Diagonalization Methods for Viscous and Hysteretic Damping Matrices

A. Cella, A. Greco
CISE S.p.A., C.P. 3986, I-20100 Milano, Italy
R. Leo
NIRA S.p.A., Via dei Pescatori 35, I-16129 Genova, Italy

A method to obtain damping matrices which are diagonalized by the undamped eigenmodes and which are equivalent to the true viscous and hysteretic damping matrices is presented.
The method is based on the assumption that the damped complex eigenmodes of a structure with non proportional damping can be expressed as a linear combination of the first few undamped modes.
In addition a simplified procedure is presented which is based on perturbation techniques. This simplified procedure allows to evaluate straightforwardly an equivalent damping matrix when the damping is sufficiently low.
A criterium is suggested to judge when this simplified procedure can be accepted.

1. INTRODUCTION

The response of linear elastic structures to various types of excitation is often performed using the modal superposition method. The application of this technique requires the definition of proper coordinate transformations to decouple the motion equations. In general the decoupling cannot be obtained using the undamped modes of the structure but it is possible to retain modal superposition by working with the complex eigenmodes of the damped structure [1]. This rigorous technique results into an expensive computational effort for structures discretized with a large number of degrees of freedom. For this reason approximate techniques are used requiring only the computation of the undamped modal characteristics of the system. Such approximate thecniques are based on the substitution of the damping matrix C with an equivalent one Ceq which is diagonalized by the undamped eigenmodes ϕ of the structure so that the following relation holds:

$$\phi^T C_{eq} \phi = 2\,\Omega^{-1} N$$

where Ω and N are diagonal matrices whose elements are the undamped frequencies and the equivalent damping ratios, respectively. Among these thecniques the most simple and straightforward procedure is the generalized Bigg's rule [2] which defines the equivalent damping ratios by neglecting the off-diagonal terms of matrix $\phi^T C \phi$. This approximation is in general sufficient when damping is not significantly high. In this paper a procedure for the evaluation of the equivalent damping ratios is proposed. Such procedure can be considered as an improvement of the generalized Bigg's rule since takes into account to some extent the contributions of the off-diagonal terms of matrix $\phi^T C \phi$. Furthemore this method is applicable both in the case of viscous and hysteretic damping. To evaluate the efficiency of the proposed procedure a numerical experimentation was performed for the containment and internals of a typical pressure water nuclear reactor [3] founded on different soil conditions.

2. PROBLEM DEFINITION

The basic dynamic equations for a multi degree of freedom system (like that obtained from a finite element idealization of a structure) is

$$\begin{cases} K\,\delta(t) + M\,\ddot{\delta}(t) + F_D(t) = F(t) \\ \delta(t_0) = \delta_0 \qquad \dot{\delta}(t_0) = \dot{\delta}_0 \end{cases} \tag{1}$$

where K,M are the stiffness, mass matrices respectively, $\delta(t)$ is the displacement vector,F(t) the external force vector and $F_D(t)$ the damping force vector. If viscous damping is considered the damping force vector is

$$F_D = C_V\,\dot{\delta}(t)$$

where Cv is the viscous damping matrix. In the case of hysteretic damping the solution of equations (1) can be obtained by expanding the external force vector in terms of a Fourier series as follows:

$$F(t) = \sum_{s=1}^{m} A_s \sin(\alpha_s t + \varphi_s)$$

and solving for each harmonic term the following equations

$$\begin{cases} K\,\delta^{(s)}(t) + M\,\ddot{\delta}^{(s)}(t) + F_D^{(s)}(t) = A_s \sin(\alpha_s t + \varphi_s) \\ \delta^{(s)}(t_o) = \delta_o^{(s)} \qquad \dot{\delta}^{(s)}(t_o) = \dot{\delta}_o^{(s)} \end{cases} \tag{2}$$

In the previous equations the damping force vector is

$$F_D^{(s)}(t) = \frac{C_H}{\alpha_s}\,\dot{\delta}^{(s)}(t)$$

being C_H the hysteretic damping matrix. The dynamic response is then computed by adding the contributions of each harmonic term

$$\delta(t) = \sum_{s=1}^{m} \delta^{(s)}(t)$$

Therefore the motion equations of the system considered can be written in the form

$$\begin{cases} K\,\delta(t) + C\,\dot{\delta}(t) + M\,\ddot{\delta}(t) = F(t) \\ \delta(t_o) = \delta_o \qquad \dot{\delta}(t_o) = \dot{\delta}_o \end{cases} \tag{3}$$

where the damping matrix **C** is:

$$C = \begin{cases} C_V & \text{for viscous damping} \\ \dfrac{C_H}{\alpha_s} & \text{for hysteretic damping} \end{cases}$$

and, in the case of hysteretic damping, equation (3) refers to each harmonic amplitude $\delta^{(s)}(t)$. The solution of equation (3) is often determined using the modal superposition method which is particularly advantageous if the response of the structure considered is dominated by contribution from a relatively small fraction of the modes. The application of this method to equation (3) requires the solution of the generalized eigenvalue problem

$$K x_i + \lambda_i C x_i + \lambda_i^2 M x_i = 0 \tag{4}$$

If matrix C satisfies the orthogonality condition

$$\phi^T C \phi = D \tag{5}$$

where ϕ is the undamped mode matrix and D is a diagonal matrix, problem (4) is reduced to the solution of the undamped equations

$$K\phi_i = \omega_i^2 M \phi_i \qquad \phi_i^T K \phi_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{6}$$

so that the eigenmodes are real and $\phi_i \equiv x_i$. If relation (5) is not verified, the eigenvectors x_i are complex and the modal superposition method can lose its advantages. In fact the solution of problem (4) results into an expensive computational effort for structure discretized with a large number of degrees of freedom. The purpose of this paper is to describe an approximate thecnique which allows the use of the undamped modal characteristics in the modal superposition method and the definition of equivalent modal damping ratios.

3. NUMERICAL METHOD

An approximate solution of equation (3) can be obtained by expanding the vector of displacements in terms of the first n few undamped modes

$$\delta(t) = \tilde{\phi}\, q(t) \tag{7}$$

where q(t) is the vector of order n of the modal amplitudes. Substitution of equation (7) in equation (3) and premultiplication by Φ^T, leads to a set of coupled equations in the modal amplitudes

$$\begin{cases} q(t) + \tilde{C}\,\dot{q}(t) + \Omega^{-2}\ddot{q}(t) = \tilde{\Phi}^T F(t) \\ q(t_o) = q_o \qquad \dot{q}(t_o) = \dot{q}_o \end{cases} \tag{8}$$

where $\tilde{C} = \tilde{\Phi}^T C \tilde{\Phi}$ and $\Omega^2 = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2)$

The solution of equations (8) can be expressed by the eingenvalues and eigenvectors of the problem

$$y_j + \tilde{\lambda}_j \tilde{C} y_j + \tilde{\lambda}^2 \Omega^{-2} y_j = 0 \qquad (j = 1, n) \tag{9}$$

Being matrix not diagonal, the eigeinmodes are complex and this drawback can be overcomed defining an equivalent diagonal damping matrix D so that equations (8) become

$$\begin{cases} q^*(t) + D\,\dot{q}^*(t) + \Omega^{-2}\ddot{q}^*(t) = \tilde{\Phi}^T F(t) \\ q^*(t_o) = q_o^* \qquad \dot{q}^*(t_o) = \dot{q}_o^* \end{cases} \tag{10}$$

and the corresponding eigenproblem is

$$y_j^* + \lambda_j^* D\, y_j^* + \lambda_j^{*2} \Omega^{-2} y_j^* = 0 \qquad (j = 1, n)$$

The elements of matrix D are computed by minimizing the modulus of the difference between the exact eigenvalues $\tilde{\lambda}_j$ and the approximate one's λ_j^*. From the relations:

$$\frac{\partial}{\partial D_j} |\tilde{\lambda}_j - \lambda_j^*|^2 = 0 \qquad (j = 1, m)$$

one obtains:

$$D_j = \frac{2\nu_j}{\Omega_j} \qquad \nu_j = -\frac{\mathrm{Re}(\lambda_j)}{|\lambda_j|} \tag{11}$$

In the case of viscous damping, relations (11) completely define the elements of matrix D. In the case of hysteretic damping, matrix D depends on the frequency α_s of the forcing function. In order to overcome this drawback, it is convenient to define a criterium establishing an equivalence between viscous damping and hysteretic damping. Equations (10) can be considered as n single degree of freedom motion equations, and for such systems the equivalence between viscous damping and hysteretic damping is defined by assuming $\alpha_s = \Omega_s$, being Ω_s the natural frequency of the system [2]. This criterium can be applied to evaluate the matrix D entries in case of hysteretic damping; the following computational rule is used. For each value $\Omega_K (k=1,n)$:

a) solve the eigenvalue problem

$$y_j^{(K)} + \tilde{\lambda}_j^{(K)} \frac{C_H}{\Omega_K} y_j^{(K)} + \tilde{\lambda}_j^{(K)2} \Omega^{-2} y_j^{(K)} = 0$$

b) define the k^{th} entry of the diagonal matrix D as

$$D_K = -\frac{2}{\Omega_K} \frac{\mathrm{Re}(\tilde{\lambda}_K^{(K)})}{|\lambda_K^{(K)}|}$$

4. PERTURBATION ANALYSIS

The numerical procedure described in the previous paragraph allows the definition of an equivalent diagonal damping matrix if the eigenvalues of problem (9) are computed. If the entries of matrix C are sufficiently small, it is possible to avoid the computation of $\tilde{\lambda}_i$ by using a method based on the perturbation theory. Problem (9) can be written in the following form

$$\left(A_o + E\right) z_i = \tilde{\lambda}_i z_i \qquad (12)$$

with the orthogonality condition

$$z_r^T \Delta z_j = \begin{cases} 0 & r \neq j \\ 1 & r = j \end{cases}$$

where

$$A_o = \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^2\tilde{C} \end{bmatrix} \quad \Delta = \begin{bmatrix} -I & 0 \\ 0 & \Omega^{-2} \end{bmatrix} \quad z_i = \begin{Bmatrix} y_i \\ \tilde{\lambda}_i y_i \end{Bmatrix}$$

Define:

$$z_j = \psi_j + \varepsilon_j \qquad \tilde{\lambda}_j = i\,\Omega_j + \mu_j \qquad (i^2 = -1)$$

where ψ_j are the eigenmodes of problem (12) with $\tilde{C} = 0$ and ε_j, μ_j are perturbation of the first order with respect to ψ_j and Ω_j respectively. Under the assumption that the entries of E are sufficiently small, it is possible to calculate ε_j, μ_j by neglecting high order infinitesimal terms. This procedure allows the computation of approximate values of z_j and $\tilde{\lambda}_j$ through the definition of

$$\mu_j = \mathrm{Re}(\tilde{\lambda}_j) = -\Omega_j^2 \frac{\tilde{C}_{jj}^2}{2} \qquad (13)$$

$$y_j = a^{(j)} + i\, b^{(j)}$$

being

$$a_K^{(j)} = \frac{4\Omega_K^2 \Omega_j \tilde{C}_{Kj}(\Omega_j + \Omega_K)}{\sqrt{2}(\Omega_K - \Omega_j)[4(\Omega_j + \Omega_K)^2 + \tilde{C}_{jj}^2 \Omega_j^4]} \qquad j \neq K$$

$$a_j^{(j)} = \frac{4\Omega_j \tilde{C}_{jj}}{\sqrt{2}[16 + \tilde{C}_{jj}^2 \Omega_j^2]}$$

$$b_K^{(j)} = \frac{e^{(j)}}{\sqrt{2}} - a_K^{(j)} \frac{\tilde{C}_{jj}\Omega_j^2}{2(\Omega_j + \Omega_K)}$$

where $e^{(j)}$ is the j^{th} unit vector. In order to evaluate the accuracy of these approximations the residual vector R_j obtained by introducing relations (13) into equation (9) has been computed, and the following acceptance criterium has been used:

- if $\frac{\|R_j\|}{\|y_j\|} < \varepsilon$ (ε prescribed tolerance) (14)

the equivalent damping ratio is computed as

$$\nu_j = \frac{\Omega_j^2 \tilde{C}_{jj}^2/2}{\lambda_j} = \frac{\Omega_j \tilde{C}_{jj}}{2\sqrt{1 + \Omega_j^2 C_{jj}^2/4}} \qquad (15)$$

- if relation (14) is not satisfied problem (9) is solved and

$$\nu_j = -\frac{\mathrm{Re}(\lambda_j)}{|\lambda_j|}$$

It can be noticed that relation (15) is coincident with the generalized Bigg's rule if $\frac{\Omega_j^2 \tilde{C}_{jj}^2}{4} \ll 1$.

5. NUMERICAL EXAMPLE

In order to evaluate the efficiency of the proposed procedure, the equivalent modal damping ratios of a simplified PWR model founded on different soil conditions have been computed. The containement structure and the internals of PWR were idealized as a 19 lumped mass model

3 . The soil-structure interaction was represented by two springs and related dampers: one set corresponding to horizontal motion and the other one to rocking. Fig.1 represents the lumped mass model of structure-foundation system and Table I summarizes the model's properties. The equivalent foundation spring-damper constants are quoted in Table II. The cases considered were chosen so that the damping ratios for the foundation

$$\begin{cases} \beta_x = \dfrac{C_x}{2\sqrt{K_x M}} \qquad \beta_\phi = \dfrac{C_\phi}{2\sqrt{K_\phi I}} \\ \text{M total mass; I total mass moment of inertia} \end{cases}$$

where β_x = 20%, β_ϕ= 7% for cases Nos 1,3,5 and β_x= 40%, β_ϕ= 20% for cases Nos 2,4,6. Moreover 4% and 3% of hysteretic damping ratio were assumed for the structure and the foundation respectivly. The results obtained are shown in table III where, for comparison's sake, also the equivalent modal damping ratios computed by relation (15) are quoted. The results presented in these tables indicate that the procedure proposed provides reasonable accurate results for all the cases considered; on the other hand, relation (15) can supplie unacceptable values of the equivalent damping ratios. However for these last cases the evaluation of the parameter defined by eq.(14), gives sufficient information to judge the validity of relation (15). Therefore, this parameter can be employed for an automatic choice of the more convenient approximation to be used in the calculation of the equivalent modal damping ratios. The numerical experimentation performed, even if limited, has shown that a value of 5% for the acceptance criterium (see eq. (14)) may be a good choice.

REFERENCES

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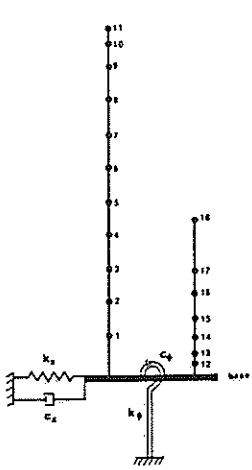

Fig. 1 Lumped mass model of structure-foundation system

Table I
Properties of the structural models of the containment building and internals

(Containement: $E = 3.30\ 10^{10}\ N\ m^{-2}$; $v = 0.27$)
(Internals: $E = 2.06\ 10^{11}\ N\ m^{-2}$; $v = 0.3$)

Joint Properties			Member Properties			
Mass No.	$m_i \times 10^{-6}$ (Kg)	$I_i \times 10^{-6}$ ($Kg.m^2$)	Location between Joint No	Element's length (m)	Shear Area (m^2)	Moment of Inertia $\times 10^{-3}$ (m^4)
base	9.07	889.15				
1	2.08	396.15	base to 1	7.163	65.03	24.16
2	1.90	358.19	1 to 2	6.188	"	"
3	"	"	2 to 3	6.096	"	"
4	"	"	3 to 4	"	"	"
5	"	"	4 to 5	"	"	"
6	"	"	5 to 6	5.736	"	"
7	2.09	396.11	6 to 7	6.456	"	"
8	1.37	248.63	7 to 8	6.553	46.45	16.40
9	1.12	155.92	8 to 9	5.822	"	12.94
10	0.96	71.64	9 to 10	4.298	"	6.90
11	0.08	4.21	10 to 11	2.591	"	1.72
12	1.27	101.14	base to 12	2.438	122.63	9.49
13	1.14	80.06	12 to 13	1.524	144.93	10.35
14	2.85	210.70	13 to 14	2.743	135.64	10.35
15	1.70	278.12	14 to 15	3.505	67.82	11.22
16	3.87	530.97	15 to 16	4.724	55.74	7.76
17	0.55	33.71	16 to 17	3.658	33.44	1.72
18	0.37	4.21	17 to 18	9.754	6.50	0.01

Table II
Equivalent foundation spring-damper constants

Case number	Translational spring Kx ($N.m^{-1}$)	Translational damper Cx ($N.s.m^{-1}$)	Rotational spring Ky (N.m)	Rotational damper (N.s.m)
1	$2.0\ 10^{10}$	$3.49\ 10^{8}$	$7.6\ 10^{12}$	$6.54\ 10^{10}$
2	$2.0\ 10^{10}$	$6.98\ 10^{8}$	$7.6\ 10^{12}$	$1.87\ 10^{11}$
3	$1.5\ 10^{11}$	$9.55\ 10^{8}$	$4.8\ 10^{13}$	$1.64\ 10^{11}$
4	$1.5\ 10^{11}$	$1.91\ 10^{9}$	$4.8\ 10^{13}$	$4.69\ 10^{11}$
5	$6.0\ 10^{11}$	$1.91\ 10^{9}$	$1.9\ 10^{14}$	$3.27\ 10^{11}$
6	$6.0\ 10^{11}$	$3.82\ 10^{9}$	$1.9\ 10^{14}$	$9.34\ 10^{11}$

Table III

Results

Case No.	Mode No.	Undamped frequencies (Hz)	Equivalent modal damping: Exact method (eq.(4)) (%)	Present method (12 modes) (%)	Present method (6 modes) (%)	Eq.(15) (%)	Residual: 12 modes (%)	6 modes (%)
1	1	2.13	9.28	9.28	9.28	9.27	1.2	1.2
	2	5.11	26.84	26.84	26.83	25.78	7.1	7.2
	3	13.15	8.63	8.63	8.64	8.55	2.6	0.3
	4	16.45	6.66	6.65	6.64	6.65	3.4	3.7
	5	19.12	5.27	5.26	5.25	5.30	4.1	4.3
	6	28.06	4.13	4.13	4.13	4.12	0.3	0.6
	7	30.65	4.64	4.63	–	4.64	2.0	–
	8	41.89	4.04	4.04	–	4.04	0.5	–
2	1	2.13	18.67	18.67	18.66	18.38	2.8	2.8
	2	5.11	53.67	53.64	53.54	46.20	21.3	21.7
	3	13.15	19.20	19.23	19.20	17.08	17.9	16.7
	4	16.45	9.55	9.45	9.28	10.08	18.8	20.1
	5	19.12	6.86	6.78	6.68	7.76	28.1	29.7
	6	28.06	4.36	4.36	4.33	4.36	3.0	5.5
	7	30.65	5.46	5.42	–	5.58	13.5	–
	8	41.89	4.09	4.09	–	4.10	3.0	–
3	1	3.99	5.66	5.66	5.66	5.66	0.5	0.5
	2	11.45	17.36	17.40	17.50	17.02	4.2	3.7
	3	15.45	7.27	7.27	7.26	7.25	3.0	2.8
	4	18.42	9.39	9.38	9.49	9.70	6.7	5.9
	5	22.01	15.74	15.71	14.96	14.53	9.7	10.5
	6	28.21	4.44	4.43	4.40	4.43	0.9	2.4
	7	31.41	7.07	7.00	–	7.39	8.6	–
	8	41.93	4.14	4.13	–	4.14	1.9	–

(Continues)

Table III

(Continued)

Results

Case No.	Mode No.	Undamped frequencies (Hz)	Equivalent modal damping: Exact method (eq.(4)) (%)	Present method (12 modes) (%)	Present method (6 modes) (%)	Eq.(15) (%)	Residual: 12 modes (%)	6 modes (%)
4	1	3.99	8.96	8.98	8.99	9.03	1.9	1.8
	2	11.45	33.46	33.56	34.85	29.74	17.4	15.6
	3	15.45	11.11	10.90	10.77	12.60	20.3	18.0
	4	18.42	10.02	9.76	9.89	15.65	36.8	32.3
	5	22.01	48.09	45.05	39.08	30.91	43.2	45.6
	6	28.21	5.19	5.06	4.73	5.27	8.4	19.6
	7	31.41	7.57	7.24	–	12.35	47.1	–
	8	41.93	4.25	4.22	–	4.36	11.2	–
5	1	4.80	4.27	4.27	4.27	4.27	0.2	0.2
	2	14.19	5.05	5.06	5.11	5.16	2.4	1.5
	3	17.26	4.78	4.80	4.81	4.87	2.3	1.8
	4	23.18	13.52	13.61	14.06	13.35	6.2	4.7
	5	27.49	6.75	6.76	6.69	7.09	3.5	5.1
	6	28.96	4.79	4.79	4.85	4.86	3.1	1.5
	7	28.51	20.97	20.68	–	17.12	14.4	–
	8	42.41	4.69	4.65	–	6.60	19.4	–
6	1	4.80	4.94	4.95	4.95	4.95	0.5	0.5
	2	14.19	5.62	5.69	5.98	6.41	9.7	6.3
	3	17.26	5.37	5.56	5.57	6.17	9.5	8.5
	4	23.18	26.52	26.26	28.36	23.90	22.6	18.8
	5	27.49	6.73	6.57	8.40	10.63	22.2	20.7
	6	28.96	5.34	5.57	6.36	6.53	15.7	7.3
	7	38.51	53.66	51.16	–	33.68	51.5	–
	8	42.41	4.70	4.27	–	10.15	68.3	–