Abstract

JIANG, ZONGLIANG. Symmetric chain decompositions and independent families of curves. (Under the direction of Dr. Carla D. Savage.)

This thesis shows that symmetric independent families of n curves with the minimum possible number of regions exist for all $n \leq 16$. Recent research has shown that such an independent family of curves exists for all prime n [GKS02]. For composite n, before this thesis, such a symmetric independent family of curves was known to exist only for $n = 2, 4, 6, 8, 9$, and 10.

An independent family of curves is a collection of simple closed curves intersecting at finite number of points. If we label the curves with $1, 2, \ldots, n$, then each region is labeled with the set of labels of the curves containing that region. If every subset of $\{1, 2, \ldots, n\}$ is a label for at least one region, then the collection of curves is an independent family of curves. If we rotate any curve around a point by an angle of $2\pi/n$ radians $(n - 1)$ times and each time it coincides with one of the other curves, then the collection of curves is a *symmetric* independent family of curves.

We solve this geometric problem by first solving a combinatorial problem of looking for symmetric chain decompositions (SCD's) of necklace-representative posets with the chain cover property (CCP). We then show the way of constructing a symmetric

independent family of curves with the minimum possible number of regions from an SCD of necklace-representative poset with the CCP.

SYMMETRIC CHAIN DECOMPOSITIONS AND INDEPENDENT FAMILIES OF CURVES

BY

ZONGLIANG JIANG

A THESIS SUBMITTED TO THE GRADUATE FACULTY OF
NORTH CAROLINA STATE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

COMPUTER SCIENCE

RALEIGH

JULY 2003

APPROVED BY:

Dr. George N. Rouskas | Dr. Matthias Stallmann

Dr. Carla D. Savage
chair of advisory committee

Biography

Zongliang Jiang was born on September 30th, 1980, in Shanghai, China. He spent all his childhood and teens there. In July 1999, he graduated from Shanghai Jiao Tong University (China) with dual Bachelor of Science degree in Engineering Mechanics and Computer Science.

He came to the United States in August 2000. He spent one year in the graduate program of Engineering Mechanics at the University of Texas at Austin, before he left in the summer of 2001. He joined the Computer Science Department at North Carolina State University in the fall of 2001, to pursue Master of Science degree. He is expecting to graduate in August 2003.

Acknowledgments

I wish to express my sincere thankfulness to Dr. Carla D. Savage, my advisor, for her patience and encouragement. Without her great support and assistance, little would have been accomplished in this thesis.

Special thanks to Dr. George N. Rouskas and Dr. Matthias Stallmann for serving on my committee.

I owe my family and friends dearly for their support and understanding during the time when I was writing this thesis.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Venn diagrams are well-known as graphical tools from discrete mathematics which show how a number of sets intersect. They form an important class of combinatorial objects applied in set theory and other fields [Wes03]. Some of their applications include "revolving door" listings, symmetric Gray codes, etc. [Rus97]. Most familiar is the Venn diagram for three sets, as shown below, usually introduced in high school math class.

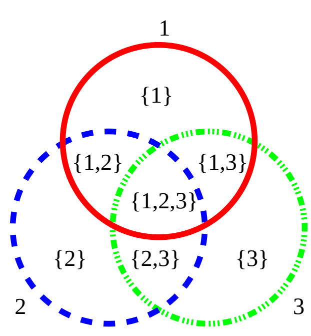

Figure 1.1: Venn diagram for $n = 3$.

Less familiar is the term *independent family of curves*, which is one of the main subjects of this paper. In fact, a Venn diagram is a special case of an independent family of curves. Informally, to obtain an independent family of n curves, we first take n non-self-intersecting closed curves in the plane and label all the curves with numbers from 1 to n. The curves, each representing a set, are going to intersect on the plane. A *region* is a maximal connected area of the plane bounded by curves. Each region will be labeled by a set of numbers, each of which corresponds to the label of a curve containing the region. (See Figure 1.1 and 1.2 for examples of labeling.) If we can arrange the topology of the curves so that every subset of $\{1, 2, \ldots, n\}$ is a label for at least one region in the diagram, where the unbounded region on the plane corresponds to the empty set, the resulting graph is an independent family of n curves. Furthermore, if every subset of $\{1, 2, \ldots, n\}$ is associated with exactly one region, then the independent family of n curves is a Venn diagram for n sets. Figure 1.2 shows an independent family of three curves which is not a Venn diagram, since set $\{1\}$ is associated with two regions, and so are sets $\{2\}$, $\{2, 3\}$, and $\{3\}$.

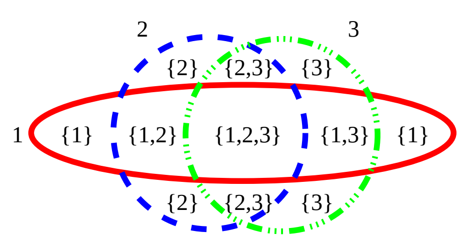

Figure 1.2: Independent family of three curves (not a Venn diagram).

In this paper, we will focus on independent families of curves rather than Venn diagrams. First we describe some background and motivation for this research.

People who love discrete mathematics and graph theory usually find independent families of curves fascinating. Because of the combinatorial meaning of independent families of curves, we sometimes transform the geometric problem of constructing the diagram into a new problem which we can solve using tools in discrete mathematics and combinatorics. Then, while applying the results back to the diagram, we will encounter some challenging questions regarding its graphical features. That's when graph theory comes into play. The research of this thesis follows this logic.

Because of its better combinatorial nature, the Venn diagram is more interesting to researchers. For the sake of aesthetics, in the past few decades, mathematicians have been taking on the challenge to look for rotational symmetry in Venn diagrams. We say a diagram consisting of n curves in the plane is rotationally symmetric when there exists a point, p, in the plane, such that if we rotate any of the curves by an angle of $2\pi/n$ for $n-1$ times in the same direction, each time it coincides with one of the other curves. Figure 1.3 shows a rotationally symmetric Venn diagram for $n = 5$ [Grü75]. The Venn diagram for $n = 3$ sets shown in Figure 1.1 also has rotational symmetry.

It was first shown by Henderson in [Hen63] that a rotationally symmetric Venn diagram for n sets cannot exist when n is not prime. But whether such rotationally

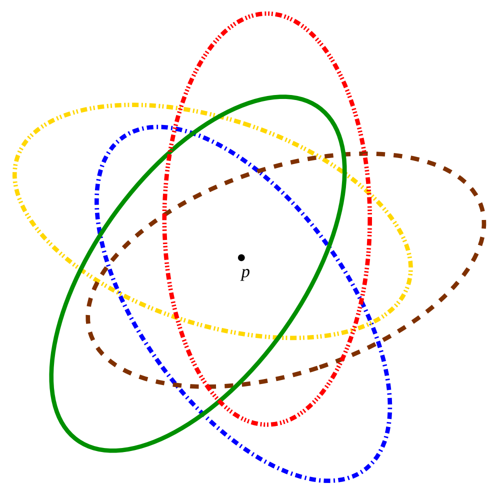

Figure 1.3: Venn diagram for $n = 5$ with rotational symmetry [Grü75].

symmetric diagrams exist for all prime n was unknown at that time. Since then, symmetric Venn diagrams have been known for small primes. Garünahum conjectured in [Grü75] that a symmetric Venn diagram for $n = 7$ doesn't exist. But in 1992, he discovered such a Venn diagram and conjectured that rotationally symmetric Venn diagrams exist for all prime n (see [Grü92]). In the same year, additional examples for $n = 7$ were found by Anthony Edwards and independently by Carla Savage and Peter Winkler. In 1997, Ruskey ([Rus97]) did a thorough survey on Venn diagrams including previous results on symmetric Venn diagrams, which has served as a resource to researchers interested in this area. Before long, in 2001 Hamburger constructed the first known symmetric Venn diagram for $n = 11$ (see [Ham02]). However, the question of whether symmetric Venn diagrams exist for all prime n was still unanswered until recently when Griggs, Killian, and Savage solved the puzzle in 2002, showing the answer to be yes. (See [GKS02].)

It is known that symmetric Venn diagrams don't exist for non-prime n. But what if we want a symmetric independent family of curves instead of a symmetric Venn diagram? In other words, does there exist a symmetric independent family of n curves for every non-prime n? Grünbaum states in [Grü99] that a symmetric independent family of n curves can be constructed for every n. In fact, since it doesn't matter in an independent family of curves how many regions a subset of $[n]$ is associated with, Grünbaum has been able to show how to keep rotational symmetry. In the same paper, Grünbaum proposes a related open question. Does there exists, for every n, a rotationally symmetric independent family of n curves with the minimum possible number of regions? (We will discuss details about this lower bound in Chapter 5.) This thesis is trying to answer Grünbaum's question. The answer to the question was known to be yes only for prime n and for $n = 2, 4, 6, 8, 9$, and 10. Most recently, Mark Weston developed a backtracking technique to generate symmetric independent families of curves and solved the open question for $n = 8, 9$, and 10 [Wes03]. In this thesis, we find that the answer is also yes for $n = 12, 14, 15$, and 16. We leave the question unresolved for $n \geq 18$, although our work shows how to substantially simplify the problem. Figure 1.4 shows an example of a symmetric independent family of four curves with eighteen (the minimum possible number for $n = 4$) regions [Grü99].

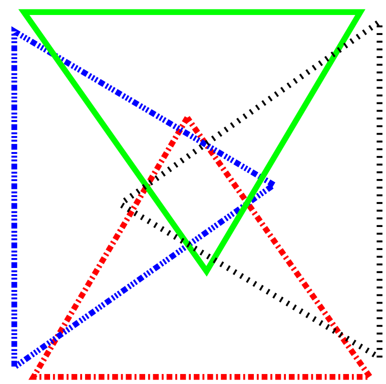

Figure 1.4: A symmetric independent family of $n = 4$ curves with 18 regions [Grü99].

In order to solve this geometric problem, we will first focus on a more general open question from [GKS02], i.e., for every n, whether there exists a symmetric chain decomposition of ordered representatives. We now explain what this means.

Suppose there is an n-bit string. We rotate it, and let the strings resulting from all rotations belong to the same class. For example, $\{10110, 01101, 11010, 10101, 01011\}$ is one such class for $n = 5$. Imagine that the set of all n-bit strings is partitioned into classes. In each class, we pick a string as a representative for the class. Arrange the representatives into levels and put representatives with the same number of 1's on the same horizontal level. Thus, there are a total of $n + 1$ levels. The levels of representatives are ordered in such a way that the number of 1's increases as the level rises. So the only representative in the lowest level is the string of all 0's, and the only representative in the highest level is the string of all 1's. We say that one string, x, is covered by another string, x', if x differs from x' only in one bit where the bit in x is '0' and the bit in x' is '1'. For example, if $x = 10100$ and $x' = 10110$, then x is covered by x'. Now suppose we choose exactly one representative from each of a number of consecutive levels so that, for any two representatives from adjacent levels, the one from the lower level is covered by the one from the higher level. Such a series of representatives forms a "chain". A single string can form a chain by itself. We call the string on the chain with the least number of 1's (on the lowest level) the *chain starter*, and the string with the most number of 1's (on the highest level) the *chain terminator*. We say that a chain, C, is covered by another chain, C', if the chain starter of C covers some string in C' and the chain terminator of C is covered by some string in C'. Figure 1.5 shows an example of two chains where C is covered by C'. 1010 is the chain starter and the chain terminator of C. 0000 and 1111 are the chain starter and terminator, respectively, of C'. It is obvious that 1010 covers 1000, and 1010 is covered by 1110. The dashed lines show the string cover relationship.

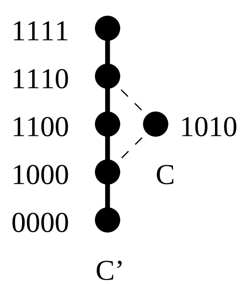

Figure 1.5: Chain cover relationship between C and C'.

What [GKS02] has accomplished is that it finds a way of choosing the representatives so that we can partition the ordered representatives into disjoint symmetric chains for all prime n. A chain is symmetric when the number of 1's of the chain starter plus the number of 1's of the chain terminator is equal to n. (The two chains in Figure 1.5 are disjoint, and they are both symmetric.) We call such a partition a *symmetric chain decomposition*. [GKS02] also makes sure that in the symmetric chain decomposition, every chain is covered by another chain, except for the longest chain whose chain starter is the string of all 0's and the chain terminator is the string of all 1's. We call this the *chain cover property*. The layout of chains in Figure 1.5 happens to be a symmetric chain decomposition of ordered representatives for $n = 4$ with the chain cover property. Figure 1.6 shows such a symmetric chain decomposition for $n = 5$.

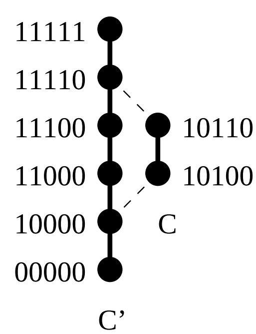

Figure 1.6: Symmetric chain decomposition of ordered representatives for $n = 5$ with the chain cover property.

In this thesis, we attempt to extend these results to non-prime n, and apply this to the construction of symmetric independent families of curves. The following results are the main contribution of this paper:

- For every $n \leq 16$, there exists a symmetric chain decomposition of ordered representatives with the chain cover property.
- For every $n \leq 16$, there exists a rotationally symmetric independent family of n curves with the minimum possible number of regions.
- When $n > 16$, the problem of searching for symmetric chain decompositions is reduced to a much smaller subproblem. This will significantly facilitate computer programming and expedite the process in any future effort to generate symmetric chain decompositions for large composite n.

The organization of the paper is as follows. In the next chapter (Chapter 2), we will give definitions of some combinatorial terms that will appear throughout this thesis. Also, we will discuss previous results in this area, current status of the problem, and the plan we will take on. In Chapter 3, we will solve part of the problem of searching for symmetric chain decompositions of ordered representatives with the chain cover property. Then we will reduce the big combinatorial problem into a much smaller subproblem. In Chapter 4, we will investigate the smaller subproblem. In Chapter 5, we will apply the results from the combinatorial problem to the geometric problem, i.e., looking for a symmetric independent family of n curves with the minimum possible number of regions.

Chapter 2

Background

2.1 Combinatorial definitions

2.1.1 Partially ordered sets (posets)

A *partially ordered set*, or *poset*, $\mathcal{A} = (A, \leq)$ is a set A with a binary relation $\leq$, on A, which is reflexive, transitive, and antisymmetric.

For $x, y \in A$, we say y *covers* x if $x \leq y$ and there is no $z \in A$, where $z \neq x$ and $z \neq y$, such that $x \leq z \leq y$. A poset $\mathcal{A}$ is *ranked* if there exists a function $r(x)$ on every $x \in A$ such that $r(x) = 0$ when there are no elements in A that are covered by x, and $r(y) = r(x) + 1$ for all $x, y \in A$ such that y covers x. In a ranked poset $\mathcal{A}$, we say that $r(x)$ is the *rank* of x, and the *rank of poset* $\mathcal{A}$ is $\max_{x \in A} r(x)$.

A *subposet* of poset $\mathcal{A} = (A, \leq)$ is a poset $\mathcal{A}' = (A', R)$, where $A' \subseteq A$ and xRy in $\mathcal{A}' \Rightarrow x \leq y$ in $\mathcal{A}$. If $R =\leq$, then $\mathcal{A}'$ is the subposet of $\mathcal{A}$ *induced* by A'.

2.1.2 The Boolean lattice

The Boolean lattice $\mathcal{B}_n$ is the poset $\{B_n, \leq\}$, where B_n consists of all subsets of $\{1, 2, \ldots, n\}$ and the ordering $\leq$ is defined for $x, y \in B_n$ by $x \leq y \leftrightarrow x \subseteq y$. So $r(x) = |x|$ for every $x \in B_n$.

Sometimes it is more convenient to view elements in the Boolean lattice $\mathcal{B}_n$ as n-bit strings. Let $b \in B_n$ correspond to $x_1 x_2 \cdots x_n$, where $x_i = 1$ $(1 \leq i \leq n)$ if $i \in b$, otherwise, $x_i = 0$. Then, the rank $r(x)$ of an element, $x \in \{0, 1\}^n$, is the number of ones in x.

2.1.3 Symmetric chain decompositions

In a ranked poset $\mathcal{A}$ of rank n, a *symmetric chain* C is a sequence of elements $x_1, x_2, \ldots, x_t \in A$, $1 \leq t \leq n$, where x_{i+1} covers x_i for $1 \leq i \leq t-1$, and $r(x_1) + r(x_t) = n$. We say x_1 is the *chain starter* denoted as $start(C)$, and x_t is the *chain terminator* denoted as $term(C)$. A *symmetric chain decomposition* (SCD) of $\mathcal{A}$ is a partition of the elements of A into symmetric chains.

2.1.4 Greene-Kleitman SCD in $\mathcal{B}_n$

In [GK76], Greene and Kleitman show a systematic way of constructing an SCD for $\mathcal{B}_n$. In this method, we view elements in $\mathcal{B}_n$ as n-bit strings. In each string, we view '0' as '(' and '1' as ')'. We scan $x = x_1 x_2 \cdots x_n$ from left to right and match parentheses in the usual way. That is, when a 0 is encountered, it becomes (temporarily) an unmatched 0. When a 1 is encountered, it is matched to the rightmost unmatched 0 to its left, if any, otherwise, it becomes an unmatched 1.

We choose strings with no unmatched ones as chain starters, and apply the following function, τ, to a string x on a chain to obtain the successor of x, $\tau(x)$:

$$\tau(x) = \begin{cases} nil & \text{if there are no unmatched 0's in } x\text{, else} \\ x_1 \cdots x_{i-1} 1 x_{i+1} \cdots x_n & \text{where } i \text{ is the position of the leftmost unmatched 0} \end{cases}$$

We also define the inverse of τ as follows:

$$\tau^{-1}(x) = \begin{cases} nil & \text{if there are no unmatched 1's in } x\text{, else} \\ x_1 \cdots x_{j-1} 0 x_{j+1} \cdots x_n & \text{where } j \text{ is the position of the rightmost} \\ & \text{unmatched 1} \end{cases}$$

Greene and Kleitman showed that if there is at least one unmatched 0 in x then $\tau^{-1}(\tau(x)) = x$, and if there is at least one unmatched 1 in x then $\tau(\tau^{-1}(x)) = x$ [GKS02].

2.1.5 Rotations

Following [GKS02], for $x = x_1 x_2 \cdots x_n \in \{0,1\}^n$, let σ denote the *rotation* of x defined by $\sigma(x) = x_2 x_3 \cdots x_n x_1$. Let $\sigma^1 = \sigma$, and $\sigma^i(x) = \sigma(\sigma^{i-1}(x))$, where $i > 1$. We say that x is *periodic* if $\sigma^l(x) = x$ for some l satisfying $0 < l < n$. Note that if x is periodic, so is any rotation of x. Otherwise, x is *aperiodic*. We can extend the same definition of periodicity to sequences.

Following [GKS02], if $b = (b_1, b_2, \ldots, b_k)$ is a sequence of integers, let $|b|$ denote the number of terms of b, $|b| = k$, and let $\|b\|$ denote the sum of the integers comprising b, $\|b\| = b_1 + b_2 + \cdots + b_k$. Analogous to the rotation σ of a string, define rotation, σ_s on sequences by $\sigma_s^i(b) = (b_{i+1}, \ldots, b_k, b_1, \ldots, b_i)$ for $i < k$.

2.1.6 Necklaces, necklace posets, and necklace-representative posets

In combinatorics, the term *necklace* is used to describe a certain equivalence class of binary strings. As defined in [GKS02], *necklaces* are the equivalence classes of the equivalence relation "$\sim$" on $\{0,1\}^n$ defined by $x \sim y$ iff $y = \sigma^i(x)$ for some integer $i \geq 0$. We denote a necklace as $[x]$, where x is an element of the equivalence class. Since rotating a string will not change the number of 1's in it, all strings in a necklace have the same rank. We define the rank of necklace $[x]$, $rank'([x])$, to be the same value as $rank(x)$, i.e., $rank'([x]) = rank(x)$.

Let N_n be the set of necklaces containing all strings in $\{0,1\}^n$. Define the *necklace poset*, $\mathcal{N}_n$, by $\mathcal{N}_n = (N_n, \preceq)$ with ordering $\preceq$ defined for $\eta_1, \eta_2 \in N_n$ by $\eta_1 \preceq \eta_2$ if and only if some $x \in \eta_1$ differs from some $y \in \eta_2$ in one bit, where the bit is a '0' in x and a '1' in y.

Given a set R_n of n-bit necklace representatives consisting of one representative string from each necklace, as described in [GKS02], the *necklace-representative poset* associated with R_n is the subposet of the Boolean lattice induced by R_n, denoted as $\mathcal{R}_n$.

2.2 Introduction of new ideas

2.2.1 Block codes

Block codes play a critical role in helping to choose necklace representatives in [GKS02]. Introduced in [GKS02], the *block code* of a string x, where $x \in \{0,1\}^n$, is a sequence over the alphabet $\{2, ..., n, \infty\}$, which is defined as follows.

If x has the form

$$x = 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}, \quad k > 0, \quad a_i > 0, \quad c_i > 0, \quad 1 \le i \le k, \tag{2.1}$$

i.e., x begins with 1 and ends with 0, then the block code of x is

$$\beta(x) = \ (a_1 + c_1,\ a_2 + c_2,\ ...,\ a_k + c_k), \tag{2.2}$$

and we say that the block code is *finite*.

Otherwise, we define $\beta(x) = (\infty)$.

In [GKS02], when n is prime, the necklace representative of a necklace is chosen to be the string with the lexicographically minimum block code among all its rotations.

For a finite block code

$$\beta(x) = (n_1, n_2, ..., n_m), \quad n_i \geq 2, \quad 1 \leq i \leq m,$$

we say that $\beta(x)$ is a *periodic block code* if $\sigma_s^l(\beta(x)) = \beta(x)$ for some l satisfying $0 < l < m$. Note that if $\beta(x)$ is periodic, so is any rotation of $\beta(x)$. Otherwise, we say that $\beta(x)$ is an *aperiodic block code.* According to this definition, all block codes with a single element, e.g. $\beta(111100000) = (9)$, are aperiodic.

2.2.2 Periodic necklaces

Define a *periodic necklace* as a necklace containing a string with periodic block code. Then an *aperiodic necklace* is a necklace containing strings all with finite aperiodic block codes. So note that $[0^n]$ and $[1^n]$ are not considered aperiodic necklaces.

2.2.3 SCD of a necklace-representative poset

In this thesis, we want to choose a set of necklace representatives, R_n, such that the necklace representatives in R_n can be partitioned into symmetric chains, which form an SCD of the necklace-representative poset $\mathcal{R}_n$.

[GKS02] proves that there exists an SCD of $\mathcal{R}_n$ for all prime n.

2.2.4 The chain cover property

Following [GKS02], let $\mathcal{C}$ be an SCD in a finite ranked poset $\mathcal{A} = (\mathcal{A}, \preceq)$. Call the longest chains in $\mathcal{C}$ the *root chains*. Say that $\mathcal{C}$ has the *chain cover property* (CCP) if for every $C \in \mathcal{C}$ except the root chains, there exists a chain $\pi(C) \in \mathcal{C}$ such that $start(C)$ covers an element of $\pi(C)$, and $term(C)$ is covered by an element of $\pi(C)$. Call such a mapping π a *chain cover mapping*. We can say that $\pi(C)$ *covers* C.

[GKS02] proves that the Greene-Kleitman SCD of $\mathcal{B}_n$ has the chain cover property (see Lemma 9 of [GKS02]). Consider such an SCD $\mathcal{C}$. For $C \in \mathcal{C}$ which is not a root chain, $\pi(C)$ is identified by its starter which is obtained by changing the rightmost 1 in $start(C)$ to 0. Note that in the Greene-Kleitman SCD of $\mathcal{B}_n$, there is only one root chain which starts with 0^n and terminates with 1^n.

A similar technique is applied in [GKS02] to find the chain cover property in SCD of necklace-representative poset $\mathcal{R}_n$ for prime n.

2.3 Overview of the problem

It was shown in [GKS02] that an SCD of $\mathcal{R}_n$ with the chain cover property exists for all prime n. It is also found in [GKS02] that rotationally symmetric Venn diagrams for n sets exist for all prime n.

As shown in [Hen63], a symmetric Venn diagram for n sets cannot exist when n

is not prime. So we were interested in knowing whether a symmetric independent family of n curves exists when n is not prime, which Grünbaum states to be true in [Grü99]. Since we don't care how many regions a subset of $[n]$ is associated with, Grünbaum has been able to show that there exists a symmetric independent family of n curves for all n. In the same paper, Grünbaum proposes a challenging question, i.e., whether there exists, for every n, a symmetric independent family of n curves with the minimum possible number of regions. That is the problem this thesis is trying to solve.

In order to solve the geometric problem regarding independent families of curves, our plan is to focus first on a combinatorial problem which is an open question from [GKS02], i.e., whether there exists a symmetric chain decomposition of the necklace-representative poset $\mathcal{R}_n$ for *every* n. When n is composite, we will divide necklaces in N_n into two subsets. One subset consists of necklaces that have no strings with periodic block codes. This part is easy to solve. The other subset consists of necklaces each of which has some string with a periodic block code. This part is rather hard. We will look, respectively, for ways of generating symmetric chain decompositions of necklace-representative subposets for these two subsets. Then, we will apply the results from the combinatorial problem back to the geometric problem of looking for symmetric independent family of n curves with the minimum possible number of regions. As a preview, please see Figures 5.1–5.4.

Chapter 3

SCD for aperiodic necklaces

3.1 Introduction

In this chapter, we look for symmetric chain decompositions of necklace-representative posets for composite n. [GKS02] sets up a whole package of rules, for prime n, in choosing necklace representatives and chain starters, as well as in generating chains. We find that, when n is composite, those rules can still be applied, on aperiodic necklaces, to generate symmetric chains. However, those rules may fail to work for periodic necklaces. We will leave such necklaces to the next chapter, and only focus on necklaces containing strings with no periodic block codes. By obtaining an SCD for a subset of the necklaces, we are able to reduce the problem of searching for SCD's to a smaller subproblem which will be investigated in the next chapter.

3.2 Definitions

3.2.1 Some subsets of the set of n-bit necklaces

Let N_n^* denote the set of all n-bit necklaces excluding $[0^n]$ and $[1^n]$. We will first look for an SCD for N_n^*.

When n is composite, we can divide N_n^* into two subsets: the set of aperiodic necklaces excluding $[0^n]$ and $[1^n]$ and the set of periodic necklaces.

Given composite n, define $N_n^*(A)$ to be the set of aperiodic necklaces excluding $[0^n]$ and $[1^n]$. We define $\mathcal{N}_n^*(A)$ as the subposet of $\mathcal{N}_n$ induced by $N_n^*(A)$.

Given composite n, define $N_n^*(P)$ to be the set of periodic necklaces. We define $\mathcal{N}_n^*(P)$ as the subposet of $\mathcal{N}_n$ induced by $N_n^*(P)$.

Given a finite block code β, define $N_n^*(\beta)$ to be the set of necklaces each of which contains some string with block code β. Define $\mathcal{N}_n^*(\beta)$ as the subposet of $\mathcal{N}_n$ induced by $N_n^*(\beta)$.

When n is composite, also define $A(n)$ to be the set of finite aperiodic block codes β with $\|\beta\| = n$, and define $P(n)$ to be the set of periodic block codes β with $\|\beta\| = n$.

3.2.2 One-count codes

When n becomes large, it is convenient sometimes to identify an n-bit string by its *one-count code* combined with its block code.

For $x \in \{0,1\}^n$, we associate a sequence $\gamma(x)$ over the alphabet $\{1, 2, ..., n-1\}$ called the *one-count code* of x as follows:

If $\beta(x) \neq \infty$, and x has the form

$$x = 1^{a_1}0^{c_1}1^{a_2}0^{c_2} \cdots 1^{a_k}0^{c_k}, \quad k > 0, \quad a_i > 0, \quad c_i > 0, \quad 1 \leq i \leq k,$$

then

$$\gamma(x) = < a_1, a_2, ..., a_k > .$$

Otherwise, we define $\gamma(x) = < -\infty >$.

Lemma 1. *There are no two different strings in $\{0,1\}^n$ that have both the same finite block code and the same one-count code.*

Proof. For $x \in \{0,1\}^n$, assume $\beta(x) = (b_1, b_2, \ldots, b_k)$, and $\gamma(x) =< a_1, a_2, \ldots, a_k >$, where $k > 0$ and $b_1 \neq \infty$. According to the definition of block code and one-count code, $\sum_{i=1}^{k} b_i = n$, $b_i \geq 2$, and $1 \leq a_i \leq b_i - 1$.

So,

$$\begin{aligned} x &= x_1 x_2 \cdots x_n \\ &= 1^{a_1}0^{b_1-a_1}1^{a_2}0^{b_2-a_2} \cdots 1^{a_i}0^{b_i-a_i} \cdots 1^{a^k}0^{b^k-a^k}. \end{aligned}$$

□

Therefore, one-count codes can be combined with block codes to represent any string. If a string x has block code $\beta(x)$ and one-count code $\gamma(x)$, we then denote x

as $\beta(x)/\gamma(x)$. For example, considering a string $x = 11010001001110$, we have $\beta(x) = (3,4,3,4)$ and $\gamma(x) =< 2,1,1,3 >$. Thus we denote x as $(3,4,3,4)/ < 2,1,1,3 >$.

3.3 Choosing necklace representatives for necklaces in $N_n^*(A)$

In this section, we want to choose necklace representatives for necklaces in $N_n^*(A)$ to be strings with the minimum block code among all its rotations. But we must first show that such a choice is unique within every necklace in $N_n^*(A)$.

Lemma 2. (Lemma 12 from [GKS02]) *If $x \in \{0,1\}^n$, and y is a rotation of x and if x and y both start with '1' and end with '0' then $\beta(y)$ is a rotation of $\beta(x)$.*

Lemma 14 of [GKS02] proves for prime n that no two different n-bit strings in the same necklace have the same finite block code. We now extend the lemma to include necklaces in $N_n^*(A)$. It is obvious that when n is prime, n-bit strings all have aperiodic block codes, so the following is a generalization of Lemma 14 for [GKS02].

Lemma 3. *For every n, no two different strings of $\{0,1\}^n$ with aperiodic block codes in the same necklace have the same finite block code.*

Proof. Assume $x \in \{0,1\}^n$ and the block code of x is finite and aperiodic. Let $b = \beta(x) = (b_1, b_2, \cdots, b_k)$. Since b is aperiodic, for every l with $0 < l < k$, $\sigma_s^l(b) \neq b$.

Suppose $y \neq x$ is a rotation of x and y starts with '1' and ends with '0'. Then by Lemma 2, $\beta(y) = \sigma_s^l(b)$ where $0 < l < k$. If $\beta(y) = \beta(x)$, then $\sigma_s^l(b) = b$, a contradiction. □

By Lemma 3, if $\eta \in N_n^*(A)$ where n is composite, then there is one and *only* one string in η, say x, that has the minimum block code among all its rotations. Thus, η has a unique representative x.

When n is composite, we define the set of necklace representatives for necklaces containing no strings with periodic block codes, as follows:

$$\begin{aligned} R_n^*(A) = \{ \quad & x \mid x \text{ is in } \eta \text{ where } \eta \in N_n^*(A), \text{ and} \\ & x \text{ has the minimum block code among all its rotations.} \} \end{aligned} \quad (3.1)$$

Then let $\mathcal{R}_n^*(A)$ be the subposet of $\mathcal{R}_n$ induced by $R_n^*(A)$. We notice that this is the same as the way representatives were chosen in [GKS02] for N_n^* where n is prime. However, we have no idea in general at this time how to choose representatives for $N_n^*(P)$. We address this problem for small n in Chapter 4.

3.4 Generating an SCD in $\mathcal{R}_n(A)$

We show in the next two lemmas that when τ is restricted to elements with the minimum block codes among all their rotations, no matter whether the block code is periodic or aperiodic, and with at least two unmatched 0's, the block code is preserved by τ.

Lemma 4. *If $x \notin \{0^n, 1^n\}$, and x has the minimum block code among all its rotations, and x has at least two unmatched 0's, then $\beta(x) = \beta(\tau(x))$.*

Proof. Lemma 10 from [GKS02] shows that in a bit string x, for a bit x_i that is neither the first nor the last in the string, if $x_i = 0$ and either $x_{i+1} = 1$ or $x_{i-1} = 0$, then x_i is not the leftmost unmatched 0. We can use this lemma here.

Since x has the minimum block code among all its rotations and $x \notin \{0^n, 1^n\}$, x starts with 1 and ends with 0. The 0 that x ends with is an unmatched 0 because there are no 1's to the right of that 0. Since it is also the last 0 in x, x_n is the rightmost unmatched 0. Now since there are at least two unmatched 0's in the string, the position of the leftmost unmatched 0, say i, must be within $[2, n-1]$. Let $y = \tau(x)$. By the definition of τ, $\tau(x)$ differs from x only in the i-th position, where $x_i = 0$ and $y_i = 1$. By the contrapositive of Lemma 10 from [GKS02], $x_{i-1}x_ix_{i+1} = 100$, and so $y_{i-1}y_iy_{i+1} = 110$. Therefore, by applying τ to x, the only change that occurs in x is that one block of x changes from $1^{a_j}0^{c_j}$ to $1^{a_j+1}0^{c_j-1}$. We notice that the block code is not changed. (It also implies that if x has the form (2.1) then so does y.) □

In a similar way, we can prove that when τ^{-1} is restricted to elements with the minimum block codes among all their rotations, with at least two unmatched 1's, the block code is preserved by τ^{-1}.

Lemma 5. *If $x \notin \{0^n, 1^n\}$, and x has the minimum block code among all its rotations, and x has at least two unmatched 1's, then $\beta(x) = \beta(\tau^{-1}(x))$.*

Proof. Lemma 11 from [GKS02] proves that in a bit string x, for a bit x_i that is neither the first nor the last in the string, if $x_i = 1$ and either $x_{i+1} = 1$ or $x_{i-1} = 0$, then x_i is not the rightmost unmatched 1. We can use this lemma here.

Since x has the minimum block code among all its rotations and $x \notin \{0^n, 1^n\}$, x starts with 1 and ends with 0. The 1 that x starts with is an unmatched 1 because there are no 0's to the left of that 1. Since it is also the first 1 in x, x_1 is the leftmost unmatched 1. Now since there are at least two unmatched 1's in the string, the position of the rightmost unmatched 1, say i, must be within $[2, n-1]$. Let $y = \tau^{-1}(x)$. By the definition of τ^{-1}, $\tau^{-1}(x)$ differs from x only in the i-th position, where $x_i = 1$ and $y_i = 0$. By the contrapositive of Lemma 11 from [GKS02], $x_{i-1}x_ix_{i+1} = 110$, and so $y_{i-1}y_iy_{i+1} = 100$. Therefore, by applying τ^{-1} to x, the only change that occurs in x is that one block of x changes from $1^{a_j}0^{c_j}$ to $1^{a_j-1}0^{c_j+1}$. We notice that the block code is not changed. (It also implies that if x has the form (2.1) then so does y.) □

Corollary 1. *If $x \in R_n^*(A)$ and x has at least two unmatched 0's, then $\tau(x) \in R_n^*(A)$. Similarly, if $x \in R_n^*(A)$ and x has at least 2 unmatched 1's, then $\tau^{-1}(x) \in R_n^*(A)$.*

Proof. As proved in Lemma 4, $\beta(x) = \beta(\tau(x))$. Since $x \in R_n^*(A)$, $\beta(x)$ is minimum, or lexicographically smallest, among all rotations of the sequence. Thus the same is true for $\beta(\tau(x))$ since $\beta(x) = \beta(\tau(x))$. By definition of $R_n^*(A)$, $\tau(x) \in R_n^*(A)$. A similar argument follows for $\tau^{-1}(x)$ from Lemma 5. □

We define $\mathcal{R}_n(A)$ as the subposet of $\mathcal{R}_n$ induced by $R_n^*(A) \cup \{0^n, 1^n\}$.

Lemma 6. *When n is composite, $\mathcal{R}_n(A)$ has an SCD with the chain cover property.*

Proof. We first look for an SCD of $\mathcal{R}_n^*(A)$, i.e. $\mathcal{R}_n(A)$ excluding 0^n and 1^n. We will add 0^n and 1^n back later.

Recall the definition of $R_n^*(A)$ in (3.1). $R_n^*(A)$ consists of strings that start with a '1' and end with a '0', and have the minimum block code among all their rotations. Choose chain starters from $R_n^*(A)$ to be strings that have only one unmatched one, which obviously is the '1' in the first bit. If a string $x \in R_n^*(A)$ has at least two unmatched 0's, apply τ, as defined in Section 2.1.4, to x, i.e., change the leftmost unmatched '0' of x to '1', to obtain the successor of x, $\tau(x)$, on the chain containing x. Therefore, from each element in $\mathcal{R}_n^*(A)$, if it has at least two unmatched 0's, we apply τ repeatedly to generate a chain until encountering a string that has only one unmatched '0', which is the terminator of the chain. Otherwise, the chain starter is also the chain terminator.

We show (1) every chain obtained in the way described above is symmetric, (2) all strings on the chains are elements of $R_n^*(A)$, and every element of $R_n^*(A)$ is on one and only one chain. Then, (3) add 0^n and 1^n back without destroying the SCD. (4)

Show the resulting SCD of $\mathcal{R}_n(A)$ has the chain cover property, i.e., every non-root chain in the resulting SCD is covered by another chain in the same SCD.

(1). Every chain starts with a string with only one unmatched '1'. Assume a chain starter s has u_0 unmatched 0's, m_1 matched 1's, and m_0 matched 0's. Clearly, $m_1 = m_0$, and $1 + u_0 + m_1 + m_0 = n$. So, $m_1 = (n - 1 - u_0)/2$, and the number of 1's in s, $n_1 = 1 + m_1 = (n + 1 - u_0)/2$. By applying τ to a string x, the leftmost unmatched '0' in x is changed to '1'. Since every '0' to the left of the current position is matched, the new '1' is therefore unmatched. The numbers of matched '0's and matched 1's are unchanged. Then, the number of unmatched 0's in $\tau(x)$ is decreased by one from x, and the number of unmatched 1's in $\tau(x)$ is increased by one from x. So assume the chain terminator t has $u_0 - l$ unmatched 0's, then the number of unmatched 1's in t is $1+l$. By definition of terminator, t has only one unmatched '0'. Thus, $l = u_0 - 1$, and t has u_0 unmatched 1's. Assume t has m'_0 matched '0', and m'_1 matched '1'. Then, $m'_0 = m'_1$, and $1 + u_0 + m'_0 + m'_1 = n$. So, $m'_1 = (n - 1 - u_0)/2$. The number of 1's in t, $n'_1 = u_0 + (n - 1 - u_0)/2 = (n - 1 + u_0)/2$. Therefore, $n_1 + n'_1 = (n + 1 - u_0)/2 + (n - 1 + u_0)/2 = n$. This proves that every chain is symmetric.

(2). By applying Corollary 1 repeatedly, every string on chains generated by τ is an element of $R^*_n(A)$. Greene and Kleitman proved in [GK76] that applying τ to two distinct n-bit strings can never result in the same string. Here, we apply τ to a restricted domain of n-bit strings, i.e. $R_n^*(A)$, so it holds that any two distinct chains are disjoint, i.e., if an element of $R_n^*(A)$ is on some chain, it can only be on exactly one chain. We still need to show that every element of $R_n^*(A)$ *is* on some chain. Consider an element of $R_n^*(A)$, x. If x has exactly one unmatched '1', then x is a chain starter and must be on some chain. If x has at least two unmatched 1's, by applying τ^{-1} to x, the rightmost unmatched '1' in x is changed to '0'. Since every '1' to the right of the current position is matched, the new '0' is therefore unmatched. The numbers of matched '0's and matched 1's are unchanged. Then, the number of unmatched 1's in $\tau^{-1}(x)$ is decreased by one from x, and the number of unmatched 0's in $\tau(x)$ is increased by one from x. Apply τ^{-1} repeatedly until reaching a string, s, with only one unmatched '1'. By applying Corollary 1, it is known that $s \in R_n^*(A)$. Therefore, s is a chain starter, and x must be on the chain which starts with s.

(3). By the way of generating chains described above, there is always a chain that starts with $100\cdots000$ and terminates with $111\cdots110$. We now add 0^n before $100\cdots000$ and add 1^n after $111\cdots110$, so that 0^n is the new chain starter and 1^n is the new chain terminator. Note that the covering relationship in the chain still holds, and the chain is still symmetric. It is obvious that this chain is the longest chain among all chains. Thus, it's the root chain. Put the root chain back. The resulting partition of elements of $R_n(A)$ is therefore an SCD of $\mathcal{R}_n(A)$. Define the set S of chain starters as $S = \{s \mid s \in R_n^*(A), \text{ and } s \text{ has only one unmatched '1'}\} - \{100\cdots000\} \cup \{0^n\}$.

We denote this SCD as $SCD_n(A)$.

(4). Consider a non-root chain C in $SCD_n(A)$ with starter s and terminator t. We identify a chain, $C' = \pi(C)$, which covers C, by specifying a string on C', s', which is obtained by changing the rightmost '1' of s to '0'. (The other strings on C' are obtained by applying τ and/or τ^{-1} from s' repeatedly.)

We first show that $C' \in SCD_n(A)$. Assume $s = 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}$, where $a_1 = 1$.

(i) If $a_k > 1$, then $s' = 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k-1}0^{c_k+1}$, and $\beta(s') = \beta(s) = (a_1+c_1, a_2+c_2, \ldots, a_k+c_k)$. Thus, the block code of s' is aperiodic and minimum among all its rotations. It follows that $s' \in R_n^*(A)$. Since s has at least two 1's and the first '1' is the only unmatched '1' in s, the rightmost '1' in s is a matched '1'. Also, there are no 1's to the right of the rightmost '1', so changing it to '0' does not change the number of unmatched 1's. Therefore, there is only one unmatched '1' in s'. It follows that $s' \in S$. So $C' \in SCD_n(A)$.

(ii) Consider $a_k = 1$. As in case (i), the rightmost '1' in s is a matched '1', and there are no 1's to the right of this '1', so changing it to '0' does not change the number of unmatched 1's. Therefore, there is only one unmatched '1' in s'. Assume $\beta(s) = (b_1, b_2, \ldots, b_{k-1}, b_k)$, then $\beta(s') = (b_1, b_2, \ldots, b_{k-2}, b_{k-1} + b_k)$. Since $\beta(s)$ is minimum among all its rotations, $(b_1, b_2, \ldots, b_k) < (b_{i+1}, \ldots, b_k, b_1, \ldots, b_i)$ for every i, where $1 \le i \le k-1$. Then, for each i, we compare the two subsequences consisting of the first $k-i-2$ elements in $\beta(s)$ and $\sigma^i(\beta(s))$. There are two possibilities: (1) $(b_1, b_2, \ldots, b_{k-i-2}) < (b_{i+1}, \ldots, b_{k-2})$; (2) $(b_1, b_2, \ldots, b_{k-i-2}) = (b_{i+1}, \ldots, b_{k-2})$ and $b_{k-i-1} \leq b_{k-1}$. In the first case,

$$\begin{aligned}\beta(s') &= (b_1, b_2, \ldots, b_{k-i-2}, b_{k-i-1}, \ldots, b_{k-2}, b_{k-1} + b_k)\\ &< (b_{i+1}, \ldots, b_{k-2}, b_{k-1} + b_k, b_1, \ldots, b_i),\end{aligned}$$

when $1 \leq i \leq k-2$. In the second case, since $b_{k-i-1} \leq b_{k-1}$, $b_{k-i-1} < b_{k-1} + b_k$. Therefore, $(b_1, b_2, \ldots, b_{k-i-2}, b_{k-i-1}) < (b_{i+1}, \ldots, b_{k-2}, b_{k-1} + b_k)$. It follows that

$$\begin{aligned}\beta(s') &= (b_1, b_2, \ldots, b_{k-i-2}, b_{k-i-1}, \ldots, b_{k-2}, b_{k-1} + b_k)\\ &< (b_{i+1}, \ldots, b_{k-2}, b_{k-1} + b_k, b_1, \ldots, b_i),\end{aligned}$$

when $1 \leq i \leq k-2$. Thus, $\beta(s')$ is minimum among all its rotations. Next, we show $\beta(s')$ is aperiodic. Assume that $\beta(s')$ is periodic. Then there exists l, where $1 \leq l \leq k-2$, such that $\sigma^l(\beta(s')) = \beta(s')$, i.e., $(b_1, \ldots, b_l) = (b_{l+1}, \ldots, b_{2l}) = \cdots = (b_{k-l}, \ldots, b_{k-2}, b_{k-1} + b_k)$. So, $(b_{k-l}, \ldots, b_{k-2}, b_{k-1}) < (b_{k-l}, \ldots, b_{k-2}, b_{k-1} + b_k) = (b_1, \ldots, b_l)$. Also note that $(b_1, \ldots, b_{k-l-1}) = (b_{l+1}, \ldots, b_{k-1})$. Therefore, it follows that

$$\begin{aligned}\sigma^{(k-l-1)}(\beta(s)) &= (b_{k-l}, \ldots, b_{k-1}, b_k, b_1, \ldots, b_{k-l-1})\\ &< (b_1, \ldots, b_l, b_{l+1}, \ldots, b_k)\\ &= \beta(s),\end{aligned}$$

i.e., $\beta(s)$ is not minimum among all its rotations. This is a contradiction, and therefore, $\beta(s')$ is aperiodic.

Recall that $\beta(s')$ is minimum among all its rotations. So, $s' \in R^*_n(A)$. Since there is only one unmatched '1' in s', (1) $s' = 100\cdots0$ which is on the root chain of $SCD_n(A)$, or (2) $s' \in S$. We conclude that $C' \in SCD_n(A)$.

It remains to show that C' covers C. Obviously, s' is covered by s. Let t be the chain terminator of C, and let t' be the string on C' with $r(t') = r(t)+1$. We need to prove that t' covers t. Recall that the rightmost '1' in s is a matched '1'. Let m_1 be the position of that '1', and let m_0 be the position of the '0' that is matched with the rightmost '1'. After changing the rightmost '1' of s to '0', that zero (in position m_1) is an unmatched '0' since there are no 1's to its right. And the '0' (in position m_0) that was matched with the rightmost '1' becomes an unmatched '0' since all 1's to its right are matched with some other 0's. Let $U_0(x), U_1(x), M_0(x)$, and $M_1(x)$ denote, respectively, the sets of the positions of unmatched 0's, unmatched 1's, matched 0's, and matched 1's in an n-bit string x. Therefore,

$$\begin{aligned}
U_0(s') &= U_0(s) + \{m_0, m_1\},\\
U_1(s') &= U_1(s),\\
M_0(s') &= M_0(s) - \{m_0\},\\
M_1(s') &= M_1(s) - \{m_1\}.
\end{aligned}$$

By the definition of τ, $\tau(x)$ is obtained by changing the leftmost unmatched '0' to '1'. Then, the new '1' becomes an unmatched '1' since there are no unmatched 0's to its left. All matched 0's and 1's are unchanged. Therefore, in chain C,

$$\begin{aligned} U_0(t) &= \{n\}, \\ U_1(t) &= U_1(s) + U_0(s) - \{n\}, \\ M_0(t) &= M_0(s), \\ M_1(t) &= M_1(s). \end{aligned}$$

In chain C',

$$
\begin{aligned}
U_0(t') &= \{n\} \\
&= U_0(t), \\
U_1(t') &= U_1(s') + U_0(s') - \{n\} \\
&= U_1(s) + U_0(s) + \{m_0, m_1\} - \{n\} \\
&= U_1(t) + \{m_0, m_1\}, \\
M_0(t') &= M_0(s') \\
&= M_0(s) - \{m_0\} \\
&= M_0(t) - \{m_0\}, \\
M_1(t') &= M_1(s') \\
&= M_1(s) - \{m_1\} \\
&= M_1(t) - \{m_1\}.
\end{aligned}
$$

So we find that t' and t only differ in position m_0, where the bit in t' is a '1' and the bit in t is a '0'. Therefore, t' covers t. We have already proved that s' is covered by s. It then follows that C' covers C. □

From Lemma 6, we can obtain a unique SCD of $\mathcal{R}_n(A)$ with the CCP for every n, with representatives picked as in (3.1), plus 0^n and 1^n, and the rule τ to generate symmetric chains. We denote the SCD as $SCD_n(A)$. By far, we have solved part of the problem of searching for an SCD of $\mathcal{R}_n$ with the CCP.

So what is left, in order to solve the complete problem, is to look for an appropriate rule to generate the set of representatives $R_n^*(P)$ for the periodic necklace set $N_n^*(P)$, so that we can find an SCD, $SCD_n(P)$, for the subposet of $\mathcal{R}_n$ induced by $R_n^*(P)$. We denote the subposet as $\mathcal{R}_n^*(P)$. After we solve this part, we have an SCD, i.e. $SCD_n(A) \cup SCD_n(P)$, of $\mathcal{R}_n$, which is the subposet of $\mathcal{B}_n$ induced by $R_n = R_n^*(A) \cup \{0^n, 1^n\} \cup R_n^*(P)$. We will discuss this part of the problem in detail in the next chapter.

3.5 Reducing the size of the problem

3.5.1 Introduction

We have solved part of the problem in the previous section and obtained an SCD for aperiodic necklaces, $SCD_n(A)$. In this section, we investigate the chain cover relationship between chains in $SCD_n(A)$ and chains for periodic necklaces. The result of the investigation enables us to reduce the problem of searching for an SCD of $\mathcal{R}_n$ with the CCP to a substantially smaller subproblem.

3.5.2 Backbone chains

When n is composite, consider a periodic block code $\beta = (b_1, b_2, \ldots, b_k)$, where $b_1 + b_2 + \cdots + b_k = n$. Let $\eta_s, \eta_t \in N_n^*(\beta)$ be defined, using one-count codes, by $\eta_s = [< a_1, a_2, \ldots, a_k > /(b_1, b_2, \ldots, b_k)]$, where $a_i = 1$ $(1 \leq i \leq k)$, and $\eta_t = [< b_1 - 1, b_2 - 1, \ldots, b_k - 1 > /(b_1, b_2, \ldots, b_k)]$.

We show that η_s is the only necklace with the minimum rank in $N_n^*(\beta)$. If $k = 1$, then $\eta_s = [< 1 > /(n)]$. For any string in η_s, switching the position of the only '1' with any other 0 is equivalent to rotating the necklace, which always yields strings in the same necklace. If $k \geq 2$, since every '1' in η_s is the only 1 in its block, switching the position of a '1' with any other 0 in η_s will change the block code, which yields strings that are no longer in any necklace in $N_n^*(\beta)$. Thus, we cannot find any other necklaces that have the same rank as η_s in $N_n^*(\beta)$. A similar approach can be followed to prove that η_t is the only necklace with the maximum rank in $N_n^*(\beta)$.

We note that $rank(\eta_s) = k$ and $rank(\eta_t) = n - k$, so $rank(\eta_s) + rank(\eta_t) = n$. Therefore, if there exists an SCD for $N_n^*(\beta)$, η_s and η_t must be on the same chain. We call such a chain a *backbone chain* denoted as C_b, where $start(C_b) = \eta_s$ and $term(C_b) = \eta_t$. So the backbone chain is the longest symmetric chain in the SCD for $N_n^*(\beta)$.

For example, when $n = 14$ and $\beta = (3, 4, 3, 4)$, the backbone chain will be

$$1\underline{0}01\underline{00}01\underline{0}01\underline{00}0 - \cdots - 1\underline{1}01\underline{11}01\underline{1}01\underline{11}0$$

So each of the six underlined 0's in the starter will need to be changed to 1 when we proceed towards the terminator. We are not sure yet about the specific order of the intermediate elements of such a chain, i.e. the rule to generate the backbone chain, as well as how to generate general symmetric chains for $N_n^*(P)$, not to mention a fixed chain cover mapping. But we will hold on to the rule used to choose a set of necklace representatives for $N_n^*(P)$, i.e., choosing the necklace representative of a necklace to be the string, in the necklace, with the minimum block code among all its rotations. We also require that every symmetric chain in $\mathcal{R}_n^*(P)$ consists of strings, i.e. necklace representatives, with the same block code.

3.5.3 Chain cover mappings between backbone chains for periodic necklaces and chains in $SCD_n(A)$

In this subsection, we discuss the chain cover relationship between backbone chains for periodic necklaces and chains in $SCD_n(A)$. Before showing the main result, first we prove the following lemma and its corollary.

Lemma 7. *Given a periodic string $x = x_1x_2 \cdots x_n$, where x_i is a non-negative integer with $1 \leq i \leq n$, the string $x' = x_1x_2 \cdots x_{n-2}(x_{n-1} + x_n)$ is not periodic.*

Proof. Let the minimum period of x be p, where $1 \leq p \leq n - 1$. Express x in terms of substrings α as follows:

$$x = \alpha^{n/p}, \text{ where } |\alpha| = p$$

We prove this lemma by contradiction. So assume that x' is also periodic with minimum period q, where $1 \leq q \leq n-2$. Express x' in terms of substrings β as follows:

$$x' = \beta^{(n-1)/q}, \text{ where } |\beta| = q$$

Since n and $n-1$ are relatively prime, so are p and q.

We first assume $p < q$. Let $\beta = \alpha^t \gamma$, where $t \geq 1$ and $1 \leq |\gamma| < p$.

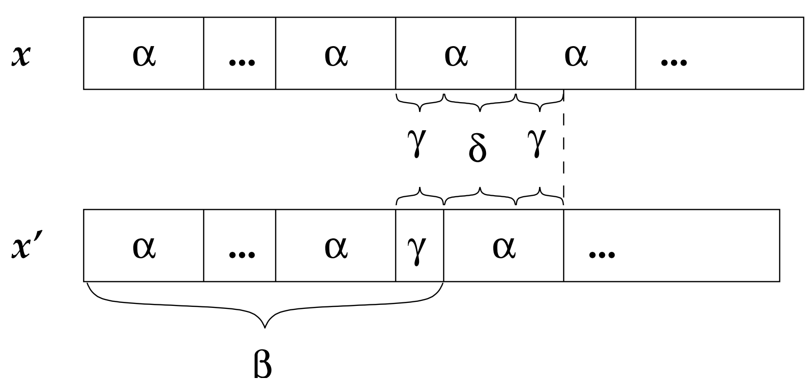

Figure 3.1: Lemma 7 ($p < q$)

As shown in Figure 3.1, there must be at least one symbol after the dashed line in x'. Thus all symbols before the dashed line in x' must have positions less than $n-1$. In other words, each of those symbols must be the same as the symbol with the same position in x. Therefore, $\alpha = \gamma\delta = \delta\gamma$, where $1 \leq |\delta| < p$. Thus,

$$\begin{aligned} x &= (\gamma\delta)^{n/p} \\ &= (\delta\gamma)^{n/p} \end{aligned}$$

Let $p' = |\delta|$. Since $\sigma^{p'}((\gamma\delta)^{n/p}) = (\delta\gamma)^{n/p}$ and $p' < p$, p is not the minimum period of x, a contradiction.

Assume $p > q$. Let $\alpha = \beta^s\epsilon$, where $s \geq 1$ and $1 \leq |\epsilon| < q$.

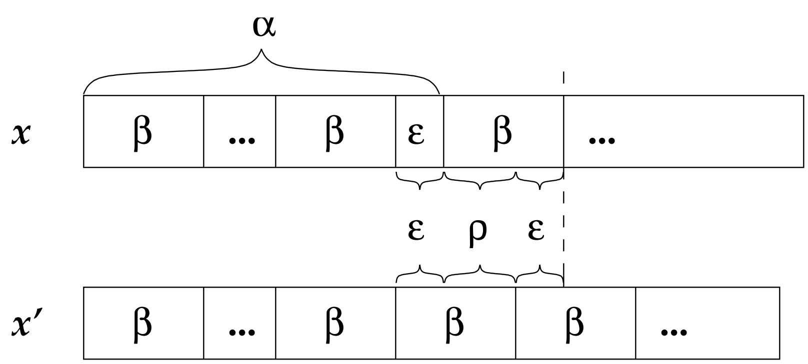

Figure 3.2: Lemma 7 ($p > q$)

As shown in Figure 3.2, there is at least one symbol after the dashed line in x.

If there is only one symbol following the dashed line in x, then $x = \beta\epsilon\beta\epsilon$ where $|\epsilon| = 1$. Thus, $|x| = 2|\beta| + 2$, and $|x'| = 2|\beta| + 1$. Since $|x'|$ is divisible by $|\beta|$, $|\beta| = 1$. So $|x'| = 3$. Let $x' = sss$ where s is any non-negative integer. Then $x = sss's''$ where $s' > 0$, $s'' > 0$, and $s = s' + s''$. It is obvious that x is not periodic, a contradiction.

If there is more than one symbol after the dashed line in x, then all symbols before the dashed line in x must have positions less than $n - 1$. In other words, each of those symbols must be the same as the symbol with the same position in x'. Therefore, $\beta = \epsilon\rho = \rho\epsilon$, where $1 \leq |\rho| < q$. Thus,

$$\begin{aligned} x' &= (\epsilon\rho)^{(n-1)/q} \\ &= (\rho\epsilon)^{(n-1)/q} \end{aligned}$$

Let $q' = |\rho|$. Since $\sigma^{q'}((\epsilon\rho)^{(n-1)/q}) = (\rho\epsilon)^{(n-1)/q}$ and $q' < q$, q is not the minimum period of x', a contradiction.

After all possibilities are considered, we obtain a contradiction to the original assumption. □

Corollary 2. *Given a periodic sequence $x = (x_1, x_2, \ldots, x_n)$ with minimum period p, where x_i is a non-negative integer with $1 \leq i \leq n$, the sequence $x' = (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_n)$ is not periodic.*

Proof. The same proof for Lemma 7 also works here. □

Now we show the main result of this section in the following lemma.

Lemma 8. *When n is composite, for every $\beta \in P(n)$, there exists $C \in SCD_n(A)$ such that the backbone chain in $\mathcal{N}_n^*(\beta)$ is covered by C.*

Proof. Let C_b be the backbone chain in $\mathcal{N}_n^*(\beta)$. Assume $\beta = (b_1, b_2, \ldots, b_k)$. Let $start(C_b) = x =< a_1, a_2, \ldots, a_k > /(b_1, b_2, \ldots, b_k)$ where $a_i = 1$ with $1 \leq i \leq k$. Since β is periodic, it must hold that $k \geq 2$. So $start(C_b)$ has at least two 1's. The bit to the left of every '1' that starts each block is 0, except in the first block. Let $x = x_1 x_2 \dots x_n$. Then x_1 is the only unmatched 1 in x.

We obtain the string x' by changing the rightmost 1 in x to 0. So x' is covered by x, and $x' =< a_1, a_2, \dots, a_{k-2}, a_{k-1} > /(b_1, b_2, \dots, b_{k-2}, b_{k-1} + b_k)$ where $a_i = 1$ with $1 \leq i \leq k-1$. We will show that x' is on a symmetric chain C where $C \in SCD_n(A)$, and C has an element that covers $term(C_b)$. As shown before, $term(C_b) =< b_1 - 1, b_2 - 1, \dots, b_k - 1 > /(b_1, b_2, \dots, b_k)$.

We first show that $\beta(x') = (b_1, b_2, \dots, b_{k-2}, b_{k-1} + b_k)$ is the minimum block code among all rotations of x'. Since we still choose the representative of a necklace in $N_n^*(P)$ to be a string, in the necklace, with the minimum block code among all its rotations, $\beta(x) = (b_1, b_2, \dots, b_{k-2}, b_{k-1}, b_k)$ is minimum among all its rotations. So $(b_1, b_2, \dots, b_k) < (b_{i+1}, \dots, b_k, b_1, \dots, b_i)$ for every i, where $1 \leq i \leq k-1$. Then, we compare the two subsequences consisting respectively of the first $k-i-2$ elements in both block codes. There are two possibilities: (1) $(b_1, b_2, \dots, b_{k-i-2}) < (b_{i+1}, \dots, b_{k-2})$; (2) $(b_1, b_2, \dots, b_{k-i-2}) = (b_{i+1}, \dots, b_{k-2})$ and $b_{k-i-1} \leq b_{k-1}$.

In the first case,

$$\begin{aligned} \beta(x') &= (b_1, b_2, \dots, b_{k-i-2}, b_{k-i-1}, \dots, b_{k-2}, b_{k-1} + b_k) \\ &< (b_{i+1}, \dots, b_{k-2}, b_{k-1} + b_k, b_1, \dots, b_i), \end{aligned}$$

when $1 \leq i \leq k-2$.

In the second case, since $b_{k-i-1} \leq b_{k-1}$, $b_{k-i-1} < b_{k-1} + b_k$. Therefore,

$$(b_1, b_2, \ldots, b_{k-i-2}, b_{k-i-1}) < (b_{i+1}, \ldots, b_{k-2}, b_{k-1} + b_k).$$

Finally,

$$\begin{aligned}\beta(x') &= (b_1, b_2, \ldots, b_{k-i-2}, b_{k-i-1}, \ldots, b_{k-2}, b_{k-1} + b_k) \\ &< (b_{i+1}, \ldots, b_{k-2}, b_{k-1} + b_k, b_1, \ldots, b_i),\end{aligned}$$

when $1 \leq i \leq k-2$.

Thus, we have proved that $\beta(x')$ is minimum among all its rotations. Following Corollary 2, $\beta(x') = (b_1, b_2, \ldots, b_{k-2}, b_{k-1} + b_k)$ is aperiodic.

Therefore, $[x'] \in N_n^*(A)$, and x' is the representative of $[x']$. So $x' \in R_n(A)$. By Lemma 6, there exists a symmetric chain, say C, in $SCD_n(A)$ such that x' is on C. Since $x' =< a_1, a_2, \ldots, a_{k-2}, a_{k-1} > /(b_1, b_2, \ldots, b_{k-2}, b_{k-1} + b_k)$, where $a_i = 1$ with $1 \leq i \leq k-1$, $r(x') = k-1$. By applying τ to x' $(n - 2k + 2)$ times, we obtain the string with rank $n - k + 1$ on C,

$$y' =< b_1 - 1, b_2 - 1, \ldots, b_{k-2} - 1, b_{k-1} + b_k - 1 > /(b_1, b_2, \ldots, b_{k-2}, b_{k-1} + b_k).$$

We find that y' differs from $term(C_b)$ only in the $(n_k - b_k)$-th position which is a '1' in y' and a '0' in $term(C_b)$. So y' covers $term(C_b)$. Therefore, $C = \pi(C_b)$. □

3.5.4 Remaining problems

To solve the problem of looking for an SCD of $\mathcal{R}_n$ with the chain cover property, we now only need to focus on periodic block codes, i.e. $\beta \in P(n)$. What is more convenient is that now we can look into each periodic block code independently.

For every $\beta \in P(n)$, we need to find a rule to choose necklace representatives from $N_n^*(\beta)$ to form set $R_n^*(\beta)$ such that there exists an SCD of $\mathcal{R}_n^*(\beta)$ with the chain cover property. Here $\mathcal{R}_n^*(\beta)$ is the subposet of $\mathcal{B}_n$ induced by $R_n^*(\beta)$.

We will explore this more in the next chapter.

Chapter 4

Looking for an SCD of $\mathcal{R}_n^*(P)$

4.1 Introduction

From the results we have obtained in the previous chapter, to solve the problem of looking for an SCD of $\mathcal{R}_n$ with the chain cover property, it suffices to focus on an especially small subproblem, i.e., only looking for a necklace-representative set, $R_n^*(P)$, for necklaces in $N_n^*(P)$, such that $\mathcal{R}_n^*(P)$ has an SCD with the chain cover property. To make the subproblem even smaller, it suffices to focus on individual periodic block codes $\beta \in P(n)$ and look for an SCD of $\mathcal{R}_n^*(\beta)$ with the chain cover property. Yet it is still far from easy, as we will see, to take care of the whole problem. In this chapter, we will focus on this critical part, where necklaces are all periodic. It is critical because finding a solution to this small subproblem means discovery of the key to the whole open question.

In the previous chapter, to generate an SCD of representatives for aperiodic necklaces, we (1) select strings with the minimum block codes among all their rotations as necklace representatives, (2) choose necklace representatives with only one unmatched '1' as chain starters, and (3) generate chains with function τ. In this chapter, we will first apply those rules to $N_n^*(P)$, i.e. the set of periodic necklaces, to see if they still work here. Then we adjust those rules and come up with new rules that apply to strings with weight less than $n/2$ that are in necklaces from $N_n^*(P)$ for $n \leq 16$. Later we will find that those new rules don't apply to some $n > 16$. Finally, we show some nice properties for necklaces with strings that have block codes of certain patterns.

4.2 Choosing chain starters and necklace representatives

In this section, we start looking for ways of choosing chain starters as well as necklace representatives for periodic necklaces, i.e. necklaces in $N_n^*(P)$.

We start our investigation by picking strings that have minimum block codes among all their rotations and have only one unmatched '1' from necklaces in $N_n^*(P)$. Those are the candidates for chain starters. We immediately find that some of the potential chain starters end up in the same necklace. For example, strings 10011001001000 and 10010001001100 both have block code $(3, 4, 3, 4)$ and have only one unmatched '1', thus both are candidates for chain starters, yet they belong to the same necklace.

For those necklace representatives that are from the same necklace, we will need to set up additional rules to choose only one of them to keep. Therefore, it is inevitable to make changes to the rules in order to make them work for $N_n^*(P)$. From this point, we will go along with those candidates for chain starters, i.e. strings with min block code and only one unmatched 1, and try generating an SCD with function τ. We will find in the SCD that some representatives other than chain starters may also belong to the same necklace. Then we will find a way of eliminating redundant representatives in the same necklace, and keep only one for each necklace eventually, while still having an SCD.

By the results obtained from the previous chapter, it suffices to focus on each periodic block code $\beta \in P(n)$ and look for an SCD within the necklaces in $N_n^*(\beta)$. For all composite n when $n \leq 16$, we write down all the periodic block codes. For each periodic block code, as we discussed earlier in this section, we pick candidates for chain starters with that block code. From each of those potential chain starters, we apply τ repeatedly until we reach a string that has only one unmatched '0', and thus form a chain. It is obvious that every chain we get here is symmetric.

By observing the symmetric chains we have generated for the block codes, we find a way of removing redundant necklace representatives without destroying the

SCD. We want to keep those chains starting with strings, with block code β, that are lexicographically largest among all rotations with block code β. For any other chain that doesn't start with such a string, we start from the starter and proceed along the chain towards the terminator. We want to find the first string with weight less than $n/2$ that is lexicographically largest among all rotations with that block code. Then that string becomes the new chain starter, which we denote as x'_s. Assume weight$(x'_s) = m$, where $m < n/2$. Then, the new terminator of the chain will be the string with weight $n - m$ on the same chain. With the new chain starter and terminator, we have obtained a new chain, and we discard all other strings of the old chain that are not on the new chain. However, if we cannot find such a string on the chain, we put aside the chain. If n is even, later, we may need that chain, or to be more accurate, the string with weight $n/2$ on the chain. After we have gone through this procedure on every chain in the same block code, we go back to those chains, if any, that have been temporarily discarded. We examine the strings in the middle of those chains, i.e., strings of weight $n/2$. If we find a string that represents a necklace that hasn't yet been represented in our reformed SCD, we add the string to the SCD. The string by itself will form a chain. The result of the exploration is shown in Appendix A. We have verified that every chain is symmetric and all necklaces that have periodic min block codes are represented in the SCD's.

We have demonstrated in this section the approach to finding chain starters to construct an SCD of $\mathcal{R}_n^*(P)$ when $n \leq 16$. Surprisingly, in such an SCD, not only chain starters but all strings (necklace representatives) with weight less than $n/2$ are lexicographically largest among all rotations with that block code. Thus, we can precisely define those necklace representatives and chain starters with weight less than $n/2$ in the SCD of $\mathcal{R}_n^*(P)$ that we have constructed. We will show that specification in the next section.

Lemma 9. *When n is composite and $n \leq 16$, there exists a set $R_n^*(P)$ such that $\mathcal{R}_n^*(P)$ has an SCD.*

Proof. We have just described how we came up with SCD's for $n \leq 16$. Appendix A presents those SCD's of $\mathcal{R}_n^*(P)$ for $n \leq 16$. We *grouped* symmetric chains by the same block code. Each *group* is an SCD of the subposet of $\mathcal{N}_n^*(\beta)$ induced by representatives of $N_n^*(\beta)$ for $\beta \in P(n)$ where $n \leq 16$.

We *grouped* representatives by the same block code. We have verified in all *groups* that every chain is symmetric and is disjoint from any other chains, if any, in the same group. Also, we have checked to make sure that all representatives were listed. (See [Dav01] for the method used to count necklaces.) □

Conjecture. *For every composite n, there exists a set $R_n^*(P)$ such that $\mathcal{R}_n^*(P)$ has an SCD.*

Corollary 3. *When n is composite and $n \leq 16$, there exists a set R_n such that $\mathcal{R}_n$*

has an SCD.

Proof. It follows from Lemma 6 that $\mathcal{R}_n(A)$ always has an SCD. By Lemma 9, $\mathcal{R}_n^*(P)$ has an SCD for composite n where $n \leq 16$. Note that $R_n(A)$ and $R_n^*(P)$ are disjoint and $R_n(A) \cup R_n^*(P)$ is the set of all representatives for n-bit necklaces. Thus, by combining $SCD_n(A)$ and the SCD from Lemma 9, we obtain an SCD for $\mathcal{R}_n$ where n is composite and $n \leq 16$. Here, $R_n = R_n(A) \cup R_n^*(P)$. □

When n becomes larger, the function τ may not be a good rule to generate symmetric chains to form SCD's for $\mathcal{N}_n^*(P)$. We will discuss this issue more later.

Corollary 4. *When n is composite and $n \leq 16$, $\mathcal{N}_n$ has an SCD.*

Proof. This is a weaker version of Corollary 3. Following the corollary, this statement is automatically true. □

Theorem 1. *When n is composite and $n \leq 16$, there exists a set R_n such that $\mathcal{R}_n$ has an SCD with the chain cover property.*

Proof. In Appendix A, we find SCD's of $\mathcal{R}_n^*(P)$ for $n \leq 16$ where n is composite. We *grouped* symmetric chains by the same block code. Each *group* is an SCD of the subposet of $\mathcal{N}_n^*(\beta)$ induced by representatives of $N_n^*(\beta)$ for $\beta \in P(n)$ where $n \leq 16$. We have verified that every chain except the backbone chain can always be covered by another chain in the same group. By Lemma 8, the backbone chain in each group can be covered by a chain in $SCD_n(A)$. By Lemma 6, $SCD_n(A)$ has the chain cover property. Thus, combining the SCD in Appendix A with $SCD_n(A)$ when $n \leq 16$, we obtain an SCD of $\mathcal{R}_n$ with the chain cover property for $n \leq 16$ where n is composite. □

We have just answered a small part of the open question from [GKS02].

4.3 Choosing necklace representatives and chain starters of weight less than $n/2$ when $n \leq 16$

It follows from the discussion of the previous section that when $n \leq 16$, we can choose necklace representatives with weight less than $n/2$ as follows:

$$\begin{aligned} R_n^*(P)|_{\text{weight}<n/2} \;=\; & \{x \mid \text{weight}(x) < n/2,\, \beta(x) = \beta_{min}([x]), \\ & \text{and } x \text{ is lexicographically largest among} \\ & \text{all rotations with block code } \beta(x).\} \end{aligned} \tag{4.1}$$

Among those representatives with weight less than $n/2$, we select chain starters for SCD of $\mathcal{R}_n^*(P)$ and define for $n \leq 16$ the set of chain starters, $\mathcal{S}^*(P)$, as follows:

$$\mathcal{S}^*(P) = \{x \mid x \in R_n^*(P)|_{\text{weight}<n/2}, \text{ and } \tau^{-1}(x) \notin R_n^*(P)|_{\text{weight}<n/2}\} \tag{4.2}$$

However, we don't yet have such a straightforward rule for picking necklace representatives of weight more than or equal to $n/2$, although we can always test whether a string is in $\mathcal{R}_n^*(P)$ for $n \leq 16$. One way is as follows. First consider a string y with weight greater than $n/2$. From y, apply τ^{-1} repeatedly until we encounter the string whose weight is $n-$weight(y). If that string we reached is in $R_n^*(P)|_{\text{weight}<n/2}$, then y is a necklace representative. Otherwise, y is not a necklace representative. If n is even, we also need to consider strings with weight equal to $n/2$. Consider a string y with weight$(y) = n/2$. If $\tau^{-1}(y)$ is in $R_n^*(P)|_{\text{weight}<n/2}$, then y is a necklace representative. Otherwise, as described in the previous section, we discard y temporarily and come back later after we have taken care of everything else to see if we still need y in our necklace representative set $R_n^*(P)$.

We are now hoping that (4.1) and (4.2) can also be applied to $n > 16$. We explore that in the next section.

4.4 Looking for chain starters for $n > 16$

When $n \leq 16$, we have simple rules as (4.1) and (4.2) to choose necklace representatives and chain starters with weight less than $n/2$ for SCD of $\mathcal{R}_n^*(P)$. We are also able to decide whether a string with weight more than or equal to $n/2$ is a representative or not, because we have found by hand all the necklace representatives that we need to construct SCD of $\mathcal{R}_n^*(P)$ for $n \leq 16$, as presented in Appendix A. The existing evidence verifies that the procedure that we use for testing will always work for $n \leq 16$.

In trying to extend such rules to general $n > 16$, we focus first on the applicability of (4.1) and (4.2). Then we need to determine whether each string with weight greater than or equal to $n/2$ is a necklace representative in this case. Therefore, even if both rules can be applied here, we will only be able to construct the "lower half" of our SCD, where every string has weight less than $n/2$. In this case, we would still need to determine what the rest of the SCD looks like, or whether a complete SCD exists or not. Note that, to look for SCD of $\mathcal{R}^*(P)$, it suffices to look for SCD's of $\mathcal{R}^*(\beta)$ for all $\beta \in P(n)$.

False Conjecture. *We can use* (4.1) *and* (4.2) *to choose necklace representatives and chain starters of weight less than $n/2$ for a partial SCD of $\mathcal{R}_n^*(\beta)$ for all $\beta \in P(n)$ for all n.*

Counterexample 1. Consider $\beta = (k, k, k)$. One counter-example for the block code is string 111000111000100000 which has weight 7. In this case $n = 18$. According to rules (4.1) and (4.2), the string is a chain starter. By applying τ to the string, we get string 111000111100100000 with weight 8 which is less than $n/2$ here. So it is supposed to be a necklace representative. But 111000111100100000 is not lexicographically largest among all its rotations with block code $(6, 6, 6)$, a contradiction.

Counterexample 2. Next, consider $\beta = (k, l, k, l)$. When $n = 22$, consider string 1100011110011000100000 with weight 9 that is less than $n/2$ here. According to rule (4.1), it is a necklace representative. By applying τ to the string, we get string 1100011110011100100000 which should be a representative. However, string 1100011110011100100000 is not lexicographically largest among all its rotations with block code $(5, 6, 5, 6)$, a contradiction.

4.5 Discoveries

Recall that, to look for an SCD of $\mathcal{R}^*(P)$, it suffices to look at SCD's of $\mathcal{R}^*(\beta)$ for every periodic block code $\beta \in P(n)$. We first prove that there exists an SCD of $\mathcal{R}_n^*(\beta)$ when $\beta = (k, k)$, where $k \geq 2$.

Theorem 2. *There exists a set of necklace representatives, $R_n^*(\beta)$, for necklaces in $\mathcal{N}_n^*(\beta)$, where $\beta = (k, k)$, $k \geq 2$, such that the subposet of $\mathcal{R}_n$ induced by $R_n^*(\beta)$, $\mathcal{R}_n^*(\beta)$, has an SCD with the chain cover property.*

Proof. Choose necklace representatives so that

$$R_n^*((k,k)) = \{x \mid \beta(x) = (k,k), \text{ and } x \text{ is lexicographically largest among all its rotations } \}.$$

Choose chain starters so that the set of chain starters is

$$S((k,k)) = \{s \mid s =< l, l > /(k,k), \text{ where } 1 \leq l \leq \lfloor k/2 \rfloor\}.$$

We still apply τ to generate chains. It is proved in Lemmas 4 and 5 that block codes are preserved for periodic block codes by both τ and τ^{-1}. Later we will show that the terminator of a chain that starts with string $< l, l > /(k, k)$, where $1 \leq l \leq \lfloor k/2 \rfloor$, is $< k - l, k - l > /(k, k)$.

Consider an element x of $R_n^*((k, k))$, where $x =< m, n > /(k, k)$ and $0 < n < m < k$.

First show that, by applying τ^{-1} to x repeatedly, we can reach a string $s =< l, l > /(k, k)$, where $1 \leq l \leq \lfloor k/2 \rfloor$, which is the chain starter of the chain containing x. (1) When $m+n \leq k$, the n 1's in the second block of x are all matched 1's. Thus, the only unmatched 1's in x are the m 1's in the first block of x. So after applying τ^{-1} to x for $m - n$ times, we encounter a string $< n, n > /(k, k)$, where $1 \leq n < \lfloor k/2 \rfloor$. (2) When $m+n > k$, there are $(m+n-k)$ 1's in the second block that are unmatched. Those 1's along with all 1's in the first block are the unmatched 1's in x. After applying τ^{-1} to x for $m + n - k$ times, we encounter string $x' =< m, k - m > /(k, k)$, which is in case (1), so applying τ^{-1} to x' for $2m - k$ times, we reach string $< k - m, k - m > /(k, k)$, where $1 \leq k - m \leq \lfloor k/2 \rfloor$.

Next show that, by applying τ to x repeatedly, we can reach string $< k-l, k-l > /(k, k)$, which is the chain terminator, when $< l, l > /(k, k)$ is the chain starter. (1) When $m + n \leq k$, as proved above, the starter of the chain containing x is $< n, n > /(k, k)$. In the first block of x, there are $(k - m - n)$ unmatched 0's. After applying τ to x for $(k - m - n)$ times, we encounter string $x' =< k - n, n > /(k, k)$, where there are no unmatched 0's in the first block. Then, applying τ to x' for $k - 2n$

times, we reach string $< k-n, k-n > /(k,k)$. (2) When $m+n > k$, as proved above, the starter of the chain containing x is $< k-m, k-m > /(k,k)$. In the first block of x, there are no unmatched 0's. Applying τ to x for $m-n$ times, we reach string $< m, m > /(k,k)$. Therefore, we've shown that the terminator of a chain starting with string $< l, l > /(k,k)$, where $1 \leq l \leq \lfloor k/2 \rfloor$, is $< k-l, k-l > /(k,k)$.

Therefore, it is proved that every element of $R_n^*((k,k))$ is on a chain, and such a chain is symmetric. It is proved by Greene and Kleitman that applying τ to two distinct n-bit strings can never result in the same string. Here, we apply τ to a restricted domain of n-bit strings, i.e. $R_n^*((k,k))$, so it holds that any two distinct chains are disjoint, i.e., if an element of $R_n^*((k,k))$ is on some chain, it can only be on exactly one chain.

We still need to prove that only elements of $R_n^*((k,k))$ are on the chains. Clearly,

$$
\begin{aligned}
N_n^*((k,k)) - R_n^*((k,k)) \quad = \quad & \{x \mid \beta(x) = (k,k), \text{ and } x \text{ is} \\
& \text{not lexicographically largest among all its rotations } \}.
\end{aligned}
$$

Assume $x \in N_n^*((k,k)) - R_n^*((k,k))$. Let $x =< n, m > /(k,k)$, where $0 < n < m < k$. (1) When $n+m \leq k$, there are no unmatched 1's in the second block of x. So the only unmatched 1's in x are all the 1's in the first block. Therefore, by applying τ^{-1} to x repeatedly, the number of 1's in the first block will decrease and will never equal the number of 1's in the second block, i.e., we will never reach a chain starter. (2) When $n+m > k$, there are no unmatched 0's in the first block. All the 0's in the second block are the unmatched 0's in x. By applying τ to x, the number 1's in the second block will increase and will never equal the number of 1's in the first block, i.e., we will never reach a chain terminator.

Therefore, only elements of $R_n^*((k,k))$ are on the chains.

Now we show the SCD has the chain cover property. Suppose a non-backbone chain C, in the SCD, starts with string $s =< l, l > /(k,k)$ and terminates with string $t =< k-l, k-l > /(k,k)$, where $2 \leq l \leq \lfloor k/2 \rfloor$. Identify a chain C' by string x' on C', which is obtained by changing the rightmost '1' in s to '0'. Thus, $x' =< l, l-1 > /(k,k)$, and x' is covered by s. Since $2 \leq l \leq \lfloor k/2 \rfloor$, $2l-1 < k$. So there are no unmatched 1's in the second block of x'. The unmatched 1's in x' are all the 1's in the first block. Therefore, $\tau^{-1}(x') =< l-1, l-1 > /(k,k)$, which is a chain starter. It follows that C' is in the same SCD as C. The terminator of C' is $t' =< k-l+1, k-l+1 > /(k,k)$. Since $2(k-l+1) > k$, there are unmatched 1's in the second block of t'. So $\tau^{-1}(t') =< k-l+1, k-l > /(k,k)$. We find that $\tau^{-1}(t')$ differs from t in the bit of position $k-l+1$, where the bit is a '1' in $\tau^{-1}(t')$ and is a '0' in t. Hence, $\tau^{-1}(t')$ covers t. Therefore, C' covers C. □

In the previous section, we found that the rules (4.1) and (4.2) will fail for some block codes when $n > 16$. Nevertheless, both rules work for periodic block codes (k,k), $k \geq 2$.

To extend Theorem 2, we now insert two 2's immediately to the left of both k's in block code (k, k), which gives us a new block code $(2, k, 2, k)$. It is easy to show that there exists an SCD of $\mathcal{R}_n^*(\beta)$ where $\beta = (2, k, 2, k)$. To make it more useful, we show the that this works for any numbers of k's.

Theorem 3. *If there exists an SCD of $\mathcal{R}_n^*(\beta)$, where $\beta = (b_1, b_2, \ldots, b_k)$, with the chain cover property, then there exists an SCD of $\mathcal{R}_n^*(\beta)$, where*

$$\beta = (2, b_1, 2, b_2, 2, \ldots, 2, b_k),$$

with the chain cover property.

Proof. Denote the SCD of $\mathcal{R}_n^*(\beta)$, where $\beta = (b_1, b_2, \ldots, b_k)$, as SCD_1. Let $b_1 + b_2 + \cdots + b_k = n$. Keeping the structure of SCD_1, we modify each necklace representative in SCD_1. For every representative $1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}$ in SCD_1, we put string '10' immediately to the left of each block and get the new string

$$101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k}.$$

We show that, after we modify SCD_1, the new partition, denoted as PAR_2, of strings into chains is an SCD with the chain cover property of $\mathcal{R}_{(2k+n)}^*(\beta')$, where $\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k)$. Note that $\|\beta'\| = 2k + n$. We also note that there exists a bijection from strings in SCD_1 to strings in PAR_2 as follows. In a string of SCD_1, the bit in position i, which is located within block $1^{a_j}0^{c_j}$ $(1 \leq j \leq k)$, is mapped to the bit in position $2j + i$ in the corresponding string of PAR_2.

First prove that, after the modification described above, all chains of SCD_1 are still chains, i.e., generated by function τ, in PAR_2. Assume that x is a non-terminator string on any chain C_1 in SCD_1. So the last '0' of x cannot be the only unmatched '0' in x. Let $x = 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}$. For every block $1^{a_j}0^{c_j}$ except the last block, the last '0' in the block is always matched with the first '1' in block $1^{a_{(j+1)}}0^{c_{(j+1)}}$ $(1 \le j \le k-1)$. Therefore, the leftmost unmatched '0' in x cannot be the last '0' in a block of x. Assume that the position of the leftmost unmatched '0' in x is u_1, which is located within block $1^{a_j}0^{c_j}$. So, the bit of position u_1 in $\tau(x)$ is a '1'. Then, after modification, the string in PAR_2, corresponding to x of C_1 in SCD_1, is

$$x' = 101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k}.$$

Also, denote the string in PAR_2, which corresponds to $y = \tau(x)$ of C_1 in SCD_1, as y'. Now, the bit '0' of position u_1 in x has a new position $u_1' = 2j + u_1$ in x'. Also, the bit '1' of position u_1 in y has a new position $u_1' = 2j + u_1$ in y'. Since x differs from $\tau(x)$ only in position u_1, after insertion of '10' blocks, x' differs from y' also in only one bit, in position $u_1' = 2j + u_1$, where the bit in x' is a '0' and the bit in y' is a '1'. Therefore, the covering relationship in chains still holds in PAR_2. We note that each '10' block, inserted in between two consecutive blocks $1^{a_j}0^{c_j}$ and $1^{a_{(j+1)}}0^{c_{(j+1)}}$ $(1 \le j \le k-1)$ of x, split the last '0' of $1^{a_j}0^{c_j}$ and the first '1' of $1^{a_{(j+1)}}0^{c_{(j+1)}}$, and this pair of '0' and '1' are no longer matched with each other. Instead, the last '0' of $1^{a_j}0^{c_j}$ is now matched with the '1' of the block '10' inserted, and the first '1' of

$1^{a_{(j+1)}}0^{c_{(j+1)}}$ is matched with the '0' of the block '10' inserted. After insertion of the '10' blocks, all unmatched 0's in x are still unmatched in x' and their order is not changed. It follows that the leftmost unmatched '0' in x' is in position u'_1, and it is changed to a '1' in y'. Therefore, τ still applies in PAR_2 to generate chains.

Now show that every chain in PAR_2 is symmetric. Obviously, every string in PAR_2 has the block code $\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k)$. For any chain C_1 in SCD_1, assume the number of 1's in the starter of C_1, s_1, is n_{1s} and the number of 1's in the terminator of C_1, t_1, is n_{1t}. Then, $n_{1s} + n_{1t} = n$. Assume that C_1 becomes C'_1 in PAR_2 after the modification as described above. Then, in PAR_2, the number of 1's, n'_{1s}, in the first element, s'_1 (modified s_1), of C'_1 is $k + n_{1s}$, and the number of 1's, n'_{1t}, in the last element, t'_1 (modified t_1), of C'_1 is $k + n_{1t}$. So, $n'_{1s} + n'_{1t} = 2k + n = \|\beta'\|$. Therefore, every chain in PAR_2 is symmetric.

Since all chain starters in SCD_1 are distinct, after insertion of '10' blocks, all chain starters in PAR_2 are also distinct. It was proved by Greene and Kleitman in [GK76] that applying τ to two distinct n-bit strings can never result in the same string. Here, we apply τ to a restricted domain of n-bit strings, so it holds that any two distinct chains are disjoint, i.e., no chains share the same strings.

Next, we show that the set of all strings in PAR_2 is a necklace representative set, $R^*_{(2k+n)}(\beta')$ for $N^*_{(2k+n)}(\beta')$, where $\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k)$. We prove this by contradiction. So assume that the set of all strings in PAR_2 is not a necklace representative set for $N^*_{(2k+n)}(\beta')$, which means either PAR_2 is missing a necklace representative or there are two strings in PAR_2 that represent the same necklace.

Assume that PAR_2 is still missing a necklace representative

$$y = 101^{d_1}0^{e_1}101^{d_2}0^{e_2}10\cdots 101^{d_k}0^{e_k}.$$

So, for any string

$$x = 101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k}$$

in PAR_2, no rotation of y is equal to x. It follows that no rotation of string

$$1^{d_1}0^{e_1}1^{d_2}0^{e_2}\cdots 1^{d_k}0^{e_k}$$

is equal to any string

$$1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}$$

in SCD_1, which means that SCD_1 is also missing a necklace representative, a contradiction.

Assume that there are two strings in PAR_2 that represent the same necklace:

$$x_1 = 101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k},$$

$$x_2 = 101^{a'_1}0^{c'_1}101^{a'_2}0^{c'_2}10\cdots 101^{a'_k}0^{c'_k}.$$

So some rotation of x_1 is equal to x_2. There are two possibilities. (1) The alternating '10' blocks in that rotation of x_1 match blocks $1^{a'_1}0^{c'_1}, 1^{a'_2}0^{c'_2}, \ldots, 1^{a'_k}0^{c'_k}$ in x_2. Then, it follows that blocks $1^{a_1}0^{c_1}, 1^{a_2}0^{c_2}, \ldots, 1^{a_k}0^{c_k}$ in x_1 are all '10's. Therefore, $x_1 = x_2 = (10)^{2k}$, and there are two strings $(10)^k$ in SCD_1, a contradiction. (2) The alternating '10' blocks in that rotation of x_1 match the alternating '10' blocks in x_2. Then, it follows that some rotation of $1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k}$ is equal to $1^{a'_1}0^{c'_1}1^{a'_2}0^{c'_2}\cdots 1^{a'_k}0^{c'_k}$, which means that there are two strings in SCD_1 that represent the same necklace, a contradiction.

Therefore, the strings in PAR_2 form a necklace representative set, $R^*_{(2k+n)}(\beta')$, for $N^*_{(2k+n)}(\beta')$, where $\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k)$.

It has been proved by far that PAR_2 is an SCD of $\mathcal{R}^*_{(2k+n)}(\beta')$, where

$$\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k).$$

We now show that the SCD, PAR_2, has the chain cover property. Because of the fixed position mapping from strings in SCD_1 to strings in PAR_2, the same position in two strings in SCD_1 are mapped to the same position in two corresponding strings in PAR_2. Consider two strings, x_1 and x_2, that differ in only one bit in SCD_1, where the bit in x_1 is a '0' and a '1' in x_2. Therefore, x_1 is covered by x_2. When mapped to strings, x'_1 and x'_2, respectively, in PAR_2, they still differ in only one bit, where the bit is a '0' in x'_1 and a '1' in x'_2. So, x'_1 is covered x'_2, which means the chain cover property still holds in PAR_2.

So we have obtained an SCD, PAR_2, of $\mathcal{R}^*_{(2k+n)}(\beta')$ with the chain cover property, where

$$\beta' = (2, b_1, 2, b_2, 2, \ldots, 2, b_k).$$

Choose necklace representatives for PAR_2 so that

$$R^*_{(2k+n)}((2, b_1, 2, b_2, 2, \ldots, b_{k-1}, 2, b_k)) = \{x \mid x = 101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k}, \text{ and } 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k} \in R^*_n((b_1, b_2, \ldots, b_k))\}.$$

Let $S^*_n((b_1, b_2, \ldots, b_k))$ denote the set of chain starters in SCD_1. Then, the set of chain starters in PAR_2 is defined as follows:

$$S^*_{(2k+n)}((2, b_1, 2, b_2, 2, \ldots, b_{k-1}, 2, b_k)) = \{x \mid x = 101^{a_1}0^{c_1}101^{a_2}0^{c_2}10\cdots 101^{a_k}0^{c_k}, \text{ and } 1^{a_1}0^{c_1}1^{a_2}0^{c_2}\cdots 1^{a_k}0^{c_k} \in S^*_n((b_1, b_2, \ldots, b_k))\}.$$

Function τ is used to generate chains. □

We also show that there exists an SCD of $\mathcal{R}_n^*(\beta)$ with the chain cover property, where β consists of all '2's.

Theorem 4. *There exists an SCD of $\mathcal{R}_n^*(\beta)$ with the chain cover property when $\beta = (2, 2, \ldots, 2)$.*

Proof. $N^*_n((2, 2, \ldots, 2)) = \{[1010\cdots 10]\}$, i.e., there is only one necklace in $N^*_n(\beta)$

when $\beta = (2, 2, \ldots, 2)$. Choose string $1010 \cdots 10$ to be the only necklace representative in $R_n^*((2, 2, \ldots, 2))$. That string by itself forms an SCD of $\mathcal{R}_n^*((2, 2, \ldots, 2))$. Since the string itself is the only chain in the SCD, the SCD of $\mathcal{R}_n^*((2, 2, \ldots, 2))$ has the chain cover property. □

Since the subproblem of looking for an SCD of $\mathcal{R}_n^*(P)$ with the chain cover property for all n can be divided into even smaller subproblems of looking for SCD's of $\mathcal{R}_n^*(\beta)$ with the chain cover property for all $\beta \in P(n)$ for all n, by the results of Theorems 2, 3, and 4, we have solved part of the subproblem of looking for an SCD of $\mathcal{R}_n^*(P)$ with the chain cover property for *all* n.

For example, when $n = 18$, the periodic block codes are $(9, 9)$, $(6, 6, 6)$, $(2, 7, 2, 7)$, $(3, 6, 3, 6)$, $(4, 5, 4, 5)$, $(2, 4, 2, 4, 2, 4)$, $(3, 3, 3, 3, 3, 3)$, $(2, 2, 2, 2, 2, 2, 2, 2, 2)$. By Theorem 2, there exists an SCD of $\mathcal{R}_{18}^*(\beta)$ with the chain cover property for $\beta = (9, 9)$. By Theorem 3, to solve for block codes $(2, 7, 2, 7)$ and $(2, 4, 2, 4, 2, 4)$, it suffices to solve for block codes $(7, 7)$ and $(4, 4, 4)$, respectively. Again, by Theorem 2, there exists an SCD of $\mathcal{R}_{14}^*(\beta)$ with the chain cover property for $\beta = (7, 7)$. Therefore, there exists an SCD of $\mathcal{R}_{18}^*(\beta)$ with the chain cover property for $\beta = (2, 7, 2, 7)$. Since $\|(4, 4, 4)\| = 12 < 16$, there exists an SCD of $\mathcal{R}_{12}^*(\beta)$ with the chain cover property for $\beta = (4, 4, 4)$. Thus, there also exists an SCD of $\mathcal{R}_{18}^*(\beta)$ with the chain cover property for $\beta = (2, 4, 2, 4, 2, 4)$. By Theorem 4, there exists an SCD of $\mathcal{R}_{18}^*(\beta)$ with the chain cover property for $\beta = (2, 2, 2, 2, 2, 2, 2, 2, 2)$. Now that we have solved part of the problem of looking for SCD of $\mathcal{R}_{18}^*(P)$ with the chain cover property, the only periodic block codes still left for $n = 18$ to be solved are $(6,6,6)$, $(3,6,3,6)$, $(4,5,4,5)$, and $(3,3,3,3,3,3)$.

Chapter 5

Symmetric Independent Families of Curves from SCD's

5.1 Introduction

This chapter shows how to obtain a rotationally symmetric independent family of curves with the minimum possible number of regions from an SCD of a necklace-representative poset with the chain cover property. We also conclude that for every $n \leq 16$, there exists a symmetric independent family of n curves with the minimum possible number of regions.

5.2 Definitions

5.2.1 Independent families of curves and Venn diagrams

Following [Grü99], an *independent family of curves* is a family of n simple closed (Jordan) curves, $\{C_1, C_2, ..., C_n\}$, intersecting in finitely many points such that each of the 2^n sets

$$X_1 \cap X_2 \cap ... \cap X_n \tag{5.1}$$

is not empty, where X_j denotes either the interior or the exterior of C_j. Furthermore, if each of the 2^n sets in (5.1) is also connected, then the independent family of curves is a *Venn diagram*.

Define a *region* as a maximal nonempty subset of $(\mathbb{R} - \bigcup_{i=1}^{n} C_i)$, where $\mathbb{R}$ denotes the plane which the collection of curves $\{C_1, C_2, ..., C_n\}$ is in.

We say that an independent family of n curves is *rotationally symmetric*, or *symmetric*, if there is a point, p, in the plane such that rotating any curve around p by an angle of $2\pi i/n$ radians, for every i $(1 \leq i \leq n-1)$, maps the curve onto one of the other curves.

5.2.2 Cover edges and chain cover graphs

Consider an SCD $\mathcal{C}$ with the chain cover property for poset $\mathcal{A} = (A, \leq)$. Let π be the chain cover mapping. Then every non-root chain $C \in \mathcal{C}$ is covered by another chain

$\pi(C) \in \mathcal{C}$, in that $start(C)$ covers an element, s, of $\pi(C)$, and $term(C)$ is covered by an element, t, of $\pi(C)$. Connect $start(C)$ and s with an edge e_s, and connect $term(C)$ and t with an edge e_t. Call e_s and e_t *cover edges*.

A *chain cover graph* $G(\mathcal{C}, \pi)$ is a graph whose vertices are the elements of A, and whose edges are the covering edges in the chains in $\mathcal{C}$ along with the cover edges e_s and e_t for all non-root chains in $\mathcal{C}$.

5.2.3 Planar graphs and a planar embedding

According to [Wes01], a graph G is *planar* if there is a drawing of the graph without crossing edges. We call such a drawing a *planar embedding* of G. In this paper, we only talk about the planar embedding of chain cover graphs.

5.2.4 Duals and geometric duals

Define a *face* in a planar embedding as a maximal connected region of the plane, not containing any points from the embedding. Let P denote a planar embedding of a planar graph G. The *dual* of P is defined as the graph with vertex set V and edge set E, defined as follows. Each element of V corresponds to a face f of P. Assume e is the boundary, in P, of two (not necessarily distinct) faces, f and g, which correspond to vertices v and w in the dual, respectively. Then, an element $e*$ of E connects vertices v and w in the dual, and corresponds to the edge e in P. Note that the dual of a planar embedding of a planar graph is also planar, which can be seen from the construction below.

As defined in [Wes01], the *geometric dual*, P^*, of P is the planar embedding of the dual of P obtained as follows. Fix a point (vertex) v_f in the interior of each face f of P. Pick a special point on each edge of P. For each face f of P and each edge e on its boundary, we connect v_f and the special point on e to form a "half-edge" of P^*. We make sure that the half-edges incident with v are internally disjoint. Two half-edges of P^* meet at the special point of each edge e of P to form the edge e^* of P^*. Then each face of P^* contains exactly one vertex of P. Since P is connected, P is isomorphic to $(P^*)^*$. We will use this method to construct independent families of curves later in this chapter.

5.3 A planar embedding of a chain cover graph

Assume we have an SCD, $\mathcal{C}$, with the chain cover property, and the chain cover mapping is denoted as π. Now we have the chain cover graph $G(\mathcal{C}, \pi)$. We want to come up with a planar embedding of $G(\mathcal{C}, \pi)$ before preceding to the next section where we will eventually obtain a symmetric independent family of curves. We show the way of constructing a planar embedding of a chain cover graph in the following lemma.

Lemma 10. (Lemma 1 from [GKS02]) *Let $\mathcal{C}$ be an SCD with the chain cover property for poset $\mathcal{A} = (A, \leq)$, and let π be a chain cover mapping for $\mathcal{C}$. The chain cover graph $G(\mathcal{C}, \pi)$ has a planar embedding $P(\mathcal{C}, \pi)$.*

Proof. We prove this lemma by describing how to construct such a planar embedding. Chains are all vertical and arranged such that vertices with the same rank are always in the same horizontal level. Successive vertices on every chain are separated by one unit.

First, for every chain C in the chain cover graph, order the chains that are covered by C, if any, from shortest chain to longest chain. Let the root chain be the current chain for now.

(1) Move the current chain, C_1, one unit to the right of the rightmost chain of the existing embedding. (If the existing embedding is empty, move C_1 so that it appears leftmost.) If there are chains covered by C_1, let the first chain that is covered by C_1, which is the shortest among all chains covered by C_1, be the current chain, and go back to the beginning of step (1). Otherwise, if C_1 is not the root chain, let the chain that covers C_1 be the current chain. If C_1 is the root chain, the embedding is done.

(2) If there are remaining chains, i.e. chains not yet embedded, covered by the current chain C_1, let the first one of those chains be the current chain, and go back to step (1). Otherwise, if C_1 is not the root chain, let the chain that covers C_1 be the current chain, and go back to the beginning of step (2). If C_1 is the root chain, then

the embedding is done.

By the way of embedding the chain cover graph, crossing of edges can always be avoided, since all chains are symmetric and are arranged parallel so that the same rank are in the same horizontal level. Therefore, the embedding of the chain cover graph $G(\mathcal{C}, \pi)$ is a planar embedding.

□

5.4 Constructing a symmetric independent family of curves

Lemma 5 of [GKS02] proves for prime n that if there exists a set R_n of necklace representatives for $\{0,1\}^*$ such that the subposet $\mathcal{R}_n = (R_n, \leq)$ of $\mathcal{B}_n$ has an SCD with the chain cover property, then a symmetric Venn Diagram for n sets can be constructed.

We now extend the scope of n to include composite numbers for which we won't be able to construct symmetric Venn diagrams even if such SCD's exist. Instead, we prove in the following lemma a weaker argument, that is, there exists a rotationally symmetric independent family of curves that is obtained from such an SCD. In addition, we discuss the new topic on the lower bound for number of regions in symmetric independent families of curves for all n.

Lemma 11. (Grünbaum [Grü99]) *The lower bound for the number of regions in an symmetric independent family of n curves is*

$$M(n) = 2 + n * (C_n - 2),$$

where C_n is the number of distinct n-bit necklaces.

Proof. In an independent family of n curves, each region is associated and labeled with a subset of $\{1, 2, \ldots, n\}$, and every subset of $\{1, 2, \ldots, n\}$ is a label of a region in the independent family of curves. We say that all labels that can be transformed into each other via cyclic permutations form an *equivalence class*. For example, when $n = 4$, labels $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, and $\{1, 2, 3\}$ form an equivalence class. When an independent family of n curves is *rotationally symmetric*, every equivalence class must be represented in n or a multiple of n regions in the independent family of curves, except (1) one equivalence class associated with the unbounded region when the center of symmetry is on some curves, or (2) two equivalence classes associated with the unbounded region and the region containing the center of symmetry when the center of symmetry is not on any curves.

We find that the equivalence classes mentioned above can be viewed as necklaces. In the first case, the number of regions is at least $1 + n * (C_n - 1)$, where C_n is the number of distinct n-bit necklaces. In the second case, the number of regions is at least $2 + n * (C_n - 2)$. Since

$$1 + n * (C_n - 1) \geq 2 + n * (C_n - 2),$$

when $n \geq 1$, we find the lower bound for the number of regions in a symmetric independent family of n curves is

$$M(n) = 2 + n * (C_n - 2),$$

where C_n is the number of distinct n-bit necklaces. □

Now we show that a symmetric independent family of n curves with the minimum possible number of regions can be constructed from an SCD of $\mathcal{R}_n$ with the chain cover property.

Lemma 12. *For all n, if there exists an SCD of $\mathcal{R}_n$ with the chain cover property, then there exists a rotationally symmetric independent family of n curves, with number of regions that reaches the lower bound $M(n)$.*

Proof. Assume that such an SCD exists.

Let $\mathcal{C}$ be an SCD in $\mathcal{R}_n$ and let π be a chain cover mapping for $\mathcal{C}$. By Lemma 10, the chain cover graph $G(\mathcal{C}, \pi)$ has a planar embedding with C_n vertices. For example, when $n = 4$, $C_4 = 6$. If $R_4 = \{0000, 1000, 1100, 1110, 1111, 1010\}$, there exists an SCD of $\mathcal{R}_4$, $\mathcal{C}$, with chain cover mapping π. A planar embedding of the chain cover graph $G(\mathcal{C}, \pi)$ is shown in Figure 5.1, where bold edges indicate chains and chain cover edges are light.

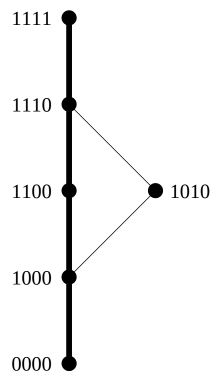

Figure 5.1: Embedding of the chain cover graph for a necklace-representative subposet of $\mathcal{B}_4$.

Partition the plane into n pie slices, or wedges, around a point p. Embed the chain cover graph in one of the wedges and let vertex 1^n coincide with p. Extend the other end of the chain cover graph so that vertex 0^n goes to infinity. Rotate the planar embedding of the chain cover graph about p by an angle of $2\pi/n$ radians for $n-1$ times, so that every wedge has a copy of the chain cover graph embedded in it. Denote the collection of all the copies of the planar embedding of the chain cover graph as a planar embedding P_n. For example, when $n = 4$, Figure 5.2 shows P_4 obtained from the planar embedding of the chain cover graph for $\mathcal{R}_4$ shown in Figure 5.1, along with the wedge boundaries.

It is clear that there are $2 + n * (C_n - 2)$ vertices in P_n. Note that when n is prime, all vertices have distinct labels, and the labels include all subsets of $[n]$. Since $C_n = 2 + (2^n - 2)/n$ in this case, the total number of vertices in P is actually 2^n for prime n. However, when n is composite, we will find, in P, vertices with the same

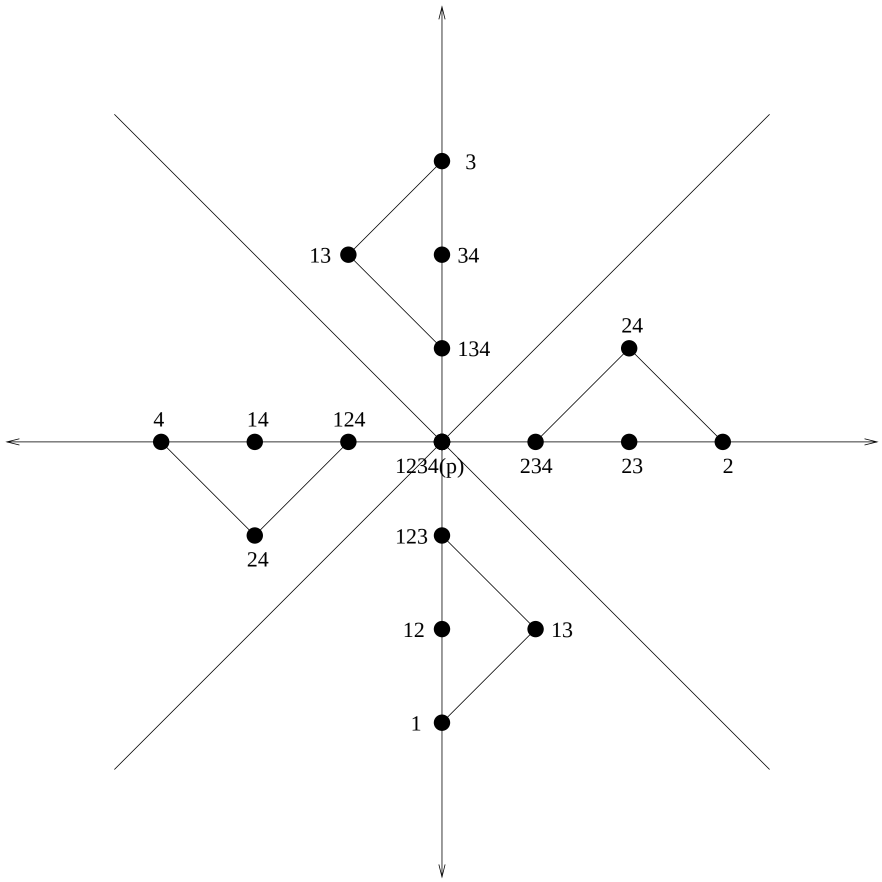

Figure 5.2: The planar graph P obtained by rotating the embedding of Figure 5.1 about vertex p by $(2\pi i)/4$ for each i where $0 \leq i \leq 3$.

label, i.e. the same subset of $[n]$.

Now we prove that, by taking the geometric dual, P_n^*, of P_n as described in Section 5.2.4, the resulting graph, P_n^*, is an independent family of n curves. Later in this paragraph, we will show how to draw the geometric dual so that the independent family of curves is rotationally symmetric. Define a j-edge e_j in P_n as an edge connecting a vertex containing j and a vertex not containing j in P_n. Then, e_j

corresponds to an edge e_j^* in P_n^*. Let G_j and $\overline{G_j}$ be the subgraphs of P_n induced by the set of vertices containing j and the set of vertices not containing j, respectively. Obviously, there are no j-edges in either G_j or $\overline{G_j}$. Removing all the j-edges will cause separation of P_n into G_j and $\overline{G_j}$, so the set of j-edges is a disconnecting edge set. Adding any one j-edge back will reconnect the graphs since there will be a path from 0^n to 1^n. Therefore, the set of j-edges in P_n is a minimal disconnecting edge set. By Theorem 6.1.14 of [Wes01], the edges e_j^* in P_n^* form a simple closed curve, Θ_j. Furthermore, sets containing j, corresponding to vertices in G_j, and sets not containing j, corresponding vertices in $\overline{G_j}$, are on different sides of the simple closed curve Θ_j in P_n^*. Since vertex 0^n is always at infinity in P_n, sets not containing j are always on the outside of Θ_j while sets containing j are always inside Θ_j in P_n^*. In the way of constructing P_n^* as described before, every face of P_n^* contains one vertex of P_n, and thus is nonempty. Therefore, P_n^* is the collection of simple closed curves $\{\Theta_1, \Theta_2, \ldots, \Theta_n\}$, which is an independent family of curves. To make the independent family of curves rotationally symmetric, we first take the geometric dual of a copy of the planar embedding of the chain cover graph in one wedge, and then rotate the resulting geometric dual around p by an angle of $2\pi/n$ radians for $n-1$ times. When taking the geometric dual, to choose vertices corresponding to the faces of P_n bounded by consecutive rotations of the chain cover graph, we first pick a point on a wedge boundary. We then rotate the point around p by an angle of $2\pi/n$ radians for $n-1$

times, so that the resulting points fall on all other wedge boundaries. Then, all the curves, in rotationally symmetric P_n^*, intersect on each wedge boundary at only one point (vertex), which corresponds to the face that the boundary runs across in P_n. Figure 5.3 shows the geometric dual, constructed upon the embedding in Figure 5.2, which is rotationally symmetric. By the way of constructing geometric duals, all edges in P_n^* in each wedge are disjoint, and there is no way for edges of P_n^* in different wedges to cross. Thus, there are no extra crosses of curves than what are specified in the definition of P_n^*, which means every face of P_n^*, or every region of the independent family of curves, contains exactly one vertex of P_n. Thus, the total number of regions in the diagram is the number of vertices of P_n, i.e., $2+n*(C_n-2)$, and 2^n specifically for prime n.

We still need to prove that in the rotationally symmetric embedding of P_n^*, each of the n rotations of Θ_1 about p by $2\pi i/n$ radians where $0 \leq i \leq n-1$ coincides with one of the curves $\Theta_1, \Theta_2, \ldots, \Theta_n$. Start from a copy of the planar embedding of the chain cover graph in one wedge, can call it P_n^0. Then go clockwise, and call the next copy of the planar embedding P_n^1. Repeat that until we reach P_n^{n-1}. We note that if we rotate the j-edges in P_n^i clockwise about p by $2\pi/n$ radians, they become the $(j-1)$-edges in $P_n^{i+1 \pmod n}$. Therefore, rotating Θ_j clockwise about p by $2\pi/n$ radians gives Θ_{j-1} if $j > 1$, or Θ_n if $j = 1$. The disjoint union of $\{\Theta_1, \Theta_2, \ldots, \Theta_n\}$ is a rotationally symmetric independent family of n curves (and specifically a symmetric Venn diagram

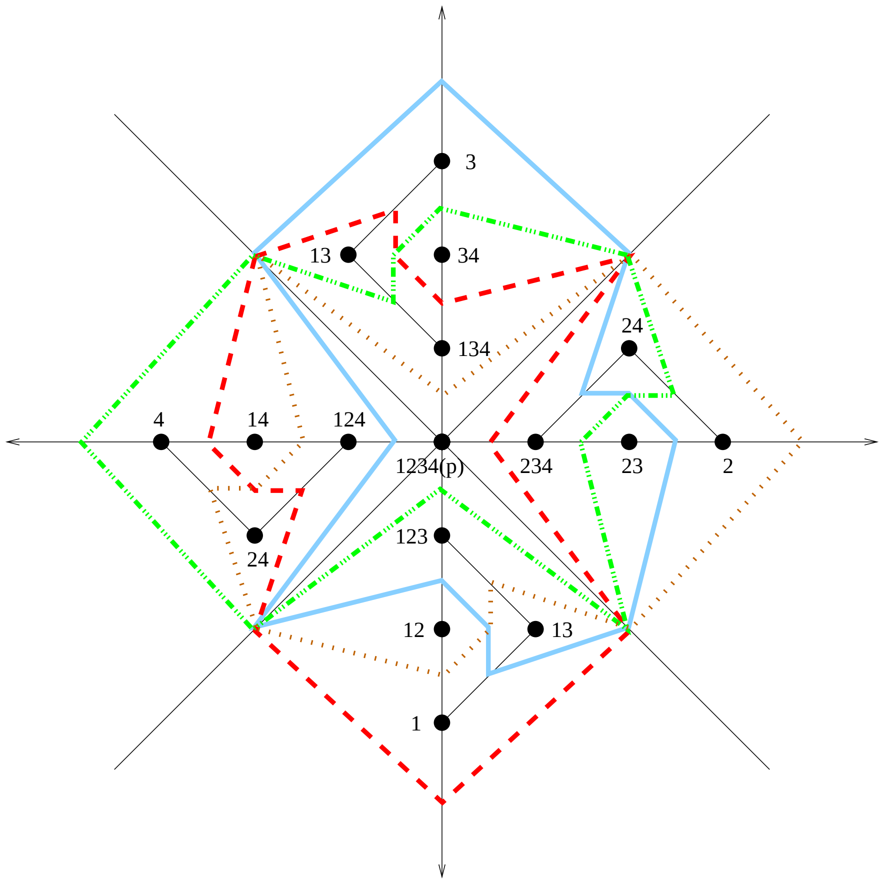

Figure 5.3: The embedding of the geometric dual of the graph of Figure 5.2, with rotational symmetry as described in the proof of Lemma 12.

when n is prime). Figure 5.4 shows the rotationally symmetric independent family of curves for $n = 4$, obtained from the geometric dual constructed in Figure 5.3.

□

By Theorem 1 combined with Lemma 12, we get the main result of this paper:

Theorem 5. *For $n \leq 16$, there exists a rotationally symmetric independent family of n curves, with number of regions that reaches the lower bound $M(n) = 2+n*(C_n-2)$,*

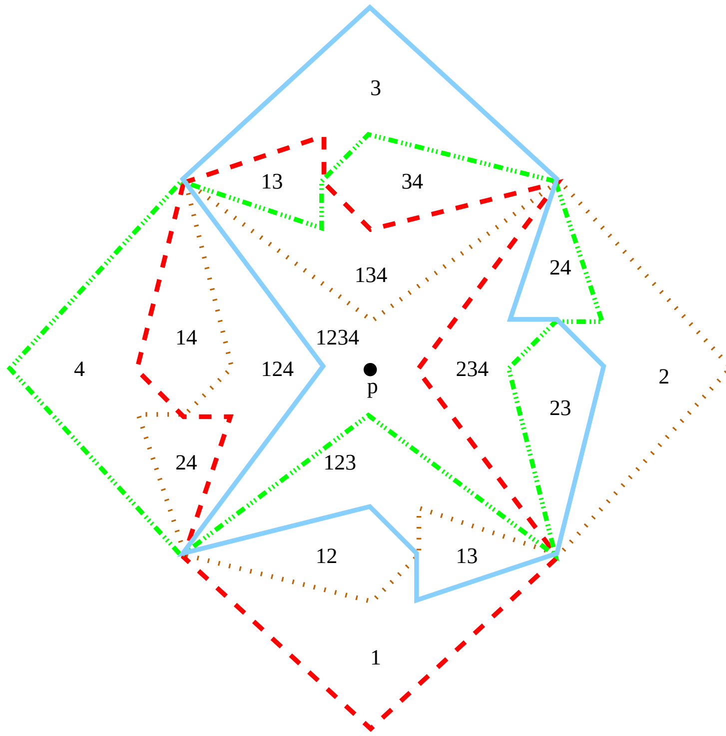

Figure 5.4: The rotationally symmetric independent family of $n = 4$ curves. Rotating any one about p by $2\pi i/n$ radians results in one of the curves.

where C_n is the number of distinct necklaces of n bits.

□

Chapter 6

Conclusions

In this thesis, we have studied the geometric problem of looking for a rotationally symmetric independent family of curves with the minimum possible number of regions. To solve it, we first looked at the combinatorial problem of searching for a symmetric chain decomposition of a necklace-representative poset with the chain cover property.

It is proved in [GKS02] that there exists an SCD of a necklace-representative poset with the CCP for all prime n. So we have focused on composite n in this research. In Chapter 3, we have shown that the answer to this problem is yes for all $n \leq 16$. We chose necklace representatives to be strings with the minimum block codes among all their rotations. When n is composite, necklaces are either periodic or aperiodic. We have proved that there exists an SCD ($SCD_n(A)$) of the necklace-representative subposet induced by representatives for *aperiodic* necklaces with the CCP for all composite n. We showed that, for every periodic block code of representatives for periodic necklaces, the longest chain of necklace representatives with that block code is always covered by a chain in $SCD_n(A)$. Therefore, we could reduce the problem to a much smaller one, i.e., looking for SCD's with the CCP in subposets of $\mathcal{N}_n$ associated with *periodic* block codes.

In Chapter 4, we found, for all $n \leq 16$, that there exists a set of necklace-representative set, R_n, such that $\mathcal{R}_n$ has an SCD with the CCP. We also discovered that there always exists an SCD with the CCP for subposets of $\mathcal{N}_n$ associated with periodic block codes of patterns (k, k) and $(2, 2, \ldots, 2)$ for all n, and the problem of solving for subposets of $\mathcal{N}_n$ associated with block codes $(2, b_1, 2, b_2, 2, \ldots, 2, b_k)$ can be reduced to solving for subposets of $\mathcal{N}_n$ associated with block codes $(b_1, b_2, \ldots, b_k)$.

In Chapter 5, we proved that an SCD of a necklace-representative poset with the CCP can always be transformed into a symmetric independent family of curves with the minimum possible number of regions. As a result, we showed that there exists such a symmetric independent family of curves for all $n \leq 16$.

Hence, the open questions are (1) whether there exists an SCD of necklace-representative poset with the CCP for all composite $n \geq 18$; and (2) whether there exists a symmetric independent family of curves with the minimum possible number of regions for all composite $n \geq 18$.

So, as mentioned earlier in this chapter, future studies should focus on looking for SCD's with the CCP within subposets of $\mathcal{N}_n$ associated with periodic block codes.

In Chapter 4, we have found SCD's with the CCP for $n = 18$ for subposets of $\mathcal{N}_n$ associated with block codes $(9, 9)$, $(2, 7, 2, 7)$, $(2, 4, 2, 4, 2, 4)$, $(2, 2, 2, 2, 2, 2, 2, 2, 2)$. The only block codes still left to be solved for $n = 18$ are $(6, 6, 6)$, $(3, 6, 3, 6)$, $(4, 5, 4, 5)$, $(3, 3, 3, 3, 3, 3)$. We would recommend that as the starting point for future research.

List of References

[BS96] Girish S. Bhat and Carla D. Savage. Balanced Gray codes. *Electron. J. Combin.*, 3(1):Research Paper 25, approx. 11 pp. (electronic), 1996.

[Dav01] Tom Davis. Pólya's counting theory, September 2001. http://www.geometer.org/mathcircles.

[Edw98] A. W. F. Edwards. Seven-set Venn diagrams with rotational and polar symmetry. *Combin. Probab. Comput.*, 7(2):149–152, 1998.

[GK76] Curtis Greene and Daniel J. Kleitman. Strong versions of Sperner's theorem. *J. Combinatorial Theory Ser. A*, 20(1):80–88, 1976.

[GKS02] Jerrold Griggs, Charles E. Killian, and Carla D. Savage. Venn diagrams and symmetric chain decompositions in the Boolean lattice. *Electronic Journal of Combinatorics*, submitted, 2002.

[Grü75] Branko Grünbaum. Venn diagrams and independent families of sets. *Math. Mag.*, 48:12–23, 1975.

[Grü92] Branko Grünbaum. Venn diagrams. II. *Geombinatorics*, 2(2):25–32, 1992.

[Grü99] Branko Grünbaum. The search for symmetric Venn diagrams. *Geombinatorics*, 8(4):104–109, 1999.

[Ham02] Peter Hamburger. Doodles and doilies, non-simple symmetric Venn diagrams. *Discrete Math.*, 257(2-3):423–439, 2002. Kleitman and combinatorics: a celebration (Cambridge, MA, 1999).

[Hen63] D. W. Henderson. Venn diagrams for more than four classes. *American Mathematical Monthly*, 70:424–426, 1963.

[NW78] Albert Nijenhuis and Herbert S. Wilf. *Combinatorial algorithms*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1978. For computers and calculators, Computer Science and Applied Mathematics.

[Rus97] Frank Ruskey. A survey of Venn diagrams. *Electron. J. Combin.*, 4(1):Dynamic Survey 5, 10 HTML documents (electronic), 1997.

[Wes01] Douglas B. West. *Introduction to graph theory - 2nd ed.* Prentice Hall Inc., Upper Saddle River, NJ, 2001.

[Wes03] Mark R. N. Weston. On symmetry in Venn diagrams and independent families. Master's thesis, University of Victoria, Department of Computer Science, 2003.

Appendix A

Chain cover graphs for $\mathcal{R}_n^*(\beta)$ for all $\beta \in P(n)$ when $n \leq 16$

A.1 $n = 4$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2,2)$	1

Table A.1: $n = 4$.

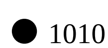

Figure A.1: Embedding of the chain cover graph for block code $(2,2)$.

A.2 $n = 6$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2, 2, 2)$	1
$(3, 3)$	$(2^2 - 2)/2 + 2 = 3$

Table A.2: $n = 6$.

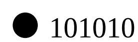

Figure A.2: Embedding of the chain cover graph for block code $(2, 2, 2)$.

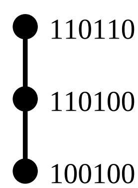

Figure A.3: Embedding of the chain cover graph for block code $(3, 3)$.

A.3 $n = 8$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2, 2, 2, 2)$	1
$(4, 4)$	$(3^2 - 3)/2 + 3 = 6$

Table A.3: $n = 8$.

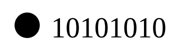

Figure A.4: Embedding of the chain cover graph for block code $(2, 2, 2, 2)$.

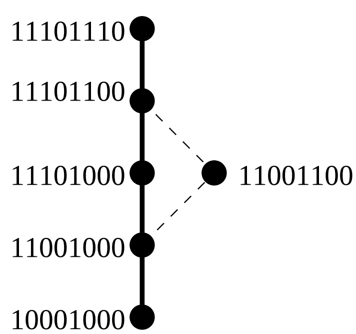

Figure A.5: Embedding of the chain cover graph for block code $(4, 4)$.

A.4 $n = 9$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(3, 3, 3)$	$(2^3 - 2)/3 + 2 = 4$

Table A.4: $n = 9$.

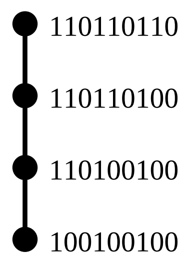

Figure A.6: Embedding of the chain cover graph for block code $(3, 3, 3)$.

A.5 $n = 10$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2, 2, 2, 2, 2)$	1
$(2, 3, 2, 3)$	$(2^2 - 2)/2 + 2 = 3$
$(5, 5)$	$(4^2 - 4)/2 + 4 = 10$

Table A.5: $n = 10$.

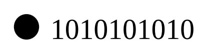

Figure A.7: Embedding of the chain cover graph for block code $(2, 2, 2, 2, 2)$.

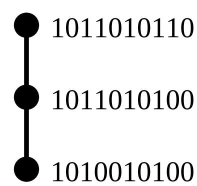

Figure A.8: Embedding of the chain cover graph for block code $(2, 3, 2, 3)$.

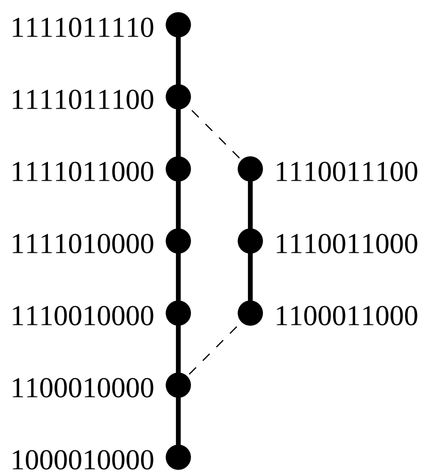

Figure A.9: Embedding of the chain cover graph for block code $(5, 5)$.

A.6 $n = 12$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2,2,2,2,2,2)$	1
$(2,4,2,4)$	$(3^2 - 3)/2 + 3 = 6$
$(3,3,3,3)$	$(2^4 - 2^2)/4 + C_6((3,3)) = 6$
$(4,4,4)$	$(3^3 - 3)/3 + 3 = 11$
$(6,6)$	$(5^2 - 5)/2 + 5 = 15$

Table A.6: $n = 12$.

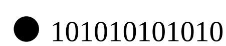

Figure A.10: Embedding of the chain cover graph for block code $(2,2,2,2,2,2)$.

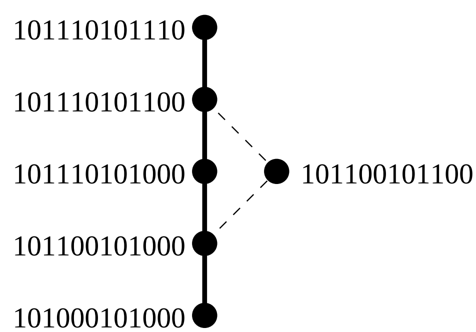

Figure A.11: Embedding of the chain cover graph for block code $(2,4,2,4)$.

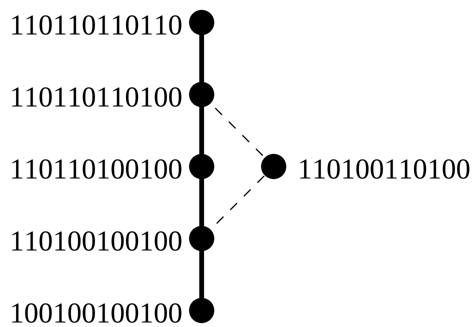

Figure A.12: Embedding of the chain cover graph for block code $(3, 3, 3, 3)$.

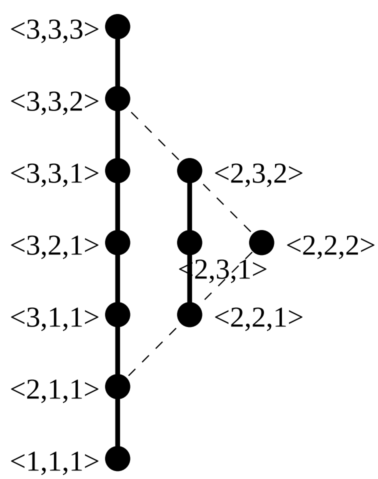

Figure A.13: Embedding of the chain cover graph for block code $(4, 4, 4)$. (Vertices are labeled with one-count codes to fit the limited space.)

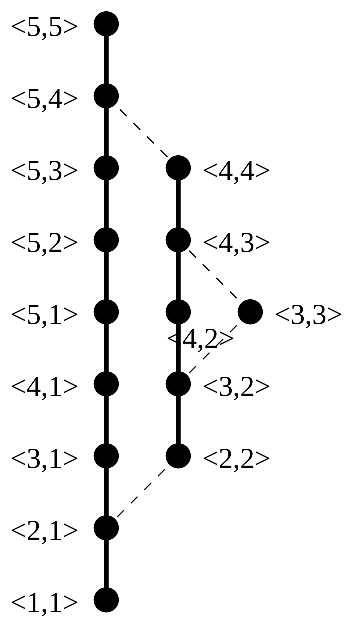

Figure A.14: Embedding of the chain cover graph for block code $(6, 6)$. (Vertices are labeled with one-count codes to fit the limited space.)

A.7 $n = 14$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2,2,2,2,2,2,2)$	1
$(2,2,3,2,2,3)$	$(2^2-2)/2+2=3$
$(2,5,2,5)$	$(4^2-4)/2+4=10$
$(3,4,3,4)$	$(2*3*2*3-2*3)/2+2*3=21$
$(7,7)$	$(6^2-6)/2+6=21$

Table A.7: $n = 14$.

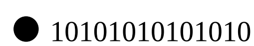

Figure A.15: Embedding of the chain cover graph for block code $(2,2,2,2,2,2,2)$.

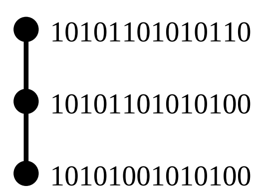

Figure A.16: Embedding of the chain cover graph for block code $(2,2,3,2,2,3)$.

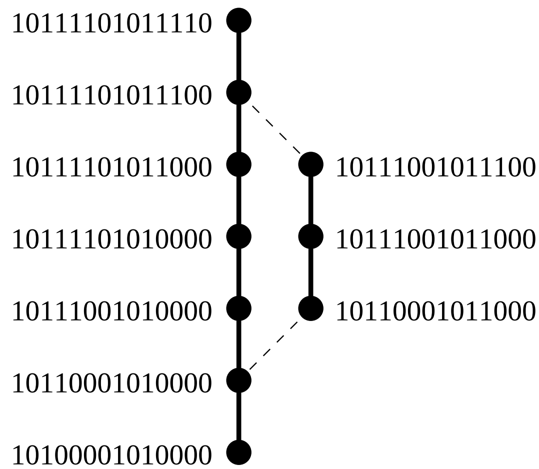

Figure A.17: Embedding of the chain cover graph for block code $(2, 5, 2, 5)$.

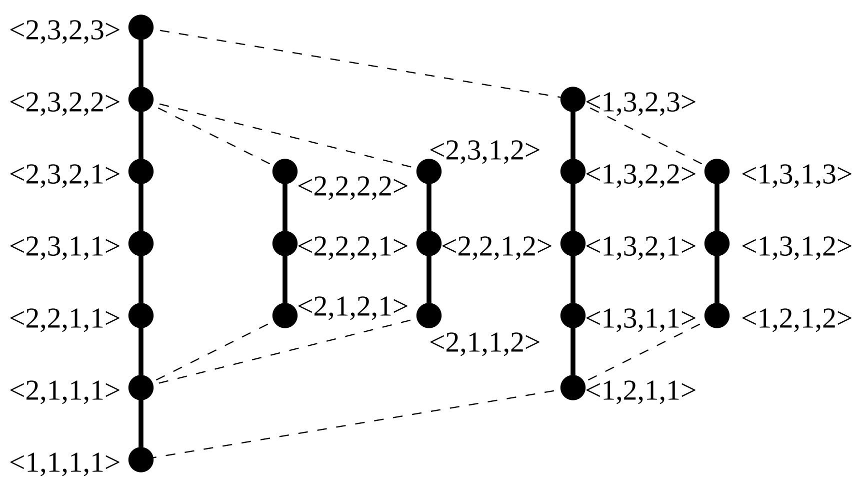

Figure A.18: Embedding of the chain cover graph for block code $(3, 4, 3, 4)$. (The graph is expanded horizontally and vertices are labeled with one-count codes to fit the limited space.)

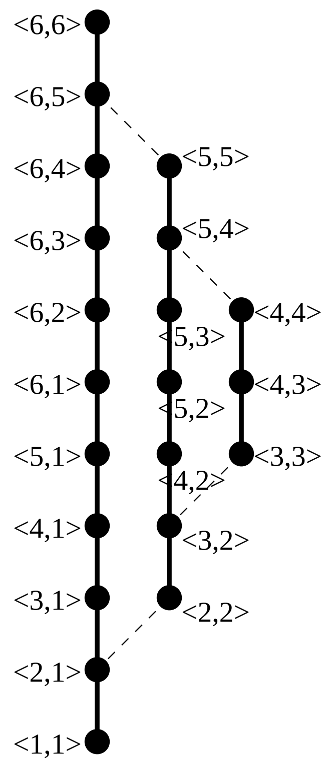

Figure A.19: Embedding of the chain cover graph for block code $(7, 7)$. (Vertices are labeled with one-count codes to fit the limited space.)

A.8 $n = 15$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2,3,2,3,2,3)$	$(2^3 - 2)/3 + 2 = 4$
$(3,3,3,3,3)$	$(2^5 - 2)/5 + 2 = 8$
$(5,5,5)$	$(4^3 - 4)/3 + 4 = 24$

Table A.8: $n = 15$.

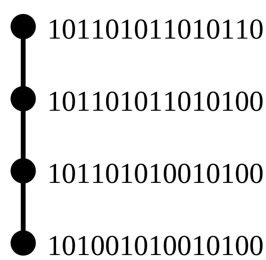

Figure A.20: Embedding of the chain cover graph for block code $(2,3,2,3,2,3)$.

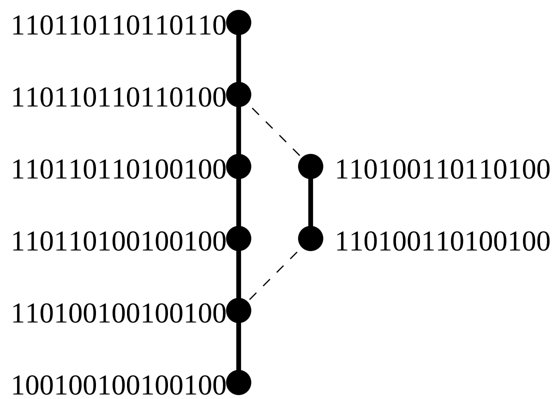

Figure A.21: Embedding of the chain cover graph for block code $(3, 3, 3, 3, 3)$.

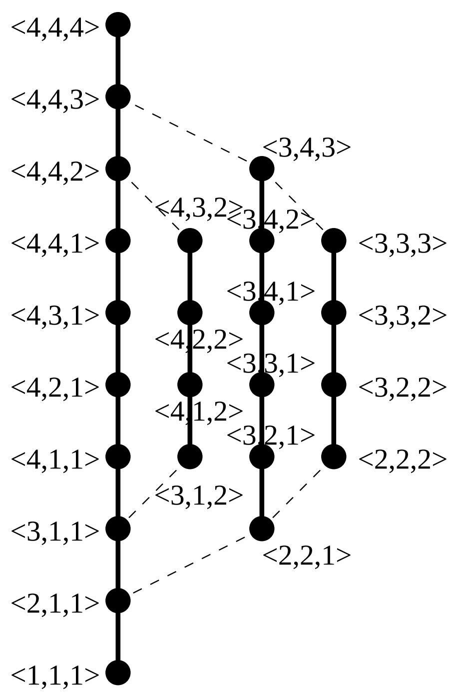

Figure A.22: Embedding of the chain cover graph for block code $(5, 5, 5)$. (Vertices are labeled with one-count codes to fit the limited space.)

A.9 $n = 16$

Periodic block code β	Number of necklaces $C_n(\beta)$
$(2,2,2,2,2,2,2,2)$	1
$(2,2,4,2,2,4)$	$(3^2-3)/2+3=6$
$(2,3,3,2,3,3)$	$(2^4-4)/2+4=10$
$(2,6,2,6)$	$(5^2-5)/2+5=15$
$(4,4,4,4)$	$(3^4-3^2)/4+C_8((4,4))=24$
$(8,8)$	$(7^2-7)/2+7=28$
$(3,5,3,5)$	$(2*4*2*4-2*4)/2+2*4=36$

Table A.9: $n = 16$.

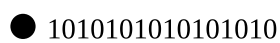

Figure A.23: Embedding of the chain cover graph for block code $(2,2,2,2,2,2,2,2)$.

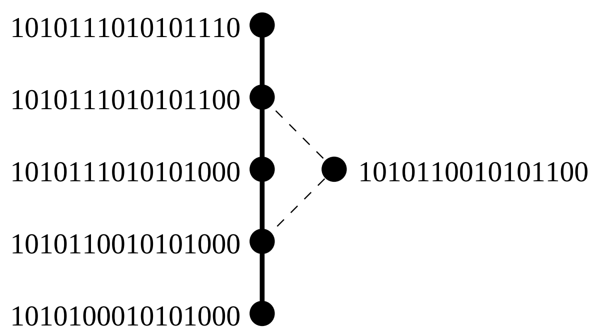

Figure A.24: Embedding of the chain cover graph for block code $(2, 2, 4, 2, 2, 4)$.

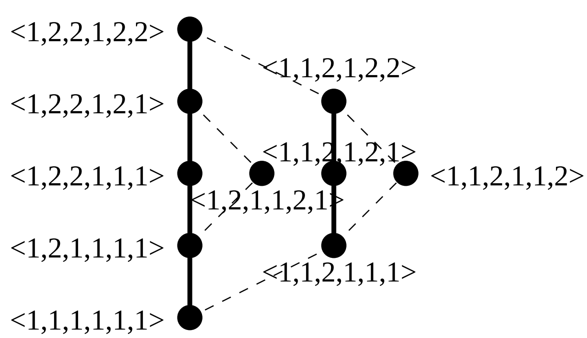

Figure A.25: Embedding of the chain cover graph for block code $(2, 3, 3, 2, 3, 3)$. (Vertices are labeled with one-count codes to fit the limited space.)

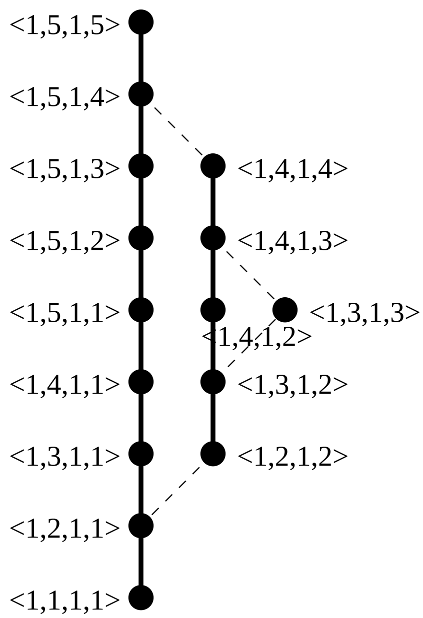

Figure A.26: Embedding of the chain cover graph for block code $(2, 6, 2, 6)$. (Vertices are labeled with one-count codes to fit the limited space.)

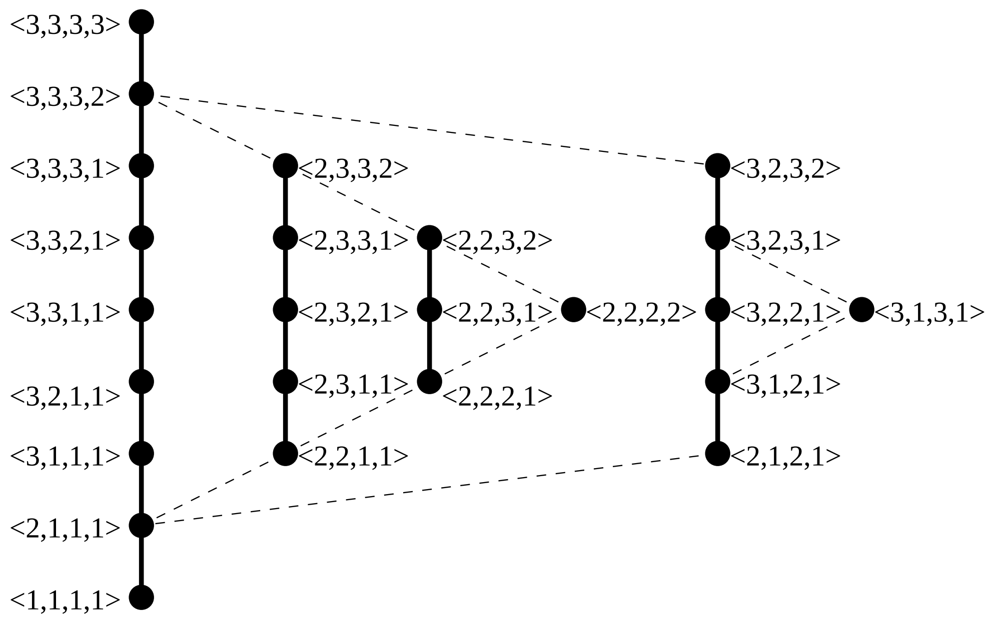

Figure A.27: Embedding of the chain cover graph for block code $(4, 4, 4, 4)$. (The graph is expanded horizontally and vertices are labeled with one-count codes to fit the limited space.)

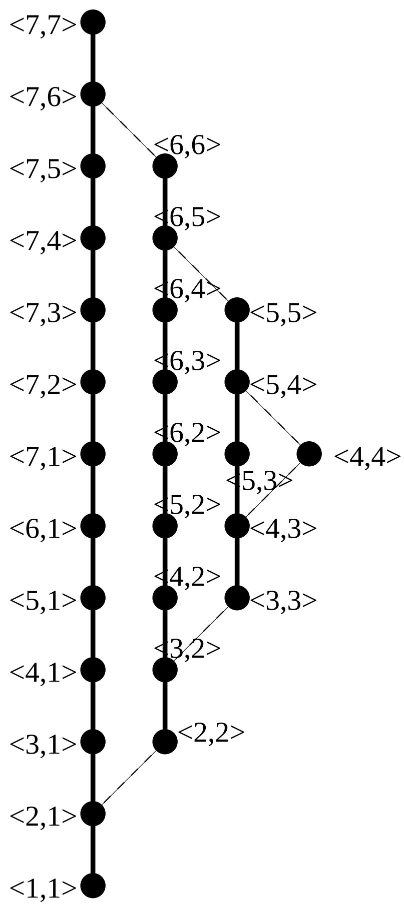

Figure A.28: Embedding of the chain cover graph for block code $(8, 8)$. (Vertices are labeled with one-count codes to fit the limited space.)

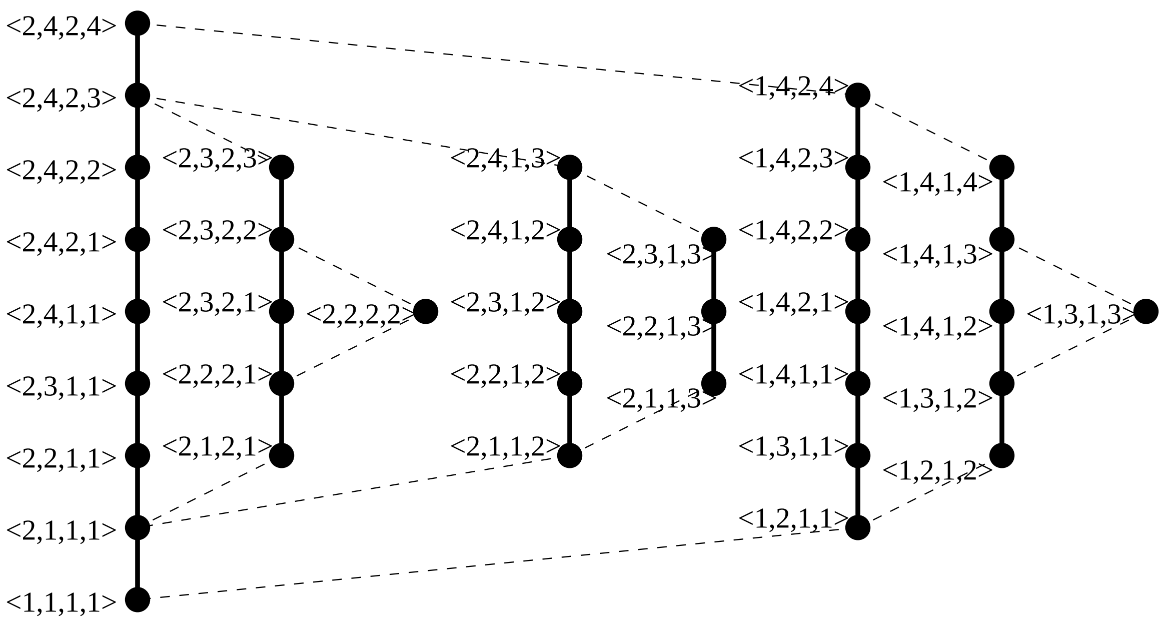

Figure A.29: Embedding of the chain cover graph for block code $(3, 5, 3, 5)$. (The graph is expanded horizontally and vertices are labeled with one-count codes to fit the limited space.)

Appendix B

Symmetric independent family of curves for $n = 4, 6, 8,$ and 9

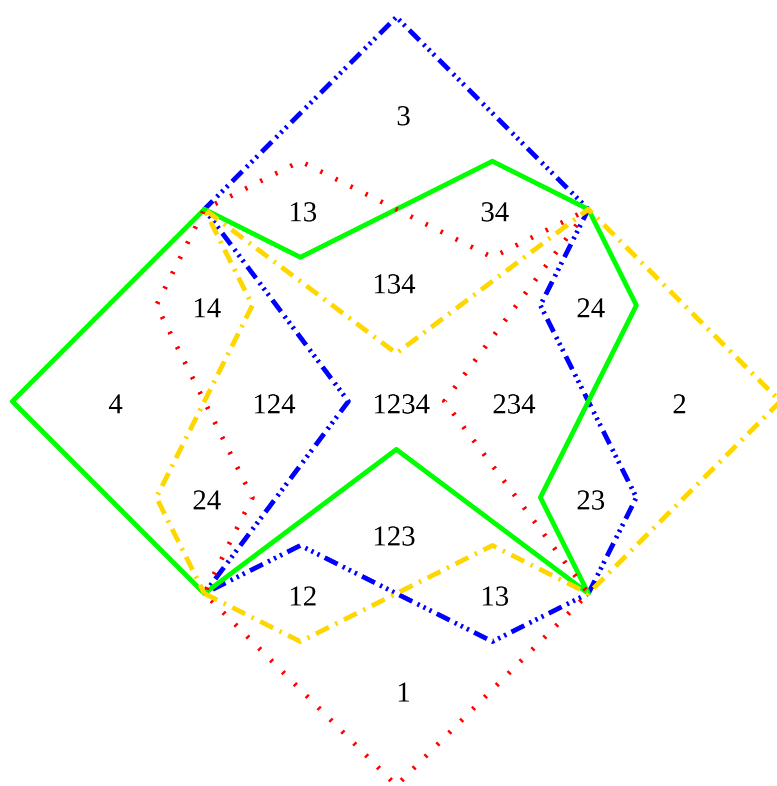

Figure B.1: Rotationally symmetric independent family of $n = 4$ curves with $M(4) = 18$ (lower bound for $n = 4$) regions.

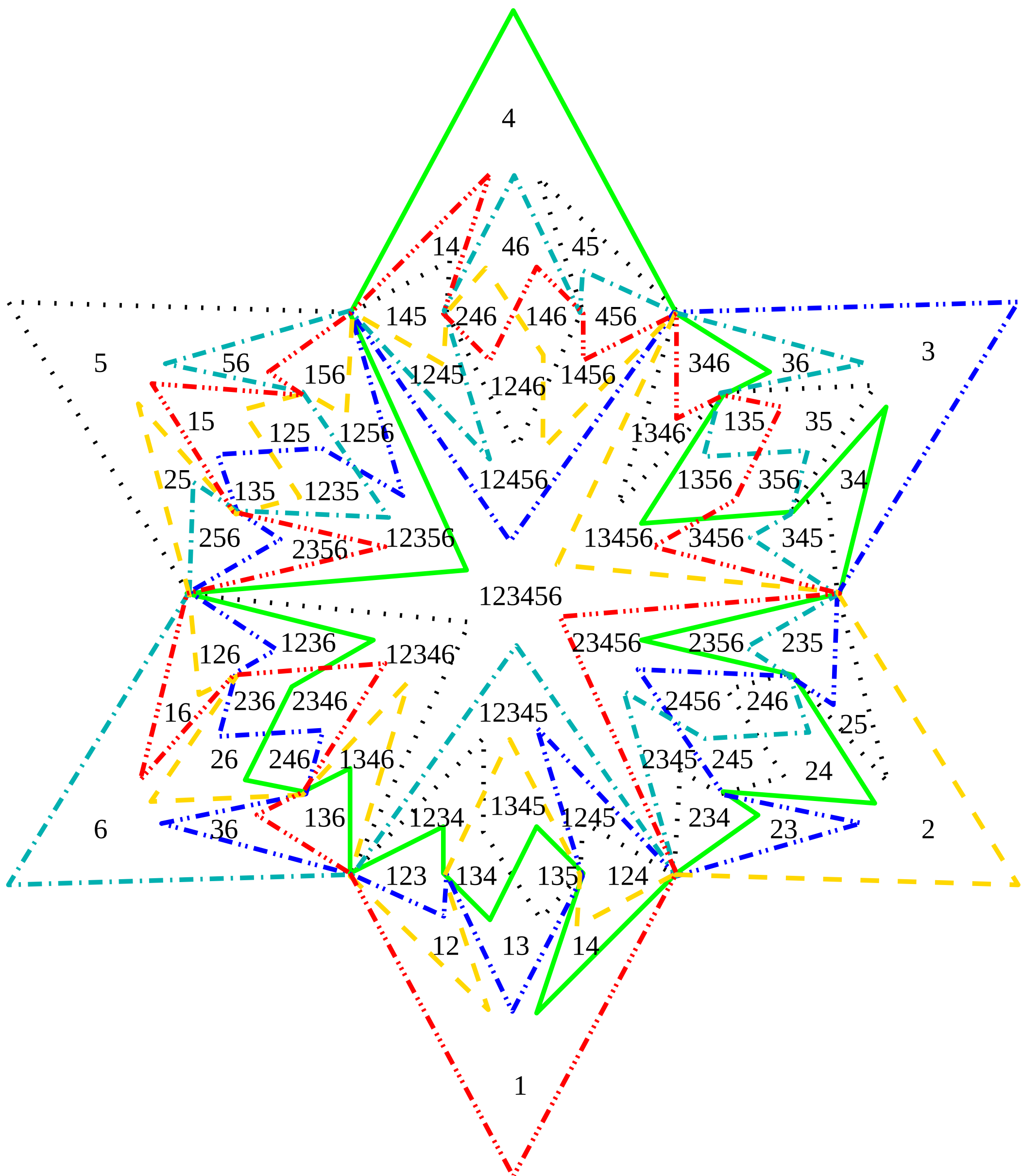

Figure B.2: Rotationally symmetric independent family of $n = 6$ curves with $M(6) = 74$ (lower bound for $n = 6$) regions.

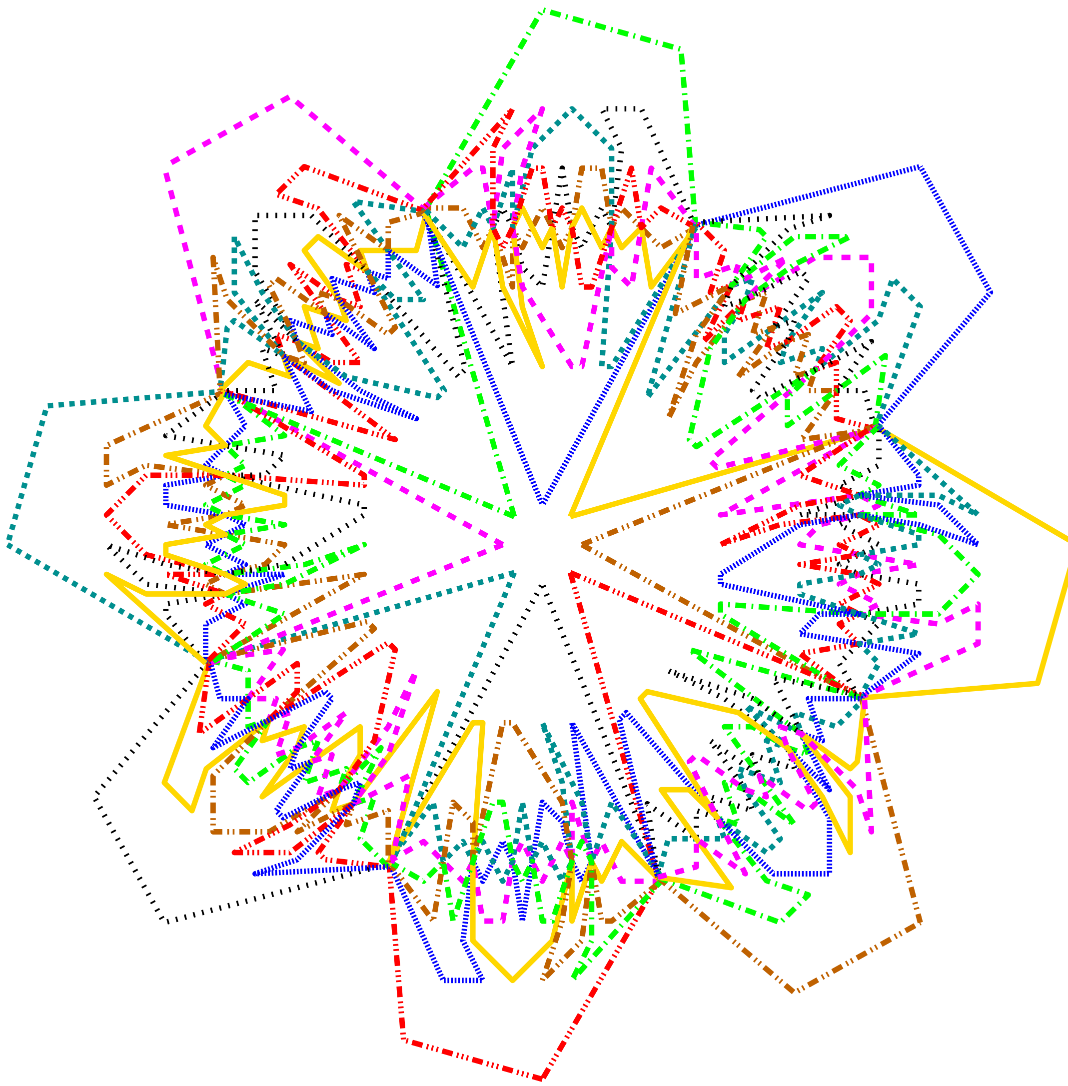

Figure B.3: Rotationally symmetric independent family of $n = 8$ curves with $M(8) = 274$ (lower bound for $n = 8$) regions.

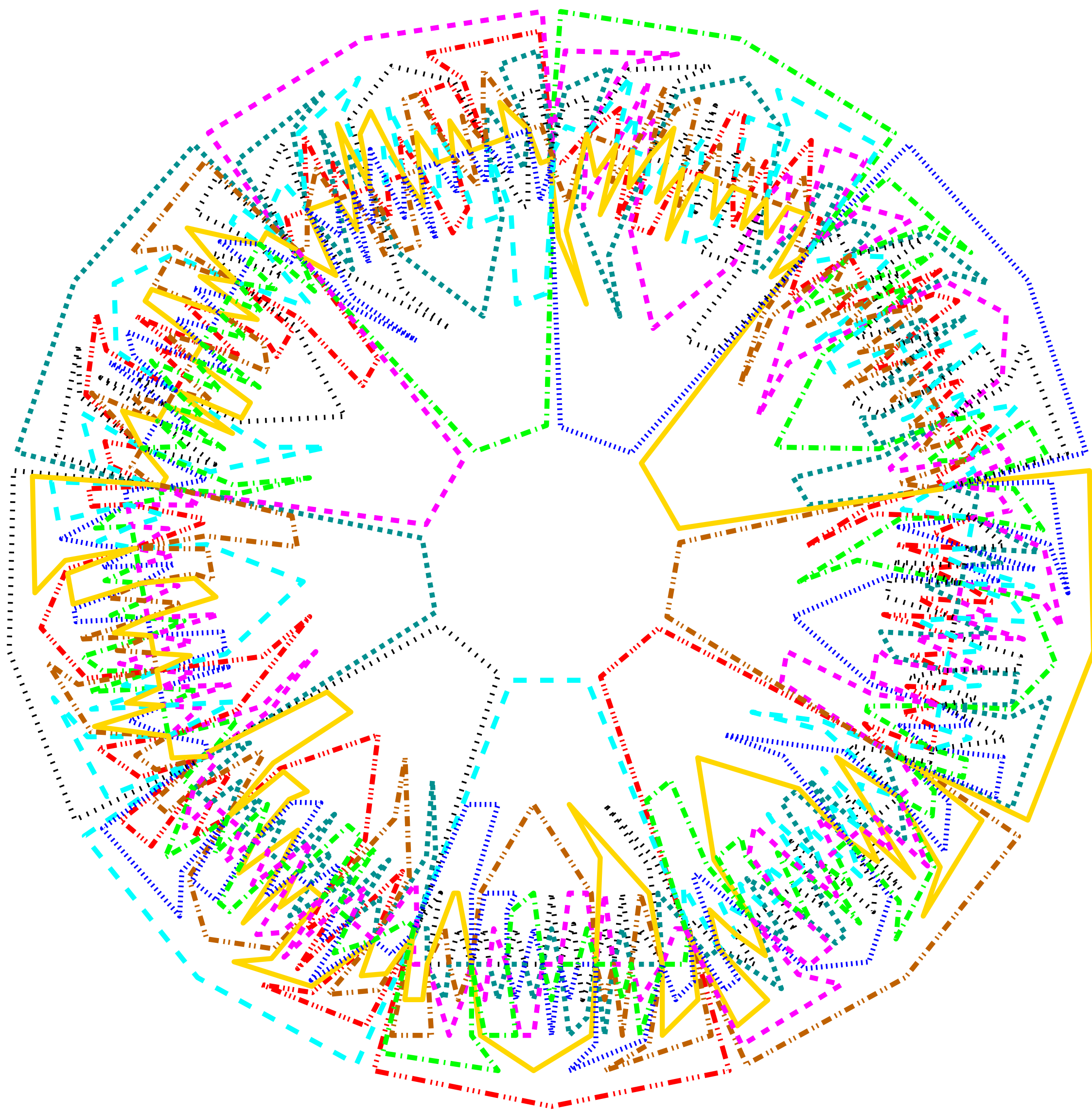

Figure B.4: Rotationally symmetric independent family of $n = 9$ curves with $M(9) = 524$ (lower bound for $n = 9$) regions.