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By: R. C. Bose and K. J. C. Smith
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5	0	2	7	8	9	3	4	6	1
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COMBINATORIAL
MATHEMATICS
YEAR

TERNARY RINGS OF A CLASS OF LINEARLY REPRESENTABLE SEMI-TRANSLATION PLANES*

by

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Abstract

The theory of linear representation of projective planes was developed by Bruck and one of the authors (Bose) in two earlier papers [*J. Algebra,* 1 (1964), 85-102 and 4 (1966), 117-172]. Bose and Barlotti obtained some new linear representations by generalizing the concept of incidence in the representation [*Ann. Mat. Pura. Appli.,* (197),]. In this paper, we show that the Δ-planes of Bose and Barlotti are semi-translation planes, and obtain the ternary rings of these planes, where the ternary function of Δ is expressed explicitly in terms of the addition and multiplication in the Veblen-Wedderburn system, coordinatizing the translation plane T, from which Δ can be obtained by dualization and derivation.

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§ 1.

1. A (right) Veblen-Wedderburn system (or quasifield) is a set of elements R, with two binary operations $+$ and $\times$ satisfying the following axioms:

(i) $(R,+)$ is an Abelian group with zero element 0.

(ii) $x \times 0 = 0 \times x = 0$ for any x in R.

(iii) If R_1 is the set of non-zero elements of R then $(R_1,\times)$ is a loop with identity element 1.

(iv) $(x+y)\times z = x\times y + y\times z$ for all x, y, z in R.

(v) If a, b, c are elements of R, with $a \neq b$, there exists one and only one element x of R such that $x \times a = x \times b + c$.

We shall denote the Veblen-Wedderburn system satisfying (i)-(v) by $(R,+,\times)$, or in short R.

Let K be the subset of all elements f of R for which

$$(1.1) \qquad f\times(x+y) = f\times x + f\times y, \quad (f\times x)\times y = f\times(x\times y)$$

for all x, y in R. Then we know that [see for example (5)], that K is a skew field, which is defined to be the left operator skew field (or kernel) of $(R,+,\times)$.

2. It is well known [8] that we can coordinatize any affine translation plane αT by using a suitable Veblen-Wedderburn system $(R,+,\times)$. The ternary ring of T is

$$(1.2) \qquad F(x,m,b) = x \times m + b.$$

The points of T are ordered pairs (x,y) where x and y are in R. The lines of T are of two types. Lines of the first type are specified by ordered pairs $[m,b]$, where m and b are in R. The lines of the second type are specified by a single element $[c]$ of R. m is called the slope and b the y-intercept of the line $[m,b]$ of the first type. c is called the x-intercept of the line $[c]$ of the second type. The point (x,y) is incident with the line $[m,b]$ of the first type if and only if

$$(1.3) \qquad y = F(x,m,b) = x \times m + b.$$

The point (x,y) is incident with the line $[c]$ of the second type if and only if $x = c$. The affine plane αT can be extended to a projective plane T in the usual manner.

We shall show that if $(R,+,\times)$ is two dimensional over some skew field contained in K and satisfies an additional condition, then we can use R to coordinatize another affine plane $\alpha\Delta$ which when extended to a projective plane Δ, is a semi-translation plane in the sense of Ostrom [9].

3. If $(R,+,\times)$ is two dimensional over F, we can without loss of generality choose the basis of R over F to be 1, i where 1 is the unit element of F and R, and i is a suitably chosen element of R, not in F. Then any element x in R can be expressed as $x = x_1+x_2i$ where x_1, x_2 are in F, and x_2i is $x_2\times i$.

To each element x of R, there corresponds the binary vector (x_1,x_2) with coordinates from F. This is a $(1,1)$ correspondence. If $x+y = z$, and the vectors corresponding to x, y and z are (x_1,x_2), (y_1,y_2) (z_1,z_2) then

$$(1.4) \qquad (z_1,z_2) = (x_1,x_2) + (y_1,y_2) = (x_1+y_1,\ x_2+y_2).$$

The vectors corresponding to the elements 0, 1 and i of R are $(0,0)$, $(1,0)$ and $(0,1)$. Let $m = m_1+m_2i$ and $im = m^*_1+m^*_2i$, where m_1, m_2, m^*_1, m^*_2 are in F. We shall say that the 2×2 matrix

$$A_m = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix},$$

corresponds to the element m of R. The first row of A_m is the vector corresponding to m, and the second row of A_m is the vector corresponding to im. Let C be the collection of all 2×2 matrices over F, which correspond in this way to the elements of R. Then C is a subset of the set of 2×2 matrices over F. Now if $y = x\times m$,

$$
\begin{aligned}
(1.5) \qquad y_1 + y_2 i &= (x_1+x_2 i) \times (m_1+m_2 i) \\
&= x_1(m_1+m_2 i) + x_2 i(m_1+m_2 i) \\
&= x_1(m_1+m_2 i) + x_2(m^*_1+m^*_2 i) \\
&= x_1 m_1 + x_2 m^*_1 + (x_1 m_2+x_2 m^*_2)i.
\end{aligned}
$$

Hence the vector co-responding to $x\times m$ is $(x_1m_1+x_2m^*_1,\ x_1m_2+x_2m^*_2)$. We can therefore alternatively represent $(R,+,\times)$ by binary vectors over F where the addition is the ordinary vector addition given by (1.4) and multiplication is defined by

$$
(1.6) \qquad (x_1,x_2) \times (m_1,m_2) = (x_1,x_2)\begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix}.
$$

In particular

$$
(0,1) \times (m_1,m_2) = (m^*_1,m^*_2).
$$

The vector corresponding to the product $x\times m$ is the ordinary matrix product of the vector corresponding to x and the matrix corresponding to m. We shall write the elements of R as $x_1+x_2 i$ or (x_1,x_2) according to convenience.

4. A 2×2 matrix

$$
(1.7) \qquad U = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},
$$

with elements from the skew field F is defined to be singular, if its row vectors are independent from the left, i.e. there exist elements r_1, r_2 of F, not both zero, such that

$$r_1a_1 + r_2b_1 = 0, \quad r_1a_2 + r_2b_2 = 0.$$

U is defined to be non-singular if U is not singular. We state here, for subsequent use, the following two obvious Lemmas.

LEMMA (1.1). The column vectors of the matrix U, given by (1.7) are dependent from the right, i.e., there exist elements s_1 and s_2 of F, not both zero, such that

$$a_1s_1 + a_2s_2 = 0, \quad b_1s_1 + b_2s_2 = 0$$

if and only if U is singular.

In other words, the dependence (independence) of the row vectors of U from the left is equivalent to the dependence (independence) of the column vectors of U from the right.

COROLLARY. If U is non-singular, it cannot have a null row or a null column.

LEMMA (1.2). If the matrix U given by (1.7) is non-singular, then the equations

$$x_1a_1 + x_2b_1 = c_1, \quad x_1a_2 + x_2b_2 = c_2$$

have a unique solution x_1, x_2 and the equations

$$a_1y_1 + a_2y_2 = d_1, \quad b_1y_1 + b_2y_2 = d_2$$

have a unique solution y_1, y_2.

5. The collection C of 2×2 matrices corresponding to the elements of R has the properties given by the following Lemma:

LEMMA (1.3). (a) The null matrix 0 and the unit matrix I belong to C.

(b) Each binary vector of R, appears as the first (second) row of a unique matrix of C.

(c) If A_m and $A_{m'}$ are two distinct matrices of C then their difference $A_m - A_{m'}$ is non-singular. In particular, if $A_m \neq 0$, then A_m is non-singular.

(d) If $(x_1,x_2) \neq (0,0)$ and (y_1,y_2) are binary vectors of R, there is a unique matrix A_m of C such that

$$(1.8) \qquad (y_1,y_2) = (x_1,x_2)A_m = (x_1,x_2)\begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix}.$$

(e) If (y_1,y_2) and $(m_1,m_2) \neq (0,0)$ are given binary vectors of R and A_m is the unique matrix of C whose first row is (m_1,m_2), then there is a unique vector (x_1,x_2) of R for which (1.8) holds.

The property (a) is obvious since the null matrix

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

corresponds to the zero element of R, and the unit matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

corresponds to the unity of R. The first part of (b) follows from definition. The second part follows by noting that if the vector (m^*_1,m^*_2) with elements from F is given, and $m^* = m^*_1+m^*_2 i$, then the relation $im = m^*$ uniquely determines the element $m = m_1+m_2 i$ of R. Since (1.8) can be written as

$$y_1 + y_2 i = (x_1+x_2 i) \times (m_1+m_2 i),$$

(d) and (e) follow from axiom (iii), which states that the non-zero elements of R form a loop with respect to multiplication. We thus only need to prove the property (c). Let

$$A_m = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix}, \quad A_{m'} = \begin{bmatrix} m'_1 & m'_2 \\ m'^*_1 & m'^*_2 \end{bmatrix}$$

be two distinct matrices belonging to C. From (a), $(m_1,m_2) \neq (m'_1,m'_2)$. If $A_m - A_{m'}$ is singular, then by definition there exists a vector $(r_1,r_2) \neq (0,0)$ with coordinates from F, such that

$$(1.9) \qquad (r_1,r_2)(A_m - A_{m'}) = 0,$$

which is equivalent to

$$(1.10) \qquad (r_1,r_2) \times (m_1,m_2) = (r_1,r_2) \times (m'_1,m'_2).$$

Since $(r_1,r_2) \neq (0,0)$, it follows from (iii), that $(m_1,m_2) = (m'_1,m'_2)$ which is a contradiction. This proves (c) and completes the proof of the Lemma.

6. We shall now assume that the Veblen-Wedderburn system R which is two dimensional over the skew field F contained in the left operator skew field (kernel) K of R, satisfies the following additional axiom:

(vi) For any given elements $x = x_1 + x_2 i \neq 0$ and $y = y_1 + y_2 i$ of R, there exists a unique element $m = m_1 + m_2 i$ of R, such that

$$(1.12) \qquad m_1 x_1 + m_2 x_2 + (m^*_1 x_1 + m^*_2 x_2)i = y_1 + y_2 i,$$

where $im = m^*_1 + m^*_2 i$, and $x_1,x_2,y_1,y_2,m_1,m_2,m^*_1,m^*_2$ are in F.

Axiom (vi) may alternatively be written as:

(vi,a) given the binary vectors $(x_1,x_2) \neq (0,0)$ and (y_1,y_2) with coordinates from F, there exists a unique matrix A_m of C such that

$$(1.13) \qquad A_m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

LEMMA (1.4). Given any binary vector with coordinates from F, it appears as the first (second) column of a unique matrix A_m of C, if axiom (vi) is satisfied.

Let the binary vector be (y_1, y_2). In (1.13), choose $x_1 = 1$, $x_2 = 0$. Then $m_1 = y_1$, $m^*_1 = y_2$. Hence there is one matrix of C with the first column $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. There cannot be two such matrices, since their difference would be singular from the corollary to Lemma (1.1), contradicting Lemma (1.3), part (c). Similarly by choosing $x_1 = 0$, $x_2 = 1$, it follows that there is a unique matrix of C whose second column is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

7. In view of Lemma (1.3) part b, we can define the functions F_1, F_2 of elements x_1, x_2 of F by setting

$$(1.14) \qquad F_1(x_1,x_2) = x^*_1, \qquad F_2(x_1,x_2) = x^*_2$$

where

$$A_x = \begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix}$$

is the uniquely defined matrix of C, whose first row is (x_1, x_2).

Similarly, when axiom (vi) is satisfied, we can define functions G_1, G_2 of elements x_1, x_2 of F by setting

$$(1.15) \qquad G_1(x_1,x^*_1) = x_2, \qquad G_2(x_1,x^*_1) = x^*_2$$

where A_x is the uniquely defined matrix of C whose first column is $\begin{pmatrix} x_1 \\ x^*_1 \end{pmatrix}$.

LEMMA (1.5). If axiom (vi) is satisfied and m_1, $m_2 \neq 0$, b_1, b_2 belonging to F are given, and

$$(1.16) \qquad (y_1,y_2) = (m_1,m_2) \times (x_1,x_2) + (b_1,b_2),$$

then any one of the pairs (x_1,y_1), (x_2,y_2) uniquely determines the other.

The relation (1.16) is equivalent to

$$(1.17) \qquad (y_1,y_2) = (m_1,m_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} + (b_1,b_2),$$

where $x^*_1+x^*_2i = i(x_1+x_2i)$, i.e. the matrix on the right in (1.17) belongs to C. Hence

$$(1.18) \qquad y_1 = m_1x_1 + m_2x^*_1 + b_1, \qquad y_2 = m_1x_2 + m_2x^*_2 + b_2.$$

Hence $x^*_1 = m_2^{-1}(y_1-m_1x_1-b_1)$, and therefore from Lemma (1.4) x_2 and x^*_2 are uniquely determined if x_1,y_1 are given. Then y_2 is determined by the second equation in (1.18). In terms of the functions F_1, F_2, G_1, G_2 given by (1.14) and (1.15) we can write

$$(1.19) \qquad x_2 = G_1[x_1,\ m_2^{-1}(y_1-m_1x_1-b_1)],$$

$$(1.20) \qquad y_2 = m_1G_1[x_1,\ m_2^{-1}(y_1-m_1x_1-b_1)] + m_2G_2[x_1,\ m_2^{-1}(y_1-m_1x_1-b_1)] + b_2.$$

In the same way. we can show that if x_2, y_2 are given, x_1, y_1 are uniquely determined.

8. We note that if F is finite, i.e., F is a Galois field $GF(q)$, then axiom (vi) is automatically satisfied. This follows by observing that the right hand side of (1.13) can assume exactly q^2 values and there are exactly q^2 matrices in C. If for some (y_1,y_2) there is no corresponding A_m, then there must exist distinct matrices $A_{m'}$ and $A_{m''}$ belonging to C such that

$$A_{m'}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_{m''}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence

$$(A_{m'}-A_{m''})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Since $(x_1,x_2) \neq (0,0)$, it follows from Lemma (1.1), that $A_{m'}-A_{m''}$ is singular, which contradicts Lemma (1.3), part (c). Hence when F is finite, and $(x_1,x_2) \neq (0,0)$ and (y_1,y_2) are given, there is a unique A_m belonging to C for which (1.13) is satisfied.

In §3, we shall give a geometrical interpretation of axiom (vi).

§ 2.

1. We shall now use the (right) Veblen-Wedderburn system $(R, +, \times)$ considered in §1, to obtain an affine plane $\alpha\Delta$ different from the affine plane αT with the ternary ring (1.2). It will be shown in §5 that $\alpha\Delta$ is a semi-translation plane in the sense of Ostrom [9]. As before, we take R to be two dimensional over the skew field F contained in the kernel K of R, and having the basis elements 1 and i, where 1 is the unit element of F and R, and i is not in F. We shall assume that R satisfies the standard axioms (i)-(v) and the additional axiom (vi).

2. Let us consider an incidence structure $[P, L, I]$ where P is a set of points, L is a set of lines and I is an incidence relation between points and lines, defined as follows:

Each point of P is coordinatized by an ordered pair of elements of R. The coordinates of a point can be taken as $\{x_1+y_1i,\ x_2+y_2i\}$, where x_1, y_1, x_2, y_2 are elements of F. When F is the Galois field GF(q), there are q^4 points.

Lines of L are of several types. Each line of type 1 is coordinatized by an ordered pair of elements of R, the first of which does not belong to F. The coordinates of a line of type 1 can be taken as $[m_1+m_2i,\ b_1+b_2i]$, where $m_2 \neq 0$. Lines of type 2a are coordinatized by an ordered pair of elements of R, the first of which belongs to F. The coordinates of a line of type 2a can be taken as $[m_1, b_1+b_2i]$. Each line of type 2b is coordinatized by a single element of R. Thus a line of type 2b has a coordinate $[c_1+c_2i]$. When F is the Galois field $GF(q)$, there are q^4-q^3 lines of type 1, q^3 lines of type 2a and q^2 lines of type 2b.

The line $[m_1+m_2i, b_1+b_2i]$ of type 1, $m_2 \neq 0$, is defined to be incident with the point $\{x_1+y_1i, x_2+y_2i\}$ if and only if

(2.1) $$y_1 + y_2i = (m_1+m_2i) \times (x_1+x_2i) + (b_1+b_2i).$$

The line $[m_1, b_1+b_2i]$ of type 2a is defined to be incident with the point $\{x_1+y_1i, x_2+y_2i\}$ if and only if

(2.2) $$x_2 + y_2i = (x_1m_1+b_1) + (y_1m_1+b_2)i.$$

The line $[c_1+c_2i]$ of type 2b is said to be incident with the point $\{x_1+y_1i, x_2+y_2i\}$ if and only if

(2.3) $$x_1 + y_1i = c_1 + c_2i.$$

We shall show that the incidence structure (P,L,I) so defined is an affine plane.

We have already noted that the elements of R can alternatively be represented by binary vectors over F through the mapping $(x_1+x_2i) \to (x_1,x_2)$. The point $\{(x_1+y_1i),(x_2+y_2i)\}$ then becomes $\{(x_1,y_1),(x_2,y_2)\}$. The line $[m_1+m_2i, b_1+b_2i]$ becomes $[(m_1,m_2),(b_1,b_2)]$, and the line $[c_1+c_2i]$ becomes the line $[c_1,c_2]$.

The condition for the line $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ of type 1 to be incident with point $\{(x_1,y_1),(x_2,y_2)\}$ then becomes

(2.4) $$(y_1,y_2) = (m_1,m_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} + (b_1,b_2),$$

where $(0,1)\times(x_1,x_2) = (x^*_1,x^*_2)$, i.e.,

$$(2.5) \qquad A_x = \begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix}$$

belongs to C.

The condition for the line $[(m_1,0),(b_1,b_2)]$ of type 2a to be incident with the point $\{(x_1,y_1),(x_2,y_2)\}$ becomes

$$(2.6) \qquad x_2 = x_1m_1 + b_1, \qquad y_2 = y_1m_1 + b_2.$$

The condition for the line $[c_1,c_2]$ of type 2b to be incident with the point $\{(x_1,y_1),(x_2,y_2)\}$ becomes

$$(2.7) \qquad x_1 = c_1, \qquad y_1 = c_2.$$

3. Two lines $[(m_1,m_2),(b_1,b_2)]$ and $[(m'_1,m'_2),(b'_1,b'_2)]$ both of type 1 or both of type 2a will be defined to belong to the same parallel class if and only if $(m_1,m_2) = (m'_1,m'_2)$. We may, therefore, speak of parallel classes (m_1,m_2). The parallel class (m_1,m_2) is said to be type 1 if $m_2 \neq 0$, and type 2a if $m_2 = 0$. All lines of type 2b are defined to belong to the same parallel class which may be denoted by (∞). Two lines of different types do not belong to the same parallel class. The two succeeding Lemmas provide the justification for this nomenclature.

LEMMA (2.1). Two distinct lines of the same parallel class are not both incident with the same point.

Case I. Let $[(m_1,m_2),(b_1,b_2)]$ and $[(m_1,m_2),(b'_1,b'_2)]$, $m_2 \neq 0$ be two distinct lines of type 1 belonging to the parallel class (m_1,m_2) of type 1. If they are incident with the same point $\{(x_1,y_1),(x_2,y_2)\}$, then from (2.4), $(b_1,b_2) = (b'_1,b'_2)$. This is a contradiction since the two lines are distinct.

Case II. Let $[(m_1,0),(b_1,b_2)]$ and $[(m_1,0),(b'_1,b'_2)]$ be two distinct lines of type 2a belonging to the parallel class $(m_1,0)$. If they are incident with the same point $\{(x_1,y_1),(x_2,y_2)\}$, then from (2.6), $b_1 = b'_1$, $b_2 = b'_2$, which is a contradiction.

Case III. From (2.7), two distinct lines $[c_1,c_2]$, $[c'_1,c'_2]$ of type 2b cannot be incident with the same point $\{(x_1,y_1),(x_2,y_2)\}$.

LEMMA (2.2). Two lines belonging to different parallel class are both incident with a single point.

Case I. It follows from (2.6) and (2.7) that the line $[c_1,c_2]$ of type 2b, and the line $[(m_1,0),(b_1,b_2)]$ of type 2a are both incident with the unique point $\{(c_1,c_2),(c_1m_1+b_1,\ c_2m_1+b_2)\}$.

Case II. If $[c_1,c_2]$ is a line of type 2b, and $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ is a line of type 1, then if $\{(x_1,y_1),(x_2,y_2)\}$ is a point common to the two lines, $x_1 = c_1$, $y_1 = c_2$ and (2.4) is satisfied. It follows from Lemma (1.5), that there is a single point with which both lines are incident.

Case III. Let $[(m_1,m_2),(b_1,b_2)]$, $[(m'_1,m'_2),(b'_1,b'_2)]$, $m_2 \neq 0$, $m'_2 \neq 0$, be two given lines of type 1, not belonging to the same parallel class. Then $(m_1,m_2) \neq (m'_1,m'_2)$. If $\{(x_1,y_1),(x_2,y_2)\}$ is a point incident with both lines, it follows from (2.4) that

$$(2.8) \qquad (0,0) = (m_1-m'_1,\ m_2-m'_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} + (b_1-b'_1,\ b_2-b'_2).$$

If $(b_1,b_2) = (b'_1,b'_2)$ then from (2.8) and Lemma (1.3) part (d), $x_1 = 0$, $x_2 = 0$ is the only solution of (2.8). Hence from (2.4), $\{(0,0),(b_1,b_2)\}$ is the unique point incident with both lines.

If $(b_1,b_2) \neq (b'_1,b'_2)$, then from (2.8), $(x_1,x_2) \neq (0,0)$. Hence from Lenna (1.3) part (c), the matrix (2.5) is non-singular. Hence from Lemma (1.3) part (d), (x_1,x_2) is uniquely determined. Then (y_1,y_2) is determined by (2.4).

Thus there is a unique point $\{(x_1,y_1),(x_2,y_2)\}$ incident with both lines.

Case IV. Let $[(m_1,0),(b_1,b_2)]$, $[(m'_1,0),(b'_1,b'_2)]$ be two lines of type 2a, not belonging to the same parallel class. Then $m_1 \neq m'_1$. If $\{(x_1,y_1),(x_2,y_2)\}$ is a point incident with both lines, it follows from (2.5), that

$$(2.9) \qquad x_2 = x_1m_1 + b_1 = x_1m'_1 + b'_1, \qquad y_2 = y_1m_1 + b_2 = y_1m'_1 + b'_2.$$

Hence $x_1 = -(b_1-b'_1)(m_1-m'_1)^{-1}$, $y_1 = -(b_2-b'_2)(m_2-m'_2)^{-1}$. Then x_2 and y_2 are uniquely determined by (2.9).

Case V. Let $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ be a line of type 1, and $[(m'_1,0),(b'_1,b'_2)]$ be a line of type 2a. If $\{(x_1,y_1,(x_2,y_2)\}$ is a point incident with both lines, it follows from (2.4) and (2.6) that

$$(2.10) \qquad y_1 = m_1x_1 + m_2x^*_1 + b_1,$$

$$(2.11) \qquad y_2 = m_1x_2 + m_2x^*_2 + b_2,$$

$$(2.12) \qquad x_2 = x_1m'_1 + b'_1,$$

$$(2.13) \qquad y_2 = y_1m'_1 + b'_2.$$

Let $\lambda = m_2^{-1}(b_1m'_1-m_1b'_1-b_2+b'_2)$. From axiom (vi,a) formula (1.13), we can find unique matrix A_x belonging to C, such that

$$(2.14) \qquad A_x\begin{pmatrix}-m'_1\\ 1\end{pmatrix} = \begin{bmatrix}x_1 & x_2\\ x^*_1 & x^*_2\end{bmatrix}\begin{pmatrix}-m'_1\\ 1\end{pmatrix} = \begin{pmatrix}b'_1\\ \lambda\end{pmatrix}$$

Then (2.12) is satisfied, and y_1,y_2 can be uniquely determined from (2.10) and (2.11). We shall show that (2.13) is automatically satisfied. Multiplying (2.10) by $-m'_1$ from the right and adding (2.11) we have

$$-y_1m'_1 + y_2 = m_1(-x_1m'_1+x_2) + m_2(-x^*_1m_1+x^*_2) - b_1m'_1 + b_2.$$

Using (2.14) and substituting for λ, this reduces to (2.13). Thus there is exactly one point incident with both the given lines.

4. LEMMA (2.3). Given any point $\{(x_1,y_1),(x_2,y_2)\}$, there is exactly one line in any given parallel class incident with the point.

Case I. In the parallel class (∞) consisting of all lines of type 2b, there is only one line, viz $[x_1,y_1]$ which is incident with the given point.

Case II. Let $[(m_1,0),(b_1,b_2)]$ be a line of type 2a belonging to the parallel class $(m_1,0)$. If it is incident with the point $\{(x_1,y_1),(x_2,y_2)\}$, then (b_1,b_2) is uniquely determined by (2.6).

Case III. Let $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ be a line of type 1 incident with the point $\{(x_1,y_1),(x_2,y_2)\}$. Then (b_1,b_2) is uniquely determined by (2.4).

LEMMA (2.4). For given (x_1,y_1), (x_2,y_2), (x'_1,y'_1), (x'_2,y'_2), where $(x_1,y_1) \neq (x'_1,y'_1)$, the equations

$$(2.15)\qquad (y_1,y_2) = (m_1,m_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} + (b_1,b_2)$$

$$(2.16)\qquad (y'_1,y'_2) = (m_1,m_2)\begin{bmatrix} x'_1 & x'_2 \\ x'^*_1 & x'^*_2 \end{bmatrix} + (b_1,b_2)$$

in which the matrices appearing on the right belong to C, uniquely determine (m_1,m_2), (b_1,b_2). Further $m_2 = 0$ if and only if the matrix

$$(2.17)\qquad \begin{bmatrix} x_1 - x'_1 & x_2 - x'_2 \\ y_1 - y'_1 & y_2 - y'_2 \end{bmatrix}$$

is singular.

From (2.15) and (2.16) we have

$$(2.18)\qquad (y_1-y'_1,y_2-y'_2) = (m_1,m_2)\begin{bmatrix} x_1 - x'_1 & x_2 - x'_2 \\ x^*_1 - x'^*_1 & x^*_2 - x'^*_2 \end{bmatrix}.$$

Now $(x_1,x_2) \neq (x'_1,x'_2)$, otherwise $(x_1,y_1) = (x'_1,y'_1)$ contradicting the hypothesis. From Lemma (1.3), part (c), the matrix appearing on the right in (2.18) is non-singular. From Lemma (1.2), m_1, m_2 are uniquely determined. Then (b_1,b_2) is determined from either (2.15) or (2.16). If m_2 so determined is zero, then

$$(2.19) \qquad m_1(x_1-x'_1) - (y_1-y'_1) = 0, \qquad m_1(x_2-x'_2) - (y_2,y'_2) = 0,$$

Then by definition, the matrix (2.17) is singular.

Conversely, if (2.17) is singular, we can find $(r_1,r_2) \neq (0,0)$ such that

$$r_1(x_1-x'_1) + r_2(x_2-x'_2) = 0, \qquad r_1(y_1-y'_1) + r_2(y_2-y'_2) = 0.$$

Since $(x_1,y_1) \neq (x'_1,y'_1)$, $r_2 \neq 0$. If we set $m_1 = -r_2^{-1}r_1$, (2.19) is satisfied. Hence (2.18) is satisfied with $m_2 = 0$. Since (m_1,m_2) is uniquely determined by (2.18), m_2 must be zero. Thus in this case (2.15), (2.16) uniquely determine (m_1,m_2), (b_1,b_2) with $m_2 = 0$.

5. LEMMA (2.5). Any two distinct points $\{(x_1,y_1),(x_2,y_2)\}$ and $\{(x'_1,y'_1),(x'_2,y'_2)\}$ are incident with exactly one line.

Case I. Let $(x_1,y_1) = (x'_1,y'_1)$. Then the line $[x_1,y_1]$ of type 2b is clearly incident with both points. If the line $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ of type 1 is incident with both points, then from (2.4) and Lemma (1.5), the two points would coincide. Similarly, if the line $[(m_1,0),(b_1,b_2)]$ of type 2a is incident with both points, then from (2.6) the two points with coincide. Hence the line $[x_1,y_1]$ of type 2b is the only line incident with both points.

Case II. Suppose $(x_1,y_1) \neq (x'_1,y'_1)$, and the matrix (2.17) is singular. From Lemma (1.1) we can find $(s_1,s_2) \neq (0,0)$ such that

$$(2.20) \qquad (x_1-x'_1)s_1 + (x_2-x'_2)s_2 = 0, \qquad (y_1-y'_1)s_1 + (y_2-y'_2)s_2 = 0.$$

Now s_2 is non-zero, otherwise $(x_1,y_1) = (x'_1,y'_1)$ contradicting the hypothesis. Let $m_1 = -s_1s^{-1}_2$, $b_1 = x_2-x_1m_1 = x'_2-x'_1m_1$, $b_2 = y_2-y_1m_1 = y'_2-y'_1m_1$. Then from (2.6), $[(m_1,0),(b_1,b_2)]$ is the only line of type 2a incident with both points. From (2.7) there is no line of type 2b incident with both points. If the line $\{(m_1,m_2),(b_1,b_2)\}$, $m_2 \neq 0$ of type 1 is incident with both points, then from (2.4), equations (2.15) and (2.16) are satisfied. Hence from Lemma (2.4), (m_1,m_2), (b_1,b_2) are uniquely determined with $m_2 = 0$ which is a contradiction. Thus there is exactly one line which is incident with both the points. It is of type 2a.

Case III. Suppose $(x_1,y_1) \neq (x'_1,y'_1)$ and the matrix (2.17) is non-singular. From Lemma (2.4), we can uniquely determine (m_1,m_2), (b_1,b_2), $m_2 \neq 0$ satisfying (2.15) and (2.16). Hence from (2.4) there is a unique line $[(m_1,m_2),(b_1,b_2)]$ of type 1 incident with both points. From (2.7) there is no line of type 2b incident with both points. Again, if there is a line of type 2a incident with both points then from (2.6), the equation (2.20) is satisfied with $s_1 = m_1$, $s_2 = -1$. Hence from Lemma (1.1), the matrix (2.17) is singular, which is a contradiction. Thus there is exactly one line incident with both points. It is of type 1.

7. THEOREM (2.1). The incidence structure $[P, L, I]$ defined in §2, paragraph 2, is an affine plane.

In view of the Lemmas (2.1), (2.2), (2.3), (2.5) it is sufficient to verify the existence of three non-collinear points. The points $\{(0,0),(0,0)\}$, $\{(0,0),(1,0)\}$ are from (2.6), incident with the line $[(1,0),(0,0)]$ of type 2a. The point $\{(1,0),(0,0)\}$ is not incident with this line.

The affine plane of Theorem (2.1) will be denoted by $\alpha\delta$. Its points will be called $\alpha\Delta$-points, and its lines will be called $\alpha\Delta$-lines. It can be completed to a projective plane Δ in the usual manner. Corresponding to each parallel class (m_1,m_2) of type 1 or 2a, we take a point at infinity denoted by $\{m_1,m_2\}$. Similarly corresponding to the parallel class (∞) which consists of all $\alpha\Delta$-lines of type 2b, we take a point at infinity denoted by $\{\infty\}$. Finally, we take a line at infinity denoted by $[\infty]$. Each point at infinity is defined to be incident with the line at infinity, and all lines of the parallel class corresponding to it (and with no other line). Thus Δ-points consist of $\alpha\Delta$-points and the points at infinity, and Δ-lines consist of $\alpha\Delta$-lines and the line at infinity.

§ 3.

1. Let F be a skewfield. Let S_4 be a four dimensional projective space based on F, regarded as a vector space V, with F as a ring of left operators. We can then assign coordinates to the points of S_4, and write down the equations of linear subspaces of S_4 in the usual manner, by choosing a specified basis e_1, e_2, e'_1, e'_2, e_0 of V. Points of S_4 correspond to one dimensional vector subspaces of V. Thus the coordinates of a point P which corresponds to the vector subspace $V(P)$ with basis vector $x_1e_1+x_2e_2+y_1e'_1+y_2e'_2+ze_0$ are (x_1, x_2, y_1, y_2, z) where x_1, x_2, y_1, y_2, z are elements of F not all zero. The coordinates of a point are arbitrary up to a non-zero multiple of an element of F from the left. Thus if $r \neq 0$, then the point with coordinates $(rx_1, rx_2, ry_1, ry_2, rz)$ is identical with P. A linear equation $\xi_1a_1+\xi_2a_2+\eta_1b_1+\eta_2b_2+\zeta c = 0$ (where a_1, a_2, b_1, b_2, c are elements of F not all zero) represents a 3-space S_3 which contains the point (x_1,x_2,y_1,y_2,z) if and only if $x_1a_1 + x_2a_2 + y_1b_1 + y_2b_2 + zc = 0$. The equation of a 3-space is arbitrary up to a non-zero multiple of an element of F from the right. Thus if $k \neq 0$, then the equation

$$\xi_1 a_1 k + \xi_2 a_2 k + \eta_1 b_1 k + \eta_2 b_2 k + \zeta ck = 0$$

also represents S_3. Dependence and independence of vectors or equations can be defined in the usual manner with the convention that multiples of vectors are to be taken from the left and multiples of equations from the right. Dependence or independence of points is defined by the dependence or independence of the corresponding coordinate vectors. A t dimensional subspace S_t of S_4 $(0 \le t \le 4)$ corresponds to a $t+1$ dimensional vector subspace $V(S_t)$ of V, and may therefore be defined as the set of points dependent on a given set of $t+1$ independent points corresponding to a basis of $V(S_t)$. Alternatively S_t can be represented by a set of $4-t$ independent equations. A point belongs to S_t if and only if its coordinates satisfy the equations of S_t, i.e., the point is contained in S_t.

2. Let Σ be a fixed 3-space of S_4. A spread $\mathcal{S}$ is defined as a set of lines contained in Σ, such that every point of Σ is contained in exactly one line of $\mathcal{S}$. A plane in Σ cannot contain more than one line of $\mathcal{S}$ (otherwise they would intersect in a point). The spread $\mathcal{S}$ is called a dual spread if each plane of Σ contains at least (and therefore exactly) one line of $\mathcal{S}$. If S_4 is a finite space, F can be taken as the Galois field $GF(q)$. In this case, the spread $\mathcal{S}$ must necessarily be a dual spread. In the infinite case, $\mathcal{S}$ may or may not be a dual spread [7]. If a plane contains the line s_∞ belonging to $\mathcal{S}$, then every other line s of $\mathcal{S}$ meets the plane in a unique point S, not in s_∞; and conversely through every point S of the plane, not in s_∞, there passes a unique line of $\mathcal{S}$. In the finite case, $\mathcal{S}$ contains exactly q^2+1 lines. For further properties of spreads, see references [4], [5], [6] and [12].

3. Let us identify F, with skew field F of §1, which is contained in the left operator skew field (kernel) K of the Veblen-Wedderburn system $(\mathcal{R}, \times, +)$ which is two dimensional over F, and which satisfies the standard axioms

(i)-(v). We shall continue to use the notation of §1, i.e., if $m = m_1+m_2i$, then $im = m^*_1+m^*_2i$, where m_1, m_2, m^*_1, m^*_2 are in F. We shall also continue to identify the element x_1+x_2i of R, where x_1 and x_2 are in F, with the binary vector (x_1,x_2).

Let E_1, E_2, E'_1, E'_2 be four independent points of Σ, and let E_0 and U be distinct points of S_4, not contained in Σ where the line E_0U meets Σ in a point U_0 which is not contained in any of the planes $E_1E_2E'_1$, $E_1E_2E'_2$, $E_1E'_1E'_2$, $E_2E'_1E'_2$. We can now choose a basis of V so that the coordinates of E_1, E_2, E'_1, E'_2, E_0 and U are $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$, $(0,0,0,0,1)$ and $(1,1,1,1,1)$ respectively. Then the equation of Σ is $\zeta = 0$. Let the planes $E_1E'_1E'_2$ and $E_2E'_1E'_2$, contained in Σ be denoted by Π and Π^* respectively. Π is represented by the equations $\xi_2 = 0$, $\zeta = 0$ and Π^* by the equations $\xi_1 = 0$, $\zeta = 0$. Π and Π^* contain the common line $s_\infty = E'_1E'_2$, represented by the equations $\xi_1 = 0$, $\xi_2 = 0$, $\zeta = 0$. Let Π_0 be the set of points in Π which are not in s_∞. Then the ξ_1-coordinate of any point Y in Π_0 is non-zero. Hence the coordinates of Y can be written in the cannonical form $(1,0,y_1,y_2,0)$. Similarly, if Π_0^* is the set of points in Π^* but not in s_∞, then the coordinates of any point Y^* in Π_0^* can be written in the cannonical form $(0,1,y^*_1,y^*_2,0)$. In particular, let U_1 be the point of Π_0 with coordinates $(1,0,1,0,0)$ and U^*_1 the point of Π_0^* with coordinates $(0,1,0,1,0)$.

Let $m = m_1+m_2i = (m_1,m_2)$ be any element of R, and let

$$(3.1) \qquad A_m = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix},$$

be the unique matrix of C, corresponding to m. Then $im = m^*_1+m^*_2i$. To each element, m of R then corresponds a unique point M of Π_0 with coordinates $(1,0,m_1,m_2,0)$ and a unique point M^* of Π_0^* with coordinates $(0,1,m^*_1,m^*_2,0)$.

To any element m of R, therefore corresponds a unique line MM^* in Σ, which we shall denote by s_m. In particular if we choose m to be the zero element of R, then M and M^* become E_1 and E_2 respectively. The line corresponding to the zero element of R is thus E_1E_2 which can be denoted by s_0. Again if we choose m to be the unit element of R, then M and M^* become U_1 and U_1^* respectively. The line corresponding to the unit element of R is thus $U_1U^*_1$ which may be denoted by s_1.

Let S_0 be the set of all lines corresponding to different elements of R, and let S be obtained from S_0 by adjoining the additional line s_∞. We shall show that S is a spread.

Clearly s_m cannot intersect s_∞, and is represented by the equations

$$(3.2) \qquad \eta_1 = \xi_1 m_1 + \xi_2 m^*_1, \qquad \eta_2 = \xi_1 m_2 + \xi_2 m^*_2, \qquad \zeta = 0,$$

which may be written as

$$(3.3) \qquad (\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} = (\xi_1, \xi_2) A_m, \qquad \zeta = 0.$$

If $s_{m'}$ is another line of S, corresponding to the element $m' \neq m$ of R, then the equations of the corresponding line $s_{m'}$ of S are

$$(3.4) \qquad (\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} m'_1 & m'_2 \\ m'_1{}^* & m'_2{}^* \end{bmatrix} = (\xi_1, \xi_2) A_{m'}, \qquad \zeta = 0,$$

where $A_{m'}$ is the matrix of C corresponding to m'. The lines s_m and $s_{m'}$ cannot have any point in common. If there is a common point P, then its coordinates (a_1, a_2, b_1, b_2, c) must satisfy both (3.3) and (3.4). Hence $c = 0$, and

$$(3.5) \qquad (b_1, b_2) = (a_1, a_2) A_m = (a_1, a_2) A_{m'} .$$

Hence

$$(3.6) \qquad (a_1,a_2)[A_m - A_{m'}] = (0,0).$$

Since P is not in s_∞, $(a_1,a_2) \neq (0,0)$. Hence the matrix $A_m - A_{m'}$ is singular, which contradicts Lemma (1.3) part (c). We have thus shown that any two lines of S are non-intersecting.

Again, let Q be any point of Σ with coordinates $(x_1,x_2,y_1,y_2,0)$. If $(x_1,x_2) = (0,0)$, then $(y_1,y_2) \neq (0,0)$ and Q is contained in the line s_∞ of S. It cannot be contained in any other line s_m of S, otherwise its coordinates would satisfy (3.3), making $(y_1,y_2) = (0,0)$. If $(x_1,x_2) \neq (0,0)$, then from Lemma (1.3) part (d), there is a unique matrix A_m of C, such that the coordinates of Q satisfy (3.3). Hence there is a unique line s_m belonging to S, which contains Q. This completes the proof that S is a spread.

We have now shown that starting from $(R, +, \times)$ we can construct a corresponding spread S in Σ. This is a special case of a result proved in [5].

Note that the necessary and sufficient condition for the point $(x_1,x_2,y_1,y_2,0)$ to be contained in the line s_m of S is

$$(3.8) \qquad (y_1,y_2) = (x_1,x_2)A_m = (x_1,x_2)\begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} = (x_1,x_2) \times (m_1,m_2)$$

where A_m is the matrix of C corresponding to m.

4. We shall now show that R will satisfy axiom (vi) of §1, precisely when S is a dual spread.

Let S be a dual spread and let Π_m be the plane represented by the equations

$$(3.9) \qquad \xi_1 y_1 + \xi_2 y_2 = \eta_1 x_1 + \eta_2 x_2, \qquad \zeta = 0.$$

If $(x_1,x_2) \neq (0,0)$, then Π_m does not contain s_∞. Hence it must contain a unique line s_m of S, corresponding to the element m of R. Therefore Π_m contains the points M and M^* with coordinates $(1,0,m_1,m_2,0)$, $(0,1,m^*_1,m^*_2,0)$, where A_m given by (3.1) is the matrix of C corresponding to m. Hence

$$(3.10) \qquad y_1 = m_1x_1 + m_2x_2, \qquad y_2 = m^*_1x_1 + m^*_2x_2.$$

Hence given the binary vectors $(x_1,x_2) \neq (0,0)$ and (y_1,y_2), with coordinates from F, there exists a unique matrix A_m of C, such that

$$(3.11) \qquad A_m\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We have thus proved the alternative form of axiom (vi). Conversely if axiom (vi) or alternatively (vi,a) holds, and Π_m is an arbitrary plane of Σ, represented by the equations (3.9), then if $(x_1,x_2) = (0,0)$, Π_m contains the line s_∞ of S. Otherwise (3.11) determines a unique matrix A_m of C, and Π_m contains the corresponding line s_m of S.

Note that the necessary and sufficient condition that the plane

$$\xi_1y_1 + \xi_2y_2 = \eta_1x_1 + \eta_2x_2, \qquad \zeta = 0$$

contains the line s_m of the spread S is

$$(3.12) \qquad \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

§ 4.

1. We have shown in §3 how to construct a spread $\mathcal{S}$ corresponding to a Veblen-Wedderburn system $(R, +, \times)$ which is two dimensional over a skew field F contained in its kernel. This process is reversible.

Let S_4 be a four dimensional projective space based on the skew field F, Σ a particular 3-space contained in S_4, and $\mathcal{S}$ a spread in Σ. Let s_∞ be a particular line of $\mathcal{S}$, and let $\mathcal{S}_0$ be the set of lines of $\mathcal{S}$ other than s_∞. Let Π be a plane contained in Σ, and passing through s_∞, and Ω a 3-space in S_4 meeting Σ in the plane Π. Let E'_1, E'_2 be any two distinct points on s_∞. Let E_1 be any point of Π not in s_∞, and U_1 any point of the line $E_1E'_1$ distinct from E_1 and E'_1. Let s_0 and s_1 be the lines of $\mathcal{S}$ passing through the points E_1 and U_1 respectively. There is a unique line ℓ passing through E'_2 which intersects s_0 and s_1. Let Π^* be the plane containing ℓ and s_∞, and let E_1 and U^*_1 be the points of intersection of ℓ with s_0 and s_1 respectively. Let E_0 by any point of Ω not in Σ, and U_0 any point of s_1 distinct from U_1 and U^*_1. Finally, let U be a point of S_4 on the line E_0U_0 distinct from E_0 and U_0. We can then choose a basis of V, such that the points E_1, E_2, E'_1, E'_2, E_0 and U have the coordinates $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$, $(0,0,0,0,1)$ and $(1,1,1,1,1)$ respectively. Then the equation of Ω is $\xi_2 = 0$, and the equations of Σ, Π, Π^*, s_∞ and the coordinates of U_1 and U^*_1 are as in paragraph 3 of §3. If s_m is any line of $\mathcal{S}_0$, and if the coordinates of the points M and M* in which s_m intersects the planes Π and Π^* are $(1,0,m_1,m_2,0)$ and $(0,1,m^*_1,m^*_2,0)$ respectively, then we can associate with the line s_m the matrix

$$(4.1) \qquad A_m = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix}.$$

Let C be the set of all possible matrices A_m corresponding in this way to all possible lines of S_0. Then the matrices of C satisfy the properties (a)-(e) of Lemma (1.3).

The property (a) is obvious since the null matrix 0 corresponds to the line s_0 and the unit matrix I corresponds to the line s_1. For (b), we observe that for any binary vector (m_1,m_2) there is unique point M of Π with coordinates $(1,0,m_1,m_2,0)$. Let s_m be the line of S_0 through M, then the matrix A_m corresponding to s_m is the unique matrix of C, with (m_1,m_2) as its first row. Similarly, any binary vector (m^*_1,m^*_2) occurs as the second row of a unique matrix of C.

If A_m and $A_{m'}$ are two distinct matrices of C, then equations of the corresponding lines s_m and $s_{m'}$ are given by (3.3) and (3.4) respectively. If $A_m - A_{m'}$ is singular, we can find $(a_1,a_2) \neq (0,0)$ such that (3.6) is satisfied. Let (b_1,b_2) be determined by (3.5). Then the point $(a_1,a_2,b_1,b_2,0)$ satisfies the equations of both s_m and $s_{m'}$. This is a contradiction since any two lines of S are non-intersecting. This establishes (c). In particular, if $A_m = 0$, then A_m is non-singular.

The property (d) follows because given $(x_1,x_2) \neq (0,0)$ and (y_1,y_2) there is a unique line s_m of S_0 which passes through the point $(x_1,x_2,y_1,y_2,0)$.

The property (e) is equivalent to the assertion that the equations

$$x_1m_1 + x_2m^*_1 = y_1, \qquad x_1m_2 + x_2m^*_2 = y_2$$

have a unique solution (x_1,x_2), given $(m_1,m_2) \neq 0$ and (y_1,y_2). This follows from Lemma (1.2) and the non-singularity of A_m.

We can now take R to be the set of binary vectors (m_1,m_2) over F. Consider the system $(R, +, \times)$ with addition and multiplication defined by

$$(4.2) \qquad (m_1,m_2) + (m'_1,m'_2) = (m_1+m'_1,\ m_2+m'_2)$$

$$(4.3) \qquad (x_1,x_2) \times (m_1,m_2) = (x_1,x_2)A_m = (x_1m_1+x_2m'_1,\ x_1m_2+x_2m^*_2)$$

where A_m is the unique matrix of C whose first row is (m_1,m_2). Then $(R, +, \times)$ is a right Veblen-Wedderburn system with zero element $(0,0)$ and unity $(1,0)$.

The axioms (i) and (ii) follow directly from definitions. Axiom (iii) follows from properties (d) and (e) of Lemma (1.3). Axiom (iv) follows directly from definition. Axiom (v) is a consequence of the property (c) of Lemma (1.3). It is also readily seen that the subset of elements $(m_1,0)$ form a skew field contained in the kernel of R, and isomorphic with F. The elements $(1,0)$ and $(0,1)$ form a basis of R over F since

$$(4.4) \qquad (m_1,m_2) = (m_1,0) \times (1,0) + (m_2,0) \times (0,1).$$

We can alternatively write $(m_1,m_2) = m_1+m_2i$. Then

$$(4.5) \qquad (1,0) = 1, \quad (0,1) = i, \quad im = (0,1) \times (m_1,m_2) = (m^*_1,m^*_2) = m^*_1 + m^*_2i.$$

Finally, as we have shown in §3, axiom (vi) or (vi,a) is satisfied by $(R, +, \times)$ if S is a dual spread. In what follows, we shall always take S to be a dual spread.

2. We shall now obtain a linear representation of the affine plane $\alpha\Delta$ obtained in §2, in the projective space S_4, where the spread S and the corresponding Veblen-Wedderburn system $(R,+,\times)$ are as in paragraph 1.

Let X and X* be points of S_4 with coordinates $(1,0,x_1,x_2,0)$ and $(0,1,x^*_1,x^*_2,0)$. Then X is a point of Π, and X* is point of Π^*. The line $s_x = XX^*$ belongs to S_0, and from (3.2) its equation is

$$(4.6) \qquad \eta_1 = \xi_1x_1 + \xi_2x^*_1, \quad \eta_2 = \xi_1x_2 + \xi_2x^*_2, \quad \zeta = 0,$$

which may be written as

$$(4.7)\qquad (\eta_1,\eta_2) = (\xi_1,\xi_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} = (\xi_1,\xi_2)A_x, \qquad \zeta = 0$$

where A_x is the matrix belonging to C, corresponding to the element $x = x_1+x_2i$ of R. Denote by (s_x,Y) the plane of S_4 passing through s_x and meeting the plane $E'_1E'_2E_0$ with equations $\eta_1 = 0$, $\eta_2 = 0$, in the point Y with coordinates $(0,0,-y_1,-y_2,1)$. Let the $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ be represented by the plane (s_x,Y). This sets up a $(1,1)$ correspondence between $\alpha\Delta$-points, and planes of S_4 passing through the lines of S_0, and not contained in S_4. The equations of the plane (s_x,Y) are

$$(4.8)\qquad (\eta_1,\eta_2) = (\xi_1,\xi_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} - \zeta(y_1,y_2).$$

Let the $\alpha\Delta$-line $[(m_1,m_2),(b_1,b_2)]$ of type 1, be represented by the point $(m_1,m_2-b_1,-b_2,1)$, $m_2 \neq 0$, of S_4. This sets up a $(1,1)$ correspondence between $\alpha\Delta$-lines of type 1, and points of S_4 not contained in Σ or Ω.

Again let the $\alpha\Delta$-line $[(m_1,0),(b_1,b_2)]$ of type 2a be represented by the plane

$$(4.9)\qquad \eta_1 m_1 - \eta_2 = -\xi_1 b_1 + \zeta b_2, \qquad \xi_2 = 0$$

of S_4. This sets up a $(1,1)$ correspondence between $\alpha\Delta$-lines of type 2a, and planes of S_4 which are contained in Ω, and which do not pass through the point E'_2.

Finally, let the $\alpha\Delta$-line $[c_1,c_2]$ of type 2b be represented by the plane

$$(4.10)\qquad \eta_1 = \xi_1 c_1 - \zeta c_2, \qquad \xi_2 = 0$$

of S_4. This sets up a $(1,1)$ correspondence between $\alpha\Delta$-lines of type 2b, and planes of S_4, which are contained in Ω, and pass through the point E'_2 but do not contain the line s_∞.

Thus the $\alpha\Delta$-lines of type 2a and 2b taken together are represented by planes of S_4 contained in Ω but not containing the line s_∞.

LEMMA (4.1). An $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is incident with the $\alpha\Delta$-line $\{(m_1,m_2),(b_1,b_2)\}$, $m_2 \neq 0$ of type 1, if and only if the plane of S_4 representing the $\alpha\Delta$-point, passes through the point of S_4, representing the $\alpha\Delta$-line.

The condition for the plane (s_x,Y) of S_4 with equations (4.8), representing the $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ to pass through the point $(m_1,m_2,-b_1,-b_2,1)$ of S_4 representing the $\alpha\Delta$-line $\{(m_1,m_2),(b_1,b_2)\}$ is

$$-(b_1,b_2) = (m_1,m_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} - (y_1,y_2)$$

where (x^*_1,x^*_2) is the uniquely determined matrix of C, with the first row (x_1,x_2). This is precisely the same as the condition (2.4) for the $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ to be incident with the $\alpha\Delta$-line $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ of type 1.

LEMMA (4.2). An $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is incident with the $\alpha\Delta$-line $[(m_1,0),(b_1,b_2)]$ of type 2a if and only if the plane of S_4 representing the $\alpha\Delta$-point intersects the plane of S_4 representing the $\alpha\Delta$-line.

The $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is represented by the plane (s_x,Y) with equations (4.8), and the $\alpha\Delta$-line $[(m_1,0),(b_1,b_2)]$ is represented by the plane with equations (4.9). These planes intersect in a line if and only if there exist elements r_1, r_2, s_1, s_2 of F, not all zero, such that the result of adding the four equations after multiplying them successively from the right by r_1, r_2, s_1, s_2 is an identity. This gives

$$(4.11) \qquad r_1 + m_1 s_1 = 0, \qquad r_2 - s_1 = 0$$

(4.12) $\quad x_1r_1 + x_2r_2 - b_1s_1 = 0, \qquad -y_1r_1 - y_2r_2 + b_2s_1 = 0$

(4.13) $\quad x^*_1r_1 + x^*_2r_2 + s_2 = 0.$

If the above conditions are satisfied, $s_1 \neq 0$, otherwise r_1,r_2,s_1,s_2 are all zero. Hence without loss of generality, we can take $s_1 = 1$. This gives $r_1 = -m_1$, $r_2 = 1$, $s_1 = 1$. Then the equations (4.12) reduce to (2.6). Conversely, if the conditions (2.6) are satisfied, we can choose $r_1 = -m_1$, $r_2 = 1$, $s_1 = 1$, $s_2 = x^*_1m_1-x^*_2$. Then (4.11), (4.12), (4.13) are satisfied. This proves the Lemma.

LEMMA (4.3). An $\alpha\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is incident with the $\alpha\Delta$-line $[c_1,c_2]$ of type 2b, if and only if the plane of S_4 representing the $\alpha\Delta$-point, intersects the plane representing the $\alpha\Delta$-line in a line.

Proof is similar to Lemma (4.2).

The affine plane $\alpha\Delta$ can be completed to the projective plane Δ. We shall now obtain linear representations for the elements at infinity in Δ, denoting them as in §2, paragraph 7.

Let the point at infinity $\{m_1,m_2\}$, $m_2 \neq 0$, corresponding to the parallel class (m_1,m_2) of type 1, be represented by the plane $\xi_1 = \zeta m_1$, $\xi_2 = \zeta m_2$ of S_4, where $m_2 \neq 0$. This sets up a (1,1) correspondence between the points at infinity in Δ which correspond to parallel classes of type 1, and planes of S_4. which pass through the line s_∞ and are not contained in Σ or Ω.

The point at infinity $\{m_1,0\}$ which corresponds to the parallel class $(m_1,0)$ of type 2a, will be represented by the point $(0,0,1,m_1,0)$ of S_4. Again the point at infinity $\{\infty\}$, which corresponds to the parallel class (∞) consisting of all lines of type 2b, will be represented by the point $(0,0,0,1,0)$ of S_4, viz. the point E'_2. We thus have a (1,1) correspondence between the points at infinity in Δ, which correspond to parallel classes of type 2a or 2b, and the points of the line s_∞ of S_4. Finally $[\infty]$ the line at infinity in Δ will be represented by the line s_∞ of S_4. A point at infinity in Δ, may also be called a Δ-point at infinity, and the line at infinity in Δ, may be called the Δ-line at infinity.

The following Lemmas are now easy to verify.

LEMMA (4.4). A Δ-point at infinity is incident with an $\alpha\Delta$-line if and only if the linear space of S_4 representing the Δ-point at infinity is contained in or contains the linear space of S_4 representing the $\alpha\Delta$ line, except in the case when both the Δ-point at infinity and the $\alpha\Delta$-line are represented by planes in which case they are incident if and only if the planes representing them intersect in a line.

LEMMA (4.5). The line s_∞ representing the Δ-line at infinity either contains or is contained in the linear space of S_4 representing any Δ-point at infinity.

Thus we have arrived at the same linear representation of a projective plane Δ, which was obtained by Bose and Barlotti in [3]. We thus have an alternative proof of the following theorem due to them.

THEOREM (4.1). Let S_4 be a projective space based on a skew field F. Let Σ and Ω be distinct 3-spaces of S_4, intersecting in the plane Π. Let S be a spread of lines in Σ, which is also a dual spread. Let s_∞ be the line of S contained in Π, and let S_0 be the set of lines of S not contained in Π. Let us consider an incidence structure Δ whose points and lines are represented by linear spaces of S_4 as follows:

Δ-points are of the following types: (i) Type 0, represented by planes of S_4 passing through lines of S_0, and not contained in Σ; (ii) Type I_1, represented by planes of S_4, passing through s_∞ and not contained in either Σ or Ω; (iii) Type I_2, represented by points of s_∞.

Δ-lines are of the following types: (i) Type 1, represented by points of S_4, not contained in Σ or Ω; (ii) Type 2, represented by planes of S_4 contained in Ω, but not containing the line s_∞; (iii) A single Δ-line represented by the line s_∞.

A Δ-point and Δ-line both of which are not represented by palnes are incident if and only if the linear space of S_4 representing the Δ-point either contains or is contained in the linear space representing the Δ-line. A Δ-point and a Δ-line both of which are represented by planes are incident, if the planes representing them intersect in a line.

Note that the Δ-points of type 0 are the $\alpha\Delta$-points. Δ-points of type I_1 are the Δ-points at infinity corresponding to parallel classes of type 1, and Δ-points of type I_2 are Δ-points at infinity corresponding to parallel classes of type 2a or 2b.

Bose and Barlotti [3] showed that the plane Δ is the derived plane in the sense of Ostrom [10,11], and Albert [1] of the dual translation plane δT which can be linearly represented as follows: Points of δT are of two types: (i) δT-points of type 1 are represented by planes of S_4 passing through the lines of $\mathcal{S}$, and not contained in Σ; (ii) There is only a single δT point of type 2 represented by Σ. Lines of δT are of two types: (i) δT-lines of type 1 are represented by points of S_4 not belonging to Σ; (ii) δT-lines of type 2 are represented by the lines of $\mathcal{S}$. Incidence in δT is given by the containing contained relation. The derivation set consists of all δT points represented by planes passing through s_∞ and contained in Ω but not in Σ, and the δT-point represented by Σ

§ 5.

1. In §2, we have obtained a projective plane Δ, based on a Veblen-Wedderburn system $(\mathcal{R}, +, \times)$ and in §4, we have obtained a linear representation of Δ, in a projective space S_4. Any linear transformation σ of S_4 which fixes Σ, Ω, s_∞ and $\mathcal{S}_0$ (σ may interchange lines of $\mathcal{S}_0$) will give a collineation of Δ, in terms of the representation in S_4.

We want to consider in particular a linear transformation of S_4 which gives a group of elations of Δ, with axis represented by s_∞ and centres represented by the points on s_∞. Consider the group G of linear transformations $\sigma(k_1,k_2)$ given by

$$(5.1) \qquad \xi'_1 = \xi_1, \quad \xi'_2 = \xi_2, \quad \eta'_1 = \eta_1 - \zeta k_1, \quad \eta'_2 = \eta_2 - \zeta k_2, \quad \zeta' = \zeta.$$

The point $(u_1,u_2,v_1,v_2,w) \to (u_1,u_2,v_1-wk_1,v_2-wk_2,w)$ and the 3-space

$$\xi_1 a_1 + \xi_2 a_2 + \eta_1 b_1 + \eta_2 b_2 + \zeta c = 0$$

to

$$\xi_1 a_1 + \xi_2 a_2 + \eta_1 b_1 + \eta_2 b_2 + \zeta(k_1 b_1 + k_2 b_2 + c) = 0.$$

Thus $\sigma(k_1,k_2)$ fixes Σ_1, Ω, s_∞ and $\mathcal{S}_0$ and transforms a linear space of S_4 representing an element of Δ into a linear space representing some element of Δ.

The equations of s_∞ are $\xi_1 = 0$, $\xi_2 = 0$, $\zeta = 0$. Hence $[\infty]$, the line at infinity in Δ remains fixed. Δ-points at infinity of type I_1 are represented by planes $\xi_1 = \zeta m_1$, $\xi_2 = \zeta m_2$, and Δ points at infinity of type I_2 are represented by points on s_∞. They remain fixed, i.e., the Δ-points $\{m_1,m_2\}$ and $\{\infty\}$ remain fixed. A Δ-point of type 0, i.e. an $\alpha\Delta$-point represented by the plane,

$$(5.2) \qquad (\eta_1,\eta_2) = (\xi_1,\xi_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} - \zeta(y_1,y_2)$$

is transformed by $\sigma(k_1,k_2)$ to the Δ-point represented by the plane

$$(5.3) \qquad (\eta_1,\eta_2) = (\xi_1,\xi_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} - \zeta(y_1+k_1,y_2+k_2).$$

Hence the Δ-point $\{(x_1,y_1),(x_2,y_2)\}$ is transformed to $\{(x_1,y_1+k_1),(x_2,y_2+k_2)\}$. It is fixed only by the identity of G.

The Δ-line $\{(m_1,m_2),(b_1,b_2)\}$, $m_2 \neq 0$ of type 1, represented by the point $(m_1,m_2,-b_1,-b_2,1)$ of S_4 is transformed to the Δ-line $\{(m_1,m_2),(b_1+k_1,b_2+k_2)\}$ represented by the point $(m_1,m_2,-b_1-k_1,-b_2-k_2,1)$. It is fixed only by the identity of G.

The Δ-line $[(m_1,0),(b_1,b_2)]$ of type 2a represented by the plane

$$(5.4) \qquad \eta_1 m_1 - \eta_2 = -\xi_1 b_1 + \zeta b_2, \qquad \xi_2 = 0$$

is transformed by $\sigma(k_1,k_2)$ to the Δ-line $[(m_1,0),(b_1,k_2-k_1m_1+b_2)]$ represented by the plane $\eta_1m_1-\eta_2 = -\xi_1b_1+\zeta(k_2-k_1m_1+b_2)$, $\xi_2 = 0$. It is fixed by the subgroup G_{m_1} of G for which $k_2 = k_1m_1$, where m_1 is fixed, but k_1 is arbitrary. Now the Δ-line $[(m_1,0),(b_1,b_2)]$ is incident with the Δ-point at infinity $\{m_1,0\}$ which is represented by the point $(0,0,1,m_1,0)$ of s_∞, and conversely every Δ-line incident with this Δ-point at infinity is of the form $[(m_1,0),(b_1,b_2)]$ where m_1 is a fixed and b_1, b_2 are arbitrary elements of F. Thus G_{m_1} is a subgroup of elations for which the line at infinity in Δ is the axis and the Δ-point at infinity $\{m_1,0\}$ is the center. Corresponding to each m_1 belonging to F, we get a subgroup G_{m_1}.

The Δ-line $[c_1,c_2]$ of type 2b represented by the plane

$$(5.5) \qquad \eta_1 = \xi_1 c_1 - \zeta c_2, \qquad \xi_2 = 0$$

is transformed by $\sigma(k_1,k_2)$ to the line $[c_1,c_2+k_1]$, represented by the plane $\eta_1 = \xi_1c_1-\zeta(k_1+c_2)$, $\xi_2 = 0$. It is fixed by the subgroup G_∞ of G for which $k_1 = 0$, but k_2 is arbitrary. Now the Δ-line $[c_1,c_2]$ is incident with the

Δ-point $\{\infty\}$, represented by the point $(0,0,0,1,0)$ of s_∞, and conversely every Δ-line incident with $[\infty]$ is a line $[c_1,c_2]$ of type 2b for some c_1, c_2 belonging to F. Hence the subgroup under consideration is a subgroup of elations for which the line at infinity in Δ is the axis and the Δ-point $\{\infty\}$ is the center.

Any two subgroups have only the identity in common. Each point of S_4 on the line s_∞ represents a Δ-point at infinity, which is the center for a subgroup of elations, for which the axis is the line at infinity in Δ, represented by s_∞. When F is the field GF(q), Δ is a finite plane. The group G is of order q^2, and there are $q+1$ subgroups each of order q. Hence Δ is a semitranslation plane with respect to the line at infinity in Δ in the sense of Ostrom [9]. Of course, the situation is completely analogous in the infinite case.

2. We shall now obtain the ternary ring of the projective plane Δ. All points and lines considered in this paragraph belong to Δ. Hence we shall speak only of points and lines instead of Δ-points and Δ-lines. Let us choose the line $[(0,0),(0,0)]$ of type 2a to be the x-axis and the line $[0,0]$ of type 2b as the y-axis. From (2.6) and (2.7), the point $\{(0,0),(0,0)\}$ is incident both with the x-axis and y-axis. This will be chosen to be the origin. The x-axis belongs to the parallel class $(0,0)$ of type 2a. Lines parallel to the x-axis have coordinates $[(0,0),(b_1,b_2)]$. The y-axis belongs to the parallel class $[\infty]$ consisting of all lines of type 2b. Hence lines parallel to the y-axis have coordinates $[c_1,c_2]$. Let us choose the line $[(1,0),(0,0)]$ of type 2a to be the unit line. From (2.6) it is incident with the origin. If the point $\{(x_1,y_1),(x_2,y_2)\}$ is an incident with the unit line, then from (2.6), $x_1 = x_2$, $y_1 = y_2$. Hence a general point incident with the unit line has coordinates $\{(x_1,y_1),(x_1,y_1)\}$. We may associate to this point the element (x_1,y_1) of R. Thus there is a $(1,1)$ correspondence between the elements of R and the points on the unit

line, the element (x,y) of R corresponding to the point with coordinates $\{(x,y),(x,y)\}$

3. **LEMMA (5.1).** Given two points $\{(x_1,y_1),(x_1,y_1)\}$ and $\{(x_2,y_2),(x_2,y_2)\}$ incident with the unit line corresponding to the elements (x_1,y_1), (x_2,y_2) of R the line through the first point parallel to the y-axis, meets the line through the second point parallel to the x-axis in the point whose coordinates are $\{(x_1,y_1),(x_2,y_2)\}$

The line parallel to the y-axis and incident with $\{(x_1,y_1),(x_2,y_2)\}$ is from (2.7), $[x_1,y_1]$. Again the line parallel to the x-axis and incident with $\{(x_2,y_2),(x_2,y_2)\}$ is from (2.6), $[(0,0),(x_2,y_2)]$. Then from (2.6) and (2.7), we see that the point incident with both $[x_1,y_1]$ and $[(0,0),(x_2,y_2)]$ is $\{(x_1,y_1),(x_2,y_2)\}$. This proves the Lemma.

Thus we may appropriately call (x_1,y_1) the x-coordinate and (x_2,y_2) the y-coordinate of the point $\{(x_1,y_1),(x_2,y_2)\}$. (See Albert [1].)

4. However, (m_1,m_2) and (b_1,b_2) are not the slope and the y-intercept of the line $[(m_1,m_2),(b_1,b_2)]$ in the standard sense. We will call (m_1,m_2) the m-coordinate, and (b_1,b_2) the b-coordinate of the line $[(m_1,m_2),(b_1,b_2)]$.

Since $(1,0)$ is the unity of R, we may take the point $\{((1,0),(1,0)\}$ on the unit line, to be the unit point. The line $[1,0]$ of type 2b, parallel to the y-axis through the unit point will be called the slope line. Let $[(m_1,m_2),(b_1,b_2)]$ be any line of type 1 or 2a. Then the line parallel to this and incident with the origin is $[(m_1,m_2),(0,0)]$. If it meets the slope line in the point $\{(1,0),(u_2,v_2)\}$, then (u_2,v_2) is defined to be the slope of the line $[(m_1,m_2),(b_1,b_2)]$. We have to consider two seperate cases.

Case I. Let the line $[(m_1,m_2),(b_1,b_2)]$ be of type 1, i.e., $m_2 \neq 0$. If its slope is (u_2,v_2), then the point $\{(1,0),(u_2,v_2)\}$ of the slope line is incident with the line $[(m_1,m_2),(0,0)]$. From (2.4) the condition for this is

$$(5.6) \qquad (0,\nu_2) = (m_1,m_2)\begin{bmatrix} 1 & \mu_2 \\ \mu_1{*} & \mu_2{*} \end{bmatrix},$$

where the matrix appearing on the right hand side of (5.6) belongs to C, i.e., $(\mu_1{*},\mu_2{*}) = (0,1)\times(1,\mu_2)$ or in terms of the functions F_1, F_2, G_1, G_2 defined by (1.14) and (1.15)

$$(5.7) \qquad \mu_1{*} = F_1(1,\mu_2), \qquad \mu_2{*} = F_2(1,\mu_2).$$

From Lemma (1.5), μ_2, ν_2 are uniquely determined. From (1.19) and (1.20),

$$(5.8) \qquad \mu_2 = G_1(1,-m_2{}^{-1}m_1), \qquad \nu_2 = m_1G_2(1,-m_2{}^{-1}m_1) + m_2G_2(1,-m_2{}^{-1}m_1).$$

Also $\nu_2 \neq 0$. If $\nu_2 = 0$, then from Lemma (1.3), part (d), the matrix on the right in (5.6) would be the null matrix which is a contradiction.

Conversely, given (μ_2,ν_2), from Lemma (1.3) part (e), (m_1,m_2) is uniquely determined, and since $m_2 \neq 0$, it follows that $\nu_2 \neq 0$. We can explicitly express m_1, m_2 as

$$(5.9) \qquad (m_1,m_2) = \nu_2 d^{-1}(\mu_1{*},-1)$$

where $\mu_1{*}$, $\mu_2{*}$ are given by (5.7) and $d = \mu_1{*}\mu_2-\mu_2{*}$.

Note that $d \neq 0$, otherwise the column vectors of the matrix on the right in (5.6), would be dependent from the right. From Lemma (1.1), the matrix would then be singular. This would contradict Lemma (1.3) part (c), since the matrix is non-null. Thus there is a (1,1) correspondence between m-coordinates (m_1,m_2), $m_2 \neq 0$ characterizing parallel classes of type 1, and the corresponding slopes (μ_2,ν_2), $\nu_2 \neq 0$.

Case II. Consider the line $[(m_1,0),(b_1,b_2)]$ of type 2a. If its slope is (μ_2,ν_2), then the point $\{(1,0),(\mu_2,\nu_2)\}$ of the slope line is incident with the line $[(m_1,0),(0,0)]$. Hence from (2.6), $\mu_2 = m_1$, $\nu_2 = 0$. Hence the slope of the line $[(m_1,0),(b_1,b_2)]$ of type 2a is $(m_1,0)$.

The slope of any line $[c_1, c_2]$ of type 2b may be defined to be ∞.

5. Given any two elements (x_1, y_1), (μ_2, ν_2) of R we can define a new multiplication $\circledcirc$ by setting

$$(5.10) \qquad (x_2, y_2) = (x_1, y_1) \circledcirc (\mu_2, \nu_2),$$

if (x_2, y_2) is the y-coordinate of the uniquely determined point with x-coordinate (x_1, y_1) and incident with the line through the origin whose slope is (μ_2, ν_2). Then this multiplication is a loop with unity $(1,0)$.

We can express $(x_1, y_1) \circledcirc (\mu_2, \nu_2)$ in terms of the operations $\times$ and $+$ in $(R, \times, +)$ as follows: Let $[(m_1, m_2), (0,0)]$ be the line through the origin with slope (μ_2, ν_2). Then if $\nu_2 = 0$, $(m_1, m_2) = (\mu_2, 0)$, from the Case II of the previous paragraph. If $\nu_2 \neq 0$, (m_1, m_2) is given by (5.9). Now the point $\{(x_1, y_1), (x_2, y_2)\}$ is incident with $[(m_1, m_2), (0,0)]$. Hence from (2.4), and formulae (1.19) (1.120) of Lemma (1.5), if $\nu_2 \neq 0$,

$$(5.11) \qquad (x_1, y_1) \circledcirc (\mu_2, \nu_2) = (x_2, y_2) = [G_1(x_1, k), m_1 G_1(x_1, k) + m_2 G_2(x_2, k)],$$

where

$$(5.12) \qquad k = m_2^{-1}(y_1 - m_1 x_1),$$

and the functions G_1, G_2 are defined by (1.15). Substituting for m_1, m_2 in terms of μ_2, ν_2 from (5.9), we have an explicit expression for $(x_1, y_1) \circledcirc (\mu_2, \nu_2)$ when $\nu_2 \neq 0$.

If $\nu_2 = 0$, then $(m_1, m_2) = (\mu_2, 0)$. Then from 2.6),

$$(5.13) \qquad (x_1, y_1) \circledcirc (\mu_2, \nu_2) = (x_1, y_1)\mu_2.$$

Note that from Lemma (2.4) and (2.5) the line through the origin with slope (μ_2, ν_2) is of type 1 or 2a according as $\nu_2 \neq 0$ or $\nu_2 = 0$. Hence if we wish to determine (μ_2,ν_2) given (x_1,y_1), (x_2,y_2), the formula (5.11) or (5.13) is valid according as the matrix

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

is non-singular or singular.

6. The equation of any line of Δ, passing through the origin and having the slope (μ_2,ν_2) is given by (5.8).

Let $[(m_1,m_2),(b_1,b_2)]$ be any line of type 1. From (2.4), the necessary and sufficient condition for the point $\{(x_1,y_1),(x_2,y_2)\}$ to be incident with it is

$$(5.14) \qquad (y_1-b_1,\ y_2-b_2) = (m_1,m_2)\begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix},$$

where $(x^*_1,x^*_2) = (0,1)\times(x_1,x_2)$. It follows that the point $\{(x_1,y_1-b_1),(x_2,y_2-b_2)\}$ is incident with the line $[(m_1,m_2),(0,0)]$ through the origin. If (μ_2,ν_2) is the slope of the line, we have

$$(5.15) \qquad (x_2,\ y_2-b_2) = (x_1,\ y_1-b_1) \circledcirc (\mu_2,\nu_2).$$

This is the equation of a line of type 1, with slope (μ_2,ν_2) and b-coordinate (b_1,b_2).

Similarly, the equation of a line of type 2a with slope $(\mu_2,0)$ and b-coordinate (b_1,b_2) is

$$(5.16) \qquad (x_2,y_2) = (x_1,y_1)\mu_1 + (b_1,b_2).$$

7. Finally we note that the b-coordinate of a line is not the y-intercept of a line in the standard sense. We may define the y-intercept of a line of type 1 or type 2a, to be the y-coordinate of the point in which the line intersects the y-axis.

Let (β_2,γ_2) be the y-intercept of the line $[(m_1,m_2),(b_1,b_2)]$. We have to consider two cases.

Case I. The line $[(m_1,m_2),(b_1,b_2)]$ is of type 1. Hence $m_2 \neq 0$. The point $\{(0,0),(\beta_2,\gamma_2)\}$ is incident with it. Hence from (2.4),

$$(5.17) \qquad (-b_1,\ \gamma_2-b_2) = (m_1,m_2)\begin{bmatrix} 0 & \beta_2 \\ \beta_1* & \beta_2* \end{bmatrix},$$

where $(\beta_1*,\beta_2*) = (0,1)\times(0,\beta_2)$ or alternatively

$$\beta_1* = F_1(0,\beta_2), \qquad \beta_2* = F_2(0,\beta_2),$$

where the functions F_1 and F_2 are given by (1.14).

From Lemma (1.5), (β_2,γ_2) is uniquely determined by (b_1,b_2) since (m_1,m_2) is given. Conversely given (β_2,γ_2), (5.17) determines (b_1,b_2).

Case II. The line $[(m_1,0),(b_1,b_2)]$ is of type 2a. Since $\{(0,0),(\beta_2,\gamma_2)\}$ is incident with it, we have from (2.6), $(\beta_2,\gamma_2) = (b_1,b_2)$. Thus in this case, the y-intercept is (b_1,b_2).

We are now in a position to write down the ternary ring of Δ. Let

$$(5.18) \qquad \begin{aligned} \Phi[(x_1,y_1),(\mu_2,\nu_2),(\beta_2,\gamma_2)] &= (x_1,\ y_1-b_1) \circledcirc (\mu_2,\nu_2) + b_2 && \text{if } \nu_2 \neq 0 \\ &= (x_1,y_1)\mu_1 + (\beta_2,\gamma_2) && \text{if } \nu_2 = 0, \end{aligned}$$

where in the case $\nu_2 \neq 0$, we determine (b_1,b_2) from (5.15) and (5.7).

Then (5.18) is the ternary ring of Δ. The equations of lines of type 1 or type 2a are given by

$$(5.19) \qquad (x_2,y_2) = \Phi[(x_1,y_1),(\mu_2,\nu_2),(\beta_2,\gamma_2)],$$

where (x_1,y_1), (x_2,y_2) are the x and y coordinates of a point incident with a line which has slope (μ_2,ν_2) and y-intercept (β_2,γ_2).

The lines of type 2b, parallel to the y-axis have equations

$$(5.20)\qquad (x_1,y_1) = (c_1,c_2).$$

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