A Two-Crown Finite Element Technique for the Determination of Tearing Modulus

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INTRODUCTION

The importance of approach to the subject of crack instability for the design of structures containing cracks has increased considerably over the last few years. The tearing modulus theory recently enunciated by Paris and co-workers [1] has emerged as one of the leading criterions for stable crack growth and for instability, and the estimation of T termed "Tearing modulus" in the theory has since been extensively investigated theoretically as well as experimentally [2, 3].
Analytical methods exist for calculating the tearing modulus of various crack configurations in simple-shaped structures under certain loading conditions. However, for arbitrary structures under general loading, more sophisticated calculation techniques are required.
Extending the virtual crack extension method introduced independently by Hellen [5] and Parks [4, 6], a new numerical approach for calculating the tearing modulus is presented hereafter and put in a form suitable for the instability analysis of structures containing one single crack or several interacting cracks. As it is well-known that the calculation of the energy release rate in elasticity by the virtual crack extension method is related to a stiffness derivative to which only a small region around the crack tip has a contribution [4,6], the technique described in the paper shows that it would be reasonable to evaluate the tearing modulus, or rather, the second derivative of potential energy with respect to the crack length, by means of two stiffness derivative calculations in two crowns around the crack tip. In particular, when one crown is strictly included in another one, computation is largely curtailed at this point with some saving of computer time, but a very accurate value of tearing modulus is obtained. As an interesting consequence, an another expression of the tearing modulus is carried out.In Section 4 : the classical tearing modulus is proved to be precisely equivalent to a line integral which is independent of integration path.
Numerical example using the proposed method is given in the final section.

ENERGY RELEASE RATE AND TEARING MODULUS CONCEPT

Consider a cracked structure subjected to a system of mechanical loads and suitable boundary conditions to produce equilibrium. In the loaded state, the total strain energy W is related to the energy release rate G by :

$$G = -\frac{dW}{da} \qquad (1)$$

for a small crack extensional opening area a, assuming the loading during this extension is not altered. Crack initiation occurs if G defined by Eq. (1) exceeds an experimentally determined critical value Gc. In such a case, without any load argument modification an investigation to the stability of the crack growth is required, i.e. to predict whether the crack propagation stops automatically or not.
Paris and al. [1] introduced a non-dimensional temperature-independent material parameter called the tearing modulus, T, that in general has the form :

$$T = \frac{E}{\sigma_0^2}\ \frac{dJ}{da} \qquad (2)$$

where E is the elastic modulus, σ_0 the flow stress and J is the path independent J-integral which is equal to G for elastic materials as well as materials following the deformation theory of plasticity. If Eq. (2) is evaluated using the J-integral R-curves of used material the resulting T is the material tearing modulus T_{mat}. If instead, dJ/da in Eq. (2) is calculated as the rate of change of crack drive or the applied J, per unit virtual crack extension, with the condition of applied forces, f, kept constant (or other similar conditions specified), the resulting T is the applied tearing modulus T_{app}. Following Ref. 1, instability will occur when :

$$T_{app} \geq T_{mat} \tag{3}$$

Therefore, crack growth stability analysis may be performed if dJ/da or the second derivative of potential energy with respect to crack length is obtained. On the other hand, by using standard finite element computer programs for cracked structures, it is possible to derive the stresses and displacements around the crack tip and a variety of methods are available for assessing the energy release rate G or J-integral, the best of which involve incontestably the virtual crack extension method. According to this method, when the loading of the cracked structure, is accomplished by surface tractions applied on other than the crack face, G or J may be evaluated by :

$$G = J = -\frac{1}{2}\ [U]^T \cdot \frac{\partial [K^A]}{\partial a} \cdot [U] \tag{4}$$

Here [U] is a vector of nodal displacements, the superscript "T" denotes transpose, $[K^A]$ a part stiffness matrix of all elements in a arbitrary chosen crown A around the crack tip and $\partial[K^A]/\partial a$ represents the change in the part stiffness matrix per unit crack advance.

TWO-CROWN FINITE ELEMENT TECHNIQUE

In the finite element model, a solid structure is represented by a system of nodes and elements with, at each node, two or three degrees of freedom. Let [K] be the master stiffness matrix of the structure, and let [U] be the total displacement vector. Then, in equilibrium, [U] is associated with [K] by equilibrium equation as :

$$[K] \cdot [U] = [f] \tag{5}$$

with [f] being the corresponding vector of equivalent nodal loads. We may differentiate Eq. (5) with respect to crack length, a, to carry out the displacement decrease rate δ [U] per unit crack advance δb :

$$\frac{\delta[K]}{\delta b} \cdot [U] + [K] \cdot \frac{\delta[U]}{\delta b} = \frac{\delta[f]}{\delta b} \tag{6}$$

However, for many problems, the crack face is free of tractions and body forces are neglected, thus [f] in Eq. (6) keeps constant during the infinitesimal crack length increments so that Eq (6) reduces, in this case, to :

$$\frac{\delta[U]}{\delta b} = -\ [K]^{-1} \cdot \frac{\delta[U]}{\delta b} \cdot [U] \tag{7}$$

where matrix $\delta[K]$ represents the change of the master stiffness matrix and $\delta[U]$ the corresponding change of the total displacements per unit crack advance. We can imagine accomodating an increment of crack advance with the mesh of Fig. 1 by rigidly translating all nodes on and within a contour Γ_{B^1} about the crack tip by an infinitesimal amount Δb in the x_1 - direction. All other nodes remain in their initial position. Thus the master stiffness matrix [K], which depends on only individual element geometries, displacement functions, and elastic material properties, remains unchanged in the regions interior to Γ_{B^1} and exterior to Γ_{B^2}, and the only contributions to Eq (7) come from the band of all elements between the contours Γ_{B^1} and Γ_{B^2} which form a new crown B.
Similarly, we differentiate Eq (4) in order to evaluate the change of G or J with respect to crack length when a infinitesimal virtual crack length increase, δb, occurs :

$$\frac{\delta G}{\delta b} = -\frac{1}{2}\left[\frac{\delta[U]^T}{\delta b} \cdot \frac{\partial[K^A]}{\partial a} \cdot [U] + [U]^T \cdot \frac{\delta\left(\frac{\partial[K^A]}{\partial a}\right)}{\delta b} \cdot [U] + [U]^T \cdot \frac{\partial[K^A]}{\partial a} \cdot \frac{\delta[U]}{\delta b}\right] \tag{8}$$

where we note that $[K^A]$ is symmetric, so that Eq. 8 become :

$$\frac{\delta G}{\delta b} = - [\frac{\delta[U]^T}{\delta b} \cdot \frac{\partial[K^A]}{\delta a} \cdot [U]] - \frac{1}{2} [[U]^T \cdot \frac{\delta(\frac{\partial[K^A]}{\partial a})}{\delta b} \cdot [U]] \quad (9)$$

The straightforward application of Eq. (9) to calculate the tearing modulus is very intricate. Let us consider, to simplify our calculation, a special situation of the crowns A and B : crown A (or B) is completely included in the rigidly translated domain of crown B (or A) (see Fig. 2). If B is located within the rigidly translated domain of A, $\partial[K^A]/\partial a$ keeps unchanged during the advance of Γ_{B1} since all of the nodes in the region exterior to Γ_{B2} remain in their initial geometrical position, while $\partial[K^A]/\partial a$ depends on only element's stiffness matrix in the crown A ; if instead, A is included in the rigidly translated region of B, $\partial[K^A]/\partial a$ remains also unaltered before/after a rigid translation of Γ_{B1} since the relative positions among the node points located in the domain of the crown A keep unchanged. Consequently, the second term of Eq (9) in these two special cases vanishs, so that the tearing modulus may be calculated by :

$$T = - \frac{E}{\sigma_0^2} [\frac{\delta[U]^T}{\delta b} \cdot \frac{\partial[K^A]}{\partial a} \cdot [U]] \quad (10)$$

From the fact that in finite element method the derivative calculation is in general approximated by a simple forward finite difference, thus by combining Eq. (7) and Eq. (10), we obtain :

$$T = \frac{E}{\sigma_0^2} [(K^{-1} \cdot \frac{\Delta[K]}{\Delta b} \cdot [U])^T \cdot \frac{\Delta[K^A]}{\Delta a} \cdot [U]] \quad (11)$$

in which we note that the only contribution to $\Delta[K]/\Delta b$ comes from the elements located in the crown B, thus we have :

$$T = \frac{E}{\sigma_0^2} \cdot \frac{1}{\Delta a \cdot \Delta b} \left[[U]^T \cdot (\Delta[K^B] \cdot [C] \cdot \Delta[K^A]) \cdot [U] \right] \quad (12)$$

where $[C] = [K]^{-1}$ represents the master compliance matrix of the cracked structure.
Obviously, Eq. (12) defines a symmetric value T with respect to the crowns A and B, i.e. T (Δa, Δb) = T (Δb, Δa), and Eq. (12) holds for any structure with only mechanical loading. Since thermal strains give rise to thermal loads, a modification is required to subtract the thermal tearing modulus. Moreover, since the compliance matrix C is seldom embeded in finite element programes for the displacement - stress solutions, a numerical calculation scheme is recommended as follows :

$$\text{Let } [C] \cdot \Delta[K^B] \cdot [U] = [\tilde{u}] \quad (13)$$

$$\text{then } [K] \cdot [\tilde{u}] = \Delta [K^B] \cdot [u] \quad (14)$$

Therefore, having the displacement solution, [u], of Eq. (5) and the part stiffness variation $\Delta[K^B]$ generated by rigidly shifting all nodes on and within the contour Γ_{B1} (fig. 1) in an amount of Δb, we can calculate the multiplication $\Delta[K^B] \cdot [u]$, and that forms a virtual force, $[f^B]$, acting on the crown B, thus Eq. (14) reduces to :

$$[K] \cdot [\tilde{u}] = [f^B] \quad (15)$$

and $[\tilde{u}]$ is obtained by resolving Eq. (15) by classical finite element programs, and, in this case, the tearing modulus is written as :

$$T = \frac{E}{\sigma_0^2} \cdot \frac{1}{\Delta a \cdot \Delta b} \left[[\tilde{u}]^T \cdot \Delta[K^A] \cdot [u] \right] \quad (16)$$

which is very similar to the Parks' formula (4) used to calculate the energy release rate.

TEARING MODULUS ASSOCIATED WITH PATH INTEGRAL

In the absence of surface tractions on the crack face, it is easy to demonstrate that, for constant strain triangular elements, the tearing modulus defined by Eq. (12) or (16) is precisely equal to a line integral termed D-integral :

$$D = \frac{E}{\sigma_0^2} \left[\frac{1}{2} \int_\Gamma \mathrm{Tr} \, (\tilde{\sigma} \cdot \frac{\partial u}{\partial x} + \sigma \cdot \frac{\partial \tilde{u}}{\partial x}) \, dy - \int_\Gamma (\sigma \cdot \vec{n} \cdot \frac{\partial \tilde{u}}{\partial \alpha_1} + \tilde{\sigma} \cdot \vec{n} \cdot \frac{\partial u}{\partial \alpha_1}) \, dS \right] \quad (17)$$

Here Γ is a curve surrounding the crack tip and situated between the crowns A and B, Tr denotes the trace operator of a matrix coming from the multiplication of stress σ (or $\tilde{\sigma}$) by deformations $\partial \tilde{u}/\partial x$ (or $\partial u/\partial x$), while $\partial \tilde{u}/\partial x$ ($\partial u/\partial x$) indicates a derivative matrix of the components of $\tilde{u}$ (or u) with respect to the coordinates x_1 and x_2, and σ, $\tilde{\sigma}$ are associated with displacement solution, [u], as :

$$[\sigma] = [D] . [B] . [u] \quad (18)$$

$$[\tilde{\sigma}] = [D] . [\Delta[B^B] - [B] . K^{-1} . \Delta [K^B]] . [u] \quad (19)$$

with :

[D] = elasticity matrix of cracked structure ;
[B] = matrix transforming the displacement fields, u, into deformation fields ;
$\Delta[B^B]$ = change of matrix [B] due to a slight rigid translation of all inner border node points of crown B in an amount of Δb.

The integral defined by Eq. (17) is evaluated in a contraclockwise sense starting from the lower flat crack surface and continuing along the path Γ to the upper flat surface with $\vec{n}$ being the unit outward normal to Γ. We have not any mathematical difficulty to prove that when Γ is placed between the two crowns A and B, the value of D-integral is null for any closed Γ in a two-dimensional body of linear elastic material free of body forces as well as of surface tractions on crack face. Therefore, D represents a line path integral, having the same value for all contours surrounding the crack tip. This point of view will also be verified by a simple exemple using the proposed method.

STABILITY OF A SYSTEM OF INTERACTING CRACKS

In the case of a structure containing several cracks, a system of straight cracks may interact, and the generalization of such a problem has been discussed by Nemat-Nasser [7]. As a consequence, it has been pointed out that in the absence of crack branching, the stability of crack growths is controlled by the sign of the second variations of the total potential energy. Let us consider a solid containing 2 cracks of l_1 and l_2, and W denotes its total potential energy being thought of as functions of each crack length. For admissible variation $\delta l_i \geq 0$ (i = 1, 2), Nemat-Nasser's criterion is writen as :

$$\frac{\partial^2 W}{\partial l_1 \, \partial l_2} . \delta l_1 \, \delta l_2 \left| \begin{array}{ll} > 0 & \text{stable} \\ = 0 & \text{critical} \\ < 0 & \text{unstable} \end{array} \right. \quad (20)$$

The two-crown method previously presented still holds for calculating the matrix of second derivatives of potential energy :

$$\frac{\partial^2 W}{\partial l_i \, \partial l_j} = - [[\tilde{u}_i] . \Delta[K_j^A] . [u]] \quad (21)$$

with $[\tilde{u}_i] = K^{-1} . \Delta [K_i^B] . [u]$, and i, j = 1, 2

NUMERICAL RESULTS

The procedure outlined in the previous section was applied to a center-cracked plan strain specimen having a unit thickness as shown in Fig. 3, and because of symmetry only one-fourth of the specimen is analyzed using a mesh of focussed isoparametric quadrilateral and constant strain triangular elements. With this example we want to :

1) Verify the independence of the tearing modulus (or D-integral) with respect to integration contour. To reach this goal, we have defined two groups of contours, and each group contains five contours surrounding the crack tip. As it is shown in Fig. 4, the first group of contours is reduced in a very small region near the crack tip, and the second is relatively remote from it. Numerical results using the proposed method for these two types of contours are summarized in Table 1 in which two small crack length increments Δa = 0.0002 mm, Δb = 0.0001 mm are taken into account in the calculations.

TABLE 1 : TWO-CROWN FINITE ELEMENT METHOD

First group of contours	Analytical solution	Ra/l x 100 = 0.20 Rb/l x 100 = 0.52	0.52 4.51	1.12 2.46	2.46 0.52	7.62 1.12
dJ/da (kgf/mm^2)	1.825	1.985	1.739	1.740	1.735	1.756
Error per cent %		8.7	4.7	4.6	4.9	3.8
Second group of contours	Analytical solution	Ra/l x 100 = 30.79 Rb/l x 100 = 47.17	47.17 72.81	58.64 47.17	2.46 0.52	7.62 1.12
dJ/da (kgf/mm^2)	1.825	1.777	1.739	1.740	1.735	1.756
Error per cent %		2.6	2.5	2.5	2.3	2.3

with l = crack length = 50 mm
Ra, Rb = average radius of crowns A and B
Because of the more accurate knowledge of the displacement vector with increasing distance from the crack tip, it is obvious from TABLE 1 that the results for the second group of contours is more stable and more accurate than the first one.

2) When the relative position of the two crowns A and B is fixed, the second aim of present example is to verify the independence of the tearing modulus defined by Eq. (12) or Eq. (16) with respect to the two slight crack length icrements Δa and Δb. Hence, a series of Δa and Δb have been assumed, and TABLE 2 gives corresponding numerical results when fixed crowns Ra/l x 100 = 90.32, Rb/l x 100 = 72.81 have been considered.

TABLE 2 : INFLUENCE OF Δa, Δb ON dJ/da CALCULATION

dJ/da; Δb mm / Δa mm	4×10^{-2}	5×10^{-3}	1×10^{-4}	1×10^{-5}
2×10^{-2}	1.773	1.779	1.779	1.779
2×10^{-3}	1.777	1.782	1.783	1.783
2×10^{-5}	1.777	1.782	1.783	1.783

This table shows that dJ/da calculated by two-crown method is independent of the selections of the two slight crack length increments Δa and Δb.

CONCLUSION

Two-crown finite element technique for the tearing modulus determination has been developed in this paper. The method, based on virtual crack extension technique, does not require a second solution of a slightly different crack length, and the present formulation is readily included as an integral part of an elastic finit element program, but it can also conveniently be used in a post-processing program which uses the displacements from a finite element analysis as input for the tearing modulus calculation, and therefore, a subsequent analysis to crack growth stability. As a consequence, it has been shown that the tearing modulus, or rather, the second derivatives of potential energy, is precisely equal to a path independent integral : D-integral.
A simple two-dimensional example of the method was presented, giving the tearing modulus within a few per cent of accepted values.

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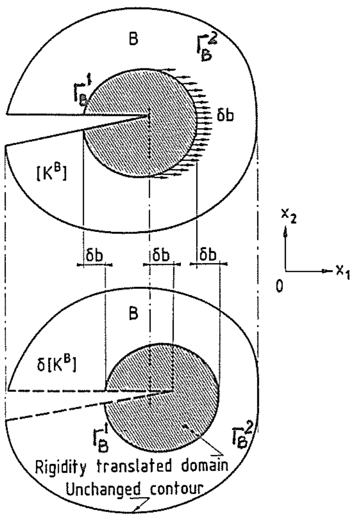

Figure 1

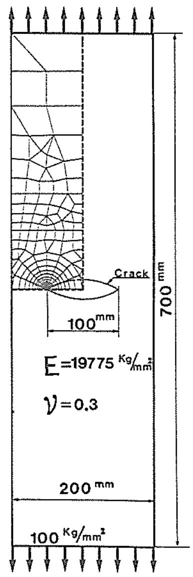

Figure 3

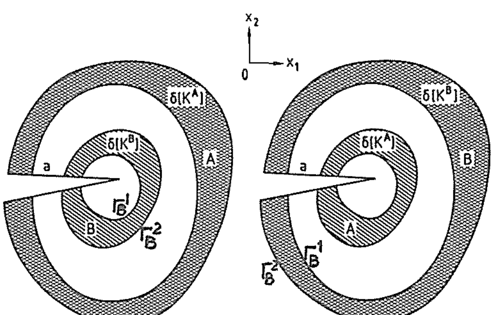

Figure 2

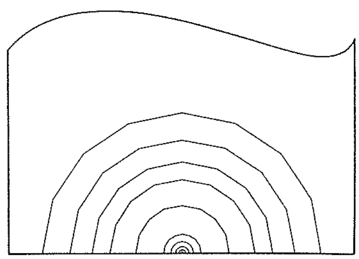

Figure 4