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ASYMPTOTICALLY EFFICIENT RANK TESTS FOR RESTRICTED ALTERNATIVES IN A MANOVA IN RANDOMIZED BLOCK DESIGNS *

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Summary

In a randomized block design MANOVA model, for tests for the homogeneity of treatments against restricted alternatives based on intra-block rankings and on ranking after alignment, asymptotic optimality is studied in the light of asymptotically most stringency and somewhere most powerful character. For ordered alternatives, the proposed tests are shown to be asymptotically most powerful within the respective rank class. Tests based on the ranking after alignment are preferred when the error distributions are homogeneous accross the blocks.

1. Introduction

We consider the usual randomized block design model consisting of n blocks each containing p plots where p treatments are assigned at random ; the response is possibly multivariate. Let $\underset{\sim}{X}_{ij} = (X_{ij}^{(1)},\ldots,X_{ij}^{(q)})'$ be the response (vector) of the $j\underline{th}$ treatment in the $i\underline{th}$ block, for j=1,...,p; i=1,...,n. We assume that $(\underset{\sim}{X}_{i1},\ldots,\underset{\sim}{X}_{ip})$ has a pq-variate continuous distribution function (d.f.) $F_i^o(x)$, where letting $\underset{\sim}{\beta}_j = (\beta_j^{(1)},\ldots,\beta_j^{(q)})'$, j=1,...,p and $\underset{\sim}{\beta} = (\underset{\sim}{\beta}_1,\ldots,\underset{\sim}{\beta}_p)$,

$$(1.1)\qquad F_i^o(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p) = F_i(\underset{\sim}{x}_1-\underset{\sim}{\beta}_1,\ldots,\underset{\sim}{x}_p-\underset{\sim}{\beta}_p)\ ,\ (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p)\in E^{pq}\ ,$$

and the d.f. F_i is symmetric in its p arguments (each a q-vector); although, the form of the F_i may not remain the same for every i(=1,...,n). Thus, we consider independent blocks, and within each block, interchangeable errors. The case of independent and identically distributed (i.i.d.) error vectors is thus included as a particular one. In (1.1), the $\underset{\sim}{\beta}_j$ stand for the vector of treatment effects

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while the block effects may not be additive and may even be stochastic in nature.

Generally, in a MANOVA(multivariate analysis of variance), the null hypothesis relates to the homogeneity of the treatments i.e.,

(1.2) $H_0 : \underset{\sim}{\beta}_1 = \dots = \underset{\sim}{\beta}_p$ ($= \underset{\sim}{0}$, without any loss of generality).

A global alternative relates to the non-homogeneity of the $\underset{\sim}{\beta}_j$. In this study, we shall be specifically interested in suitable restricted alternatives. For the univariate case (i.e., q = 1), the most common form of such a restricted alternative is the so called <u>ordered alternative</u>

(1.3) $H^< : \beta_1 \leq \beta_2 \leq \dots \leq \beta_p$, with at least one strict inequality.

A general account of rank tests for ordered alternatives in randomized blocks is given in Chapter 7 of Puri and Sen (1971). Later on, De (1976) extended the method of Sen (1968) by effectively incorporating the union-intersection (UI-) principle of Roy (1953) to form an aligned rank test for H_0 against $H^<$ when the r.v.'s X_{ij} have i.i.d. error components. Boyd and Sen (1984) used the concept of locally most powerful rank (LMPR) tests of Hoeffding (1950) and incorporated the UI-principle to construct UI-LMPR tests based on intra-block rankings as well as on ranking after alignments. However, in either of these studies, no attempt was made to establish any possible optimal properties of the proposed tests. For the ordered alternative problem, Araki and Shirahata (1981) and Shirahata (1984) considered some rank tests (for q = 1) which are analogues of the usual likelihood ratio tests. Interestingly, their proposed tests may also be characterized as UI-LMPR tests, and hence, the question on their asymptotic optimality properties remains open. The main objectives of the current study are the following:

(i) To establish the asymptotic superiority (in terms of power) of the ranking after alignment procedure to the intra-block ranking procedure for general forms of restricted alternatives ,

(ii) to show that the UI-LMPR tests are asymptotically UMP against the class of ordered alternatives within the respective class of rank tests, and

(iii) to characterize that the UI-LMPR tests have the property of asymptotically most stringency and somewhere most powerful character within the respective class of rank tests for testing H_0 in (1.2) against a general form of restricted alternative

(1.4) $\quad H^*: \underset{\sim}{\beta} \in \Gamma = \{ \underset{\sim}{\beta} \in E^{pq}: \underset{\sim}{A}\,\mathrm{vec}\underset{\sim}{\beta} \geq \underset{\sim}{0}\ ,\ \underset{\sim}{A} \in \mathcal{C}(a,pq) \}$,

where $\mathrm{vec}\underset{\sim}{\beta}$ denotes the pq-vector obtained by stacking the rows of $\underset{\sim}{\beta}$ under each other and $\mathcal{C}(a,pq)$ is the set of a×pq matrix of rank a : $1 \leq a \leq pq$.

The basic regularity conditions and preliminary notions are presented in Section 2. UI-LMPR tests based on intra-block rankings and ranking after alignment (for testing H_0 against H^*) are considered in Sections 3 and 4. Asymptotic compariosons of the power properties of these two procedures are made in the last section.

2. Preliminary notions

We make the following regularity assumptions :

[A1] Let O be an open set containing $\underset{\sim}{0}$ and define $f_i(\underset{\sim}{x}-\delta\underset{\sim}{\gamma})$ as in (1.1) with $\underset{\sim}{\beta}=\delta\underset{\sim}{\gamma}$, $i \geq 1$. Then (i) for every i (≥ 1), $f_i(\underset{\sim}{x}-\delta\underset{\sim}{\gamma})$ is absolutely continuous for almost all $\underset{\sim}{x}$ and $\underset{\sim}{\gamma} \in O$, so that $\dot{\underset{\sim}{f}}_{i\underset{\sim}{\beta}}(\underset{\sim}{x}-\underset{\sim}{\beta}) = \frac{\partial}{\partial\ \mathrm{vec}\ \underset{\sim}{\beta}} f_i(\underset{\sim}{x}-\delta\underset{\sim}{\gamma})$ and $\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x}-\delta\underset{\sim}{\gamma}) = \frac{\partial}{\partial\delta} f_i(\underset{\sim}{x}-\delta\underset{\sim}{\gamma})$, then for $\underset{\sim}{\beta} = \delta\underset{\sim}{\gamma}$, $\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x}-\delta\underset{\sim}{\gamma}) = (\mathrm{vec}\ \underset{\sim}{\gamma})'\ \dot{\underset{\sim}{f}}_{i\underset{\sim}{\beta}}(\underset{\sim}{x}-\underset{\sim}{\beta})$, $\underset{\sim}{\gamma} \in O$, $\underset{\sim}{x} \in E^{pq}$. (ii) For almost all $\underset{\sim}{x}$ and $\underset{\sim}{\gamma}$, the limit $\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x}) = \lim_{\delta\to 0} \delta^{-1}[f_i(\underset{\sim}{x}-\delta\underset{\sim}{\gamma}) - f_i(\underset{\sim}{x})]$ exists. (iii) For every $\underset{\sim}{\gamma} \in O$, $\lim_{\delta\to 0} \int_{-\infty}^{\infty} |\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x}-\delta\underset{\sim}{\gamma})|d\underset{\sim}{x} = \int_{-\infty}^{\infty} |\dot{f}_{i\underset{\sim}{\gamma}}(\underset{\sim}{x})|d\underset{\sim}{x}$ is finite. (iv) The largest characteristic root of $\underset{\sim}{I}(f) = \mathcal{E}_0\left\{\left[\frac{\partial}{\partial\ \mathrm{vec}\ \underset{\sim}{X}} \log f(\underset{\sim}{X})\right]\left[\frac{\partial}{\partial\ \mathrm{vec}\ \underset{\sim}{X}}\right]' \log f(\underset{\sim}{X})\right\}$ is finite, where $\mathcal{E}_0$ stands for the expectation under the null hypothesis.

[A2] For the pq-variate p.d.f. $f_i(\underset{\sim}{x}-\underset{\sim}{\beta})$, we denote the conditional p.d.f. of $(j\ell)$th coordinate, given the others by $f_{j\ell}(x_{ij}^{(\ell)} - \beta_j^{(\ell)}|\underset{\sim}{x}_0)$ and let

(2.1) $$g_{[j]\ell}(x) = \frac{\partial}{\partial x} \log f_{j\ell}(x)\ ,\ j=1,\dots,p;\ \ell=1,\dots,q.$$

Also, let $f_{[j]\ell}$ denote the marginal p.d.f. of the (j,ℓ)th coordinate, and let

(2.2) $$f^*_{[j]\ell}(x) = (\partial/\partial x)\log f_{[j]\ell}(x)\ ,\ \ j=1,\dots,p;\ \ \ell=1,\dots,q\ .$$

We denote the pq-vectors with the elements in (2.1) and (2.2) by $\underset{\sim}{g}(\underset{\sim}{x})$ and $\underset{\sim}{f}^{*}(\underset{\sim}{x})$, respectively, and assume that there exists a p.d. matrix $\underset{\sim}{H}$, such that

(2.3) $$\underset{\sim}{g}(\underset{\sim}{x}) = \underset{\sim}{H}\,\underset{\sim}{f}^{*}(\underset{\sim}{x}) \quad , \text{ for almost all } \underset{\sim}{x} .$$

[A3] We assume that for each $j(=1,\ldots,p)$ and $\ell\ (=1,\ldots,q)$, $f^{*}_{[j]\ell}(x)$ is differentiable on the support of the p.d.f. $f_{[j]\ell}$. We denote by $f^{*'}_{[j]\ell}(x) = (\partial/\partial x) f^{*}_{[j]\ell}(x)$ and assume that there exists a positive r : $0 < r < 1/2$,such that

(2.4) $$\sup_{1\le i\le n}\ \sup_{1\le j\le p}\ \sup_{1\le \ell\le q} |f^{*'}_{[j]\ell}(X^{(\ell)}_{ij})| = o(n^{r}) \text{ a.s., as } n\to\infty \quad .$$

3. UI-LMPR tests based on intra-block rankings

Intra-block rank tests for MANOVA against global alternatives were considered by Gerig (1969), and we shall follow the same notations along with general scores. Let $R^{(\ell)}_{ij}$ be the rank of $X^{(\ell)}_{ij}$ among $X^{(\ell)}_{i1},\ldots,X^{(\ell)}_{ip}$, for $j=1,\ldots,p$; $i=1,\ldots,n$ and $\ell=1,\ldots,q$. By virtue of the assumed continuity of the d.f,'s, ties among the observations may be neglected, with probability 1. Let then

(3.1) $$\underset{\sim}{R}_{i} = \begin{pmatrix} R^{(1)}_{i1} & \cdots & R^{(1)}_{ip} \\ \cdots & \cdots & \cdots \\ R^{(q)}_{i1} & \cdots & R^{(q)}_{ip} \end{pmatrix} \quad , \text{ for } i=1,\ldots,n,$$

and define $\underset{\sim}{R}^{*}_{i}$ to be the matrix derived from $\underset{\sim}{R}_{i}$ by permuting the columns in such a way that the top row is in the natural order. Let $S(\underset{\sim}{R}^{*}_{i})$, $i=1,\ldots,n$ be the set of $[(p!)^{n}]$ matrices which are permutationally equivalent to the $\underset{\sim}{R}^{*}_{i}$, $i=1,\ldots,n$. Since under H_0, $\underset{\sim}{X}_{i1},\ldots,\underset{\sim}{X}_{ip}$ are interchangeable random vectors, their joint distribution remains invariant under any permutation of these p-vectors among themselves. This implies that under H_0, given a particular realization of $\underset{\sim}{R}^{*}_{i}$, the (conditional) distribution of $\underset{\sim}{R}_{i}$ will be uniform over the p! possible realizations in $S(\underset{\sim}{R}^{*}_{i})$. That is, for any $\underset{\sim}{r}_{i} \in S(\underset{\sim}{R}^{*}_{i})$, we have

(3.2) $$P\{\underset{\sim}{R}_{i} = \underset{\sim}{r}_{i} \mid S(\underset{\sim}{R}^{*}_{i}), H_0\} = (p!)^{-1} \quad , \text{ for all } F_i \ , \ i=1,\ldots,n \ .$$

Moreover, the rankings are made independently for the different blocks. Hence

(3.3) $$P\{\underset{\sim}{R}_{i} = \underset{\sim}{r}_{i},\ i=1,\ldots,n \mid S(\underset{\sim}{R}^{*}_{i}),\ i=1,\ldots,n,\ H_0\} = (p!)^{-n} \ ,$$

for every $\underset{\sim}{r}_{i} \in S(\underset{\sim}{R}^{*}_{i})$, $i=1,\ldots,n$, whatever be the F_i , $i=1,\ldots,n$.

We denote by $\mathcal{P}_n^{(1)}$ the probability measure over the $(p!)^n$ conditionally equally likely realizations in (3.3), and define the matrix of linear rank statistics by

$$\underset{\sim}{T}_N^0 = \underset{j=1,\ldots,p;\ \ell=1,\ldots,q}{((T^0_{Nj\ell}))} = ((\frac{1}{n}\sum_{i=1}^{n} a_{Nj\ell}(R_{ij}^{(\ell)}))) \tag{3.4}$$

where

$$a_{Nj\ell}(r) = \mathcal{E}\{-f^*_{[j]\ell}(U_{Ni})\},\quad i=1,\ldots,n;\ j=1,\ldots,p;\ \ell=1,\ldots,q. \tag{3.5}$$

with $U_{N1},\ldots,U_{Nn}$ being the ordered random variables of a sample of size n from (0,1) uniform d.f. It is thus easily seen that

$$\mathcal{E}\{T^0_{Nj\ell}|\mathcal{P}_n^{(1)}\} = 0 \quad \forall\ j = 1,\ldots,p;\ \ell = 1,\ldots,q \tag{3.6}$$

Also, it is easy to verify that

$$\mathrm{Cov}(T^0_{Nj\ell},\ T^0_{Nj'\ell'}|\mathcal{P}_n^{(1)}) = (\delta_{jj'} - p^{-1})\upsilon_{n\ell\ell'} \tag{3.7}$$

for $j,j' = 1,\ldots,p$; $\ell,\ell' = 1,\ldots,q$, where $\delta_{jj'}$ is the usual Kronecker delta and

$$\underset{\sim}{V}_{n1} = ((\upsilon_{n\ell\ell'})) \tag{3.8}$$

is defined by

$$\upsilon_{n\ell\ell'} = \frac{1}{n(p-1)}\sum_{i=1}^{n}\sum_{j=1}^{p} a_{Nj\ell}(R_{ij}^{(\ell)})a_{Nj\ell'}(R_{ij}^{(\ell')}) \tag{3.9}$$

Thus, if we write

$$\underset{\sim}{T}_{N1} = \mathrm{vec}\ \underset{\sim}{T}_N^{0'} \tag{3.10}$$

then

$$\mathcal{E}\{\underset{\sim}{T}_{N1}\underset{\sim}{T}_{N1}'|\mathcal{P}_n^{(1)}\} = \underset{\sim}{V}_{n1} \otimes (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{1}_p\underset{\sim}{1}_p') = \underset{\sim}{\Sigma}_{N1},\ \text{say}. \tag{3.11}$$

where $\otimes$ denotes the Kronecker product.

Now we consider a sequence $\{K_n;\ \underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\gamma},\ \underset{\sim}{\gamma}$ is fixed$\}$ of alternative hypothesis where K_n specifies that the random vectors $\underset{\sim}{X}_{ij} - \underset{\sim}{\alpha}_i - n^{-\frac{1}{2}}\underset{\sim}{\gamma}_j$, $j = 1,\ldots,p$ are interchangeable for $i = 1,\ldots,n$ and $\underset{\sim}{\gamma}_1,\ldots,\underset{\sim}{\gamma}_p$ are p real q vectors with $\sum_{j=1}^{p} \underset{\sim}{\gamma}_j = \underset{\sim}{0}$. Thus under $\{K_n\}$

$$F_{i[j]\ell}(x) = F_{i[\cdot]\ell}(x-n^{-\frac{1}{2}}\gamma_j^{(\ell)}) \quad \forall\ j=1,\ldots,p;\ \ell=1,\ldots,q. \tag{3.12}$$

$$F_{i[j]\ell\ell'}(x,y) = F_{i[\cdot]\ell\ell'}(x - n^{-\frac{1}{2}}\gamma_j^{(\ell)},\ y - n^{-\frac{1}{2}}\gamma_j^{(\ell')}) \tag{3.13}$$

Moreover, we assume that the following limits exist

$$F_{[\cdot]\ell}(x) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} F_{i[\cdot]\ell}(x), \tag{3.14}$$

$$F_{[\cdot]\ell\ell'}(x,y) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} F_{i[\cdot]\ell\ell'}(x,y), \tag{3.15}$$

and let

$$\nu_{\ell\ell'}^{(1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-\dot{f}_{[\cdot]\ell}(x)/f_{[\cdot]\ell}(x)][-\dot{f}_{[\cdot]\ell}(y)/f_{[\cdot]\ell}(y)]dF_{[\cdot]\ell\ell'}(x,y) \tag{3.16}$$

Then, under assumption [A1] and following Lemma 7.3.10 of Puri and Sen (1971), we have, as $n \to \infty$,

$$\underset{\sim}{V}_{n1} \xrightarrow{P} \underset{\sim}{V}_1 = ((\nu_{\ell\ell'}^{(1)})) \tag{3.17}$$

and

$$\underset{\sim}{T}_{N1} \xrightarrow[\{K_n\}]{\mathfrak{D}} \Phi_{pq}(\cdot;\ \underset{\sim}{\Sigma}_1\underset{\sim}{\lambda}, \underset{\sim}{\Sigma}_1),\ \underset{\sim}{\lambda} = \text{vec}\ \underset{\sim}{\gamma} \tag{3.18}$$

where

$$\underset{\sim}{\Sigma}_1 = \underset{\sim}{V}_1 \otimes (\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{1}_p\underset{\sim}{1}_p') \tag{3.19}$$

Based on $\underset{\sim}{T}_{N1}$, we then can derive a suitable test statistic for testing H_0 against H^* defined in (1.4). First we note that the set Γ in (1.4) is positively homogeneous in the sense that for every $\underset{\sim}{\gamma} \in \Gamma$ and $\delta > 0$, $\delta\underset{\sim}{\gamma} \in \Gamma$. So for a given $\underset{\sim}{\gamma} > \underset{\sim}{0}$, under the assumptions [A1] through [A3], by mimicking the proof of Theorem 4.2 of Tsai and Sen (1987), the LMPR test for testing H_0 against $H_{\underset{\sim}{\gamma}}$ is based on $\underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{T}_{N1}$, where $\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})$ denotes the block diagonal matrix of $\underset{\sim}{\Sigma}_{N1}$. Namely,

$$\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1}) = \text{Diag } \underset{\sim}{V}_{n1} \otimes (\underset{\sim}{I}_p - \frac{1}{p} \underset{\sim}{1}_p \underset{\sim}{1}_p') \tag{3.20}$$

where Diag $\underset{\sim}{V}_{n1}$ is the diagonal matrix of $\underset{\sim}{V}_{n1}$, and $\underset{\sim}{\Sigma}_{N1}^{-1}$ stands for the generalized inverse matrix of $\underset{\sim}{\Sigma}_{N1}$. Furthermore, for every $\underset{\sim}{\gamma} \in \Gamma$, we may write

$$T_N(\underset{\sim}{\gamma}) = \underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{T}_{N1} \;/\; \sqrt{\underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\lambda}} \tag{3.21}$$

By noting that $H^* = \bigcup_{\underset{\sim}{\gamma}\in\Gamma} H_{\underset{\sim}{\gamma}}$ and making the use of UI-principle, then the overall test statistic for H_0 versus H^* is granted as

$$Q_{N1} = \sup\{T_N(\underset{\sim}{\gamma}),\ \underset{\sim}{\gamma} \in \Gamma\} \tag{3.22}$$

For the computation of Q_{N1} in (3.22), we need to maximize $\underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{T}_{N1}$ subject to $\underset{\sim\sim}{A\lambda} \geq \underset{\sim}{0}$ and $\underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\lambda} = 1$. If we let $h(\underset{\sim}{\lambda}) = -\underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\underset{\sim}{T}_{N1}$, $\underset{\sim}{h}_1(\underset{\sim}{\lambda}) = -\underset{\sim\sim}{A\lambda}$ and $h_2(\underset{\sim}{\lambda}) = \underset{\sim}{\lambda}'\underset{\sim}{B}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}^{-1}\ \underset{\sim}{B}(\underset{\sim}{\Sigma}_1)\underset{\sim}{\lambda}-1$, then for this non-linear programming problem, the Kuhn-Tucker-Lagrange (K.T.L.) point formula theorem can be used to arrive at the following result: Let

$$\underset{\sim}{U}_{N1} = \underset{\sim}{A}\ \underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{T}_{N1} \tag{3.23}$$

and

$$\underset{\sim}{\Delta}_{N1} = \underset{\sim\sim}{AB}^{-1}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{\Sigma}_{N1}\underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_{N1})\underset{\sim}{A}' \tag{3.24}$$

also, let J be any subset of $A_0 = \{1,2,\ldots,a\}$ and J' be its complement. For each of the 2^a set J, we partition $\underset{\sim}{U}_{N1}$ and $\underset{\sim}{\Delta}_{N1}$ as

$$\underset{\sim}{U}_{N1} = \begin{bmatrix} \underset{\sim}{U}_{N1(J)} \\ \underset{\sim}{U}_{N1(J')} \end{bmatrix} \begin{matrix} k(J) \\ k(J') \end{matrix} \quad \text{and} \quad \underset{\sim}{\Delta}_{N1} = \begin{bmatrix} \underset{\sim}{\Delta}_{N1(JJ)} & \underset{\sim}{\Delta}_{N1(JJ')} \\ \underset{\sim}{\Delta}_{N1(J'J)} & \underset{\sim}{\Delta}_{N1(J'J')} \end{bmatrix} \tag{3.25}$$

where k(J) denotes the cardinality of set J. Also, for each $J(\phi \subseteq J \subseteq A_0)$, we let

$$\underset{\sim}{U}_{N1(J:J')} = \underset{\sim}{U}_{N1(J)} - \underset{\sim}{\Delta}_{N1(JJ')}\ \underset{\sim}{\Delta}_{N1(J'J')}^{-1}\ \underset{\sim}{U}_{N1(J')} \tag{3.26}$$

$$\underset{\sim}{\Delta}_{N1(JJ:J')} = \underset{\sim}{\Delta}_{N1(JJ)} - \underset{\sim}{\Delta}_{N1(JJ')}\ \underset{\sim}{\Delta}_{N1(J'J')}^{-1}\ \underset{\sim}{\Delta}_{N1(J'J)} \tag{3.27}$$

Then, we have

$$Q_{N1}^2 = \underset{\sim}{T}'_{N1} \underset{\sim}{\Sigma}_{N1}^{-1} \underset{\sim}{T}_{N1} - \underset{\sim}{U}'_{N1} \underset{\sim}{A}_{N1}^{-1} \underset{\sim}{U}_{N1} + \sum_{\phi \subseteq J \subseteq A_0} \{\underset{\sim}{U}'_{N1(J:J')} \underset{\sim}{A}_{N1(JJ:J')}^{-1} \underset{\sim}{U}_{N1(J:J')}\}$$

$$1\{\underset{\sim}{U}'_{N1(J:J')} > \underset{\sim}{0}, \underset{\sim}{A}_{N1(J'J')}^{-1} \underset{\sim}{U}_{N1(J')} \le \underset{\sim}{0}\} \quad (3.28)$$

where 1(B) stands for the indicator function of the set B.

Theorem 3.1. Let $\underset{\sim}{\eta}_1 = \underset{\sim}{A}\underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_1)\underset{\sim}{\Sigma}_1\underset{\sim}{\lambda}$ and $\underset{\sim}{A}_1 = \lim_{n\to\infty} \mathcal{E}(\underset{\sim}{A}_{N1}|H_0)$. For each J ($\phi \subseteq J \subseteq A_0$), assume $\Gamma_J^{(1)} = \{\underset{\sim}{\eta}_1 \in E^{+a};\ \underset{\sim}{\eta}_{J:J'}^{(1)} = \underset{\sim}{\eta}_{1J} - \underset{\sim}{A}_{1(JJ')}\underset{\sim}{A}_{1(J'J')}^{-1}\underset{\sim}{\eta}_{1J'} \ge \underset{\sim}{0}\}$ and $\Gamma_0^{(1)} = \bigcap_{\phi \subseteq J \subseteq A_0} \Gamma_J^{(1)}$, where E^{+a} is a-dimensional positive orthant space. If the family $f_i(\underset{\sim}{x}-\underset{\sim}{\beta})$ satisfy the assumptions [A1] through [A3], then for testing H_0 vs. H^* : $\underset{\sim}{\beta} \in \Gamma$ (under $\{K_n\}$) Q_{N1}^2 in (3.28) is asymptotically most stringent for Γ and asymptotically UMP for $\Gamma_0^{(1)}$ within the class of intrablock rank tests at the respective level of significance α.

Proof. The proof follows directly from Theorem 4.4 of Tsai and Sen (1987) and hence is omitted.

Corollary 3.2. For q = 1, under $\{K_n\}$ and Assumptions [A1] and [A2], within the class of intra-block rank tests, the UI-LMPR test based on (3.28) is asymptotically UMP at the significance level α .

Outline of Proof. The testing (1.2) against (1.3) can be written as for testing (1.2) against (1.4), where $\underset{\sim}{A} = \underset{\sim}{I}_q \otimes \underset{\sim}{L}$ with

$$\underset{\sim}{L} = \begin{bmatrix} -1 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 1 & . & . & . & 0 & 0 \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ 0 & 0 & 0 & . & . & . & -1 & 1 \end{bmatrix} \quad (3.29)$$

Then we have

$$\underset{\sim}{\eta} = \begin{bmatrix} \underset{\sim}{\gamma}_2^* - \underset{\sim}{\gamma}_1^* \\ \underset{\sim}{\gamma}_3^* - \underset{\sim}{\gamma}_2^* \\ \vdots \\ \underset{\sim}{\gamma}_p^* - \underset{\sim}{\gamma}_{p-1}^* \end{bmatrix} \quad (3.30)$$

with

$$\underset{\sim}{\gamma}_j^* = (\text{Diag } \underset{\sim}{V}_1^{-1})\underset{\sim}{V}_1 \underset{\sim}{\gamma}_j, \quad j=1,2,\ldots,p \tag{3.31}$$

And

$$\underset{\sim}{\Delta}_1 = (\text{Diag } \underset{\sim}{V}_1^{-1}) \underset{\sim}{V}_1(\text{Diag } \underset{\sim}{V}_1^{-1}) \otimes \underset{\sim}{\Delta}^* \tag{3.32}$$

where

$$\underset{\sim}{\Delta}^* = ((\sigma_{ij}^*))_{i,j=1,\ldots,p-1} \quad \text{with} \quad \sigma_{ij}^* = \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases} \tag{3.33}$$

Let $\mathcal{P} = \{q, 2q, \ldots, (p-1)q\}$ and J be any subset of $\mathcal{P}$, J' be its complement, then for any J ($\phi \subseteq J \subseteq \mathcal{P}$)

$$\underset{\sim}{\eta}_{J:J'}^{(1)} \geq \underset{\sim}{0} \quad \text{if } \underset{\sim}{\eta}_1 \geq \underset{\sim}{0} \,. \tag{3.34}$$

By noting that if q=1, then $\underset{\sim}{\gamma}_j^* = \underset{\sim}{\gamma}_j$, $\forall$ j=1,...,p, hence the corollary is proved.

Note that for q = 1 and i.i.d. errors, the condition (2.3) is not needed (as there H is an identity matrix); however, for q > 1, (2.3) is a sufficient condition to ensure the stated optimality properties in Theorem 3.1.

4. UI-LMPR tests and ranking after alignment

In intra-block ranking , because of the lack of information from the inter-block comparisons, the tests are generally less efficient than the aligned rank tests which incorporate the inter-block information through the alignment procedure. For the standard MANOVA model in two-way layouts, Sen (1969) formulated aligned rank tests based on general scores and studied their asymptotic efficiency in a unified manner. Here, we extend the results to testing against general forms of restricted alternatives [as in (1.4)] and show that under fairly general regularity conditions, aligned rankings lead to more efficient tests for such alternatives too. To do this, we need to eliminate the (nuisance) block effects by simple alignment procedures; namely, we subtract suitable estimates of the block effects (vectors) from the respective $X_{ij}^{(\ell)}$ and on the residuals, we make an overall ranking (ignoring blocks) of all the treatments (in a coordinatewise manner). We may use any translation-equivariant estimator of the block effects; for simplicity, we take them as the block averages. Thus, we define the aligned random vectors as

$$\underset{\sim}{Y}_{ij} = \underset{\sim}{X}_{ij} - \frac{1}{p} \sum_{j=1}^{p} \underset{\sim}{X}_{ij}, \quad i = 1,\dots,n;\ j = 1,\dots,p. \tag{4.1}$$

For convenience, we assume that $\underset{\sim}{Y}_{ij}$ has a q-variate continuous c.d.f F_{ij}^{0}, $\forall\ i = 1,\dots,n;\ j = 1,\dots,p$. Let $S_{ij}^{(\ell)}$ be the rank of $Y_{ij}^{(\ell)}$ among the N (= np) observations $Y_{11}^{(\ell)}, Y_{12}^{(\ell)}, \dots, Y_{np}^{(\ell)}$ for $j = 1,\dots,p$, $i = 1,\dots,n$, $\ell = 1,\dots,q$. Thus, corresponding to the aligned observation $\underset{\sim}{Y}_{ij}$, we have a rank vector $\underset{\sim}{S}_{ij} = (S_{ij}^{(1)},\dots,S_{ij}^{(q)})'$, $i = 1,\dots,n$, $j = 1,\dots,p$. We also define the rank collection matrix $\underset{\sim}{S}_N$ by $(\underset{\sim}{S}_{11},\dots,\underset{\sim}{S}_{np})$. Note that under H_0, $\underset{\sim}{Y}_{i1}, \underset{\sim}{Y}_{i2},\dots,\underset{\sim}{Y}_{ip}$ are interchangeable random vectors, so the joint distribution of $\underset{\sim}{Y}_N = (\underset{\sim}{Y}'_{11},\dots,\underset{\sim}{Y}'_{1p},\dots,\underset{\sim}{Y}'_{np})$ remains invariant under the finite group $\mathscr{G}_n$ of transformation $\{g_n^0\}$ (which maps the sample space onto itself). Thus for any $g_n^0 \in \mathscr{G}_n$, there exists $\underset{\sim}{Y}_N^{*} = g_n^0 \underset{\sim}{Y}_N$ which is permutationaslly equivalent to $\underset{\sim}{Y}_N$. If we denote $\underset{\sim}{S}_N^{*}$ the rank collection matrix corresponding to $\underset{\sim}{Y}_N^{*}$, then $\underset{\sim}{S}_N^{*} = g_n^0 \underset{\sim}{S}_N$ and is permutationally equivalent to $\underset{\sim}{S}_N$. Thus, under H_0, the conditional distribution of $\underset{\sim}{S}_N$ over the $(p!)^n$ realization $\{\underset{\sim}{S}_N^{*} = g_n^0 \underset{\sim}{S}_N;\ g_n^0 \in \mathscr{G}_n\}$ is uniform, each realization having the conditional probability $(p!)^{-n}$. Let us denote this conditional probability measure by $\mathscr{P}_N^{(2)}$ and define the scores $\underset{\sim}{a}_N^{0}(k)$, $k=1,\dots,N$, as in (3.5) with $F_{[j]\ell}$ being replaced by $F_{[j]\ell}^{0}$. Similarly define Q_{N2}^{2}, $\underset{\sim}{T}_{N2}$, $\underset{\sim}{V}_{N2}$, $\underset{\sim}{\Sigma}_{N2}$, $\underset{\sim}{U}_{N2}$ and $\underset{\sim}{\Delta}_{N2}$ the same as in Q_{N1}^{2}, $\underset{\sim}{T}_{N1}$, $\underset{\sim}{V}_{n1}$, $\underset{\sim}{\Sigma}_{N1}$, $\underset{\sim}{U}_{N1}$ and $\underset{\sim}{\Delta}_{N1}$ respectively with $a_{Nj\ell}(k)$ being replaced by $a_{Nj\ell}^{0}(k)$ for $j=1,\dots,p$; $\ell=1,\dots,q$, $k=1,\dots,N$. Furthermore, consider a restricted (contiguous) alternative

$$\{K_N:\ \underset{\sim}{\beta} = N^{-\frac{1}{2}}\underset{\sim}{\gamma},\ \underset{\sim}{\gamma} \in \Gamma,\ \sum_{j=1}^{p} \underset{\sim}{\gamma}_j = \underset{\sim}{0}\}. \tag{4.2}$$

Then we have

$$F_{i[j]\ell}^{0}(x) = F_{i[\cdot]\ell}^{0}(x - N^{-\frac{1}{2}}\gamma_j^{(\ell)}), \tag{4.3}$$

and

$$F^0_{i[j]\ell\ell'}(x,y) = F^0_{i[\cdot]\ell\ell'}(x - N^{-\frac{1}{2}}\gamma_j^{(\ell)},\ y - N^{-\frac{1}{2}}\gamma_j^{(\ell')}). \tag{4.4}$$

Finally, we define $\underset{\sim}{V}_2$ and $\underset{\sim}{\Sigma}_2$ the same as in $\underset{\sim}{V}_1$ and $\underset{\sim}{\Sigma}_1$ with $F_{i[\cdot]\ell}(x)$ and $F_{i[\cdot]\ell\ell'}(x,y)$ being replaced by $F^0_{i[\cdot]\ell}$ and $F^0_{i[\cdot]\ell\ell'}(x,y)$ respectively. Then under parallel arguments as in the previous section we have

$$\underset{\sim}{T}_{N2} \xrightarrow[\{K_N\}]{\mathcal{D}} \Phi_{pq}(\cdot;\ \underset{\sim}{\Sigma}_2\underset{\sim}{\lambda}, \underset{\sim}{\Sigma}_2). \tag{4.5}$$

Theorem 4.1. Let $\underset{\sim}{\eta}_2 = \underset{\sim}{A}\underset{\sim}{B}^{-1}(\underset{\sim}{\Sigma}_2)\underset{\sim}{\Sigma}_2\underset{\sim}{\lambda}$ and $\underset{\sim}{\Delta}_2 = \lim_{N\to\infty} \mathcal{E}\{\underset{\sim}{\Delta}_{N2}|H_0\}$. For each J ($\phi \subseteq J \subseteq A_0$), assume $\Gamma_J^{(2)} = \{\underset{\sim}{\eta}_2 \in E^{+a};\ \underset{\sim}{\eta}^{(2)}_{J:J'} = \underset{\sim}{\eta}_{2J} - \underset{\sim}{\Delta}_{2(JJ')}\underset{\sim}{\Delta}^{-1}_{2(J'J')}\underset{\sim}{\eta}_{2J'} \geq \underset{\sim}{0}\}$ and $\Gamma_0^{(2)} = \bigcap_{\phi\subseteq J\subseteq A_0} \Gamma_J^{(2)}$. If the family $F_i^0(\underset{\sim}{y}-\underset{\sim}{\beta})$ satisfy the assumption [A1] through [A3], then for testing (1.2) against (1.4) (under $\{K_N\}$ defined in (4.2)), the UI-LMPR test Q^2_{N2} is asymptotically most stringent for Γ and asymptotically UMP for $\Gamma_0^{(2)}$ within the class of aligned rank tests at the respective level of significance α.

Corollary 4.2. Under the same regularity assumptions as in Corollary 3.2, the corresponding UI-LMPR test is asymptotically UMP (under $\{K_N\}$) within the class of aligned rank tests for testing (1.2) against (1.3) when q=1 at the same level of significance α.

Note that both the proofs of Theorem 4.1 and Corollary 4.2 are very similar to the proof of Theorem 3.1 and Corollary 3.2, and hence, are omitted. Also note that the last condition in (2.4) is needed for the aligned rank tests but not for the intra-block rank tests (only in the context of the asymptotic optimality study).

5. Asymptotic power comparison of Q^2_{N1} and Q^2_{N2}

The optimal property of Q^2_{N1} may not generally hold when we extend the domain to the class of tests based on rankings after alignment. Since the rankings of Q^2_{N2} bases on the aligned observations which disregard blocks, so Q^2_{N2} should have the optimal property within a more broad class. For testing homogeneity against global alternatives, Hodges and Lehmann (1962) were successful in establishing the superiority of the rankings after alignment procedure over the intrablock ranking procedure when the underlying distribution is normally distributed. For a wider class of distribution, this result was extended by Sen (1968) and studied in detail by Puri and Sen (1971). The assertion of the superiority of the aligned rank tests to the intrablock rank tests can be easily extended to the testing against alternatives which put constraints on the parameters in the linear form of lower dimensional hyperspace. However, for a wider class of restricted alternatives, this assertion may not be true. The following Theorem provides a partial answer when the aligned rank tests are asymptotically power-superior to the intrablock rank tests in the restricted alternative space.

<u>Theorem 5.1</u>. Let $\eta^* = \Lambda_1^{-\frac{1}{2}}\eta_1$, $\beta_1(\gamma,\Sigma_1) = \lim_{n\to\infty} P\{Q^2_{N1} \geq x^{(1)}_\alpha | K_n\}$ and $\beta_2(\gamma,\Sigma_2) = \lim_{N\to\infty} P\{Q^2_{N2} \geq x^{(2)}_\alpha | K_N\}$. If $\lim_{n\to\infty} P\{Q^2_{N1} \geq x^{(1)}_\alpha | H_0\} = \lim_{N\to\infty} P\{Q^2_{N2} \geq x^{(2)}_\alpha | H_0\} = \alpha$, then $\beta_1(\gamma,\Sigma_1) \leq \beta_2(\gamma,\Sigma_2)$ whenever $\gamma \in \Omega_0 = \{\eta^* \in E^{+a} ; \eta^*_j \geq \frac{1}{3}\sqrt{x^{(1)}_\alpha},\ j = 1,\ldots,a\}$

<u>Outline of the proof</u>. Let us define

$$\Lambda_{1i} = \mathrm{Var}\, X_{ij},\ \Lambda_{2i} = \mathrm{Cov}(X_{ij}, X_{ij'}) \tag{5.1}$$

for $j \neq j' = 1,\ldots,p$; $i = 1,\ldots,n$, and

$$\Lambda_1 = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\Lambda_{1i},\ \Lambda_2 = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\Lambda_{2i} \tag{5.2}$$

Then via Theorems 3.1 and 4.1, we have

$$V_1^{-1} = \Lambda_1 - \Lambda_2 \quad \text{and} \quad V_2^{-1} = \frac{p-1}{p}(\Lambda_1 - \Lambda_2)\ . \tag{5.3}$$

Namely,

$$\Sigma_1 = \frac{p-1}{p}\Sigma_2\ ,\quad \Lambda_1 = \frac{p}{p-1}\Lambda_2 \quad \text{and} \quad \eta_1 = \eta_2 = \eta,\ \text{say.} \tag{5.4}$$

Next, we define

$$Q^2_k = \sum_{\phi\subseteq J\subseteq A_0}\{Z'_{k(J:J')}\Lambda^{-1}_{k(JJ:J')}Z_{k(J:J')}\}1\{Z_{k(J:J')} > 0,\ \Lambda^{-1}_{k(J'J')}Z_{k(J')} \leq 0\} \quad \forall\ k = 1,2, \tag{5.5}$$

where

$$Z_k \sim \Phi_a(\cdot;\ \eta, \Lambda_k).$$

It suffices to show that if $P\{Q_k^2 \geq x_\alpha^{(k)} | H_0\} = \alpha$, $\forall$ k = 1,2, then $P\{Q_1^2 \geq x_\alpha^{(1)} | K_n\} \leq P\{Q_2^2 \geq x_\alpha^{(2)} | K_N\}$. Without any loss of generality we assume $\underset{\sim}{A}_1 = \underset{\sim}{I}$ and note that

$$P\{Q_2^2 \geq x_\alpha^{(2)} | K_N\} - P\{Q_1^2 \geq x_\alpha^{(1)} | K_n\} \tag{5.6}$$
$$= \sum_{\phi \subseteq J \subseteq A_0} \left[\int_{B_2(J)} d\Phi_a(\underset{\sim}{z};\ \sqrt{\tfrac{p}{p-1}}\,\underset{\sim}{\eta}, \underset{\sim}{I}) - \int_{B_1(J)} d\Phi_a(\underset{\sim}{z};\ \underset{\sim}{\eta}, \underset{\sim}{I}) \right]$$

where

$$B_k(J) = \{\underset{\sim}{z} \in E^a;\ \underset{\sim}{z}_{J'} \leq \underset{\sim}{0},\ \underset{\sim}{z}_J > \underset{\sim}{0},\ \|\underset{\sim}{z}_J\| \geq x_\alpha^{(k)}\}\ \forall\ k = 1,2. \tag{5.7}$$

Since $P\{Q_k^2 \geq x_\alpha^{(k)} | H_0\} = \alpha$, $\forall$ k = 1,2, therefore it is obvious that $x_\alpha^{(1)} = x_\alpha^{(2)}$. Furthermore, we write $x_\alpha^{(1)} = x_\alpha$ and $\underset{\sim}{\eta}^\delta = \underset{\sim}{\eta} + \delta \underset{\sim}{1}$, where $\delta > 0$. For $\underset{\sim}{\eta} \in \Omega_0$, (i) if a = 1, then we have

$$\sum_{\phi \subseteq J \subseteq A_0} \int_{B_1(J)} \left[d\Phi_a(\underset{\sim}{z};\ \underset{\sim}{\eta}^\delta, \underset{\sim}{I}) - d\Phi_a(\underset{\sim}{z};\ \underset{\sim}{\eta}, \underset{\sim}{I}) \right] \tag{5.8}$$
$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{z \leq 0} [e^{-\frac{1}{2}(z-\eta^\delta)^2} - e^{-\frac{1}{2}(z-\eta)^2}]dz + \int_{z > \sqrt{x_\alpha}} [e^{-\frac{1}{2}(z-\eta^\delta)^2} - e^{-\frac{1}{2}(z-\eta)^2}]dz \right\} \geq 0;$$

(ii) if a = 2, by regrouping the sums and symmetric arguments, we have

$$\sum_{\phi \subseteq J \subseteq A_0} \int_{B_1(J)} \left[d\Phi_a(\underset{\sim}{z};\ \underset{\sim}{\eta}^\delta, \underset{\sim}{I}) - d\Phi_a(\underset{\sim}{z};\ \underset{\sim}{\eta}, \underset{\sim}{I}) \right]$$

$$= \frac{1}{2\pi} \left\{ \int_{\sqrt{x_\alpha}-\eta_1-\delta}^{\sqrt{x_\alpha}-\eta_1} \int_{-\infty}^{-\eta_2-\delta} e^{-\frac{1}{2}\|\underset{\sim}{z}\|} dz_2 dz_1 - \int_{-\eta_1-\delta}^{-\eta_1} \int_{-\infty}^{-\eta_2-\delta} e^{-\frac{1}{2}\|\underset{\sim}{z}\|} dz_2 dz_1 \right.$$

$$\left. + \int_{\sqrt{x_\alpha}-\delta-\eta_2}^{\sqrt{x_\alpha}-\eta_2} \int_{-\infty}^{-\delta-\eta_1} e^{-\frac{1}{2}\|\underset{\sim}{z}\|} dz_1 dz_2 - \int_{-\eta_2-\delta}^{-\eta_2} \int_{-\infty}^{-\eta_1-\delta} e^{-\frac{1}{2}\|\underset{\sim}{z}\|} dz_1 dz_2 \right\}$$

$$+ o(1) \geq 0. \tag{5.9}$$

By mathematical induction, thus $P\{Q_1^2 \geq x_\alpha | K_n\}$ is non-decreasing in $\underset{\sim}{\eta}^*$ whenever $\underset{\sim}{\gamma} \in \Omega_0$. Therefore we have, $\underset{\sim}{\gamma} \in \Omega_0$,

$$P\{Q_2^2 \geq x_\alpha^{(2)} | K_N\} \geq P\{Q_1^2 \geq x_\alpha^{(1)} | K_n\}$$

and hence the theorem follows.

Remark. For a = 2, let $\Omega^* = \{\underset{\sim}{\eta}^* \in E^2;\ \eta_1^* \geq \frac{1}{3}\sqrt{x_\alpha},\ \eta_2^* \leq 0\} \cup \{\underset{\sim}{\eta}^* \in E^2;\ \eta_1^* \leq 0,\ \eta_2^* \geq \frac{1}{3}\sqrt{x_\alpha}\} \cup \{\underset{\sim}{\eta}^* \in E^2;\ \eta_1^* \leq 0,\ \eta_2^* \leq 0\} \cup \Omega_0$, then under parallel arguments as in (ii) of Theorem 5.1, we have

$$\beta_1(\underset{\sim}{\gamma},\underset{\sim}{\Sigma}_1) \leq \beta_2(\underset{\sim}{\gamma},\underset{\sim}{\Sigma}_2),\ \forall\ \underset{\sim}{\gamma} \in \Omega^*. \tag{5.10}$$

Generalizing this result to the higher dimensions is still an open problem.

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