v_2}) = (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\
&\quad \hat{f}_1(0,2)(e_{v_2}, e_{v_1}) = (\mathbf{0}_\mathrm{p}, \mathbf{1}_\mathrm{c}) \\
&\quad \hat{f}_1(0,3)(e_{v_1}, e_{v_2}) = (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}) \\
&\quad \hat{f}_1(0,3)(e_{v_2}, e_{v_1}) = (\mathbf{0}_\mathrm{p}, \mathbf{0}_\mathrm{c}),
\end{aligned} \tag{12.137}$$

and, except where implied by the above and the definition of a force system, for any instant $t \in \Gamma_M$ and bodies $A, B \in \Omega$,

$$f_t(A,B) = \mathbf{0}. \tag{12.138}$$

As usual, this definition ignores forces on subbodies of N. Note that the above forces are precisely the accelerations defined for $[(0,0),(0,2)]$ in (12.123) and the following lines, and that all bodies involved have unit physical or controller mass. In other words, for every instant $t \in J$, body $B \in \Omega$, and index

$i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.139}$$

That is, χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during J_1.

For any subsequent interval $J' \in \mathscr{I}_{\top_c}$, the proof that χ is an induced motion during J' is functionally identical to that presented in Theorem 72. Then χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$ during all intervals in $\mathscr{I}_{\top_c}$, and so χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$.

Prove equal traces: The proof of equal traces follows directly from the construction of χ given in (12.113), in particular from the fact that for any instant $t \in \Gamma_A$,

$$\chi_{\mathrm{p}}(G, t) = \chi_{\mathrm{p}}(G, \hat{t}_{\mathrm{p}M}(t)) = \Pi_X(\alpha(t)); \tag{12.140}$$

by definition,

$$\mathit{trace}(\chi) = \mathit{trace}(\alpha). \tag{12.141}$$

Prove endpoints in relation: Finally, show that $(\alpha.\mathit{lstate}, (\chi(\top, \hat{t}_M(\alpha.\mathit{ltime})), \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime}))))$ is in R. For ease of notation, let

$$\begin{aligned} q' &= (v_2, \mathbf{x}') = \alpha.\mathit{lstate} \\ w' &= \chi(\top, \hat{t}_M(\alpha.\mathit{ltime})) \\ \dot{w}' &= \dot{\chi}(\top, \hat{t}_M(\alpha.\mathit{ltime})). \end{aligned} \tag{12.142}$$

Clearly there exists some trajectory and instant for which the automaton system state of the trajectory is q', since in particular $\tau_2.\mathit{lstate} = q'$. By the definition of χ given in (12.117),

$$\begin{aligned} {w_{\mathrm{p}}}'(G) &= \Pi_X(\tau_2.\mathit{lstate}) \\ {w_{\mathrm{c}}}'(e_{v_2}) &= 1 \\ \forall v \neq v_2 \in V, {w_{\mathrm{c}}}'(e_v) &= 0 \\ \forall v \in V, {w_{\mathrm{c}}}'(s_v) &= 0. \end{aligned} \tag{12.143}$$

and, by (12.122),

$$\begin{aligned} \dot{w}'_{\mathrm{p}}(G) &= \Pi_X(\dot{\tau}_2.\mathit{lstate}) \\ \forall v \in V, \dot{w}'_{\mathrm{c}}(e_v) &= 0 \\ \forall v \in V, \dot{w}'_{\mathrm{c}}(s_v) &= 0. \end{aligned} \tag{12.144}$$

But these are precisely the requirements in the construction of R given in (12.2). Thus, $(q', (w', \dot{w}'))$ is in R.

Conclusion: Then χ is a related motion of M for α. Since the original choice of $(q_1, (w_1, \dot{w}_1)) \in R$, trajectory $\tau \in \mathcal{T}$, and point-initiated extension of τ are unconstrained, there exists such a related motion for every such execution fragment. Thus, M has related motions for the discrete transitions of A. $\square$

12.2.5 Relation of execution fragments

I now prove that A has related execution fragments for the motions of M.

12.2.5.1 Mechanical Zeno behavior

First, recall that Chapter 3 described the concept of *Zeno behavior*, that is, of automata that make an infinite number of automaton controller state changes in some finite interval of time. Given a hybrid system with hybrid mechanical counterpart A, automaton controller states correspond naturally to configurations of the effector; recall that a mechanical controller state is a substate of ψ_v iff e_v is the sole effector body point whose position and velocity disagree.

Formally, for any induced motion χ, let $\equiv_\chi$ be an equivalence relation defined over Γ_M such that two instants $t_1, t_2 \in \Gamma_M$ are related iff the universal mechanical controller states in both instants, and in all instants between the two, are subsets of the same set ψ_v. In other words,

$$t \equiv_\chi t' \text{ iff } \exists v \in V \left(\forall t \in [min\{t_1, t_2\}, max\{t_1, t_2\}] \left(\mathtt{state}(t) \subseteq \psi_v \right) \right). \tag{12.145}$$

I call $\equiv_\chi$ the *automaton controller-state relation* on Γ_M (for χ). Let $\Gamma_M / \equiv_\chi$ be the partition of Γ_M imposed by $\equiv_\chi$. For any interval $J \in \Gamma$, let $J/\equiv_\chi$ be the set of all elements of $\Gamma_M/\equiv_\chi$ contained in J. Then the system exhibits Zeno behavior iff, for any such induced motion χ, there is any finite interval $J \in \Gamma$ such that $|J/\equiv_\chi|$ is infinite.

It would be ideal to show formally that, if A has no Zeno behaviors, it necessarily follows that M likewise has no Zeno behaviors. Intuitively, it seems reasonable that, if A and M truly do bisimulate each other, this condition should hold, since a Zeno behavior of one system would necessarily imply a Zeno behavior of the other. Such a proof is more onerous than it might initially appear, however, as one cannot appeal to the very bisimulation which this chapter attempts to prove. The only clear route would seem to be to identify those properties of a hybrid automaton which are necessary and sufficient for non-Zeno behavior, and to show that those properties, when translated into a mechanical system, necessarily prevent mechanical Zeno behavior.

Sufficient conditions for automaton Zeno behavior can be expressed relatively straightforwardly [16]; so, for instance, an automaton exhibits Zeno behavior if there is any cycle of discrete transitions. A full guarantee of no Zeno behavior, however, poses a substantially greater problem; thus, in Section 3.2.3, I restricted attention to non-Zeno automata without specifying a set of conditions to provide such a guarantee. It is therefore difficult to formally demonstrate that M must likewise lack Zeno behaviors.

Thus, following my earlier pattern, I restrict attention to hybrid mechanical counterparts that are non-Zeno, saving any proof from existing properties for future work.

Assumption 11. *Any hybrid mechanical system M (counterpart to a hybrid automaton A) considered in this paper is non-Zeno. Formally, for any induced motion χ and finite interval $J \in \Gamma$, the cardinality of $|J/\equiv_\chi|$ is finite.*

It follows immediately from this restriction that, for any induced motion χ, it must be the case that $|\Gamma_M/\equiv_\chi|$ is countable, and so the elements of $\Gamma_M/\equiv_\chi$ can be labeled in increasing temporal order $J_0, J_1, \ldots$ For any such equivalence class, the infimum of J_0 is the first instant in hybrid time, that is, $(0,0)$. As I show in the following theorem, every other interval in $\Gamma_M/\equiv_\chi$ begins at some $(r,3)$, and every nonfinal interval in $\Gamma_M/\equiv_\chi$ ends at some $(r,2)$.

Theorem 74. *Let A be a hybrid automaton, with hybrid mechanical counterpart M. Let χ be any induced motion for $\hat{\mathfrak{f}}[\cdot]$ such that all controller bodies are initially at rest, with all controller body points in position $\mathbf{0}_\mathrm{c}$ except for some $e \in E$ with position $\mathbf{1}_\mathrm{c}$. Let* `state` *be the state-time function for χ, and let $\Gamma_M/\equiv_\chi$ be the partition of Γ_M defined by the automaton controller-state relation $\equiv_\chi$ for χ. For any interval $J \in \Gamma_M/\equiv_\chi$, the following properties hold:*

- *If J is nonfinal, then it is right-closed, with final instant $(r,2)$ for some $r \in \mathbb{R}^{\geq 0}$.*
- *If J is noninitial, then it is left-closed, with first instant $(r,3)$ for some $r \in \mathbb{R}^{\geq 0}$.*

Proof. By the definition of mechanical controller states, a change from some substate of some ψ_v to some $\psi_{v'}$ is only possible via a change in the position of the effector body. If all controller bodies are at rest at some point, any subsequent such change is only possible via some nonzero acceleration of effector body points. Since χ is an induced motion, such nonzero acceleration requires a simultaneous nonzero force on effector body points. By (8.135), for any $\psi \in \Psi(\top_\mathrm{c})$, the only force variations in $\hat{\mathfrak{f}}[\psi]$ with any nonzero force on effector point bodies are those of the form $\hat{f}_{v\text{–}v',r}$, or the sum of that force variation with some other force variation. By the definition of $\hat{f}_{v\text{–}v',r}$, the only nonzero effector forces for any such system are those at $(r,1)$ and $(r,2)$; by an argument identical to that in Theorem 24, if all controller bodies are at rest at any such instant $(r,0)$, with a single effector body point in position $\mathbf{1}_\mathrm{c}$ and all other controller body points in position $\mathbf{0}_\mathrm{c}$, then all such bodies are again at rest in any such instant $(r,3)$, with a single (different) effector body point in position $\mathbf{1}_\mathrm{c}$ and all other controller body points in position $\mathbf{0}_\mathrm{c}$. By the same arguments,

$$\mathtt{state}(r,0) = \mathtt{state}(r,1) = \mathtt{state}(r,2) \subseteq \psi_v, \tag{12.146}$$

and

$$\mathtt{state}(r,3) \subseteq \psi_{v'}. \tag{12.147}$$

Then the the only possible changes from some substate of ψ_v to a substate of any $\psi_{v'}$, with $v' \neq v$, are at the instant of change from some such $(r, 2)$ to some such $(r, 3)$. Then if any interval $J \in \Gamma_M/\equiv_\chi$ is nonfinal, it is right-closed, and its last instant must be some $(r, 2)$; if it is noninitial, it is left-closed, and its first instant must be some $(r, 3)$. □

Theorem 75. *Let A be a hybrid automaton, with hybrid mechanical counterpart M. Let χ be any induced motion for $\hat{\mathfrak{f}}[\cdot]$. If, at any instant $(r, 0) \in \Gamma_M$ all controller bodies are at rest, with all controller body points in position $\mathbf{0}_\mathrm{c}$ except for some $e \in E$ with position $\mathbf{1}_\mathrm{c}$, then in the instant $(r, 3)$ all controller bodies are at rest, with all controller body points in position $\mathbf{0}_\mathrm{c}$ except for some $\mathbf{x}'$ with position $\mathbf{1}_\mathrm{c}$.*

Proof. Consider any such instant $(r, 0)$. Let $\mathbf{x}, \dot{\mathbf{x}} \in X$ be the values such that

$$\begin{aligned} \chi_\mathrm{p}(G, r) &= \mathbf{x} \\ \dot{\chi}_\mathrm{p}(G, r) &= \dot{\mathbf{x}}, \end{aligned} \tag{12.148}$$

and let $v \in V$ be the value such that

$$\begin{aligned} \chi_\mathrm{c}(e_v, (r, 0)) &= \mathbf{1}_\mathrm{c} \\ \forall c \neq e_v \in \Omega_\mathrm{c}\big[\chi_\mathrm{c}(c, (r, 0)) &= \mathbf{0}_\mathrm{c}\big] \\ \forall c \in \Omega_\mathrm{c}\big[\dot{\chi}_\mathrm{c}(c, (r, 0)) &= \mathbf{0}_\mathrm{c}\big]. \end{aligned} \tag{12.149}$$

Let `state` be the state-time function for χ, and let $\mathscr{I}_{\top_\mathrm{c}}$ be the maximal interval set for `state`. Let $J \in \mathscr{I}_{\top_\mathrm{c}}$ be the maximal interval containing $(r, 0)$, and let $\psi \in \Psi(\top_\mathrm{c})$ be the mechanical controller state such that $\texttt{state}(J) = \psi$. Since χ is an induced motion, there exists some $\hat{f}_J \in \hat{\mathfrak{f}}[\psi]$ such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.150}$$

In particular, for any controller point body $c \in \Omega_\mathrm{c}$,

$$\hat{f}_\mathrm{c}(t)(c, \overline{c}) = m_\mathrm{c}(c)\ddot{\chi}_\mathrm{c}(c, t). \tag{12.151}$$

Either there exist $\tau \in \mathfrak{T}$ and $v' \neq v \in V$ such that

$$\hat{f}_J = \hat{f}^\tau + \hat{f}_{v \multimap v', r} \tag{12.152}$$

or there do not. In the latter case, by the definition of $\hat{\mathfrak{f}}[\cdot]$ in (8.135), for any (r, z) with $z \in \mathbb{Z}_4$, for all $c \in \Omega_\mathrm{c}$,

$$\hat{f}_\mathrm{c}(r, z)(c, \overline{c}) = \mathbf{0} \tag{12.153}$$

and so $\ddot{\chi}_{\rm c}(c,(r,z)) = \mathbf{0}_{\rm c}$. In other words, all controller bodies remain at rest. Clearly in this case, $\chi_{\rm c}(\top_{\rm c},(r,0)) = \chi_{\rm c}(\top_{\rm c},(r,3))$ and $\dot{\chi}_{\rm c}(\top_{\rm c},(r,0)) = \dot{\chi}_{\rm c}(\top_{\rm c},(r,3))$, and so the proof is satisfied.

Otherwise, if $\hat{f}_J = \hat{f}^\tau + \hat{f}_{v\text{-}v',r}$, then by definition χ is a $(v,\mathbf{x},\dot{\mathbf{x}})$-motion active at r, and by arguments identical to those in Theorem 24, at instant $(r,3)$ all controller bodies are at rest, with all controller body points in position $\mathbf{0}_{\rm c}$ except for $\mathbf{x}_{v'}$, which has position $\mathbf{1}_{\rm c}$.

Then in any event, the theorem holds. □

Corollary 76. *Let A be a hybrid automaton, with hybrid mechanical counterpart M. Let χ be any induced motion for $\hat{\mathfrak{f}}[\cdot]$ such that all controller bodies are initially at rest, with all controller body points in position $\mathbf{0}_{\rm c}$ except for some e with position $\mathbf{1}_{\rm c}$. Then at any instant $(r,0)$, for any $r \in \mathbb{R}^{\geq 0}$, all controller bodies are at rest, with all controller body points in position $\mathbf{0}_{\rm c}$ except for some $e' \in E$ with position $\mathbf{1}_{\rm c}$. In other words, for any $(r,0) \in \Gamma_M$ and controller body points $c \in \top_{\rm c}$,*

$$\dot{\chi}_{\rm c}(c,(r,0)) = \mathbf{0}_{\rm c}, \tag{12.154}$$

and, for some effector point $e \in E$,

$$\chi_{\rm c}(e,(r,0)) = \mathbf{1}_{\rm c}, \tag{12.155}$$

while for any other controller point body $c' \neq e \in \top_{\rm c}$,

$$\chi_{\rm c}(c',(r,0)) = \mathbf{0}_{\rm c}. \tag{12.156}$$

Proof. This theorem follows immediately from the proof of Theorem 75. □

As a useful result for later theorems, I show that, for any induced motion of M and automaton controller states $v, v' \in V$, the mechanical universal controller state changes from a subset of ψ_v to a subset of $\psi_{v'}$ in some instant $(r,2)$ only if some force variation $f_{v\text{-}v',r}$ determines the controller acceleration from $(r,0)$ to $(r,2)$.

Theorem 77. *Let A be a hybrid automaton, with hybrid mechanical counterpart M. Let χ be any induced motion of $\top$ beginning in some configuration $(w,\dot{w}) \in \mathfrak{X}_0(\top,\mathcal{S})$, let* `state` *be the state-time function for χ, and let $\Gamma_M/\equiv_\chi$ be the partition of Γ_M imposed by $\equiv_\chi$, as defined in (12.145).*

For any real value $r \in \mathbb{R}^{\geq 0}$ and states $v, v' \in V$, if there exist two sequential intervals $J, J' \in \Gamma_M/\equiv_\chi$ such that

$$\begin{aligned} \sup(J) &= (r,2) \\ \inf(J') &= (r,3), \end{aligned} \tag{12.157}$$

and

$$\begin{aligned}\texttt{state}(r,2) &\subseteq \psi_v \\ \texttt{state}(r,3) &\subseteq \psi_{v'},\end{aligned} \tag{12.158}$$

then v' is in $\mathcal{D}_v(\mathbf{x})$ and, for any instant $t \in [(r,0),(r,2)]$ and controller $C \in \Omega_\mathrm{c}$,

$$\left(\hat{f}_{v\text{–}v',r}\right)_\mathrm{c}(t)(C,\overline{C}) = m_\mathrm{c}(C)\ddot{\chi}_\mathrm{c}(C,t). \tag{12.159}$$

Proof. Suppose there exist two sequential intervals $J, J' \in \Gamma_M/\equiv_\chi$, with J the earlier of the two; by Theorem 74, there exists some $r \in \mathbb{R}^{\geq 0}$ such that

$$\begin{aligned}\sup(J) &= (r,2) \\ \inf(J') &= (r,3),\end{aligned} \tag{12.160}$$

and by the definition of $\Gamma_M/\equiv_\chi$, there exist states $v, v' \in V$ such that

$$\begin{aligned}\texttt{state}(r,2) &\subseteq \psi_v \\ \texttt{state}(r,3) &\subseteq \psi_{v'}.\end{aligned} \tag{12.161}$$

Let $\mathbf{x}, \dot{\mathbf{x}} \in X$ be the values such that

$$\begin{aligned}\chi_\mathrm{p}(G,r) &= \mathbf{x} \\ \dot{\chi}_\mathrm{p}(G,r) &= \dot{\mathbf{x}}.\end{aligned} \tag{12.162}$$

It follows by the definition of states in (8.127) that $\texttt{state}(r,2) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$. By Corollary 76, at $(r,0)$ all controller bodies are at rest, with all controller body points in position $\mathbf{0}_\mathrm{c}$ except for some $e \in E$ with position $\mathbf{1}_\mathrm{c}$; by (8.127) again, it must be the case that $e = e_v$. By the same reference, since $\texttt{state}(r,3) \subseteq \psi_{v'}$, there must be some change in the positions or velocities of e_v and $e_{v'}$ during $[(r,0),(r,2)]$. Since all effector point bodies are at rest in $(r,0)$, there must also be some acceleration of those effector point bodies during that time. Since χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$, there must be some force variation in $\hat{\mathfrak{f}}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]$ with nonzero forces on those effector point bodies during that period. But by (8.135), the only such forces in $\hat{\mathfrak{f}}[\psi_{v,\mathbf{x},\dot{\mathbf{x}}}]$ are of the form $\hat{f}^\tau + \hat{f}_{v\text{–}v',r}$, for some $\tau \in \mathcal{T}$. Then some such force variation is proportionate to the acceleration of those bodies during some part of $[(r,0),(r,2)]$; it remains to show that it is proportionate for all instants in that interval.

Let $\mathscr{I}_{\top_\mathrm{c}}$ be the maximal interval set for $\texttt{state}$. Either $\inf(J) = (0,0)$, or, by Theorem 74, $\inf(J) =$

$(r', 3)$ for some $r' < r$. In either case, $\inf(J) \leq (r, 0)$, and so

$$\texttt{state}(r,0) = \texttt{state}(r,1) = \texttt{state}(r,2). \tag{12.163}$$

Then there is some maximal interval $J'' \in \mathscr{I}_{\top_c}$ such that $[(r,0),(r,2)]$ is in J'', and such that $\texttt{state}(J'') = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$. Since χ is an induced motion, then, for every instant $t \in J''$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\left(\hat{f}^{\tau} + \hat{f}_{v\text{--}v',r}\right)_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.164}$$

In particular, for any instant $t \in [(r,0),(r,2)]$ and controller $C \in \Omega_c$,

$$\left(\hat{f}^{\tau} + \hat{f}_{v\text{--}v',r}\right)_c(t)(C, \overline{C}) = m_c(C)\ddot{\chi}_c(C,t). \tag{12.165}$$

But by (8.135), $\hat{f}_c^{\tau}$ is $\mathbf{0}$, for any choice of instants and bodies, and so this reduces to

$$\left(\hat{f}_{v\text{--}v',r}\right)_c(t)(C, \overline{C}) = m_c(C)\ddot{\chi}_c(C,t). \tag{12.166}$$

□

12.2.5.2 Mechanical system never enters $\overline{\psi}_V$

Again as a useful result for later theorems, I show that for any induced motion of M, the mechanical system never enters any placement in the set $\overline{\psi}_V$.

Theorem 78. *Let A be a hybrid automaton, with hybrid mechanical counterpart M. Let $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$ be any choice of configuration, and χ any choice of motion, such that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$. Let* `state` *be the state-time function for χ. Then there is no instant $t \in \Gamma_M$ such that*

$$\texttt{state}(t) = \overline{\psi}_V. \tag{12.167}$$

Proof. Note first that $\overline{\psi}_V \not\subseteq \psi_v$, for any $v \in V$.

By the construction of R in (12.2), the configuration $(w, \dot{w})$ is such that all controller point bodies are at rest, and all controller point bodies are in position $\mathbf{0}_c$ except for some $e \in E$ with position $\mathbf{1}_c$. By Corollary 76, at every instant $(r, 0) \in \Gamma_M$, there exists some $e \in E$ such that this description holds.

Consider any such instant $(r, 0)$. Let $\mathbf{x}, \dot{\mathbf{x}} \in X$ be the values such that

$$\begin{aligned} \chi_p(G, r) &= \mathbf{x} \\ \dot{\chi}_p(G, r) &= \dot{\mathbf{x}}, \end{aligned} \tag{12.168}$$

and let $v \in V$ be the value such that

$$\begin{aligned} \chi_{\mathrm{c}}(e_v,(r,0)) &= \mathbf{1}_{\mathrm{c}} \\ \forall c \neq e_v \in \Omega_{\mathrm{c}}\big[\chi_{\mathrm{c}}(c,(r,0)) &= \mathbf{0}_{\mathrm{c}}\big] \\ \forall c \in \Omega_{\mathrm{c}}\big[\dot{\chi}_{\mathrm{c}}(c,(r,0)) &= \mathbf{0}_{\mathrm{c}}\big]. \end{aligned} \tag{12.169}$$

Let $\mathtt{state}$ be the state-time function for χ, and let $\mathscr{I}_{\top_{\mathrm{c}}}$ be the maximal interval set for $\mathtt{state}$. Let $J \in \mathscr{I}_{\top_{\mathrm{c}}}$ be the maximal interval containing $(r,0)$, and let $\psi \in \Psi(\top_{\mathrm{c}})$ be the mechanical controller state such that $\mathtt{state}(J) = \psi$. Since χ is an induced motion, there exists some $\hat{f}_J \in \hat{\mathfrak{f}}[\psi]$ such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.170}$$

In particular, for any controller point body $c \in \Omega_{\mathrm{c}}$,

$$\hat{f}_{\mathrm{c}}(t)(c,\overline{c}) = m_{\mathrm{c}}(c)\ddot{\chi}_{\mathrm{c}}(c,t). \tag{12.171}$$

Either there exist $\tau \in \mathfrak{T}$ and $v' \neq v \in V$ such that

$$\hat{f}_J = \hat{f}^{\tau} + \hat{f}_{v \to v',r} \tag{12.172}$$

or there do not. In the latter case, by the definition of $\hat{\mathfrak{f}}[\cdot]$ in (8.135), for any (r,z) with $z \in \mathbb{Z}_4$, for all $c \in \Omega_{\mathrm{c}}$,

$$\hat{f}_{\mathrm{c}}(r,z)(c,\overline{c}) = \mathbf{0} \tag{12.173}$$

and so $\ddot{\chi}_{\mathrm{c}}(c,(r,z)) = \mathbf{0}_{\mathrm{c}}$. In other words, all controller bodies remain at rest. Clearly in this case,

$$\chi_{\mathrm{c}}(\top_{\mathrm{c}},(r,0)) = \chi_{\mathrm{c}}(\top_{\mathrm{c}},(r,1)) = \chi_{\mathrm{c}}(\top_{\mathrm{c}},(r,2)) = \chi_{\mathrm{c}}(\top_{\mathrm{c}},(r,3)). \tag{12.174}$$

and so there is no (r,z), for any $z \in \mathbb{Z}_4$, such that $\mathtt{state}(r,z) = \overline{\psi}_V$.

Otherwise, if $\hat{f}_J = \hat{f}^{\tau} + \hat{f}_{v \to v',r}$, then by definition χ is a $(v,\mathbf{x},\dot{\mathbf{x}})$-motion active at r, and by Theorem 24,

$$\mathtt{state}(r,0) = \mathtt{state}(r,1) = \mathtt{state}(r,2) \subseteq \psi_v \tag{12.175}$$

and

$$\mathtt{state}(r,3) \subseteq \psi_{v'}. \tag{12.176}$$

Then again, there is no instant (r,z) such that $\mathtt{state}(r,z) = \overline{\psi}_V$.

Since the above hold for any $(r,0) \in \Gamma_M$, there is no $t \in \Gamma_M$ such that $\mathtt{state}(t) = \overline{\psi}_V$. □

12.2.5.3 Trajectories matching initial trajectory values and derivatives are in $\mathcal{T}$

Again, as a helper theorem for later, I show that a trajectory such that, at every instant, its value and derivative match the initial value and derivative of some trajectory in $\mathcal{T}$ is, itself, in $\mathcal{T}$.

Lemma 24. *Let A be a hybrid automaton with trajectory set $\mathcal{T}$. Let τ be a trajectory such that, for every instant $t \in dom(\tau)$, there exists some trajectory $\tau_t \in \mathcal{T}$ such that*

$$\begin{aligned} \tau(t) &= \tau_t(0) \\ \dot{\tau}(t) &= \dot{\tau}_t(0). \end{aligned} \tag{12.177}$$

Then τ is in $\mathcal{T}$. In other words, a trajectory such that, at every instant, its value and derivative match the initial value and derivative of some trajectory in $\mathcal{T}$ is, itself, in $\mathcal{T}$.

Proof. Recall from Assumption 6 that there exists some time-invariant activity labeling function Act such that a trajectory is in $\mathcal{T}$ iff it is a solution of Act. Let τ' be any trajectory in $\mathcal{T}$; then for any instant $t \in dom(\tau)$, there is an activity function $a \in \mathsf{Act}(\tau'(t))$ such that

$$\Pi_X(\dot{\tau}'(t)) = a(t). \tag{12.178}$$

In particular, letting $t = 0$, there must be an activity function $a \in \mathsf{Act}(\tau'(0))$ such that

$$\Pi_X(\dot{\tau}'(0)) = a(0). \tag{12.179}$$

Since Act is time-invariant, for any instant $t' \in \Gamma$, there must exist some activity $a' \in \mathsf{Act}(\tau'(0))$ such that

$$\Pi_X(\dot{\tau}'(0)) = a'(t'). \tag{12.180}$$

Then suppose there exists some trajectory τ such that, for every instant $t \in dom(\tau)$, there exists a trajectory $\tau_t \in \mathcal{T}$ such that

$$\begin{aligned} \tau(t) &= \tau_t(0) \\ \dot{\tau}(t) &= \dot{\tau}_t(0). \end{aligned} \tag{12.181}$$

It follows from (12.180) that, for each such instant t, there must exist some activity $a_t \in \mathsf{Act}(\tau_t(0))$ such that

$$\Pi_X(\dot{\tau}_t(0)) = a_t(t). \tag{12.182}$$

But by (12.181), $\tau(t) = \tau_t(0)$ and $\dot{\tau}(t) = \dot{\tau}_t(0)$. Thus, there exists some activity $a_t \in \mathsf{Act}(\tau(t))$ such that

$$\Pi_X(\dot{\tau}(t)) = a_t(t). \tag{12.183}$$

But in this case, τ is a solution of Act. Again by Assumption 6, then, τ is in $\mathfrak{T}$. $\square$

12.2.5.4 A has related execution fragments for the motions of M

I now use the preceding results to show that A has related execution fragments for the motions of M.

Theorem 79. *Let A be a hybrid automaton, with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then A has related execution fragments for the motions of M.*

Proof. Let $q \in Q$, $(w, \dot{w}) \in \mathfrak{X}(\top, \mathcal{S})$, and χ be any choice of system state, configuration, and motion such that $(q, (w, \dot{w}))$ is in R, and such that χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$. If no such system state, configuration, and motion exist, the proof holds trivially. Otherwise, A has related execution fragments for the motions of A iff for any such choices, A has related execution fragments for χ, that is, if there exists some execution fragment $\alpha = \tau_0 a_1 \tau_1 \ldots$ such that

- for all $i \geq 0$, τ_i is in $\mathfrak{T}$,
- for all $i > 0$, a_i is in D,
- $\alpha.\mathit{fstate} = q$, and
- $\mathit{trace}(\alpha) = \mathit{trace}(\chi)$.

Let $\Gamma_M / \equiv_\chi$ be the partition of Γ_M imposed by $\equiv_\chi$, as defined in (12.145).

Construction of α: Let `state` be the state-time function for χ, and let $\mathscr{I}_{\top_\mathrm{c}}$ be the maximal interval set for `state`. Since χ is an induced motion, for each interval $J \in \mathscr{I}_{\top_\mathrm{c}}$, there exists some force variation $\hat{f} \in \hat{\mathfrak{f}}[\mathtt{state}(J)]$ such that, for every instant $t \in J$, body $B \in \Omega$, and index $i \in \mathcal{I}(B)$,

$$\hat{f}_i(t)(B \sqcap \top_i, \overline{B \sqcap \top_i}) = m_i(B \sqcap \top_i)\ddot{\chi}_i(B \sqcap \top_i, \hat{t}_i(t)). \tag{12.184}$$

Let each such force variation for each such interval $J \in \mathscr{I}_{\top_\mathrm{c}}$ be labeled $\hat{f}_J$, and let U be the set of all such force variations; that is,

$$U = \{\hat{f}_J \mid J \in \mathscr{I}_{\top_\mathrm{c}}\}. \tag{12.185}$$

By (8.104),

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^\tau + \hat{f}_{v \multimap v', r} \,\middle|\, [\tau \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^\tau \,\middle|\, \tau \in \mathfrak{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_\mathbf{0}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{12.186}$$

and so for each force variation $\hat{f}_J \in U$, one of the following must be true:

1. $\texttt{state}(J) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ for some $v \in V$ and $\mathbf{x}, \dot{\mathbf{x}} \in X$, and $\hat{f}_J = \hat{f}^\tau + \hat{f}_{v\text{–}v',r}$ for some $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$ and some $v' \in \mathcal{D}_v(\mathbf{x})$ and some $r \in \mathbb{R}^{\geq 0}$.
2. $\texttt{state}(J) = \psi_{v,\mathbf{x},\dot{\mathbf{x}}}$ for some $v \in V$ and $\mathbf{x}, \dot{\mathbf{x}} \in X$, and $\hat{f}_J = \hat{f}^\tau$ for some $\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$.
3. $\texttt{state}(J) = \overline{\psi}_V$, and $\hat{f}_J = \hat{f}_{\mathbf{0}}$.

By Theorem 78, however, there is no interval $J \in \mathscr{I}_{\top_c}$ such that $\texttt{state}(J) = \overline{\psi}_V$, and so the third of these possibilities can be ignored. In other words, for every instant $t \in \hat{t}_M$, there exists some automaton controller state $v \in V$ such that

$$\texttt{state}(t) \subseteq \psi_v. \tag{12.187}$$

Then the construction of α is as follows: By Assumption 11, $\Gamma_M/\equiv_\chi$ is a countable set, and so its elements can be labeled $J_0, J_1, \ldots$ in increasing temporal order. Let J_i be any interval in $\Gamma_M/\equiv_\chi$. By Theorem 74, if J_i is non-initial, then it is left-closed, and its first instant is some $(r, 3)$. Likewise, if J_i is non-final, then it is closed, and its last instant is some $(r, 2)$. For each such interval J_i, if J_i is non-initial and non-final, let r_i, r_i' be the values in $\mathbb{R}^{\geq 0}$ such that

$$\begin{aligned} \inf(J_i) &= (r_i, 3) \\ \sup(J_i) &= (r_i', 2). \end{aligned} \tag{12.188}$$

Otherwise, if J_i is initial and non-final, let r_i' be the value in $\mathbb{R}^{\geq 0}$ such that

$$\sup(J_i) = (r_i', 2);$$

clearly in this case, $\inf(J_i) = (0, 0)$, and so let $r_i = 0$. Otherwise, if J_i is non-initial and final, let r_i be the value in $\mathbb{R}^{\geq 0}$ such that

$$\inf(J_i) = (r_i, 3),$$

and let $r_i' = \infty$. Otherwise, if $\Gamma_M/\equiv_\chi$ contains only a single initial, final interval, let $r_i = 0$ and $r_i' = \infty$.

For any interval J_i, let $\mathbf{x}_i \in X$ be the automaton environment state such that

$$\mathbf{x}_i = \chi_{\mathrm{p}}(G, r_i) \tag{12.189}$$

and, if r_i' is non-infinite, let $\mathbf{x}_i' \in X$ be the automaton environment state such that

$$\mathbf{x}_i' = \chi_{\mathrm{p}}(G, r_i'). \tag{12.190}$$

Let $v_i \in V$ be the automaton controller state such that

$$\texttt{state}(J_i) = \psi_{v_i}. \tag{12.191}$$

Let τ_i be the trajectory with domain $[0, r_i' - r_i]$ such that, for any $t \in dom(\tau_i)$,

$$\tau_i(t) = (v_i, \chi_{\text{p}}(G, t + r_i). \tag{12.192}$$

Note that, given the above construction, for any two sequential intervals J_i, J_{i+1},

$$r_i' = r_{i+1}, \tag{12.193}$$

and so

$$\chi_{\text{p}}(G, r_i') = \chi_{\text{p}}(G, r_{i+1}). \tag{12.194}$$

For any two such intervals, let a_i be the transition such that

$$a_i = (v_i, \mathbf{x}_i') \rightarrowtail (v_{i+1}, \mathbf{x}_{i+1}). \tag{12.195}$$

By (12.194), this implies that

$$a_i = (v_i, \mathbf{x}_i') \rightarrowtail (v_{i+1}, \mathbf{x}_i'). \tag{12.196}$$

Then α is the execution fragment such that

$$\alpha = \tau_0 a_1 \tau_1 a_1 \tau_2 \ldots \tag{12.197}$$

Prove τ_i in $\mathcal{T}$: Consider any component trajectory τ_i. By (12.192), for any instant $t \in dom(\tau_i)$,

$$\tau_i(t) = \big(v_i, \chi_{\text{p}}(G, t + r_i)\big)\,. \tag{12.198}$$

Restricting consideration to the environment component,

$$\Pi_X(\tau_i(t)) = \chi_{\text{p}}(G, t + r_i). \tag{12.199}$$

Since χ is twice differentiable, so is the X-projection of τ_i. Taking the first and second derivatives, then,

$$\begin{aligned} \Pi_X(\dot{\tau}_i(t)) &= \dot{\chi}_{\text{p}}(G, t + r_i) \\ \Pi_X(\ddot{\tau}_i(t)) &= \ddot{\chi}_{\text{p}}(G, t + r_i). \end{aligned} \tag{12.200}$$

Consider any such arbitrary instant $t \in dom(\tau_i)$. Let

$$\begin{aligned}\mathbf{x} &= \chi_{\mathrm{p}}(G, t + r_i) = \Pi_X(\tau_i(t)) \\ \dot{\mathbf{x}} &= \dot{\chi}_{\mathrm{p}}(G, t + r_i) = \Pi_X(\dot{\tau}_i(t)).\end{aligned} \tag{12.201}$$

It follows that

$$\mathtt{state}(t + r_i, 0) \subseteq \psi_{\mathbf{x},\dot{\mathbf{x}}}. \tag{12.202}$$

But by (12.191),

$$\mathtt{state}(t + r_i, 0) \subseteq \psi_{v_i}. \tag{12.203}$$

Then

$$\begin{aligned}\mathtt{state}(t + r_i, 0) &\subseteq \psi_{\mathbf{x},\dot{\mathbf{x}}} \cap \psi_{v_i} \\ &= \psi_{v_i,\mathbf{x},\dot{\mathbf{x}}}.\end{aligned} \tag{12.204}$$

By (8.127), $\psi_{v_i,\mathbf{x},\dot{\mathbf{x}}}$ is in $\Psi(\top_{\mathrm{c}})$, and so

$$\mathtt{state}(t + r_i, 0) = \psi_{v_i,\mathbf{x},\dot{\mathbf{x}}}. \tag{12.205}$$

As argued above, either

$$\hat{f}_{J_i} = \hat{f}^{\tau} + \hat{f}_{v\text{–}v',r} \tag{12.206}$$

or

$$\hat{f}_{J_i} = \hat{f}^{\tau} \tag{12.207}$$

for some $\tau \in \mathcal{T}_{v_i,\mathbf{x},\dot{\mathbf{x}}}$ and some $v' \in V$ and $r \in \mathbb{R}^{\geq 0}$. Since τ is in $\mathcal{T}_{v_i,\mathbf{x},\dot{\mathbf{x}}}$, it must be the case by the construction of that set that

$$\begin{gathered}\forall t \in dom(\tau)\big[\Pi_V(\tau(t)) = v_i\big] \\ \Pi_X(\tau.\mathit{fstate}) = \mathbf{x} = \Pi_X(\tau_i(t)) \\ \Pi_X(\dot{\tau}.\mathit{fstate}) = \dot{\mathbf{x}} = \Pi_X(\dot{\tau}_i(t)).\end{gathered} \tag{12.208}$$

In other words, at every instant in the domain of τ_i, there exists some trajectory $\tau \in \mathcal{T}$ whose first state agrees with $\tau_i(t)$ in its X-projection value and derivative, as well as in its V-projection. It follows that, for any $t \in dom(\tau_i)$, there exists some trajectory $\tau \in \mathcal{T}$ such that

$$\begin{aligned}\tau.\mathit{fstate} &= \tau_i(t) \\ \dot{\tau}.\mathit{fstate} &= \dot{\tau}_i(t).\end{aligned} \tag{12.209}$$

By Lemma 24, then, τ_i is in $\mathcal{T}$.

Prove $\alpha.\mathit{fstate} = q$: Since χ is an induced motion for $\hat{\mathfrak{f}}[\cdot]$ beginning in $(w, \dot{w})$,

$$\begin{aligned} \chi(\top, 0) &= w \\ \dot{\chi}(\top, 0) &= \dot{w}. \end{aligned} \tag{12.210}$$

By assumption, $(q, (w, \dot{w}))$ is in R. By construction, $q = (v, \mathbf{x})$ for some $v \in V$ and $\mathbf{x} \in X$. Since $(q, (w, \dot{w}))$ is in R, by (12.1), there exists some $\tau \in \mathcal{T}$ such that $\tau.\mathit{fstate} = q$ and

$$\begin{aligned} w_{\text{p}}(G) &= \mathbf{x} \\ \dot{w}_{\text{p}}(G) &= \Pi_X(\dot{\tau}(t)) \\ \forall v' \in V\ [w_{\text{c}}(s_{v'}) &= 0] \\ \forall v' \in V\ [\dot{w}_{\text{c}}(s_{v'}) &= 0] \\ w_{\text{c}}(e_v) &= 1 \\ \forall v' \neq v \in V\ [w_{\text{c}}(e_{v'}) &= 0] \\ \forall v' \in V\ [\dot{w}_{\text{c}}(e_{v'}) &= 0]. \end{aligned} \tag{12.211}$$

By the definition of ψ_v in (8.127), $(w, \dot{w})$ is an element of ψ_v, and so

$$\mathtt{state}(0, 0) \subseteq \psi_v. \tag{12.212}$$

Consider interval $J_0 \in \Gamma_M/\equiv_\chi$. Since J_0 is the chronologically first interval in $\Gamma_M/\equiv_\chi$, it must contain $(0, 0)$, and indeed $(r_0, z_0) = (0, 0)$. Then since $\mathtt{state}(0, 0) \subseteq \psi_v$, it must be the case that

$$v_0 = v. \tag{12.213}$$

Then by the construction of τ_i, it must be the case that τ_0 contains 0, and indeed $\tau_0.\mathit{ftime} = 0$. Clearly

$$\alpha.\mathit{fstate} = \tau_0.\mathit{fstate} = \tau_0(0). \tag{12.214}$$

By (12.198),

$$\begin{aligned} \tau_0(0) &= (v_0, \chi(G, 0 + r_0)) \\ &= \big(v_0, \chi_{\text{p}}(G, 0 + 0)\big) \\ &= \big(v_0, \chi_{\text{p}}(G, 0)\big) \\ &= (v_0, w(G)). \end{aligned} \tag{12.215}$$

Substituting via (12.213) and (12.211),

$$\begin{aligned} \tau_0(0) &= (v, \mathbf{x}) \\ &= q. \end{aligned} \tag{12.216}$$

Then by (12.214), $\alpha.fstate = q$.

Prove $trace(\alpha) = trace(\chi)$**:** By (3.9), for any instant $t \in dom(\alpha)$ such that t is i-bounded for some integer $i \geq 0$,

$$\alpha(t) = \begin{cases} \tau_i(t), & i = 0 \\ \tau_i'(t), & i > 0 \end{cases} \tag{12.217}$$

where

$$\tau_i' = \tau_i + \sum_{j=0}^{i-1} dur(\tau_j). \tag{12.218}$$

For any trajectory τ_j,

$$\begin{aligned} dur(\tau_j) &= \tau_j.ltime - \tau_j.ftime \\ &= \tau_j.ltime - 0 \\ &= \tau_j.ltime. \end{aligned} \tag{12.219}$$

By the construction of τ_i,

$$\tau_j.ltime = r_j' - r_j. \tag{12.220}$$

Then

$$\sum_{j=0}^{i-1} dur(\tau_j) = \sum_{j=0}^{i-1} (r_j' - r_j). \tag{12.221}$$

But by (12.193), for any nonfinal index j,

$$r_j' = r_{j+1}. \tag{12.222}$$

Then

$$\begin{aligned} \sum_{j=0}^{i-1} dur(\tau_j) &= \sum_{j=0}^{i-1} (r_j' - r_j) \\ &= \sum_{j=0}^{i-1} (r_{j+1} - r_j) \\ &= r_{(i-1)+1} - r_0 \end{aligned} \tag{12.223}$$

$$
\begin{aligned}
&= r_i - r_0 \\
&= r_i - 0 \\
&= r_i.
\end{aligned}
$$

Then for any i-bounded instant t,

$$\alpha(t) = \begin{cases} \tau_i(t), & i = 0 \\ \tau_i'(t), & i > 0 \end{cases} \tag{12.224}$$

where

$$\tau_i' = \tau_i + r_i. \tag{12.225}$$

But by (12.192),

$$\tau_i(t) = \left(v_i, \chi_{\mathrm{p}}(G, t + r_i)\right). \tag{12.226}$$

Then

$$\tau_i'(t) = \left(v_i, \chi_{\mathrm{p}}(G, t)\right). \tag{12.227}$$

Likewise,

$$
\begin{aligned}
\tau_0(t) &= \left(v_0, \chi_{\mathrm{p}}(G, t + r_0)\right) \\
&= \left(v_0, \chi_{\mathrm{p}}(G, t + 0)\right) \\
&= \left(v_0, \chi_{\mathrm{p}}(G, t)\right).
\end{aligned} \tag{12.228}
$$

Then for every i-bounded instant $t \in dom(\alpha)$,

$$\alpha(t) = \left(v_i, \chi_{\mathrm{p}}\left(G, t\right)\right). \tag{12.229}$$

Equivalently, restricting consideration to the environment components, for every instant $t \in dom(\alpha)$,

$$\Pi_X(\alpha(t)) = \chi_{\mathrm{p}}(G, t). \tag{12.230}$$

Then by definition, $trace(\alpha) = trace(\chi)$.

Prove a_i in D: Consider any transition $a_i = (v_i, \mathbf{x}_i') \rightarrowtail (v_{i+1}, \mathbf{x}_i')$. By the construction of a_i, there must exist intervals $J_i, J_{i+1} \in \Gamma_M/\equiv_\chi$.

Let t_i be the final instant in $dom(\alpha)$ such that t_i is i bounded; then by arguments similar to those in the section "Prove $trace(\alpha) = trace(\chi)$,"

$$t_i = \sum_{j=0}^{i} dur(\tau_j) \tag{12.231}$$

$$
\begin{aligned}
&= \sum_{j=0}^{i} (r'_j - r_j) \\
&= \sum_{j=0}^{i} (r_{j+1} - r_j) \\
&= r_{i+1} - r_0 \\
&= r_{i+1} i - r_0 \\
&= r_{i+1} - 0 \\
&= r_{i+1} \\
&= r'_i.
\end{aligned}
$$

Note that, by (12.188),

$$(r'_i, 0) < (r'_i, 2) = \sup(J_i). \tag{12.232}$$

Then $(t_i, 0)$ is in J_i, and o $\texttt{state}(t_i, 0) \subseteq \psi_{v_i}$. However, no subsequent instant $(r, 0) \in \Gamma_M$ is in J_i, since if $r > r'_i$,

$$(r, 0) > (r'_i, 2) = \sup(J_i). \tag{12.233}$$

By Theorem 74, it must be the case that $\sup(J_i) = (t_i, 2)$ and that $\inf(J_{i+1}) = (t_i, 3)$. By the construction of v_i, v_{i+1},

$$
\begin{aligned}
\texttt{state}(t_i, 2) &\subseteq \psi_{v_i} \qquad (12.234)\\
\texttt{state}(t_i, 3) &\subseteq \psi_{v_{i+1}}.
\end{aligned}
$$

By Theorem 77, then, for any mechanical controller body $C \in \Omega_{\mathrm{c}}$,

$$\left(\hat{f}_{v \text{–} v', r}\right)_{\mathrm{c}}(t)(C, \overline{C}) = m_{\mathrm{c}}(C) \ddot{\chi}_{\mathrm{c}}(C, t). \tag{12.235}$$

By (12.194),

$$\mathbf{x}'_i = \chi_{\mathrm{p}}(G, t_i). \tag{12.236}$$

Let $\dot{\mathbf{x}}'_i = \dot{\chi}_{\mathrm{p}}(G, t_i)$. Then by (8.127),

$$\texttt{state}(t_i, 2) \subseteq \psi_{\mathbf{x}', \dot{\mathbf{x}}'}. \tag{12.237}$$

Then by (12.237) and (12.234),

$$
\begin{aligned}
\texttt{state}(t_i, 2) &\subseteq \psi_{v_i} \cap \psi_{\mathbf{x}', \dot{\mathbf{x}}'} \qquad (12.238)\\
&= \psi_{v_i, \mathbf{x}', \dot{\mathbf{x}}'}.
\end{aligned}
$$

By (8.127), $\psi_{v_i,\mathbf{x}',\dot{\mathbf{x}}'}$ is a mechanical controller state in $\Psi(\top_{\mathrm{c}})$, and so

$$\mathtt{state}(t_i, 2) = \psi_{v_i,\mathbf{x}',\dot{\mathbf{x}}'}. \tag{12.239}$$

But then since χ is an induced motion, by the definition of induced motion there must be some force variation $\hat{f} \in \hat{\mathfrak{f}}[\psi_{v_i,\mathbf{x}',\dot{\mathbf{x}}'}]$ such that for any mechanical controller body $C \in \Omega_{\mathrm{c}}$,

$$\hat{f}_{\mathrm{c}}(t_i, 2)(C, \overline{C}) = m_{\mathrm{c}}(C)\ddot{\chi}_{\mathrm{c}}(C, (t_i, 2)), \tag{12.240}$$

and so, substituting via (12.235),

$$\hat{f}_{\mathrm{c}}(t_i, 2)(C, \overline{C}) = \left(\hat{f}_{\mathrm{c}}\right)_{v_i \multimap v_{i+1}, t_i}(C, \overline{C}, (t_i, 2)). \tag{12.241}$$

Again by (8.104),

$$\hat{\mathfrak{f}}[\psi] = \begin{cases} \left\{ \hat{f}^{\tau} + \hat{f}_{v \multimap v', r} \,\middle|\, [\tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}] \wedge [v' \in \mathcal{D}_v(\mathbf{x})] \wedge \left[r \in \mathbb{R}^{\geq 0}\right] \right\} \cup \left\{ \hat{f}^{\tau} \,\middle|\, \tau \in \mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}} \right\}, & \psi = \psi_{v,\mathbf{x},\dot{\mathbf{x}}} \\ \{\hat{f}_{\mathbf{0}}\}, & \psi = \overline{\psi}_V, \end{cases} \tag{12.242}$$

Then the force variation $\hat{f}^{\tau} + f_{v_i \multimap v_{i+1}, t_i}$ is only an element of $\hat{\mathfrak{f}}[\psi_{v_i,\mathbf{x}',\dot{\mathbf{x}}'}]$ iff

$$v_{i+1} \in \mathcal{D}_{v_i}(\mathbf{x}'). \tag{12.243}$$

Then by the definition of $\mathcal{D}_{v_i}$, a is in D.

Conclusion: Then by definition, A has related execution fragments for χ. Since the choice of q, $(w, \dot{w})$, and χ were arbitrary, A has related execution fragments for the motions of M. □

12.2.6 Final proof

The proof of bisimulation follows immediately from the preceding results. First, M simulates A.

Theorem 80. *Let A be a hybrid automaton, with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then R is a simulation; that is, M simulates A.*

Proof. R is a simulation iff M is a comparable mechanical system with related initial states, related motions for the closed trajectories of A, and related motions for the discrete transitions of A. By Theorem 69, Theorem 70, Theorem 72, and Theorem 73, all of these requirements are met. Thus, R is a simulation. □

Likewise, A simulates M.

Theorem 81. *Let A be a hybrid automaton, with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then R^{-1} is a simulation; that is, A simulates M.*

Proof. R^{-1} is a simulation iff M is a comparable mechanical system and A has related initial states and related execution fragments for the motions of M. By Theorem 69, Corollary 71, and Theorem 79, all of these requirements are met. Thus, R^{-1} is a simulation. □

Then by definition, M and A bisimulate each other.

Theorem 82. *Let A be a hybrid automaton, with hybrid mechanical counterpart M, and let R be the relation over $Q \times \mathfrak{X}(\top, \mathcal{S})$ defined as per (12.1). Then R is a bisimulation; that is, M and A bisimulate each other.*

Proof. R is a bisimulation iff both R and R^{-1} are simulations. By Theorem 80, R is a simulation. By Theorem 81, R^{-1} is a simulation. Thus, R is a bisimulation. □

12.3 Summary of notation

$\equiv_\chi$	equivalence relation on instants mapping to substates of the same ψ_v
$\Gamma_M/\equiv_\chi$	equivalence classes imposed by $\equiv_\chi$

Chapter 13

Applications and Further Work

The fundamental goal of this thesis has been to answer the question, "Can hybrid automata be converted into mechanical systems, while preserving their behaviors?" By definition, a hybrid automaton consists of both continuous environment motion and discrete automaton controller behavior; mechanics, by contrast, provides a single, unified set of concepts and laws. The above goal, then, amounts to determining whether one can translate an automaton from a form that cannot be readily described or characterized via physical concepts to one that can be described in such terms.

As the preceding pages have shown, the answer to the above question is "Yes, automata can be so converted." Subject to the assumptions and restrictions outlined in previous chapters, one can cast a hybrid automaton in purely mechanical terms; in particular, the construction provided in Chapters 7 and 8 do so in a way that preserves the behaviors of the original automaton. In this chapter, I present a number of possible extensions and applications for the preceding work.

13.1 Refined interpretation of automaton environment space

Throughout this paper, a number of simplifying assumptions have been made to reduce the length and complexity of the discussions involved. Perhaps the most unnatural of these is the requirement that physical space be $\mathbb{R}^n$, that is, that physical space has the same number of dimensions as the automaton environment space. If one takes ordinary, real-world physical space to be $\mathbb{R}^3$, this poses something of a problem for any automaton with more than three environment variables!

The fundamental problem that forces this awkward construction, as discussed in Section 7.1, is that one cannot readily decompose an automaton environment state into its original variables. As I discussed there, no information regarding state decomposition is available in the automaton tuple, and so there is no clear way to provide such a translation in the general case. That there is no clear means, however, does not imply that no means at all may be possible. As a simple example, consider the inverted pendulum in [17], among others. The environment space of that automaton is the product of two variables, X_1 and X_2, representing the position and velocity of the pendulum bob, respectively. It follows that, at any reachable state of the automaton, $\dot{x}_1 = x_2$, that is, that the bob's velocity is the derivative of its acceleration. By the assumptions present in this paper, both X_1 and X_2 must be second-degree semidifferentiable; in other words, the acceleration of both variables must be defined at all points. Since X_2 is already a measure of velocity, however, it seems sufficient for mechanical realizability that X_2 is left- and right-differentiable, and no requirements must be made of its derivative. Under my current construction, such a recognition is not possible, and so this implementation of the inverted pendulum cannot be modeled.

As demonstrated in Chapter 9, however, if one is aware of the relation between the environment state components in advance, it is relatively straightforward to adjust the construction of the mechanical system to account for them. So, for instance, in that example I was able to remove all references to variables that did not satisfy my assumptions, replacing them with constants or derivatives of other variables as appropriate. It seems plausible that at least such straightforward analysis might be directly incorporated into a refined version of the mechanical transformation.

Any mechanical construction incorporating such distinctions requires one to begin, not only with the automaton, but with the underlying physical system, and with a description of how the components of the automaton environment state correspond to features of that physical system. Given such a description, one might let the corresponding body G consist of multiple bodies, one for each body of interest in the physical system; one might likewise fold some dimensions of the environment space into velocities, accelerations, or other mechanical properties. Since all of the bodies in the original physical system presumably take placements in normal, three-dimensional space, the same should be true for any resulting hybrid mechanical bodies.

Such refinements might also extend the model beyond pure forces. Since I treat the automaton environment state purely as a position, it is sufficient to consider the velocities, the accelerations, and so the forces that produce such positions. More generally, one might construct a mechanical system that included non-positional aspects such as temperature, electromagnetic charge, and so on, which would require the development of mechanical concepts not treated here.

One might similarly attempt to remove other assumptions made through this paper. A partial list of such assumptions includes: the environment state space is $\mathbb{R}^n$, rather than some more complex manifold; the controller and environment universes are entirely separate; and nothing is known regarding the construction of the remainder body N. An area for future expansion, then, is to re-evaluate these restrictions and determine whether, in fact, the proofs demonstrated here do hold for such broader descriptions.

13.2 Composition of automata

Another major area for expansion is in the *composition* of hybrid automata, as discussed by Lynch [18] among others. As discussed in Section 3.3.1, Lynch's composition relies on a number of concepts not otherwise considered here, such as division of the automaton into *input* and *output* variables. Still, some interpretation more consistent with the terminology used here may be possible, and so one could define the mechanical transformation for the composition of arbitrary automata. In practical terms, automaton composition is a valuable subject given the increasing prevalence of interoperating autonomous systems. One may, for instance, have many different hybrid systems operating in a shared environment, whose effects may interrelate. A simple example of such behavior would be an assembly line with many independent devices, each with its own controllers, that collectively move, manipulate, and modify some shared physical object. Modeling of a single hybrid system in such an environment might be of some use, but it would be far more helpful to be able to combine such separate systems into a single whole. Thus, a better handling of composition seems like a worthwhile extension.

13.3 Simplifying the transformation

Regardless of broadening the scope of the mechanical transformation, another avenue for future work is to re-evaluate the existing transformation for some simpler alternative. In Section 7.2.2, I propose and dismiss a number of alternative constructions as insufficient, but that list is hardly exhaustive. Several technical details of the current transformation are less than ideal; the requirement that controller bodies colocate, for instance, seems counter-intuitive. As well, it may prove to be unnecessary to have $|V|$ separate point bodies for the effector, with a like number of point bodies for the effector. Perhaps one of these sets can be merged into a single body, or perhaps the division into sensor and effector is itself unnecessary. Chapter 10, at least, suggests that each pair of effector and sensor point bodies derived from a common automaton controller state can be combined into a single whole, with the added benefit that such a combination would allow for controller-mediation of all forces. Again, such exploration was beyond the scope of this work, but would seem to be an area for future exploration.

More generally, then, there may be ways to simplify the transformation, producing simpler constructions of bodies, mechanical controller states, or forces, for instance.

13.4 Applications for philosophy

13.4.1 Intentionality and theory of mind

One of the original inspirations for this paper was a study of Franz Brentano's theory of *intentionality* [8]. Intentionality may be loosely described as the property of "aboutness" possessed by some mental objects; if one believes that he is holding an apple (and so, if one has a belief about an apple), then he has a belief exhibiting intentionality. In particular, the belief's *intentional object* is a mental representation of an apple. It should be noted that intentionality in this sense is unrelated to the idea of being *intentional*, that is, of acting with deliberateness.

In Brentanian intentionality, observations of the outside world cause a mind to develop corresponding intentional mental objects. Returning to the example of the apple, an observer might note that the apple is red, of a particular size and weight, and so on, forming corresponding intentional beliefs. Each of these beliefs has, as its object, a mental representation of the apple. Intentional beliefs need not be complete or even accurate descriptions of the original object; one might, on closer observation of the apple, realize that it is actually a plastic model and much lighter than initially believed.

Once adopted, intentional mental objects can inspire the adoption of other such objects. These new thoughts may exhibit *higher-order* intentionality; that is, they may have as their intentional objects other thoughts or beliefs. Eventually, some chain of thoughts lead to the adoption of a thought that is acted on. Actions modify the world, creating a new set of observations, and the cycle repeats.

Doyle [12] hypothesizes that minds may be modeled as mechanical systems. In conjunction, these two ideas suggested the possibility of mechanically modeling intentionality. More specifically, mechanical forces seemed something of a parallel for intentionality; as with intentionality, a force is directed towards some object, and a force on one object may produce a chain of forces on other objects, resulting in some change in the state of the world. The primary goal of this thesis, to answer the question, “Can a hybrid automaton be represented in mechanical terms?” is a distant offspring of the original question, “Can intentionality be represented in mechanical terms?” One might hope that answering the former in the affirmative will assist in answering the latter as well.

In hybrid systems terms, a mind exhibiting intentionality can be thought of as the controller portion of a system, with the physical environment forming the environment. The various states of the controller then correspond to different combinations of intentional thoughts. Just as the state of a controller depends both on the controller’s design and the state of the continuous system, the particular thoughts adopted by an intentional mind might depend on both the nature of the mind and the arrangement of the environment.

Once converted to hybrid mechanics, the parallels become stronger. The sensor bodies of a hybrid mechanical system can be taken to correspond to the senses of sight, sound, or whatever other means are used to observe the environment. Individual sub-controllers might correspond to individual intentional thoughts, with the state of each sub-controller corresponding to the particular belief, desire, and so forth. Higher-order intentionality, with lower-order thoughts as its object, could be described by higher-order controllers whose states are defined by the states of lower-order ones; the effector bodies, then, would match to intentions-in-action.

More recent works suggest further correspondences. Peter Gärdenfors [13] describes each property of an intentional object - for instance, its perceived weight, color, size, and so on - as a single value from a set of possibilities. He argues that many of these sets can reasonably be treated as the basis a metric space, that is, a space for whom any two points have an associated numerical distance. In particular, he argues that *similarity* forms a natural metric over these spaces, with the distance between any two points proportionate to how dissimilar they are to a human observer. The product of all such dimensions would then also be a metric space, with every possible set of properties for a particular intentional object defining a particular point within this *conceptual space*. Gärdenfors claims that conceptual spaces form a foundation for a theory of natural "kinds" or categories, with each category corresponding to some connected region within the space. "Black," for instance, would be a natural kind and a contiguous region in the color dimension; "duck" might be a much more complex region, defined across multiple dimensions of a conceptual space, but it would ultimately be possible to associate a single area with that label.

In a hybrid mechanical view, Gärdenfors's conceptual spaces seem like a natural fit for the state spaces of various sensor or controller bodies. Natural kinds might then correspond to states or sets of similar states of different controllers. The work in this paper has been sufficient to show that many hybrid automata may be converted to hybrid mechanical systems; it may be that similar constructions could produce mechanical systems that were meaningfully representative of something like Brentannian intentionality ofr Gärdenfors-style conceptual spaces. Obviously such hopes are extremely tentative, but the benefits to such a transformation could be substantial, with mechanical theorems used to offer mathematical analysis of changes in belief. As in the discussion of computer science applications, above, the application of existing mechanical results to theories of mind might provide a novel line of mathematical characterization of mental processes and behaviors.

13.4.2 Descartes and metaphysical conservativism

Previous sections have made no claims regarding whether mechanical controller bodies must be (or, for that matter, must not be) physical bodies, in the everyday, material sense of the word "physical." The work on mechanical controllers in Chapters 6 and 10 requires only that the controllers and controlled bodies be separate, and even that requirement is primarily for simplicity of notation. This paper, then, makes no ontological claims regarding the existence or non-existence of non-physical bodies, and this current section is not intended to alter that fact.

It seems appropriate, however, to consider whether the technical constructions in preceding chapters have any relevance to classic mind-body problems. In particular, Chapter 10 discusses the possibility of alternate assignments of controller forces. That is, rather than controllers simply determining forces between separate physical bodies, under certain reasonable constraints one can produce descriptions of such systems which have identical motions, but in which controller-determined forces are instead forces of the controller on the physical system.

These alternate physical constructions suggest questions beyond the scope of this work. In classical philosophy, Cartesian dualism asserted that human beings were composed of both material and mental substance, both bodies and minds, and that human behavior could not be understood without recourse to both substances. More recent work has tended to reject Descartes' view, in part on the grounds that the physical universe appears to be a closed system. To the present, no changes in the human body suggest intervention by any force not explainable under a purely physical ontology; even if nonphysical material exists, it would seem necessarily to be epiphenomenal.

The alternate force constructions in this paper, however, suggest that epiphenomenality is not a necessary conclusion. As noted, both the construction in which all forces are purely physical and the construction in which forces mediate between physical and non-physical bodies can produce identical motions of all bodies; in practical terms, to an observer in the physical universe, there is no means of determining which of the two force systems is actually in operation. Thus, the existence of a description of the system in which controllers have no direct influence does not necessitate that the controllers do not, in fact, have any direct influence. By practical extension, the existence of a physical-only description of human behavior does not guarantee that some alternative hybrid description is not more accurate, and it may be impossible to tell the difference between the two based on pure observation.

Again, nothing in the above is evidence for the existence of something like Cartesian dualism; that one cannot tell the difference between the single- and hybrid-system models based on physical observation can hardly be used to argue that the hybrid-system model is correct. At best, then, the existence of these mechanically equivalent ontologies is reason to not automatically dismiss the Cartesian view, and to note that one can reach identical mechanical positions in either case without introducing epiphenomenal components. Some skepticism in committing absolutely to either position on mechanical grounds would thus seem to be appropriate. In previous work [10], I term such skepticism *metaphysical conservativism*. A broader consideration of the extent to which such conservativism is justified, while not the primary point of this paper, seems worthwhile.

Chapter 14

Conclusion

While hybrid automata provide a solid framework for analyzing hybrid systems, their division of the world into controller and environment portions, with different interactions for each, can conceal any mechanical character of the underlying system. My objective in the preceding work has been to preserve the behavior of a hybrid automaton while transforming it to a system in which mechanical inquiry is more readily possible. At the same time, I hope to provide a mechanical descriptive framework that might be used to characterize more metaphysical problems, such as the relationship between minds and bodies. Thus, I provide the following contributions:

- I formally define the notion of a conditional force system, including a mechanical definition of controller states. I also define supporting concepts such as state independence, force independence, and controller-additivity, permitting many conditional force systems to be expressed as the sum of a set of base forces and, for each controller, a state-dependent differential force system.

- I formally define the notion of force mediation, in which differential forces are assigned to the controller on whose state they depend. Given reasonable conditions, I show that force mediation preserves resultant forces on all bodies.

- I provide a hybrid mechanical transformation of a hybrid automaton, producing a hybrid mechanical system whose behaviors are isomorphic to those of the original automaton. The resulting mechanical system relies on a set of sensor and effector controller bodies to control the motion of the physical correspondent body. Conceptually, sensor states correspond to automaton dynamic transitions, effector states correspond to automaton controller states, and the position of the correspondent body is equivalent to the automaton environment state.

- I formally define the simulation of a hybrid automaton by a hybrid mechanical system, and vice-versa. I show that, if a hybrid automaton and a hybrid mechanical system bisimulate each other, then the sets of their traces are isomorphic. In other words, if automaton A and mechanical system

M bisimulate each other, then there is an isomorphism between the environment-state projections of the trajectories of A and the induced motions of some body of M.

- I show that the provided transformation produces a hybrid mechanical system that bisimulates the original hybrid automaton, and so that the traces of the automaton and the system are isomorphic. I likewise show that the effector-sensor pair bodies and their joins form a subuniverse of bodies that satisfy the requirements for force mediation.

While much room for refinement and expansion remains, these results suggest a range of novel inquiries that apply the theorems of mechanics to the behavior of hybrid automata.

REFERENCES

[1] R. Abraham and J. Marsden. *Foundations of Mechanics, Second Edition*. Westview Press, 1994.

[2] R. Alur, C. Courcoubetis, N. Halbwachs, T. A. Henzinger, P. H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138(1):3–34, 1995.

[3] R. Alur, C. Courcoubetis, T. A. Henzinger, and P. H. Ho. Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems. *Hybrid systems*, pages 209–229, 1993.

[4] R. Alur and D. L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.

[5] R. Alur, T. A. Henzinger, G. Lafferriere, and G. J. Pappas. Discrete abstractions of hybrid systems. *Proceedings of the IEEE*, 88(7):971–984, 2000.

[6] M.S. Branicky. Stability of switched and hybrid systems. *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, 4:3498–3503, 1994.

[7] M.S. Branicky. *Studies in Hybrid Systems: Modeling, Analysis, and Control*. Massachussetts Institute of Technology, 1995.

[8] F. Brentano. *Psychology from an Empirical Standpoint*. Routledge, 1995.

[9] B. Dellinger and J. Doyle. Mechanically equivalent ontologies I: Hybrid and conditional forces. January 2012.

[10] B.J. Dellinger and J. Doyle. Metaphysical conservativism and mechanical characteristics of human nature. *2014 AAAI Fall Symposium Series*, 2014.

[11] J. Doyle. Matter, mind, and mechanics: New models for dynamogenesis and rationality.

[12] J. Doyle. *Extending Mechanics to Minds: The Mechanical Foundations of Psychology and Economics*. Cambridge University Press, New York, NY, 2006.

[13] P. Gardenfors. *Conceptual spaces: The geometry of thought*. MIT Press, 2004.

[14] R. Ghosh and C. Tomlin. Symbolic reachable set computation of piecewise affine hybrid automata and its application to biological modelling: Delta-Notch protein signaling. *Systems Biology*, 1(1), 20044.

[15] T. A. Henzinger. The theory of hybrid automata. *Logic in Computer Science, 1996. LICS'96. Proceedings, Eleventh Annual IEEE Symposium on*, pages 278–292, 1996.

[16] K. H. Johansson, J. Lygeros, S. Sastry, and M. Egerstedt. Simulation of Zeno hybrid automata. *38th IEEE Conference on Decision and Control*, pages 3538–3543, 1999.

[17] K.H. Johansson, J. Lygeros, and S. Sastry. Modeling of hybrid systems. *Encyclopedia of Life Support Systems*, 2004.

[18] N. Lynch, R. Segala, and F. Vaandrager. Hybrid I/O automata. *Information and Computation*, 185(1):105–157, 2003.

[19] W. Noll. The foundations of classical mechanics in the light of recent advances in continuum mechanics. *The Axiomatic Method, with Special Reference to Geometry and Physics (Berkeley, 1957/58)*, pages 266–281, 1959.

[20] P. J. Ramadge and W. M. Wonham. Supervisory control of a class of discrete event processes. *SIAM journal on control and optimization*, 25(1):206–230, 1987.

[21] P. J. Ramadge and W. M. Wonham. The control of discrete event systems. *Proceedings of the IEEE*, 77(1):81–98, 1989.

[22] V. Suman and P. K. Pandya. Foundations of timed and hybrid automata: A survey. 2006.

[23] C. Truesdell. *A first course in rational continuum mechanics, Volume 1: General concepts*. Academic Press, 1991.

[24] P. Varaiya. Smart cars on smart roads: problems of control. *Automatic Control, IEEE Transactions on*, 38(2), 1993.

[25] G.R. Venayagamoorthy. Potentials and promises of computational intelligence for smart grids. *Power and Energy Society General Meeting, 2009*, pages 1–6, 2009.

APPENDIX

Appendix A

Summary of Notation

Chapter 2: Mathematical Foundations

U	a set
$\leq$	a total order on a set
J	a interval in a total ordering
$[j_1, j_2]$	the interval from j_1 to j_2
$min(J)$	minimal element of an interval (if exists)
$max(J)$	maximum element of an interval (if exists)
$J + j'$	interval translated up by j'
$J - j'$	interval translated down by j'
f	a function
$dom(f)$	domain of function f
$\dot{f}$	derivative of function f
$\ddot{f}$	second derivative of function f
$\dot{f}_-$	left-derivative of function f
$\dot{f}_+$	right-derivative of function f
$\lim x \to j^-$	limit from the left
$\lim x \to j^+$	limit from the right
$f\lceil U$	restriction of function f to set U
$\mathfrak{M}$	a differentiable manifold
$T_x\mathfrak{M}$	tangent space to manifold $\mathfrak{M}$ at point x
$T\mathfrak{M}$	tangent bundle for manifold $\mathfrak{M}$
$\mathbf{v}$	vector field on a manifold

$\mathfrak{X}(\mathfrak{M})$	set of all vector fields on manifold $\mathfrak{M}$
χ	deformation of a subset of a manifold
τ	a trajectory
$\tau.\mathit{ftime}$	the infimum of the domain of trajectory τ
$\tau.\mathit{ltime}$	the supremum of the domain of trajectory τ
$\tau.\mathit{fstate}$	the value of a left-closed trajectory τ at its infimum
$\tau.\mathit{lstate}$	the value of a right-closed trajectory τ at its supremum
$\leq$	prefix relation between two trajectories
$\tau + c$	trajectory translated up by c
$\tau - c$	trajectory translated down by c
$\tau \trianglerighteq j$	suffix of trajectory τ beginning at j
$\mathfrak{T}$	set of trajectories
$\tau \frown \tau'$	concatenation of trajectories τ and τ'
Π_i	projection of a tuple onto its ith component
$\Pi_i(\tau)$	projection of a trajectory τ onto its ith component
$\Pi(U)$	set containing the projection of all members of the set U
$\ddot{f}_L$	left-favoring second derivative of function f

Chapter 3: Hybrid Automata

Γ	time
$\mathbb{R}$	the real numbers
A	a hybrid automaton
V	automaton controller state space
v	an automaton controller state
X	automaton environment state space
$\mathbf{x}$	automaton environment state
Q	automaton system states
q	an automaton system state
Q_0	initial states of a hybrid automaton
D	transition relation of a hybrid automaton
(q, q')	a discrete transition from system state q to q'
$q \rightarrowtail q'$	a discrete transition from system state q to q'
α	an execution fragment

$frags_A$	set of all execution fragments for an automaton A
$execs_A$	set of all executions for an automaton A
Q_{reach}	set of all reachable system states in an automaton
$trace(\alpha)$	trace of an execution fragment α
$tracefrags_A$	set of the traces of all execution fragments of an automaton A
$traces_A$	set of all traces of all executions of an automaton A
$dur(\tau)$	duration of trajectory τ
$\dot{\alpha}$	derivative of a differentiable execution fragment α
$\alpha.fstate$	first state of the first trajectory in execution fragment α
$\alpha.lstate$	last state of a right-closed execution fragment α
$\dot{\alpha}.fstate$	derivative of an execution fragment α at its first instant
$\dot{\alpha}.lstate$	derivative of a right-closed execution fragment α at its last instant
$\alpha \frown \alpha'$	concatenation of execution fragments α and α'
a	an activity
$\mathfrak{A}$	set of all possible activities
Act	activity labeling function
Inv	invariant set
(v, v')	edge of a hybrid automaton
E	set of all edges of a hybrid automaton
$Guard$	guard set
$Reset$	reset function

Chapter 4: Mechanics

Ω	the universe of bodies
B	a body
$B \sqcup B'$	union of bodies B and B'
$B \sqcap B'$	intersection of bodies B and B'
$\sqsubseteq$	subbody relation on $\Omega \times \Omega$
$\top$	the universal body
$\bot$	the null body
Θ	set of all open subsets of $\top$
$(\top, \Theta)$	a topology of $\top$

$\overline{B}$	exterior of body B
$\texttt{clo}\, B$	closure of body B
$\texttt{int}\, B$	interior of body B
$\mathcal{S}$	a mechanical space
t	an instant in time
Γ	time
χ	placement function
$\mathcal{C}(B, \mathcal{S})$	set of all placements of body B in mechanical space $\mathcal{S}$
$\cong$	isomorphic to
χ_t	placement function for instant t
b	a body point
$\dot{\chi}$	velocity function
$\ddot{\chi}$	acceleration function
$\mathfrak{X}(\top, \mathcal{S})$	configuration space of the universal body
w	one placement of a body
$\dot{w}$	one assignment of velocity to a body
$(w, \dot{w})$	a configuration of the universal body
$\mathcal{M}$	set of mass values
m	mass function
p	momentum function
$\mathcal{V}$	set of force values
$\mathbf{0}$	the null force
f	force system
$f(A, B)$	force of a body B on body A
$f(B, \overline{B})$	resultant force on body B
$\mathcal{F}(\Omega, \mathcal{V})$	set of all force systems for universe Ω
$\hat{f}$	force variation
f_t	force system associated with instant t
$\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$	set of all force variations for universe Ω
$\mathfrak{L}$	system law function
$\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$	equivalence class of configurations mapped to the same force systems by system law function $\mathfrak{L}$
$\mathfrak{X}\text{-time}(\chi, \mathfrak{L})$	configuration-time function associating every instant with an equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$
$\mathscr{I}_{\chi, \mathfrak{L}}$	partition of Γ into intervals imposed by $\mathfrak{X}\text{-time}(\chi, \mathfrak{L})$
$\mathfrak{X}_0(\top, \mathcal{S})$	initial configuration set for universal body $\top$

Chapter 5: Hybrid Mechanics

$\mathcal{I}$	set of indices for a universe
Ω_i	universe of bodies for ith factor mechanical system
Ω	hybrid universe of bodies
$\top_i$	universal body for ith factor mechanical system
$\top$	hybrid universal body
$\mathcal{I}(B)$	spray of hybrid body B
$\mathcal{S}_i$	mechanical space for ith factor mechanical system
$\mathcal{S}$	hybrid mechanical space
Γ_i	time for ith factor mechanical system
Γ	hybrid time
$\hat{t}_i$	mapping from hybrid time to ith factor time
χ_i	placement function for ith factor mechanical system
$\chi_{\mathcal{I}(B)}$	spray placement for body B
$\mathcal{C}(B, \mathcal{S}_i)$	placement space of body B in ith factor mechanical system
$\mathcal{C}(B, \mathcal{S})$	hybrid placement space of body B
$\mathbf{v}_i$	material velocity field for ith factor mechanical system
$\mathbf{v}$	hybrid material velocity field
$\mathfrak{X}(B, \mathcal{S})$	hybrid configuration space of body B
$\mathfrak{X}(\top, \mathcal{S})$	hybrid configuration space of the hybrid universal body
$\mathfrak{X}(B, \mathcal{S}_i)$	set of configurations of body B in ith factor mechanical system
m_i	mass function for ith factor mechanical system
m	hybrid mass function
p_i	momentum function for ith factor mechanical system
p	hybrid momentum function
$\mathcal{V}_i$	force values for ith factor mechanical system
$\mathcal{V}$	hybrid force values
f	hybrid force system
$\mathcal{F}(\Omega, \mathcal{V})$	set of all hybrid force systems for universe Ω
$\hat{f}$	hybrid force variation
f_t	hybrid force system associated with hybrid instant t
$\hat{\mathcal{F}}(\Omega, \mathcal{V}, \Gamma)$	set of all hybrid force variations for hybrid universe Ω
$\mathfrak{L}$	hybrid system law function

$\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$	equivalence class of configurations mapped to the same force systems by hybrid system law function $\mathfrak{L}$
$\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$	configuration-time function associating every hybrid instant with an equivalence class in $\mathfrak{X}(\top, \mathcal{S})/\equiv_{\mathfrak{L}}$
$\mathscr{I}_{\chi,\mathfrak{L}}$	partition of Γ into intervals imposed by $\mathfrak{X}\text{-}\mathsf{time}(\chi, \mathfrak{L})$
w	one hybrid placement of a body
$\dot{w}$	one assignment of hybrid velocity to a body
$(w, \dot{w})$	hybrid configuration for a body
$\mathfrak{X}_0(\top, \mathcal{S})$	initial hybrid configuration set for hybrid universal body $\top$

Chapter 6: Mechanical Controllers and Conditional Force Systems

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C	a mechanical controller
$\Psi(C)$	states of controller C
Ω_{p}	physical universe
Ω_{c}	controller universe
$\top_{\mathrm{p}}$	universal physical body
$\top_{\mathrm{c}}$	universal controller body
$\perp_{\mathrm{p}}$	physical null body
$\perp_{\mathrm{c}}$	controller null body
$\overline{B}^{\mathrm{p}}$	physical exterior of body B
$\overline{B}^{\mathrm{c}}$	controller exterior of body B
$f[\cdot]$	conditional force system
$f[\psi]$	force system conditional on mechanical controller state ψ
$C(\psi)$	projection of mechanical controller state ψ onto mechanical controller C
$\mathfrak{f}[\cdot]$	conditional force inclusion
$\mathfrak{f}[\psi]$	force inclusion conditional on mechanical controller state ψ
$f^*[\cdot, \cdot]$	conditional difference
$f^*[C, \psi]$	conditional difference given mechanical controller C in state ψ
$\mathtt{state}$	state-time function
$\mathtt{state}(t)$	state of universal mechanical controller in instant t
$\mathtt{state}_C$	state-time function for mechanical controller C

$\mathtt{state}_C(t)$	state of mechanical controller C in instant t
$\mathtt{state}(C,t)$	state of mechanical controller C in instant t
$\mathtt{state}(J)$	state of universal mechanical controller during interval J (if it exists)
$\mathtt{state}(C,J)$	state of mechanical controller C during interval J (if it exists)
$\mathscr{I}_C$	maximal intervals for mechanical controller C
$\hat{f}[\cdot]$	conditional force variation
$\hat{f}_t[\psi], \hat{f}[\psi](t)$	force system conditional on mechanical controller state ψ at instant t
$\hat{f}^*[\cdot,\cdot]$	conditional differential force variation
$\hat{f}^*[C,\psi]$	conditional differential force variation given mechanical controller C in state ψ
$f_{\mathbf{0}}$	passive force system
$\hat{f}_{\mathbf{0}}$	passive force variation
$\mathfrak{f}$	force inclusion
$\mathfrak{f}[\cdot]$	conditional force inclusion
$\mathfrak{f}^*[\cdot,\cdot]$	conditional differential force inclusion
$\mathfrak{f}^*[C,\psi]$	conditional differential force inclusion given mechanical controller C in state ψ
$\hat{\mathfrak{f}}$	force variation inclusion
$\hat{\mathfrak{f}}[\cdot]$	conditional force variation inclusion
$\hat{\mathfrak{f}}^*[\cdot,\cdot]$	conditional differential force variation inclusion
$\hat{\mathfrak{f}}^*[C,\psi]$	conditional differential force variation inclusion given mechanical controller C in state ψ
$\equiv_C$	equivalence class over $\mathfrak{X}(\top,\mathcal{S})$ based on membership in the same state of controller C
$\mathtt{state}_\chi$	state-time function for motion χ

Chapter 7: Transformation from Automata to Mechanics

M	a hybrid mechanical system
Γ_M	hybrid mechanical time
Γ_A	automaton time

Γ_{p}	physical time
$\mathcal{S}_{\mathrm{p}}$	physical space
G	the physical correspondent body
N	physical exterior of G
$\mathcal{C}(B, \mathcal{S}_{\mathrm{p}})$	physical placement space of physical body B
$\mathfrak{X}(B, \mathcal{S}_{\mathrm{p}})$	physical configurations of physical body B
$\cong$	isomorphic
$\mathbf{x}$	automaton environment state, or the isomorphic placement of G
L	reachability image
L^{-1}	reachability preimage
$\equiv_V$	reachability equivalence relation over X
$X/\equiv_V$	partition imposed by $\equiv_V$
$[\mathbf{x}]$	element of the partition $X/\equiv_V$
$v_{[\mathbf{x}]}$	mechanical controller state defined in terms of $[\mathbf{x}]$
$\mathcal{S}_{\mathrm{c}}$	mechanical controller space
Γ_{c}	mechanical controller time
e	an effector body point, or an effector point body
E	the effector controller
s	a sensor body point, or a sensor point body
S	the sensor controller
$\mathcal{C}(C, \mathcal{S}_{\mathrm{c}})$	controller placement space of controller body C
$\mathfrak{X}(C, \mathcal{S}_{\mathrm{c}})$	controller configurations of controller body C
$\hat{t}_{\mathrm{p}}$	mapping from Γ_M to Γ_{p}
$\hat{t}_{\mathrm{c}}$	mapping from Γ_M to Γ_{c}
$\hat{t}_M$	mapping from Γ_A to Γ_M
$\mathcal{M}_{\mathrm{p}}$	set of physical mass values
$\mathcal{M}_{\mathrm{c}}$	set of controller mass values
m_{p}	physical mass function
m_{c}	controller mass function
$0_{\mathrm{c}}, 1_{\mathrm{c}}$	controller mass values
$0_{\mathrm{p}}, 1_{\mathrm{p}}$	physical mass values
ψ_v	mechanical controller state defined in terms of placement of e_v
$\overline{\psi}_v$	complement of ψ_v with respect to $\mathfrak{X}(\top, \mathcal{S})$
ψ_V	union of all mechanical controller states ψ_v
$\overline{\psi}_V$	complement of ψ_V with respect to $\mathfrak{X}(\top, \mathcal{S})$

$\equiv_G$	equivalence relation over $\mathfrak{X}(\top, \mathcal{S})$ defined in terms of placement and velocity of G
$\mathfrak{X}(\top, \mathcal{S})/\equiv_G$	partition imposed by $\equiv_G$
$\psi_{\mathbf{x},\dot{\mathbf{x}}}$	element of the partition $\mathfrak{X}(\top, \mathcal{S})/\equiv_G$
$\psi_{v,\mathbf{x},\dot{\mathbf{x}}}$	mechanical controller state, intersection of ψ_v and $\psi_{\mathbf{x},\dot{\mathbf{x}}}$
$\mathcal{D}$	transition image of an automaton system state
$\mathcal{D}_v$	transition image of an automaton environment state, given automaton controller state v
$\equiv_{\mathcal{D},v}$	equivalence relation over X defined in terms of $\mathcal{D}_v$
$X/\equiv_{\mathcal{D},v}$	partition of X imposed by $\equiv_{\mathcal{D},v}$
$[\![\mathbf{x}]\!]$	element of the partition $X/\equiv_{\mathcal{D},v}$ containing automaton environment state $\mathbf{x}$
$\psi_{[\![\mathbf{x}]\!]}$	mechanical controller state defined in terms of $[\![\mathbf{x}]\!]$
$\psi_{v,[\![\mathbf{x}]\!]}$	mechanical controller state, intersection of ψ_v and $\psi_{[\![\mathbf{x}]\!]}$
$\mathtt{Mech}$	mapping from trajectories to configurations

Chapter 8: Mechanical Forces for the Transformation

$\hat{\mathfrak{f}}_{e,v}[\cdot]$	conditional force variation inclusion, conditional on state of effector point body e_v
$\hat{\mathfrak{f}}_{s,v}[\cdot]$	conditional force variation inclusion, conditional on state of sensor point body s_v
$\hat{\mathfrak{f}}_E[\cdot]$	conditional force variation inclusion, conditional on state of effector
$\hat{\mathfrak{f}}_S[\cdot]$	conditional force variation inclusion, conditional on state of sensor
$\hat{f}_{\mathbf{0}}$	passive force variation
$\hat{f}^{\tau}$	the force variation for trajectory τ
$\mathcal{T}_v$	set of all trajectories in $\mathcal{T}$ with constant automaton controller state v
$\mathcal{T}_{v,\mathbf{x},\dot{\mathbf{x}}}$	set of all trajectories in $\mathcal{T}$ with constant controller state v, environment state projection $\mathbf{x}$, and environment state projection derivative $\dot{\mathbf{x}}$

$f_{v\text{–}v',r}$	v–v' transition force variation active at r

Chapter 10: Interpreting Controller-Dependent Forces

$h[\cdot]$	conditional hybrid force system
$h[0]$	null hybrid force system
$h^*[\cdot,\cdot]$	conditional differential hybrid force system
$\hat{h}[\cdot]$	conditional hybrid force variation
$\hat{h}[0]$	null hybrid force variation
$\hat{h}^*[\cdot,\cdot]$	conditional differential hybrid force variation
$\mathfrak{h}[\cdot]$	conditional hybrid force inclusion
$\mathfrak{h}[0]$	null hybrid force inclusion
$\mathfrak{h}^*[\cdot,\cdot]$	conditional differential hybrid force inclusion
$\hat{\mathfrak{h}}[\cdot]$	conditional hybrid force variation inclusion
$\hat{\mathfrak{h}}[0]$	null hybrid variation inclusion
$\hat{\mathfrak{h}}^*[\cdot,\cdot]$	conditional differential hybrid force variation inclusion
es_v	effector-sensor pair body for automaton controller state v
es_U	join of all effector-sensor pair bodies es_v, with $v \in U$
$\Omega_{\mathrm{c},es}$	subuniverse of Ω_{c} containing all es_U bodies

Chapter 11: Bisimulation

$\hat{t}_M(\Gamma_A)$	image of Γ_A under $\hat{t}_M$
$\hat{t}_{\mathrm{p}M}$	mapping from Γ_A to Γ_{p}
$traces_M$	mechanical traces of hybrid mechanical system M
$trace$	mechanical trace

Chapter 12: Proof of Bisimulation

$\equiv_\chi$	equivalence relation on instants mapping to substates of the same ψ_v
$\Gamma_M/\equiv_\chi$	equivalence classes imposed by $\equiv_\chi$