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Axisymmetric elastoplasticity with body forces by the boundary element method

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ABSTRACT

This work is concerned with the application of the Boundary Element Method to axisymmetric problems with inelastic behavior of materials, including body force loads. The generalized body forces considered are the centrifugal load due to rotation about a fixed axis and the effect of a steady-state thermal loading. The results of some illustrative examples are compared with the results obtained using the Finite Element Method, including the elastic solution.

INTRODUCTION

The analysis of three-dimensional (3-D) problems with inelastic behavior of materials in which the geometry and loading conditions are axisymmetric can accommodate generalized body forces as well. In this work the Boundary Element Method (BEM) is used to treat this kind of problems and the generalized body forces considered are centrifugal load due to rotation about a fixed axis and the effect of a steady-state thermal loading. In these cases, domain integrals have to be computed. However, the Galerkin vector corresponding to the three-dimensional Kelvin fundamental solution is employed to transform domain integrals into surface ones. Following the procedure presented by *Danson* [1], who introduced the Kernels of the generalized body force contributions in terms of Legendre functions, a limitation is produced by a singularity in the rotation axis. In the present work, the Kernels of the generalized body force contributions are presented in terms of complete elliptic integrals of superior order, then the formulation is capable of handling not only hollow bodies of revolution but also solids, in which the boundary intercepts the rotation axis. So, it is possible to analyze thick walled reactor vessels where the thermal stresses are important. To consider the contributions due to elastoplasticity it is necessary to divide the domain of the body into internal cells for integration, but in this case simple cells (triangular cells) can be used, restricted to the plasticity region only.

BASIC EQUATIONS

The total strain rate for inelastic problems can be represented by

$$\dot{\varepsilon}_{ij}=\dot{\varepsilon}_{ij}^{e}+\dot{\varepsilon}_{ij}^{T}+\dot{\varepsilon}_{ij}^{p} = \frac{1}{2}\left[\dot{u}_{i,j}+\dot{u}_{j,i}\right] \quad (1)$$

where $\dot{\varepsilon}_{ij}^{e}$, $\dot{\varepsilon}_{ij}^{T}$ and $\dot{\varepsilon}_{ij}^{p}$ are the elastic, thermal and plastic parts of the total strain rate and $\dot{u}_i$ is the displacement rate components. In this work, the comma represent space derivatives and the dot is associated with pure incremental quantity instead of time derivative.

The equilibrium conditions within the domain Ω of the body can be written in rate form as

$$\dot{\sigma}_{ij,j}+\dot{b}_i=0 \quad (2)$$

where $\dot{\sigma}_{ij}$ and $\dot{b}_i$ represent the stress rate components and the body force rate, respectively.

The stress-strain equations, for an initial strain approach, including the plastic and temperature rates are given by

$$\dot{\sigma}_{ij}=2G\dot{\varepsilon}_{ij}+\frac{2G\nu}{1-2\nu}\dot{\varepsilon}_{kk}\,\delta_{ij}-2G\dot{\varepsilon}_{ij}^{p}-\frac{2G\nu}{1-2\nu}\dot{\varepsilon}_{kk}^{p}\,\delta_{ij}-2G\left(\frac{1+\nu}{1-2\nu}\right)\alpha\,\dot{T}\,\delta_{ij} \quad (3)$$

where ν is the Poisson ratio, δ_{ij} is the kronecker delta, G é is the shear modulus, α is the coefficient of linear thermal expansion and $\dot{T}$ is the difference in temperature.

Equilibrium at the boundary Γ of the body requires the satisfaction of the following stress boundary conditions

$$\dot{p}_i=\dot{\sigma}_{ij}\,n_j \quad (4)$$

where $\dot{p}_i$ are the equivalent tractions and n_j represents the direction cosines of the outward normal to the boundary surface.

For axisymmetric problems, the strain rate expressions remain valid if referred to a cylindrical coordinate system (r, θ, z). Due to symmetry of the geometry, they simplify to:

$$\dot{\varepsilon}_{rr}=\frac{\partial \dot{u}_r}{\partial r}\quad ;\quad \dot{\varepsilon}_{\theta\theta}=\frac{\dot{u}_r}{r}\quad ;\quad \dot{\varepsilon}_{zz}=\frac{\partial \dot{u}_z}{\partial z}\quad ;\quad \dot{\varepsilon}_{z\theta}=\dot{\varepsilon}_{\theta z}=\frac{1}{2}\frac{\partial \dot{u}_\theta}{\partial z}$$

$$\dot{\varepsilon}_{r\theta}=\dot{\varepsilon}_{\theta r}=\frac{1}{2}\left[\frac{\partial \dot{u}_\theta}{\partial r}-\frac{\dot{u}_\theta}{r}\right]\quad ;\quad \dot{\varepsilon}_{rz}=\dot{\varepsilon}_{zr}=\frac{1}{2}\left[\frac{\partial \dot{u}_z}{\partial r}+\frac{\partial \dot{u}_r}{\partial z}\right] \quad (5)$$

FUNDAMENTAL SOLUTION FOR AXISYMMETRIC LOADS

Starting with the (3-D) fundamental solution due to Kelvin in the Cartesian system, one can make use of a Cylindrical coordinate system to express the fundamental displacements $\hat{\underset{\sim}{u}}^{*}(p,q)$, as shown in details by *Cisternas* [2],

$$u^{*}_{\alpha\beta}(p,q)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{u}^{*}_{\alpha\beta}(p,q)\,d\theta\quad ;\quad \alpha,\beta=r,\theta,z\ ;\ \theta=\theta q-\theta p \quad (6)$$

where $r=r(p,q)$ is the distance between the source point p and the field point q related with the Cylindrical coordinate system.

INTEGRAL EQUATIONS

Considering a closed domain Ω with boundary Γ, the starting integral equation to axisymmetric elastoplasticity referred to a cylindrical coordinate system is as follows,

$$C_{ij}(p)\,\dot{u}_j(p) + 2\pi \int_\Gamma p^*_{ij}(p,q)\,\dot{u}_j(q)\ rq\ d\Gamma(q) = 2\pi \int_\Gamma u^*_{ij}(p,q)\,\dot{p}_j(q)\ rq\ d\Gamma(q) +$$
$$+ 2\pi \int_\Omega u^*_{ij}(p,q)\,\dot{b}_j(q)\ rq\ d\Omega(q) + 2\pi \int_\Omega \sigma^*_{\alpha\beta i}(p,q)\,\dot{\varepsilon}^p_{\alpha\beta}(q)\ rq\ d\Omega(q) +$$
$$+ 2\pi\, 2G\alpha \left(\frac{1+\nu}{1-2\nu}\right) \int_\Omega u^*_{ij,j}(p,q)\,\dot{T}(q)\ rq\ d\Omega(q) \quad ; \quad i,j = r,z \quad ; \quad \alpha,\beta = r,\theta,z \tag{7}$$

where $u^*_{ij}(p,q)$ and $p^*_{ij}(p,q)$ are the components of the fundamental solution. In equation (7) the factor $2\pi rq$ appears into integrals because integration with respect to θ was already accomplished and consequently, Γ and Ω correspond to a (2-D) region only; $C_{ij}(p)$ is equal $\delta_{ij}/2$ at a smooth boundary and in other cases including rp = 0, $C_{ij}(p)$ is defined as presented by *Telles* [3].

Equation (7) can be differentiated to obtain the stresses at internal points on condition that the space derivatives be taken with respect to the coordinates of the source point p. In this case $C_{ij}(p) = \delta_{ij}$ and the final equation can be expressed as

$$\dot{\sigma}_{\alpha\beta}(p) = 2\pi \int_\Gamma u^*_{\alpha\beta k}(p,q)\,\dot{p}_k(q)\ rq\ d\Gamma(q) - 2\pi \int_\Gamma p^*_{\alpha\beta k}(p,q)\,\dot{u}_k(q)\ rq\ d\Gamma(q) +$$
$$+ 2\pi \int_\Omega u^*_{\alpha\beta k}(p,q)\,\dot{b}_k(q)\ rq\ d\Omega(q) + 2\pi \int_\Omega \sigma^*_{\alpha\beta\gamma\rho}(p,q)\,\dot{\varepsilon}^p_{\gamma\rho}(q)\ rq\ d\Omega(q) +$$
$$+ 2\pi\, 2G\alpha \left(\frac{1+\nu}{1-2\nu}\right) \int_\Omega u^*_{\alpha\beta,\beta k}(p,q)\,\dot{T}(q)\ rq\ d\Omega(q) + h_{\alpha\beta}\left(\dot{\varepsilon}^p_{\gamma\rho}\right) \quad k = r,z\ ;\alpha,\beta,\gamma,\rho = r,\theta,z \tag{8}$$

The domain integral involving the plastic strains is in the principal value sense and the terms of $h_{\alpha\beta}$ were discussed by *Telles* [3].

GENERALIZED BODY FORCES

In order to simplify this presentation, the generalized body force integrals of equation (7) will be represented as

$$B_i = 2\pi \int_\Omega u^*_{ij}(p,q)\ b_j(q)\ rq\ d\Omega(q) + 2\pi\, 2G\alpha \left(\frac{1+\nu}{1-2\nu}\right) \int_\Omega u^*_{ij,j}(p,q)\ T(q)\ rq\ d\Omega(q) = B_i^C + B_i^T \tag{9}$$

To transform the domain integrals into surface integrals, the Galerkin tensor G^*_{ij} is employed, as shown by Fung [4] and it is related to the 3-D fundamental solution $u^*_{ij}(p,q)$ by the following expression:

$$u^*_{ij} = G^*_{ij,kk} - \frac{1}{2(1-\nu)} G^*_{ik,kj} \quad ; \quad G^*_{ij} = \frac{1}{8\pi G}\, r\, \delta_{ij} \tag{10}$$

KERNELS FOR CENTRIFUGAL LOADS

Considering a body rotating with angular velocity ω_i, about fixed axes. The problem is equivalent to a prescribed body force of the form

$$b_j = g_{ij}\, x_i \quad ; \quad g_{ij} = \rho\left(\delta_{ij}\,\omega_k\,\omega_k - \omega_i\,\omega_j\right) \tag{11}$$

where ρ is the mass density.

Making use of the divergence theorem, the first domain integral of equation (9) can be written as follows,

$$B_i^C = 2\pi \int_\Gamma P_i \; rq \; d\Gamma \; ; \; i = r,\theta,z \tag{12}$$

where

$$P_r = \frac{2\,\rho\, w^2\, c1}{\pi}\left\{ \frac{c2}{rr}\int_0^{\pi/2}\frac{d\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}} + \frac{c3}{rr}\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}}\,d\phi \; + \right.$$

$$\left. - \, c4\; rr\int_0^{\pi/2}\left[1 - fk^2\cos^2\phi\right]^{1/2}\cos 2\phi\; d\phi + \frac{c2}{rr}\int_0^{\pi/2}\frac{\cos 4\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}}\,d\phi \right\} \tag{13}$$

$$P_\theta = 0 \; ; \; P_z = -\frac{2\,c1\,c5}{\pi}\, n_z\,\rho\, w^2\left\{ c6\int_0^{\pi/2}\frac{d\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}} - c7\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}}\,d\phi \right\} \tag{14}$$

The Kernel P_i can be naturally expressed in terms of complete elliptic integrals of superior order. In this work these integrals were reduced to complete elliptic integrals of the first and second kind and are shown in details by *Cisternas* [2].

Expressions (13) and (14) are valid for general axisymmetric problems. For the case rp = 0, limiting relations have to be employed and the nonzero Kernels are given by

$$P_r = 0 \; ; \; P_\theta = 0 \text{ and } P_z = -\frac{1}{16\pi G(1-\nu)}\,\frac{rq^2}{rr}\,\rho\,\omega^2\, n_z \tag{15}$$

KERNELS FOR THERMAL LOADS

Considering the steady state heat conduction, the second domain integral of equation (9) can be written as follows as shown in details by *Cisternas* and *Telles* [5],

$$B_i^T = 2\pi\int_\Gamma P_i\; T\; rq\; d\Gamma - 2\pi\int_\Gamma Q_i\; T_{,k}\; n_k\; rq\; d\Gamma \; ; \; i = r,\theta,z \tag{16}$$

where

$$P_r = \frac{(1+\nu)\,\alpha}{4\pi^2(1-\nu)}\left\{ \frac{n_r}{rr}\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1 - fk^2\cos^2\phi\right]^{1/2}}\,d\phi - c10\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1 - fk^2\cos^2\phi\right]^{3/2}}\,d\phi \; + \right.$$

$$\left. + \, c11\int_0^{\pi/2}\frac{d\phi}{\left[1 - fk^2\cos^2\phi\right]^{3/2}} + c12\int_0^{\pi/2}\frac{\cos 4\phi}{\left[1 - fk^2\cos^2\phi\right]^{3/2}}\,d\phi \right\} \; ; \; P_\theta = 0 \tag{17}$$

$$P_z = \frac{(1+\nu)\,\alpha}{4\pi^2(1-\nu)}\left\{\frac{n_z}{rr}\int_0^{\pi/2}\frac{d\phi}{\left[1-fk^2\cos^2\phi\right]^{1/2}} - \frac{(zq-zp)\,c8}{rr^3}\int_0^{\pi/2}\frac{d\phi}{\left[1-fk^2\cos^2\phi\right]^{3/2}} + \right.$$

$$\left. + \frac{(zq-zp)\,c9}{rr^3}\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1-fk^2\cos^2\phi\right]^{3/2}}\,d\phi\right\} \tag{18}$$

$$Q_r = \frac{(1+\nu)\,\alpha}{4\pi^2(1-\nu)\,rr}\left\{rq\int_0^{\pi/2}\frac{\cos 2\phi}{\left[1-fk^2\cos^2\phi\right]^{1/2}}\,d\phi - rp\int_0^{\pi/2}\frac{d\phi}{\left[1-fk^2\cos^2\phi\right]^{1/2}}\right\} \tag{19}$$

$$Q_\theta = 0 \;\; ; \;\; Q_z = \frac{(1+\nu)\,\alpha\,(zq-zp)}{4\pi^2(1-\nu)\,rr}\int_0^{\pi/2}\frac{d\phi}{\left[1-fk^2\cos^2\phi\right]^{1/2}} \tag{20}$$

where,

$$c1 = \frac{1}{8\pi G} \;\; ; \;\; c2 = \frac{rq\,rp}{4(1-\nu)}\,n_r - \frac{rq\,c9}{2} \;\; ; \;\; c3 = rq\,c8 - \frac{rq^2}{2(1-\nu)}\,n_r \;\; ; \;\; c4 = \frac{(1-2\nu)}{2(1-\nu)}\,n_r$$

$$c5 = \frac{1}{2(1-\nu)} \;\; ; \;\; c6 = \frac{rq^2}{rr} \;\; ; \;\; c7 = \frac{rq\,rp}{rr} \;\; ; \;\; c8 = rq\,n_r + (zq-zp)\,n_z \;\; ; \;\; c9 = rp\,n_r$$

$$c10 = \frac{(c8\,rq + c9\,rp)}{rr^3} \;\; ; \; c11 = \frac{\left(c8\,rp + \frac{c9\,rq}{2}\right)}{rr^3} \;\; ; \; c12 = \frac{c9\,rq}{2\,rr^3} \;\; ; \; rr = \left[(rq+rp)^2 + (zq-zp)^2\right]^{1/2} \tag{21}$$

In equation (1 6), T is the temperature at point q and $T_{,k}\,n_k$ is the heat flux.

Expressions (17), (18), (19) and (20) are valid for general axisymmetric problems. For the case rp = 0, limiting relations have to be employed and the nonzero Kernels are given by

$$P_r = 0 \;\; ; \;\; P_\theta = 0 \;\; ; \;\; P_z = \frac{(1+\nu)\,\alpha}{8\pi(1-\nu)}\left\{\frac{n_z}{rr} - \frac{(zq-zp)(rq\,n_r + (zq-zp)\,n_z)}{rr^3}\right\} \tag{22}$$

$$Q_r = 0 \;\; ; \;\; Q_\theta = 0 \;\; \text{and} \;\; Q_z = \frac{(1+\nu)\,\alpha\,(zq-zp)}{8\pi(1-\nu)\,rr} \tag{23}$$

KERNELS TO STRESSES IN INTERNAL POINTS

The body force integral and the contribution of thermal strains of equation (8), can be represented as

$$B'_{ij} = 2\pi\int_\Gamma \bar{S}_{ij}\,rq\,d\Gamma + 2\pi\int_\Gamma S_{ij}T\,rq\,d\Gamma - 2\pi\int_\Gamma V_{ij}\,T_{,k}n_k\,rq\,d\Gamma - 2G\left(\frac{1+\nu}{1-2\nu}\right)\alpha T\delta_{ij} \tag{24}$$

where $\bar{S}_{ij}$ is due to centrifugal load, S_{ij} and V_{ij} are due to thermal loads and they are obtained from Hooke's law.

NUMERICAL IMPLEMENTATION

For the numerical implementation of equations (7) and (8), linear elements and linear internal cells have been adopted. The regular integrals ($p_i \notin \Gamma_j$) are obtained by standard Gaussian quadrature and the number of points employed is a function of the dimensionless distance between the singular node or internal point and the element under consideration. In the case rp = 0, $S_{rr} = S_{\theta\theta}$ and $V_{rr} = V_{\theta\theta}$, which provide less computational effort. In order to properly simulate traction discontinuities over the boundary, the concept of double nodes was employed, as shown by *Brebbia, Telles* and *Wrobel* [6].

APPLICATIONS

APPLICATION 1 - The first application of inelastic behavior of materials consists of the analysis of a solid disk rotating about its center with angular velocity ω, as shown in figure 1. An idealized stress-strain curve is assumed with the von Mises yield criterion.

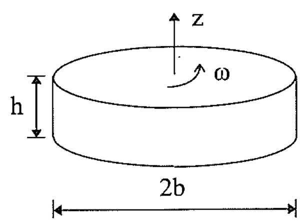

Figure 1 - Solid disk subjected to centrifugal load

Geometry and Material Properties:

$E = 2.06 \times 10^5$ N/mm^2
$\nu = 0.3$
$\rho = 7.85 \times 10^{-9}$ N s^2/mm^4
h = 20 mm ; b = 100 mm
$\omega = 3300$ rad/s
$\sigma_o = 240$ N/mm^2 ; $E_T = 515$ N/mm^2

Figure 2, shows the stress-strain curve used in this application.

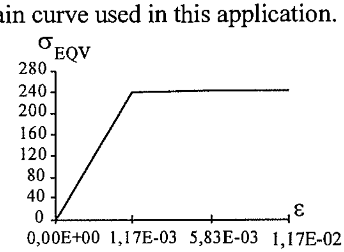

Figure 2 - Stress-strain curve

The theoretical angular velocity ω_0 that produces complete plastification of the rotating disk (collapse) is $\omega_0 = 1/b\sqrt{(3\sigma_0/\rho)} = 3028{,}53$ rad/s. The adopted angular velocity was 3300 rad/s, the boundary element discretization will be shown in figure 3.

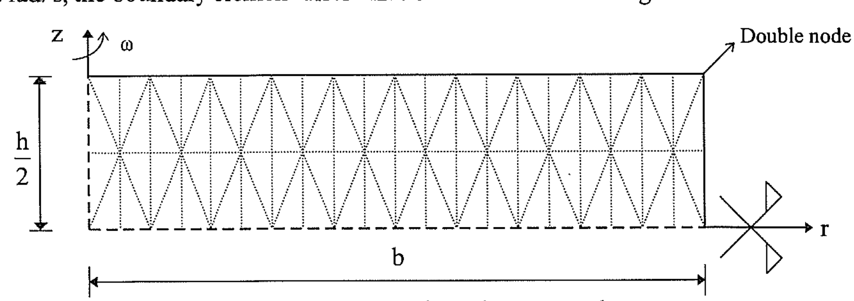

Figure 3 - Boundary element mesh.

A comparison between the BEM and FEM, including the elastic solution is shown in figure 4:

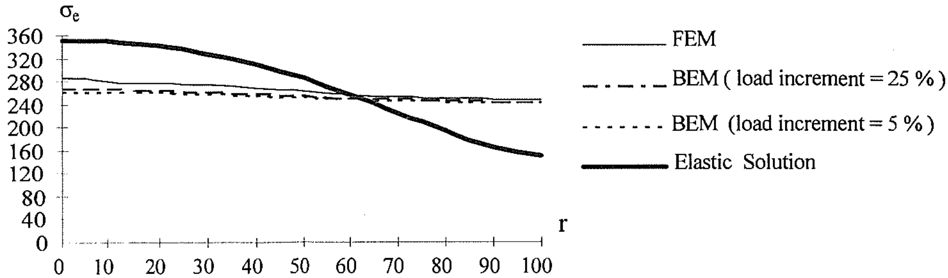

Figure 4 - Comparison BEM x FEM.

APPLICATION 2 - The second application of inelastic behavior of materials, consists of the analysis of a steady-state temperature distribution in a hollow sphere of inner radius a and outer radius b. In this application, the inner and outer surfaces are kept at fixed temperatures of value T_i and zero, respectively and the temperature at any point is equal to

$$T = \frac{T_i\, a}{b-a}\left(\frac{b}{r} - 1\right) \tag{20}$$

The geometry and material properties are:

$E = 2{,}06 \times 10^5$ N/mm^2 ; $\nu = 0{,}3$; $T_i = 400$ °C ; $\alpha = 1{,}2 \times 10^{-5}$ °C^{-1}
a = 100 mm ; b = 200 mm ; $\sigma_0 = 240$ N/mm^2 ; $E_T = 515$ N/mm^2

The idealized stress-strain curve shown in figure 2 is assumed valid with the von Mises yield criterion. In this application, two plastic zones are formed due to the temperature gradient. In addition, it should be noted that when the inner plastic zone onsets the sphere can only become completely plastic if the temperature difference is infinite. Figure 5 shows the two plastic zones due to the temperature gradient.

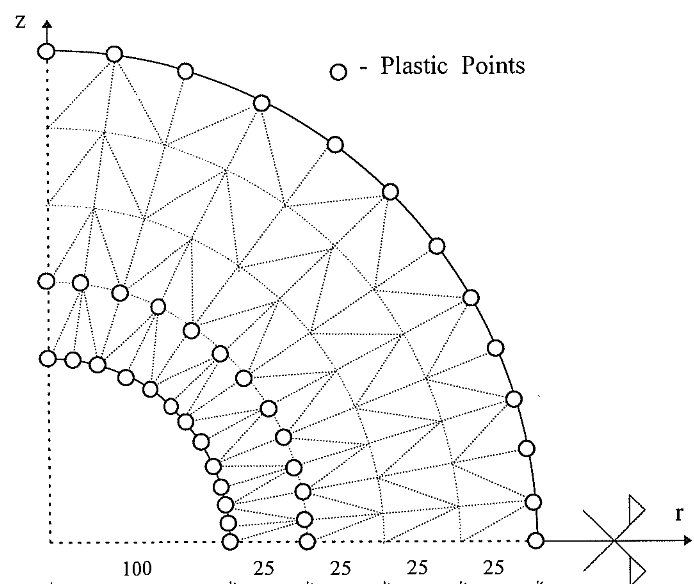

Figure 5 - Plastic points in the hollow sphere.

Figure 6 shows a comparison between BEM ɑ FEM, including the elastic solution.

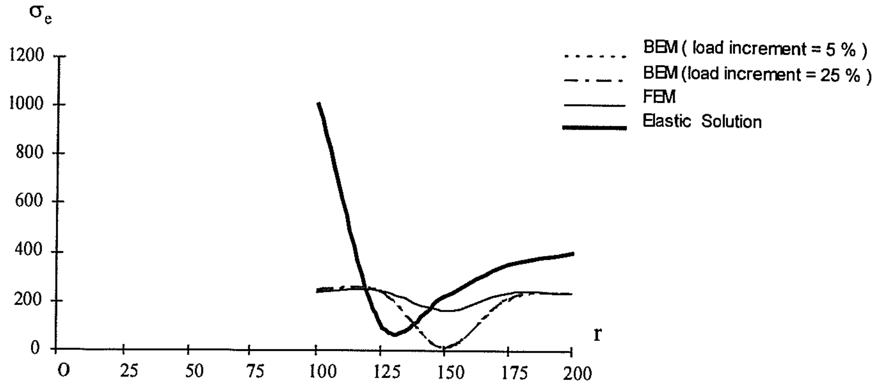

Figure 6 - von Mises stresses (N/mm^2) x radius (mm).

CONCLUSIONS

The Kernels of the generalized body force contributions have been implemented in terms of complete elliptic integrals of superior order. This permits the consideration of generalized body forces, in hollow or solid bodies of revolution, through surface integrals only, with equal ease. In addition the examples indicated that the Boundary Element Method can not only be an alternative for the Finite Element Method, but may also produce the best solution for many difficult problems.

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