

COMPLEX STIFFNESS FORMULATION FOR THE FINITE ELEMENT ANALYSIS OF ANISOTROPIC AXISYMMETRIC SOLIDS SUBJECTED TO NONSYMMETRIC LOADS

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SUMMARY

This paper describes the development of an improved semi-analytical finite element for the stress analysis of anisotropic axisymmetric solids subjected to nonsymmetric loads. Orthogonal functions, in the form of finite Fourier exponential transforms, which satisfy the equations of equilibrium of the theory of elasticity for an anisotropic solid of revolution, are used to expand the imposed loadings and displacement field. It is found that the orthogonality conditions for the assumed solution reduce the θ -dependency, thus reducing the three dimensional problem to an infinite series of two dimensional problems.

The classical finite element approach has traditionally involved the expansion of the imposed loadings and the displacement field into either a Fourier sine or cosine series. Even though both series are orthogonal, it is demonstrated that when substituted into the equations of equilibrium of the theory of elasticity for an anisotropic solid of revolution, the θ -dependency cannot be eliminated due to the presence of the D_{15} , D_{16} , D_{25} , D_{26} , D_{35} , D_{36} , D_{45} and D_{46} terms in the constitutive matrix. Therefore, it is shown that the semi-analytical approach using the half Fourier series expansion is only correct when the above terms are identically zero. This is true for all isotropic and orthotropic materials. Although the above condition is not new, the classical constant strain element has been derived and used even though the stiffness matrix is devoid of terms containing D_{15} , D_{16} , D_{25} , D_{26} , D_{35} , D_{36} , D_{45} , and D_{46} . These terms do not appear in the stiffness matrix due to the orthogonality of the sine and cosine functions. Thus, the assumed solution does not satisfy the equations of equilibrium for an anisotropic solid of revolution but does produce a stiffness matrix which is independent of the constitutive coefficients associated with the anisotropy.

This paper describes a formulation which satisfies the equations of equilibrium for an anisotropic solid of revolution and produces a stiffness matrix which depends on the constitutive coefficients associated with the anisotropy. The formulation uses a full Fourier series in terms of exponential functions of imaginary arguments to expand the loadings and displacement field. With this series, it is shown that the stiffness matrix is given by the volume integral of $[\bar{B}]^T [D] [B]$ where $[\bar{B}]$ is the complex conjugate of $[B]$. The resulting matrix although not symmetric is Hermitian. Appropriate transformations to expand the Hermitian matrix to a form suitable for use in a standard structural analysis program is given.

Quadrilateral elements formed from the constant strain triangles developed for the classical and improved elements were compared in a series of numerical experiments. It is shown that the results for the isotropic and orthotropic cases are identical. However, for anisotropic materials the results between the classical and the improved theory are shown to be different by as much as 52.8 percent. Numerical studies were made for a glass/epoxy composite, and for a graphite/epoxy composite. Angles of anisotropy of 0, 15, 30, 45, 60, 75 and 90 degrees were studied for harmonic numbers of one through five for a series of cylindrical and conical shells. Numerical experiments, involving the refinement of the mesh size, indicate that the classical element displacements do not converge to those for the improved theory presented in this paper.

The last element considered in this study was obtained from the application of the finite Fourier exponential transform to an isoparametric element. Numerical experiments show that this element compares very favorably with the results from a shell analysis which includes the transverse shear effects.

1. Introduction

The stress analysis of many pressure vessels is performed by using computer programs based on axisymmetric finite element theory. The popularity of this approach is due to the fact that this method provides the stress analyst with a design tool that simplifies the modeling of nonsymmetrically loaded solid of revolution structures having a complex geometry in the meridional plane. The finite element method used to perform these analyses was initially formulated by Wilson [1]. Wilson's pioneering work is the basis for many computer programs that have been used to perform structural analysis of axisymmetric solids with symmetric loads, [2, 3] and nonsymmetric loads [4, 6].

Wilson's solution of axisymmetric solids with nonsymmetric loads was performed by expanding the assumed displacement field into a half Fourier series. Due to the orthogonality of the sine and cosine functions, the θ dependency is eliminated from the finite element equations resulting in a two dimensional analysis. This approach has had considerable application for structures fabricated from isotropic or orthotropic materials as well as anisotropic materials. However, for anisotropic materials this formulation does not satisfy the equilibrium equations of the theory of elasticity. For anisotropic materials, it will be shown that significant errors can occur in the analysis and that the refinement of the element mesh does not produce convergence to the true solution.

This problem is not a new one, it is reported in the finite element book by Cook [7]. For anisotropic circular plates and shells of revolution, Padovan and Lestingi [8-10] made the same observation and solved the problem by expanding the assumed displacement field in a full Fourier series in the form of the finite exponential Fourier transform. Padovan further extended this work to the analysis of stresses, heat conduction, dynamics and stability for various finite element applications [11-13].

The main emphasis of this paper is twofold: (1) to compare two quadrilateral finite elements formed from four constant strain triangles to determine the magnitudes of the error introduced by not satisfying the equilibrium equations and to assess the relative convergence characteristics of each element; (2) to compare a higher order finite element based on the finite Fourier exponential transform with the results from a shell theory solution. In addition, a short description of the background theory for the finite Fourier exponential transform formulation will be given. More detailed accounts can be found in [14] and [8-13].

2. Formulation

In the finite element method, equations which approximate the equilibrium configuration of a structure are formulated using techniques such as the minimum potential energy or virtual work theorems. It follows that the best approximations are those which satisfy the equations of equilibrium. For the anisotropic solid of revolution subjected to nonsymmetrical loadings, the stress-strain relationship is given by the following:

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \\ \tau_{r\theta} \\ \tau_{z\theta} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{z\theta} \end{bmatrix} \quad (1)$$

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For isotropic and orthotropic materials, D_{15} , D_{16} , D_{25} , D_{26} , D_{35} , D_{36} , D_{45} , and D_{46} are identically zero.

Under general loading conditions, the equilibrium equations [14] are functions of the D_{ij} in eq. (1) as well as of r , z , and θ , i.e.,

$$f(r, z, \theta) = 0 \tag{2}$$

The usual approach in solving boundary value problems of the type represented by eq. (2) is to assume a separating function, $\phi(r, z, \theta)$, such that when it is substituted into eq. (2), the result is

$$F_1(\theta) F_2(r, z) = 0 \tag{3}$$

If it can be shown that

$$F_1(\theta) \neq 0, \tag{4}$$

then it follows that

$$F_2(r, z) = 0, \tag{5}$$

thereby reducing the complexity of the analysis by one degree.

Wilson [1] proposed the following function for $\phi(r, z, \theta)$:

$$\begin{aligned} u(r, z, \theta) &= \sum_{n=0}^N u_n(r, z) \cos n \theta \\ v(r, z, \theta) &= \sum_{n=0}^N v_n(r, z) \cos n \theta \\ w(r, z, \theta) &= \sum_{n=0}^N w_n(r, z) \sin n \theta \end{aligned} \tag{6}$$

Letting $n=k$ in eq. (6) and then substituting into the equilibrium equations results in equations of the form

$$\phi_1(r, z) \cos k\theta + \phi_2(r, z) \sin k\theta = 0 \tag{7}$$

Eq. (7) cannot be separated in the manner described above due to the presence of the anisotropic terms D_{15} , D_{16} , D_{25} , D_{26} , D_{35} , D_{36} , D_{45} and D_{46} . If these terms are set equal to zero, the equilibrium equations reduce to the form of eq. (2). This result was recognized by Crose [15] and Cook [7].

Padovan and Lestingi [8-10] and Padovan [11-13] suggested a solution in the following form for plates and shells problems and various finite element applications for anisotropic materials:

$$\begin{aligned} u(r, z, \theta) &= \sum_{n=-N}^N u_n(r, z) e^{in\theta} \\ v(r, z, \theta) &= \sum_{n=-N}^N v_n(r, z) e^{in\theta} \\ w(r, z, \theta) &= \sum_{n=-N}^N w_n(r, z) e^{in\theta} \end{aligned} \tag{8}$$

Substitution of the k^{th} harmonic of eq. (8) into the equilibrium equations results in equations of the form

$$\phi_1(r, z) e^{ik\theta} = 0 \tag{9}$$

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Since eq. (9) is in the form of eq. (3), eq. (9) automatically satisfies the equilibrium equation for any material having a fully populated material stiffness matrix as given in eq. (1).

3. Solid of Revolution Elements

In this paper, three finite elements for the stress analysis of axisymmetric solids subjected to nonsymmetric loads are considered:

1. A quadrilateral formed from four constant strain triangles based on the half Fourier series.
2. A quadrilateral formed from four constant strain triangles based on the finite exponential Fourier transform.
3. An incompatible element based on the finite exponential Fourier transform.

The first of these elements is the one which was proposed by Wilson [1] and has been extensively used in the ASAAS computer program which was written by Crose [16]. The derivation of this element and each term in the stiffness matrix of the triangular element is given in [15] and [16]. The element stiffness matrix is derived using the equation

$$[K^e] = \int_V [B]^T [D] [B] dV \quad (10)$$

In this derivation Crose used an elasticity matrix as given in eq. (1) with all anisotropic terms set to zero. Frater [14] derived the same stiffness matrix using the elasticity matrix of eq. (1). In this derivation all integrals that contained a D_{15} , D_{16} , D_{25} , D_{26} , D_{35} , D_{36} , D_{45} or D_{46} are zero because of the orthogonality property of the sine and cosine functions.

The second finite element is that proposed by Padovan [11]. The detailed analysis and derivation of the stiffness matrix and each term in the stiffness matrix are given by Frater [14]. For this element, the stiffness matrix is given by

$$[K^e] = \int_V [\bar{B}]^T [D] [B] dV \quad (11)$$

where $[\bar{B}]$ is the complex conjugate of $[B]$. The resulting equilibrium equations are

$$\{F\} = [K]\{\delta\} \quad (12)$$

where the stiffness matrix is Hermitian and the force and displacement vectors are, in general, complex variables. The real and imaginary parts represent the symmetric and anti-symmetric loadings and displacements.

The third element is an improved version of the second element. Rather than using four constant strain triangles to form the stiffness matrix, this element is an isoparametric element having two incompatible displacement modes as described by Wilson, et al [17].

4. Numerical Results

In this section, numerical results are presented which illustrate the previously discussed theory. For these numerical examples, the cantilevered cylindrical shell shown in Figure 1 is used with the following materials:

through 14 show the radial displacement at the free end of the cylinder as a function of the angle of anisotropy. The first five of these figures are for each of the five Fourier harmonics using the glass/epoxy material while the remaining five are for the graphite epoxy material. In all cases, these graphs illustrate the displacements for elements 1 and 2 and the percent difference between them using element 2 as a reference.

From these results, it can be observed that in all cases the 0° and 90° solutions are identical. Since these orientations represent orthotropic material properties, this observation agrees with the theory presented earlier in this paper.

In all cases the curves given by element 2 predict larger displacements than those given by element 1. This agrees with the potential energy theory. The largest percent difference for the glass/epoxy and graphite/epoxy cylinders was 26.4 and 52.8 percent respectively. From this observation, it can be concluded that the larger the ratio of E_{22} to E_{11} , the larger the error that will be introduced by using the classical finite element (element 1) rather than the theory presented in this paper (element 2).

5 Conclusions

The results given in this paper demonstrate that for solids of revolution constructed of anisotropic materials it is necessary to use a finite element formulation based on the finite Fourier exponential transform. It has been shown that as the ratio of E_{22} to E_{11} is increased, the error increases. It is also apparent that no degree of mesh refinement for the classical half Fourier series can produce correct results. In addition to these observations it should be mentioned that the computer code used in this analysis has the following advantages:

1. Half the computer storage requirements for the local and global stiffness matrices when the formulation is based on a complex formulation;
2. Significantly lower running costs, particularly for larger problems;
3. Higher numerical resolution for the same word size machine;
4. Overall program coding is simpler.

6. References

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| <p>1. Glass/Epoxy</p> <p>$E_{11}=E_{33}=1,425,000$ psi</p> <p>$E_{22}=5,255,000$ psi</p> <p>$\nu_{23}=\nu_{21}=0.261$</p> <p>$\nu_{12}=\nu_{32}=0.071$</p> <p>$\nu_{13}=0.19$</p> <p>$G_{23}=G_{12}=497,000$ psi</p> <p>$G_{13}=644,000$ psi</p> | <p>2. Graphite/Epoxy</p> <p>$E_{11}=E_{33}=1,700,000$ psi</p> <p>$E_{22}=25,000,000$ psi</p> <p>$\nu_{21}=\nu_{23}=0.300$</p> <p>$\nu_{12}=\nu_{32}=0.020$</p> <p>$\nu_{13}=0.014$</p> <p>$G_{23}=G_{12}=650,000$ psi</p> <p>$G_{13}=849,000$ psi</p> |
|---|--|

A number of numerical calculations were performed to compare:

1. The finite element results with shell theory results.
2. The convergence characteristics of elements 1 and 2.
3. Tip deflection with varying angles of anisotropy for Fourier harmonics 1 through 5.

4.1 Comparison with Shell Theory

Figure 2 shows the radial displacement as a function of axial position at $\theta=0$ for the cantilevered cylinder composed of a glass/epoxy material which is oriented at an angle of 30° with the shell axes. The shell solutions were obtained by Kennedy and Lestingi [18]. These results indicate that the deflections predicted by element 1 are, in all cases, less than those predicted by the other elements and are in general stiffer than the actual structure. These results are in accordance with the potential energy theory. The deflections for element 2 represent a definite improvement over element 1. In fact, most of the points plotted are so close to the higher order element result that it is difficult to distinguish between the two sets of points.

Finally, the displacements for element 3 follow the shell theory curves better than the previous two elements. This result is to be expected of a higher order element.

4.2 Element Convergence Characteristics

This study was performed to determine if the displacements obtained with element 1 would converge to those obtained with element 2. Three finite element meshes were considered:

1. The finite element mesh defined in Figure 1.
2. The 0.4 in length of the element was changed to 0.2 in and 100 elements were used.
3. The 0.2 in was maintained, however, two elements were used through the shell thickness resulting in 200 elements being used.

For this study the graphite/epoxy material was used with anisotropic angles of 30° and 60° degrees with a harmonic number of 1. Figures 3 and 4 demonstrate that two elements do not converge to the same solution.

4.3 Parametric Study

An important question to be answered in this study was what are the effects of the material properties, the Fourier harmonic number, and the angle of anisotropy on the results as given by element 1 (half Fourier series) and element 2 (finite Fourier exponential transform).

The Fourier harmonics numbers which were considered are one through five. Figures 5

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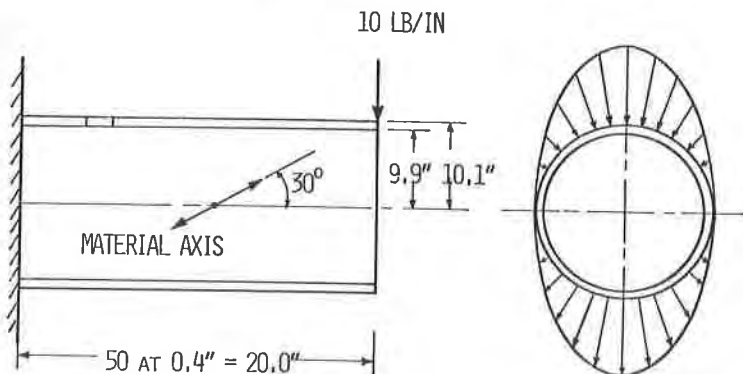


Fig. 1 Anisotropic Cantilevered Cylindrical Shell

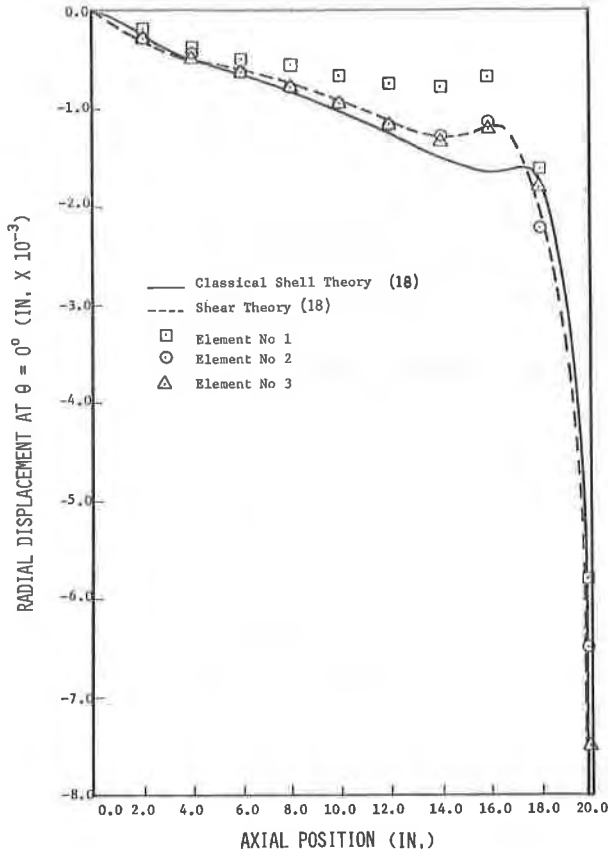


Fig. 2 Radial Displacement for Shell in Figure 1

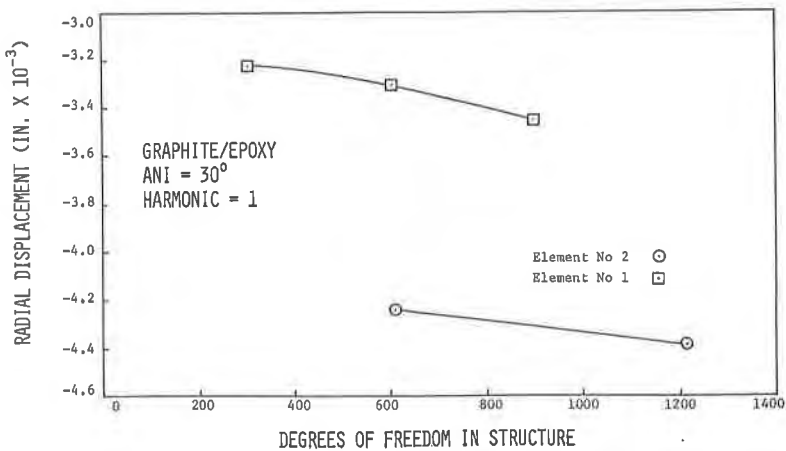


Fig. 3 Element Convergence Characteristics for ANI = 300

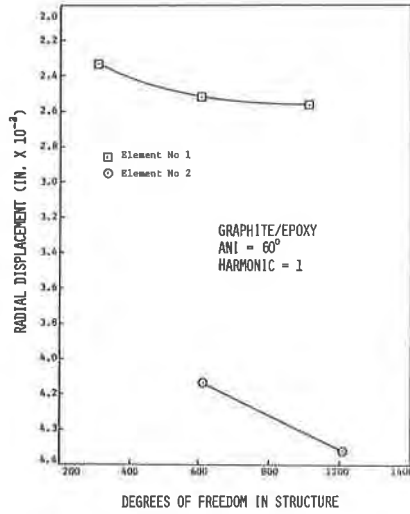


Fig. 4 Element Convergence Characteristics for ANI = 60°

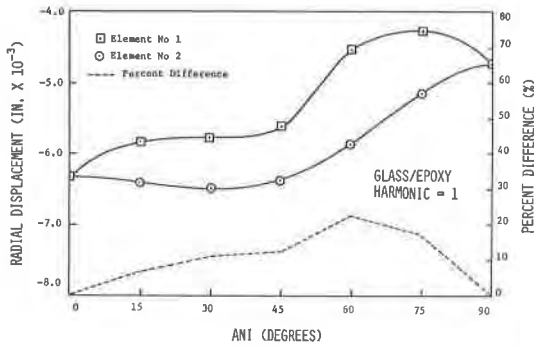


Fig. 5 Free End Radial Displacement versus Angle of Anisotropy for Glass/Epoxy Material with N = 1

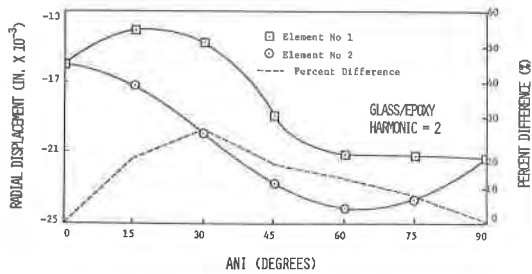


Fig. 6 Free End Radial Displacement versus Angle of Anisotropy for Glass/Epoxy Material with N = 2

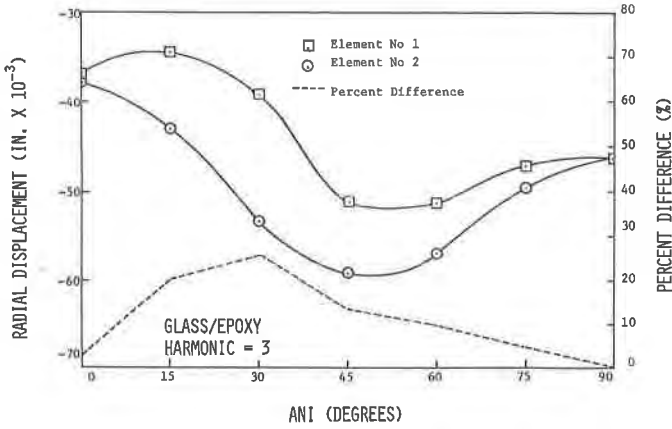


Fig. 7 Free End Radial Displacement versus Angle of Anisotropy for Glass/Epoxy Material with N = 3

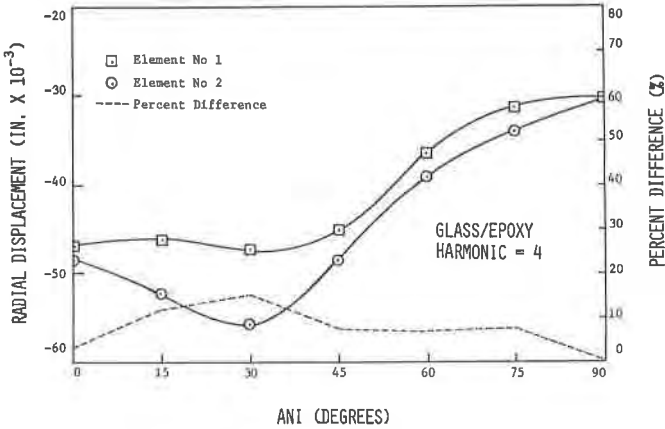


Fig. 8 Free End Radial Displacement versus Angle of Anisotropy for Glass/Epoxy Material with N = 4

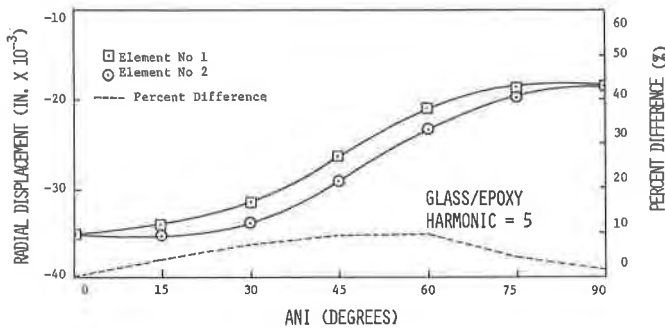


Fig. 9 Free End Radial Displacement versus Angle of Anisotropy for Glass/Epoxy Material with N = 5

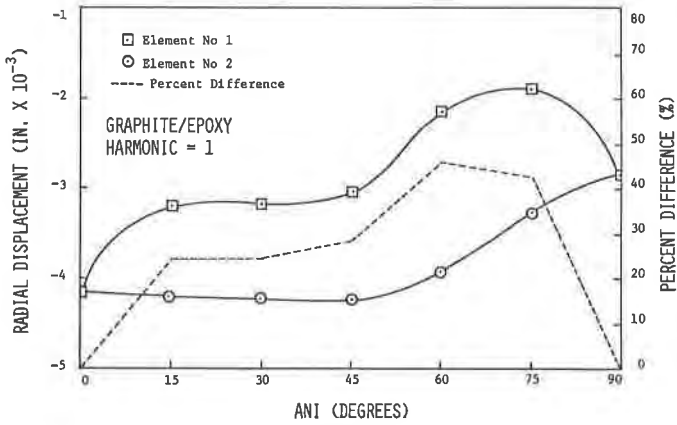


Fig. 10 Free End Radial Displacement versus Angle of Anisotropy for Graphite/Epoxy Material with N = 1

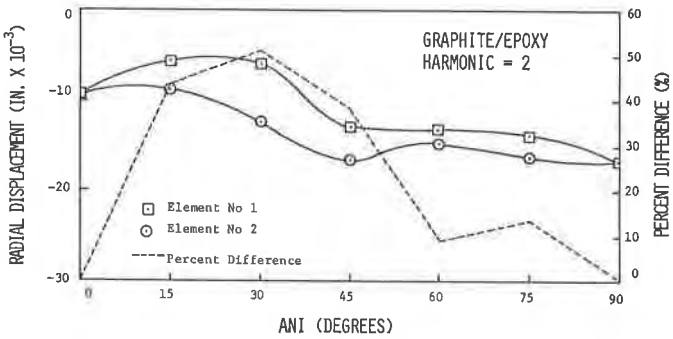


Fig. 11 Free End Radial Displacement versus Angle of Anisotropy for Graphite/Epoxy Material with N = 2

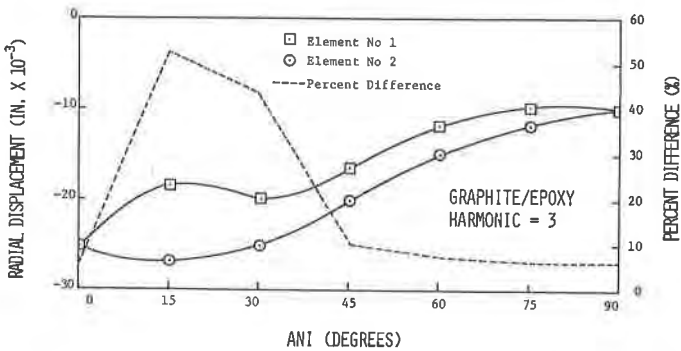


Fig. 12 Free End Radial Displacement versus Angle of Anisotropy for Graphite/Epoxy Material with N = 3

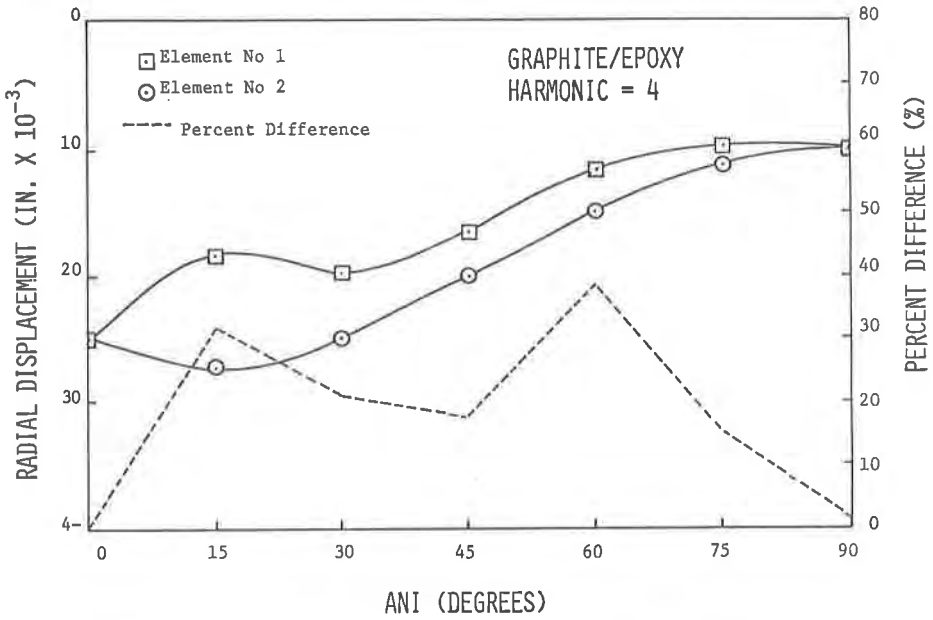


Fig. 13 Free End Radial Displacement versus Angle of Anisotropy for Graphite/Epoxy Material with N = 4

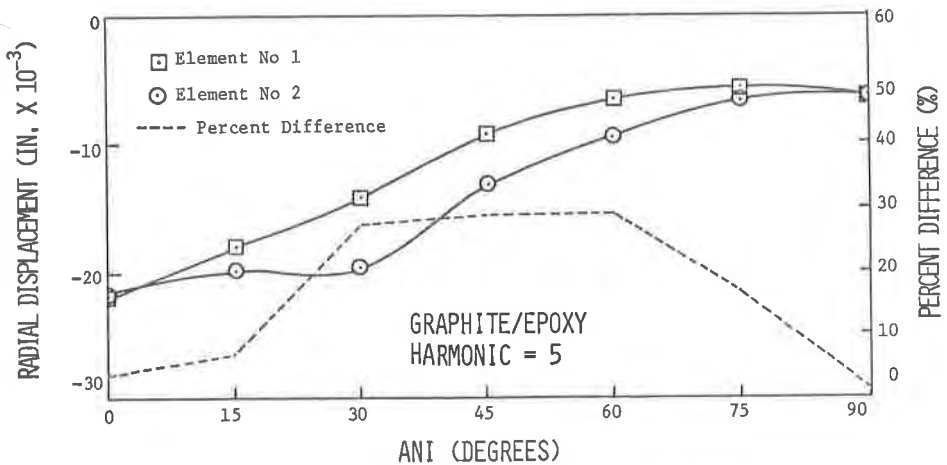


Fig. 14 Free End Radial Displacement versus Angle of Anisotropy for Graphite/Epoxy Material with N = 5