MUHAMMAD KAMEL HABIB. Sampling Representations and Approximations for Certain Functions and Stochastic Processes. (Under the direction of STAMATIS CAMBANIS.)

Sampling representations and approximations of deterministic and random signals are important in communication and information theory. Finite sampling approximations are derived for certain functions and processes which are not bandlimited, as well as bounds on the approximation errors. Also derived are sampling representations for bounded linear operators acting on certain classes of bandlimited functions and processes; these representations enable one to reconstruct the image of a function (or a process) under a bounded linear operator using the samples of the function (or the process) rather than the samples of the image. Finally, sampling representations for distributions and random distributions are obtained.

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TABLE OF CONTENTS

CHAPTER I

Introduction

1.1. Rationale.

In many areas of communication engineering, such as radio and television broadcasting and satellite communications, one is concerned with the transmission of several signals (functions) through a "channel." Since it is inefficient to transmit one signal at a time over the channel, it is natural to ask if it is possible to "reconstruct" a continuous-time signal from its samples under certain conditions on the signal and the sampling scheme. If the answer is in the affirmative, one may transmit only the samples of the signal, thus occupying the channel only at the instants of sampling. Between these instants the samples of other signals can be transmitted.

Another interesting application of "sampling" is in the field of sound recording (see, e.g., Vitushkin, 1974). The most widely used technique is the analogue method where the signal is recorded without any preceding transformations. However, the signals recorded by this method suffer distortion due to the defects of recording and reproduction devices. A more promising method of recording is called the digital recording technique. By this method, the signal is first transformed into a discrete code. In other words, the signal is sampled and the samples are coded; then the code of the signal is recorded, and finally the discrete code is read off the recording and is transformed into its continuous-time form in order to reproduce the signal. More precisely, the signal is reconstructed from the (decoded) samples read off the recording. This technique has been recently employed with remarkable results.

Clearly, sampling representations and approximations are very significant in communication and information theory, especially in the era of digital computers.

1.2. Sampling Representations.

The sampling representation (expansion, theorem)

$$(1.2.1) \qquad f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \quad , \quad t \in \mathbb{R}^1 ,$$

was originated by E.T. Whittaker (1915). J.M. Whittaker (1929, 1935), Kotelnikov (1933), Shannon (1949), and others have studied extensively the sampling theorem and its extensions in developing communication and information theory. For a review of the sampling theorem, see Jerri (1977).

A function f which can be represented, for some $W_0 > 0$, by

$$(1.2.2) \qquad f(t) = \int_{-W_0}^{W_0} e^{2\pi itu} F(u)\,du \ , \quad t \in \mathbb{R}^1 ,$$

is called L^1-bandlimited to W_0 if $F \in L^1[-W_0,W_0]$, and is called conventionally or L^2-bandlimited to W_0 if $F \in L^2[-W_0,W_0]$. In both cases the sampling representation (1.2.1) is valid for all $W \geq W_0$. The series in (1.2.1) converges uniformly on compact sets for L^1-bandlimited functions, and for conventionally bandlimited functions it converges in $L^2(\mathbb{R}^1)$ as well as uniformly on $\mathbb{R}^1$.

In reconstructing a function (signal) f from a periodic set of samples (sampling at a constant rate), errors of the following types may arise:

(1) f is bandlimited but it is observed only over a finite interval, and hence only a finite number of samples can be used for its reconstruction. This type of error is called a truncation error.

(2) f is bandlimited, but there are observation errors, so that the observed samples are not $f(\frac{n}{2W})$ but $f(\frac{n}{2W}) + \varepsilon_n$, where $\{\varepsilon_n\}$ are random variables.

(3) f is not bandlimited (or not bandlimited to the frequency it is sampled at), and yet a reconstruction of the type of the sampling theorem is attempted.

1.3. Summary.

In this study the samples are assumed to be error free, and only errors of type (1) and (3) (possibly combined) are considered. Furthermore, the area of inquiry is limited to constant rate (or uniform) sampling schemes. It should be mentioned, though, that non-uniform sampling schemes, as well as random sampling schemes, are of considerable interest in communication and information processing.

Chapter II deals with sampling approximations as well as error estimates of functions and stochastic processes which are not bandlimited. In Section 2.2 a sampling approximation is derived for processes which are not necessarily weakly stationary. In Section 2.3 the rate of convergence in the finite sampling approximation for time limited processes is estimated; the convergence holds both in the mean square sense and with probability one. Finite sampling approximations for functions which are Fourier transforms of finite measures are derived in Section 2.4, along with error estimates under various conditions. These results are then extended to various types of stochastic processes. In Section 2.5 Walsh functions are used to derive sampling approximations for functions which are not necessarily continuous and for processes which are not necessarily mean square continuous, as well as error estimates under further conditions.

In Chapter III we turn to the problem of sampling expansions of bounded linear operators acting on various classes of bandlimited functions and stochastic processes. The merit of these representations lies in the fact that the image of a function under the operator is expressed or represented in terms of the samples of the function rather than the samples of its image. Section 3.2 deals with bounded linear operators acting on classes of functions bandlimited in the sense of Zakai (1965) and of Lee (1976a), and Section 3.3 considers bounded linear operators acting on classes of functions with wandering spectra in Lloyd's sense (Lloyd, 1959).

Finally, in Chapter IV, distributions and random distributions are considered. Section 4.3 deals with sampling representations for distributions with compact spectra and shows that a distribution with compact spectrum can be reconstructed using samples of its Fourier transform (regarded as a function). In Section 4.4 similar results are derived for certain types of random distributions.

1.4. Notation.

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, $n \geq 1$, and the complex numbers by $\mathbb{C}$. The class of all absolutely integrable or square integrable functions on $\mathbb{R}^n$, with respect to the measure μ is denoted by $L^1(\mu)$ or $L^2(\mu)$, and when the measure is Lebesgue measure by $L^1(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$. The complement of a set A is denoted by A^c. Finally, the symbol $\square$ is used to signal the end of each proof.

CHAPTER II

Sampling Approximation for Non-Bandlimited Functions and Processes

2.1. Introduction.

In this chapter we consider the problem of deriving sampling approximations and their rate of convergence for functions and stochastic processes which are not necessarily bandlimited. The merit of these approximations lies in the fact that in many practical engineering systems, such as causal systems and time-limited systems, the signals under consideration are not bandlimited. It is thus of interest to consider non-bandlimited signals.

In Section 2.2 we derive a sampling approximation for stochastic processes which are not necessarily stationary or bandlimited. Section 2.3 deals with the rate of convergence in the finite sampling approximation for time-limited stochastic processes. In Section 2.4 a finite sampling approximation is derived for functions which are the Fourier transforms of finite signed measures, as well as the rate of convergence. This result is extended to weakly stationary, harmonizable, and certain stable processes. In Section 2.5 a sampling approximation using Walsh functions is derived for functions which are not necessarily continuous and for processes which are not necessarily mean-square continuous.

2.2. Sampling Approximations for Non-Stationary Stochastic Processes.

In this section we prove the stochastic process analogue of the sampling approximation theorem proved for (deterministic) non-band-limited functions of one variable by Brown (1967). The n-variable version of Brown's result is stated below as Theorem 2.2.1 .

For fixed $t \in \mathbb{R}^1$ and $W > 0$, denote by $e_W^{2\pi itu}$ the 2W-periodic extension of the function $e^{2\pi itu}$, $-W < u \leq W$, to the real line. Its Fourier series is given in the following well known result.

Lemma 2.2.1. For any fixed $t \in \mathbb{R}^1$ and $W > 0$

$$e_W^{2\pi itu} = \sum_{n=-\infty}^{\infty} e^{\pi i \frac{n}{W} u} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \quad \text{(a.e. (u))}$$

and the Fourier series converges boundedly.

Theorem 2.2.1. (Brown, 1967). If $F \in L^1(\mathbb{R}^n)$, and $f(t) = \int_{\mathbb{R}^n} F(u) e^{2\pi itu} d\mu_n(u)$, $t \in \mathbb{R}^n$, where $tu = t_1u_1 + \ldots + t_nu_n$, and $d\mu_n(u) = du_1 \ldots du_n$, then for each $t \in \mathbb{R}^n$ and $W > 0$,

$$f_W(t) := \sum_{k_1,\ldots,k_n=-\infty}^{\infty} f\left(\frac{k_1}{2W}, \ldots, \frac{k_n}{2W}\right) \prod_{j=1}^{n} \frac{\sin \pi(2Wt_j - k_j)}{\pi(2Wt_j - k_j)}$$

$$= \int_{\mathbb{R}^n} e_W^{2\pi itu} F(u) d\mu_n(u)$$

where $e_W^{2\pi itu} = \prod_{j=1}^{n} e_W^{2\pi it_ju_j}$, and

$$|f(t) - f_W(t)| \leq 2^n \int_{A(W)} |F| d\mu_n$$

where $A(W) = \{u \in \mathbb{R}^n : |u_j| > W;\ j=1,2,\ldots,n\}$. Thus for each $t \in \mathbb{R}^n$,

$$f(t) = \lim_{W\to\infty} f_W(t) \ .$$

If, in addition, for some $W > 0$, $F(u) = 0$ for almost all $u \in \mathbb{R}^n$ with $|u_j| > W$, $j=1,2,\ldots,n$, then we have the classical sampling expansion

$$f(t) = f_W(t) \ .$$

The following will be needed in stating the stochastic process analogue of Theorem (2.2.1). Let $\{x(t),\ t \in \mathbb{R}^1\}$ be a second order mean square continuous stochastic process with correlation function $R(t,s) = E[x(t)\overline{x}(s)]$. Assume that $R \in L^1(\mathbb{R}^2)$ and $R_t(\cdot) \triangleq R(t,\cdot) \in L^1(\mathbb{R}^1)$. The Fourier transform of R is denoted by

$$\hat{R}(u,v) = \iint_{-\infty}^{\infty} R(t,s)e^{-2\pi i(tu+sv)}dtds \ , \quad u,v \in \mathbb{R}^1 \ .$$

Since R is continuous and Lebesgue integrable on $\mathbb{R}^2$ it is also Riemann integrable on $\mathbb{R}^2$, and thus the quadratic mean integral

$$y(u) = R\int_{-\infty}^{\infty} x(t)e^{-2\pi itu}dt \ , \ u \in \mathbb{R}^1 \ ,$$

exists and defines a mean square continuous stochastic process (since $E[y(u)\overline{y}(v)] = \hat{R}(u,-v)$ is continuous in both u and v). Since $R_t \in L^1(\mathbb{R}^1)$ for all $t \in \mathbb{R}^1$, its Fourier transform

$$\hat{R}_t(v) \triangleq \int R_t(s)e^{-2\pi isv}ds \ , \ v \in \mathbb{R}^1 \ ,$$

is well defined for all $t \in \mathbb{R}^1$. In fact, $\hat{R}_t(v) = E[x(t)\overline{y}(-v)]$ for all $t,v \in \mathbb{R}^1$, and since both $x(t)$ and $y(v)$ are mean square continuous, then

$\hat{R}_t(v)$ is continuous in both t,v. Now we also assume that $\hat{R}\epsilon L^1(\mathbb{R}^2)$, and $\hat{R}(\cdot,v)\epsilon L^1(\mathbb{R}^1)$ for all $v\epsilon\mathbb{R}^1$. Since $\hat{R}_t(v)$ is continuous in both t and v, it follows that

$$\hat{R}_t(v) = \int \hat{R}(u,v)e^{2\pi itu}du ,$$

for all $t,v\epsilon\mathbb{R}^1$. Hence

$$\int |\hat{R}_t(v)|d\mu_1(v) \leq \iint |\hat{R}(u,v)|dudv < \infty ,$$

for every $t\epsilon\mathbb{R}^1$, i.e. $\hat{R}_t(v)$ is a continuous function in both t,v, and is integrable with respect to v for every $t\epsilon\mathbb{R}^1$.

Remark. It should be noted that if $R\epsilon L^1(\mathbb{R}^2)$ and $R(t,s) \geq 0$ for all $t,s\epsilon\mathbb{R}^1$, then the condition $R_t\epsilon L^1(\mathbb{R}^1)$ is satisfied for all $t\epsilon\mathbb{R}^1$. Indeed, the two conditions, $R\epsilon L^1(\mathbb{R}^2)$ and R is continuous on $\mathbb{R}^2$, imply that the Riemann integral $R\iint R(t,s)$ exists and is finite. It follows that the Riemann mean integral $R\int E[\xi\cdot\overline{x}(s)]ds$ exists and is finite, and

$$E[\xi\cdot R\int\overline{x}(s)ds] = R\int E[\xi\cdot\overline{x}(s)]ds .$$

Taking $\xi = x(t)$ gives that for each $t\epsilon\mathbb{R}^1$, the Riemann integral $R\int R(t,s)ds$ exists and is finite. Since $R(t,\cdot) \geq 0$ for all $t\epsilon\mathbb{R}^1$, it follows that the integral exists as a Lebesgue integral as well, i.e.

$$\int|R(t,s)|ds < \infty \quad \text{for all } t\epsilon\mathbb{R}^1 .$$

We now establish the analogue of Brown's result (Theorem 2.2.1) for a non-stationary (non-bandlimited) second order stochastic process.

<u>Theorem 2.2.2.</u> Let $\{x(t),\ t\in\mathbb{R}^1\}$ be a second order mean square continuous stochastic process with correlation function R. Assume that $R\in L^1(\mathbb{R}^2)$, $R(t,\cdot)\in L^1(\mathbb{R}^1)$ for all $t\in\mathbb{R}^1$, and $\hat{R}\in L^1(\mathbb{R}^2)$, $\hat{R}(\cdot,v)\in L^1(\mathbb{R}^1)$ for all $v\in\mathbb{R}^1$. Then for each fixed $t\in\mathbb{R}^1$ and $W > 0$,

$$(2.2.1)\qquad x_W(t) := \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right)\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)} = R\int e_W^{2\pi itu}\, y(u)\,du$$

where the equality is a.s., the series converges in quadratic mean, and $y(u)$ is defined by the quadratic mean integral

$$y(u) = R\int e^{-2\pi itu}x(t)\,dt\ ,\quad u\in\mathbb{R}^1\ .$$

Also for each $t\in\mathbb{R}^1$ and $W > 0$, the error

$$(2.2.2)\qquad e_W^2(t) := E|x(t)-x_W(t)|^2 \le 4\int_{|u|>W}\int_{|v|>W}|\hat{R}(u,v)|\,du\,dv\ ,$$

and thus for each $t\in\mathbb{R}^1$,

$$(2.2.3)\qquad x(t) = \lim_{W\to\infty} x_W(t)$$

is quadratic mean. If, in addition, $\hat{R}(u,v) = 0$ for almost all $|u|,\ |v| > W$, for some $W > 0$, then $x(t) = x_W(t)$, which is the classical sampling theorem for non-stationary bandlimited processes.

<u>Proof.</u> Let us denote $\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$ by $g_n(t;W)$. It has already been noted that the quadratic mean integral $y(u)$ defines a mean square continuous stochastic process with correlation function $E[y(u),\overline{y}(v)] = \hat{R}(u,-v)$. Also, the quadratic mean integral $R\int e_W^{2\pi itu}y(u)\,du$, $t\in\mathbb{R}^1$, exists since the integral

$$\iint e_W^{2\pi itu}\, e_W^{-2\pi itv} E[y(u)\overline{y}(v)]\,du\,dv\ ,$$

exists (and is finite) as a Riemann and as a Lebesgue integral (for $\hat{R}(y,v)$ is continuous and Lebesgue integrable on $\mathbb{R}^2$). Let

$$e_N^2(t;W) := E\left|\sum_{n=-N}^{N} x(\tfrac{n}{2W})g_n(t;W) - R\!\int e_W^{2\pi itu} y(u)\,du\right|^2$$

$$= \sum_{n=-N}^{N}\sum_{m=-N}^{N} R(\tfrac{n}{2W}, \tfrac{m}{2W})g_n(t;W)g_m(t;W)$$

$$- \sum_{n=-N}^{N} g_n(t;W)\ R\!\int e_W^{-2\pi itu} E[x(\tfrac{n}{2W})\cdot\overline{y}(u)]\,du$$

$$- \sum_{n=-N}^{N} g_n(t;W)\ R\!\int e_W^{2\pi itu}\ E[\overline{x}(\tfrac{n}{2W})y(u)]\ du$$

$$+ E\left|R\!\int e_W^{2\pi itu} y(u)\,du\right|^2 . \tag{2.2.4}$$

We notice that

$$E\left|R\!\int e_W^{2\pi itu} y(u)\,du\right|^2 = R\!\iint e_W^{2\pi itu}\ e_W^{-2\pi itv}\hat{R}(u,-v)\,du\,dv$$

$$= R\!\iint e_W^{2\pi itu}\ e_W^{2\pi itv}\ \hat{R}(u,v)\ du\ dv\ .$$

Since $\hat{R}$ is continuous and Lebesgue integrable, then by Theorem (2.2.1)

$$\sum_{n=-N}^{N}\sum_{m=-N}^{N} R(\tfrac{n}{2W}, \tfrac{m}{2W})g_n(t;W)g_m(t;W) \xrightarrow[N\to\infty]{} \iint e_W^{2\pi itu} e^{2\pi itv}\hat{R}(u,v)\,du\ dv\ .$$

Since $\hat{R}_{\frac{n}{2W}}$ is continuous and integrable, we have

$$\sum_{n=-N}^{N} g_n(t;W)R\!\int e_W^{-2\pi itv}\ E[x(\tfrac{n}{2W})\overline{y}(v)]\,dv$$

$$= \sum_{n=-N}^{N} g_n(t;W)\int e^{-2\pi itv} R_{\frac{n}{2W}}(-v)\,dv$$

$$= \iint \Big[\sum_{n=-N}^{N} e^{\pi i \frac{n}{W} u} g_n(t;W)\Big] e_W^{2\pi itv} \hat{R}(u,v)\,du\,dv$$

$$\longrightarrow \iint e_W^{2\pi itu} e_W^{2\pi itv} \hat{R}(u,v)\,du\,dv \ ,$$

where we used Lemma (2.2.1) and the dominated convergence theorem. Similarly the same is true for the third term in expression (2.2.4), and thus $e_N^2(t;W) \to 0$ as $N \to \infty$ for all $t \in \mathbb{R}^1$, $W > 0$, proving (2.2.1).

To prove (2.2.2), we have

$$e_W^2(t) = E|x(t) - \mathcal{R}\int e_W^{2\pi itu} y(u)\,du|^2$$

$$= R(t,t) - \mathcal{R}\int e_W^{-2\pi itu} E[x(t)\overline{y}(u)]\,du$$

$$- \mathcal{R}\int e_W^{2\pi itv} E[\overline{x}(t)y(v)]\,dv$$

$$+ \mathcal{R}\iint e_W^{2\pi itu} e_W^{-2\pi itv} \hat{R}(u,-v)\,du\,dv \ . \tag{2.2.5}$$

But

$$\mathcal{R}\int e_W^{-2\pi itu} E[x(t)\overline{y}(u)]\,du = \int e_W^{-2\pi itu} \hat{R}_t(-u)\,du$$

$$= \iint e_W^{2\pi itu} e^{2\pi itv} \hat{R}(u,v)\,du\,dv \ ,$$

and similarly for the third term in (2.2.5). Hence

$$e_W^2(t) = \iint (e^{2\pi itu} e^{2\pi itv} - e_W^{2\pi itu} e^{2\pi itv} - e^{2\pi itu} e_W^{2\pi itv}$$

$$+ e_W^{2\pi itu} e_W^{2\pi itv})\ \hat{R}(u,v)\,du\,dv$$

$$= \iint_{|u|,|v|>W} (e^{2\pi itu} - e_W^{2\pi itu})(e^{2\pi itv} - e_W^{2\pi itv})\ \hat{R}(u,v)\,du\,dv \ . \tag{2.2.6}$$

Now the inequality $|e^{2\pi itu} - e^{2\pi itv}| \le 2$, we obtain

$$e_W^2(t) \le 4 \int_{|u|,|v|>W} \int |\hat{R}(u,v)|du\,dv\ ,$$

proving (2.2.2), and (2.2.3) follows from the fact that $\hat{R}\epsilon L^2(\mathbb{R}^2)$.

□

Remark. Theorem 2.2.2 holds for multiparameter processes $\{x(t),\ t\epsilon\mathbb{R}^n\}$, $n \ge 1$.

Remark. The bound in (2.2.2) may be written in a different form as follows. From (2.2.6) we have

$$e_W^2(t) \le \iint |e^{2\pi itu} - e_W^{2\pi itu}|\cdot|e^{2\pi itv} - e_W^{2\pi itv}|\cdot|\hat{R}(u,v)|du\,dv$$

$$= 4 \sum_{k,j=-\infty}^{\infty} |\sin 2\pi kWt|\cdot|\sin 2\pi jWt|\cdot|\hat{R}(u,v)|du\,dv\ .$$

The term $k = 0 = j$ is always zero, so that only the integral of $|\hat{R}|$ over the remaining squares enters into the sum. Also, for any integer n, $e_W^2(\frac{n}{2W}) = 0$ and thus $x(\frac{n}{2W}) = x_W(\frac{n}{2W})$.

2.3. The Rate of Convergence in the Finite Sampling Approximation for Time-Limited Stochastic Processes.

Butzer and Splettstösser (1977) derived a sampling approximation for time-limited functions which, in fact, is a special case of Brown's result (Theorem 2.2.1), and determined the rate of convergence of the approximating series under certain conditions. These results are summarized in the following theorem.

Theorem 2.3.1. (Butzer and Splettstösser, 1977). Let f be a continuous function defined on $\mathbb{R}^1$ such that, for some (fixed) $T > 0$, $f(t) = 0$ for all $|t| > T$ and $\hat{f}\epsilon L^1(\mathbb{R}^1)$. Then

$$f(t) = \lim_{W\to\infty} \sum_{n=-N(W)}^{N(W)} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}, \quad t\in\mathbb{R}^1, \tag{2.3.1}$$

where $N(W) = [2WT]$, the largest integer less than or equal to $2WT$. If, in addition, $f^{(r)}\in\text{Lip } \alpha$, $0 < \alpha \le 1$ for some (fixed) $r\in\{1,2,\ldots\}$, with constant L_r, then for $W > \frac{r+1}{2}$

$$\left|f(t) - \sum_{n=-N(W)}^{N(W)} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|$$

$$\le \frac{(T+1)L_r}{2^{r-1}(r+\alpha-1)} \frac{1}{W^{r+\alpha-1}}, \quad t\in\mathbb{R}^1 . \tag{2.3.2}$$

Here Lip α, $0 < \alpha \le 1$, is the Lipschitz class of continuous functions f on $\mathbb{R}^1$ for which there exists a constant $0 < L < \infty$ such that $\sup_{t\in\mathbb{R}^1} |f(t+h)-f(t)| \le L|h|^\alpha$, $h\in\mathbb{R}^1$.

The following result is the analogue of Theorem (2.3.1) for second order processes which are not necessarily stationary. First we introduce some notation. Let $C(\mathbb{R}^2)$ be the class of all continuous functions on $\mathbb{R}^2$, and $||R|| = \sup_{t,s\in\mathbb{R}^1} |R(t,s)|$. Let $\text{Lip}^{(2)}\alpha$, $\alpha\in(0,1]$, be the class of all functions $R\in C(\mathbb{R}^2)$ for which there exists a constant $0 < L < \infty$ such that

$$||\Delta_{h,g}R|| \le L|h|^\alpha|g|^\alpha, \quad h,g\in\mathbb{R}^1,$$

where $\Delta_{h,g}R(t,s) = R(t+h, s+g) - R(t+h,s) - R(t,s+g) + R(t,s)$. The first modulus of continuity is defined by

$$\omega_1(\delta,\lambda;R) = \sup\{||\Delta_{h,g}R||: |h| \le \delta, |g| \le \lambda\}$$

where $\delta > 0$, $\lambda > 0$. Similarly, the r-th modulus of continuity of R, r is a positive integer, is defined by

$$\omega_r(\delta,\lambda;R) = \sup\{||\Delta^r_{h,g}R||: \ |h|\le\delta, |g|\le\lambda\},$$

where

$$\Delta^r_{h,g}R(t,s) = \sum_{k=0}^{r}\sum_{\ell=0}^{r}(-1)^{k+\ell}\binom{r}{k}\binom{r}{\ell}R(t+kh,\ s+\ell g) \ .$$

It should be noted that for $\alpha\in(0,1]$,

$$\mathrm{Lip}^{(2)}\alpha = \{R\in C(\mathbb{R}^2):\ \omega_1(\delta,\lambda;R) \le L\delta^\alpha\lambda^\alpha;\ \delta > 0,\ \lambda > 0\}.$$

<u>Theorem 2.3.2.</u> Let $x = \{x(t),\ t\in\mathbb{R}^1\}$ be a second order mean-square continuous stochastic process with correlation function R such that, for some (fixed) $T > 0$, $R(t,t) = 0$ for all $|t| > T$ (time limited process) and $\hat{R}\in L^1(\mathbb{R}^2)$. Then for all $t\in\mathbb{R}^1$ and $W > 0$,

$$e^2_W(t):= E|x(t) - \sum_{n=-N(W)}^{N(W)} x(\tfrac{n}{2W})\ \frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}|^2$$

$$\text{(2.3.3)} \qquad \le 4\int_{|u|,|v|>W}\int |\hat{R}(u,v)|du\ dv\ ,$$

and thus

$$\text{(2.3.4)} \qquad x(t) = \lim_{W\to\infty}\sum_{n=-N(W)}^{N(W)} x(\tfrac{n}{2W})\ \frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}\ ,$$

in the mean square sense, where $N(W) = [2WT]$. If, in addition, for some positive integer r and some $\alpha\in(0,1]$,

$$\frac{\partial^{2r}R(t,s)}{\partial t^r\partial s^r}\in\mathrm{Lip}^{(2)}\alpha$$

with constant L_r, then for every $t\in\mathbb{R}^1$ and $W > 0$ we have

$$\text{(2.3.5)} \qquad e^2_W(t) \le \frac{L_r}{2^{2(r+\alpha-1)}(r+\alpha-1)^2}\,(2T + \frac{r+1}{2W})^2\frac{1}{W^{2(r+\alpha-1)}}\ .$$

<u>Proof.</u> (2.3.3) and (2.3.4) are special cases of (2.2.2) and (2.2.3). To show (2.3.5) notice that, for all $u,v \neq 0$, we have

$$(-1)^{n+m}\hat{R}(u,v) = \iint R(t + \frac{n}{2u}, s + \frac{m}{2v})e^{-2\pi i(tu+sv)}dt\, ds ,$$

and thus for all $u,v > W$,

$$2^{2(r+1)}\hat{R}(u,v) = \sum_{n=0}^{r+1}\sum_{m=0}^{r+1}\binom{r+1}{n}\binom{r+1}{m}\hat{R}(u,v)$$

$$= \int_{-T-\frac{r+1}{2W}}^{T}\int_{-T-\frac{r+1}{2W}}^{T}\sum_{n=0}^{r+1}\sum_{m=0}^{r+1}(-1)^{n+m}\binom{r+1}{n}\binom{r+1}{m}$$

$$\cdot\, R(t + \frac{n}{2u}, s + \frac{m}{2v})e^{-2\pi i(tu+sv)}dt\, ds$$

$$= \int_{-T-\frac{r+1}{2W}}^{T}\int_{-T-\frac{r+1}{2W}}^{T}\{\Delta^{r+1}_{\frac{1}{2u},\frac{1}{2v}} R(t,s)\}e^{-2\pi i(tu+sv)}dt\, ds,$$

which yields the inequality

$$2^{2(r+1)}|\hat{R}(u,v)| \le \int_{-T-\frac{r+1}{2W}}^{T}\int_{-T-\frac{r+1}{2W}}^{T}|\Delta^{r+1}_{\frac{1}{2u},\frac{1}{2v}}R(t,s)|dt\, ds$$

$$\le (2T + \frac{r+1}{2W})^2\omega_{r+1}(\frac{1}{2n}, \frac{1}{2v};R) .$$

Now since $\frac{\partial^{2r}R(t,s)}{\partial^r\, \partial s^r} \in \mathrm{Lip}^{(2)}\alpha$, it can be easily shown that, for any positive integer j,

$$\omega_{r+j}(\delta,\lambda;R) \le \delta^r\lambda^r\omega_j(\delta,\lambda, \frac{\partial^{2r}R(t,s)}{\partial t^r\, \partial s^r}) ,$$

and hence for all $u,v > W$,

$$|\hat{R}(u,v)| \le 2^{-2(r+1)}L_r(2T + \frac{r+1}{2W})^2\frac{1}{(2u)^{r+\alpha}}\frac{1}{(2v)^{r+\alpha}} . \tag{2.3.6}$$

From (2.3.3) and (2.3.6) we have

$$e_W^2(t) \le 4 \int_{|u|,|v|>W} \int |\hat{R}(u,v)|\,du\,dv$$

$$\le 2^{-2(2r+\alpha)} L_r \left(2T + \frac{r+1}{2W}\right)^2 \left(2 \int_{u>W} \frac{du}{u^{r+\alpha}}\right)^2$$

$$= 2^{-2(2r+\alpha-1)} L_r \left(2T + \frac{r+1}{2W}\right)^2 \frac{1}{(r+\alpha-1)^2} \frac{1}{W^{2(r+\alpha-1)}}$$

$$= \frac{L_r}{2^{2(2r+\alpha-1)}(r+\alpha-1)^2} \left(2T + \frac{r+1}{2W}\right)^2 \frac{1}{W^{2(r+\alpha-1)}} ,$$

proving (2.3.5). □

Under the conditions of Theorem 2.3.2 the approximating sequence $x_W(t)$, in fact, converges to $x(t)$ with probability one for each fixed $t \in \mathbb{R}^1$. The rate of convergence is given in the following corollary.

Corollary 2.3.1. Let x be as in Theorem 2.3.2 and assume that $r \ge 1$ when $1/2 < \alpha < 1$ and that $r \ge 2$ when $0 < \alpha \le 1/2$. Then for a separable version of x and each $t \in \mathbb{R}^1$,

$$(2.3.7) \qquad W_0^\gamma \sup_{W>W_0} \left| x(t) - \sum_{n=-N(W)}^{N(W)} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \right| \to 0 \text{ a.s. as } W_0 \to \infty,$$

where $0 < \gamma < r+\alpha - \frac{3}{2}$.

Proof. Consider a separable version of x. For $W \ge 1$ put $W = \frac{1}{u}$ and for each fixed $t \in \mathbb{R}^1$, define X_u, $u \in [0,1]$, by

$$X_u = \begin{cases} x(t) & , u = 0 \\ x_{\frac{1}{u}}(t) & , 0 < u \le 1 , \end{cases}$$

where $x_{\frac{1}{u}}(t) = x_W(t) = \sum_{n=-N(W)}^{N(W)} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$. Then X is separable (in u). From (2.3.5), we have

$$E|X_0-X_u|^2 = E|x(t) - x_W(t)|^2 \le Cu^{1+\beta},$$

where $C = 2^{-2(r+\alpha-1)}(2T+1)^2 L_r(r+\alpha-1)^{-2}$ and $\beta = 2(r+\alpha) - 3 > 0$. Thus, by Kolmogorov's theorem (see Neveu (1965, p. 97)),

$$\frac{1}{h^\gamma} \sup_{0<u<h} |X_0-X_u| \to 0 \text{ a.s. as } h \downarrow 0 , \quad 0 < \gamma < \frac{\beta}{2} ,$$

and (2.3.7) follows by putting $h = \frac{1}{W_0}$. □

2.4. Finite Sampling Approximations for Non-Bandlimited Functions and Stochastic Processes.

Theorem 2.2.1 (the case n = 1) states that, if f is the Fourier transform of an $L^1(\mathbb{R}^1)$-function, then the infinite sum

$$\sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$$

converges to f(t) pointwise everywhere as $W \to \infty$. It is of practical interest to investigate the possibility that a finite sum of the form

$$\sum_{n=-N(W)}^{N(W)} f(\tfrac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} , \tag{2.4.1}$$

where N(W) is a positive integer-valued function of W > 0, would converge to f(t) under suitable conditions on N(W), and also to determine the speed of convergence. Such a result is obtained in Theorem 2.4.3 , and its analogue for an appropriate modification of the finite sum (2.4.1) is obtained in Theorems 2.4.1 and 2.4.2, and then extended to weakly stationary, harmonizable, and certain stable processes. In all these results N(W) is required to tend to infinity, as the sampling rate W tends to infinity, fast enough so that $\frac{N(W)}{W} \to \infty$.

Theorem 2.4.1. If f is the Fourier transform of a f inite signed (or complex) measure μ on the Borel sets of the real line, i.e,,

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi itu} d\mu(u) \quad , \quad t \in \mathbb{R}^1 \; , \tag{2.4.2}$$

and if N(W) is a positive integer valued function of $W > 0$ such that $\frac{N(W)}{W} \to \infty$ as $W \to \infty$, then for each $t \in \mathbb{R}^1$,

$$f(t) = \lim_{W \to \infty} \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) f\left(\frac{n}{2W}\right) \frac{\sin \pi(2W-t-n)}{\pi(2W-t-n)}, \tag{2.4.3}$$

and the convergence is uniform on compact sets.

Proof. Consider the error

$$e_W(t) := \left| f(t) - \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \right|$$

$$= \left| \int_{-\infty}^{\infty} e^{2\pi itu} d\mu(u) - \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) \left[\int_{-\infty}^{\infty} e^{\pi i \frac{n}{W} u} d\mu(u) \right] \cdot \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \right|$$

$$\leq \int_{-\infty}^{\infty} \left| e^{2\pi itu} - \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) e^{\pi i \frac{n}{W} u} \cdot \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \right| d|\mu|(u) \tag{2.4.4}$$

$$= \int_{-\infty}^{\infty}\left|e^{2\pi itu} - \frac{1}{N(W)+1}\sum_{k=0}^{N(W)}\sum_{n=-k}^{k} e^{\pi i\frac{n}{W}u}\left[\frac{1}{2W}\int_{-W}^{W} e^{2\pi i(t-\frac{n}{2W})v}\,dv\right]\right| d|\mu|(u)$$

$$= \int_{-\infty}^{\infty}\left|e^{2\pi itu} - \frac{1}{2W}\int_{-W}^{W} e^{2\pi itv}\left[\frac{1}{N(W)+1}\sum_{k=0}^{N(W)}\sum_{n=-k}^{k} e^{-2\pi in(\frac{v-u}{2W})}\right]dv\right| d|\mu|(u)$$

$$= \int_{-\infty}^{\infty}\left|e^{2\pi itu} - \frac{1}{2W}\int_{-W}^{W} e^{2\pi itv}K_{N(W)}\left(\frac{v-u}{2W}\right)dv\right| d|\mu|(u)$$

$$= \int_{-\infty}^{\infty}\left|e^{2\pi itu} - \int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} e^{2\pi it(2Wx+u)}K_{N(W)}(x)\,dx\right| d|\mu|(u)$$

$$= \int_{-\infty}^{\infty}\left|1 - \int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} e^{4\pi iWtx}K_{N(W)}(x)\,dx\right| d|\mu|(u)$$

where $K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N}\sum_{n=-k}^{k} e^{-2\pi inx} = \frac{1}{N+1}\left[\frac{\sin \pi(N+1)x}{\sin \pi x}\right]^2$ is the Fejér kernel. Since $|\mu|(\mathbb{R}^1) < \infty$ (see Rudin (1974), p. 126), given $\varepsilon > 0$, there exists an $a = a(\varepsilon)\in(0,\infty)$ such that if $A = [-a,a]$, then $|\mu|(A^c) < \frac{\varepsilon}{2}$. Since $K_N \geq 0$ is periodic with period 1 and $\int_{-\frac{1}{2}}^{\frac{1}{2}} K_N(x)\,dx = 1$, (2.4.4) can be written as

(2.4.5) $\quad e_W(t) \leq 2|\mu|(A^c) + |\mu|(A)Q_N(t,W) \leq \varepsilon + |\mu|(\mathbb{R}^1)Q_N(t,W)$

where

(2.4.6) $\quad Q_N(t,W) = \sup_{|u|\leq a}\left|1 - \int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} e^{4\pi iWtx}K_{N(W)}(x)\,dx\right|.$

By writing $\int_{-\frac{1}{2}-y}^{\frac{1}{2}-y} = \int_{-\frac{1}{2}-y}^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{1}{2}-y}$, $(y = \frac{u}{2W})$, we obtain

$$Q_N(t,W) \leq \left| 1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{4\pi iWtx} K_{N(W)}(x)dx \right|$$

$$+ \sup_{|u|\leq a} \left| \int_{\frac{1}{2}-\frac{|u|+u}{4W}}^{\frac{1}{2}+\frac{|u|-u}{4W}} e^{4\pi iWtx} K_{N(W)}(x)dx \right|$$

$$\text{(2.4.7)} \qquad + \sup_{|u|\leq a} \left| \int_{-\frac{1}{2}-\frac{|u|+u}{4W}}^{-\frac{1}{2}+\frac{|u|-u}{4W}} e^{4\pi iWtx} K_{N(W)}(x)dx \right|$$

Let us denote by $I_k(t,W)$, $k = 1,2,3$, the three terms on the right hand side of (2.4.7) in the order they appear. For $I_2(t,W)$ we have

$$I_2(t,W) \leq \int_{\frac{1}{2}-\frac{a}{2W}}^{\frac{1}{2}+\frac{a}{2W}} K_{N(W)}(x)dx$$

$$= 2 \int_{\frac{1}{2}-\frac{a}{2W}}^{\frac{1}{2}} K_{N(W)}(x)dx \; .$$

But for $0 < x \leq \frac{1}{2}$, $K_N(x) \leq \frac{C}{\pi^2(N+1)x^2}$, where $C(\simeq \frac{\pi^2}{4})$ is a constant (see Zygmund (1959), p. 90) and thus for $W > a$

$$I_2(t,W) \leq \frac{2C}{\pi^2[N(W)+1]} \int_{\frac{1}{2}-\frac{a}{2W}}^{\frac{1}{2}} \frac{1}{x^2}\,dx$$

$$= \frac{2C}{\pi^2[N(W)+1]} \left[\frac{-1}{x}\right]_{\frac{1}{2}-\frac{a}{2W}}^{\frac{1}{2}}$$

$$= \frac{2C}{\pi^2[N(W)+1]} \frac{2a}{(W-a)} \to 0 \quad \text{as } W \to \infty .$$

Similary,

$$I_3(t,W) \leq \frac{2C}{\pi^2[N(W)+1]} \frac{2a}{(W-a)} \to 0 \quad \text{as } W \to \infty .$$

Now for $I_1(t,W)$, using the fact that $\int_0^{\frac{1}{2}} K_N(x)\,dx = \frac{1}{2}$, we have

$$I_1(t,W) = 2 \int_0^{\frac{1}{2}} (1 - \cos 4\pi Wtx) K_{N(W)}(x)\,dx .$$

Choosing any $\delta(W)$ such that $0 < \delta(W) < \frac{1}{2}$ and $\delta(W) = 0(\frac{1}{N(W)})$, we obtain

$$I_1(t,W) = 2\left(\int_0^{\delta(W)} + \int_{\delta(W)}^{\frac{1}{2}} \right) (1 - \cos 4\pi Wtx) K_{N(W)}(x)\,dx .$$

Since for all x, $K_N(x) \leq 2N+1$ (see Zygmund (1959, p. 90), then

$$\int_0^{\delta(W)} (1 - \cos 4\pi Wtx) K_{N(W)}(x)\,dx \leq 2[2N(W)+1]\delta(W) \to 0 \quad \text{as } W \to \infty.$$

Also

$$\int_{\delta(W)}^{\frac{1}{2}} (1 - \cos 4\pi Wtx) K_{N(W)}(x)\,dx = 2 \int_{\delta(W)}^{\frac{1}{2}} \sin^2 2\pi Wtx \cdot K_{N(W)}(x)\,dx$$

$$\leq \frac{2C}{\pi^2[N(W)+1]} \int_{\delta(W)}^{\frac{1}{2}} \sin^2 2\pi Wtx \cdot \frac{1}{x^2}\,dx$$

$$= \frac{4CW|t|}{\pi[N(W)+1]} \int_{2\pi W|t|\delta(W)}^{\pi W|t|} (\frac{\sin y}{y})^2 dy$$

$$\leq 2C|t| \cdot \frac{W}{N(W)+1} \to 0 \quad \text{as } W \to \infty .$$

Notice that this is the only bound which depends on t and that the dependence is linear in $|t|$. It follows that for $W > a$,

$$(2.4.8)\quad Q_N(t,W) \leq \frac{8aC}{\pi^2[N(W)+1](W-a)} + 2[2N(W)+1]\delta(W) + \frac{2C|t|W}{N(W)+1},$$

and thus $Q_N(t,W) \to 0$ as $W \to \infty$. Hence, for each fixed t and $\varepsilon > 0$, we have by (2.4.5) that $\limsup_{W\to\infty} e_W(t) \leq \varepsilon$ which implies that $e_W(t) \to 0$ as $W \to \infty$. It is clear from (2.4.8) that the convergence is uniform on compact sets. □

The following theorem gives a more concrete bound on the error committed by considering only a finite number of samples in reconstructing a function which has the representation (2.4.2).

Theorem 2.4.2. Let $f, \mu, W, N(W)$, and $e_W(t)$ be as in Theorem (2.4.1). Then for an arbitrary (but fixed) $W > 1$ and every $|t| < \frac{N(W)}{2W}$ with $t \neq \frac{n}{2W}$, $n \in \mathbb{N} = \{0, \pm 1, \pm 2, \ldots\}$ we have

$$(2.4.9)\quad e_W(t) \leq 2|\mu|(\mathbb{R}^1)\left[|t| + \frac{|\sin 2\pi Wt|}{\pi(1-e^{-\pi})N(W)}\right]\frac{W}{N(W)} + 2|\mu|\{|u|>W-1\}.$$

Proof. Fix $W > 1$ and $t \neq \frac{n}{2W}$, $n \in \mathbb{N}$. From (2.4.4) we have

$$(2.4.10)\qquad e_W(t) \leq \int_{-\infty}^{\infty} |H_{N,W}(t,u)|\, d|\mu|(u),$$

where

$$(2.4.11)\qquad H_{N,W}(t,u) = e^{2\pi itu} - \sum_{n=-N(W)}^{N(W)} \left(1 + \frac{|n|}{N(W)+1}\right) e^{\pi i \frac{n}{W} u} \cdot \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}.$$

If $A(W) = [-W+1, W-1]$, then the inequality (2.4.10) may be written as

$$(2.4.12) \qquad e_W(t) \le \left(\int_{A(W)} + \int_{A^c(W)} \right) |H_{N,W}(t,u)|\, d|\mu|(u) \ .$$

By (2.4.4) we notice that $|H_{N,W}(t,u)| = |1 - \int_{-\frac{1}{2}-(u/2W)}^{\frac{1}{2}-(u/2W)} e^{4\pi iWtx} K_{N(W)}(x)\,dx|$,

and thus $|H_{N,W}(t,u)| \le 2$ for all $t \in \mathbb{R}^1$, $u \in \mathbb{R}^1$, $N \ge 1$, and $W > 0$. From (2.4.12) we obtain

$$(2.4.13) \qquad e_W(t) \le \int_{A(W)} |H_{N,W}(t,u)|\, d|\mu|(u) + 2|\mu|(A^c(W))$$

$$\le |\mu|(\mathbb{R}^1) P_N(t,W) + 2|\mu|(A^c(W)) \ ,$$

where $P_N(t,W) = \sup_{u \in A(W)} |H_{N,W}(t,u)|$.

Now consider the integral

$$I_{N,W}(t,u) = \frac{1}{2\pi i} \int_{C_{N,W}} \left(1 - \frac{2W|z|}{N(W)+1}\right) \frac{e^{2\pi iuz}}{(z-t)\sin 2\pi Wz}\, dz, \quad |t| < \frac{N(W)}{2W}$$

where $C_{N,W} = \{z \in \mathbb{C}: |z| = \frac{1}{2W}[N(W)+1]\}$. From Cauchy's residue theorem, we have

$$(2.4.14) \qquad I_{N,W}(t,u) = \left(1 - \frac{2W|t|}{N(W)+1}\right) \frac{e^{2\pi itu}}{\sin 2\pi Wt}$$

$$+ \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) \frac{e^{\pi i \frac{n}{W} u}}{\left(\frac{n}{2W} - t\right)} \frac{(-1)^n}{2\pi W} \ ,$$

and thus from (2.4.11)

$$H_{N,W}(t,u) = \frac{2W|t|}{N(W)+1} e^{2\pi itu} + \sin 2\pi Wt \cdot I_{N,W}(t,u) \ , \quad |t| < \frac{N(W)}{2W} \ .$$

Hence

$$(2.4.15) \qquad |H_{N,W}(t,u)| \le \frac{2W|t|}{N(W)+1} + |\sin 2\pi Wt| \cdot |I_{N,W}(t,u)|, \quad |t| < \frac{N(W)}{2W} .$$

Since $e^{2\pi iuz}$ is an entire function of exponential type $2\pi|u|$, then from a result by Piranashvili (1967) we have for $u\epsilon A(W)$ that,

$$|I_{N,W}(t,u)| \le \frac{1}{2[N(W)+1]} \frac{4}{\pi(1-e^{-\pi})N(W)} \frac{W}{W-|u|} ,$$

and from (2.4.15), that for $W > 1$

$$|H_{N,W}(t,u)| \le \frac{2W|t|}{N(W)} + \frac{2W|\sin 2\pi Wt|}{\pi(1-e^{-\pi})[N(W)]^2} \frac{1}{(W-|u|)} \quad , u\epsilon A(W), \ |t| < \frac{N(W)}{2W}$$

and hence for $W > 1$ and $\frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W}$,

$$(2.4.16) \qquad P_N(t,W) \le \frac{2W|t|}{N(W)} + \frac{2W|\sin 2 \ Wt|}{\pi(1-e^{-\pi})[N(W)]^2} \quad .$$

Thus from (2.4.13), we obtain

$$e_W(t) \le 2|\mu|(\mathbb{R}^1)[|t| + \frac{|\sin 2\pi Wt|}{(1-e^{-\pi})N(W)}][\frac{W}{N(W)}] + 2|\mu|(A^c(W)) \ ,$$

for $W > 1$ and $\frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W}$, $\quad n\epsilon \mathbb{N}$. □

<u>Remark 2.4.1.</u> We now comment on the two bounds obtained in Theorems 2.4.1 and 2.4.2 . From (2.4.5) and (2.4.8) in the proof of Theorem 2.4.1 we have,

$$e_W(t) \le 2|\mu|\{|u|>a\} + 2|\mu|(\mathbb{R}^1)\{\frac{C|t|W}{N(W)+1} + \frac{4Ca}{\pi^2[N(W)+1](W-a)}$$

$$+ [N(W)+1]\delta(W)\}$$

$$= 2|\mu|\{|u|>a\} + 2|\mu|(\mathbb{R}^1)\{C|t|\frac{N(W)}{N(W)+1} + \frac{4C}{\pi^2}\frac{N(W)}{N(W)+1}\cdot\frac{a}{W(W-a)}$$

$$+ \frac{N(W)[N(W)+1]}{W}\,\delta(W)\}\frac{W}{N(W)}$$

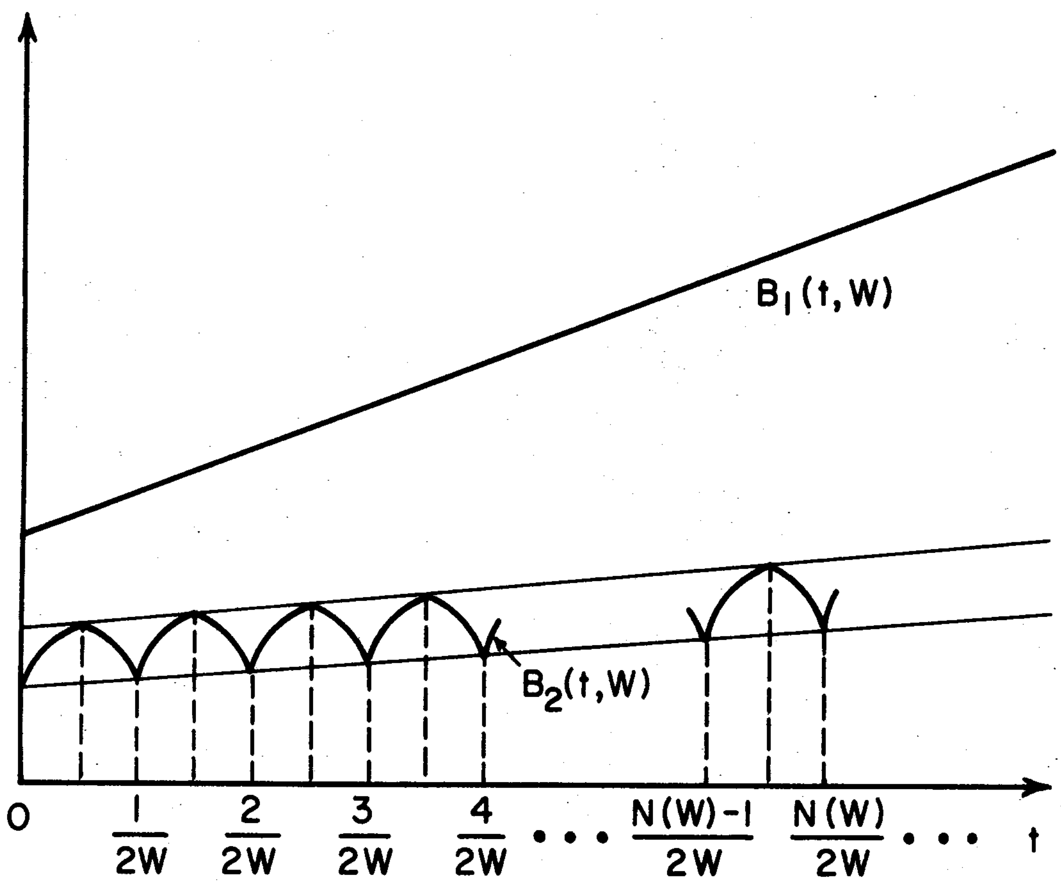
$$=: B_1(t,W)$$

for all $W > a$ and $t\in\mathbb{R}^1$. From Theorem (2.4.2) we have that for each fixed $W > 1$ and t with $\frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W}$,

$$e_W(t) \leq 2|\mu|\{|u|>W-1\} + 2|\mu|(\mathbb{R}^1)\{|t| + \frac{|\sin 2\pi Wt|}{\pi(1-e^{-\pi})N(W)}\}\frac{W}{N(W)}$$

$$=: B_2(t,W)\ .$$

In the bound B_1, C could be taken to be $\frac{\pi^2}{4}$ and $\delta(W)$ may be chosen as small as desirable so that the fourth term in the expression of B_1 may be omitted (since it can be made arbitrarily smaller than the other three terms by an appropriate choice of $\delta(W)$). The bound B_1 has the advantage over the bound B_2 of holding for all $t\in\mathbb{R}^1$. It should be noted that even though for each fixed t, $B_1(t,W) \to 0$ as $W \to \infty$, for t's large relative to W, namely for $|t| > \frac{1}{2C}\frac{N(W)+1}{W}$, the bound B_1 (through its second term) is larger than $|\mu|(\mathbb{R}^1)$ which is an upper bound on $|f(t)|$. Large as this bound on the error may seem, there is no smaller bound available for such t's and, in fact, it seems quite likely that at least for certain t's the approximation error would be of the magnitude of $|f(t)|$. The bound B_2 holds only on the interval $|t| < \frac{N(W)}{2W}$ excluding the sample points $\frac{n}{2W}$ (at which we know anyway that the approximation error $e_W(t)$ vanishes). However, it has the following advantages over the bound B_1. It contains one term which vanishes at the sampling points and is thus very small near the sampling points; thus the bound B_2 is more sensitive than B_1 near sampling points. Also, the second term of B_2 is always smaller than the second term of B_1, and putting $a = W-1$ to equalize the first term of both bounds, it is easily seen that the third term of B_2 is always smaller than the third term of B_1. In conclusion, in the restricted region where it holds the bound B_2 is lower and more sensitive to the location of the sampling points than the bound B_1, while, of course, B_1 provides a bound even where B_1 does not hold. A typical plot of the two bounds for fixed $W > 1$ (and $a = W-1$) is as follows.



Under further conditions a function of the form (2.4.2) can be approximated by the simpler finite sums in (2.4.1) rather than those in (2.4.3).

Theorem 2.4.3. Let f be the Fourier transform of a finite signed (or complex) measure μ, on the Borel sets of the real line, which satisfies

$$\int_{-\infty}^{\infty} \phi(u)\,d|\mu|(u) < \infty \quad , \tag{2.4.17}$$

where ϕ is a symmetric non-negative function, strictly increasing on $(0,\infty)$, and such that $\phi(u) \to \infty$ as $u \to \infty$. Let $N(W)$ be a positive integer valued function of $W > 0$. Then for an arbitrary (but fixed) $W > 1$, and $|t| < \frac{N(W)}{2W}$ such that $t \neq \frac{n}{2W}$, $n \in \mathbb{N}$, we have

$$e_W(t) = \left| f(t) - \sum_{n=-N(W)}^{N(W)} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \right|$$

$$\leq 2\left\{\frac{1+\ln \pi[N(W)+\frac{1}{2}]}{\phi(W-1)}\right\}\Phi(W) + 4|\mu|(\mathbb{R}^1)\frac{|\sin 2\pi Wt|}{\pi(1-e^{-\pi})} \cdot \frac{W}{N(W)} =: B(t,W), \tag{2.4.18}$$

where $\Phi(W) = \int_{|u|>W-1} \phi(u)\,d|\mu|(u)$. If, in addition,

(i) $\frac{N(W)}{W} \to \infty$ as $W \to \infty$, and

(ii) $N(W) = O(e^{c\phi(W-1)})$ for some $c > 0$,

then for all $t \in \mathbb{R}^1$, $B(t,W) \to 0$ and $e_W(t) \to 0$ as $W \to \infty$.

Proof. Fix $W > 1$ and $t \neq \frac{n}{2W}$, $n \in \mathbb{N}$. As in (2.4.4) we obtain

$$e_W(t) \le \int_{-\infty}^{\infty} |G_{N,W}(t,u)|\, d|\mu|(u)$$

$$= \Big(\int_{A(W)} + \int_{A^c(W)}\Big) |G_{N,W}(t,u)|\, d|\mu|(u) \ ,$$

where $A(W) = [-W+1, W-1]$ and

$$G_{N,W}(t,u) = e^{2\pi i t u} - \sum_{n=-N(W)}^{N(W)} e^{\pi i \frac{n}{W} u} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \ . \tag{2.4.20}$$

Also as in the proof of Theorem (2.4.1), we notice that

$$|G_{N,W}(t,u)| \le \left| 1 - \int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} e^{4\pi i W t x} D_{N(W)}(x)\, dx \right|$$

where $D_N(x) = \sum_{n=-N}^{N} e^{-2\pi i n x} = \frac{\sin \pi(2N+1)x}{\sin \pi x}$ is the Dirichlet kernel. D_N is an even periodic function with period 1 and $\int_{-\frac{1}{2}}^{\frac{1}{2}} |D_N(x)|\, dx \le 1 + \ell n\, \pi[N+\tfrac{1}{2}]$ (see Kufner and Kadlec (1971), p. 230). Then for all $t \in \mathbb{R}^1$

$$\int_{A^c(W)} |G_{N,W}(t,u)|\, d|\mu|(u) \le 2 \int_{A^c(W)} \frac{1}{\phi(u)} \left[\int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} |D_{N(W)}(x)|\, dx \right] \phi(u)\, d|\mu|(u)$$

$$\le \frac{2[1+\ell n[N(W)+\frac{1}{2}]]}{\phi(W-1)} \int_{A^c(W)} \phi(u)\, d|\mu|(u) \ . \tag{2.4.21}$$

Now, for $u \in A(W)$ and $\frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W}$, consider the integral

$$I_{N,W}(t,u) = \frac{1}{2\pi i} \int_{C_{N,W}} \frac{e^{2\pi i u z}}{(z-t)\sin 2\pi W z}\, dz \ ,$$

where $C_{N,W} = \{z \in \mathbb{C} : |z| = \frac{1}{2W}[N(W)+\frac{1}{2}]\}$. Proceeding as in the proof of Theorem (2.4.2), we obtain

$$|G_{N,W}(t,u)| = |\sin 2\pi Wt| \cdot |I_{N,W}(t,u)|$$

$$\leq \frac{4|\sin 2\pi Wt|}{\pi(1-e^{-\pi})N(W)} \cdot \frac{W}{W-|u|}\ ,\ u\in A(W),\ \frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W}$$

$$\leq \frac{4\ \sin 2\pi Wt}{\pi(1-e^{-\pi})}\ \frac{W}{N(W)}\ ,\ \frac{|n|}{2W} \neq |t| < \frac{N(W)}{2W} \tag{2.4.22}$$

and (2.4.8) follows from (2.4.1) and (2.4.22). It is clear from conditions (i) and (ii) that $B(t,W) \to 0$ as $W \to \infty$ for all $t\in\mathbb{R}^1$. Now fix $t\in\mathbb{R}^1$ and let $W \to \infty$. Whenever W is such that $N(W) > 2W|t|$ and $t = \frac{n}{2W}$ for some $n\in\mathbb{N}$, then we clearly have $e_W(t) = 0$ from its very definition. As $W \to \infty$ along any other values $(t \neq \frac{n}{2W}\ ,\ n\in\mathbb{N})$, then $e_W(t) \leq B(t,W) \to 0$. It follows that $e_W(t) \to 0$ as $W \to \infty$. □

<u>Remark 2.4.2</u>. It is clear that the approximation of $f(t)$ by a finite sum of the form (2.4.1) is a much more delicate problem than its approximation by the modified finite sum of the form (2.4.3). Hence the additional assumptions on the function f required in Theorem 2.4.3, compared with Theorem 2.4.1. As in Theorem 2.4.1, condition (i) of Theorem 2.4.3 puts a lower bound on the growth of $N(W)$ as the sampling rate tends to infinity. Such a condition is quite natural and anticipated, as one intuitively expects that unless enough terms are employed, the approximation may be inadequate. Condition (ii) on the other hand puts an upper bound on the growth of $N(W)$ and in this sense it may seem somewhat counterintuitive. Condition (ii) could be improved, i.e. the restriction on the growth of $N(W)$ could be weakened, if a better bound than that used in the

proof of Theorem 2.4.3 could be found for the function

$$G_{N,W}(t,u) = e^{2\pi itu} - \frac{1}{2W} \int_{-W}^{W} e^{2\pi itv} D_{N(W)}\left(\frac{u-v}{2W}\right) dv$$

as a function of W for fixed $t,u \in \mathbb{R}^1$ (or after some algebra, if the rate of growth of $\left|\int_0^{\pi/2} D_u(x)\,dx\right|$ as $u \to \infty$ can be found, instead of the rate of growth of $\int_0^{\pi/2} |D_u(x)|\,dx$ which is used in the proof of Theorem 2.4.3, where D_u is the Kirichlet kernel $D_u(x) = \frac{\sin ux}{\sin x}$). It is not known at present whether some restriction on the growth of $N(W)$ is necessary for the approximation error to tend to zero as $W \to \infty$, or whether this result holds with no upper limit on the growth of $N(W)$ as one may intuitively be tempted to expect, and condition (ii) arises only because of the specific proof used here. Also, it is not known at present whether a bound similar to B_1 (see Remark 2.4.1) holds, in this case, for all $t \in \mathbb{R}^1$.

The preceding results are now extended to weakly stationary stochastic processes (Theorem 2.4.4), harmonizable processes (Theorem 2.4.5), and to certain stable processes (Theorem 2.4.6).

Theorem 2.4.4. If $x = \{x(t),\ t \in \mathbb{R}^1\}$ is a mean square continuous weakly stationary process, and if $N(W)$ is a positive integer-valued function of $W > 0$ such that $\frac{N(W)}{W} \to \infty$ as $W \to \infty$, then for each $t \in \mathbb{R}^1$

$$x(t) = \lim_{W\to\infty} \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right) x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}, \tag{2.4.23}$$

where the convergence is in the mean square sense uniformly on compact sets. Furthermore, for any fixed $W > 1$ and $|t| < \frac{N(W)}{2W}$ such that $t \neq \frac{n}{2W}$, $n \in \mathbb{N}$, we have

$$E\left|x(t) - \sum_{n=-N(W)}^{N(W)} \left(1 - \frac{|n|}{N(W)+1}\right)x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|^2$$

$$\leq 4|\mu|(\mathbb{R}^1)\left[|t| + \frac{|\sin 2\pi Wt|}{\pi(1-e^{-\pi})N(W)}\right]^2\left[\frac{W}{N(W)}\right]^2 + 4|\mu|\{|u|>W-1\} , \tag{2.4.24}$$

and

$$E\left|x(t) - \sum_{n=-N(W)}^{N(W)} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|^2$$

$$\leq 4\Phi(W)\frac{(1+\ell n\ \pi[N(W)+\frac{1}{2}])^2}{\phi(W-1)} + 16|\mu|(\mathbb{R}^1)\left[\frac{\sin 2\pi Wt}{\pi(1-e^{-\pi})N(W)}\right]^2\left[\frac{W}{N(W)}\right]^2 , \tag{2.4.25}$$

where ϕ and Φ are as defined in Theorem 2.4.3. If, in addition, $N(W)$ and the spectral measure μ of the process satisfy the condition in Theorem 2.4.3, then for all $t \in \mathbb{R}^1$

$$x(t) = \lim_{W\to\infty} \sum_{n=-N(W)}^{N(W)} x\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} , \tag{2.4.26}$$

is mean square.

<u>Proof</u>. Since x is mean square continuous weakly stationary, we have for all $t \in \mathbb{R}^1$

$$R(t) = E[x(t)\overline{x(0)}] = \int_{-\infty}^{\infty} e^{2\pi it\lambda}d\mu(\lambda) ,$$

and

$$x(t) = \int_{-\infty}^{\infty} e^{2\pi it\lambda}dz(\lambda) ,$$

where μ (the spectral measure of x) is a finite measure defined on $(\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$, $\{Z(\lambda), \lambda \in \mathbb{R}^1\}$ is a process with orthogonal increments, and for all $-\infty < a \leq b < \infty$, $E|Z(b) - Z(a)|^2 = \mu\{(a,b]\}$.

Consider the mean square error

$$e_W^2(t) := E\left|x(t) - \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N(W)+1}\right)x\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|^2$$

$$= E\left|\int_{-\infty}^{\infty}[e^{2\pi it\lambda} - \sum_{n=-N(W)}^{N(W)}(1 - \frac{|n|}{N(W)+1})e^{\pi i \frac{n}{W}\lambda}\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}]dZ(\lambda)\right|^2$$

$$\leq \int_{-\infty}^{\infty}\left|e^{2\pi it\lambda} - \sum_{n=-N(W)}^{N(W)}(1 - \frac{|n|}{N(W)+1})e^{\pi i \frac{n}{W}\lambda}\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}]\right|^2 d\mu(\lambda)$$

$$(2.4.27) \quad = \int_{-\infty}^{\infty}\left|1 - \int_{-\frac{1}{2}-\frac{\lambda}{2W}}^{\frac{1}{2}-\frac{\lambda}{2W}} e^{4\pi iWtu} K_{N(W)}(u)du\right|^2 d\mu(\lambda) \ .$$

Since this expression is similar to (2.4.4), (2.4.23) follows as in the proof of Theorem 2.4.1. The proofs of (2.4.24) and (2.4.25), (2.4.26) are similar to those of Theorems 2.4.2 and 2.4.3 respectively, and hence they are omitted. □

A second order stochastic process $x = \{x(t), t \in \mathbb{R}^1\}$ is called a harmonizable process if its correlation function $R(t,s) = E[x(t)\overline{x}(s)]$ is of the form

$$R(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(tu-sv)}d\mu(u,v) \ , \ t,s \in \mathbb{R}^1 \ ,$$

where μ (the spectral measure of x) is a complex measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Theorem 2.4.5. If $x = \{x(t), t \in \mathbb{R}^1\}$ is a harmonizable process, and if $\frac{N(W)}{W} \to \infty$ as $W \to \infty$, then for each $t \in \mathbb{R}^1$ (2.4.23) holds, where the convergence is in the mean square sense uniformly on compact sets. Furthermore, under the conditions of Theorems 2.4.2 and 2.4.3, bounds on the mean square error similar to (2.4.24) and (2.4.25) hold as well as the approximation (2.4.26).

Proof. Consider the mean square error

$$e_W^2(t) := E\left|x(t) - \sum_{n=-N(W)}^{N(W)}(1 - \frac{|n|}{N(W)+1})x(\frac{n}{2W})\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big[e^{2\pi itu} - \sum_{n=-N(W)}^{N(W)} \Big(1 - \frac{|n|}{N(W)+1}\Big) e^{\pi i \frac{n}{W} u} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\Big]$$

$$\cdot \Big[e^{-2\pi itv} - \sum_{n=-N(W)}^{N(W)} \Big(1 - \frac{|n|}{N(W)+1}\Big) e^{-\pi i \frac{n}{W} v} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\Big] d\mu(u,v)$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big|1 - \int_{-\frac{1}{2}-\frac{u}{2W}}^{\frac{1}{2}-\frac{u}{2W}} e^{4\pi iWtu} K_{N(W)}(u)\,du\Big|$$

$$\cdot \Big|1 - \int_{-\frac{1}{2}-\frac{v}{2W}}^{\frac{1}{2}-\frac{v}{2W}} e^{4\pi iWtv} K_{N(W)}(v)\,dv\Big| \, d|\mu|(u,v) \ .$$

By the familiar technique used in the proof of Theorem 2.4.1, we have

$$e_W^2(t) \leq |\mu|(\mathbb{R}^2)Q^2(t,W) + 4|\mu|(A^c \times A^c) \ ,$$

where $A = [-a,a]$, $a > 0$, and the proof is completed as in Theorems 2.4.1 to 2.4.3. □

We finally consider certain harmonizable, but non-stationary, stable processes. A random variable X is symmetric α-stable (SαS), $0 < \alpha < 2$, if its characteristic function is of the form $E(e^{itX}) = \exp(-b_X|t|^\alpha)$ for some positive constant b_X. A stochastic process $x = \{x(t),\ t \in \mathbb{R}^1\}$ is called SαS, if every finite linear combination of its random variables is SαS. The following can be found in Shilder (1970). If X is a SαS random variable, then, for $1 < \alpha < 2$, the map $X \mapsto b_X^{1/\alpha}$ defines a norm on a linear space of SαS random variables: $||X||_\alpha = b_X^{1/\alpha}$. If the process $\{Z(\lambda)): \lambda \geq 0\}$ is SαS with independent increments, then the function F defined on $[0,\infty)$ by $F(\lambda) = ||Z(\lambda)||_\alpha^\alpha$, $\lambda \geq 0$, is nondecreasing and thus defines a Lebesgue-Stieltjes measure μ_F on the Borel sets of $[0,\infty)$. If the family of functions $\{f_t(\cdot),\ t\in\mathbb{R}^1\}$ belongs to $L^2(\mu_F)$, then the integral

$$x(t) = \int_0^\infty f_t(\lambda)dZ(\lambda) \quad , \ t\in\mathbb{R}^1 \ ,$$

defines an SαS process and for every $t\in\mathbb{R}^1$,

$$||x(t)||_\alpha^\alpha = \int_0^\infty |f_t(\lambda)|^\alpha d\mu_F(\lambda) \quad .$$

<u>Theorem 2.4.6</u>. Let a stochastic process x be defined by

$$x(t) = \int_0^\infty \cos 2\pi t\lambda \ dZ(\lambda) \quad , \ t\in\mathbb{R}^1 \ ,$$

where $\{Z(\lambda),\ \lambda\geq 0\}$ is an SαS process with independent increments and finite measure μ_F, and $1 < \alpha < 2$. If $\frac{N(W)}{W} \to \infty$ as $W \to \infty$, then for every $t\in\mathbb{R}^1$, (2.4.23) holds, where the convergence is in the $||\cdot||_\alpha$-norm. Furthermore, under the conditions of Theorems 2.4.2 and 2.4.3, bounds on the $||\cdot||_\alpha$-norm error similar to (2.4.24) and (2.4.25) hold as well as the approximation given in (2.4.26).

<u>Proof</u>. Consider the α-mean error

$$e_W^\alpha(t) := ||x(t) - \sum_{n=-N(W)}^{N(W)} (1-\frac{|n|}{N(W)+1})x(\frac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}||_\alpha^\alpha$$

$$= \int_0^\infty \left|\cos 2\pi t\lambda - \sum_{n=-N(W)}^{N(W)} (1 - \frac{|n|}{N(W)+1})\cos \frac{n\pi}{W}\lambda \ \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\right|^\alpha d\mu_F(\lambda)$$

$$= \int_0^\infty |f_{N,W}(t,\lambda)|^\alpha d\mu_F(\lambda) \quad , \text{ say} \ .$$

As in (2.4.4), we have

$$f_{N,W}(t,\lambda) = \cos 2\pi t\lambda - \int_{-\frac{1}{2}-\frac{\lambda}{2W}}^{\frac{1}{2}-\frac{\lambda}{2W}} \cos 2\pi t(2Wy+\lambda) K_{N(W)}(y)dy$$

$$= \mathrm{Re}[e^{2\pi it\lambda} - \int_{-\frac{1}{2}-\frac{\lambda}{2W}}^{\frac{1}{2}-\frac{\lambda}{2W}} e^{2\pi it(2Wy+\lambda)} K_{N(W)}(y)dy] ,$$

and the proof is completed as in Theorems 2.4.1 to 2.4.3. □

The same results hold for SαS processes

$$x(t) = \int_0^\infty \cos 2\pi t\lambda \, dZ_1(\lambda) + \int_0^\infty \sin 2\pi t\lambda \, dZ_2(\lambda) , \quad t \in \mathbb{R}^1 ,$$

where Z_1 and Z_2 are independent processes as in Theorem 2.4.6. These processes are the SαS (non-stationary) analogues ($1 < \alpha < 2$) of the real stationary Gaussian ($\alpha = 2$) processes.

2.5. Sampling Approximation Using Walsh Functions.

Recently, Walsh functions have been increasingly used in digital communication systems: they are easily generated by semiconductor devices, their pulse shape (1,-1) conforms with operations of digital computers, and they play, for discontinuous signals, the role complex exponential functions (and Fourier transforms) play for continuous signals. In addition, they have been used in experimental sequency-multiplex systems, image coding and enhancement, general two-dimensional filtering, etc.

In this section, using Walsh functions, we derive a sampling approximation and error estimates for functions which are not necessarily continuous, and for stochastic processes which are not necessarily mean square continuous. These results are the Walsh-analogues for

W-continuous functions of the Fourier results of Sections 2.2 and 2.3 for continuous functions. Results similar to those of Section 2.4 should be feasible for W-continuous functions, but complete proofs have not as yet been obtained.

The following notation and definitions will be used in the sequel. $\mathbb{R}_+ = [0,\infty)$, $\mathbb{N}$ is the set of all integers, $\mathbb{N}_+$ is the set of all non-negative integers, and D_+ is the set of all non-negative dyadic rationals. Each $t > 0$ has the dyadic expansion

$$t = \sum_{j=-N(t)}^{\infty} t_j 2^{-j} \ , \ t_j \in \{0,1\} \tag{2.5.1}$$

for all j, where $N(t)$ is such that $2^{N(t)} \leq t < 2^{N(t)+1}$, and we put $t_j = 0$ for $j < -N(t)$. If $t \in D_+$, there are two expansions and the finite one is chosen so that expansion (2.5.1) becomes unique. The componentwise addition modulo 2 (dyadic addition) of $t,s \in \mathbb{R}_+$ is defined by $t \oplus s = \sum_{j=-\infty}^{\infty} |t_j - s_j| 2^{-j}$.

The Walsh functions can be defined in several ways. The following definition is based on the system of Rademacher functions $\{R_n(t)\}_{n\in\mathbb{N}_+}$, $t \in [0,1)$, where

$$R_0(t) = e^{\pi i [2t]}$$

$$R_n(t) = R_0(2^n t) \ , \ n \geq 1 \ ,$$

and by

$$R_n(t) = e^{\pi i t_{n+1}} \ , \ n \in \mathbb{N}_+ \ , \ t \in [0,1) \ .$$

The set of Walsh functions $\{\psi_n(t)\}_{n\in\mathbb{N}_+}$ on $[0,1)$ is defined for each non-negative integer $n = \sum_{n=-N(n)}^{0} n_j 2^{-j}$ by

$$\psi_n(t) = \prod_{j=1}^{N(n)} (R_j(t))^{n_{-j}} = \exp\{\pi i \sum_{j=1}^{N(n)+1} n_{1-j} t_j\} \ , \tag{2.5.2}$$

and is orthonormal and complete in $L^2[0,1)$. The Walsh functions are extended to $\{\psi_u(t)\}_{u,t\in\mathbb{R}_+}$ by

$$\psi_t(u) = \psi_u(t) = \exp\{\pi i \sum_{j=-N(t)}^{N(u)+1} u_{1-j}t_j\}, \quad t, u\in\mathbb{R}_+ ,$$

and they have the property that, for all $u\in\mathbb{R}_+$, whenever $t \oplus s \notin D_+$,

$$\psi_u(t\oplus s) = \psi_u(t)\psi_u(s) .$$

A function f on $\mathbb{R}_+$ is called W-continuous if f is continuous on $\mathbb{R}_+\backslash D_+$ and right continuous on D_+. It is clear that the Walsh functions are W-continuous. If $f\in L^1(\mathbb{R}_+)$, then the Walsh-Fourier transform (WFT) f^W of f is defined by

$$f^W(u) = \int_0^\infty f(t)\psi_u(t)dt , \quad u\in\mathbb{R}_+ , \tag{2.5.3}$$

and f^W is bounded and W-continuous. If f, $f^W\in L^1(\mathbb{R}_+)$ and f is W-continuous, then the WFT can be inverted to give

$$f(t) = \int_0^\infty f^W(u)\psi_t(u)du , \quad t\in\mathbb{R}_+ . \tag{2.5.4}$$

(See Butzer and Splettstösser, 1978.) The Walsh modulus of continuity of a function $f\in L^1(\mathbb{R}_+)$ is defined by

$$\omega(f;\delta) = \sup_{0\le h<\delta} ||f(\cdot) - f(\cdot\oplus h)||_{L^1(\mathbb{R}^1)} , \quad \delta > 0 .$$

For $\alpha > 0$ and a constant $L > 0$, the Lipschitz class $\mathrm{Lip}_L\alpha$ is defined by

$$\mathrm{Lip}_L\alpha = \{f\in L^1(\mathbb{R}_+): \ \omega(f;\delta) \le L\delta^\alpha , \quad \delta > 0\} .$$

A function $f\in L^1(\mathbb{R}_+)$ is said to be dyadic differentiable if there exists a $g \in L^1(\mathbb{R}_+)$ such that

$$\lim_{m\to\infty} \| \tfrac{1}{4} \sum_{j=-m}^{m} 2^j [f(\cdot) - f(\cdot \oplus 2^{-j-1})] - g(\cdot) \|_{L^1(\mathbb{R}_+)} = 0 ,$$

g is called the first strong dyadic derivative of f, and is denoted by $D^{[1]}f$. For $r > 1$, $D^{[r]}f$ is defined iteratively by $D^{[r]}f = D^{[1]}(D^{[r-1]}f)$. If f and $D^{[r]}f$ belong to $L^1(\mathbb{R}_+)$, there exists a constant M such that $\omega(f;\delta) \leq M\delta^r \omega(D^{[r]}f;\delta)$, $\delta > 0$.

A complex function f on $\mathbb{R}_+$ of the form

$$\text{(2.5.5)} \qquad f(t) = \int_0^{2n} F(u)\psi_t(u)du , \quad t \in \mathbb{R}_+$$

for some $n \in \mathbb{N}$ and some $F \in L^1(0,2^n)$, is called sequency limited to 2^n. A sequency limited function f which is W-continuous and in $L^1(\mathbb{R}_+)$ has a sampling expansion of the form:

$$\text{(2.5.6)} \qquad f(t) = \sum_{k=0}^{\infty} f(\frac{k}{2^n}) J(1;\ 2^n t \oplus k) , \quad t \in \mathbb{R}_+ ,$$

where $J(v;t) = \int_0^v \psi_t(u)du$, $t,v \in \mathbb{R}_+$ (Fine, 1950). As it was pointed out by Kak (1970) and Butzer and Splettstösser (1978),

$$J(1;2^n t \oplus k) = 1_{[2^{-n}k,\ 2^{-n}(k+1))}(t) ,$$

and thus (under the stated conditions) the functions that are sequency limited to 2^n are precisely the functions that are constant on each interval $[2^{-n}k, 2^{-n}(k+1))$, a rather small class (unlike the class of bandlimited functions).

A (dyadic) sampling approximation for time-limited functions (which are not necessarily continuous) was derived by Butzer and Splettstösser (1978):

Theorem 2.5.1. (Butzer and Splettstösser, 1978). Let f be a W-continuous function on $\mathbb{R}_+$ such that $f(t) = 0$ for all $t \geq T$, for some $T > 0$, and $f, f^W \in L^1(\mathbb{R}_+)$. Then

$$(2.5.7) \qquad f(t) = \lim_{n\to\infty} \sum_{k=0}^{N(n)} f(\tfrac{k}{2^n}) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) \quad , \; t \in \mathbb{R}_+$$

where $N(n) = [2^n T]$. If, in addition, either (i) $D^{[r]}f$ exists and $D^{[r]}f \in \mathrm{Lip}_{L/M}\alpha$ or (ii) $f \in \mathrm{Lip}_L(\alpha + r)$, for some fixed $\alpha > 0$ and $r \in \{1,2,\ldots\}$, then

$$(2.5.8) \quad \sup_{t \in \mathbb{R}_+} \left| f(t) - \sum_{k=0}^{N(n)} f(\tfrac{k}{2^n}) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) \right| \leq \frac{L2^{r+\alpha}}{r+\alpha-1} 2^{-n(r+\alpha-1)} \; .$$

We now derive a sampling approximation as well as error estimates for functions which are the WFT of finite (or complex) measures and thus not necessarily time-limited nor sequency limited.

Theorem 2.5.2. If f is the WFT of a finite (or complex) measure μ on the Borel sets of $\mathbb{R}_+$, i.e.

$$f(t) = \int_0^\infty \psi_t(u) d\mu(u) \quad , \; t \in \mathbb{R}_+$$

then for every $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$(2.5.9) \qquad f_n(t) = \sum_{k=0}^{\infty} f(\tfrac{k}{2^n}) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) = \int_0^\infty \psi_t^{[n]}(u) d\mu(u) \; ,$$

where for each (fixed) $t \in \mathbb{R}_+$, $\psi_t^{[n]}(u)$ is the 2^n-periodic extension of the function $\psi_t(u)$, $0 \leq u < 2^n$ to $\mathbb{R}_+$, and

$$(2.5.10) \qquad |f(t) - f_n(t)| \leq 2|\mu|[2^n, \infty) \; .$$

Thus for each $t \in \mathbb{R}_+$,

$$(2.5.11) \qquad f(t) = \lim_{n\to\infty} f_n(t) \ .$$

If, in addition, μ is absolutely continuous with respect to Lebesgue measure, and $f \in \mathrm{Lip}_L(\alpha+r)$ for some $\alpha > 0$, $L > 0$, and $r \in \{1,2,\ldots\}$, then for every $t \in \mathbb{R}_+$

$$(2.5.12) \qquad |f(t) - f_n(t)| \leq \frac{L2^{r+\alpha-1}}{r+\alpha-1}\, 2^{-n(r+\alpha-1)} \ .$$

<u>Proof</u>. Since $\psi_t^{[n]} \in L^1[0,2^n)$ is periodic with period 2^n, W-continuous, and of bounded variation on $[0,2^n)$, then the partial sums of its Walsh Fourier series converge everywhere to $\psi_t^{[n]}$ (see Chrestenson, 1955, Theorem 2), i.e. for each $t \in \mathbb{R}_+$

$$(2.5.13) \qquad \psi_t^{[n]}(u) = \sum_{k=0}^{\infty} a_{n,k}(t)\psi_k(2^{-n}u) \ , \ u \in \mathbb{R}_+$$

where

$$\begin{aligned} a_{n,k}(t) &= 2^{-n}\int_0^{2^n} \psi_t(v)\psi_k(2^{-n}v)\,dv \\ &= \int_0^1 \psi_t(2^n v)\psi_k(v)\,dv \\ &= \int_0^1 \psi_v(2^n t)\psi_v(k)\,dv \\ &= \int_0^1 \psi_v(2^n t \oplus k)\,dv \\ &= J(1;2^n t \oplus k) \\ &= 1_{[2^{-n}k,2^{-n}(k+1))}(t) \ . \end{aligned}$$

Let

$$\varepsilon_K(t;n) = \left|\sum_{k=0}^{K} f(\frac{k}{2^n}) 1_{[2^{-n}k,2^{-n}(k+1))}(t) - \int_0^\infty \psi_t^{[n]}(u)d\mu(u)\right|$$

$$\le \int_0^\infty \left|\psi_t^{[n]}(u) - \sum_{k=0}^{K} \psi_k(2^{-n}u) 1_{[n^{-n}k,2^{-n}(k+1))}(t)\right| d|\mu|(u) \ .$$

Using (2.5.13) and the fact that for each $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$,

$$\left|\psi_t^{[n]}(u) - \sum_{k=0}^{K} \psi_k(2^{-n}u) 1_{[2^{-n}k,2^{-n}(k+1))}(t)\right| \le 2$$

for all $u\in\mathbb{R}_+$ and $K > 1$, it follows that

$$\varepsilon_K(t;n) \to 0 \quad \text{as } K \to \infty, \quad \text{for each } t\in\mathbb{R}_+ \text{ and } n\in\mathbb{N} \ ,$$

proving (2.5.9). Now

$$|f(t) - f_n(t)| = \left|\int_0^\infty \psi_t(u)d\mu(u) - \int_0^\infty \psi_t^{[n]}(u)d\mu(u)\right|$$

$$\le \int_{2^n}^{\infty} |\psi_t(u) - \psi_t^{[n]}(u)|\, d|\mu|(u)$$

$$\le 2|\mu|[2^n,\infty) \ ,$$

hence (2.5.10) and (2.5.11). To prove (2.5.12), notice that if $\frac{d\mu(u)}{dt} = F(u)$ and $f\in \mathrm{Lip}_L(r+\alpha)$, then

$$|F(u)| \le L\ 2^{r+\alpha-1} u^{-r-\alpha}, \quad u > 0 \ , \tag{2.5.14}$$

(see, Butzer and Splettströsser (1978)). From (2.5.10) and (2.5.14) we have

$$|f(t) - f_n(t)| \le L\ 2^{r+\alpha-1} \int_{2^n}^{\infty} \frac{1}{u^{r+\alpha}}\, du$$

$$= \frac{L\ 2^{r+\alpha-1}}{(r+\alpha-1)}\, 2^{-n(r+\alpha-1)} \ . \qquad \square$$

We end this section by extending the results in Theorem 2.5.2 to stochastic processes which are not necessarily continuous. We first introduce the following notation. A function R on $\mathbb{R}_+^2 = [0,\infty)\times[0,\infty)$ is called W_2-continuous, if R is continuous on $\mathbb{R}_+^2\backslash D_+^2$ and continuous from above on D_+^2 (in the sense of Neuhaus, 1971). If $R\in L^1(\mathbb{R}_+^2)$, the first modulus of continuity is defined by

$$\omega(\delta,\lambda;R) = \sup\{||\Delta_{h,g}R||_{L^1(\mathbb{R}_+^2)},\ 0 \le h < \delta,\ 0 \le g < \lambda\},\ \delta,\lambda > 0 ,$$

where $\Delta h,gR(t,s) = R(t\oplus h, s\oplus g) - R(t\oplus h,s) - R(t, s\oplus g) + R(t,s)$. Also the class $\mathrm{Lip}_L^{(2)}\alpha$ is defined by

$$\mathrm{Lip}_L^{(2)}\alpha = \{R\in L^1(\mathbb{R}_+^2): \omega(\delta,\lambda;R) \le L\delta^\alpha\lambda^\alpha\ ,\quad \delta > 0,\ \lambda > 0\} .$$

The WFT of a function $R\in L^1(\mathbb{R}_+^2)$ is defined by

$$R^W(u,v) = \int_0^\infty \int_0^\infty R(t,s)\psi_u(t)\psi_v(s)dt\, ds\ ,\ u,v\in\mathbb{R}_+ .$$

Finally, if R is W_2-continuous and $R,R^W\in L^1(\mathbb{R}_+^2)$, then (as in (2.5.4)

$$(2.5.15)\qquad R(t,s) = \int_0^\infty \int_0^\infty R^W(u,v)\psi_t(u)\psi_t(v)du\, dv\ ,\ t,s\in\mathbb{R}_+ .$$

<u>Theorem 2.5.3</u>. Let $\{x(t),\ t\in\mathbb{R}_+\}$ be a second order stochastic process with correlation function R. Assume that R is W_2-continuous, $R,R^W\in L^1(\mathbb{R}_+^2)$, $R(t,\cdot)\ L^1(\mathbb{R}_+)$ for all $t\in\mathbb{R}_+$, and $R^W(\cdot,v)\in L^1(\mathbb{R}_+)$ for all $v\in\mathbb{R}_+$. Then for each $t\in\mathbb{R}^1$, $n\in\mathbb{N}$

$$(2.5.16)\qquad x_n(t):= \sum_{k=0}^\infty x(\tfrac{k}{2^n})1_{[2^{-n}k,2^{-n}(k+1))}(t) = R\int_0^\infty \psi_t^{[n]}(u)y(u)du ,$$

where the equality is a.s., the series converges in quadratic mean, and y is defined by the quadratic mean integral

$$y(u) = R\int_0^\infty \psi_u(t)x(t)dt \ , \ u\in\mathbb{R}_+ \ .$$

Also, for each $t\in\mathbb{R}_+$ and $n\in\mathbb{N}$,

(2.5.17) $$e_n^2(t) := E|x(t)-x_n(t)|^2 \le 4\int_{2^n}^{\infty}\int_{2^n}^{\infty} |R^W(u,v)|du\,dv \ ,$$

and thus for each $t\in\mathbb{R}_+$

(2.5.18) $$x(t) = \lim_{n\to\infty} x_n(t)$$

is quadratic mean. If, in addition, $R\in\mathrm{Lip}_L^{(2)}(r+\alpha)$, for some $\alpha > 0$ and $r\in\{1,2,\ldots\}$, then

(2.5.19) $$e_n^2(t) \le \frac{L\ 2^{2(r+\alpha)}}{(r+\alpha-1)^2}\ 2^{-2n(r+\alpha-1)} \ .$$

Proof. The proofs of (2.5.16), (2.5.17), and (2.5.18) are similar to those in Theorem 2.2.2 and hence omitted. To show (2.5.19), notice that for any $u > 0$ we have the dyadic expansions

$$u = \sum_{j=-N(u)}^{\infty} u_j 2^{-j} \ , \quad u^{-1} = \sum_{j=N(u)+1}^{\infty} (u^{-1})_j\ 2^{-j} \ ,$$

and thus

$$\psi_u(u^{-1}) = \exp\{\pi i \sum_{j=-N(u)}^{N(u)} (u^{-1})_j\ u_j\} = -1 \ ,$$

(see Butzer and Splettstösser, 1978, p. 102). From (2.5.15), we have that for any $u > 0$

$$R^W(u,v) = -\int_0^\infty\int_0^\infty R(t,s)\psi_u(t\oplus u^{-1})\psi_v(s)dt\ ds$$

$$= -\int_0^\infty\int_0^\infty R(t\oplus u^{-1},s)\psi_u(t)\psi_v(s)dt\ ds \ , \ v\in\mathbb{R}_+ \ ,$$

Similarly for any $v > 0$

$$R^W(u,v) = -\int_0^\infty\int_0^\infty R(t,s\oplus v^{-1})\psi_u(t)\psi_v(s)dt\ ds\ ,\ u\in\mathbb{R}_+$$

and for $u,v > 0$

$$R^W(u,v) = \int_0^\infty\int_0^\infty R(t\oplus u^{-1},\ s\oplus v^{-1})\psi_u(t)\psi_v(s)dt\ ds\ .$$

Hence for $u,v > 0$

$$R^W(u,v) = \frac{1}{4}\int_0^\infty\int_0^\infty [R(t\oplus u^{-1},\ s\oplus v^{-1}) - R(t\oplus u^{-1},s)$$

$$- R(t,s\oplus v^{-1}) + R(t,s)]\psi_u(t)\psi_v(s)dt\ ds$$

and

$$|R^W(u,v)| \le \frac{1}{4}\,\omega(2u^{-1},2v^{-1};R) \le \frac{L}{4}\,(\tfrac{2}{u})^{r+\alpha}(\tfrac{2}{v})^{r+\alpha}\ ,\ u,v > 0\ .$$

Now from (2.5.17) we have

$$e_n^2(t) \le L\ 2^{2(r+\alpha)}\left(\int_{2^n}^\infty \frac{1}{u^{r+\alpha}}\,du\right)^2 = \frac{L\ 2^{2(r+\alpha)}}{(r+\alpha-1)}\ 2^{-2n(r+\alpha-1)}\ . \quad \square$$

The following corollary shows that the approximating sequence $x_n(t)$ converges to $x(t)$ with probability one for each $t\in\mathbb{R}^1$, and gives the rate of convergence.

Corollary 2.5.1. Let x be as in Theorem (2.5.3) and assume that $r + \alpha > 2$. Then for each $t\in\mathbb{R}^1$,

$$2^{2\gamma n_0}\sup_{n>n_0}|x(t)-x_n(t)| \to 0 \quad \text{a.s. as } n_0 \to \infty\ , \tag{2.5.20}$$

where $0 < \gamma < \frac{r+\alpha}{2} - 1$.

Proof. For each fixed t, define X_u, $0 \le u \le 1$, by

$$X_u = \begin{cases} x(t) & \text{for } u = 0 \\ x_n(t) & \text{for } \frac{1}{2^{2n}} < u \le \frac{1}{2^{2(n-1)}}\ ,\ n \ge 1, \end{cases}$$

where $x_n(t) = \sum_{k=0}^{\infty} x(\frac{k}{2^n}) 1_{[2^{-n}k, 2^{-n}(k+1))}(t)$. Then X is separable in u and from (2.5.19), we have (with n such that $2^{-2n} < u \le 2^{-2(n-1)}$)

$$E|X_0 - X_u|^2 = E|x(t)-x_n(t)|^2 \le C(2^{-2n})^{1+\beta} < Cu^{1+\beta} \ , \ u > 0 \ ,$$

where $C = \frac{L\ 2^{2(r+\alpha)}}{(r+\alpha-1)^2}$ and $\beta = r+\alpha-2 > 0$. Thus, by Kolmogorov's theorem (Neveu, 1965, p. 97),

$$\frac{1}{h^\gamma} \sup_{0<\frac{1}{2^{2n}}<h} |x(t)-x_n(t)| = \frac{1}{h^\gamma} \sup_{0<u<h} |X_0 - X_u| \to 0 \text{ a.s. as } h \downarrow 0 \ ,$$

and (2.5.20) follows by putting $h = 2^{-2n_0}$, $n_0 \to \infty$. □

CHAPTER III

Sampling Expansions for Operators Acting on Certain Classes of Functions and Processes

3.1. Introduction.

In this chapter the problem of reconstructing bounded linear operators acting on classes of functions bandlimited in the sense of Zakai (1965) and of Lee (1976a) will be considered. Sampling expansions for bounded linear operators acting on classes of functions with wandering spectra will also be investigated.

Recall that a function of the form $f(t) = \int_{-W_0}^{W_0} e^{2\pi i t u}\hat{f}(u)du$, where $W_0 > 0$ and $\hat{f} \in L^2[-W_0,W_0]$ is called conventionally bandlimited to W_0. The class of all such functions will be denoted by $B_0(W_0)$ and is a Hilbert subspace of $L^2(\mathbb{R}^1)$. Every $f \in B_0(W_0)$ has the following sampling expansion and convolution representation

$$f(t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$$

$$= \int_{-\infty}^{\infty} f(u) \frac{\sin 2\pi W(t-u)}{2\pi W(t-u)} du \quad , \quad t \in \mathbb{R}^1 , \tag{3.1.1}$$

where $W \geq W_0$, and the series in (3.1.1) converges uniformly on $\mathbb{R}^1$ and also in $L^2(\mathbb{R}^1)$. The functions

$$\phi_n(t;W) = \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} , \quad n = 0, \pm 1, \pm 2,\ldots,$$

form a complete orthogonal set in $B_0(W)$ which is strictly larger than $B_0(W_0)$. It is thus interesting to notice that when $W > W_0$ the partial sums of the series in (3.1.1) belong to $B_0(W)$ and yet its pointwise (or $L^2(\mathbb{R}^1)$) limit belongs to the smaller subspace $B_0(W_0)$. Of course, when $W = W_0$, (3.1.1) is the expansion of f in terms of the basis $\{\phi_n(t;W_0)\}$ of $B_0(W_0)$.

Zakai (1965) extended the classical concept of conventional bandlimitedness to a broader class in which the functions need not be square integrable. He also proved that if $f \in B_0(W_0)$ and $W > W_0$, then

$$\sum_{n=-\infty}^{\infty} (-1)^n f(\tfrac{n}{2W}) = 0 \;, \tag{3.1.2}$$

and if, in addition, $\hat{f}(u)(1 - e^{\pi i \frac{u}{W_0}})^{-1}$ belongs to $L^1[-W_0,W_0]$, then (3.1.2) is also valid for $W = W_0$.

3.2. <u>The Bandlimited Case</u>.

Kramer (1973) derived a sampling expansion for bounded linear operators acting on conventionally bandlimited functions, and Mugler (1976) derived a convolution representation for such operators.

<u>Theorem 3.2.1</u>. (Kramer (1973) and Mugler (1976)). If T is a bounded linear operator on $B_0(W_0)$, then for every $f \in B_0(W_0)$

$$[Tf](t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W_0})\, T[\frac{\sin \pi(2W_0(\cdot)-n)}{\pi(2W_0(\cdot)-n)}](t)$$

$$= \int_{-\infty}^{\infty} f(u)\, T[\frac{\sin 2\pi W_0(\cdot-u)}{2\pi W_0(\cdot-u)}](t)du \;, \quad t \in \mathbb{R}^1 \;. \tag{3.2.1}$$

When T is time invariant, i.e. $[Tf(\cdot-a)](t) = [Tf](t-a)$, (3.2.1) takes the simpler form

$$[Tf](t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2W_0})\ T[\frac{\sin 2\pi\ W_0(\cdot)}{2\pi W_0(\cdot)}](t - \frac{n}{2W_0})$$

$$= \int_{-\infty}^{\infty} f(t-u)\ T[\frac{\sin 2\pi W_0(\cdot)}{2\pi W_0(\cdot)}](u)du\ ,\ t\in \mathbb{R}^1\ .$$

The significance of Kramer's expansion is that Tf may be reconstructed from samples of f itself rather than from samples of Tf. For instance, the differentiation operator $[Df](t) = \frac{d}{dt} f(t)$ is a time invariant bounded linear operator on $B_0(W_0)$, hence from (3.2.1) we have

$$f'(t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2W_0})\frac{d}{dt}\ [\frac{\sin \pi(2W_0t-n)}{\pi(2W_0t-n)}]\ .$$

Kramer's sampling expansion (3.2.1) will be generalized to broader classes of functions. The following notation will be used in the sequel. For a non-negative integer k, $L^2(\mu_k)$ is the class of all complex valued functions defined on $\mathbb{R}^1$ that are square integrable with respect to the measure $d\mu_k(t) = \frac{dt}{(1+t^2)^k}$. If $f\in L^2(\mu_k)$, then f defines a tempered distribution (denoted also by f) on the class S of rapidly decreasing functions by

$$f(\theta) = \int_{-\infty}^{\infty} f(t)\theta(t)dt\ ,\ \theta\in S\ .$$

(See Chapter 4 for relevant definitions.) The distributional Fourier transform of f is the tempered distribution $\hat{f}$ defined by $\hat{f}(\theta) = f(\hat{\theta})$, $\theta\in S$. The spectrum of f is the support of $\hat{f}$. For k = 0,1,2,... and $W_0 > 0$, $B_k(W_0)$ is the class of all continuous functions $f\in L^2(\mu_k)$ whose (distributional) spectrum is contained in $[-W_0,W_0]$, and is called the class of W_0-bandlimited functions in $L^2(\mu_k)$. It is clear that $B_0(W_0)$ is the class of W_0-bandlimited functions in $L^2(\mathbb{R}^1)$, and $B_k(W_0) \subset B_{k+1}(W_0)$. Also, $B_0(W_0)$ is dense in $B_k(W_0)$ for every positive integer k (see Lee, 1976b).

Zakai (1965) obtained a sampling representation for functions in $B_1(W_0)$. Cambanis and Masry characterized Zakai's class $B_1(W_0)$ and as a consequence sharpened Zakai's sampling expansion (see also Piranashvili (1967) and Lee (1976a)).

Theorem 3.2.2. (Zakai (1965), and Cambanis and Masry (1976)). If $f \in B_1(W_0)$ and $W > W_0$, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}, \quad t \in \mathbb{R}^1, \tag{3.2.2}$$

and the series converges uniformly on compact sets.

Thus, functions in $B_1(W_0)$ are reconstructed from their samples using functions in $B_0(W)$, $W > W_0$.

Remark 3.2.1. It should be noted that (3.2.2) holds for $W = W_0$ if the Fourier transform $\hat{g}$ of $g(t) \triangleq [f(t)-f(0)]t^{-1}$ is such that $\hat{g}(u)(1 + e^{\pi i \frac{u}{W_0}})^{-1}$ belongs to $L^1[-W_0, W_0]$.

Lee (1977) proved an analogue of Theorem 3.2.2 for functions $f \in B_k(W_0)$, $k \geq 0$.

Theorem 3.2.3. (Lee, 1977). If $f \in B_k(W_0)$, $W > W_0$, $0 < \beta < W-W_0$, and ψ is an arbitrary but fixed C^∞-function with support in $[-1,1]$ and $\int_{-\infty}^{\infty} \psi(t)dt = 1$, then

$$(3.2.3) \qquad f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \hat{\psi}\left(\beta\left(t - \frac{n}{2W}\right)\right), \quad t\in\mathbb{R}^1,$$

and the series converges uniformly on compact sets.

It should be pointed out that the function $\frac{\sin 2\pi W(\cdot)}{2\pi W(\cdot)} \hat{\psi}(\beta(\cdot))$ belongs to $B_0(W+\beta)$, $W > W_0$. Thus functions in $B_k(W_0)$ are reconstructed from their samples using functions in $B_0(W+\beta)$, $W > W_0$ and $0 < \beta < W-W_0$. It should also be noted that the presence of the (damping) factor $\hat{\psi}$ in (3.2.3) cannot be eliminated, as (3.2.3) is not valid for $f\in B_k$, $k \geq 2$. As a counter example consider $f(t) = t(f\in B_2(W_0))$; then $f(\frac{n}{2W}) = \frac{n}{2W}$ and the series in (3.2.2) does not converge.

Campbell (1968) derived sampling expansions for the Fourier transforms (as functions) of tempered distributions with compact supports. If a tempered distribution F has a compact support and $e_u(t) = e^{2\pi itu}$, then $F(e_u)$ is well defined, since $e_u \in C^\infty$ for all $u\in\mathbb{R}^1$. In this case the Fourier transform $\hat{F}$ of F may be thought of as a function defined on $\mathbb{R}^1$ by $\hat{F}(u) = F(e_u)$, $u\in\mathbb{R}^1$ (see Section 4.2).

Theorem 3.2.4. (Campbell, 1968). Let F be a tempered distribution with compact support and with Fourier transform f as a function on $\mathbb{R}^1$, i.e. $f(t) = F(e_t)$, $t\in\mathbb{R}^1$. Let ψ be a test function such that $\psi(u) = 1$ on some open set containing supp(F), and let $W > 0$ be such that the translates $\{\text{supp}(\psi) + 2nW\}_{n\neq 0}$ are disjoint from supp(F). Then

$$(3.2.4) \qquad f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) K\left(t - \frac{n}{2W}\right),$$

where $K(t) = \frac{1}{2W} \int e^{2\pi itu} \psi(u) du$, and the series converges for every $t \in \mathbb{R}^1$.

Campbell's result does not require the support of F to be symmetric with respect to the origin and is more general than Lee's (Theorem 3.2.3) as functions in $B_k(W_0)$ have (distributional) Fourier transform with compact support in $[-W_0, W_0]$. Campbell considered also a sampling expansion similar to (3.2.3) when $\text{supp}(F) \subset (-W_0, W_0)$, $W_0 > 0$, (for specific ψ) and derived an upper bound for the truncation error which was recently made more explicit by Lee (1979).

To establish notation, let ψ be a fixed but arbitrary function in S whose Fourier transform $\hat{\psi}$ satisfies the conditions

(i) $\hat{\psi}$ is a symmetric test function supported by $[-W_0-\delta, W_0+\delta]$, $0 < \delta < W_0$,

(ii) $\hat{\psi}(t) = 1$ for all $t \in [-W_0, W_0]$,

(iii) $\hat{\psi}(t) < 1$ for all $t \notin [-W_0, W_0]$.

Then for all $f \in L^2(\mu_k)$, the convolution $f * \psi$ exists and is a C^∞-function in $L^2(\mu_k)$. If $f \in B_k(W_0)$, then

$$f(t) = (f*\psi)(t) \quad , \quad t \in \mathbb{R}^1 \; ,$$

(see Lee (1976a), also Cambanis and Masry (1976)). The following characterization of B_k will be needed.

<u>Lemma 3.2.1</u>. (Lee, 1976b). If $f \in L^2(\mu_k)$ is continuous, then the following are equivalent,

(a) $f(t) = (f*\psi)(t)$, $t \in \mathbb{R}^1$,

(b) $f(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}_{(0)}}{j_0^1} t^j + \frac{t^k}{k_0^1} g(t)$, $t \in \mathbb{R}^1$, and some $g \in B_0(W_0)$,

(c) the spectrum of f is contained in $[-W_0, W_0]$,

(d) f has an extension to an entire function satisfying

$|f(z)| \leq C_k(1+|z|)^k e^{2\pi W_0 |\mathrm{Im}\, z|}$, $z \in \mathbb{C}$ and some $C_k > 0$.

The following result is a generalization of Kramer's sampling expansion (3.2.1), in the sense that (3.2.1) is established for band-limited functions $f \in B_1(W_0)$ $(\supset B_0(W_0))$, as well as of Zakai's sampling expansion (3.2.2).

Theorem 3.2.5. Let $f \in B_1(W_0)$ and $W > W_0$. Then for any bounded linear operator T on $B_1(W)$

$$[Tf](t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})[T\phi_n](t) \ , \ t \in \mathbb{R}^1 \ , \tag{3.2.5}$$

where $\phi_n(t;W) = \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$, and the series converges in $L^2(\mu_1)$ and also uniformly on compact sets.

Proof. Fix $f \in B_1(W_0)$ and $W > W_0$. We first show that the convergence in (3.2.2) is in $L^2(\mu_1)$ as well. From (b) of Lemma (3.2.1) we have that $f(t) = f(0) + tg(t)$ for all $t \in \mathbb{R}^1$, and some $g \in B_0(W_0)$; hence by (3.1.1) we have

$$0 = \int |g(t) - \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W})\phi_n(t;W)|^2 dt$$

$$= \int \left|\frac{f(t)-f(0)}{t} - f'(0)\phi_0(t;W) - \sum_{n\neq 0} \frac{f(\frac{n}{2W})-f(0)}{n/2W}\phi_n(t;W)\right|^2 dt$$

$$\geq \int \Big|f(t)-f(0)-f'(0)\phi_0(t;W) - \sum_{n\neq 0} \frac{2Wt[f(\frac{n}{2W})-f(0)]}{n\pi(2Wt-n)} \sin \pi(2Wt-n)\Big|^2 \frac{dt}{1+t^2} \ . \tag{3.2.6}$$

Since $\dfrac{t}{n(t-\frac{n}{2W})} = \dfrac{1}{n} + \dfrac{1}{2Wt-n}$, by writing

$$\frac{t[f(\frac{n}{2W})-f(0)]}{\pi n(t-\frac{n}{2W})} \sin \pi(2Wt-n) = (-1)^n \frac{f(\frac{n}{2W})-f(0)}{n/2W} \phi_0(t;W)$$

$$+ [f(\tfrac{n}{2W})-f(0)]\ \phi_n(t;W)$$

in (3.2.6) we obtain

$$0 = \int \Big| f(t)-f(0)-f'(0)\phi_0 t;W) - \sum_{n\neq 0} [(-1)^n \frac{f(\frac{n}{2W})-f(0)}{n/2W} \phi_0(t;W)$$

$$\text{(3.2.7)} \qquad + f(\tfrac{n}{2W})\phi_n(t;W) - f(0)\phi_n(t;W)]\Big|^2 d\mu_1(t) \ .$$

Now (3.2.2) applied to g gives

$$\text{(3.2.8)} \qquad f'(0) + \sum_{n\neq 0} (-1)^n \frac{f(\frac{n}{2W})-f(0)}{n/2W} = 0 \ .$$

Also from the classical sampling theorem (see Zakai, 1965) we have

$$\sum_{n=-\infty}^{\infty} \phi_n(t;W) = 1 \ , \ t \in \mathbb{R}^1 \ .$$

For any positive integer N, let $S_{-N}^{N}(t) \triangleq \sum_{n=-N}^{N} \phi_n(t;W)$. Then for any $t \geq 0$

$$\text{(3.2.9)} \qquad |S_{-N}^{N}(t)| \leq |S_{-N}^{0}(t)| + |S_{0}^{[2Wt]}(t)| + |S_{[2Wt]+1}^{N}(t)| \ .$$

Consider $S_{-N}^{0}(t)$ and let t be such that $\sin 2\pi Wt \geq 0$. The successive terms of $S_{-N}^{0}(t)$ can be written as

$$\frac{\sin 2\pi Wt}{2\pi Wt} \ , \ -\frac{\sin 2\pi Wt}{\pi(2Wt+1)} \ , \ \frac{\sin 2\pi Wt}{\pi(2Wt+2)} \ , \ldots , (-1)^N \frac{\sin 2\pi Wt}{\pi(2Wt+N)} \ ,$$

and are thus alternating in sign and decreasing in magnitude. It follows that $|S_{-N}^{0}(t)| \leq |\frac{\sin 2\pi Wt}{2\pi Wt}| \leq 1$ for all $t \geq 0$, and similarly for $S_0^{[2Wt]}$ and $S_{[2Wt]+1}^{N}$. Thus for all $N > 0$ and $t \geq 0$, $|S_{-N}^{N}(t)| \leq 3$. The same result holds for $t < 0$. Thus $|S_{-N}^{N}(t)-1| \leq 4$, and since $S_{-N}^{N}(t) \to 1$ for all $t \in \mathbb{R}^1$, and μ_1 is a finite measure, it follows by the bounded convergence theorem that $S_{-N}^{N}(t) \to 1$ in $L^2(\mu_1)$. It then follows from (3.2.7) that

$$0 = \int |f(t) - \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})\phi_n(t;W)|^2 d\mu_1(t) ,$$

proving the convergence of (3.2.2) in $L^2(\mu_1)$. Since T is a bounded linear operator on $B_1(W)$, and hence continuous, (3.2.5) follows with the series converging in $L^2(\mu_1)$. Masry and Cambanis (1976) showed that if $h_1, h_2 \in B_1(W)$, then

$$|h_1(t)-h_2(t)| \leq C(1+t^2)^{\frac{1}{2}} ||h_1-h_2||_{L^2(\mu_1)} , \quad t \in \mathbb{R}^1 ,$$

which implies that (2.5.2) converges uniformly on compact sets. □

<u>Remark 3.2.2</u>. As in Remark 3.2.1, we notice that if the Fourier transform $\hat{g}$ of $g(t) = [f(t)-f(0)]t^{-1}$ satisfies the condition: $\hat{g}(u)(1 + e^{\pi i \frac{u}{W_0}})^{-1} \in L^1[-W_0, W_0]$, then (3.2.8) and thus Theorem 3.2.5 remain true for $W = W_0$.

For $k \geq 2$, (3.2.5) is not valid, since it is not valid when T is the identity operator (a counter example is $f(t) = t$ which was mentioned earlier). The following expansion for $k \geq 2$ involves the derivatives of f at zero and the functions $t^j \in B_{j+1}(W)$, $0 \leq j \leq k-2$, and $t^{k-1} \frac{\sin 2\pi Wt}{2\pi Wt} \in B_{k-1}(W)$, so that functions $f \in B_k(W_0)$ are reconstructed from their samples and the values of their derivatives at zero using functions in $B_j(W)$, $1 \le j \le k-1$, $W > W_0$, i.e. in $B_{k-1}(W)$.

Theorem 3.2.6. Let $f \in B_k(W_0)$, $k \ge 2$, and $W > W_0$. If T is a bounded linear operator on $B_k(W)$, then

$$[Tf](t) = [Tf_{k-1}](t) + \sum_{n \neq 0} \{f(\tfrac{n}{2W}) - f_{k-2}(\tfrac{n}{2W})\} [T\phi_{n,k}](t), \quad t \in \mathbb{R}^1 , \tag{3.2.10}$$

where

$$f_{k-2}(t) = \sum_{j=0}^{k-2} \frac{f^{(j)}(0)}{j!} t^j, \quad f_{k-1}(t) = f_{k-2}(t) + \frac{f^{(k-1)}(0)}{(k-1)!} t^{k-1} \frac{\sin 2\pi Wt}{2\pi Wt},$$

$$\phi_{n,k}(t) = (\frac{2Wt}{n})^{k-1} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} ,$$

and the series converges in $L^2(\mu_k)$, as well as uniformly on compact sets.

Proof. From (b) of Lemma 3.2.1 we have

$$f(t) \triangleq \frac{k!}{t^{k-1}} \left[f(t) - \sum_{j=0}^{k-2} \frac{f^{(j)}(0)}{j!} t^j \right] = kf^{(k-1)}(0) + tg(t) , \tag{3.2.11}$$

for some $g \in B_0(W_0)$. Since $F(0) = kf^{(k-1)}(0)$, then by part (a) of Lemma 3.2.1 we have that $F \in B_1(W_0)$ and then by (3.2.5)

$$F(t) = \sum_{n=-\infty}^{\infty} F(\tfrac{n}{2W}) \phi_n(t;W) , \quad t \in \mathbb{R}^1 , \tag{3.2.12}$$

where the series converges in $L^2(\mu_1)$ and uniformly on compact sets. Then from (3.2.11) and (3.2.12), it follows that for all $t \in \mathbb{R}^1$,

$$f(t) = f_{k-1}(t) + \sum_{n \neq 0} \{f(\tfrac{n}{2W}) - f_{k-2}(\tfrac{n}{2W})\} (\frac{2Wt}{n})^{k-1} \phi_n(t;W) , \tag{3.2.13}$$

and the series converges in $L^2(\mu_k)$ and uniformly on compact sets.

We notice that $t^{j-1} \epsilon B_j(W)$, $1 \le j \le k$, since $t^{j-1} \epsilon L^2(\mu_k)$ and for any function $\psi \epsilon S$ whose Fourier transform $\hat{\psi}$ is a test function with $\hat{\psi}(t) = 1$ on $[-W,W]$, and $\hat{\psi}(t) < 1$ for all $t \epsilon [-W,W]$ we have

$$\begin{aligned}
\int (t-u)^j \psi(u)\,du &= \int t^j \psi(u)\,du + \int [\sum_{r=1}^{j} \binom{j}{r} t^j (-u)^{j-r}] \psi(u)\,du \\
&= t^j \hat{\psi}(0) + \sum_{r=1}^{j} \binom{j}{r} t^r (-1)^{j-r} \int u^{j-r} \psi(u)\,du \\
&= t^j + \sum_{r=1}^{j} \binom{j}{r} t^r (-1)^{j-r} \hat{\psi}^{(j-r)}(0) \\
&= t^j \ .
\end{aligned}$$

We also notice that $t^{k-1}\phi_n(t;W)$ belongs to $B_k(W)$ by (b) of Lemma 3.2.1 since $\phi_n(t;W)\ B_0(W)$ for all n. Now since T is a bounded linear operator on $B_k(W)$, and hence continuous, (3.2.10) holds where the convergence is in $L^2(\mu_k)$, and the uniform convergence on compact sets follows by (c) of Lemma 2 of Masry and Cambanis (1976).

Example 3.2.1. The m-th derivative operator $[D^{(m)}f](t) = f^{(m)}(t)$, $f \epsilon B_k(W_0)$, $m,k \ge 1$, is a bounded linear operator on $B_k(W_0)$.

Proof. Since $f \epsilon B_k(W_0)$, then by (a) of Lemma 3.2.1 we have

$$f(t) = \int_{-\infty}^{\infty} f(u) \psi(t-u)\,du$$

for any $\psi \epsilon S$ such that $\hat{\psi}$ is a symmetric test function supported by $[-W-\delta, W+\delta]$ for some $0 < \delta < W_0$, $\hat{\psi}(t) = 1$ on $[-W_0, W_0]$, and $\hat{\psi}(t) < 1$ on $[-W_0, W_0]^c$. Hence, for all $t \epsilon \mathbb{R}^1$,

$$[D^{(m)}f](t) = \int_{-\infty}^{\infty} f(u) [D^{(m)}\psi](t-u)\,du$$

and

$$|[D^{(m)}f](t)|^2 \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(u)|\cdot|f(v)|\cdot|[D^{(m)}\psi](t-u)|\cdot|[D^{(m)}\psi](t-v)\ du\ dv\ .$$

Since $D^{(m)}\psi\epsilon S$, $m\geq 1$, then $|[D^{(m)}\psi](x)| \leq \dfrac{C_{m,n}}{(1+x^2)^n}$, for every $n \geq 0$,

and thus for $n = k$,

$$|[D^{(m)}f](t)|^2 \leq C_{m,k}^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(u)|\cdot|f(v)|}{(1+(t-u)^2)^k(1+(t-v)^2)^k}\ du\ dv$$

$$\leq \frac{C_{m,k}^2}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(u)|^2+|f(v)|^2)}{(1+(t-u)^2)^k(1+(t-v)^2)^k}\ du\ dv$$

$$\leq C'_{m,k}\int_{-\infty}^{\infty} \frac{|f(u)|^2}{(1+(t-u)^2)^k}\ du\ ,\quad t\epsilon\mathbb{R}^1\ , \tag{3.2.14}$$

where $C'_{m,k} = C_{m,k}^2\int_{-\infty}^{\infty}\dfrac{dx}{(1+x^2)^k}$, $m,k \geq 1$. From (3.2.14) we obtain

$$||D^{(m)}f||^2_{L^2(\mu_k)} \leq C'_{m,k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(u)|^2}{(1+t^2)^k(1+(t-u)^2)^k}\ du\ dt\ . \tag{3.2.15}$$

Now we argue as in Lee (1973). Letting

$$I_k(u) := \int_{-\infty}^{\infty}\frac{dt}{(t+t^2)^k(1+(t-u)^2)^k}\ ,$$

and using $(1+u^2) \leq 2(1+t^2)(1+(t-u)^2)$, we obtain

$$I_{k+1}(u) \leq \frac{2}{1+u^2}\ I_k(u)\ ,$$

and thus

$$I_k(u) \leq \frac{2^{k-1}}{(1+u^2)^{k-1}}\ I_1(u)$$

$$\leq \frac{\pi 2^{k-1}}{(1+u^2)^k}\ , \tag{3.2.16}$$

since $I_1(u) \le \frac{\pi}{1+u^2}$ (Zakai, 1965). From (3.2.15) and (3.2.16), it follows that

$$||D^{(m)}f||^2_{L^2(\mu_k)} \le 2^{k-1}\pi C^1_{m,k}||f||^2_{L^2(\mu_k)} . \qquad \square$$

From Example 3.2.1 we notice that if $f\epsilon B_k(W_0)$, then for any $W > W_0$, $m \ge 1$

$$f^{(m)}(t) = f^{(m)}_{k-1}(t) + \sum_{n\neq 0}\{f(\tfrac{n}{2W}) - f_{k-2}(\tfrac{n}{2W})\}\phi^{(m)}_{n,k}(t) \;, \; t\epsilon \mathbb{R}^1 ,$$

where f_{k-1}, f_{k-2}, and $\phi_{n,k}$ are as defined in Theorem 3.2.6.

We now obtain a convolution representation (which is a variation of part (a) of Lemma 3.2.1) and an alternative proof of the sampling expansion (3.2.3) for functions in $B_k(W_0)$, $k \ge 1$. This result is the analogue of (3.1.1) for functions in $B_k(W_0)$, $k \ge 1$.

<u>Theorem 3.2.7</u>. Let $f\epsilon B_k(W_0)$, $W > W_0$, and $0 < \beta < W-W_0$. If ψ is an arbitrary (but fixed) test function with support $(\psi)\subset[-1,1]$ and $\int_{-\infty}^{\infty}\psi(t)dt = 1$, then

$$f(t) = \int_{-\infty}^{\infty} f(u)\,\frac{\sin 2\pi W(t-u)}{2\pi W(t-u)}\,\hat{\psi}(\beta(t-u))du$$

$$= \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})\,\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\,\hat{\psi}(\beta(t-\tfrac{n}{2W})) \;, \; t\epsilon\mathbb{R}^1 ,$$

and the series converges pointwise everywhere.

<u>Proof</u>. Fix $f\epsilon B_k(W_0)$, $W > W_0$, and $0 < \beta < (W-W_0)$. Then for all $s\epsilon\mathbb{R}^1$, $f(\cdot)\hat{\psi}(\beta(s-\cdot))$ belongs to $L^2(\mathbb{R}^1)$. Indeed, for all $z\epsilon\mathbb{C}$,

$$\hat{\psi}(\beta z) = \int_{-1}^{1} \psi(u) e^{-2\pi i\beta zu}\, du ,$$

and integrating by parts n times, we obtain

$$(-2\pi i\beta z)^n \hat{\psi}(\beta z) = \int_{-1}^{1} \psi^{(n)}(u) e^{-2\pi i\beta zu}\, du .$$

Thus for any $n \geq 1$ we have

$$|z|^n \hat{\psi}(\beta z) \leq \frac{1}{(2\pi\beta)^n} e^{2\pi\beta|\operatorname{Im} z|} \int_{-1}^{1} |\psi^{(n)}(u)|\, du ,$$

and hence

$$|\hat{\psi}(\beta z)| \leq \frac{C_{n,\beta} e^{2\pi\beta|\operatorname{Im} z|}}{(1+|z|)^n} . \tag{3.2.18}$$

It follows that for every $s \in \mathbb{R}^1$,

$$\int_{-\infty}^{\infty} |f(t)\hat{\psi}(\beta(s-t))|^2 dt \leq C_{k,\beta}^2 \int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+|s-t|)^{2k}}\, dt$$

$$\leq 2^k C_{k,\beta}^2 (1+|s|)^{2k} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^k}\, dt < \infty$$

(since $(1+t^2)^k \leq 2^k(1+|s|)^{2k}(1+|t-s|)^{2k}$ for all $t,s \in \mathbb{R}^1$). Also, by by (d) of Lemma 3.2.1 we have

$$|f(z)| \leq C_k(1+|z|)^k e^{2\pi W_0 |I_m z|} , \quad z \in \mathbb{C}$$

and by (3.2.18)

$$|f(z)\hat{\psi}(\beta(s-z))| \leq C_k C_{k,\beta} \frac{(1+|z|)^k}{(1+|s-z|)^k} e^{2\pi(W_0+\beta)|I_m z|}$$

$$\leq C_k C_{k,\beta} (1+|s|)^k e^{2\pi(W_0+\beta)|I_m z|} ,$$

for all $s \in \mathbb{R}^1$ and $z \in \mathbb{C}$ (since $(1+|z|)^k \leq (1+|s|)^k(1+|s-z|)^k$). Thus by the Paley-Wiener Theorem, we have that for all $s \in \mathbb{R}^1$, $f(\cdot)\hat{\psi}(\beta(s-\cdot))$ belongs to $B_0(W_0+\beta)$. Since $W_0+\beta < W_0+W-W_0 = W$, by (3.1.1) we have that, for all $t,s \in \mathbb{R}^1$,

$$f(t)\hat{\psi}(\beta(s-t)) = \int_{-\infty}^{\infty} f(u) \frac{\sin 2\pi W(t-u)}{2\pi W(t-u)} \hat{\psi}(\beta(s-u))du$$

$$= \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \hat{\psi}(\beta(s-\tfrac{n}{2W})) , \tag{3.2.19}$$

and the series converges uniformly on $-\infty < t < \infty$. Now putting $s = t$ in (3.2.19), the required representation (3.2.17) follows. □

3.3. Bandlimited in Lloyd's Sense.

Our goal in this section is to obtain sampling expansions for bounded linear operators acting on classes of functions and stochastic processes bandlimited in Lloyd's sense. Lloyd (1959) extended the concept of "bandlimitedness" by allowing a "bandlimited" function to have a wandering, rather than compact, spectrum. An open set $V \subset \mathbb{R}^1$ is called a wandering set if there exists a real number $W > 0$ such that all its translates $\{V+2nW\}$, $n \in \mathbb{N}$, are disjoint. Lloyd derived a sampling expansion for wide-sense stationary processes whose spectral distributions F have wandering supports, and also proved that if $2W_0$ is the Lebesgue measure of supp(F), then $W_0 \leq W$.

Theorem 3.3.1. (Lloyd, 1959). Let $\{x(t), t \in \mathbb{R}^1\}$ be a measurable, second order, mean-square continuous, wide-sense stationary stochastic process, and V be the support of its spectral distribution F. If, for some fixed number $W > 0$, the translates $\{V+2nW\}$ of V are all disjoint then

$$(3.3.1)\quad x(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) x\left(\frac{n}{2W}\right) K\left(t - \frac{n}{2W}\right) \ , \ t\in\mathbb{R}^1 \ ,$$

where $K(t) = \frac{1}{2W} \int_V e^{2\pi itu} du$, $t\in\mathbb{R}^1$, and if, furthermore $\lim\sup_{t\in\mathbb{R}^1} |tK(t)| < \infty$, then

$$(3.3.2)\qquad x(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} x\left(\frac{n}{2W}\right) K\left(t - \frac{n}{2W}\right) \ , \ t\in\mathbb{R}^1 \ ,$$

where the convergence in both (3.3.1) and (3.3.2) is in the mean square sense.

Lee (1978) extended Lloyd's result to functions in $L^2(\mu_k)$ with wandering (distributional) spectra, and to non-stationary processes whose correlation functions have (distributional) spectra. Before stating Lee's results we introduce the following notation. Let $x = \{x(t),\ t\in\mathbb{R}^1\}$ be a measurable stochastic process with correlation function $R(t,s) = E[x(t)\overline{x}(s)]$, $t,s\in\mathbb{R}^1$, which satisfies

$$(3.3.3)\qquad \int_{-\infty}^{\infty} R(t,t)\, d\mu_k(t) < \infty \ , \ k \geq 0 \ ,$$

where $d\mu_k(t) = \frac{1}{(1+t^2)^k}\, dt$. We may define an operator R on $L^2(\mu_k)$ by $[Rf](t) = \int_{-\infty}^{\infty} R(t,s) f(s)\, d\mu_k(s)$. R is a trace class operator, with non-zero eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and corresponding eigenvectors $\{f_k\}_{k=1}^{\infty}$. Cambanis and Masry (1971) obtained the following representations for x and R:

$$(3.3.4)\qquad x(t) = \sum_{k=1}^{\infty} f_k(t)\xi_k \ , \ t\in\mathbb{R}^1 \ ,$$

where the series converges in the mean square sense and also in $L^2(\mu_k)$ a.s., and $\{\xi_k\}_k$ forms an orthogonal basis with $E|\xi_k|^2 = \lambda_k$ for the

Hilbert space $H(x)$ generated (in the mean-square sense) by the random variables of the process x, and

$$R(t,s) = \sum_{k=1}^{\infty} \lambda_k f_k(t)\bar{f}_k(s) \ , \ t,s \in \mathbb{R}^1 \ , \tag{3.3.5}$$

where the series converges absolutely and in $L^2(\mu_k)$.

Remark 3.3.1. Since R satisfies (3.3.3), then

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|R(t,s)|^2}{(1+t^2+s^2)^{2k}}\, dt\, ds < \infty$, and hence the Fourier transform $\hat{R}$ of R exists as a tempered distribution in the Sobolev space $H^{2,-2k}(\mathbb{R}^2)$ (see Trèves, 1967). If the support of $\hat{R}$ is contained in an open set V whose translates by $(2nW,2nW)$, for some $W > 0$, are all disjoint, then (Lee, 1978) for all k, $\mathrm{supp}(\hat{f}_k) = \{y\colon (y,y)\in \mathrm{supp}(\hat{R})\}\subset V_0 = \{v\colon (v,v)\in V\}$, and the translates of V_0 by $2nW$ are all disjoint. Let U_0 be an open set such that $\mathrm{supp}(\hat{f}_k)\subset U_0\subset \bar{U}_0\subset V_0$, let ψ be a C^∞-function that equals 1 on $\bar{U}_0$ and 0 on V_0^c, and $|\psi(t)| \le 1$ for all $t\in\mathbb{R}^1$, and let the function K be defined by

$$K(t) = \frac{1}{2W}\int_{-\infty}^{\infty} e^{2\pi i t u}\psi(u)\,du \ . \tag{3.3.6}$$

Theorem 3.3.2. (Lee, 1978). Let $f\in L^2(\mu_k)$, and suppose that there exists an open set $V\supset \mathrm{supp}(\hat{f})$ such that for some fixed $W > 0$ the translates $\{V+2nW\}$, $n\in\mathbb{N}$, of V are all disjoint. Let U be an open set such that $\mathrm{supp}(\hat{f})\subset U\subset\bar{U}\subset V$, ψ be a C^∞-function that is 1 on $\bar{U}$ and 0 on V^c, and K be the function $K(t) = \frac{1}{2W}\int_{-\infty}^{\infty} e^{2\pi i t u}\,\psi(u)\,du$. Then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)K\left(t - \frac{n}{2W}\right) \ , \ t\in\mathbb{R}^1 \ , \tag{3.3.7}$$

where the series converges pointwise everywhere.

Theorem 3.3.3. (Lee, 1978). Let $\{x(t),\ t\in\mathbb{R}^1\}$ be a measurable, second order, mean-square continuous process with correlation function R which satisfies (3.3.3) for some non-negative integer k. Let V be an open set such that $\mathrm{supp}(\hat{R})\subset V$, and suppose that, for some $W > 0$, the translates $\{V + (2nW, 2nW)\}$, $n\in\mathbb{N}$, are all disjoint. Then for each $t\in\mathbb{R}^1$,

$$(3.3.8) \qquad x(t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})K(t - \tfrac{n}{2W}) \ ,$$

where the series converges in quadratic mean and almost surely, and K is defined as in Remark 3.3.1.

We now generalize Kramer's expansion (3.2.1) to bandlimited functions with wandering spectra and Lee's expansion (3.3.7) (the case $k = 0$) to bounded linear operators acting on the space $L_k(U,V;W)$ which is defined as follows. Let U be an open set in $\mathbb{R}^1$ such that for some open set $V \supsetneq U$ and $W > 0$ all the translates $\{V + 2nW\}$, $n\in\mathbb{N}$ are disjoint. The class of all functions $f\in L^2(\mu_k)$ with $\mathrm{supp}(\hat{f})\subset U$ is denoted by $L_k(U,V;W)$, and simply by $L_k(U)$ when the set V is not required to have the properties stated above. Since the translates of $\bar{U}$ are disjoint, its Lebesgue measure is finite (Lloyd, 1959), for $k = 0$, $\hat{f}\in L^1(\mathbb{R}^1) \cap L^2(\mathbb{R}^1)$. It follows that

$$(3.3.9) \qquad f(t) = \int_{-\infty}^{\infty} e^{2\pi itu}\, \hat{f}(u)du \ , \text{ a.e.} \ ,$$

and thus every function in $L_0(U;V,W)$ has a continuous version. Only continuous versions will be considered in this section.

Theorem 3.3.4. If T is a bounded linear operator in $L^2(\mathbb{R}^1)$ such that T maps $L_0(U_0,V_0;W)$ into $L_0(U_0,V_0;W)$, and K is defined as in Theorem 3.3.2, then for every $t\in\mathbb{R}^1$,

$$[Tf](t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t) \ , \tag{3.3.10}$$

and the series converges uniformly on $\mathbb{R}^1$ as well as in $L^2(\mathbb{R}^1)$.

Proof. Define the function F by

$$F(u) = \sum_{m=-\infty}^{\infty} \hat{f}(u-2mW) \ , \ u \in \mathbb{R}^1 \ .$$

F is the 2W-periodic extension of $\hat{f}$, and $\int_{I_W} |F(u)|^2 du = ||f||^2_{L^2(\mathbb{R}^1)}$ where $I_W = (-W,W)$. Thus from the L^2-theory of Fourier series, F has the Fourier expansion

$$F(u) = \lim_{N\to\infty} \sum_{n=-N}^{N} C_n e^{-\pi i \frac{n}{W} u} \ , \tag{3.3.11}$$

in $L^2(-W,W)$, where

$$\begin{aligned} C_n &= \frac{1}{2W} \int_{I_W} F(u) e^{\pi i \frac{n}{W} u} du \\ &= \frac{1}{2W} \int_{I_W \cap (U_m \overline{U}_m)} F(u) e^{\pi i \frac{n}{W} u} du \quad (U_m = U + 2mW) \\ &= \frac{1}{2W} \sum_m \int_{I_W \cap \overline{U}_m} \hat{f}(u-2mW) e^{\pi i \frac{n}{W} u} du \\ &= \frac{1}{2W} \sum_m \int_{I_W \cap \overline{U}_m - 2mW} \hat{f}(u) e^{\pi i \frac{n}{W} u} du \ . \end{aligned}$$

But the sets $(I_W \cap \overline{U}_m) - 2mW = (I_W - 2mW) \cap \overline{U}$ are disjoint and their union is $\overline{U}$, so that

$$C_n = \frac{1}{2W} \int_{\overline{U}} \hat{f}(u) e^{\pi i \frac{n}{W} u} du = \frac{1}{2W} f(\tfrac{n}{2W}) \ . \tag{3.2.12}$$

We notice that for any $g \in L_0(U,V;W)$,

$$|g(t)| \le \int_U |\hat{g}(u)e^{2\pi itu}| du \le |U|^{\frac{1}{2}} ||g||_{L^2(\mathbb{R}^1)}, \tag{3.3.13}$$

where $|U|$ is the Lebesgue measure of U; i.e., the evaluation map on $L_0(U;V,W)$ is bounded with norm $\le |U|^{\frac{1}{2}}$.

Now consider the error

$$\begin{aligned}
e_N(t) :&= |[Tf](t) - \sum_{n=-N}^{N} f(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t)| \\
&\le |U|^{\frac{1}{2}} ||Tf - \sum_{n=-N}^{N} f(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})]||_{L^2(\mathbb{R}^1)} \\
&\le |U|^{\frac{1}{2}} ||T|| \{ \int_{-\infty}^{\infty} |f(t) - \sum_{n=-N}^{N} f(\tfrac{n}{2W})K(t - \tfrac{n}{2W})|^2 dt\}^{\frac{1}{2}} \\
&\le |U|^{\frac{1}{2}} ||T|| \{\int_{\bar{U}} |\hat{f}(u) - \sum_{n=-N}^{N} \tfrac{1}{2W} f(\tfrac{n}{2W}) e^{-\pi i \frac{n}{W} u}|^2 du \\
&\qquad + \int_{V-\bar{U}} |\psi(u)|^2 | \sum_{n=-N}^{N} \tfrac{1}{2W} f(\tfrac{n}{2W}) e^{-\pi i \frac{n}{W} u}|^2 du\}^{\frac{1}{2}} \\
&\le |U|^{\frac{1}{2}} ||T|| \{\int_{\bar{U}} |F(u) - \sum_{n=-N}^{N} C_n e^{-\pi i \frac{n}{W} u}|^2 du \\
&\qquad + \int_{V-\bar{U}} | \sum_{n=-N}^{N} C_n e^{-\pi i \frac{n}{W} u}|^2 du\}^{\frac{1}{2}} .
\end{aligned} \tag{3.3.14}$$

By (3.3.11) and noting that $F(u) = 0$ on $V-\bar{U}$, it follows that the right hand side of the last inequality in (3.3.14) tends to zero independently of t, then $e_N(t)$ converges to zero uniformly in t on $\mathbb{R}^1$. □

We now consider the stochastic analogue of Theorem 3.3.4 for processes bandlimited in Lloyd's sense which are not necessarily stationary. To fix notation, let $x = \{x(t), t\in\mathbb{R}^1\}$ be a measurable, second order, mean-square continuous process with correlation function R which satisfies (3.3.3) for $k = 0$. Assume that V is an open set in $\mathbb{R}^2$ such that, for some fixed $W > 0$, the translates $\{V+(2nW,2nW)\}$, $n\in\mathbb{N}$, of V are all disjoint, and let $U \subsetneqq V$ be a fixed open set in $\mathbb{R}^2$. If $\mathrm{supp}(\hat{R}) \subset U$, then almost all sample paths of x belong to $L_0(U_0;V_0,W)$, where $U_0 = \{u: (u,u)\in U\}$ and $V_0 = \{v: (v,v)\in V\}$ (Lee, 1978). It then follows from Theorem 3.3.4 that if T is a bounded linear operator in $L^2(\mathbb{R}^1)$ such that T maps $L_0(U_0,V_0;W)$ into $L_0(U_0,V_0;W)$, then with probability one

$$[Tx](t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})[Tk(\cdot - \tfrac{n}{2W})](t) \quad , \; t\in\mathbb{R}^1 , \tag{3.3.15}$$

where the series converges uniformly on $\mathbb{R}^1$, as well as in $L^2(\mathbb{R}^1)$.

We show that under appropriate conditions on the operator T (Theorem 3.3.5) or on the correlation function R of the process x (Theorem 3.3.6) the expansion (3.3.15) converges also in quadratic mean.

Let T be a bounded linear operator in $L^2(\mathbb{R}^1)$ of integral type with kernel $S\in L^2(\mathbb{R}^2)$:

$$[Tf](t) = \int_{-\infty}^{\infty} S(t,u)\overline{f}(u)du \quad \text{a.e.}, \; f\in L^2(\mathbb{R}^1) , \tag{3.3.16}$$

and assume that the kernel S satisfies the following conditions:

(i) $S(t,\cdot)\in L^2(\mathbb{R}^1)$ for all $t\in\mathbb{R}^1$ and $t \mapsto S(t,\cdot)$ is a continuous map from $\mathbb{R}^1$ into $L^2(\mathbb{R}^1)$, so that each Tf has a continuous version for which (3.3.16) holds for all $t\in\mathbb{R}^1$.

(ii) $S\in[L_0(U_0;V_0,W) \otimes L_0(U_0;V_0,W)] \oplus [L^2(\mathbb{R}^1) \otimes L_0(U_0^c)]$; so that T maps $L_0(U_0;V,W)$ into $L_0(U_0;V,W)$, (note that $L_0(U_0^c) = L_0^{\perp}(U_0;V_0,W)$).

(iii) $\int_{-\infty}^{\infty} (1+u^2)|S(t,u)|du < \infty$ for all $t \in \mathbb{R}^1$.

Since the sample paths of x belong to $L_0(U_0;V_0,W)$ a.s., we have with probability one

$$[Tx(\cdot,\omega)](t) = \int_{-\infty}^{\infty} S(t,u)x(u,\omega)du\ , \quad \text{for all } t \in \mathbb{R}^1, \tag{3.3.17}$$

and we now show that under condition (iii) on the kernel S of T, the series in (3.3.15) converges also in quadratic mean for each $t \in \mathbb{R}^1$.

Theorem 3.3.5. Let $x = \{x(t),\ t \in \mathbb{R}^1\}$ and T be as defined above. Then

$$[Tx](t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t) \tag{3.3.18}$$

where the series converges in the mean square sense for every $t \in \mathbb{R}^1$, and K is as defined in Remark 3.3.1.

Proof. Recall the expansion (3.3.4) of x where the series converges in $L^2(\mathbb{R}^1)$ a.s. Since almost all sample paths of x as well as all f_k belong to $L_0(U_0;V,W)$, (Lee, 1978), and since T is a continuous linear operator on $L_0(U_0;V_0,W)$, it follows that with probability one,

$$[Tx](t) = \sum_{k=1}^{\infty} [Tf_k](t)\xi_k \tag{3.3.19}$$

in $L^2(\mathbb{R}^1)$ and also pointwise everywhere by (3.3.13). Thus, for every $t \in \mathbb{R}^1$, we have

$$\begin{aligned} E\left|[Tx](t) - \sum_{k=1}^{N}[Tf_k](t)\xi_k\right|^2 &= E\left|\sum_{k=N+1}^{\infty}[Tf_k](t)\xi_k\right|^2 \\ &= \sum_{k=N+1}^{\infty}\ \sum_{p=N+1}^{\infty} |[Tf_k](t)|\cdot|[Tf_p](t)|E(\xi_k\bar{\xi}_p) \\ &= \sum_{k=N+1}^{\infty}\lambda_k|[Tf_k](t)|^2 . \end{aligned}$$

But by (3.3.13)

$$|[Tf_k](t)|^2 \le |U|\cdot||Tf_k||^2_{L^2(\mathbb{R}^1)} \le |U|\cdot||T||^2||f_k||^2_{L^2(\mathbb{R}^1)}$$

$$= |U|\cdot||T||^2 .$$

Thus for every $t\in\mathbb{R}^1$,

$$E|[Tx](t) - \sum_{k=1}^{N}[Tf_k]\xi_k|^2 \le |U|\cdot||T||^2\sum_{k=N+1}^{\infty}\lambda_k \to 0 \text{ as } N\to\infty,$$

and the series in (3.3.19) converges in the mean square sense, for each $t\in\mathbb{R}^1$.

Now consider the mean square error

$$e_N^2(t) := E|[Tx](t) - \sum_{n=-N}^{N}x(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t)|^2 , \quad t\in\mathbb{R}^1 .$$

Making use of the convergence of the series in (3.3.19) in quadratic mean for each $t\in\mathbb{R}^1$, we obtain

$$\begin{aligned}
e_N^2(t) &= |[Tx](t)|^2 - \sum_{n=-N}^{N}E([Tx](t)\overline{x}(\tfrac{n}{2W}))[\overline{TK}(\cdot - \tfrac{n}{2W})](t) \\
&\quad - \sum_{n=-N}^{N}E([\overline{Tx}](t)x(\tfrac{n}{2W}))[TK(\cdot - \tfrac{n}{2W})](t) \\
&\quad + \sum_{n=-N}^{N}\sum_{m=-N}^{N}E(x(\tfrac{n}{2W})\overline{x}(\tfrac{m}{2W}))[TK(\cdot - \tfrac{n}{2W})](t)[\overline{TK}(\cdot - \tfrac{m}{2W})](t) \\
&= \sum_{k=1}^{\infty}\lambda_k\{|[Tf_k](t)|^2 - \sum_{n=-N}^{N}[Tf_k](t)\overline{f}_k(\tfrac{n}{2W})[\overline{TK}(\cdot - \tfrac{n}{2W})](t) \\
&\quad - \sum_{n=-N}^{N}[\overline{Tf}_k](t)f_k(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t) \\
&\quad + \sum_{n=-N}^{N}\sum_{m=-N}^{N}f_k(\tfrac{n}{2W})\overline{f}_k(\tfrac{m}{2W})[TK(\cdot - \tfrac{n}{2W})](t)[\overline{TK}(\cdot - \tfrac{m}{2W})](t)\}
\end{aligned}$$

$$(3.3.20) \qquad = \sum_{k=1}^{\infty}\lambda_k|[Tf_k](t) - \sum_{n=-N}^{N}f_k(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t)|^2 .$$

We have

$$\varepsilon_{N,k}(t) := |[Tf_k](t) - \sum_{n=-N}^{N} f_k(\tfrac{n}{2W})[TK(\cdot - \tfrac{n}{2W})](t)|$$

$$\leq \int_{-\infty}^{\infty} |S(t,u)|\cdot|f_k(u) - \sum_{n=-N}^{n} f_k(\tfrac{n}{2W})K(u - \tfrac{n}{2W})|du$$

$$= \int_{-\infty}^{\infty} |S(t,u)|\cdot|\int_{\overline{U}_0} \hat{f}_k(v)e^{2\pi iuv}dv$$

$$- \sum_{n=-N}^{N} (\int_{\overline{U}_0} \hat{f}_k(v)e^{\pi i \frac{n}{W} v}dv)K(u - \tfrac{n}{2W})|du$$

$$\leq \int_{-\infty}^{\infty} |S(t,u)|\cdot(\int_{\overline{U}_0} |\hat{f}_k(v)|\cdot|e^{2\pi iuv}$$

$$- \sum_{n=-N}^{N} e^{\pi i \frac{n}{W} v} K(u - \tfrac{n}{2W})|dv)du \,. \tag{3.3.21}$$

For every $u \in \mathbb{R}^1$, define $\phi_u(v) = \sum_{n=-\infty}^{\infty}\psi(v+2nW)e^{2\pi i(v+2nW)u}$ (Lee, 1978). Then for each $u \in \mathbb{R}^1$, ϕ_u is a periodic C^∞-function with Fourier expansion

$$\phi_u(v) = \sum_{n=-\infty}^{\infty} K(u - \tfrac{n}{2W})e^{\pi i \frac{n}{W} v} \,,$$

where the series converges uniformly on $-\infty < v < \infty$. We have

$$\varepsilon_N(u) := \sup_{v\in\overline{U}_0} |e^{2\pi iuv} - \sum_{n=-N}^{N} K(u - \tfrac{n}{2W})e^{\pi i \frac{n}{W} v}|$$

$$\leq \sup_{v\in\mathbb{R}^1} |\phi_u(v) - \sum_{n=-N}^{N} K(u - \tfrac{n}{2W})e^{\pi i \frac{n}{W} v}|$$

$$= \sup_{v\in\mathbb{R}^1} |\sum_{|n|>N} K(u - \tfrac{n}{2W})e^{\pi i \frac{n}{W} v}|$$

$$\le \sum_{|n|>N} |K(u - \tfrac{n}{2W})| \ ,$$

and since $K \in S$ implies $|K(u)| \le \frac{C}{1+u^2}$ for some $C > 0$,

$$\varepsilon_N(u) \le C \sum_{|n|>N} \frac{1}{1 + (u - \frac{n}{2W})^2}$$

$$\le 2C(1+u^2) \sum_{|n|>N} \frac{1}{1 + (\frac{n}{2W})^2}$$

$$\le \frac{16W^2C}{N} (1+u^2) \ , \ u \in \mathbb{R}^1 \ .$$

From (3.3.21) we thus have,

$$\varepsilon_{N,k}(t) \le \int_{-\infty}^{\infty} |S(t,u)| \varepsilon_N(u) du \int_{U_0} |\hat{f}_k(v)| dv$$

$$\le \frac{16W^2C}{N} |U_0|^{\frac{1}{2}} \int_{-\infty}^{\infty} (1+u^2) |S(t,u)| du \ .$$

It follows from (3.3.20) that for all $t \in \mathbb{R}^1$,

$$e_N^2(t) \le \frac{|U_0| (16W^2C)^2}{N^2} \Big(\sum_{k=1}^{\infty} \lambda_k \Big) \int_{-\infty}^{\infty} (1+u^2) |S(t,u)| du \to 0 \quad \text{as } N \to \infty$$

by property (iii) of S, and thus the series in (3.3.18) converges in the mean square sense. □

It should be noted that the integral type operator T was defined on all $f \in L^2(\mathbb{R}^1)$ since $K \notin \mathcal{L}_0(U_0;V_0,W)$. However, one could take the C^∞-function ψ (of Remark 3.3.1) equal to zero on A^c for some open set A such that $U_0 \subsetneq A \subsetneq V_0$. In this case $K \in \mathcal{L}_0(A;V_0,W)$, and it would suffice to define the integral type operator T on $\mathcal{L}_0(A;V_0,W)$, rather than on $L^2(\mathbb{R}^1)$ (also conditions (i) and (ii) would need the obvious modifications).

Under further conditions on the process x, (3.3.15) converges in quadratic mean for all bounded linear operators.

Theorem 3.3.6. Let $x = \{x(t), t\in \mathbb{R}^1\}$ be as in Theorem 3.3.5, and in addition assume that

$$\text{(i)} \quad \int_{-\infty}^{\infty} (1+t^2)\sqrt{R(t,t)}\, dt < \infty$$

$$\text{(ii)} \quad \sum_{n=-\infty}^{\infty} \left(1 + \left(\frac{n}{2W}\right)^4\right) \sqrt{R((n/2W),(n/2W))} < \infty .$$

Then if T is any bounded linear operator in $L^2(\mathbb{R}^1)$ which maps $L_0(U_0;V_0,W)$ into $L_0(U_0;V_0,W)$, (3.3.15) holds where the series converges in quadratic mean for every $t\in\mathbb{R}^1$.

Proof. Notice that by (3.3.20) we have

$$e_N^2(t) \le ||T||^2 \sum_{k=1}^{\infty} \lambda_k || f_k - \sum_{|n|\le N} f_k\left(\frac{n}{2W}\right)K\left(\cdot - \frac{n}{2W}\right)||_{L^2(\mathbb{R}^1)}$$

$$= ||T||^2 \sum_{k=1}^{\infty} \lambda_k \int_{-\infty}^{\infty} dt\{|f_k(t)|^2 - f_k(t) \sum_{|n|\le N} \overline{f}\left(\frac{n}{2W}\right)\overline{K}\left(t - \frac{n}{2W}\right)$$

$$- \overline{f}_k(t) \sum_{|n|\le N} f_k\left(\frac{n}{2W}\right)K\left(t - \frac{n}{2W}\right)$$

$$+ \sum_{|n|,|m|\le N} f_k\left(\frac{n}{2W}\right)\overline{f}_k\left(\frac{m}{2W}\right)K\left(t - \frac{n}{2W}\right)\overline{K}\left(t - \frac{m}{2W}\right)\} .$$

It follows by (3.3.5) and monotone convergence that $\sum_{k=1}^{\infty}\lambda_k \int_{-\infty}^{\infty} |f_k(t)|^2 dt = \int_{-\infty}^{\infty} R(t,t)dt$. By (3.3.4) we obtain

$$\int_{-\infty}^{\infty} R\left(t,\frac{m}{2W}\right)\overline{K}\left(t - \frac{m}{2W}\right)dt = E\{ \int_{-\infty}^{\infty} x(t)\overline{K}\left(t - \frac{m}{2W}\right)dt\cdot\overline{x}\left(\frac{m}{2W}\right)\}$$

$$= E\{[\sum_{k=1}^{\infty} \xi_k \int_{-\infty}^{\infty} f_k(t)\overline{K}\left(t - \frac{m}{2W}\right)dt][\sum_{k=1}^{\infty} \overline{\xi}_k\overline{f}_k\left(\frac{m}{2W}\right)]\}$$

$$= \sum_{k=1}^{\infty} \lambda_k \overline{f}_k(\tfrac{m}{2W}) \int_{-\infty}^{\infty} f_k(t) K(t - \tfrac{m}{2W}) dt ,$$

and thus (3.3.22) can be written as follows:

$$e_N^2(t) \le ||T||^2 \int_{-\infty}^{\infty} |[R(t,t) - \sum_{|n|\le N} R(t, \tfrac{n}{2W}) K(t - \tfrac{n}{2W})]$$

$$- \sum_{|n|\le N} [R(\tfrac{n}{2W}, t) - \sum_{|m|\le N} R(\tfrac{n}{2W}, \tfrac{m}{2W}) K(t - \tfrac{m}{2W})]$$

$$\cdot K(t - \tfrac{n}{2W})| dt . \tag{3.2.23}$$

But by Theorem 3.3.3 we have for every $t,s \in \mathbb{R}^1$

$$R(t,s) = \sum_{n=-\infty}^{\infty} R(t, \tfrac{n}{2W}) K(s - \tfrac{n}{2W}) ;$$

substituting in (3.3.23) we obtain

$$e_N^2(t) \le ||T||^2 \{ \int_{-\infty}^{\infty} \sum_{|n|>N} |R(t, \tfrac{n}{2W})| |K(t - \tfrac{n}{2W})| dt$$

$$+ \sum_{|n|\le N} \sum_{|m|>N} |R(\tfrac{n}{2W}, \tfrac{m}{2W})| \int_{-\infty}^{\infty} |K(t - \tfrac{n}{2W})| \cdot |K(t - \tfrac{m}{2W})| dt\} .$$

Since $K \epsilon S$ implies $|K(t)| \le \dfrac{C_k}{(1+t^2)^k}$ for each $k \ge 0$ and some $C_k > 0$, and $|R(t,s)| \le \sqrt{R(t,t)} \cdot \sqrt{R(s,s)}$, we have that the first term on the right hand side of (3.3.24) is less than

$$2C_1 ||T||^2 (\int_{-\infty}^{\infty} (1+t^2) \sqrt{R(t,t)}\, dt) \sum_{|n|>N} \frac{\sqrt{R((n/2W),(n/2W))}}{1 + (\frac{n}{2W})^2} \to 0 \text{ as } N \to \infty,$$

by (ii). For the second term, notice that

$$\int_{-\infty}^{\infty} |K(t - \tfrac{n}{2W})| \cdot |K(t - \tfrac{m}{2W})|\, dt \le 8C_1C_2 \int_{-\infty}^{\infty} \frac{[1+(\frac{n}{2W})^2]^2}{(1+t^2)^2} \cdot \frac{(1+t^2)}{[1+(\frac{m}{2W})^2]}\, dt$$

$$= 8\pi C_1C_2 \frac{[1+(\frac{n}{2W})^2]^2}{[1+(\frac{m}{2W})^2]} \quad .$$

Thus the second term on the right hand side of (3.3.24) is less than

$$8C_1C_2\left(\sum_{|n|\le N}(1 + (\tfrac{n}{2W})^2)^2\sqrt{R((n/2W),(n/2W))}\right)$$

$$\cdot \left(\sum_{|m|>N} \frac{\sqrt{R((m/2W),(m/2W))}}{1 + (\frac{m}{2W})^2}\right) \to 0 \text{ as } N \to \infty ,$$

by (ii) and the result follows. □

It should be noted that when $R(t,t)$ is asymptotically monotonic at $\pm\infty$ then conditions (i) and (ii) of Theorem 3.3.6 may be replaced by

$$\int_{-\infty}^{\infty} (1+t^4)\sqrt{R(t,t)}\, dt < \infty .$$

CHAPTER IV

Sampling Expansions for Distributions and Random Distributions

4.1. Introduction.

This chapter is concerned with sampling representations for distributions and random distributions. The need to consider distributions (beyond classical functions) arises from the fact that in many physical situations it may be impossible to observe the instantaneous values f(t) (of a physical phenomenon) at the various values of t. For instance, if t represents time or a point in space, any measuring instrument would merely record the effect that f produces on it over non-vanishing intervals of time I: $\int_I f(t)\phi(t)dt$, where ϕ is a "smooth" function representing the measuring instrument, i.e. the physical phenomenon is specified as a functional rather than a function. Furthermore, it is becoming exceedingly clear that the tools and techniques of the theory of distributions are useful in investigating certain problems in many applied areas. It is thus of interest to consider distributions beyond functions.

4.2. Notation of Basic Definitions.

Let $C_c^\infty = C_c^\infty(\mathbb{R}^1)$ be the class of all infinitely differentiable functions with compact support. A topology τ is introduced on the linear space C_c^∞ which makes it into a complete space: $C_c^\infty \ni \phi_n \to 0$ in τ if there exists a compact set $A \subset \mathbb{R}^1$ which contains the support of every ϕ_n, and for every non-negative integer k, $\phi_n^{(k)}(t) \to 0$ uniformly as $n \to \infty$. C_c^∞ with the topology τ is denoted by $\mathcal{D}$, and its elements are called test functions. The members of the dual $\mathcal{D}'$ of $\mathcal{D}$ are called distributions, and the value of a distribution $f\in\mathcal{D}'$ at a test function $\phi\in\mathcal{D}$ is denoted by $f(\phi)$. A (weak-star) topology on $\mathcal{D}'$ is defined by the seminorms $|f(\phi)|$, $f\in\mathcal{D}'$, as ϕ varies over all elements of $\mathcal{D}$; thus, $\mathcal{D}'\ni f_n \to 0$ weakly whenever $f_n(\phi) \to 0$ for all $\phi\in\mathcal{D}$.

The class S of rapidly decreasing functions consists of all infinitely differentiable functions ($\phi\in C^\infty$) for which

$$|t^m\phi^{(k)}(t)| \le C_{m,k} , \quad -\infty < t < \infty,$$

for all non-negative integers m,k. A topology on S is defined by the seminorms

$$||\phi||_{m,k} = \sup_{0\le n\le m} \sup_{t\in\mathbb{R}^1} \{(1+|t|)^k|\phi^{(n)}(t)|\} , \quad m,k = 0,1,2,\ldots,$$

i.e., a sequence $\{\phi_n\}_{n=1}^\infty$ is of functions in S is said to converge in S, if for every set of non-negative integers, the sequence $\{(1+|t|)^m\phi_n^{(k)}(t)\}_{n=1}^\infty$ converges uniformly on $\mathbb{R}^1$. S is complete, and the dual S' of S is called the class of tempered distributions. Similarly, a (weak-star) topology is defined on S' by the seminorms $|f(\phi)|$, $f\in S'$, as ϕ varies over all elements of S, i.e., f_n converges in S' if $f_n(\phi)$ converges for all $\phi\in S$. The space $\mathcal{D}'(S')$ is (weak-star) sequentially complete, that is, if $\{f_n\}_n$ is a sequence in $\mathcal{D}'(S')$ such that $\{f_n(\phi)\}_n$ is a Cauchy sequence for every $\phi\in\mathcal{D}(S)$, then there exists a distribution $f\in\mathcal{D}'(S')$ such that $f_n \to f$ in $\mathcal{D}'(S')$.

Finally, the space C^∞ with the topology defined by the seminorms

$$P_{m,A}(\phi) = \sum_{0\le n\le m} \sup_{t\in A}|\phi^{(n)}(t)| \quad , \quad \phi\in C^\infty ,$$

where A ranges over all compact sets in $\mathbb{R}^1$ and m over all non-negative integers, is denoted by E.

The Fourier transform $F(F(\phi) = \hat{\phi}, \phi\epsilon S)$ is a one-to-one bicontinuous mapping from S onto itself. If $f\epsilon S'$, the Fourier transform $\hat{f}$ of f is defined by $\hat{f}(\phi) = f(\hat{\phi})$, $\phi\epsilon S$, and is a tempered distribution. If $f\epsilon S'$ and $\phi\epsilon S$, the convolution $f*\phi$ is defined as a function on $\mathbb{R}^1$ by

$$(f*\phi)(t) = f(\tau_t\check{\phi}) \ , \ t\epsilon\mathbb{R}^1 \ ,$$

where $\check{\phi}(t) = \phi(-t)$ and the shift operator τ_t is defined by $(\tau_t\phi)(u) = \phi(u-t)$. $f*\phi\epsilon C^\infty$ has a polynomial growth and thus determines a tempered distribution.

Suppose $f\epsilon D'$, f is said to vanish in an open set $U\subset\mathbb{R}^1$ if $f(\phi) = 0$ for every $\phi\epsilon D$ with $\mathrm{supp}(\phi)\subset U$. Let V be the union of all open sets $U\epsilon\mathbb{R}^1$ in which f vanishes. The complement of V is the support of f. Distributions with compact supports are tempered distributions. Now, if f is a distribution with compact support (i.e., $f\epsilon S'$), then f extends uniquely to a continuous linear functional on E. If $\psi\epsilon D$ is such that $\psi(u) = 1$ on some open set containing supp(f), then $\psi f = f$ i.e. $(\psi f)(\phi) = f(\psi\phi) = f(\phi)$ for all $\phi\epsilon S$, but since $e_t(u) = e^{2\pi itu}$ is a C^∞-function, $f(e_t) = f(\psi e_t)$ exists, and the distribution $\hat{f}$ is generated by the function $\hat{f}(t)$ defined on $\mathbb{R}^1$ by

(4.2.1) $$\hat{f}(t) = f(e_t) \ .$$

Indeed

(4.2.2) $$\hat{f} = (\psi f)^\wedge \ ,$$

and $(\psi f)^\wedge$ (and therefore $\hat{f}$) is generated by the C^∞-function $(\hat{f}*\hat{\psi})(t)$

which has a polynomial growth (see Rudin, 1973, p. 179). By choosing $\phi \in S$ such that $\hat{\phi} = \psi$, we have

$$(\hat{f} * \hat{\psi})(t) = (\hat{f} * \check{\phi})(t) = \hat{f}(\tau_t \phi) = f((\tau_t \phi)^\wedge)$$

$$= f(e_t \hat{\phi}) = f(\psi e_t) = f(e_t) ,$$

and from (4.2.2), (4.2.1) is justified. Hence the Fourier transform of a distribution with compact support may be thought of as a function defined on $\mathbb{R}^1$ by (4.2.1).

Let (Ω, F, P) be a probability space. A random distribution is a continuous linear operator from $\mathcal{D}$ (or S) into a topological vector space of random variables. Specifically, a second order random distribution is a continuous linear operator from $\mathcal{D}$ (or S) onto $L^2(\Omega) = L^2(\Omega, F, P)$, the Hilbert space of all finite second moment random variables.

4.3. Sampling Expansions for Certain Distributions.

In this section we establish a sampling theorem for tempered distributions whose Fourier transforms have compact supports. A distribution $f \in S'$ is said to be W-bandlimited, $W > 0$, if $\mathrm{supp}(\hat{f}) \subset (-W,W)$. The class of all W-bandlimited distributions will be denoted by $B^d(W)$.

Let $\mathcal{D}[-W,W]$, $W > 0$, be the class of all C^∞-functions ϕ with $\mathrm{supp}(\phi) \subset [-W,W]$, and define $Z(W) \triangleq \hat{\mathcal{D}}[-W,W] = \{\hat{\phi} \in S: \phi \in \mathcal{D}[-W,W]\}$. Pfaffelhuber (1971) stated that if $H \in B^d(W)$ and h is its Fourier transform (defined as a function on $\mathbb{R}^1$), then

$$h(t) = \sum_{n=-\infty}^{\infty} h\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \tag{4.3.1}$$

and the series converges absolutely in $Z'(W)$ (the dual of $Z(W)$). (4.3.1) means precisely that, for every $\phi \in Z(W)$,

$$\int_{-\infty}^{\infty} h(t)\phi(t)dt = \sum_{n=-\infty}^{\infty} h(\frac{n}{2W}) \int_{-\infty}^{\infty} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \phi(t)dt ,$$

and the series converges absolutely. Campbell (1968) had already noted that (4.3.1) does not hold pointwise for arbitrary bandlimited distributions. Though (4.3.1) is correct, the arguments presented in its proof are not convincing.

The following lemma is a modification of Lemma 1 of Pfaffelhuber (1971), and will be needed in the proof of theorem 4.3.1.

<u>Lemma 4.3.1</u>. Let $f \in S'$ be such that $\hat{f}$ has compact support. Let E be a closed set properly containing $\mathrm{supp}(\hat{f})$, and ψ any test function with support E and $\psi = 1$ on some open set containing $\mathrm{supp}(\hat{f})$. Then f is uniquely determined by its restriction to $\hat{\mathcal{D}}(E)$, i.e., the values $f(\theta)$, $\theta \in \hat{\mathcal{D}}(f)$, by

$$f(\phi) = f(\hat{\psi} * \phi) , \quad \phi \in S . \tag{4.3.2}$$

The shift operator τ_ℓ is defined on $\mathcal{D}'(S')$, for every $\ell \in \mathbb{R}^1$, by

$$(\tau_\ell f)(\phi) = f(\tau_{-\ell}\phi) , \quad \phi \in \mathcal{D}(S) .$$

A distribution $f \in \mathcal{D}'(S')$ is said to be periodic with period $T > 0$, if

$$(\tau_T f)(\phi) = f(\phi) , \text{ for every } \phi \in \mathcal{D}(S) , \tag{4.3.3}$$

and T is the smallest positive number for which (4.3.3) holds.

We now state and prove our result.

Theorem 4.3.1. Let $f \in S'$ be a tempered distribution such that $\hat{f}$ has compact support, and let the closed set E and W > 0 be such that $E \supsetneqq \text{supp}(\hat{f})$ and the translates $\{E+2nW\}$, $n \neq 0$, are disjoint from $\text{supp}(\hat{f})$. Let α and ψ be any test functions such that ψ has support E, and $\alpha = 1$, $\psi = 1$ each on some open set containing $\text{supp}(\hat{f})$. Then

$$(4.3.4) \qquad f(\phi) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}K_W)(\phi) \; , \qquad \phi \in S \; ,$$

where $K_W(t) = \frac{1}{2W}\int_E e^{2\pi itu}\psi(u)du$ and $K_W(\phi) = \int_{-\infty}^{\infty} K_W(t)\phi(t)dt$, $\phi \in S$.
If $f \in B^d(W)$, then

$$(4.3.5) \qquad f(\phi) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}G_W)(\hat{\psi}*\phi) \; , \; \phi \in S \; ,$$

where $G_W(t) = \frac{\sin 2\pi Wt}{2\pi Wt}$, and $G_W(\phi) = \int_{-\infty}^{\infty} G_W(t)\phi(t)dt$, $\phi \in S$.

Proof. We first show that the sequence of partial sums $S_N = \sum_{n=-N}^{N}\tau_{-2nW}\hat{f}$, $N \geq 1$, converges in S'. For any $\phi \in S$,

$$S_N(\phi) = \sum_{n=-N}^{N} (\tau_{-2nW}\hat{f})(\phi)$$

$$= \sum_{n=-N}^{N} \hat{f}(\tau_{2nW}\phi)$$

$$= \hat{f}\Big(\sum_{n=-N}^{N} \tau_{2nW}\phi\Big)$$

$$(4.3.6) \qquad = \hat{f}\Big(\xi\sum_{n=-N}^{N} \tau_{2nW}\phi\Big) \; ,$$

where $\xi \in \mathcal{D}$ is a test function such that $\xi(t) = 1$ on some open set containing $\text{supp}(\hat{f})$. We now show that the sequence

$\Phi_N(t) = \xi(t)\sum_{n=-N}^{N} \phi(t-2nW)$, $N \geq 1$, converges in S. Since $\phi \in S$, there exists a constant $B > 0$ such that $|\phi(t)| \leq B(1+t^2)^{-1}$ for all $t \in \mathbb{R}^1$, and thus

$$|\phi(t-2nW)| \leq \frac{B}{1+(t-2nW)^2} \leq 2B\,\frac{1+t^2}{1+(2nW)^2} \quad .$$

Since $\xi \in \mathcal{D}$, we have $\mathrm{supp}(\xi) \subset [-C,C]$ for some $C > 0$ and $|\xi(t)| \leq A$ for some $A > 0$. It then follows that for all $t \in \mathbb{R}^1$ and non-negative integers m,

$$(4.3.7) \qquad (1+|t|)^m |\xi(t)| \sum_{n=-N}^{N} |\phi(t-2nW)|$$

$$\leq 2AB(1+C)^m(1+C^2) \sum_{n=-\infty}^{\infty} \frac{1}{1+(2nW)^2} < \infty \ ,$$

i.e., the sequence of partial sums on the left hand side of (4.3.7) converges uniformly on $\mathbb{R}^1$. Hence the sequence $(1+|t|)^m \Phi_N(t)$, $N \geq 1$, converges uniformly on $\mathbb{R}^1$ for every $m \geq 0$. Similarly, it can be shown that for every $m,k \geq 0$, the sequence $(1+|t|)^m \Phi_N^{(k)}(t)$, $N \geq 1$, converges uniformly on $\mathbb{R}^1$, i.e. $\{\Phi_N\}$, $N \geq 1$, converges in S, and since S is complete, its limit Φ belongs to S, and $\Phi_N \to \Phi$ in S. It follows from (4.3.6) that

$$S_N(\phi) = \hat{f}(\Phi_N) \to \hat{f}(\Phi) \ , \text{ as } N \to \infty$$

and since S' is (weak-star) sequentially complete, then there exists a tempered distribution $F \in S'$ such that $S_N \to F$ in S'.

Therefore, $F = \lim_{N\to\infty} S_N = \sum_{n=-\infty}^{\infty} \tau_{-2nW}\hat{f}$ is a periodic tempered distribution with period 2W. It follows that F has the Schwartz-Fourier series (Zemanian, 1965, p. 332)

$$(4.3.8) \qquad F = \sum_{n=-\infty}^{\infty} \tau_{-2nW}\hat{f} = \sum_{n=-\infty}^{\infty} a_{\frac{n}{2W}} e_{\frac{n}{2W}}, \text{ in } \mathcal{S}',$$

where $e_t(u) = e^{2\pi itu}$, and

$$a_{\frac{n}{2W}} = \frac{1}{2W} F(Ue_{-\frac{n}{2W}}) \quad ,$$

where $U\epsilon\mathcal{U}_{2W}$ is a unitary function (Zemanian, 1965, p. 315), i.e. $U\epsilon\mathcal{D}$ and $\sum_{n=-\infty}^{\infty}U(t-2nW) = 1$ for all $t\epsilon\mathbb{R}^1$. From (4.3.8) we have

$$2W\, a_{\frac{n}{2W}} = \sum_{m=-\infty}^{\infty} (\tau_{-2mW}\hat{f})(Ue_{-\frac{n}{2W}})$$

$$= \sum_{m=-\infty}^{\infty} \hat{f}([\tau_{2mW}U]e_{-\frac{n}{2W}}) \ .$$

Since $\hat{f}$ has a compact support and $U\epsilon\mathcal{D}$, then there is only a finite number of non-zero terms in the last summation, and hence

$$2W\, a_{\frac{n}{2W}} = \hat{f}([\sum_{m=-\infty}^{\infty}\tau_{2mW}U]e_{-\frac{n}{2W}})$$

$$(4.3.9) \qquad = \hat{f}(e_{-\frac{n}{2W}}) = \hat{f}(\alpha e_{-\frac{n}{2W}}) = f(\tau_{-\frac{n}{2W}}\hat{\alpha}) \ .$$

From (4.3.8) and (4.3.9), we have that

$$(4.3.10) \qquad \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2W} f(\tau_{-\frac{n}{2W}}\hat{\alpha})e_{\frac{n}{2W}}(\theta) \ , \quad \theta\epsilon\mathcal{D}(E) \ ,$$

where $e_{\frac{n}{2W}}(\theta) = \int_{-\infty}^{\infty} e^{\pi i \frac{n}{W} u}\theta(u)du = \hat{\theta}(-\frac{n}{2W})$. Thus

$$(4.3.11) \qquad f(\hat{\theta}) = \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2W} f(\tau_{\frac{n}{2W}}\hat{\alpha})\hat{\theta}(\frac{n}{2W}) \ , \quad \hat{\theta}\epsilon\hat{\mathcal{D}}(E) \ ,$$

and by Lemma 4.3.1 we have that for every $\phi \epsilon S$

$$f(\phi) = f(\hat{\psi}*\phi) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\hat{\psi}*\phi)(\tfrac{n}{2W}) \tag{4.3.12}$$

(since $\hat{\psi}*\phi = (\psi\check{\hat{\phi}})^{\wedge} \epsilon \hat{\mathcal{D}}(E)$). But

$$(\hat{\psi}*\phi)(\tfrac{n}{2W}) = \int_{-\infty}^{\infty} \hat{\psi}(\tfrac{n}{2W} - t)\phi(t)dt$$

$$= 2W \int_{-\infty}^{\infty} K_W(t - \tfrac{n}{2W})\phi(t)dt$$

$$= 2W(\tau_{\frac{n}{2W}} K_W)(\phi) \quad , \quad \phi \epsilon S \quad , \tag{4.3.13}$$

and (4.3.4) follows from (4.3.12) and (4.3.13).

To prove (4.3.5) notice that when $\theta \epsilon \mathcal{D}[-W,W]$,

$$e_{\frac{n}{2W}}(\theta) = \int_{-W}^{W} e^{\pi i \frac{n}{W} u} \theta(u)du$$

$$= 2W \int_{-W}^{W} \frac{\sin \pi(2Wt+n)}{\pi(2Wt+n)} \hat{\theta}(t)dt$$

$$= 2W(\tau_{-\frac{n}{2W}} G_W)(\hat{\theta}) \ .$$

It follows from (4.3.10) that for $\hat{\theta} \epsilon \hat{\mathcal{D}}[-W,W]$,

$$f(\hat{\theta}) = \hat{f}(\theta) = \sum_{n=-\infty}^{\infty} f(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\hat{\theta}) \ ,$$

and (4.3.5) follows by Lemma 4.3.1. □

Theorem 4.3.1 shows that a tempered distribution f with compact spectrum can be reconstructed via (4.3.4) from its values (samples) at the translates of an arbitrary but fixed test function α which equals one on some open set containing $\text{supp}(\hat{f})$. On the other hand, if we denote $\hat{f}(e_t)$ by $f(t)$, then from (4.3.9) we have $f(\tau_{\frac{n}{2W}}\hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = f(\frac{n}{2W})$, and (4.3.4) reads

$$f(\phi) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W})(\tau_{\frac{n}{2W}}K_W)(\phi) \quad , \quad \phi\epsilon S \ ,$$

so that a tempered distribution f with compact spectrum can be reconstructed using the samples of the function $f(t) = \hat{f}(e_t)$.

We now show that the sampling theorem for tempered distributions with compact spectrum includes as special cases the sampling theorems for conventionally bandlimited functions (Example 4.3.1) as well as for bandlimited functions in $L^2(\mu_k)$ (Example 4.3.2).

<u>Example 4.3.1</u>. (Conventionally bandlimited functions). Let $f\epsilon L^2(\mathbb{R}^1)$ be a continuous function such that $\hat{f}$ has compact support E. Then f determines a tempered distribution:

$$f(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)dt \quad , \quad \phi\epsilon S \ , \tag{4.3.14}$$

and its distributional Fourier transform (denoted also by $\hat{f}$) is defined by $\hat{f}(\phi) = f(\hat{\phi})$, $\phi\epsilon S$, or equivalently by

$$\hat{f}(\phi) = \int_{-\infty}^{\infty} \hat{f}(u)\phi(u)du \quad , \quad \phi\epsilon S \ .$$

$\hat{f}$ (as a tempered distribution) is supported by E. Hence (4.3.4) applies and, if $W > 0$ is defined as in Theorem 4.3.1, we have from (4.3.14)

$$f(\tau_{\frac{n}{2W}}\hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = \int_{-W}^{W} \hat{f}(u)e^{\pi i \frac{n}{W} u} du = f(\tfrac{n}{2W}) \ .$$

For $\nu > 0$, define the function

$$\phi_\nu(t) = \begin{cases} C_\nu^{-1} \exp\{\frac{-1}{1-(\frac{t}{\nu})^2}\} & \text{for } |\frac{t}{\nu}| \le 1 \\ 0 & \text{for } |\frac{t}{\nu}| > 1 , \end{cases}$$

where $C_\nu = \int_{-\infty}^{\infty} \exp\{\frac{-1}{1-(\frac{t}{\nu})^2}\}dt$. For each $\nu > 0$, $\phi_\nu \in \mathcal{D}$ and $\int_{-\infty}^{\infty} \phi_\nu(t)dt = 1$,

and for each continuous function g and every $t \in \mathbb{R}^1$,

$\int_{-\infty}^{\infty} g(u)\phi_\nu(t-u)du \to g(t)$ as $\nu \downarrow 0$. From (4.3.4) we thus have that for each $t \in \mathbb{R}^1$ and $\nu > 0$

$$(4.3.15) \qquad \int_{-\infty}^{\infty} f(u)\phi_\nu(t-u)dt = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u-\tfrac{n}{2W})\phi_\nu(t-u)du .$$

Since f and K_W are uniformly continuous, we have for each fixed $t \in \mathbb{R}^1$ and $n \in \mathbb{N}$

$$\int_{-\infty}^{\infty} f(u)\phi_\nu(t-u)du \to f(t) \quad \text{as } \nu \downarrow 0 \quad ,$$

$$\int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)du \to K_W(t - \tfrac{n}{2W}) .$$

Now by Theorem 24 of Lighthill (1958, p. 64), if for any sequence $\{b_n\}$ which is $O(n)$ as $n \to \infty$, $\sum_{n=-\infty}^{\infty} b_n a_{n,\nu}$ is absolutely convergent and tends to a finite limit as $\nu \to 0$, then

$$(4.3.16) \qquad \lim_{\nu\to 0} \sum_{n=-\infty}^{\infty} a_{n,\nu} = \sum_{n=-\infty}^{\infty} \lim_{\nu\to 0} a_{n,\nu} .$$

But, for each fixed $t \in \mathbb{R}^1$,

$$\sum_{n=-\infty}^{\infty} |b_n\ f(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)\,du|$$

$$\leq 2^k C_k B \left\{ \sum_{n=-\infty}^{\infty} \frac{|n|}{(1+ (\frac{n}{2W})^2)^k} \right\} \int_{-\infty}^{\infty} (1+u^2)^k \phi_\nu(t-u)\,du$$

$$\underset{\nu\to 0^+}{\longrightarrow} 2^k C_k B(1+t^2)^k \left\{ \sum_{n=-\infty}^{\infty} \frac{|n|}{(1+ (\frac{n}{2W})^2)^k} \right\} < \infty \ ,$$

since f is bounded, $|b_n| \leq B|n|$, and for $k > 1$,

$$|K_W(u - \tfrac{n}{2W})| \leq \frac{C_k}{(1+(u - \frac{n}{2W})^2)^k} \leq 2^k C_k \frac{(1+u^2)^k}{(1+(\frac{n}{2W})^2)^k} \ .$$

It follows that the right hand side of (4.3.15) satisfies the conditions leading to (4.3.16), and hence by letting $\nu\downarrow 0$, we obtain

$$(4.3.17) \qquad f(t) = \sum_{n=-\infty}^{\infty} f(\tfrac{n}{2W}) K_W(t - \tfrac{n}{2W}) \ , \ t\in \mathbb{R}^1 \ ,$$

which is the sampling theorem for a conventionally bandlimited function with compact spectrum.

Example 4.3.2. (Bandlimited functions in $L^2(\mu_k)$). Let $f\in L^2(\mu_k)$, $k \geq 0$, be a continuous function. Then f determines a tempered distribution by (4.3.14). If its distributional Fourier transform $\hat{f}$ has a compact support, then (4.3.4) applies and we have

$$f(\tau_{\frac{n}{2W}}\hat{\alpha}) = \hat{f}(e_{\frac{n}{2W}}) = f(\tfrac{n}{2W})$$

(see Lee, 1979). Since f is a C^∞-function and $|f(t)| \leq C_k(1+|t|)^k$, for constant $C_k > 0$ (Lee, 1977), then (4.3.15) holds and following the arguments used in Example 4.3.1, we obtain (4.3.17) which is similar to (3.2.3) and is identical to (3.2.4). It should be noted, though, that (4.3.4) cannot be obtained from Campbell's result (Theorem 3.2.4), since the convergence in (3.2.4) is not uniform on compact sets.

4.4. Sampling Expansions for Random Distributions.

In this section sampling expansions for stationary random distributions are considered. Let $X = \{X(\phi),\ \phi\in S\}$ be a second order random distribution. X is said to be weakly stationary, if for every $h > 0$ and $\phi,\psi\in S$,

$$E(\tau_h X(\phi)\cdot\tau_h X(\psi)) = E(X(\phi)\cdot X(\psi)) .$$

If X is a weakly stationary random distribution (WSRD), then there exists a unique tempered distribution $\rho\in S'$ such that for every $\phi,\psi\in S$,

$$R(\phi,\psi) = E(X(\phi)\cdot X(\psi)) = \rho(\phi*\check{\psi}) , \tag{4.4.1}$$

where $\check{\psi}(t) = \psi(-t)$ (Itô, 1954), and ρ has the spectral representation

$$\rho(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(u)\,d\mu(u) , \quad \phi\in S , \tag{4.4.2}$$

where μ is a non-negative measure on $\mathbb{R}^1$ such that $\int_{-\infty}^{\infty} \frac{d\mu(u)}{(1+u^2)^k} < \infty$ for some integer k. In this case X is said to be of type k, and μ is called the spectral measure of X.

Let B^* be the set of all Borel sets with finite μ-measure. An $L^2(\Omega)$-valued function Z defined on B^* is called a random measure with respect to μ if

$$E(Z(B_1)\cdot Z(B_2)) = \mu(B_1 \cap B_2) \ , \ B_1, B_2 \in \mathcal{B}^* \ .$$

Hence $E(Z^2(B)) = \mu(B)$, and $Z(B_1) \perp Z(B_2)$ if B_1 and B_2 are disjoint. Since μ is σ-additive, then $Z(B) = \sum_{n=1}^{\infty} Z(B_n)$, whenever $B_1, B_2, \ldots$ are disjoint sets in $\mathcal{B}^*$ with $\bigcup_{n=1}^{\infty} B_n = B$. It follows by (4.4.1) and (4.4.2) that there exists a random measure Z with respect to μ such that

$$X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(u)\,dZ(u) \ , \quad \phi \in \mathcal{S} \ .$$

If $H(X)$ is the linear subspace of $L^2(\Omega)$ generated by $\{X(\phi), \phi \in \mathcal{S}\}$, then $H(X)$ and $L^2(\mu)$ are isometrically isomorphic under the correspondence $X(\phi) \mapsto \hat{\phi}$, $\phi \in \mathcal{S}$. A WSRD X is said to be W_0-bandlimited, $W_0 > 0$, if $\mu\{[-W_0, W_0]^c\} = 0$.

Theorem 4.4.1. (a) If $X = \{X(\phi), \phi \in \mathcal{S}\}$ is a W_0-bandlimited WSRD, $W > W_0$, $\alpha \in \mathcal{D}$ and $\psi \in \mathcal{D}[-W, W]$ with $\alpha(t) = 1 = \psi(t)$ on $[-W_0, W_0]$, then for every $\phi \in \mathcal{S}$,

$$(4.4.3) \qquad X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}} \hat{\alpha})(\tau_{\frac{n}{2W}} G_W)(\hat{\psi} * \phi)$$

in mean-square, where $G_W(\phi) = \int_{-\infty}^{\infty} \frac{\sin 2\pi Wt}{2\pi Wt} \phi(t)\,dt$.

(b) Let $X = \{X(\phi), \phi \in \mathcal{S}\}$ be a WSRD with spectral measure μ which has compact support. Let the closed set E and $W > 0$ be such that $E \supsetneq \text{supp}(\mu)$ and the translates $\{E + 2nW\}$, $n \neq 0$, are disjoint from $\text{supp}(\mu)$. Let α and ψ be any test functions such that ψ has support E, and $\alpha(t) = 1 = \psi(t)$ on $\text{supp}(\mu)$. Then

$$(4.4.4)\qquad X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}K_W)(\phi)\ ,\ \phi\epsilon S\ ,$$

in mean-square, where $K_W(t) = \frac{1}{2W}\int_E \psi(u)e^{2\pi itu}du.$

Proof. To prove (a), first let $\phi\epsilon S$ be such that $\hat{\phi}\epsilon\mathcal{D}[-W,W]$. Then $\Phi(u) = \sum_{n=-\infty}^{\infty}\hat{\phi}(u+2nW)$ is a C^∞-function which is periodic with period 2W and has the Fourier series

$$(4.4.5)\qquad \Phi(u) = \sum_{n=-\infty}^{\infty} \frac{1}{2W}\,\phi(\tfrac{n}{2W})e^{\pi i \frac{n}{W} u}\ ,\ u\epsilon\mathbb{R}^1\ ,$$

which converges uniformly on $\mathbb{R}^1$. Since $\hat{\phi}\epsilon\mathcal{D}[-W,W]$,

$$\begin{aligned}(\tau_{\frac{n}{2W}}G_W)(\phi) &= \int_{-\infty}^{\infty} \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}\,\phi(t)dt \\ &= \frac{1}{2W}\int_{-W}^{W} e^{\pi i \frac{n}{W} u}\hat{\phi}(u)du \\ &= \frac{1}{2W}\,\phi(\tfrac{n}{2W})\ ,\end{aligned}$$

we have for the mean square error

$$\begin{aligned}e_N^2(\phi) &= E|X(\phi) - \sum_{n=-N}^{N} X(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}G_W)(\phi)|^2 \\ &= \int_{-W_0}^{W_0} \left|\hat{\phi}(u) - \sum_{n=-N}^{N}\frac{1}{2W}\,\phi(\tfrac{n}{2W})e^{\pi i \frac{n}{W} u}\right|^2 d\mu(u)\ .\end{aligned}$$

There exists a constant $M > 0$ such that for all N and $u\epsilon\mathbb{R}^1$,

$$|\hat{\phi}(u) - \sum_{n=-N}^{N}\frac{1}{2W}\,\phi(\tfrac{n}{2W})e^{\pi i \frac{n}{W} u}| \le M\ .$$

Since, by (4.4.5), $\sum_{n=-N}^{N}\frac{1}{2W}\,\phi(\frac{n}{2W})e^{\pi i \frac{n}{W} u}$ converges to $\hat{\phi}(u)$ on $[-W_0,W_0]$, by the dominated convergence theorem, $e_N^2(\phi) \to 0$ as $N \to \infty$.

Thus for every $\phi\epsilon\hat{\mathcal{D}}[-W,W]$, we have

$$X(\phi) = \sum_{n=-\infty}^{\infty} X(\tau_{\frac{n}{2W}}\hat{\alpha})(\tau_{\frac{n}{2W}}G_W)(\phi) \ . \tag{4.4.6}$$

Now for every $\phi\epsilon S$ and ψ as in part (a) of the statement of the theorem, we have

$$X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)dZ(u) = \int_{-W_0}^{W_0} \psi(u)\hat{\phi}(u)dZ(u)$$

$$= \int_{-W_0}^{W_0} (\hat{\psi}*\phi)^\wedge(u)dZ(u) = X(\hat{\psi}*\phi) \ , \tag{4.4.7}$$

where $\hat{\psi}*\phi = (\psi\hat{\phi})^{\vee\wedge}\epsilon\hat{\mathcal{D}}[-W,W]$, and (4.4.3) follows from (4.4.6) and (4.4.7). The proof of part (b) is similar to that of (a) with the obvious modification and hence is omitted. □

It should be noted that, since $\alpha = 1$ on $[-W_0,W_0]$,

$$X(\tau_{\frac{n}{2W}}\hat{\alpha}) = \int_{-W_0}^{W_0} e^{-\pi i \frac{n}{W}u} dZ(u) \ , \quad n\epsilon\mathbb{N} \ .$$

If we define

$$x(t) = \int_{-W_0}^{W_0} e^{-2\pi itu}dZ(u) \ , \quad t\epsilon\mathbb{R}^1 \ ,$$

then $\{x(t), t\epsilon\mathbb{R}^1\}$ is a weakly stationary W_0-bandlimited stochastic process, $X(\tau_{\frac{n}{2W}}\hat{\alpha}) = x(\frac{n}{2W})$, and (4.4.3) reads

$$X(\phi) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})(\tau_{\frac{n}{2W}} G_W)(\hat{\psi}*\phi) \ , \quad \phi\epsilon S \ , \tag{4.4.8}$$

i.e., the random distribution X is reconstructed using the samples of the ordinary stochastic process x. Hence there is a one-to-one correspondence between W_0-bandlimited weakly stationary random distributions X and W_0-bandlimited weakly stationary processes x determined by $X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)\,dZ(u)$ and $x(t) = \int_{-W_0}^{W_0} e^{2\pi itu}dZ(u)$ and satisfying (4.4.8).

We now show that the sampling theorem for bandlimited weakly stationary random distributions includes as a particular case the sampling theorem for bandlimited weakly stationary processes.

Example 4.4.1. Let $x = \{x(t),\ t\in \mathbb{R}^1\}$ be a measurable, mean-square continuous, weakly stationary process which is W_0-bandlimited, i.e.,

$$x(t) = \int_{-W_0}^{W_0} e^{-2\pi itu}dZ(u) \ , \tag{4.4.9}$$

where Z is a random measure with respect to the spectral measure μ of x with $\mu\{[-W_0,W_0]^c\} = 0$. Then x determines a W_0-bandlimited WSRD by

$$X(\phi) = \int_{-W_0}^{W_0} \hat{\phi}(u)\,dZ(u) \ , \ \phi\in S \ ,$$

which can also be written as

$$X(\phi) = \int_{-W_0}^{W_0} \left(\int_{-\infty}^{\infty} e^{-2\pi itu}\phi(t)\,dt\right)dZ(u)$$

$$= \int_{-\infty}^{\infty} x(t)\hat{\phi}(t)\,dt \ ,$$

where the latter integral exists both with probability one as well as in quadratic mean. Then by (4.4.4) we have for each $t\in\mathbb{R}^1$ and $\nu > 0$,

$$(4.4.10) \qquad \int_{-\infty}^{\infty} x(u)\phi_\nu(t-u)\,du = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W}) \int_{-\infty}^{\infty} K_W(u - \tfrac{n}{2W})\phi_\nu(t-u)\,du$$

in quadratic mean. As in Example 4.3.1,

$$\int_{-\infty}^{\infty} x(u)\phi_\nu(t-u)\,du \to x(t) \quad \text{as } \nu\downarrow 0$$

in quadratic mean, $\int_{-\infty}^{\infty} K_W(u - \frac{n}{2W})\phi_\nu(t-u)\,du \to K_W(t - \frac{n}{2W})$ as $\nu\downarrow 0$, and the right hand side of (4.4.10) converges in quadratic mean to $\sum_{n=-\infty}^{\infty} x(\frac{n}{2W})K_W(t - \frac{n}{2W})$. We thus obtain

$$x(t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2W})K_W(t - \tfrac{n}{2W}) \ , \quad t\in \mathbb{R}^1$$

in quadratic mean.

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Sampling representations and approximations of deterministic and random signals are important in communication and information theory. Finite sampling approximations are derived for certain functions and processes which are not bandlimited, as well as bounds on the approximation errors. Also derived are sampling representations for bounded linear operators acting on certain classes of bandlimited functions and processes; these representations enable one to reconstruct the image of a function (or a process) under a bounded linear

20. operator using the samples of the function (or the process) rather than the samples of the image. Finally, sampling representations for distributions and random distributions are obtained.