



## Static Responses of Elasto–plastic Nonuniform Plates of Arbitrary Shape

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*Abstract: A simple method for the analysis of the elasto-plastic plates of arbitrary shape of nonuniform thickness is developed. The method is based on the concept of isodeflection contour lines in conjunction with Ilyushin's theory for small plastic deformation. The method is applied to study the nonlinear behaviour of circular and elliptical plates and the results obtained are compared with the available results for structures of uniform thickness.*

### INTRODUCTION

Beyond elastic limit every elastic material exhibits plastic behaviour. For large amplitude vibration of structure under dynamic load and for a large flexure under static extreme load both elastic and plastic behaviour of substances are to be taken into consideration. Thin plates of arbitrary shape have a great practical importance in many engineering fields. The applications of nonuniform plates in mechanical engineering specially in aerospace engineering are very wide. However, the analysis of such structures appears to be quite difficult to yield adequately satisfactory results. Several approximate methods are in use for the analysis of such problems but these are restricted mainly to uniform structures having uniform thickness and of uniform flexural rigidity. The method of constant deflection contour lines have been used by different authors [1–6] to study the behaviours of plates and shells having uniform thickness and rigidity. The present paper aims at extending the method of constant deflection contour lines to study the nonuniform plate problems. The bending of an elliptic plate of elasto-plastic material with uniaxially varying thickness has been considered. The results obtained are presented in graphs and those are compared as far as possible with

### DERIVATION OF BASIC DIFFERENTIAL EQUATIONS

A thin elasto-plastic plate of variable thickness and of variable flexural rigidity subjected to a continually distributed lateral load  $q(x,y)$  is considered. The middle surface of the plate remains in the  $x$ - $y$  plane whereas the  $z$ -axis is perpendicular to that plane along the direction of centre of curvature. The intersections between the deflected surface  $z = w(x,y)$  and the planes  $z = \text{constant}$  yield contours which after projection onto  $z = 0$  surface are the level curves called “Lines of isodeflection”. The family of such curves is denoted by  $u(x,y) = \text{constant}$ . If the boundary of the plate is subjected to any combination of clamping and of simply support, then clearly it will belong to the family of lines of equal deflection and without loss of generality one may consider that  $u = 0$  on the boundary of the plate. Denoting the family of curves  $u = \text{constant}$  by  $C_u$ ,  $0 \leq u \leq u^*$ , it is clear that  $C_0 = C$  is the boundary of the plate. It has been assumed that the value of ‘ $u$ ’ increases as one moves towards the centre of the plate.

Considering the equilibrium of an element  $\Omega$  of the plate bounded by any closed contour  $C_u$  as shown in figure 2. Equating the total downward load acting on an element to the resultant tractions exerted on this portion by the remainder of the plate, one obtains

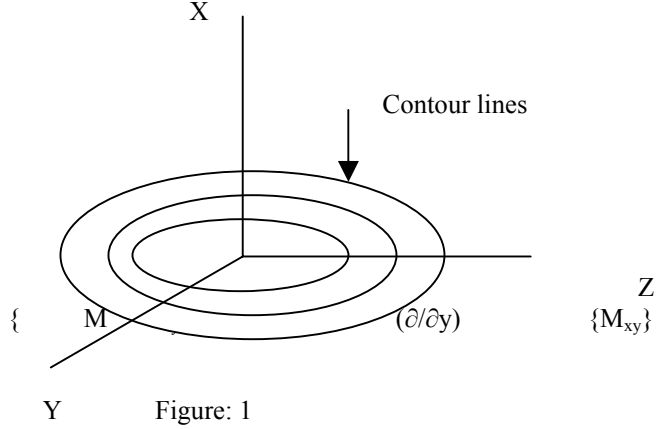
$$\int V_n ds = \iint q dx dy \dots \dots \dots (1)$$

where the transverse reaction forces .

$$V_n = Q_n - \partial/\partial s (M_{nt}) \dots \dots \dots (2)$$

represents the effect of the shearing force  $Q_n$  and the edge-rate of change of twisting moment  $M_{nt}$  along the contour  $C_u$ . According to Ilyushin's theory of the elastic plastic deformation (1948), the bending moments  $M_x, M_y, M_{xy}$  and their shear forces  $Q_x, Q_y$  are given by the following relations:

$$\begin{aligned}
M_x &= -D(1-\nu) \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right) + \nu \left( \frac{\partial^2 w}{\partial y^2} \right) \right\} \\
M_y &= -D(1-\nu) \left\{ \left( \frac{\partial^2 w}{\partial y^2} \right) + \nu \left( \frac{\partial^2 w}{\partial x^2} \right) \right\} \\
M_{xy} &= D(1-\nu)(1-\Omega) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \dots \dots \dots (3) \\
Q_x &= \frac{\partial}{\partial x} \{ M_x \} \\
Q_y &= \frac{\partial}{\partial y} \{ M_x \} - \frac{\partial}{\partial x} \{ M_{xy} \}
\end{aligned}$$



Where,  $\Omega = 0$  when  $e \leq 1$ , the region is elastic ; when  $e > 1$  the region is plastic. Also,

$$\Omega = \lambda \left[ 1 - \left( \frac{3}{2e} \right) + \left( \frac{1}{2e^3} \right) \right] \dots \dots \dots (4)$$

$$\text{and } e^2 = \left( \frac{h^2}{3e_s} \right) \left[ \left( \frac{\partial^2 w}{\partial x^2} \right) + \left( \frac{\partial^2 w}{\partial y^2} \right) + \left( \frac{\partial^2 w}{\partial x \partial y} \right) + \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) \right] \dots \dots \dots (5)$$

in which  $e_s$  is the yield strain,  $\nu$  is the poisson's ratio,  $D$  is the flexural rigidity of the plate material,  $\lambda$  is a material constant

The expressions for  $Q_n, M_{nt}$  are well known and given by,

$$\begin{aligned}
Q_n &= Q_x \cos \alpha + Q_y \sin \alpha \\
M_{nt} &= M_{xy} (\cos^2 \alpha - \sin^2 \alpha) + (M_x - M_y) \sin \alpha \cos \alpha \dots \dots \dots (6)
\end{aligned}$$

Where,  $\cos \alpha = (dy / ds)$  and  $\sin \alpha = -(dx / ds)$ .

Making use of relations (3), (5) and carrying out several steps of computation equation (1) finally takes the form:

$$\begin{aligned}
& \left( \frac{\partial^3 w}{\partial u^3} \right) \int (1-\Omega) [ R + (3m/a) Rx + 3(m/a)^2 Rx^2 + (m/a)^3 Rx^3 ] ds \\
& \left( \frac{\partial^2 w}{\partial u^2} \right) \int (1-\Omega) [ F + (3m/a) Fx + 3(m/a)^2 Fx^2 + (m/a)^3 Fx^3 ] ds \\
& \left( \frac{\partial w}{\partial u} \right) \int (1-\Omega) [ G + (3m/a) Gx + 3(m/a) \left( \frac{\partial u}{\partial x} \right) + (m/a)^3 Fx^3 ] ds \\
& \left( \frac{\partial^2 w}{\partial u^2} \right) \int D [ (\partial \Omega / \partial x) \left( \frac{\partial u}{\partial x} \right) + (\partial \Omega / \partial y) \left( \frac{\partial u}{\partial y} \right) ] \sqrt{t} ds + \\
& \left( \frac{\partial w}{\partial u} \right) \int (D/\sqrt{t}) [ (\partial \Omega / \partial x) K + (\partial \Omega / \partial y) L ] ds - \int (\partial D / \partial x) (1-\Omega) \\
& \left( \frac{\partial^2 w}{\partial x^2} \right) + \nu \left( \frac{\partial^2 w}{\partial y^2} \right) \left( \frac{dy}{ds} \right) ds - \int (\partial D / \partial x) (1-\Omega) (1-\nu) \\
& \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{dy}{ds} \right) ds = \iint q dx dy \dots \dots \dots (7)
\end{aligned}$$

where,

$$\begin{aligned}
t &= u_x^2 + u_y^2, \quad R = -D_0 (\sqrt{t})^3 \\
F &= -(D_0/\sqrt{t}) [ 3 u_{xx} u_x^2 + 3 u_{yy} u_y^2 + u_{xx} u_y^2 + u_{yy} u_x^2 + 4 u_{xy} u_x u_y ] \\
G &= -D_0/(\sqrt{t})^3 [ u_{xxx} u_x^3 + u_{yyy} u_y^3 + (2-\nu) (u_{xxx} u_x u_y^2 + u_{yyy} u_y u_x^2 + u_{xyy} u_x^3 + \\
& u_{xxy} u_y^3) + (2\nu-1) (u_{xyy} u_x u_y^2 + u_{xxy} u_y u_x^2) - 2(1-\nu) (u_{xy} u_{xx} u_x u_y - u_{xy}^2 u_x^2 - \\
& u_y^2 - u_{xy}^2 u_x^2 + u_x u_y u_{xy}) + (1-\nu) (u_{xx} - u_{yy}) (u_{xx} u_x^2 - u_{yy} u_y^2) ] + \\
& 2 D_0 (1-\nu) / (\sqrt{t})^5 [ u_{xy} (u_x^2 - u_y^2) - (u_{xx} u_x u_y - u_{yy} u_x u_y) ] \\
K &= (u_{xx} u_x + \nu u_{yy} u_x) + [(1-\nu) u_{xy} u_y - u_y (H/D)], \\
L &= u_y (u_{yy} + \nu u_{xx}) + u_x [(1-\nu) u_{xy} + H/D], \\
H/D &= (1-\nu)/t [ u_{xy} (u_x^2 - u_y^2) - u_x u_y (u_{xx} - u_{yy}) ]
\end{aligned}$$

$$h = h_0 (1 + mx/a), \quad D = D_0 (1 + mx/a)^3, \quad D_0 = Eh_0^3/12(1-\nu^2) \dots \dots \dots (7)$$

In the elastic region where  $\Omega$  is identically Zero and for a plate of uniform thickness ( $m = 0$ ), equation (6) reduces to the usual ordinary differential equation.

$$W_{,uuu} \int_{cu} R ds + W_{,uu} \int_{cu} F ds + W_{,u} \int_{cu} G ds = \iint_{\Omega} q dx dy \dots \dots \dots (8)$$

In the elastic region and plastic region  $e^2$  is given by  

$$e^2 = (h_0 / 3e_s^2) [ M W_{,u}^2 + N W_{,u} W_{,uu} + t^2 W_{,uu}^2 ] \dots\dots\dots(9)$$

Where  $M = (1 + mx/a)^2 [ u_{xx}^2 + u_{yy}^2 + u_{xx} u_{yy} + u_{xy}^2 ]$   
 $N = (1 + mx/a)^2 [ 2 u_{xx} u_x^2 + 2 u_{yy} u_y^2 + u_{xx} u_y^2 + u_{yy} u_x^2 + 2 u_{xy} u_x u_y ] \dots\dots\dots(10)$

Consequently, whenever the contour lines  $u(x,y) = \text{constant}$  are known the problem reduces to the solutions of equations (8) and (9) with continuity of the solution at the boundary of the elastic and plastic regions.

It is to be noted that the precise form of isodeflection contour lines may be obtained in many cases by symmetry consideration or by intuition otherwise assumption may be made as a first approximation.

## ILLUSTRATIONS

### Uniformly loaded elliptic plate

For example, we consider the bending of a simply supported elliptic plate of variable thickness under uniformly distributed transverse load.

The first approximation to the lines of equal deflection for elastic case may reasonably be taken as

$$u(x,y) = 1 - x^2/a^2 - y^2/b^2 \dots\dots\dots(11)$$

where 'a' and 'b' are the semi major and minor axes of the ellipse respectively

Since  $\Omega$  is a function of 'e' which in turn is a function of 'u' and considering the form of u given by equation (11) and carrying out the necessary calculations as explained by Mazumder (1970), the equation (6) takes the form :

$$(1-u)(1-\Omega) W_{,uuu} H_1 - 2(1-\Omega) H_2 W_{,uu} - (1-u) H_1 W_{,uu} + H_3 W_{,u} \Omega_{,u} + (1-\Omega) H_4 = qa^4 b^4 / (2D_0 P^*) \dots\dots\dots(12)$$

Where,  $P^* = 3a^4 + 2a^2 b^2 + 3b^4$

$$H_1 = 1 + 16m(1+u)^{1/2} (3a^4 + 4a^2 b^2 + 3b^4) / (5 \Pi P^*) + 3m^2 (1-u) (a^4 + 2a^2 b^2 + 5b^4) / (2P^*) + 32m^3 (1-u)^{3/2} (a^4 + a^2 b^2 + 8b^4) / (35 \Pi P^*)$$

$$H_2 = 1 + 6m(1+u)^{1/2} (a^4 + 2a^2 b^2 + b^4) / (\Pi P^*) + 3m^2 (1-u) (3a^4 + 4a^2 b^2 + 9b^4) / (4P^*) + 8m^3 (1-u)^{3/2} (3a^4 + 5a^2 b^2 + 12b^4) / (15 \Pi P^*)$$

$$H_3 = [ \{ 2 + 8m(1+u)^{1/2} + 3m^2 (1-u) / 2 + 16m^3 (1-u)^{3/2} / 15 \Pi \} \{ (a/b) (v/a^2 + b^2) \} + \{ 2 + 32m(1+u)^{3/2} / \Pi + 9m^2 (1+u) + 32m^3 (1-u)^{3/2} / 15 \Pi \} \{ (b/a) (1/a^2 + v/b^2) \} a^2 b^3 / P^*$$

$$H_4 = [ (1/ab) \{ 8m v (1-u)^{1/2} / (\Pi) + 3m^2 v (1-u) + 16m^3 (1-u)^{3/2} / (5 \Pi) \} + (b/a^3) \Pi \{ 16m(1-u)^{1/2} / (\Pi) + 9m^2 (1-u) + 16m^3 (1-u)^{3/2} / (5 \Pi) \} - (b/a) (1/a^2 + v/b^2) \{ 12m(1-u)^{1/2} / \Pi + 6m^2 + 8m^3 (1-u)^{1/2} / (\Pi) \} ] (a^3 b^3 / P^*)$$

Using the following non dimensional parameters

$$W = wh_0 / e_s a^2 ; \quad Q = qa^2 h_0 / 2D_0 e_s ; \quad \beta = b^4 / P^*$$

Equation (12) reduces to

$$(1-u)(1-\Omega) H_1 W_{,uuu} - 2(1-\Omega) H_2 W_{,uu} - (1-u) H_1 \Omega_{,u} W_{,uu} + H_3 \Omega_{,u} W_{,u} + (1-\Omega) H_4 = -\beta Q \dots\dots\dots(13)$$

and the expressions for  $e^2$  and  $\Omega_{,u}$  are as follows :

$$e^2 = (a^4/3) [ M W_{,u}^2 + N W_{,u} W_{,uu} + t^2 W_{,uu}^2 ] \dots\dots\dots(14)$$

$$\Omega_{,u} = P [ 2M W_{,u} W_{,uu} + N W_{,uu}^2 + N W_{,u} W_{,uuu} + 2t^2 W_{,uu} W_{,uuu} ] \dots\dots\dots(15)$$

$$P = [ (\lambda/4) (e^2 - 1) / e^5 ] / a^4 \dots\dots\dots(15, a)$$

The set of boundary conditions in nondimensional form are as follows:

$$W|_{u=0} = 0, \quad (1-u) W_{,uu} - (1+\nu)/2 W_{,u}|_{u=0} = 0, \quad (1-u) W_{,u}|_{u=1} = 0 \quad (16)$$

**METHOD OF SOLUTION**

The non linear governing differential equation (13) can be solved numerically using Finite difference and Range-kutta methods (BURDEN et al,1986). But here a completely new approach towards series solution for better approximation has been presented.

Let  $W = \sum_{j=3}^n A_j u^j$  .....(17)

The series starts at j = 3 as the approximate solution is to satisfy the boundary conditions. Substitution of the series solution in equation (13) yields the following residual:

$$R_1 = (1-u) k_1 H_1 \sum_{j=3}^n j(j-1)(j-2) A_j u^{j-3} - 2 k_1 H_2 \sum_{j=3}^n j(j-1) A_j u^j - k_2 H_1 (1-u) \sum_{j=3}^n j(j-1) A_j u^{j-2} + k_2 H_3 \sum_{j=3}^n j A_j u^{j-1} - k_1 H_4 + \beta Q \quad (18)$$

where  $k_1 = (1 - \Omega)$ ,  $k_2 = \Omega_u$

Also, the expressions for  $e^2$  and  $\Omega_u$  takes the forms:

$$e^2 = (a^4/3) [ M \{ \sum_{j=3}^n A_j u^{j-1} \}^2 + N \{ \sum_{j=3}^n A_j u^{j-1} \} \{ \sum_{j=3}^n j(j-1) A_j u^{j-2} \} + t^2 \{ \sum_{j=3}^n j(j-1) A_j u^{j-2} \}^2 ] \quad (19)$$

$$\Omega_u = P [ 2 M \{ \sum_{j=3}^n j(j-1) A_j u^{j-2} \} \{ \sum_{j=3}^n A_j u^{j-1} \} + N \{ \sum_{j=3}^n j(j-1) A_j u^{j-2} \}^2 + N \{ \sum_{j=3}^n A_j u^{j-1} \} \{ \sum_{j=3}^n A_j (j-1)(j-2) u^{j-3} \} + 2 t^2 \{ \sum_{j=3}^n A_j u^{j-2} j(j-1) \} \{ \sum_{j=3}^n A_j u^{j-3} j(j-1)(j-2) \} ]$$

To find the coefficients  $A_j$  for  $j = 3,4,5,.....n$ , we solve the following sets of equations in accordance with Galerkin's orthogonality procedure

$$\int_0^1 R_1(u) du = 0, \quad j = 3,4,5,.....n. \quad (21)$$

It is necessary to find the average value of  $e^2$  and  $\Omega_u$  over the domain of the ellipse given in equations (11) if the equations in (21) are to be solved. As a demonstration of the solution for a simple but dominating case (as it is converging)  $n=3$  and  $n=4$  are substituted into the above equations.

The results corresponding to (21) are as follows:

$$Q ( 11.9 K_1 - 1.83 K_2 )$$

$$A_3 = \frac{308 K_1^2 - 8.96 K_1 K_2 + 3.2 K_2}{Q (1.55 k_2 - 4.8 k_1)} \dots\dots\dots (22)$$

$$A_4 = \frac{(308 K_1^2 - 8.96 K_1 K_2 + 3.2 K_2^2)}{\dots\dots\dots} \dots\dots\dots (23)$$

The central deflections (u = 1) for this case is

$$W_0 = [(0.2 Q - 0.001) / (18.8 - 2.7 Q)]^{0.5} + [(0.02 Q - 0.008) / (18.8 - 1.09 Q)]^{0.5} \dots\dots\dots (24)$$

The numerical results for various aspect ratio (R = a/b) and the thickness variation parameter (m) are computed and shown in graphs (1 – 6). The results obtained using Runge-Kutta as well as Finite difference are in close agreement with the present results. Also, the results are in good agreement with those obtained by Mazumder for m = 0. The present results exhibits little variation with the results of TIMOSHENKO & WOINOWSKY (1953) for higher values of 'R'. For higher values of taper constant (m) the assumption of isodeflection contour lines may be a greater approximation for simply supported elliptic plate.

### Observations and conclusions

In figure 1, central deflection (W<sub>0</sub>) responses to static load (Q) for simply supported circular plate has been shown and it is concluded that the taper thickness has a considerable contributions to the static load which increases with the increase of the value of 'm', the taper thickness.

In figures 2 & 3 the same characteristics as in figure 1 have been drawn for an elliptic plate and it is observed that here also the required load for certain deflection increases with the increase in the value of 'm'.

In figure 4, the central plastic strain (e<sub>0</sub>) responses to static load for simply supported elliptic plate and circular plate have been drawn and found that in all cases central plastic strain increases with the increase of the value of 'm'.

In figures 5 & 6 the variation of plastic load with the changeover radius has been depicted and it is observed that the taper thickness has great influence on the response and increases with the value of the changeover radius.

From a literature survey it seems that the application of constant deflection contour lines method in I ii iii nonhomogeneous plastic plate is completely new. Though the validity of the method is long established till its application in solving nonuniform elastic and plastic plates and shells problems seem to be new.

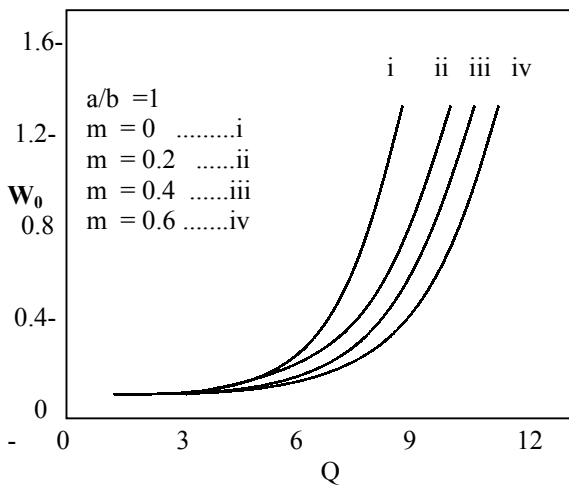


Figure - 1

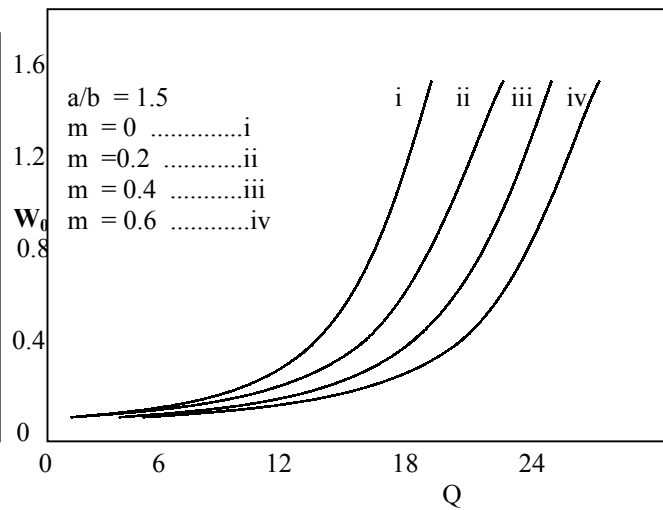


Figure - 2

Central deflection responses to static load for simply supported circular (Fig. -1) and elliptic plates (Fig. -2).

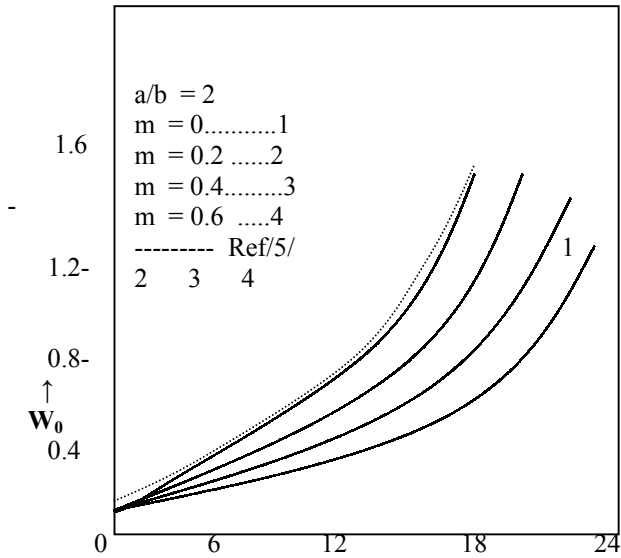


Figure -3 : Central deflection responses to static load (Q) for simply supported elliptic plate.

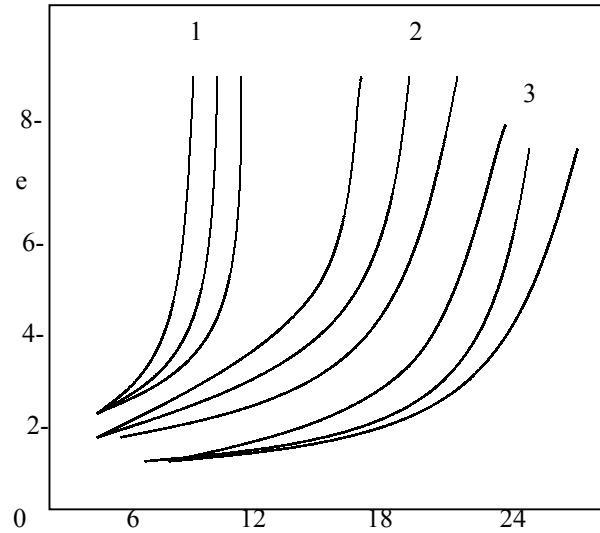


Figure -4: Central plastic strain ( $e_0$ ) responses to Static load for simply supported circular and elliptic plate.

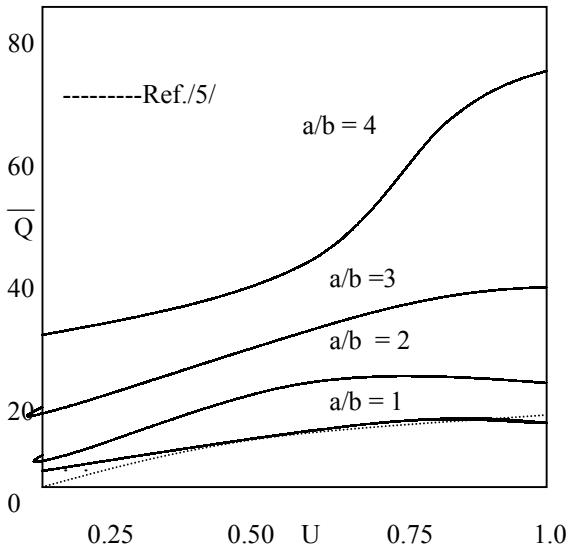


Figure - 5: Variation of plastic load (Q) with Changeover radius radius (U) for simply supported circular and elliptic plates

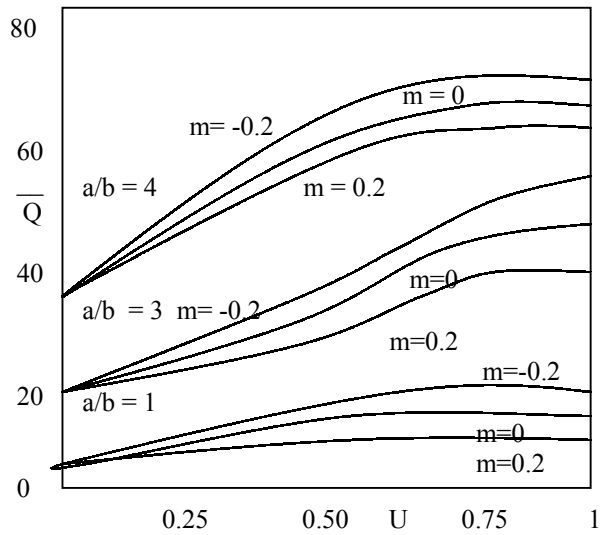


Figure 6: Variation of plastic load (Q) with changeover radius and taper constant 'm'.

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